Wang Tiles

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Abstract

Wang tiles are a type of theoretical domino, proposed by Hao Wang in 1961. Two related problems were associated with them when they were first proposed. Firstly, deciding whether an arbitrary set could tile the plane (the *Domino Problem*), and secondly whether any sets existed which could tile the plane only aperiodically. These were solved in 1966 by Robert Berger, who showed that the Domino Problem was undecidable, and produced the first set of aperiodic tiles. This proof involved devising a method by which Wang tiles could simulate Turing machines, hence also demonstrating that Wang tiles are capable of computation. We show some examples of computations done with Wang tiles, then consider hexagonal Wang tiles, and how they can be used to carry out such computations.

1 Introduction

Wang tiles are a deceivingly simple concept. Wang tiles were first designed by the mathematician Hao Wang in 1961 [1]. A Wang tile, as show in Figure 1, is a square tile with a colour assigned to each side. A Wang tiling, as shown in Figure 3, is a planar area covered by Wang tiles arranged in a grid. We will use a coordinate system to specify which tile in this grid we refer to. Each tile has a

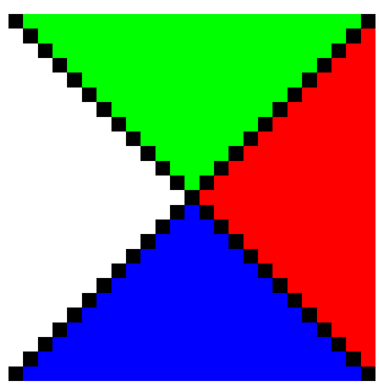

Figure 1: A Wang tile.

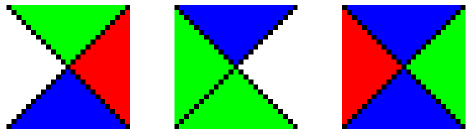

Figure 2: A set of Wang tiles.

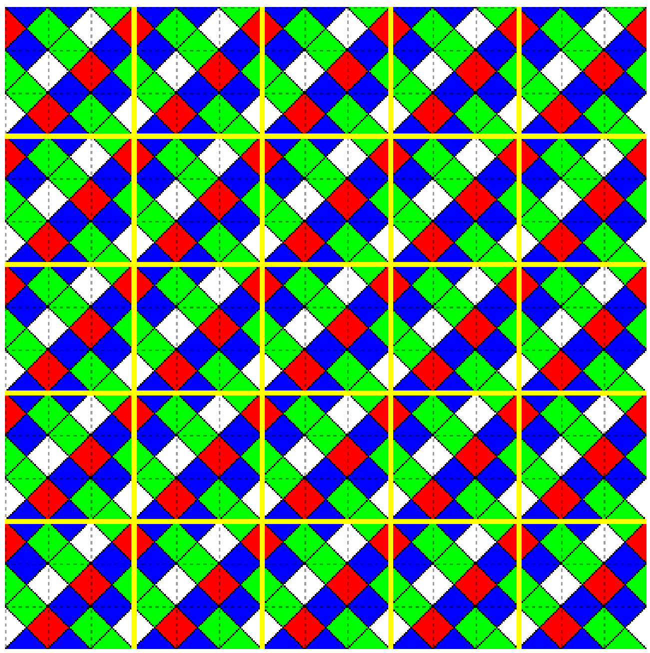

Figure 3: A Wang tiling, from the set shown in Figure 2 with cyclic rectangles highlighted.

position (x, y) where x and y are integers. x increases moving left within a row, and y increases moving down a column. (The y axis here is flipped compared to convention, to better match the way we shall define certain sets.) Tiles in a Wang tiling must follow certain rules:

- Tiles in a tiling must all be part of a predetermined set;
- Tiles in the set can occur any number of times in the tiling;
- Tiles cannot be rotated or reflected;
- A tiling may not have any unfilled gaps;
- Adjacent tiles must have matching colours on adjacent edges.

Wang tilings can be finite or infinite. An infinite tiling 'tiles the plane' if it covers the entire plane. Tilings of the plane can be classified as periodic or aperiodic. A tiling over the plane is *periodic* if there exist some integers, i, j such that the tiles in positions (x, y) and $(x + i, y + j)$ are the same for any integers x, y. This is equivalent to the tiling being made up of repeating $i \times j$ 'cyclic rectangles', with matching edge colours along the top and bottom and along the sides, for example the identical 3×3 cyclic rectangles shown in Figure 3. If a tiling of the plane is not periodic, then it is *aperiodic*. A set of Wang tiles is periodic if there is any tiling of the plane with that set which is periodic. If a set of Wang tiles can tile the plane, but not periodically, then that set is aperiodic.

2 Aperiodicity and Turing Machines

When Wang first proposed his tiles, he also proposed a conjecture (referring to tiles as plates, and a set as solvable if it tiles the plane):

Conjecture 1 *A finite set of plates is solvable (has at least one solution) if and only if there exists a cyclic rectangle of the plates; or, in other words, a finite set of plates is solvable if and only if it has at least one periodic solution.*

He also stated that

> "If [the conjecture] is true, we can decide effectively whether any given finite set of plates is solvable."[1]

This decision problem of whether a given set of Wang tiles can tile the plane became known as the Domino Problem. In 1966, Robert Berger, a student of Hao Wang's at the time, proved that the Domino Problem was undecidable [2]. He did this by formulating a method of simulating Turing machines with Wang tiles.

A Turing machine is a theoretical computer, first proposed in 1936 by Alan Turing [5], made up of an infinite tape, divided into cells with symbols on them, a read/write head, and a control device. The head reads a symbol on the tape, then the control device determines what the head should write, how it should move, and what state the control device should be in next, based on the read symbol, and the current state of the control device [5]. Turing machines were designed to be a simple universal computer, meaning that with enough time, they can perform any possible computation. This means that anything which can simulate a Turing machine can also perform any possible computation.

Simulation of a Turing machine with Wang tiles can be done by storing the tape of a Turing machine as a line along the edge of a row of tiles, with a colour representing each tape symbol. A colour also represents each possible combination of tape symbol and state, acting as the head. A tile can then be used to represent each transition of the machine, and tiles designed to accept the head as it moves to them.

Berger adjusted this method to produce a set of Wang tiles for any Turing machine, such that the set will tile the plane if and only if the Turing machine does not halt, showing that the Domino Problem was equivalent to the Halting Problem. The Halting Problem is the question of whether an arbitrary Turing machine running on an arbitrary input would eventually halt or not. The Halting Problem was already known to be undecidable, so the equivalent Domino Problem must be as well. In the process of this proof, Berger also produced the first aperiodic set of Wang tiles, with 20,426 tiles in it. Much smaller aperiodic sets have since been discovered. The smallest was found in 2015, with 11 tiles and 4 colours, shown in Figure 4 [4]. A section of this infinite aperiodic tiling is shown in Figure 5. In the same paper, it was shown that 10 tiles, or 3 colours are insufficient to force aperiodicity.

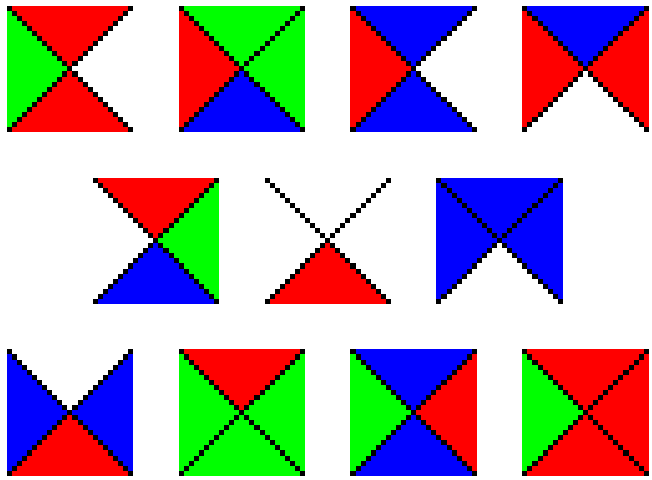

Figure 4: The smallest aperiodic set of Wang tiles.

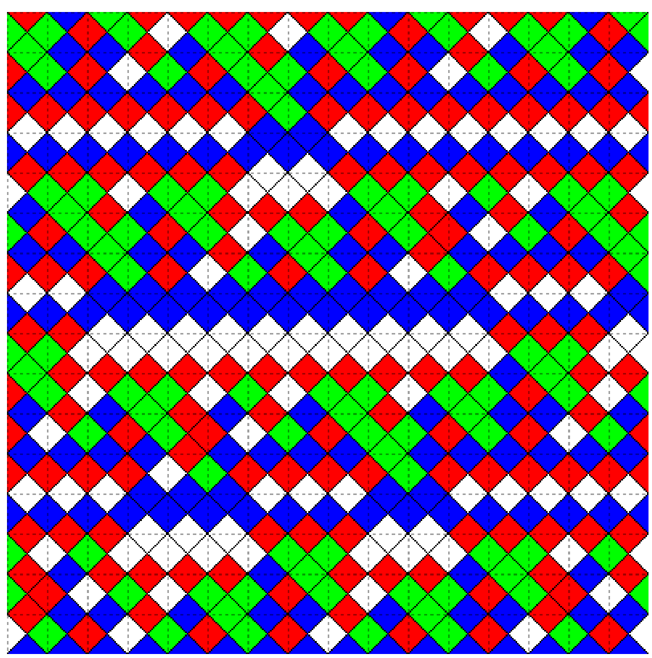

Figure 5: A section of the infinite aperiodic tiling from the set of Wang tiles shown in Figure 4.

3 Square Computation

While sets of Wang tiles can do computation by simulating a Turing machine, there are other ways Wang tiles can be used for computation, many of them more specific to Wang tiles. A simple example of how this can work is addition. The set of Wang tiles shown in Figure 6 can compute the addition of 2 different positive integers. For such computation, we lay down tiles in a way that encodes our inputs, then the only possibly valid tiling given these tiles is the computation. The tiling used for computation here does not tile the plane, and thus is neither periodic, or aperiodic. When not used for computation however, the set of tiles is periodic. This 'Wang addition' algorithm, to add two integers m and n, with $0 < m < n$, runs as follows, described from left to right in the tiling. We will describe a tile or collection of tiles as *forced* if they are the only possible tile or tiles that can be placed in a given position while maintaining a valid tiling.

1. We initialise the addition, by first laying Tile 1 to mark the origin, $(0, 0)$. We then lay Tile 4 at the positions $(m, 0)$ and $(n, 0)$. These 3 laid tiles we join with lines of Tile 13, in position i where $0 < i < n$, $i \neq m$. From here the computation is initialised. For example, the initialisation of the computation $7 + 13$ is shown in Figure 7.

2. Due to the red at $(0, 0)$, a descending diagonal line along (i, i) and $(i, i+1)$, where $0 < i < m$, is forced, formed of Tiles 2 and 3, and a vertical line, formed of Tile 6, descending from the blue axis along (m, i), where $0 < i < m$, is forced due to the first green Tile 4, at $(m, 0)$. These lines meet at position (m, m), with Tile 5. This is shown in our $7 + 13$ example, in Figure 8, the lines meeting at $(7, 7)$.

3. Due to the yellow on the right edge of Tile 5, a horizontal yellow line of Tile 7, along (i, m), where $m < i < n$, is forced, and a vertical line, formed of Tile 6, descending from the blue axis along (n, i), where $0 < i < m$, is forced due to the second green Tile 4, at $(n, 0)$. These two lines then meet at (n, m), with Tile 8. This is shown our $7+13$ example, in Figure 9, with the lines meeting at $(13, 7)$.

4. Due to the pink on the right edge of Tile 8, an ascending diagonal pink line, along $(n+i, m-i)$ and $(n+i, m-i+1)$, where $0 < i \leq m$, formed from Tiles 9 and 10 is forced. An extension of the blue line along the 0 row is also forced, extending to where the lines meet at $(m + n, 0)$. The column number of Tile 11, forced where these lines meet, then gives the output of the computation. This is shown in our $7 + 13$ example, in Figure 10. There we can see Tile 11 at position $(20, 0)$, giving us $7 + 13 = 20$.

During each section of this algorithm, we also fill spaces where nothing else is forced with Tile 12. This plays no part in the computation, but means that the final tiling does not have any spaces without tiles, thus is valid.

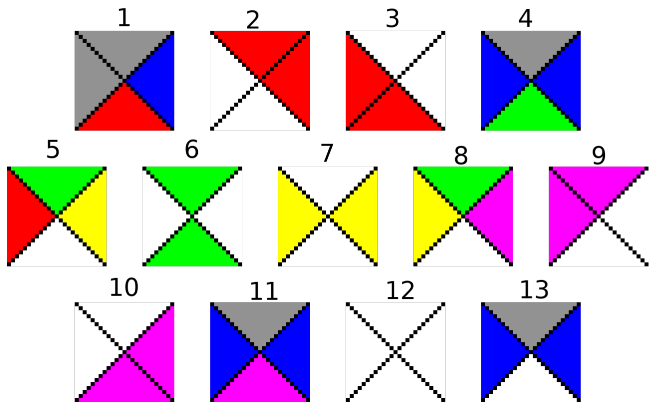

Figure 6: A set of Wang tiles to compute addition.

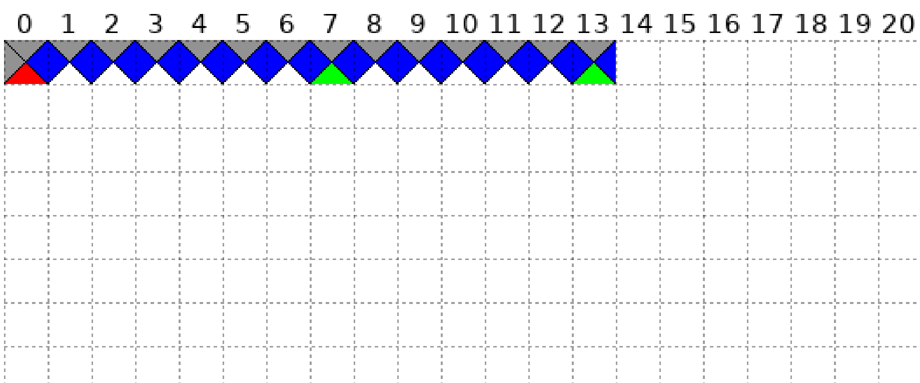

Figure 7: The initialised state.

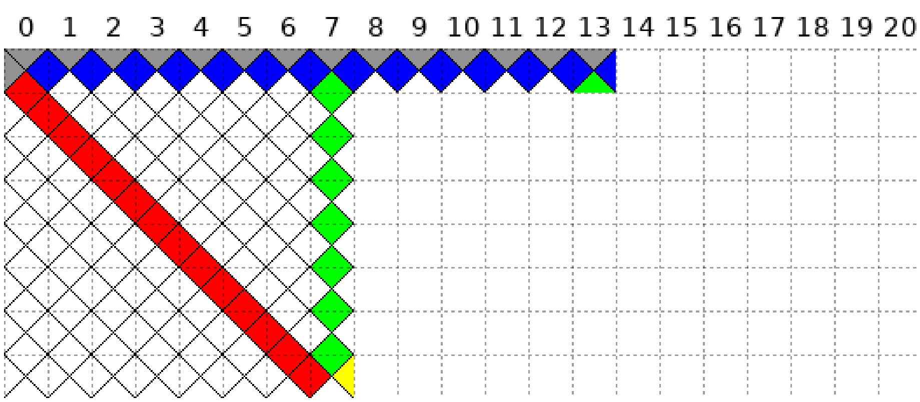

Figure 8: Green and red lines extend, to where the red and first green lines meet.

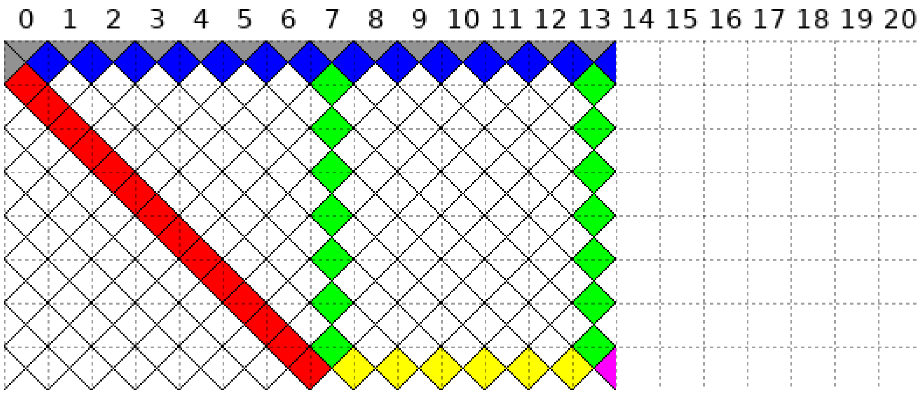

Figure 9: A yellow line is formed, and extends across to where it meets the second green line.

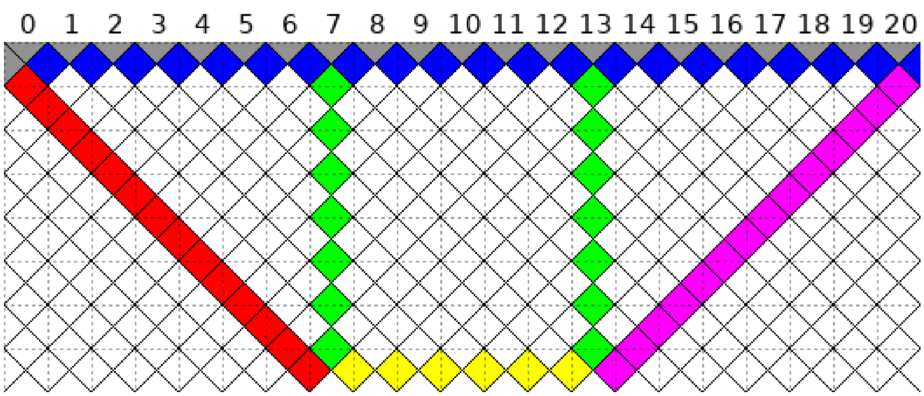

Figure 10: A pink line is formed, and it and the blue line extends to where they meet.

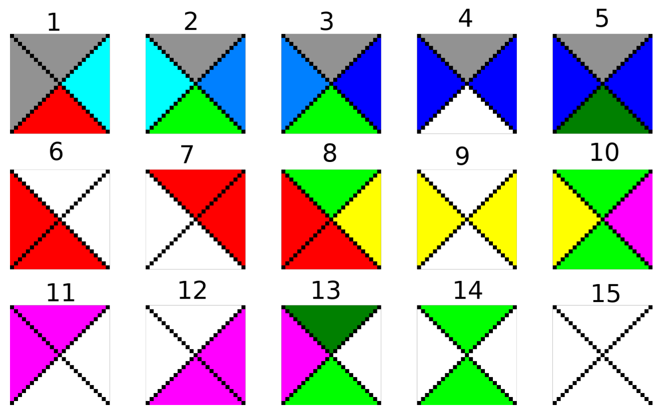

Figure 11: A set of Wang tiles to generate the Fibonacci sequence.

The calculation works by first 'storing' the value of m in the vertical position of the yellow line. This is obtained from where the red diagonal line meets the vertical green line, at (m, m). Then, the yellow line lies across the m row, to where it meets the second green line, at (n, m). As the pink line lies along $(n+i, m-i)$, once it reaches the 0 row, $i = m$, so it is at the position $(n+m, 0)$, which is where Tile 11 is placed, and where we can read the output of the completion; the column of this tile, $m+n$.

We can also use Wang tiles to compute the Fibonacci sequence, using a set slightly modified from the set used for addition, shown in Figure 11.

The basic principle is the same as for the set that computes addition. In this case, the only needed initialisation is Tile 1, then everything is generated, as shown in Figure 12. In this case, the output is Tile 5, with blue and dark green, and the sequence starts at 3, as 1 and 2 are strictly inputs, and 1 cannot be input twice as it usually would for the Fibonacci sequence. The main modification from the addition set is that the red line does not stop upon upon reaching a green line, and outputs generate new green 'input' lines. The dark green used for outputs here is to force the pink lines to reach the top before terminating. If these were simply green, nothing would prevent green output tiles from being placed where they should not, and the green line terminating the pink one early.

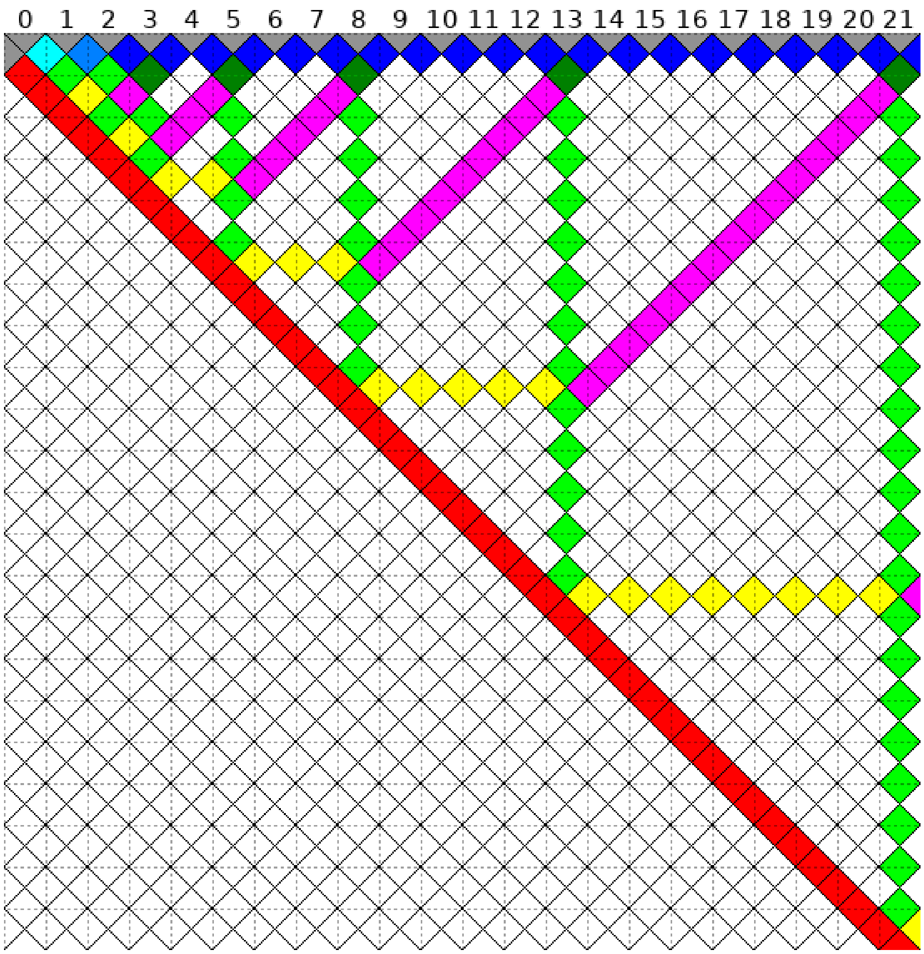

Figure 12: The Fibonacci sequence, generated using Wang tiles.

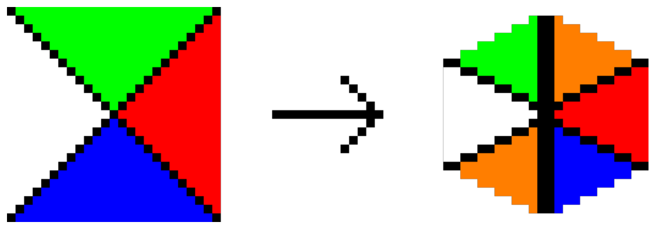

Figure 13: An example of conversion from a square to a hexagonal Wang tile.

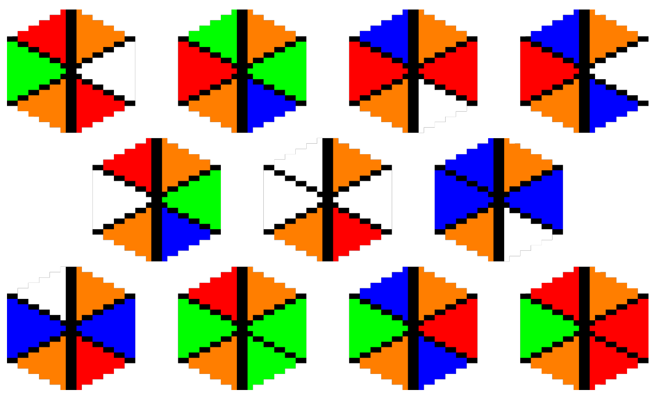

Figure 14: An aperiodic set of 11 hexagonal Wang tiles, converted from the set of square tiles shown in Figure 4.

4 Hexagons and Conversion

While square Wang tiles may offer significant depth as they are, it is not hard to extend the concept to other shapes. While more complex shapes, and sets with multiple different shapes are possible, the most obvious shape is the only other regular polygon to tile the plane without rotation or reflection; hexagons. One easy way to begin to explore the capabilities of hexagonal Wang tiles is to borrow from what we already know about square Wang tiles. For us to do this, we need a way to convert the sets of square Wang tiles to hexagons. This is far simpler than it would seem, and can be achieved by simply 'ignoring' 2 sides of the hexagons, by putting a single colour on those sides of all hexagons and transferring the colours from a square, as demonstrated in Figure 13 [3]. As an example, we can easily find an aperiodic set of hexagonal Wang tiles, by converting the square set we already know of. The set of tiles produced by this process is shown in Figure 14, and a section of the infinite aperiodic tiling it produces is shown in Figure 15.

While this conversion method could be used for the computation we have already seen with the square tiles. This is shown for addition in Figure 16 and Figure 17 (again, demonstrating $7 + 13 = 20$), and for the Fibonacci sequence

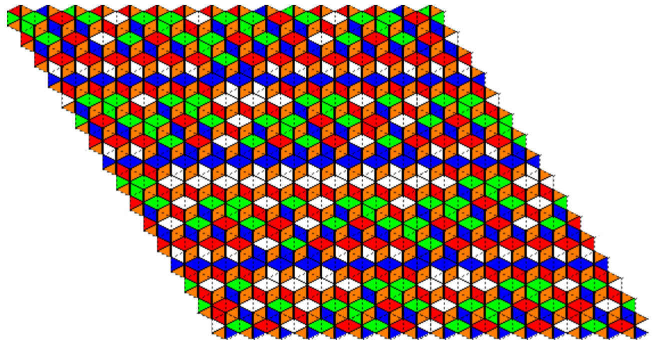

Figure 15: A section of the infinite aperiodic tiling produced by the conversion of the smallest aperiodic square set of Wang tiles to Hexagonal tiles.

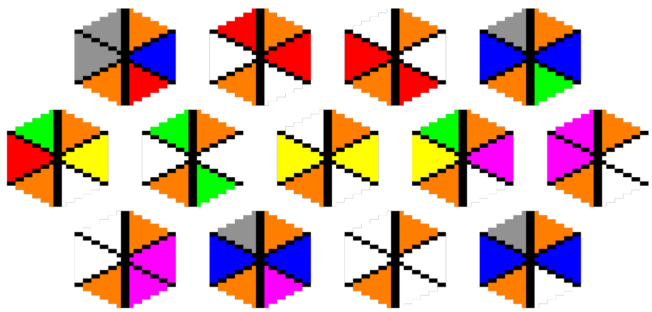

Figure 16: A set of hexagonal Wang tiles adapted from square Wang tiles to compute addition.

in Figure 18 and Figure 19, the ideas behind the square sets can simply be converted to work with hexagons.

The same computations can be done with sets of tiles utilising all 6 sides, following the ideas behind the square sets, instead of just modifying existing sets to work. A set for addition is demonstrated in Figure 20.

This set computing $7+13$ is demonstrated in Figure 21. It works similarly to the square set, though due to the way the vertical green lines alternate between tiles, they would block the pink line. To solve this, orange is used where an edge needs to be both green and pink.

Similarly to how the square set for computing addition can be modified to produce the Fibonacci sequence, so too can the equivalent hexagonal set. This hexagonal set is shown in Figure 22.

The Fibonacci sequence being generated by this set is demonstrated in Figure 23. The conversion from addition to Fibonacci sequence is much the same with hexagons as with squares, with one major difference; as the hexagon output tiles can fit 2 colours, both pink and blue, the pink line cannot end anywhere but the base line, so no additional colour is needed to fix this problem.

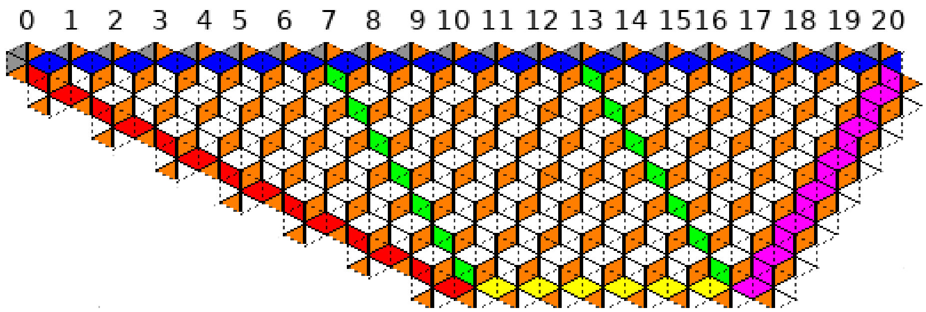

Figure 17: Hexagonal Wang tiles adapted from square Wang tiles computing addition.

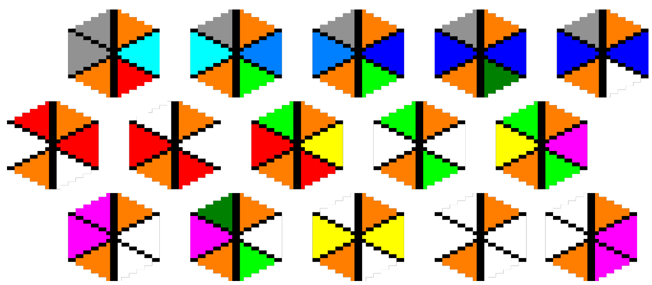

Figure 18: A set of hexagonal Wang tiles adapted from square Wang tiles to compute the Fibonacci sequence.

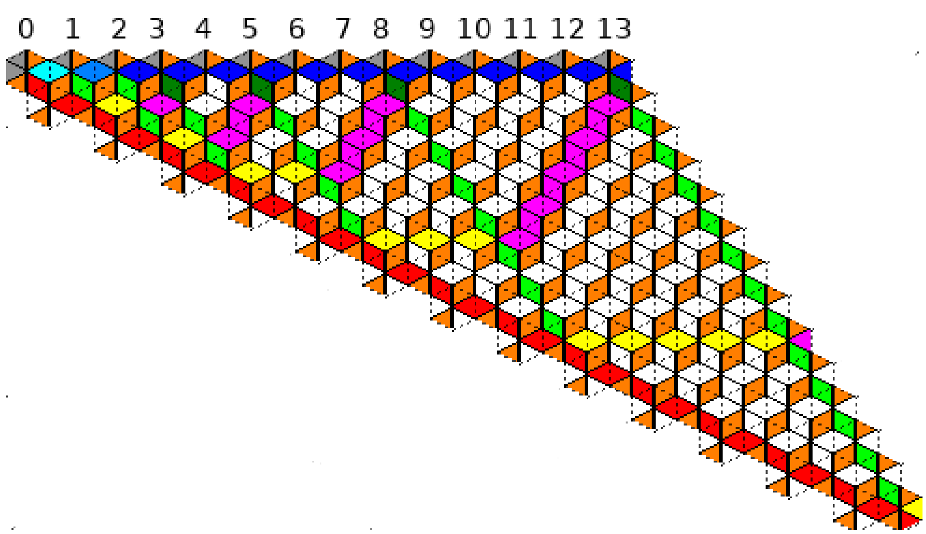

Figure 19: Hexagonal Wang tiles adapted from square Wang tiles computing the Fibonacci sequence.

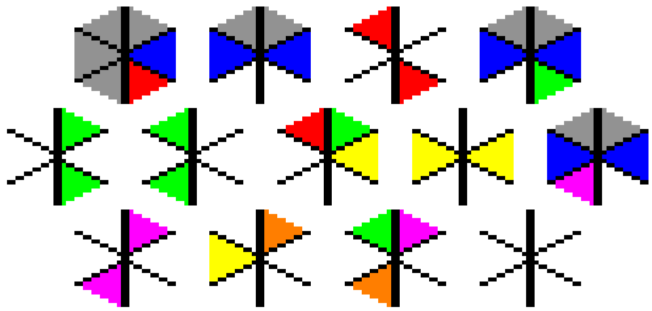

Figure 20: A set of hexagonal Wang tiles to compute addition.

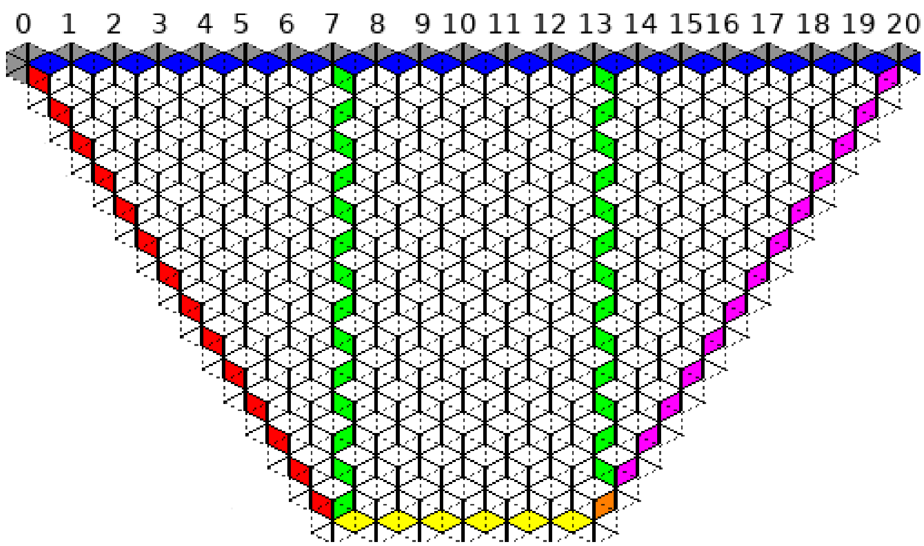

Figure 21: Hexagonal Wang tiles computing addition.

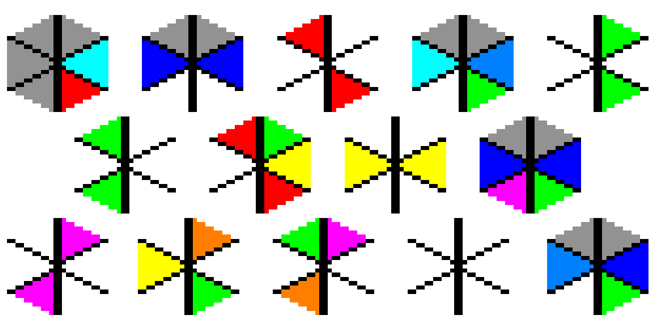

Figure 22: A set of hexagonal Wang tiles to compute the Fibonacci sequence.

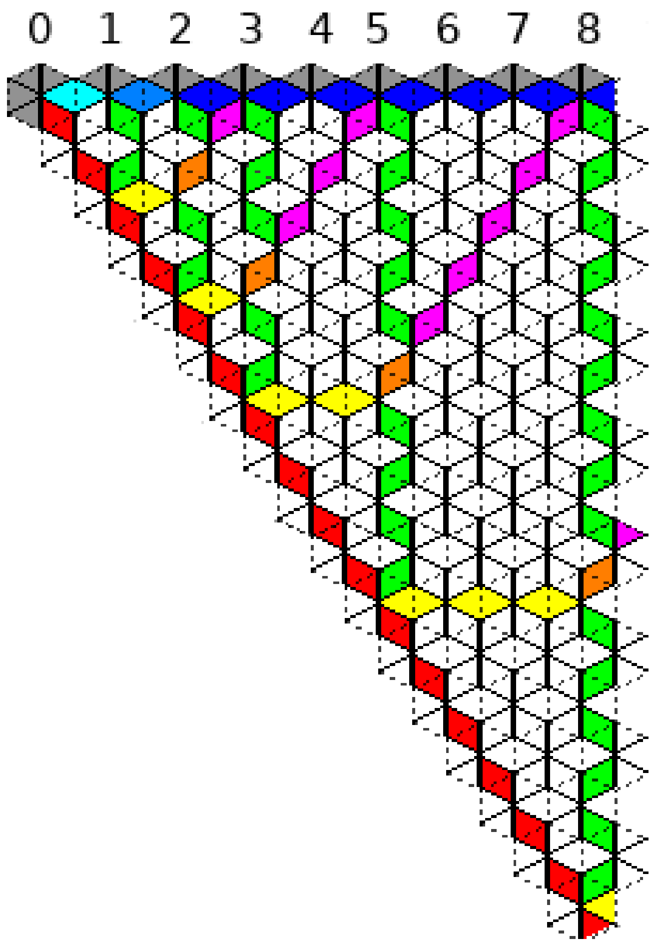

Figure 23: Hexagonal Wang tiles computing the Fibonacci sequence.

5 Conclusion

Wang tiles are a type of theoretical domino, proposed by Hao Wang in 1961. Two related problems were associated with them when they were first proposed: Deciding whether an arbitrary set could tile the plane (the Domino Problem), and whether any sets existed which could tile the plane only aperiodically. These were solved in 1966 by Robert Berger, who showed that the Domino Problem was undecidable, and produced the first set of aperiodic tiles. Berger's proof involved devising a method by which Wang tiles could simulate Turing machines, hence also demonstrating that Wang tiles are capable of computation. in this report we examined some examples of computation using Wang tiles, then examined how those methods could be used for hexagonal Wang tiles. Future research could look into different possible computations that can be done with hexagonal Wang tiles, including those not directly tied to square methods. Another possible area of research is Wang tiles of different shapes, or sets of Wang tiles with multiple shapes.

6 Acknowledgements

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7 Bibliography

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