Normalized Naive Set Theory

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Abstract

The broad goal of this thesis is the realization of a mathematically useful formal theory that contains a truth predicate, or as it's called in the literature, a "naive theory". To realize this goal, we explore the prospects for a naive set theory which can define a truth predicate.

We first consider some of the promising developments in naive set theories using various non-classical logics that have come before. We look at two classes of non-classical logics: weak relevant logics [58, 59] and light linear logics [52]. Both of these have been used in the development of naive set theories. We review the naive set theories using these logics then discuss the strengths and weaknesses of these approaches.

We then turn to the primary contribution of this thesis: the development of a robust naive set theory by accepting only normalized proofs, an idea first proposed by logician Dag Prawitz [25, 39]. It is demonstrated that this theory meets our need of logical strength in a system, while possessing more expressiveness in a foundational system than has come before. All of Heyting Arithmetic is recovered in this theory using a type-theoretic translation of the proof theory and other unique features of the theory are discussed and explored. It is further asserted that this theory is in fact the best case scenario for realizing informal proof in a formal system.

Contents

Chapter 0

Preliminaries

This thesis considers a lot of different logics, and each of these comes with its own traditional notation. If not specified, the connectives can be assumed to be classical in a chapter. The notation we will use for the connectives of classical logic are

- $\wedge$ for "and",
- $\vee$ for "or",
- $\rightarrow$ for "implication",
- $\neg$ for "not",
- $\perp$ for "absurdity".

It will be useful to have "generic" connectives. Logical statements written with these are meant to convey the general semantics or shape of an intended logical formula. These formula can then be realized in any logic of choice with the appropriate connective of the logic. For these we use the following notation

- $\underline{\wedge}$ for "and",
- $\underline{\vee}$ for "or",
- $\underline{\rightarrow}$ for "implication",
- $\underline{\neg}$ for "not".

Letters A, B, C, ... denote formulas. Capital greek letters, primarily Γ, Δ and Σ, are used to denote possibly empty sets of formulas. Sometimes subscripts are used to denote distinct sets of formulas, as in Γ_0. We also have the turnstile $\vdash$ as in $\Gamma \vdash B$, which states that there is a proof of B from assumptions Γ. If no particular logical system is specified, the turnstile asserts the existence of a proof according to classical logic. If a particular logical theory is being discussed, like relevant logic, then the turnstile asserts a proof in that particular logical theory.

Logics are presented as Hilbert-style axiomatic systems, sequent calculi or natural deduction systems. Propositional classical logic is briefly presented here in each style to provide an example of each type of formal system.

A Hilbert-style system for classical logic has three axioms and the rule of modus ponens. Note that in this presentation the turnstile is not part of the language; $\Gamma \vdash A$ is shorthand for the assertion that from the assumptions Γ we can derive A.

Axioms:

A1. $A \to (B \to A)$

A2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

A3. $(\neg A \to \neg B) \to ((\neg A \to B) \to A)$

Rules:

R1. $A, A \to B \vdash B$

The other connectives $\wedge$ and $\vee$ can be defined as $\neg(A \to \neg B)$ and $(\neg A) \to B$ respectively. When discussing naive set theory, the notation $\{x|A(x)\}$ will be used as shorthand for denoting the set y formed by the comprehension instance

$$\forall x(x \in y \leftrightarrow A(x))$$

where x may or may not occur free in A.

Next we present the multiconclusion sequent calculus for propositional classical logic.

$$\frac{}{A \vdash A}\ \text{Axiom} \qquad\qquad \frac{}{\bot \vdash A}\ \bot$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{WL} \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{WR}$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}\ \text{Cut}$$

$$\frac{\Gamma, A, B, \Delta \vdash \Sigma}{\Gamma, B, A, \Delta \vdash \Sigma}\ \text{EL} \qquad\qquad \frac{\Gamma \vdash \Delta, A, B, \Sigma}{\Gamma \vdash \Delta, B, A, \Sigma}\ \text{ER}$$

$$\frac{\Gamma, A, A, \Delta \vdash \Sigma}{\Gamma, A, \Delta \vdash \Sigma}\ \text{CL} \qquad\qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{CR}$$

$$\frac{\Gamma \vdash A, \Sigma \qquad B, \Delta \vdash \Pi}{A \to B, \Gamma, \Delta \vdash \Sigma, \Pi}\ \to_L \qquad\qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ \to_R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_{R1} \qquad\qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_{R2}$$

$$\frac{\Gamma, A, \Delta \vdash \Sigma \qquad \Gamma, B, \Delta \vdash \Pi}{\Gamma, A \vee B, \Delta \vdash \Sigma, \Pi}\ \vee_L$$

$$\frac{\Gamma, A, B, \Delta \vdash \Sigma}{\Gamma, A \wedge B, \Delta \vdash \Sigma} \wedge_L \qquad \frac{\Gamma \vdash A, \Sigma \qquad \Delta \vdash B, \Pi}{\Gamma, \Delta \vdash A \wedge B, \Sigma, \Pi} \wedge_R$$

Finally we present the natural deduction system for propositional classical logic. The presentation here differs from a more conventional natural deduction system. We use "sequents" to track assumptions throughout a proof rather than having to return to the top of a tree to discharge assumptions. There is no logical difference between a natural deduction system presented in this way and the more usual; it will be more convenient to track assumptions in this way. This presentation also helps make the later type theory translation theorems more obvious as this is essentially how type theories are presented. This presentation is further elaborated on and explained in the Chapter 6.

$$\frac{}{\Gamma, A \vdash A} \text{Ax}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_0 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_1$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee I_0 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee I_1 \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C} \vee E$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to I \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B} \to E$$

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash A} \perp$$

For both sequent calculus' and natural deduction's respective $\perp$ rules, A is to be different from $\perp$; this restriction is made for later convenience and does not change what is provable. We define $\neg A$ as an abbreviation for $A \to \perp$. We use the notation $[N/x]$ to specify that N replaces all free x's. For sequent calculus and natural deduction set theory, set terms are given by $\{x|A(x)\}$. These refer to the set of x's which satisfy the property $A(x)$. Any names like C or R for a is only a convenience. For example, when we write $C \in C$ where $C := \{x|x \in x \to A\}$, what is actually captured in the formal system is $\{x|x \in x \to A\} \in \{x|x \in x \to A\}$.

We also set binding precedence for logical formulas in this thesis in the following order: negation, conjunction, disjunction, universal quantification, existential quantification, and then finally the conditional. All logical formulas with omitted parenthesis can be assumed to be right associative. The types of our λ-terms in later chapters will bind similarly. As for λ-terms, application has precedence over abstraction, application is left associative and successive abstraction is right associative.

Chapter 1

Why Naive Set Theory?

While "naive" once served as a negative judgment [56] of naive set theory, the term "naive" has since been used as a technical definition by theorists in the field.

Definition 1.1. Let $\underline{\leftrightarrow}$ represent an arbitrary biconditional. A *naive* theory has a *transparent truth schema*: a predicate Tr such that for all closed formula A, $\vdash A \underline{\leftrightarrow} \mathrm{Tr}(\underline{A})$.

The primary technical goal of this thesis is the development of a naive theory built on an intuitive axiomatic base which is also capable of producing interesting mathematics. We seek such a theory in this thesis by exploring the possibilities for a naive **set** theory. A naive set theory has the ability to define a truth predicate with its few intuitive axioms. Naive set theory is the mathematical theory defined by the following two axioms:

Axiom 1.2 (Axiom Schema of Comprehension)**.** *Let $\Phi(x)$ be a well-formed formula. Then* $\exists y \forall x (x \in y \leftrightarrow \Phi(x))$.

Axiom 1.3 (Axiom of Extensionality)**.** $\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x = y$.

These two axioms appear to characterize our intuitive notion of a set: the Axiom Schema of Comprehension[1] says that any objects having some specific property can be collected into a set and the Axiom of Extensionality says that two sets are equal when they have equal elements. However, as was discovered by Bertrand Russell, this theory is inconsistent and thus trivial, i.e. the theory validates everything it can express [19]. The contradiction that Russell discovered is the following:

Proposition 1.4 (Russell's Paradox)**.** *Let R be the set which contains all sets which do not contain themselves. Formally, $\forall x (x \in R \leftrightarrow x \notin x)$. Then $R \in R$ and $R \notin R$.*

Proof. This set exists in naive set theory by the axiom of comprehension using the property $\Phi(x) := x \notin x$ and naming the set R.

Assume that $R \in R$. Then by definition of R and modus ponens, we have that $R \notin R$. Thus, by proof by contradiction, we must have that $R \notin R$. But by definition of R, this also implies that $R \in R$. Therefore $R \notin R$ and $R \in R$. $\square$

[1]Also referred to as unrestricted comprehension or simply the axiom of comprehension.

This discovery was viewed as a proof of the inconsistency of the axiom of comprehension and the subsequent response during the twentieth century was the rejection of this axiom. Thus the mathematical theory using these axioms came to be known as "naive" set theory and has since been replaced with different sets of axioms believed to be consistent. This methodology of changing our inconsistent theories' axioms has generated interesting and valuable mathematical theories: a family of different formal set theories, model theory, proof theory, and so on.

In tandem with these developments in the foundations of mathematics, there has been the development of a variety of non-classical (and non-intuitionistic) logics. To name a few families of non-classical logics, we now have: relevant logic, many-valued logic, linear logic, and paraconsistent logic [40]. Most of these logics still derive triviality from the axioms of naive set theory but some do not.

Light linear logics can house naive comprehension on its own without triviality [52, 22], and weak relevant logics are non-trivial with both naive comprehension and extensionality [58, 59, 8]. This thesis follows this work and proposes a solution to Russell's paradox not by replacing the axiom of comprehension but by replacing the logic which allows its derivation and/or the resulting derivation of triviality. This allows us to hold on to the high level of expressivity granted by the axiom of comprehension. This approach also challenges the universal applicability of limitative results in the foundations of mathematics. In the following chapters, we'll look at some of the work that's been done so far on naive set theories, discern what they can and can not do, and offer an alternative system.

My primary motivation for doing this is that of the pure mathematician: it's there to study and it looks interesting.[2] If that motivation suffices, then the next section can be seen as supplementary. For the reader who is not satisfied with this motivation, the next section discusses the utility such a theory might have. The philosophical implications and justification of our particular naive theory is offered in Appendix A.

1.1 Why naivety?

A theory is naive if it can encode a predicate which successfully captures its own truth. We may have hoped formal theories could express such a predicate but Alfred Tarski's Undefinability Theorem showed this to be impossible at the beginning of the twentieth century [50]. This theorem showed that our "sufficiently expressive" theories cannot have a truth predicate on pain of triviality. A proof sketch of the theorem follows, following the rules of classical or intuitionistic logic.[3]

Proposition 1.5. *In classical or intuitionistic logic,* $A \leftrightarrow \neg A \vdash A \wedge \neg A$*.*

Proof. Assume that A and $A \leftrightarrow \neg A$. Modus ponens with A gives $\neg A$ and thus

$$\vdash A \wedge \neg A$$

[2] For a defence of this motivation see Hardy's "A Mathematician's Apology" [27].

[3] A more complete presentation of this result can be found in [48].

. By proof by contradiction, $\neg A$ holds. Use modus ponens again to derive that A. Thus

$$\vdash A \wedge \neg A.$$

$\square$

Definition 1.6 (Name-forming Operation). Let ϕ be a sentence. Then $\underline{\phi}$ refers to the name of the sentence ϕ.[4]

Definition 1.7. A theory T is called *sufficiently expressive* if it is consistent, effectively axiomatized, and extends Robinson Arithmetic.[5]

Lemma 1.8 (Diagonalization Lemma). *If a theory is sufficiently expressive and $\phi(x)$ is a well-formed formula of its language with one free variable, then there is a sentence γ such that $\vdash \gamma \leftrightarrow \phi(\underline{\gamma})$.*

Theorem 1.9 (Tarski's Undefinability Theorem). *No sufficiently expressive theory can define truth for its own theory.*

Proof. Assume there is a predicate $\mathrm{Tr}(x)$ such that

$$\vdash \mathrm{Tr}(\underline{x}) \leftrightarrow x$$

for every sentence x. By the Diagonalization Lemma applied to $\neg\mathrm{Tr}(x)$, there is a sentence L such that

$$\vdash L \leftrightarrow \neg\mathrm{Tr}(\underline{L}).$$

From the truth schema instance

$$\vdash \mathrm{Tr}(\underline{L}) \leftrightarrow L,$$

we immediately get the biconditional that

$$\vdash \mathrm{Tr}(\underline{L}) \leftrightarrow \neg\mathrm{Tr}(\underline{L}).$$

This derives

$$\vdash \mathrm{Tr}(\underline{L}) \wedge \neg\mathrm{Tr}(\underline{L}).$$

Thus we reject the existence of the predicate $\mathrm{Tr}(x)$. $\square$

Tarski's Undefinability Theorem results from a theory being both expressive[6] and strong[7]. A theory has to have the expressivity given a truth predicate to prove a diagonalization result

[4] Set terms themselves, i.e. $\{x|\Phi\}$ are a kind of naming operation.

[5] Robinson arithmetic is the theory formed by taking all of Peano's axioms with classical logic except for induction [48].

[6] Where expressivity roughly means how many different logical/mathematical objects can the theory represent.

[7] Where strength roughly means how many theorems can be derived about the expressed mathematical objects.

which delivers the existence of a self-contradictory *liar sentence*, a sentence that says "this sentence is false". It also has to have the logical strength to derive triviality from that sentence.[8]

Since unrestricted comprehension allows any well-formed formula to define a set and since our quantication in naive set theory ranges over every set in the universe, we implicitly code in a diagonalization result at the very beginning of a our theory. This is what yields the paradoxes. Unrestricted comprehension is even expressive enough to generate a truth predicate, or "truth witness" set. This set behaves exactly as a truth predicate would.

Definition 1.10. The *truth witness set*, denoted Tr, is the comprehension instance

$$\forall x(x \in \mathrm{Tr} \underline{\leftrightarrow} \exists z(z \in x)).$$

We can use this set's membership relation as a truth predicate: most logics we consider will prove something like "for any closed formula A, $\vdash A \underline{\leftrightarrow} \{z|A\} \in Tr$". The following "propositions" are to be taken informally since no particular logic is being assumed.

Proposition 1.11. *For any closed formula A, $\vdash A \underline{\leftrightarrow} \{z|A\} \in Tr$.*[9]

Proof. For the forward direction we first conclude that for any y, we have $y \in \{z|A\}$ as we've assumed A holds. Thus, it is the case that $\exists x(x \in \{z|A\})$. By definition of the truth witness set, we have that $\{z|A\} \in \mathrm{Tr}$. For the other direction, we use this argument in reverse. This works since A is assumed to be closed. Thus we have the desired $A \underline{\leftrightarrow} \{z|A\} \in Tr$. □

From here we can see that Russell's set induces inconsistency of the truth witness set like the liar sentence induces inconsistency of a truth predicate.

Proposition 1.12. *The truth-witness set is inconsistent.*

Proof. Let $R := \{x|x \notin x\}$. By Proposition 1.11, we have

$$\vdash R \in R \underline{\leftrightarrow} \{x|R \in R\} \in \mathrm{Tr}$$

and its contraposition

$$\vdash R \notin R \underline{\leftrightarrow} \{x|R \in R\} \notin \mathrm{Tr}.$$

By definition of R,

$$\vdash R \in R \underline{\leftrightarrow} R \notin R,$$

and by transitivity of implication

$$\{x|R \in R\} \in \mathrm{Tr} \underline{\leftrightarrow} \{x|R \in R\} \notin \mathrm{Tr}.$$

[8] The terms "expressive" and "strong" are not being used in any technical/formal way here. They are meant to serve as informal terms describing the nature of the phenomenon that yields Tarski's results.

[9] When a generic connective is used in a proof, it can be seen as restricting the necessary properties of the conditional for the particular logic of choice that realizes the connective and a naive set theory. For example, in the following, for the proof to hold it must have something like transitivity and thus a conditional which failed to have this property may evade the paradox.

Then by Proposition 1.5,

$$\vdash \{x|R \in R\} \in \mathrm{Tr}\underline{\land}\{x|R \in R\} \notin \mathrm{Tr}.$$

□

1.2 How to be naive?

The goal of this thesis is thus to find a mathematically useful and non-trivial naive set theory through technical adjustments to the logic and thus to our methods of formal and informal proof. Further, the perspective of the thesis is that the significance of these adjustments to the logic is merely in their preservation of the tenability of naive set theory. Any greater philosophical speculation about the favored system of this thesis is reserved for Appendix A.

To facilitate these adjustments, I work under the assumption that a minimum desideratum for a naive theory is that it is non-trivial.

Definition 1.13. A theory is *trivial* if for every formula A of the theory, then A is a theorem of the theory.

Definition 1.14. The logical constant $\bot$ denotes triviality. For any formula A, $\bot \vdash A$.

A trivial theory is one in which there are no distinctions to be drawn between propositions since everything (and nothing) is falsifiable. In classical and intuitionistic logic, an inconsistency, like Russell's paradox, will imply triviality. Triviality can be derived from an inconsistency, $A\underline{\land}\neg A$, in a few ways. It may be derived directly with modus ponens when $\underline{\neg}A$ is defined as an abbreviation for $A\underline{\to}\bot$. It may also be derived by leveraging other principles in the logic like disjunctive syllogism.

Definition 1.15. *Disjunctive syllogism* is the inference from A and $\underline{\neg}A\underline{\lor}B$ to B.

Proposition 1.16 ([33])**.** *Let A be inconsistent, i.e. $A\underline{\land}\underline{\neg}A$. Then disjunctive syllogism along with disjunction introduction implies B for any formula B.*

Proof. We have that $\underline{\neg}A$ holds. Thus, $\underline{\neg}A\underline{\lor}B$. Since A also holds, we may use disjunctive syllogism to derive B. □

It's possible to define non-classical logics in which an inconsistency does not necessarily imply triviality; these logics are called paraconsistent [23]. This thesis considers two paraconsistent approaches to naive set theory, DKQ naive set theory and the system NNST.[10]

Definition 1.17. Let $\underline{\neg}$ denote a generic logical negation.[11] A theory or logic is *transparently paraconsistent*, or t-paraconsistent, with respect to $\underline{\neg}$ if there exists a proposition A for which $A, \underline{\neg}A \nvdash B$ for some B.

[10] However, NNST does not have to be seen as paraconsistent. It is my view that the negation will merely be a distraction from how NNST is behaving and its paraconsistent nature is of secondary concern.

[11] Recall this is not necessarily a negation that is defined as $\neg A := A \to \bot$.

Definition 1.18. Let $\underline{\neg}$ denote a generic logical negation. A theory or logic is *operationally paraconsistent*, or o-paraconsistent, with respect to $\underline{\neg}$ if there is some formula A such that $\vdash A \underline{\wedge} \neg A$ but there is also some B for which $\nvdash B$.

Definition 1.19. A logic is *paraconsistent* if it is either t-paraconsistent or o-paraconsistent.

In the general literature on paraconsistent logics, paraconsistency is just t-paraconsistency [21]. However, the final and preferred variant of naive set theory we'll discuss, NNST, does not satisfy t-paraconsistency while still ostensibly appearing to be paraconsistent in some broader sense. Operational paraconsistency has been defined here to capture the type of paraconsistency that NNST provably exhibits. The other paraconsistent solution we'll look at, DKQ naive set theory, will be t-paraconsistent and o-paraconsistent with respect to its primary negation. A paraconsistent logic will also sometimes be defined as one that does not validate the principle of explosion.

Definition 1.20. The principle of explosion is the inference that given any B, $A \wedge \neg A \vdash B$ for any proposition A.[12]

A paraconsistent logic distinguishes between local contradiction and global triviality. This allows the possibility that a naive set theory could prove the inconsistency of Russell's set without losing the rest of its ability to draw distinctions between propositions. However, being paraconsistent is not enough to prevent triviality in the context of naive comprehension. Another problem is Curry's set.

Definition 1.21. Let A be an arbitrary proposition. A Curry set with respect to A is the set C_A such that

$$\forall x(x \in C_A \leftrightarrow (x \in x \to A)).$$

When referring to Curry's, we will write C rather than the additional subscripted notation C_A. The A is irrelevant as Curry's paradox will follow from any we choose.

Proposition 1.22 (Curry's Paradox)**.** *Let A be any formula. Then the existence of Curry's set, defined with respect to A, can be used to show that A is a theorem.*

Proof. Assume for hypothetical proof $C \in C$ to prove $C \in C \to A$. Then by definition of Curry's set and our assumption of $C \in C$, we have that $C \in C \to A$. And again by our assumption of $C \in C$, we can detach that A holds. Thus we've proved that $C \in C \to A$.

Then by $C \in C \to A$ we have that $C \in C$ also holds. Then modus ponens derives A. □

Thus we can use Curry's set to prove any proposition. The novel feature of Curry's is that it derives triviality without reference to negation. It is hard to determine how to block this paradox simply from inspecting the informal proof. Thus, we now turn to a formal proof in a classical sequent calculus extended with comprehension inferences. These take the form of $\in_L$ and $\in_R$ inferences.

[12]The principle of explosion is also sometimes expressed as $\perp \vdash A$ and thus since $A, \neg A \vdash \perp$ due to modus ponens, the previous definition follows.

$$\frac{A[t/x], \Gamma \vdash C}{t \in \{x|A\}, \Gamma \vdash C} \in_L \qquad\qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash t \in \{x|A\}} \in_R$$

Now the formal proof of Curry's.

Proposition 1.23 (Curry's Paradox)**.** *Let A be any formula. Then the existence of Curry's set, defined with respect to A, can be used to show that A is a theorem.*

Proof. Let C be defined as the set such that $\forall x(x \in C \leftrightarrow (x \in x \to A))$. Then let $\mathbf{D}$ denote the following part of the proof:

$$\dfrac{\dfrac{\dfrac{C \in C \vdash C \in C \qquad A \vdash A}{C \in C, C \in C \to A \vdash A} \to_L}{C \in C, C \in C \vdash A} \in_L}{C \in C \vdash A} \text{Contraction}$$

Then the proof of A is as follows:

$$\dfrac{\dfrac{\dfrac{\mathbf{D}}{\vdash C \in C \to A} \to_R}{\vdash C \in C} \in_R \qquad \mathbf{D}}{\vdash A} \text{Cut}$$

□

If we want to block this proof, we have to block a principle of the logic, and all of them in the proof look innocuous. But if we want a naive set theory, we can't get rid of $\in_L$ or $\in_R$, and the conditional rules of $\to_L$ and $\to_R$ are also likely needed as written. This leaves us to suspect either cut or contraction. The usual approach in the literature has been to avoid contraction as a means of dealing with Curry's paradox [44, 43].[13]

Definition 1.24. Contraction is the inference

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{C}$$

Informally, contraction allows us to treat multiple uses of an assumption as indistinguishable from a single use. Rules that are not inferences which involve a specific logical connective like contraction, weakening, exchange, associativity are *structural* rules. Logics which remove one or more of these rules form another subclass of non-classical logics.

Definition 1.25. A *substructural logic* is a logic that is missing one or more of classical logic's structural rules.

Completely removing contraction is more difficult than it might first appear. It does not suffice to only remove the structural rule of contraction. Some of the connective rules like $\wedge_R$ can reintroduce it.

Example 1.26. $\dfrac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C} \wedge_R$

The two copies of A were treated as one to derive two different propositions. We'll see how this problem is avoided when looking at the logics used for naive set theory.

[13]Rejecting cut is a broadly a non-transitive approach, which this thesis ultimately argues for.

1.3 Three Approaches

In the following chapters we will see three different approaches to naive set theory. None of these need to be taken as insisting on a particular logic for naive set theory but they do suggest different guidelines to follow to achieve a non-trivial naive set theory.

The first approach we will look at involves weak relevant logics. These logics take both a paraconsistent and a contraction-free perspective. The paraconsistent negation that comes with these logics allows the law of excluded middle which won't be possible in our second approach. The downside to having this negation around is that it also requires a significant loss of strength in our primary conditional. This makes the system difficult to work with; this is explicated in the discussion of the weak relevant logic DKQ and its set theory.

Next we'll see an approach that is robustly contraction-free. These logics are constructed by taking a sequent calculus for a logic, and then removing the explicit contraction rule and all implicit contraction that may be hidden in rules for other connectives. This approach culminates in light linear logics which allow us some restricted forms of contraction and a little more flexibility. These logics can be shown to be consistent with unrestricted comprehension using cut elimination arguments. The downside in using these systems is that they become trivial when the axiom of extensionality is added. Nonetheless, the work done on these theories is instructive since it helps us to understand how to effectively use unrestricted comprehension.

Finally we'll discuss an approach which restricts the set of proofs. This works by using the concept of normality in natural deduction systems to achieve non-triviality. This approach, first noted by Prawitz [39], has not been extensively developed. Particularly, there is an absence in the literature of anyone attempting to develop mathematics in the theory. Some work that exists in the literature proceeds using a model theoretic type approach, this can be found in [18]. This is not sufficient if we hope to understand how to reason with the theory.

The lack of interest in this theory to date is likely due to it challenging long held beliefs about formal systems: it is paraconsistent and rejects universal transitivity of proof. To make matters worse, proving theorems in the theory is computationally difficult. However, these alterations to our formal systems and the computational difficulties can be understood by leveraging type theoretic insights. NNST will offer more logical strength than the other known approaches to naive set theory while still admitting natural mathematical reasoning.

The primary novel developments in this thesis are the work in NNST. The system is first laid out as a restriction of proofs from formal naive set theory. We implement a Curry-Howard correspondence with a simple lambda calculus to help with determining normal proofs. This is used in two different ways: we compute formal proofs by hand to determine normal proofs for Peano's Axioms, and we compute formal proofs by a computer program to determine proofs for ordered pair properties and a fixpoint theorem.[14] I also prove that from the normal proofs of Peano's axioms, we can conclude that we recover all of Heyting Arithmetic in NNST with a type theoretic argument. We also briefly consider an analog of Gödel's Incompleteness results

[14]The latter computer program is relegated to an appendix due to the complexity of the approach and a lack of software engineering expertise at the time this method was implemented.

to demonstrate how NST handles these differently and ultimately does not succumb to the same problems. On the other hand, the limitative result that will apply to NNST will be the Halting Problem.[15]

I also lay the ground work for justifying work in NNST by discussing the loss of transitivity and how it should be understood. As this is a mathematics thesis, I argue the bulk of the justification and consequences of accepting NNST in the appendix. I argue that this approach represents a natural computational understanding of mathematical proof. I conclude with an argument based on the Church-Turing thesis that NNST has an interesting avenue into discussing informal proof. The final hope of this thesis is to demonstrate that NNST is a natural mathematical theory worthy of further study. [16]

[15] The usual connection between the Halting problem and Gödel's theorems becomes much more nuanced (and confusing) in NNST.

[16] There will be plenty of open questions to be pursued at the end of this thesis.

Chapter 2

Weak Relevant Logic

The first class of logics we will discuss are the weak relevant logics. For these we will focus on one of the strongest logics in that framework: DKQ. The logic's primary conditional is a relevant contraction-free conditional and the logic's primary negation is paraconsistent and obeys the law of excluded middle. We first discuss some of the initial motivations of relevant logic. Then we look in more detail at some of the inferences that cause problems in naive set theory. We'll then conclude this chapter by defining DKQ, which is provably non-trivial with naive set theory [9, 8, 28], and discuss the advantages and disadvantages of the logic for mathematical reasoning.

2.1 The Relevant Critique of Classical Logic

In discussing the relevant logicians' critique of classical logic, I only hope to give the general flavor of arguments in this domain. Our critique will focus on the conditional of classical logic.

Arguments against classical logic are given by providing "counterexamples". This takes the form of an argument which seems to intuitively evaluate as a valid or invalid argument but which classical logic invalidates or validates respectively. Such an argument on its own is not a defeat of classical logic. A logic is still useful if we can restrict the domain of its use so that "most of the time" it still gives the "right" answer.

Now for our counterexample. The following is from Read [42].[1] Assume the following state of affairs: Let us suppose that James has claimed that Tim was in New Zealand on a certain day, and that William has denied it. Now consider the following three propositions:

1. If Tim was in New Zealand, then James was right.

2. If William was right, then so is James.

3. If Tim was in New Zealand, then William was right.

[1] [42] also provides a much more complete motivation of relevant logic and exposition of its properties. It also describes the general semantic framework which I omit. The classic texts for relevant logic are the works from Anderson and Belnap [3, 2]

It seems clear that (1) is true, for that is precisely what Tim asserted. However, (2) must be false as their assertions are contrary ones and we have never known people to be in two places at once. Finally, (3) is also false as that is not what William asserted. Now constructing an argument from these propositions:

1. If Tim was in New Zealand, then James was right.

2. It's not the case that if William was right, then so is James.

3. Hence, if Tim was in New Zealand, then William was right.

Thus we have an argument with true premises and a false conclusion. It seems then that this argument should be invalid. However, classical logic proves

$$P \supset Q, \neg(R \supset Q) \vdash P \supset R \tag{2.1}$$

Therefore we have an argument which we would like to say is invalid but which classical logic validates.

A relevant logician might point to other flaws in classical logic collectively known as the "paradoxes of material implication": unintuitive properties the classical conditional obeys but which seem wrong [26]. One such example is $(p \supset q) \vee (q \supset r)$ for arbitrary propositions p, q, r. This seems false in our every day understanding of implication which assumes there must be some connection between the premise and the conclusion. Another such example is the weakening rule, that from q we may derive that $p \supset q$. The relevant logician can point at this and say this violates a need for relevance between p and q.

2.2 Relevant Logic Responds

What does the relevantist propose will remedy these faults of classical logic?[2]

2.2.1 Changes to the Conditional and Disjunction

A primary change from classical logic is that not all of relevant logic's connectives remain extensional. Each connective from classical logic will have an intensional and extensional counterpart, and the intensional version will be a "stronger version" which implies the extensional.[3]

Definition 2.1. A connective is called *extensional*, or *truth-functional* if the sentence built from the connective can be determined to be true or false depending on the truth values of the proposition(s) it is operating on. A connective is *intensional* if it is not extensional.

[2]Of course, even if one thought that classical logic was flawed in some fundamental way and also believed that there was only "one correct logic", this does not we must endorse relevant logic as the one correct logic. There are many different non-classical logics which all have different answers for what might be causing the "paradoxes of material implication" [40].

[3]cf. the way that intuitionistic logic or modal logic connectives are no longer extensional. The semantics that underlie these changes are different for each logic.

In particular, the semantics for relevant logic's conditional are significantly different from the extensional conditional of classical logic. The relevant conditional is no longer equivalent to $\neg A \vee B$ and its truth can no longer be derived from an input of truth and falsity to the propositions. The semantics of the relevant conditional which align with its proof theory are much harder to determine and yield a much more complicated semantic theory than classical logic. In practice, when using relevant logics it'll be helpful to rely on our intuitive notion of relevance. For more details on the semantics, Read [42] is again a good resource.

As we are concerned with mathematical practice, it is more important for us to evaluate the changes to the proof theory. This perspective understands the shift to relevant logic as rejecting inferences that are used to derive equivalence between the intensional connectives and extensional connectives and also rejecting inferences on the grounds that they invoke irrelevance. I'll provide a few examples below of logical inferences that relevant logic forgoes.

A common rule that the relevant logician will do without is weakening. This rule is out on the grounds of irrelevance. In order for some proposition A to imply another B, there must be some relevant connection between them but weakening allows us to derive that A entails B as soon as we have a proof of B.

One of the more perplexing changes in the proof theory to someone trained in classical logic is that disjunctive syllogism no longer universally holds.[4]

Definition 2.2. Let A, B be logical formulas. As a formula schema, disjunctive syllogism (DS) is the inference

$$(\neg A \vee B) \rightarrow (A \rightarrow B).$$

DS can also be expressed as

$$A, \neg A \vee B \vdash B.$$

However, for relevant logic, there are good reasons to reject DS. First, DS is essentially a form of detachment for $\vee$. This rule is inherited from $\vee$'s equivalence to the conditional of the logic. Since that no longer holds for relevant logic, it should also lose its claim to detachment.

Second, DS as an axiom schema in the logic allows us to recover weakening, which is an irrelevant inference.

Proposition 2.3. *Weakening is provable in a logic with modus ponens as a rule and the axiom schemas for disjunction introduction, disjunctive syllogism and transitivity of the conditional.*

Proof. Assume A is some formula which has no relevant connection with B. We have that

$$B \rightarrow (\neg A \vee B)$$

by disjunction introduction. Then DS as an axiom schema gives that

$$(\neg A \vee B) \rightarrow (A \rightarrow B).$$

[4]For some theories using relevant logic, disjunctive syllogism turns out to be admissible [16]. That is, it doesn't prove any new theorems when added. It won't be for DKQ naive set theory.

Thus by transitivity of the conditional,

$$B \to (A \to B)$$

which is weakening. $\square$

Another argument for the failure of DS applies in inconsistent situations. In such a situation, DS is demonstrably not true.[5] Consider a situation in which we have $A \wedge \neg A$. We could still have A and $\neg A \vee A$ both be true without simultaneously having the truth of B.

Finally, the failure of disjunctive syllogism is also good news for the logic's paraconsistency. It means that a common proof for the principle of explosion is blocked.[6]

1.	$A \wedge \neg A$	Assumption
2.	A	Conjunction Elimination using 1
3.	$A \vee B$	Disjunction Introduction using 2
4.	$\neg A$	Conjunction Elimination using 1
5.	B	Disjunctive Syllogism using 3, 4

2.2.2 Adding Fusion and Changes to Negation

The move to an intensional conditional also naturally suggests an intensional counterpart for conjunction called fusion. This is denoted by $\circ$. A useful way to motivate the differences between $\circ$ and $\wedge$ is Curry's paradox.[7]

Let C be defined as the set such that $\forall x(x \in C \leftrightarrow (x \in x \to A))$. Then the pertinent part of the proof for Curry's is as follows:

$$\dfrac{\dfrac{\dfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C, C \in C \to p \vdash p} \to_L}{C \in C, C \in C \vdash p} \in_L}{C \in C \vdash p} \text{C}$$

It's here that we wish to reject contraction, but how do we go about explicating that change? The left rule of extensional conjunction reveals the premises are joined by the equivalent of $\wedge$. The left and right rules for extensional conjunction are as follows.

$$\dfrac{\Gamma, X, Y \vdash \Delta}{\Gamma, X \wedge Y \vdash \Delta} \wedge_L \qquad\qquad \dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R$$

Written this way, the right rule for $\wedge$ allows us to derive $A \vdash A \wedge A$ from two instances of $A \vdash A$. Then, due to $\wedge_L$ and Cut, we can prove contraction. For if we have $\Gamma, A, A \vdash B$ then $\wedge_L$ gives $\Gamma, A \wedge A \vdash B$ and Cut with $A \vdash A \wedge A$ gives $\Gamma, A \vdash B$. If we wish to get rid of contraction, it's clear we need to alter our conjunction in some way. A way to do this is to introduce the intensional conjunction fusion, $\circ$. The rules for this connective follow.[8]

[5] Of course, in such a situation it could also be true. But such a move leads us to triviality which we know can be avoided with paraconsistent logics.

[6] cf. Proposition 1.16

[7] For more information on the differences between fusion and ordinary conjunction, see [41].

[8] This sequent calculus presentation is taken mainly from [41] and adjusted in presentation to fit better with

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \circ B \vdash \Delta} \circ_L \qquad\qquad \frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \circ B, \Delta_1, \Delta_2} \circ_R$$

Of course, this isn't enough if $\wedge$ maintains its $\wedge_L$ rule and we would thus drop it from the system.[9] It's worth noting that the differences between these conjunctions disappears when both structural contraction and weakening are available.

Proposition 2.4. *If contraction and weakening are allowed, then*

$$A \wedge B \dashv\vdash A \circ B.$$

Proof. That $A \circ B \vdash A \wedge B$ holds without contraction.

$$\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B} \wedge_R}{A \circ B \vdash A \wedge B} \circ_L$$

The other direction requires contraction.

$$\dfrac{\dfrac{\dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash A} \wedge_{E1} \qquad \dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash B} \wedge_{E2}}{A \wedge B, A \wedge B \vdash A \circ B} \circ_R}{A \wedge B \vdash A \circ B} \mathrm{C}$$

□

We can also prove that $A \vdash A \wedge A$ with one application of $\wedge_R$. This is another dangerous form of contraction (see 2.2.4). We also find that $A \nvdash A \circ A$. Another change this forces on us is that "pseudo modus ponens", $A \wedge (A \to B) \to B$, is no longer provable, but $A \circ (A \to B) \to B$ is [46]. As we saw in discussing disjunctive syllogism, relevant logic requires stronger intensional connections between propositions before modus ponens can be used.

What about negation in relevant logics? Negation in relevant logics tries to be as classical as possible while avoiding the principle of explosion. It still contraposes with the relevant arrow, has double negation introduction and elimination, and even reductio is still provable in the special case of self application, $(A \to \neg A) \to \neg A$. The negation will also validate De Morgan's laws over the extensional conjunction and disjunction in the language.

2.2.3 How much is left?

How much of classical logic does relevant logic keep?

our later look at logics as in [37].

[9]This presentation of fusion and conjunction does obscure some of the complexity. A relevant logic sequent calculus presentation would usually differentiate between intentional and extensional collections of assumptions to the left of the turnstile. These are in turn built by using intensional or extensional connective rules. The presentation provided in this thesis ignores the extensional collections; we'll soon be turning to a Hilbert-style system as its essential for presenting weak relevant logics and thus the extra time on the sequent calculus system of relevant logics isn't helpful. The intent of this section is to demonstrate the possibility of a conjunction type connective that escapes its implicitly contractive nature. See [41] for more details on this and fusion versus conjunction.

Theorem 2.5 (ZDF Theorem)**.** *The zero-degree formulas (those containing only the connectives* $\wedge$, $\vee$, $\neg$*) provable in* R *are precisely the theorems of classical logic [16][p. 31].*

At first glance it might seem we haven't really given up that much. But this only applies to the strongest relevant logic R and that logic still proves triviality for naive set theory due to contraction being present. We must lose some of the logical strength of R before we can use it in naive set theory.

2.2.4 Further Inferences To Avoid

We'll turn to a Hilbert-style presentation of Curry's paradox to see what other inferences need to be avoided before we have a logic consistent with naive set theory. These proofs are shown with classical $\rightarrow$ and $\wedge$ connectives. These proofs could be blocked by removing any of the inferences used to derive them, but we detail here the standard approach in the literature.[10]

Definition 2.6. Implication contraction is the inference

$$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B).$$

Conjunction contraction is the inference

$$A \rightarrow (A \wedge A).$$

Proposition 2.7. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B) \dashv\vdash A \rightarrow (A \wedge A)$.

Proof. That implication contraction implies conjunction contraction is straightforward. We take the axiom schema with A and $A \wedge A$. This gives $(A \rightarrow (A \rightarrow (A \wedge A))) \rightarrow (A \rightarrow A \wedge A)$. The hypothesis is an instantiation of $A \rightarrow (B \rightarrow (A \wedge B)$. Modus ponens gives conjunction contraction.

That conjunction contraction implies implication contraction is more complicated. A formal proof is given. We do need the schema $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C)$. for this proof.

1.	$A \rightarrow (A \wedge A)$	Conjunction Contraction
2.	$(A \rightarrow (A \wedge A)) \rightarrow ((A \wedge A) \rightarrow B) \rightarrow (A \rightarrow B)$	Conjunctive Syllogism
3.	$((A \wedge A) \rightarrow B) \rightarrow (A \rightarrow B)$	Modus Ponens w/ 1, 2
4.	$(A \rightarrow (A \rightarrow B)) \rightarrow ((A \wedge A) \rightarrow B)$	Assumed Schema
5.	$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$	Hypothetical Syllogism, MP with 3, 4

$\square$

The proof of Curry's paradox using implication contraction is straightforward.

Proposition 2.8. *Implication contraction implies Curry's paradox.*

[10] In the current context it may not be absurd to doubt some of other principles in the proof instead. However, these removals are usually even harder to justify because the principles are almost definitional of the connective in question, e.g. $A \rightarrow (B \rightarrow (A \wedge B))$.

Proof.

1.	$C \in C \leftrightarrow (C \in C \to p)$	Comprehension
2.	$C \in C \to (C \in C \to p)$	Def of $\leftrightarrow$, Conj Elim, MP w/ 1
3.	$C \in C \to p$	Contraction, MP with 2
4.	$C \in C$	MP w/ 1, 3
5.	p	MP w/ 3, 4

□

As this proof is simple and relies on a lot of hard to reject rules of logic, the only weak point here is implication contraction. This is the same as we've seen in the sequent calculus formulations. As for conjunction contraction, this causes problems in the presence of the inference pseudo modus ponens.

Definition 2.9. Pseudo modus ponens is the inference

$$(A \wedge (A \to B)) \to B.$$

Proposition 2.10. *Conjunction contraction and pseudo modus ponens implies Curry's paradox [46].*[11]

Proof.

1.	$C \in C \leftrightarrow (C \in C \to p)$	Comprehension
2.	$(C \in C \wedge (C \in C \to p)) \to p$	Pseudo MP
3.	$(C \in C \wedge C \in C) \to p$	Substitution
4.	$C \in C \to p$	Conjunction Contraction, Hyp. Syllogism
5.	$C \in C$	MP with 1, 4
6.	p	MP with 4, 5

□

The standard response at this point is to reject pseudo modus ponens rather than conjunction contraction. If we did want to remove conjunction contraction, we would also have to remove conjunctive syllogism which can prove conjunction contraction.

Definition 2.11. Conjunctive syllogism is the inference

$$(A \to B) \to ((A \to C) \to (A \to (B \wedge C))).$$

Proposition 2.12. *Conjunctive syllogism implies conjunction contraction.*

Proof. This is an instantiation of conjunctive syllogism. Take the instance

$$(A \to A) \to ((A \to A) \to (A \to (A \wedge A))).$$

[11] We also use intersubstitivity of equivalent logical formula, i.e. if $A \leftrightarrow B$ and $C(A)$ is some formula that contains A, then $C(B)$ is provable exactly when $C(A)$ is.

Then two uses of modus ponens with $A \to A$ complete the proof. $\square$

Finally, another problematic inference is self-distribution which can reprove implication contraction.

Definition 2.13. Self-distribution is the inference

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)).$$

Proposition 2.14. *Self-distribution proves implication contraction.*

Proof. Take the self-distribution instance $(A \to (A \to B)) \to ((A \to A) \to (A \to B))$. Then we use permutation to give $(A \to A) \to ((A \to (A \to B)) \to (A \to B))$. Then modus ponens with $A \to A$ completes the proof. $\square$

2.3 A Weak Relevant Logic: DKQ

We'll now look at a logic that falls under the framework known as the "weak relevant logics" which avoids the problematic inferences.[12] The particular logic of this type we'll be looking at is DKQ. Let A, B, C denote logical formulas. The language is composed of lower case letters as variables and the logical symbols $\to, \vee, \wedge, \exists, \forall, \neg$.

Remark 2.15. By convention, we mainly use x, y and z as variables and subscript these to denote distinct variables when needed.

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A$

A3. $(A \wedge B) \to B$

A4. $A \to (A \vee B)$

A5. $B \to (A \vee B)$

A6. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$ (conjunctive syllogism)

A7. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ (proof by cases)

A8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$ (distribution)

A9. $\neg\neg A \to A$ (double negation elimination)

A10. $(A \to \neg B) \to (B \to \neg A)$ (contraposition)

[12] This family of logics is covered in detail in [9].

A11. $((A \to B) \wedge (B \to C)) \to (A \to C)$ (hypothetical syllogism)

A12. $A \vee \neg A$ (law of excluded middle)

A13. $\forall x A \to A[y/x]$, y free for x in A

A14. $\forall x(A \to B) \to (A \to \forall x B)$, x not free in A

A15. $\forall x(A \vee B) \to (A \vee \forall x B)$, x not free in A

A16. $A[y/x] \to \exists x A$, where y is free for x in A

A17. $\forall x(A \to B) \to (\exists x A \to B)$, where x is not free in B

A18. $(A \wedge \exists x B) \to \exists x(A \wedge B)$, where x is not free in A

Rules:

R1. A, $A \to B \vdash B$ (modus ponens)

R2. A, $B \vdash A \wedge B$

R3. $A \to B$, $C \to D \vdash (B \to C) \to (A \to D)$

R4. $A \vdash \forall x A$ (universal generalization)

R5. $x = y \vdash A(x) \to A(y)$ (substitution)

Meta-rules:

MR1. If $A \vdash B$, then $C \vee A \vdash C \vee B$.

MR2. If $A \vdash B$, then $\exists x A \vdash \exists x B$.

The meta-rules have the additional restriction that they can not be used on $A \vdash B$ if universal generalization has been used on a free variable of A. The biconditional, $A \leftrightarrow B$ is shorthand for $(A \to B) \wedge (B \to A)$.

In our presentation of Hilbert-style proofs, proofs are a linear sequence of instances of the axiom schemas as well as the application of valid rules to those instances. In these proofs, each line has an attached annotation, often abbreviated, which details which axioms schemas and lines of the proof are used to reach each line in the proof. We are also allowed to use assumptions by annotating a line as an assumption. Each line of a proof is of the following form.

$$1. \quad A \;\big|\; \text{Annotation}$$

The turnstile, $\Gamma \vdash A$, then represents the fact that there is a proof of A from the assumptions of Γ. More generally, the turnstile captures a notion of "deducibility". We can read $A \vdash B$ as "if we assume A, we can deduce B". Formally, to "deduce" means we've reached a valid line in a proof which has the formula B after assuming some A. We further call any formula that can be deduced without assumptions, that is $\vdash A$, is a theorem of the theory.

Our presentation of DKQ will fail to validate a deduction theorem[13] and thus it is the case that the turnstile captures a different relationship than the conditional of our logic represents, that is A "implies" B for $A \to B$ means something different then "deduces". We can now define what a "deduction theorem" for a Hilbert-style system is.

Definition 2.16. A *deduction theorem* is the proposition that from the Hilbert-style derivation of $\Gamma, A \vdash B$, we can construct a proof of $A \to B$ from Γ, i.e. $\Gamma \vdash A \to B$.

Proofs of the deduction theorem for a logic are constructive. They provide an explicit method for converting a deduction of B from A to a proof that $A \to B$.

The relation the turnstile defines then happens to have the following properties: reflexivity, transitivity and monotonicity.[14] The key thing to remember here is that the properties of the turnstile, of deducibility, are things we believe because we have proofs of them.

Proposition 2.17. *The turnstile is reflexive: for a given formula A, $A \vdash A$.*

Proof. Let A be an arbitrary formula. Take the proof that assumes A on line one.

$$1. \quad A \;\Big|\; \text{Assumption}$$

By definition of the turnstile, we have proved that $A \vdash A$. □

Thus, it is obvious that for any formula A we can write the proof that has the single line assuming A and thus prove $A \vdash A$.

Proposition 2.18. *The turnstile is monotonic: for any set of formulas Γ and formulas A and B, if $\Gamma \vdash B$ then $\Gamma, A \vdash B$.*

Proof. Assume that we have the proof

$$\Gamma \vdash B.$$

Take this proof and add a line that assumes A. □

That is, given any proof of B from assumptions Γ, we can extend the set of assumptions by another formula and still derive the proof of B.

Proposition 2.19. *The turnstile is transitive: for any set of formulas Γ and Δ and formulas A and B, if $\Gamma \vdash A$ and $\Delta, A \vdash B$ then $\Gamma, \Delta \vdash B$.*

Proof. Assume that

$$\Gamma \vdash A$$

[13] cf. the discussion of Theorem 2.26

[14] This is nothing new. The reason for my attention to routine details here will be clear when considering the meta-rules of DKQ and later in Appendix A when discussing justification for NNST.

and that

$$\Delta, A \vdash B.$$

Then construct the new proof that first assumes the formulas of Γ to derive A, by using the first assumed proof. Then assume the formulas of Δ and use the derived A to derive B, by using the second assumed proof. Thus we've deduced B from the assumptions Γ and Δ, that is we've proven

$$\Gamma, \Delta \vdash B.$$

□

Finally, we have that given any two proofs in which one proof has a conclusion that is the assumption of the other, we can stitch them together.

Thus, the turnstile does not ordinarily exist in Hilbert-style systems but expresses a relation about deducibility. However, this understanding is blurred in DKQ by the addition of the so-called Meta-Rules. The first one says that if we have a proof of B from an assumption A, then we also get a proof of $C \vee B$ from $C \vee A$. Normally this rule follows automatically from the deduction theorem and the axiom schema for proof by cases.

Proposition 2.20. *MR1 follows from the axiom schema for proof by cases and the deduction theorem.*[15]

Proof. Assume that $A \vdash B$. If from A we can construct a proof of B, then we can construct a proof of $C \vee B$ by a few additional steps involving disjunction introduction, hypothetical syllogism and modus ponens. Thus we have a proof that

$$A \vdash C \vee B.$$

Then the deduction theorem constructs a proof that

$$\vdash A \to C \vee B.$$

We can also construct a proof that

$$\vdash C \to C \vee B$$

by the axiom schema for disjunction introduction. Then conjunction introduction, the proof by cases schema gives that

$$\vdash C \vee A \to C \vee B.$$

An assumption of $C \vee A$ and modus ponens gives us the result:

$$C \vee A \vdash C \vee B.$$

□

[15]MR2 can also be derived in a similar way, using the deduction theorem, A16, and A17.

This proof crucially relies on the assumption that we can construct a proof of $A \to C \vee B$ from $A \vdash C \vee B$. Without this fact, the content of the meta-rules is a little mysterious. The meta-rules assert the existence of a proof, without guaranteeing that a proof can actually be constructed.[16]

Remark 2.21. As an abuse of language, it will sometimes be said that a proposition is proved "over a turnstile" or "using a turnstile" This will always mean that we're deriving a proof of some B from some A, i.e. $A \vdash B$, rather than deriving $A \to B$.

We may also abuse notation and treat the turnstile as a logical connective moreso than a mathematical relation, even though it does not actually exist as such. We have already done this above when proving the properties of the turnstile. For example, we say that **the** turnstile is monotonic, i.e. from $A \vdash B$ we can derive that $C, A \vdash B$. However, what we really mean by this statement is that relationship exhibited by the turnstile is monotonic: that is, given a deduction of B from the assumption A, we can also deduce B from assumptions A and C.

Brady proves that DKQ with the naive set theory axioms is non-trivial.[17] The only question that remains is what can DKQ do as a logic for naive set theory and mathematics. We'll look at the former here, and examine the latter in a later chapter.

2.3.1 DKQ as a Logic for Mathematics

To evaluate DKQ, our primary concern is whether it allows useful mathematical reasoning. For example, consider the inclusion law of excluded middle (LEM). As our goal is mathematical reasoning, I have no reason to reject its validity if we allow that classical mathematics offers interesting mathematical theories. Questioning its validity may be useful for mathematical practice[18] but our perspective here is not that there is one true mathematics but whether the logic facilitates mathematical discovery.[19]

With this view, what the logic does have isn't that controversial. The axioms establish fundamental principles for reasoning in mathematics, like proof by cases (A7), or contraposition (A10). The rules give us modus ponens and universal generalization. We have a "classical-like" negation, so we should expect this logic to give rise to mathematics with many classical qualities. We can prove the De Morgan negation laws.

Proposition 2.22. *The following De Morgan laws are valid:*

- $\neg(A \wedge B) \to \neg A \vee \neg B$

[16] This does perhaps venture into intuitionism/classical debate territory. Though it is also perhaps a bit different. Here it is not a question of whether the content of the proof is constructive, but whether the proof itself has been constructed.

[17] The proof is not reproduced here and can be found in the original [9] or the more readable version in [28].

[18] For example, questioning LEM led to intuitionistic mathematics which has produced rich mathematical theories.

[19] We are also in the process of simply evaluating the viability of a DKQ naive set theory. While it's possible that it's exclusion may make more semantic or philosophical sense, it's a valuable tool to classical mathematical practice. Further, it will turn out to be quite useful in developing the theory, and it is in fact hard to imagine developing a weak relevant logic naive set theory with interesting theorems which did not have LEM.

- $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$

Proof. We prove the contraposition of the first law.

1.	$\neg A \rightarrow (\neg A \vee \neg B)$	Disjunction Intro
2.	$(\neg A \rightarrow (\neg A \vee \neg B)) \rightarrow (\neg(\neg A \vee \neg B) \rightarrow \neg\neg A)$	Contraposition
3.	$\neg(\neg A \vee \neg B) \rightarrow \neg\neg A$	MP with 1, 2
4.	$\neg\neg A \rightarrow A$	Double Neg Elim
5.	$\neg(\neg A \vee \neg B) \rightarrow A$	Hypo Syl, MP with 3, 4
6.	$\neg(\neg A \vee \neg B) \rightarrow B$	Similar to 5
7.	$(\neg(\neg A \vee \neg B) \rightarrow A) \wedge (\neg(\neg A \vee \neg B) \rightarrow B) \rightarrow (\neg(\neg A \vee \neg B) \rightarrow (A \wedge B))$	Conjunctive Syl
8.	$\neg(\neg A \vee \neg B) \rightarrow (A \wedge B)$	MP with 5, 6, 7

The second law is more straightforward.

1.	$\neg(A \vee B) \rightarrow \neg A$	Disjunction Intro, Contraposition
2.	$\neg(A \vee B) \rightarrow \neg B$	Disjunction Intro, Contraposition
3.	$((\neg(A \vee B) \rightarrow \neg A) \wedge (\neg(A \vee B) \rightarrow \neg B)) \rightarrow (\neg(A \vee B) \rightarrow (\neg A \wedge \neg B))$	Conjunctive Syl
4.	$\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)$	MP with 1, 2, 3

□

As an example of using the meta-rules and of some use later, we can derive proof by cases over the turnstile without reference to the axiom form. This is mainly useful when we cannot establish relevant implications between propositions but have managed to establish turnstile deductions.

Proposition 2.23 (Proof by Cases). *If $A \vdash C$ and $B \vdash C$, then $A \vee B \vdash C$.*

Proof. This is a result of Meta-rule 1. From $A \vdash C$ we can derive that $A \vee B \vdash B \vee C$. From $B \vdash C$ we can derive that $B \vee C \vdash C \vee C$. Transitivity gives us that $A \vee B \vdash C \vee C$. The equivalence of $C \vee C$ and C finishes the proof. □

We also still have proof by contradiction in a slightly modified form.

Proposition 2.24. $A \rightarrow B, A \rightarrow \neg B \vdash \neg A$

Proof.

1.	$A \rightarrow B$	Assumption
2.	$A \rightarrow \neg B$	Assumption
3.	$(A \rightarrow B) \wedge (A \rightarrow \neg B) \rightarrow (A \rightarrow (B \wedge \neg B))$	Conj Syl
4.	$A \rightarrow (B \wedge \neg B)$	MP w/ 1, 2, 3
5.	$(A \rightarrow (B \wedge \neg B)) \wedge ((B \wedge \neg B) \rightarrow \neg(\neg B \vee B)) \rightarrow (A \rightarrow \neg(\neg B \vee B))$	Conj Syl
6.	$A \rightarrow \neg(\neg B \vee B)$	MP w/ 4, 5, DeMorgan's Law
7.	$A \rightarrow \neg(\neg B \vee B) \rightarrow (\neg B \vee B) \rightarrow (\neg A)$	Contraposition
8.	$\neg B \vee B$	LEM
9.	$\neg A$	MP w/ 6, 7, 8

□

Another useful form of proof by contradiction results from a formula implying its own negation.

Proposition 2.25. *$A \to \neg A \vdash \neg A$.*

Proof.

1.	$A \to \neg A$	Assumption
2.	$\neg A \to \neg A$	Axiom 1
3.	$((\neg A \to \neg A) \wedge (A \to \neg A)) \to ((A \vee \neg A) \to \neg A)$	Proof by Cases
4.	$(A \vee \neg A) \to \neg A$	MP with 1, 2, 3
5.	$A \vee \neg A$	LEM
6.	$\neg A$	MP with 4, 5

□

We now shift our focus to what DKQ does not have. The following discussion centers on problems with the conditional and problems with the negation.

A larger problem with DKQ that complicates usual mathematical reasoning is the loss of a deduction theorem and our usual method of conditional proof.

Theorem 2.26 (Deduction Theorem)**.** *$\vdash A \to B$ if $A \vdash B$*

The failure of the deduction theorem is due to the turnstile being capable of different inferences than the conditional, e.g. the turnstile allows weakening. We also have that modus ponens is defined as occurring over the turnstile, since we can't have the pseudo modus ponens inference in the logic. If DKQ did have a deduction theorem, that would mean that pseudo modus ponens would be provable and thus the pseudo modus ponens proof of Curry's would go through, Prop 2.10. There's also the problem that the usual proofs of a deduction theorem for a logic rely on the conditional being capable of contraction.

That said, we could make the turnstile mirror the relevant conditional and push some kind of a deduction theorem through.[20] But this would not fix the core problem. Our usual understanding of what the turnstile represents seems to match our usual understanding of mathematical deducibility. To lose the deduction theorem with that particular turnstile indicates the loss of our usual methods of conditional proof in mathematics. It no longer suffices to assume A and then find that B holds, i.e.

$$A \vdash B,$$

to conclude that A "implies" B, or

$$\vdash A \to B.$$

[20]This means changing what the turnstile relation is representing to something stronger than "assume A, then deduce B".

This is why the presentation of DKQ is easier in a Hilbert-style axiomatic system: a natural deduction formalization of DKQ would contain a $\to_I$ rule that did not intuitively correspond to our usual understanding of conditional proof.[21]

This will be further exemplified in constructing DKQ naive set theory. We will only be able to prove some essential propositions over a turnstile-like deducibility. This creates an interesting problem in using DKQ for naive set theory. This means that even though our relevant conditional gives us a strong connection between propositions with useful negation properties, we very rarely get to take advantage of these.

A further issue of not having a deduction theorem means that properties which are best expressed over the turnstile's notion of deducibility can not be written in the object language. This makes it impossible to collect objects satisfying these properties into sets. As an example, injectivity of a function will rarely be provable with the relevant conditional, so we will need to define it using a turnstile.[22] (See definition 3.21.) However, having it defined with the turnstile means that we can't define any set which is the collection of injective functions. This seems like something we should be capable of doing in a naive set theory.

Some of what DKQ loses makes it a lot harder to reason about set equality and inequality. Recall the axiom of extensionality,

$$\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y.$$

And consider its contraposed form

$$\neg\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x \neq y.$$

To get that two sets are not equal requires proving

$$\exists z(\neg(z \in x \to z \in y) \vee \neg(z \in y \to z \in x)).$$

To be able to prove this requires a counterexample inference $(A \wedge \neg B) \to \neg(A \to B)$. DKQ as defined does not have this, but we can add

$$A \wedge \neg B \vdash \neg(A \to B)$$

and still be covered by the non-triviality proof in [8].

The larger loss is that $\neg(A \to B)$ is no longer informative in the theory. We can't have the inference $\neg(A \to B) \to (A \wedge \neg B)$ in the logic as contraposition would imply the conditional was material, i.e.

$$\neg A \vee B \to (A \to B).^{23}$$

[21] Such a natural deduction system can be found in [9, Ch. 3].

[22] This results from the rule for substitution with equal sets also only applying over a turnstile.

[23] It's not clear what the status of $\neg(A \to B) \vdash (A \wedge \neg B)$ is. A naive notion of relevance implies that $A \to B$ can fail to hold because A was not relevant to B rather than it being because A was true and B was false. This rule is not included in DKQ.

This is a problem when we start a proof on the assumption that two sets are not equal. Returning to the contraposition of the axiom extensionality, this assumption of two non-equal sets amounts to a disjunction of two negated conditionals. This would ordinarily imply that the sets have a different element, but we can't draw that conclusion with DKQ.

There are also difficulties in establishing equality, and these come from a surprising place: the loss of weakening, $B \to (A \to B)$. We would not normally think weakening had much to do with equality. But it has a unique role in set theory that we do not see in other theories where equality is atomic. For example, consider an axiomatic theory given for Peano Arithmetic, which asserts among its axioms propositions like

$$Sx = Sy \to x = y$$

or

$$\neg \exists x (Sx = 0).$$

There is never a point in PA in which an equality is proved in regards to something more primitive than equality, and equality can always be derived as the consequence of some axioms regarding what's equal in PA. Weakening is never needed in PA to prove something is equal as it would amount to the assertion that from the equality of

$$a = b$$

we need to detour through some other proven equality

$$c = d$$

to prove some other equality

$$e = f.^{24}$$

However since equality can be proven in set theory through the axiom of extensionality, the provability of equality relies more on the properties of the conditional and being able to construct conditionals between the other atomic formula of set theory, namely

$$x \in y.$$

Then without weakening in DKQ naive set theory, it will no longer suffice for two sets to "have the same stuff" but they need to also have "similar reasons for having the same stuff". Weakening is what allowed us to infer equality from the mere co-incidence of membership. This is due to set equality being tied tightly to a conditional formula by the axiom of extensionality and by set instantiation being tied so tightly to properties.[25]

[24] Relevant peano arithmetic has been explored and seems capable of proving most of what we need in doing mathematics. However, it was found to not recover all of PA and was thus largely abandoned [20].

[25] Since we later discuss weaker conditionals that mitigate the downsides of the relevant conditional, one might wonder why we do not use one of those to rewrite the axiom of extensionality. This apparently would weaken

First, we'll look at this informally. Consider sets $A = \{0, 1, 2\}$ and $B = \{0, 1, 2\}$. We would like to be able to identify these sets merely by their co-incidence of elements. The way this co-incidence can be formalized requires weakening. For given the particular $0 \in A$, we use weakening to conclude that $0 \in B \to 0 \in A$. We don't have to rely on the particular reasons that A or B contain 0. Without weakening, we do.[26]

This means that any proofs of equality in this system, except in some trivial cases, have to use the defining properties of the sets. In the introduced informal case, we can prove equality if we have defined A and B in the same way. But the lack of weakening admits a way to construct infinite classes of sets for which we can not demonstrate equality but for which we also cannot demonstrate elements for which they differ.

Definition 2.27 ([9])**.** Let the formula A contain only the connectives $\neg$, $\wedge$, $\vee$, and $\to$. Then *depth* is inductively defined as follows.

1. The subformula A of the formula A is of depth 0 in A.
2. If $\neg B$, $B \wedge C$, or $B \vee C$ is a subformula occurrence of A of depth d in A then B, and C in the cases of $\wedge$ and $\vee$, are of depth d in A.
3. If $B \to C$ is a subformula occurrence of A of depth d in A, then both of these occurrences of B and C are of depth $d + 1$ in A.

Definition 2.28. A logic is *depth relevant* if for all formulas A and B, if

$$\vdash A \to B$$

then A and B share a sentential variable in a subformula at the same depth.

Theorem 2.29. *DK is depth relevant [9, p. 164].*

Corollary 2.30. *Let A and B be atomic formula, i.e. not containing any logical connectives, which need not be distinct. Then DKQ can not derive $A \to (B \to B)$ or $\neg(B \to B) \to \neg A$.*

Proof. Neither of the offered forms share a sentential variable. Since these are atomic formula, we do not need to be concerned that a quantifier inference allows this as well. Thus they can not be derived in DKQ.[27] □

the strong connection needed between sets to make them equal. But it turns out that this strict form of the axiom of extensionality is also keeping triviality at bay. cf. Proposition 3.30.

[26] This also suggests we intuitively think of the conditional that manages set equality as material and truth-functional. For a particular $z \in B$ that A shares, we have $z \in A$ and use disjunction introduction to get $z \notin B \vee z \in A$. On the other hand, for any $z \notin B$ we can still get $z \notin B \vee z \in A$. These two observations together gives $\forall z(z \notin B \vee z \in A)$.

[27] This implication is not a valid tautology for any relevant logic [42]. The relational semantics can be used to produce the following counterexample: take a world 0 where A holds and B does not hold, and a world I where B holds. Take the relation to have $RII0$ and $R0II$. The former relation gives that $B \to B$ does not hold at I, and the latter then gives that $A \to (B \to B)$ does not hold at 0.

However, this does not rule out all cases of the form nor that this does not hold in our theory. We can use the axiom of comprehension to establish this form for some cases, as in $x \in A \to (x \in B \to x \in B)$.

Now, consider A and B defined by $A := \{x|x = \emptyset\}$ and $B := \{x|x = \emptyset \wedge (C \to C)\}$ where C is any atomic formula. We can find infinitely many such C's by taking atomic membership formula for sets defined by valid formula. For example take an atomic formula D and consider the valid forms $(D \to D)$ and $(D \to D) \to (D \to D)$ and so on. Using these formula, take a set like $\{x|(D \to D)\}$, and then use the atomic membership formula $z \in \{x|(D \to D)\}$ for C. To derive that $A = B$ would require deriving that $x \in A \to x \in B$. Deriving this would require a direct relevant proof of $x = \emptyset$ to $C \to C$.[28] While Proposition 2.30 cannot completely rule these out as being provable in DKQ naive set theory, it does help to demonstrate that proving this is not likely to be possible in every instance.[29]

We also can't possibly find something that is a member of A that is not a member of B. For if we find any $x \in A$, that means $x = \emptyset$ must be provable. And $C \to C$ is always provable, and so the conjunction $x = \emptyset \wedge (C \to C)$ is also provable. That is, we can prove

$$x \in A \vdash x \in B.$$

But without a deduction theorem we can't guarantee $x \in A \to x \in B$ is also provable. Thus we can not find a proof that these sets are equal, nor can we prove they are not equal.[30]

The loss of weakening leads to another problem. Mathematical proofs usually take advantage of assumptions which operate as "contextual information", like in the statement "if x and y are ordinals, $x \in On$ and $y \in On$, then $x \subseteq y$ or $y \subseteq x$". That x and y are ordinals is meant to be used to derive the result. There are a few different paths we could take for such a proof, but the following exhibits the problem.

Assume we attempt to prove to show that $x \subseteq y$ directly after assuming some instances of LEM. This requires proving that $z \in x \to z \in y$.[31] Since x was arbitrary, we don't know the set property that defined x and so we can't move forward from $z \in x$. We need to use that x is an ordinal to imply some formula $A(z)$, i.e.

$$x \in On \vdash A(z).$$

Suppose that this property about z is enough to get $A(z) \to z \in y$. It is weakening which would allow us to say that

$$x \in On \vdash z \in x \to A(z)$$

and thus that

$$x \in On \vdash z \in x \to z \in y.$$

Without weakening, we can make no such inference. The use of the assumed information

[28] And the contraposition fairs no better as it would require a direct proof from $\neg(C \to C)$ to $x \neq \emptyset$.

[29] It also seems difficult in DKQ to produce proofs of propositions of the form $A \to (B \to C)$.

[30] With counterexample, there is the outlier case of B defined in reference to Russell's paradox. Take $R \in R \to R \in R$, which would be inconsistent by counterexample. Then B isn't equal to itself, which implies it isn't equal to anything by 3.25.

[31] We also can't redefine subset. cf. discussion of the axiom of infinity in 3.1.1.

$x \in On$ breaks the connection needed for the relevant implication.[32] (See Conjecture 3.49 for more discussion of this.)

2.3.2 Working Around These Problems

To help mitigate some of these problems, we can add an additional conditional and negation connective to the theory.

In the case of negation, we can define another connective $\perp$ which yields a more classical negation defined as $A \to \perp$. Any properties that this negation satisfies then come from the singular added rule that $\perp \vdash A$ for arbitrary A, and the properties of the conditional it's used with. This negation can in fact be defined inside naive set theory and thus there is no need to "add" it. This means the non-triviality result still applies.

In the case of the conditional, we can attempt to add a conditional which behaves more normally, i.e. has weakening and allows a more usual version of conditional proof.[33] The idea of a separate weakening conditional in relevant logics is not new. One place it appears in the literature is in the development of relevant restricted quantification.[34] In this work it takes the form of the "enthymematic" conditional. This conditional works by first adding a constant t, understood as the "conjunction of all theorems", to the language. This constant t satisfies

$$A \dashv\vdash t \to A.$$

The enthymematic conditional is defined as $A \mapsto B := (A \wedge t) \to B$. We can then prove the enthymematic conditional weakens.

Proposition 2.31. $B \vdash A \mapsto B$

Proof. By the properties of t,

$$B \vdash t \to B.$$

Further, by transitivity of the conditional, from $(A \wedge t) \to t$ and $t \to B$ we get that $(A \wedge t) \to B$. This is what we want by definition of the enthymematic conditional, i.e.

$$B \vdash A \mapsto B.$$

□

The nice thing about the enthymematic conditional is that it resembles the relevant conditional as much as possible. This probably means that the naive set theory with it included is

[32]This appears to be another example where weakening plays a unique role in set theory.

[33]Another conditional will take us beyond what is covered in Brady's non-triviality proof.

[34]Relevant restricted quantification would likely be useful for relevant naive set theory, however the proper framework for it isn't yet clear and adding it in would take us further from Brady's non-triviality result. However, we can achieve some of the properties that would be wanted from relevant restricted quantification with a new arrow. More information on restricted relevant quantification can be found in [6].

still non-trivial but this has not been proven.[35] The main difference between the enthymematic conditional and the relevant condition is that we lose access to the strong negation properties of the relevant conditional. For example, consider contraposition of $A \mapsto B$. This becomes $\neg B \to \neg A \vee \neg t$. We have no inferences with $\neg t$ and so this is deductively inert.

It does suffice to prove some theorems outside the scope of the relevant conditional. However these theorems are not necessarily that much of an improvement. All we have added is weakening and this is not enough to get a deduction theorem with it. Standard proofs of the deduction theorem use contraction or the equivalent inference of self-distribution. The enthymematic conditional is contraction-free. Thus we still have the issue that if proving a property requires using assumptions, i.e. working over the turnstile, then we are unable to represent that property in the object language. Thus, the enthymematic conditional is still unsatisfactory for our purposes.

To get a conditional for DKQ naive set theory which has both weakening and a deduction theorem requires moving away from the relevant conditional.[36] I refer to work from Weber which defines such a "naive" conditional [57, 12]. This conditional is denoted $\Rightarrow$. In this context it gets all of the axioms associated with the relevant arrow, except for those involving negation and conjunctive syllogism.[37] It also gets its own version of modus ponens, $A, A \Rightarrow B \vdash B$. We further assume that it weakens, $B \vdash A \Rightarrow B$.

The more complicated issue with including this as a primitive connective is that we have to encode a deduction theorem. Since the arrow is also meant to be contraction-free, this means the turnstile relationship as we have defined it will not suffice since we consider collections of assumptions to be sets. This follows from the principle that multiple assumptions or "uses" of A can all be identified as a single instance in the relationship exhibited by $A \vdash B$.

Definition 2.32. The *use* of a formula A means that it occurs as a premise in the application of a rule.

A deduction theorem for the naive conditional can not hold with *this* particular turnstile. Thus, we consider the turnstile relation in which the assumed collections of formulas are multisets, which are capable of tracking multiple instances of a single object.

Definition 2.33. A *multiset* is a set which contains multiple distinct copies of the same element, e.g. where the following brackets represent multisets, $\{A, A\} \neq \{A\}$.[38]

[35] This is not to say that including it does not expose us to *any* danger of finding ourselves with a trivial naive set theory. Standard relevant semantics suggest that the addition of the t constant with its usual introduction of rule of $\vdash t$ may make it possible to prove $((A \to B) \wedge A \wedge t) \to B$. Then one could use a pseudo modus ponens type proof of Curry's paradox with a slight adjustment to the definition of the Curry set: $\{x | (x \in x \to p) \wedge t\}$. Cf. Proposition 2.10. Thanks to Edwin Mares for showing me this revenge Curry.

[36] The hardest part in adding such a conditional in the present context is that it must be contraction free. cf. [7] for further exposition on how completely we need to avoid contraction.

[37] We can't have conjunctive syllogism without also validating the proof of Curry's that uses pseudo modus ponens.

[38] These can be formalized in the usual set theory as by tagging each element of a set as distinct. For example the multiset $\{A, A\}$ would be $\{\langle 0, A\rangle, \langle 1, A\rangle\}$ as a normal set.

Then this turnstile relationship, which is denoted $\vdash_M$ to separate it from the original, can sensibly track multiple uses or assumptions of the same formula in a proof. Thus we may write

$$A, A \vdash_M B$$

to mean that A was used or assumed twice. Note that this turnstile relationship still satisfies the properties of reflexivity, transitivity, and monotonicity and that if $A \vdash_M B$ holds then $A \vdash B$ also holds by definition. It is for this turnstile we assume a deduction theorem for the naive conditional, which can be understood as adding another meta-rule.

Definition 2.34 (Meta-Rule 3)**.** Let Γ be a multiset of formulas and A and B formulas. Let $\Rightarrow$ be the naive conditional. We assume the following meta-rule for this conditional: if $\Gamma, A, A \vdash_M B$ then $\Gamma, A \vdash_M A \Rightarrow B$.[39]

Remark 2.35. These turnstiles represent real proof relationships between propositions for this theory. However, the turnstiles are not actually connectives in the logic.

Notation 2.36. When referring to "the turnstile" we will omit which in particular it is as it should be clear from context. Our notation will stay differentiated. In the other logical systems, there will not be multiple turnstiles to consider and so those will be unsubscripted even if they more resemble $\vdash_M$ than $\vdash$.

Though the naive conditional does not have much interaction with the paraconsistent negation, it can derive some properties with the defined negation $A \Rightarrow \perp$ that make it look intuitionistic.

Proposition 2.37. *The following hold for* $\Rightarrow$ *and* $\vdash_M$*:*

- $A \Rightarrow B \vdash_M (B \to \perp) \Rightarrow (A \Rightarrow \perp)$
- $A \vdash_M (A \Rightarrow \perp) \Rightarrow \perp$
- $A, A \vdash_M (A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow \perp)) \Rightarrow (A \Rightarrow \perp))$

Proof. The first proposition is the result of hypothetical syllogism: take the instance

$$((A \Rightarrow B) \wedge (B \Rightarrow \perp)) \Rightarrow (A \Rightarrow \perp).$$

Then modus ponens and the deduction theorem gets the desired result.

The second proposition is the result of modus ponens. That is, we have

$$A, A \Rightarrow \perp \vdash \perp.$$

The deduction theorem completes the result.

[39] As with the other meta-rules, this meta-rule has an essentially non-constructive nature.

The third follows from repeated modus ponens, transitivity and the deduction theorem. We have

$$A, A \Rightarrow B \vdash B$$

and

$$A, A \Rightarrow (B \Rightarrow \bot) \vdash B \Rightarrow \bot.$$

Further since

$$\bot \vdash A \Rightarrow \bot,$$

we have that

$$B, B \Rightarrow \bot \vdash A \Rightarrow \bot.$$

Thus

$$A, A, A \Rightarrow B, A \Rightarrow (B \Rightarrow \bot) \vdash A \Rightarrow \bot.$$

The deduction theorem completes the result. □

The primary technical benefits of the naive conditional is that it can use contextual information in proofs and then absorb that information into a formula in the object language and that anything defined with it can be represented in the object language. This makes it more familiar to work with.

Chapter 3

DKQ Naive Set Theory

This chapter discusses some of the work on DKQ Naive Set Theory. Our focus will be the work done in the papers “Transfinite Numbers in Paraconsistent Set Theory” and “Transfinite Cardinals in Paraconsistent Set Theory” by Zach Weber [58, 59]. We will go through the basic set theory covered in those papers.

Along the way we will see various ways this thesis has contributed to the literature in this area. First, a small contribution is that the work in both papers is placed on a common logical basis. The logic originally used in “Transfinite Number” is TLQ, a weak relevant logic closely related to DKQ. However this logic is not covered by the non-triviality result from Brady.[1] A larger contribution is that we will also translate the results that use the enthymematic conditional in Weber’s original papers to use the naive conditional instead.[2] And the final contribution to the literature here will be an attempted extension of the system into non-well-founded set theory.

I will then argue that DKQ naive set theory falls short of what we want.[3] Due to the nature of foundational research, the argument that this theory falls short mostly relies on anecdotal evidence and the resulting intuitions. Examples of this evidence have been provided and formalized results whenever possible. For non-classical foundational research, these sorts of arguments are all we have for considering the validity of these systems since we are not necessarily beholden to another theory as the paradigm of validity.

3.1 The Theory

The axiomatic system DKQ-NST is defined by the logical axioms of DKQ with the naive conditional plus the axioms of naive set theory written with the relevant conditional: the unrestricted axiom schema of comprehension and the axiom of extensionality.

Axiom 3.1 (Unrestricted Comprehension). $\exists y \forall x (x \in y \leftrightarrow A(x))$

[1] Weber notes in “Transfinite Cardinals” that the results of that paper carry over to DKQ which is why we use that here. Transferring the results requires very little extra work.

[2] cf. Section 2.3.2

[3] While I conclude that DKQ naive set theory doesn’t get us the naive set theory we want, the exploration of the “naive universe” can still be illustrative and inform our intuitions.

Axiom 3.2 (Axiom of Extensionality). $\forall x \forall y \forall z((z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$

The language of DKQ-NST is thus the logical connectives $\rightarrow, \Rightarrow, \wedge, \vee, \exists, \forall, \neg$ and the extra-logical symbols $\in$ and $=$. We also add a counterexample rule as $A \wedge \neg B \vdash \neg(A \rightarrow B)$.[4] This is within the bounds of a different non-triviality proof from Brady [8]. We use $\{x|A(x)\}$ to denote the set term defined by the appropriate instantiation of the unrestricted axiom of comprehension, $\forall x(x \in \{z|A(z)\} \leftrightarrow A(x))$. The negated equality, $a \neq b$, is shorthand for $\neg(a = b)$.

Our path through the theory will be to first discuss the recovery of basic set theoretic desiderata and then move into different topics in more depth. We will use some formal axiomatic proofs, though omitting some steps for brevity, in the beginning to establish a sense of how the theory works. We then only return to them when necessary to be explicit.

3.1.1 Basics

The theory recovers basic facts about set theory.

Proposition 3.3. *Identity is an equivalence relation: reflexive, symmetric and transitive. That is, the following holds:*

- $x = x$,
- $x = y \rightarrow y = x$,
- $x = y \wedge y = z \rightarrow x = z$.

Proof. None of these are particularly difficult, but due to the nature of the relevant arrow and the lack of a deduction theorem, they can be a bit tricky.

1.	$z \in x \rightarrow z \in x$	Identity
2.	$z \in x \rightarrow z \in x$	Identity
3.	$z \in x \leftrightarrow z \in x$	Conjunction Intro w/ 1, 2
4.	$\forall z(z \in x \leftrightarrow z \in x)$	Universal Gen w/ 3
5.	$\forall z(z \in x \leftrightarrow z \in x) \leftrightarrow x = x$	Axiom of Ext
6.	$x = x$	Modus Ponens w/ 4, 5

The second property exhibits how careful we need to be to maintain the relevant connection from $x = y$ to $y = x$. Inferences like universal generalization in this theory break that connection since we do not have a deduction theorem.

[4] Adding this rules comes with both advantages and disadvantages. Without it, we have trouble proving that sets are not equal with the paraconsistent negation. On the other hand, it'll end up producing a lot of inconsistent "noise", or extraneous inconsistent facts, in the theory. For example, Russell's paradox, $R \in R \wedge R \notin R$ will imply that $\neg(R \in R \rightarrow R \in R)$. This would extend to any inconsistent set.

1.	$\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$	Axiom of Ext
2.	$\forall z(z \in x \leftrightarrow z \in y) \rightarrow (z \in x \leftrightarrow z \in y)$	Universal Inst
3.	$x = y \rightarrow (z \in x \leftrightarrow z \in y)$	Hypo Syl, MP w/ 1, 2
4.	$(z \in x \leftrightarrow z \in y) \rightarrow (z \in x \rightarrow z \in y)$	Conjunction Elim
5.	$x = y \rightarrow (z \in x \rightarrow z \in y)$	Hypo Syl, MP w/ 3, 4
6.	$x = y \rightarrow (z \in y \rightarrow z \in x)$	Similar
7.	$x = y \rightarrow (z \in y \leftrightarrow z \in x)$	Conj Syl, MP w/ 5, 6
8.	$\forall z(x = y \rightarrow (z \in y \leftrightarrow z \in x)$	Universal Gen
9.	$\forall z(x = y \rightarrow (z \in y \leftrightarrow z \in x)) \rightarrow (x = y \rightarrow \forall z(z \in y \leftrightarrow z \in x))$	A14
10.	$x = y \rightarrow \forall z(z \in y \leftrightarrow z \in x)$	MP w/ 8, 9
11.	$\forall z(z \in y \leftrightarrow z \in x) \leftrightarrow y = x$	Axiom of Ext
12.	$x = y \rightarrow y = x$	Hypo Syl, MP w/ 10, 11

Finally, we prove transitivity which also requires careful steps so as not to break the relevant arrow connection.

1.	$x = y \wedge y = z \rightarrow x = y$	Conjunction Elim
2.	$\forall a(a \in x \leftrightarrow a \in y) \leftrightarrow x = y$	Axiom of Ext
3.	$x = y \rightarrow \forall a(a \in x \leftrightarrow a \in y)$	Conjunction Elim
4.	$\forall a(a \in x \leftrightarrow a \in y) \rightarrow (a \in x \leftrightarrow a \in y)$	Universal Inst
5.	$x = y \wedge y = z \rightarrow (a \in x \leftrightarrow a \in y)$	Hypo Syl, MP w/ 1, 2, 3, 4
6.	$x = y \wedge y = z \rightarrow (a \in y \leftrightarrow a \in z)$	Similar
7.	$(a \in x \rightarrow a \in y) \wedge (a \in y \rightarrow a \in z) \rightarrow (a \in x \rightarrow a \in z)$	Hypo Syl
8.	$x = y \wedge y = z \rightarrow (a \in x \rightarrow a \in z)$	Con Elim, Hyp Syl, MP w/ 5, 6, 7
9.	$x = y \wedge y = z \rightarrow (a \in z \rightarrow a \in x)$	Similar
10.	$x = y \wedge y = z \rightarrow (a \in x \leftrightarrow a \in z)$	Conj Syl, MP w/ 8, 9
11.	$x = y \wedge y = z \rightarrow \forall a(a \in x \leftrightarrow a \in z)$	Uni Gen, A14, MP w/ 10
12.	$x = y \wedge y = z \rightarrow x = z$	Ax of Ext, Hypo Syl, MP w/ 11

□

Definition 3.4. x is a *subset* of y if each element of x is an element of y. Formally, $x \subseteq y := \forall z(z \in x \rightarrow z \in y)$. x is a proper subset of y if each element of x is an element of y and there exists an element of y that isn't an element of x. Formally $x \subset y := x \subseteq y \wedge \exists z(z \in y \wedge z \notin x)$.

Proposition 3.5. *Subsets form a partial order over sets: reflexive, antisymmetric, and transitive. That is, the following holds:*

- $x \subseteq x$,
- $x \subseteq y \wedge y \subseteq x \rightarrow x = y$,

- $x \subseteq y \wedge y \subseteq z \to x \subseteq z$.

Proof. The first is easy.

1.	$z \in x \to z \in x$	Identity
2.	$\forall z(z \in x \to z \in x)$	Universal Gen

As we saw with the equality proofs in Proposition 3.3, there's some care to maintain the relevant connection in the next two proofs.

1.	$x \subseteq y \wedge y \subseteq x \to \forall z(z \in x \to z \in y)$	Conj Elim, Def
2.	$x \subseteq y \wedge y \subseteq x \to \forall z(z \in y \to z \in x)$	Conj Elim, Def
3.	$\forall z(z \in x \to z \in y) \to (z \in x \to z \in y)$	Universal Inst
4.	$\forall z(z \in y \to z \in x) \to (z \in y \to z \in x)$	Universal Inst
5.	$(z \in x \to z \in y) \wedge (z \in y \to z \in x) \to (z \in x \leftrightarrow z \in y)$	Conj Intro
6.	$x \subseteq y \wedge y \subseteq x \to (z \in x \leftrightarrow z \in y)$	Conj Syl, Hypo Syl, MP w/ 1-5
7.	$x \subseteq y \wedge y \subseteq x \to \forall z(z \in x \leftrightarrow z \in y)$	Uni Gen, A14, Hypo Syl
8.	$x \subseteq y \wedge y \subseteq x \to x = y$	Ax of Ext, Hypo Syl, MP w/ 6, 7

1.	$x \subseteq y \wedge y \subseteq z \to \forall a(a \in x \to a \in y)$	Conj Elim, Def
2.	$x \subseteq y \wedge y \subseteq z \to \forall a(a \in y \to a \in z)$	Conj Elim, Def
3.	$(a \in x \to a \in y) \wedge (a \in y \to a \in z) \to (a \in x \to a \in z)$	Hyp Syl
4.	$x \subseteq y \wedge y \subseteq z \to (a \in x \to a \in z)$	Conj Syl, Hyp Syl, MP w/ 1-3
5.	$x \subseteq y \wedge y \subseteq z \to x \subseteq z$	Uni Gen, A14, MP w/ 4

□

Definition 3.6. The complement of a set x is $\bar{x} := \{y | y \notin x\}$.

Proposition 3.7. *A set is equal to the complement of its complement. Formally,* $x = \bar{\bar{x}}$.

Proof. Follows from the chain of inferences, transitivity of the conditional and double negation elimination:

$$z \in \bar{\bar{x}} \leftrightarrow z \notin \bar{x} \leftrightarrow z \in x.$$

Formal proof for the forward direction $z \in \bar{\bar{x}} \to z \in x$.

1.	$z \in \bar{\bar{x}} \leftrightarrow z \notin \bar{x}$	Axiom of Comp
2.	$z \in \bar{x} \leftrightarrow z \notin x$	Axiom of Comp
3.	$z \notin \bar{x} \leftrightarrow \neg(z \notin x)$	Contraposition, MP w/ 2
4.	$\neg(z \notin x) \to z \in x$	Double Neg Elim
5.	$z \notin \bar{x} \to z \in x$	Hyp Syl, MP w/ 3, 4
6.	$z \in \bar{\bar{x}} \to z \in x$	Hyp Syl, MP w/ 1, 5

Formal proof for the other direction.

1.	$z \in \bar{x} \leftrightarrow z \notin x$	Axiom of Comp
2.	$z \in x \leftrightarrow z \notin \bar{x}$	Contraposition, MP w/ 1
3.	$z \in \bar{\bar{x}} \leftrightarrow z \notin \bar{x}$	Axiom of Comp
4.	$z \in x \rightarrow z \in \bar{\bar{x}}$	Hyp Syl, MP w/ 2, 3

□

As should be clear now, the formal proofs in this theory are long. Whereas we could previously use a deduction theorem to simplify things, we have to work purely with the axiom schemas to construct the inferences we need. We will now start using a more traditional and informal method of proof presentation.

We can recover the axioms of Zermelo-Frankael set theory (ZF) except for the axiom of foundation.[5] The first few are instantiations of unrestricted comprehension with the appropriate properties.

Proposition 3.8 (Separation). $\exists y \forall x(x \in y \leftrightarrow x \in a \wedge B(x))$

Separation allows us to take any existing set a and form a subset of its elements selected by satisfying a particular property $B(x)$.

Proof. Any instance of Separation for $x \in A \wedge B(x)$ can be formed from the Axiom Schema for Comprehension by taking that as the instantiating property. □

Proposition 3.9 (Powerset). $\exists y \forall x(x \in y \leftrightarrow x \subseteq a)$

Powerset admits the existence of the powerset of any set, that is the collection of subsets of a set.

Proof. Take the defining property of $x \subseteq a$ for the Axiom of Comprehension. □

Proposition 3.10 (Pairing). $\exists y \forall x(x \in y \leftrightarrow x = a \vee x = b)$

Pairing allows the creation of paired sets from existent sets, as in $\{A, B\}$ for sets A and B.

Proof. Take the defining property of $x = a \vee x = b$. □

Proposition 3.11 (Union). $\exists y \forall x(x \in y \leftrightarrow \exists z(z \in a \wedge x \in z))$

Union grants the existence of the union of a possibly infinite collection of sets, which is the set consisting of all elements of that collection of sets.

Proof. Take $\exists z(z \in a \wedge z \in z)$ as the defining property. □

This takes care of four axioms of ZF plus extensionality which is common to the two systems. Not normally taken as an axiom of ZF, but for comparison to union, we also take intersection defined in the usual way.

[5] More on this later, but it is not surprising that unrestricted comprehension gives non-well-founded sets.

Proposition 3.12 (Intersection). $\exists y \forall x(x \in y \leftrightarrow \forall z(z \in a \rightarrow x \in z))$

However, this no longer necessarily dual to union since the conditional isn't material anymore. On the other hand, we can maintain duality in the finite case. For example, if we have two sets a and b and explicitly take $a \cup b := \{x|x \in a \vee x \in b\}$ and $a \cap b := \{x|x \in a \wedge x \in b\}$, these remain dual by De Morgan's Laws.

The remaining axioms of ZF are the axiom of infinity and the axiom of replacement. We'll start with the axiom of infinity. The axiom of infinity states that there is a set that contains the empty set and all of its successors. To derive this we first need the existence of an empty set and a working definition of successor.

Definition 3.13. The empty set is $\emptyset := \{x|\forall y(x \in y)\}$.

The proposition $\forall y(x \in y)$ is equivalent to triviality. If we find an $x \in \emptyset$, then x is a member of any set, including sets defined on closed formula $\{x|A\}$, and thus any closed formula is true. Thus we can define $\perp := \forall y(x \in y)$.

Proposition 3.14. *The empty set is empty,* $\forall x(x \notin \emptyset)$.

Proof. Either $x \notin \emptyset$ or $x \in \emptyset$. From $x \in \emptyset$, we can derive that $x \in \bar{\emptyset}$. Thus $x \notin \emptyset$ in either case. □

Proposition 3.15. *The empty set is a subset of every set,* $\forall x(\emptyset \subseteq x)$.

Proof. Immediate from the definition of $\emptyset$. If $z \in \emptyset$ then $z \in y$ for any arbitrary y. □

The relevant conditional complicates the defining of "successor" more difficult. The most common definitions of successor for a set x, $Sx = \{x\}$ or $Sx = x \cup \{x\}$ need a notion of singleton. The normal definition of singleton, i.e. $\{a\} := \{x|x = a\}$, does not work well in this theory. Consider attempting to prove the singleton was a subset, i.e. that $z \in \{a\} \rightarrow z \in y$ for some y.[6] Direct proofs of this would use the axiom of comprehension to give us that $z \in \{a\} \rightarrow z = a$, but using $z = a$ for intersubstitution then requires using the rule form

$$z = a \vdash B(a) \rightarrow B(z).$$

The inference over the turnstile will break the relevant connection from $z \in \{a\}$ and thus we can proceed no further. (c.f. Example 3.18) We can work around this limitation by "relativising" singletons to the larger set which we intend to contain them.[7]

Definition 3.16. Let a be a set. The *relevant singleton* of a is $\{a\}_b := \{x|x = a \wedge x \in b\}$.

The relevant singleton forces the subset relation between a set $\{a\}_b$ and b regardless of whether $a \in b$ and thus $\{a\}_b$ contains a or if $\{a\}_b$ turns out to be empty due to $a \notin b$. The

[6] There would be of course trivial cases in which this would be possible irregardless of the next argument. The concern is not with those but the general usability.

[7] This is reminiscent of Separation.

proof will only need to use conjunction elimination. In this theory, this is novel behavior since having $\forall x(x \notin y)$ for some "empty" y, does not suffice to prove that $y \subseteq z$ for any z. This is due to our need for relevance. The classical proof that from $\forall x(x \notin y)$ we derive $y \subseteq z$ results from weakening:

$$x \notin y \vdash x \notin z \to x \notin y.$$

Thus, the sort of emptiness exhibited by the relevant singleton is stronger than a more general notion of emptiness.

We can use relevant singletons to define a few different notions of successor. We'll use one of these now to prove the axiom of infinity. We take the set $i := \{x : x \subseteq i\}$ and define successor relative to the set i as $S_i(x) := \{x\}_i$. We also have our first occurrence of the naive conditional, $\Rightarrow$.

Proposition 3.17 (Axiom of Infinity)**.** *There is a non-empty set i such that when $x \in i$, also $\{x\}_i \in i$. Formally,* $\vdash \emptyset \in i \land \forall x(x \in i \Rightarrow \{x\}_i \in i)$.

Proof. We have $\emptyset \in i$ since $\emptyset \subseteq i$. Then, $S_i(x) \in i$ follows by definition since $z \in S_i(x) \to z = x \land z \in i \to z \in i$ and thus $S_i(x) \subseteq i$. We weaken in $x \in i$ using the naive conditional. □

While this proof is technically correct, it exhibits some further non-intuitive behavior in this theory. We would expect that $x \in i$ had a mathematically relevant connection to $\{x\}_i$ also being a member of i. However, its membership is only a result of the definition of the relevant singleton. If we were to attempt it with a the more normal singleton notion, we find a concrete example of the difficulties mentioned with that definition.

Example 3.18. Assume we're trying to prove the axiom of infinity with the usual definition of singleton, $\{x\} := \{z|z = x\}$. That proof would assume $x \in i$ and then attempt to prove that $\{x\} \subseteq i$.

1.	$x \in i$	Assumption
2.	$z \in \{x\} \to z = x$	Axiom of Comp
3.	$z = x$	Assumption
4.	$x \in i \to z \in i$	Substitution w/ 3

Which allows us to conclude

$$x \in i, z = x \vdash x \in i \to z \in i.$$

We could then use the transitivity of the turnstile and modus ponens to derive that

$$x \in i, z \in \{x\} \vdash x \in i \to z \in i.$$

This still isn't what we need though. We need to conclude

$$x \in i \vdash z \in \{x\} \to z \in i$$

for the theorem. This requires using a deduction theorem. Our need to transition over the turnstile for the rule of substitution makes the relevant connection unprovable. Thus we can't get that $\{x\} \subseteq i$.

We also can not weaken subset to the naive conditional without the relation losing antisymmetry and thus the partial ordering of the subset relation. Antisymmetry is granted by being defined in the same way as the axiom of extensionality requires to prove equality. Consider the proof of Proposition 3.5. In step 5, we use conjunction introduction on the definitions of subset to form the necessary premise for the axiom of extensionality. If those definitions were with the naive conditional, that step wouldn't work.[8] And we'll see later that we can't define extensionality in a looser way with something like the naive conditional. (See Proposition 3.30.)

To prove the axiom of replacement, we'll first need to define what we mean by ordered pairs and functions. Ordered pairs are defined in the standard way:

Definition 3.19. An ordered pair is $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$.

Proposition 3.20. $\vdash \langle a, b\rangle = \langle c, d\rangle \Leftrightarrow a = c \wedge b = d$.[9]

Due to the lack of information we gain from inequalities, we likely can't prove the contraposition, i.e. $\vdash \langle a, b\rangle \neq \langle c, d\rangle \Leftrightarrow a \neq c \vee b \neq d$.

With ordered pairs we can move on to functions. As per usual, the notation $f(x)$ refers to the second member of the ordered pair $(x, y) \in f$.

Definition 3.21. A function is a relation $f \subseteq X \times Y = \{(x, y) : x \in X \wedge y \in Y\}$ such that

1. $\vdash x \in X \Rightarrow \exists y(y \in Y \wedge (x, y) \in f)$
2. $\vdash (x, u) \in f \wedge (x, v) \in f \Rightarrow u = v$.

A function is a contra-function if $\vdash f(x) \neq f(y) \Rightarrow x \neq y$.
A function is injective if $\vdash f(x) = f(y) \Rightarrow x = y$.
A function is contra-injective if $\vdash x \neq y \Rightarrow f(x) \neq f(y)$.
A function is surjective if $\vdash y \in Y \Rightarrow \exists x(x \in X \wedge (x, y) \in f)$.[10]

The general form of a function definition from naive comprehension will look like

$$p \in f \leftrightarrow \exists x \exists y(x \in X \wedge y \in Y \wedge p = \{\{x\}, \{x, y\}\} \wedge \Phi(x, y)).$$

Definition 3.22. The image of a under a function f is $f[a] := \{y | \exists x(x \in a \wedge \langle x, y\rangle \in f)\}$.

[8]In some sense the naive conditional is weaker than the relevant conditional, but it isn't strictly so since the naive conditional has weakening. We can be sure there cannot be a theorem which states that $A \Rightarrow B \vdash A \rightarrow B$ for any A and B as this would imply triviality since it would imply weakening for the relevant arrow. (c.f. Proposition 3.30)

[9]The usual proof works but is long, tedious, and tangential to our focus. It can be found in [58].

[10]It will be useful to have weakening and the detached assumptions when proving facts about functions, and thus all of these function properties are defined with the turnstile.

Proposition 3.23 (Replacement). *Let f be a function with domain a. Then $f[a]$ exists. Formally*[11]*,*

$$\forall x \forall y \forall z (A(x, y, p) \wedge A(x, z, p) \Rightarrow y = z) \Rightarrow \forall z \exists y \forall x (x \in y \leftrightarrow \exists b (b \in z \Rightarrow A(b, x, p))).$$

Proof. The required set existence happens regardless of whether f is a function by instantiating the property $\exists b(b \in z \Rightarrow A(b, x, p)$. We can then weaken in the premise. □

We have finished recovering the axioms of ZF (minus foundation). This has also served to illustrate some of the theorems that this theory can prove. We'll now discuss particular aspects of this theory in more depth to gain a better understanding of what is and is not possible.

3.1.2 Russell's Paradox

Russell's paradox is derivable with the paraconsistent negation.

Proposition 3.24. *Let $R := \{x | x \notin x\}$. Then $R \in R \wedge R \notin R$.*

Proof. Uses the derived rule from Proposition 2.25.

1.	$R \in R \leftrightarrow R \notin R$	Axiom of Comp
2.	$R \in R \rightarrow R \notin R$	Conj Elim w/ 1
3.	$R \notin R$	Reductio w/ 2
4.	$R \notin R \rightarrow R \in R$	Conj Elim w/ 1
5.	$R \in R$	Modus Ponens w/ 3, 4
6.	$R \in R \wedge R \notin R$	Conj Intro

□

Using contraposition of the axiom of extensionality and the counter example rule, we can then derive $\exists x(x \neq x)$.

Proposition 3.25. *Let $x \neq x$. Then $y \neq x$ for all y.*

Proof. Either $y = x$ or not. In the latter case, we're done. In the former, $y = x \vdash x \neq x \rightarrow y \neq x$ by substitution. Then by modus ponens $y = x \vdash y \neq x$. Proof by cases gives that $x \neq x, y = x \vee y \neq x \vdash y \neq x$ and thus $x \neq x \vdash y \neq x$ since LEM holds. □

This result states nothing else can be equal to a non-self-identical object like Russell's set. Of course, since we're inconsistent something can be equal to Russell's set while still being not equal. But this yields a deeper question: "What does $\neg(x = y)$ actually demonstrate?"

It isn't clear what proof theoretic utility a negated equality has in this theory. One thing we could usually infer from the sentence that $a \neq b$ is that we have two distinct objects. Assuming

[11]The formalization of replacement in ZF cannot quantify over functions, and so it has to make due without reference to functions. Thus, the following formalization is what is usually given. The y in the conclusion is the image of z under the function f.

this was the case in this theory would produce some non-intuitive results. Consider the set $\{R\} := \{x|x = R\}$ Since $R = R$, we have $R \in \{R\}$. But we also have that $R \neq R$. Thus if $R \neq R$ meant that R was distinct from R, our notion of cardinality would need to find that $\{R\}$ was of infinite size.

Cardinality is defined in a traditional way in this theory. We'll still "count" by injective functions. Thus an injective mapping will need to use the positive theorem that $R \in \{R\}$ to establish that $\{R\}$ has a bijection with the canonical set of size 1. The fact that $R \neq R$ does not matter.

What does seem to be required to produce distinctness in this theory is the stronger and harder to prove claim that $(x = y) \to \perp$ or maybe $(x = y) \Rightarrow \perp$. In this case, if we had an injective function which mapped x and y to the same object, our theory would become trivial. By implying $\perp$, there is no sense in which we are allowed to identify such sets.

An open question to consider for this theory is how do inconsistent objects affect the notion of bijections (and isomorphisms) in general? Does a bijection require that two inconsistent objects be mapped to each other? That is, if we have two sets $\{a, b\}$ and $\{c, d\}$ which appear to fit the normal notion of bijection but a is inconsistent, can we succeed in forming a paraconsistent bijection between these two sets?[12] It depends on whether we think that the paraconsistently negated information needs to match in the domain and codomain. Given that negated equalities do not yield any information, it seems unlikely that any non-trivial bijection could be proved if we thought this was necessary.

3.1.3 Universal Sets

Unrestricted comprehension allows us to construct universal sets.

Definition 3.26. A set U is *universal* if $\forall x(x \in U)$.[13]

The main universal set used in the theory is

$$V := \{x|\exists y(x \in y)\}.$$

This exists by comprehension. It has the property of being universal. It also has the property of being as consistent as possible: if anything were found to not be a member of V, we would have triviality. These theorems are proved next.

Proposition 3.27. $\forall x(x \in V)$.

Proof. Either $x \in V$ or $x \notin V$. In the latter case, $x \in \bar{V}$ and thus $\exists y(x \in y)$. Thus, in either case $x \in V$. □

[12] This is complicated by the fact that the theory proves the law of non-contradiction, $\neg(A \wedge \neg A)$ as a result of De Morgan's laws. And thus the theory itself can not actually identify any objects which are behaving consistently with respect to the paraconsistent negation.

[13] Since we are paraconsistent, it could be the case that a universal set also does not contain some things, like $U := \{x|x = x\}$. Or it may contain everything and nothing, as with the peculiar set $Z = \{x : x \notin Z\}$.

Proposition 3.28. $\forall x(x \subseteq V)$.

Proof. If $z \in x$ then $\exists y(z \in y)$. Thus $z \in V$. $\square$

Proposition 3.29. $V = \bar{\emptyset}$ *and* $\bar{V} = \emptyset$.

Proof. Using previous propositions 3.28 and 3.15 we're half way there on each equality.

We only need $V \subseteq \bar{\emptyset}$ to finish the first proposition. We'll prove the contrapositive. Assume $x \notin \bar{\emptyset}$. Then $x \in \emptyset$ by double negation elimination and thus $x \in \bar{V}$. So $x \notin V$ as needed. This completes $V = \bar{\emptyset}$.

For $\bar{V} = \emptyset$, we only need that $\bar{V} \subseteq \emptyset$. If $x \in \bar{V}$ then $\forall y(x \notin y)$. Thus $x \notin \bar{\emptyset}$ which gives that $x \in \emptyset$. This completes $\bar{V} = \emptyset$. $\square$

That $\bar{V} = \emptyset$ implies if any set x was found to not be a member of V, then $x \in \emptyset$. This implies triviality.

This theory produces an infinite number of universal sets that cannot be proved equal to each other. We can take any true closed formula of the form $(B \to B)$ and form the universal set $U := \{x | \exists y(x \in y) \wedge (B \to B)\}$. We can't prove that any of these U's equal V by Proposition 2.30.[14]

These "dopplegängers" appear to be an inevitable feature of the theory. If we try to erase the need for the strong connections between formula by adding weakening, we end up in triviality.

Proposition 3.30. *DKQ-NST with the extensionality axiom as* $\forall z(z \in x \Leftrightarrow z \in y) \Leftrightarrow x = y$ *is trivial.*

Proof. Consider the set $U := \{x : x = x\}$. This set is also universal by Proposition 3.3.

If $x \in V$ then weakening gives $x \in V \to x = x$. With the axiom of comprehension giving $x = x \to x \in U$, we can conclude that $x \in V \to x \in U$. For the other direction, we use that V is universal. Since $x \in V$ for any x, by weakening we have $x \in U \to x \in V$. Thus $U = V$.

But for Russell's set we have that $R \neq R$. Thus $R \notin U$ and since $U = V$, $R \notin V$. Then $\bar{V} = \emptyset$ implies that $R \in \emptyset$. $\square$

3.1.4 Ordinals

The ordinals are normally formed by taking all sets which are well-ordered and transitive. This definition induces the rich structure which extends far beyond those two properties. The primary obstacle to the ordinals in this theory is the same: setting up the definition correctly.[15]

Notation 3.31. $x_1, x_2, \ldots, x_n \in a \vdash B$ is an abbreviation for $x_1 \in a, x_2 \in a, \ldots, x_n \in a \vdash B$.[16]

[14] In the same way, we can produce infinitely many empty sets.

[15] A large portion of the alterations I've made to the theory are in this section.

[16] Note that without contraction, this is a stronger claim then $x_1 \in a \wedge x_2 \in a \wedge \ldots \wedge x_n \in a$. Each use of one of the conjuncts requires a use of the larger conjunction. Assuming each of them is detached allows us to avoid requiring multiple instances of that conjunction.

Definition 3.32. A set a is strictly ordered with respect to $\in$ if

$$\begin{array}{ll} x \in a \vdash & x \notin x, \\ x, y \in a \vdash & (x \in y \wedge x \notin x \Rightarrow y \notin x), \\ x, y, z \in a \vdash & (y \in z \Rightarrow (x \in y \Rightarrow x \in z). \end{array}$$

This is mostly standard, except that we have the addition of $x \notin x$ to the antisymmetry clause. Omitting this in the definition would result in us being unable to prove that the set of ordinals was itself strictly ordered. We would instead be able to prove that

$$\alpha, \beta \in On \wedge \alpha \in On \Rightarrow (\alpha \in \beta \wedge \alpha \notin \alpha \Rightarrow \beta \notin \alpha),$$

but we don't wish to have the extra $\alpha \in On$ in our hypothesis.[17] Also, since we do not have contraction or conjunctive syllogism, strictly ordered is split as the conjunction of three naive implications starting with elements in a.

Definition 3.33. A set a is linearly ordered by $\subseteq$ if a is strictly ordered with respect to $\in$ and

$$x, y \in a \Rightarrow x \subseteq y \vee y \subseteq x.$$

Definition 3.34. A set a is well-founded if

$$y \subseteq a \wedge \exists z(z \in y) \Rightarrow \exists z(z \in y \wedge \neg\exists x(x \in z \wedge x \in y)).$$

Definition 3.35. A set a is well-ordered, $Wo(a)$, if it is strictly ordered by $\in$, linearly ordered by $\subseteq$, and well-founded.

Definition 3.36. A set a is transitive, $Tr(a)$, if $x \in a \to x \subseteq a$.

A strict interpretation of the classical ordinals would take all sets that are well-ordered and transitive. In this theory, this is too weak to prove many of the properties we'll want. To prove some propositions which seem essential to the structure of the ordinals, we need to define these properties into what it means to be an ordinal. Doing so makes proving that a set is an ordinal more difficult, but also means that we know more about sets that are assumed to be ordinals.

The properties we'll add to the definition of ordinals are that an ordinal α is itself composed of ordinals, $\alpha \subseteq On$, that for any other ordinal β, $\alpha \subseteq \beta \vee \beta \subseteq \alpha$, and that ordinals are not self-membered, $\alpha \notin \alpha$.

[17] cf. Proposition 3.43

Proposition 3.37. *There is a set of all ordinals, On, such that*

$$\begin{aligned} x \in On \leftrightarrow \quad & Wo(x) \wedge \\ & Tr(x) \wedge \\ & x \subseteq On \wedge \\ & y \in On \Rightarrow (x \subseteq y \vee y \subseteq x) \wedge \\ & x \notin x \end{aligned}$$

Proof. The existence of this set is from unrestricted comprehension. □

We consider each additional property in turn to explore why there is not or likely not a proof for each from the properties of well-ordering and transitivity. First, such a proof would need to be of the form $x \in On \rightarrow A$.[18] For $x \in On \rightarrow (x \subseteq On)$ which is equivalent to $x \in On \rightarrow (z \in x \rightarrow z \in On)$, this is of the logical form $A \rightarrow (B \rightarrow C)$ where A, B, and C are atomic formula. Inspection of the axioms of DKQ suggests that proving something of this form is difficult, if not impossible.[19]

The difficulty in proving that $x \in On \rightarrow y \in On \Rightarrow (x \subseteq y \vee y \subseteq x)$ is again a result of the logical form. It is not clear how such a proof would work without adding additional axioms detailing the interactions of $\rightarrow$ and $\Rightarrow$.[20] And finally irreflexivity, $x \in On \rightarrow x \notin x$, would likely rest on the use of the law of excluded middle. LEM is provided in axiomatic form as detached and not connected to anything else by a relevant implication. The primary axiom scheme that LEM is involved with is proof by cases, and producing the form $x \in On \rightarrow x \notin x$ from this requires proving both

$$(x \in x \rightarrow (x \in On \rightarrow x \notin x)) \wedge (x \notin x \rightarrow (x \in On \rightarrow x \notin x)).$$

It is assumed that to discuss some proper form of the ordinals implies that we have these properties, and so we must proceed with the extended ordinal definition offered in Proposition 3.37. Further properties of the ordinals easily follow from this definition.

Proposition 3.38. $\emptyset \in On$.

Proof. This follows from the triviality properties of the empty set. That is each property needed follows from the antecedent $x \in \emptyset$ implying $\perp$. □

Proposition 3.39. $Wo(\alpha), \beta \subseteq \alpha \vdash Wo(\beta)$.

[18] It may be possible to prove $x \in On \Rightarrow A$ for some of these, like $x \in On \Rightarrow x \subseteq On$, though that is in doubt as well. The hardest one to prove in this form would likely be $x \in On \Rightarrow x \notin x$ since it uses the paraconsistent negation which the naive conditional has little interaction with.

[19] That said, I was unable to find a proof that deriving such forms was completely impossible, of course omitting the trivial instances provided by the axiom of comprehension.

[20] Such an exploration should be pursued if this naive set theory is desired. It takes us even further from the non-triviality results and thus requires caution. As this thesis was concerned with naive set theory in general, this avenue of research was not carried out.

Proof. Follows directly using transitivity of the conditional and turnstile. For an example, we prove that

$$x, y, z \in \beta \vdash (y \in z \Rightarrow (x \in y \Rightarrow x \in z)).$$

We assume $Wo(\alpha)$, $x, y, z \in \beta$ and $\beta \subseteq \alpha$. Thus we have $x, y, z \in \beta \rightarrow x, y, z \in \alpha$ by our subset assumption. We use modus ponens to derive each of $x \in \alpha$, $y \in \alpha$ and $z \in \alpha$. Then by $Wo(\alpha)$, we have that

$$x, y, z \in \alpha \vdash (y \in z \Rightarrow (x \in y \Rightarrow x \in z).$$

Transitivity of the turnstile gives the desired result. □

Remark 3.40. This proposition looks a bit different with the other turnstile. The assumption $\beta \subseteq \alpha$ is used quite a few times, at least nine.

Proposition 3.41. $\alpha \in On \rightarrow \alpha \subseteq On$.

Proof. By the definition of On. □

Proposition 3.42. $\alpha \in On \Rightarrow \alpha \notin \alpha$.

Proof. By definition of On. □

Proposition 3.43. $\alpha, \beta \in On \Rightarrow (\alpha \in \beta \wedge \alpha \notin \alpha \Rightarrow \beta \notin \alpha)$.

Proof. Since $\alpha \in On$, we have that $Tr(\alpha)$, $\beta \in \alpha \rightarrow \beta \subseteq \alpha$. We can take the contraposition $\beta \nsubseteq \alpha \rightarrow \beta \notin \alpha$. Then using counterexample with $\alpha \in \beta \wedge \alpha \notin \alpha$ we get that $\beta \nsubseteq \alpha$. Modus ponens gets us $\beta \notin \alpha$ as we need. □

Proposition 3.44. $\alpha, \beta, \gamma \in On \Rightarrow (\beta \in \gamma \Rightarrow (\alpha \in \beta \Rightarrow \alpha \in \gamma)$.

Proof. This is the definition $Tr(\gamma)$ and this holds since $\gamma \in On$. □

Proposition 3.45. *Any two ordinals are $\subseteq$-connected,*

$$\alpha \in On \rightarrow (\beta \in On \Rightarrow \alpha \subseteq \beta \vee \beta \subseteq \alpha).$$

Proof. By definition of On. □

Proposition 3.46. *On is well-founded.*

Proof. Let $\theta \subseteq On$ and $\beta \in \theta$. Either $\beta \cap \theta$ is empty or not.

If $\beta \cap \theta$ is empty, i.e. $\forall y(y \notin \beta \cap \theta)$ then we have $\beta \in \theta \wedge \forall y(y \notin \beta \vee y \notin \theta)$. We can weaken and use existential introduction to get

$$\theta \subseteq On \wedge \exists z(z \in \theta) \Rightarrow \exists z(z \in \theta \wedge \forall y(y \notin z \vee y \notin \theta)).$$

If $\beta \cap \theta$ is not empty, then $\exists y(y \in \beta \cap \theta)$. Since $\beta \cap \theta \subseteq \beta$, we have that $\beta \cap \theta$ is well-founded by a Proposition 3.39. Thus since $\beta \cap \theta$ is well-founded, there is some γ such that

$$\gamma \in \beta \wedge \gamma \in \theta \wedge \forall y(y \notin \gamma \vee y \notin \beta \vee y \notin \theta).$$

Since $\beta \in On$ and $\gamma \in \beta$, we have that $\gamma \subseteq \beta$. By contraposition $y \notin \beta \to y \notin \gamma$. And thus the disjunction in the above formula reduces to $\forall y(y \notin \gamma \vee y \notin \theta)$. All together we have

$$y \in \theta \wedge \forall y(y \notin \gamma \vee y \notin \theta).$$

Using weakening and existential introduction gets us to

$$\theta \subseteq On \wedge \exists z(z \in \theta) \Rightarrow \exists z(z \in \theta \wedge \forall y(y \notin z \vee y \notin \theta)).$$

We've shown by proof by cases that an instance of the law of excluded middle, $\forall y(y \notin \beta \cap \theta) \vee \exists y(y \in \beta \cap \theta)$ implies that On and thus we are done. □

Proposition 3.47. *$On \notin On$.*

Proof. Either $On \in On$ or $On \notin On$. In the former case, $On \in On$ implies $On \notin On$ by definition of On. □

All of these previous propositions together imply that $On \in On$. Thus we have the Burali-Forti paradox.

Theorem 3.48 (Burali-Forti Paradox)**.** *$On \in On$ and $On \notin On$.*

The difficulties with the ordinals in this theory are past this point. While we were able to get the properties we want from the ordinals by adding to the definition, we have also made it much harder to prove that something is an ordinal. For example, it's not clear there's even a proof that $\{\emptyset\} \in On$. We would need to try to prove that $\{\emptyset\}$ is transitive, but this doesn't seem possible due to the substitution rule breaking the relevant connection between $z \in \{\emptyset\} \to z = \emptyset$ and $\emptyset \in On$. We could try $\{\emptyset\}_{On} \in On$ but it's not clear we can prove that $y \in On \Rightarrow \{\emptyset\}_{On} \subseteq y \vee y \subseteq \{\emptyset\}_{On}$.[21]

The inability of this is indicative of a larger problem. Basically, we hope to be able to prove the following conjecture to establish more sets are ordinals.

Conjecture 3.49. *Let $\alpha, \beta \in On$, let θ be a well-ordered, transitive set of ordinals, and let $\alpha \subseteq \theta \subseteq \beta$. Then $\theta \in On$.*[22]

However, this conjecture remains unsolved and doesn't seem possible to prove. To prove this we need to establish that $\forall y(y \in On \Rightarrow y \subseteq \theta \vee \theta \subseteq y)$. We can't proceed by contraposition since we're using the naive conditional. Attempting a proof by reductio results in the negated subset relation, which produces negated conditionals which produce no further information. If we have to proceed directly, then we need to prove one of the disjuncts, which means assuming some z in either y or θ. The problem then is that we don't know the defining condition of y or θ since they're arbitrary, and so comprehension can't help. Even having that $y \in On$ isn't very

[21] This primarily seems to be a result of having strict ordering apply over $\in$ but linear order over $\subseteq$. But it also doesn't seem possible to bring those two concepts together over $\in$ and have the other results go through.

[22] Note that this conjecture is never needed classically since once we know that θ is a well-ordered and transitive set, we also know it's an ordinal.

helpful since it only tells us about y's relationship with other sets we know to be ordinals, and we do not yet know θ to be an ordinal.

Without this conjecture, an essential result we can not derive is that the successor of an ordinal is an ordinal itself.

Definition 3.50. The *successor* of a set α is $\alpha^+ = \alpha \cup \{\alpha\}_{On}$.

Conjecture 3.51. *$\alpha \in On \vdash \alpha^+ \in On$.*

This means that we only know $\emptyset$ and On are ordinals and cannot prove the set is any bigger than that. This creates a problem in constructing arithmetic using the ordinals. If we attempt to take the natural numbers as a subset of On we do not get very many numbers. There may be an alternate method to constructing them as

$$\forall x(x \in \mathbf{N} \leftrightarrow \forall \alpha(\forall y(y \in \alpha \to y^+ \in \alpha) \to (\emptyset \in \alpha \to x \in \alpha))).$$

This construction is used in the other naive set theories we consider.

3.1.5 Non-well-founded Set Theory

Definition 3.52. A *non-well-founded* set is a set that has an infinite descending membership chain. That is, S is non-well-founded if a sequence of elements can be found such that

$$S \ni a_1 \ni a_2 \ni a_3 ...$$

We already have a few examples of non-well-founded sets from naive comprehension: Russell's set and universal sets are non-well-founded. We also have the "prototypical" non-well-founded set, the Quine atoms $\Omega = \{\Omega\}$, by the set instantiation

$$x \in \Omega \leftrightarrow x = \Omega.$$

Our work on non-well-founded sets in DKQ naive set theory is mostly guided by Peter Aczel's work on classical non-well-founded set theory [1]. Aczel uses ZFC but replaces the axiom of foundation with four different "Anti-Foundation Axioms". These axioms can be understood as granting the existence of progressively larger classes of non-well-founded sets.

The four different axioms Peter Aczel considers are named as follows: "Aczel's Anti-foundation Axiom", "Scott's Anti-foundation Axiom", "Finsler's Anti-foundation Axiom", and "Boffa's Anti-foundation Axiom".[23] We will only be focused on Aczel's and Boffa's. Aczel's AFA grants existence of the least amount of non-well-founded sets of the four, while Boffa's AFA grants the existence of the most of the four [1].

Both Aczel's and Boffa's AFA can be motivated by generalizing a standard classical result, Mostowski's Collapsing Lemma. We are briefly considering classical set theory and thus the

[23] Aczel's work doesn't name the first one as Aczel's Anti-Foundation Axiom, but it makes it easier to differentiate by appending his name there.

logic of these definitions is assumed to be classical. We use $\supset$ to notate the classical conditional and $\equiv$ for the classical biconditional to separate it from the other conditionals of the chapter.

Definition 3.53. A relation R on set A is well-founded if

$$\forall S(S \subseteq A \wedge \exists z(z \in S) \supset \exists z(z \in S \wedge \forall x(xRz \supset x = z)).$$

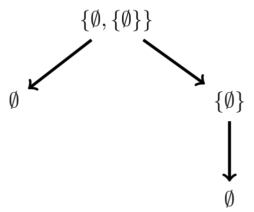
That is for each subset of A, there is a z such that z is an R-minimal element. There is nothing which R-precedes z, i.e. no x such that xRz.

Definition 3.54. A relation R is extensional if

$$\forall x \forall y(\forall z(zRx \equiv zRy) \equiv x = y).$$

Theorem 3.55 (Mostowski's Collapsing Lemma [31])**.** *Suppose R is a binary relation on a set S such that R is extensional and well-founded. Then there exists an unique transitive set whose structure under the membership relation is isomorphic to (S, R), and the isomorphism is unique. Further, the isomorphism is such that $f(x) = \{f(y)|yRx\}$.*

Mostowski's Collapsing Lemma allows us to use well-founded and extensional relations to find sets whose membership structure is the same as the relation structure. For an example, consider the set $\{a, b, c\}$. Define a relation on the set as aRc, aRb and bRc. This is a well-founded and extensional relation so we get a corresponding set. That corresponding set is $\{\emptyset, \{\emptyset\}\}$. This can be nicely visualized in the following picture where the edges correspond to a relationship between nodes.



To derive Aczel's and Boffa's axioms, we then consider generalizations of this result. Aczel's is derived by removing the need for a well-founded relation.

Axiom 3.56 (Aczel's Anti-foundation Axiom)**.** *Suppose R is a binary relation on a set S such that R is extensional. Then there exists an unique transitive set whose structure under the membership relation is isomorphic to (S, R), and the isomorphism is unique. Further, the isomorphism is such that $f(x) = \{f(y)|yRx\}$.*

Boffa's goes a step further and removes the requirement that the transitive set mapped to is unique.

Axiom 3.57 (Boffa's Anti-Foundation Axiom)**.** *Suppose R is a binary relation on a set S such that R is extensional. Then there exists a transitive set whose structure under the membership relation is isomorphic to (S, R), and the isomorphism is unique. Further, the isomorphism is such that $f(x) = \{f(y)|yRx\}$.*

Which Anti-Foundation axiom do we aim for? Aczel's axiom implies a stronger version of extensionality on the set theoretic universe. This stronger version of extensionality relies on the uniqueness of the corresponding transitive set and this is unlikely to hold in DKQ naive set theory since the theory produces few sets of unique structure, if any at all.[24] On the other hand, Boffa's produces a much larger universe of non-well-founded sets than Aczel's Anti-Foundation axiom. We'll set that axiom as our target. Can we show that DKQ naive set theory produces at least as many non-well-founded sets as Boffa's grants?

Before that we should first try to recapture something like Mostowski's Collapsing Lemma. That the needed function exists is straightforward. This is a simple instantiation of naive comprehension.

Lemma 3.58. *Let R be a relation on A. Then there exists a f such that*

$$f(x) = \{y|\exists z(zRx \wedge f(z) = y)\}.$$

Proof. By naive comprehension we may instantiate f as follows

$$\langle x, y\rangle \in f \leftrightarrow y = \{z|\exists a(aRx \wedge f(a) = z)\}.$$

□

However, we can't get much further than this. Translating the extensionality condition is problematic. We'll consider using both of the conditionals we have available.

Definition 3.59. A relation R is relevantly extensional iff $\forall z(zRx \leftrightarrow zRy) \leftrightarrow x = y$.

Definition 3.60. A relation R is deducibly extensional iff $\forall z(zRx \Leftrightarrow zRy) \Leftrightarrow x = y$.

If we use the former definition, Mostowski's is unprovable. The latter definition makes Mostowski's useless.

In the former case, the problem will arise when attempting to prove the function is injective. We would need to derive from our assumption $f(x) = f(y)$ that $\forall z(zRx \leftrightarrow zRy)$ to take advantage of our assumption that R is extensional and thus get $x = y$. Given the definition of f, we would need to start by first showing that if some aRx then $f(a) \in f(x)$ as a relevant implication. This move is not possible because it requires $aRx \rightarrow aRx \wedge f(a) = f(a)$ but the second conjunct is irrelevant to aRx.

Remark 3.61. This seems to be a case where weakening is actually needed for mathematical reasoning. Weakening would allow us to prove $\vdash aRx \rightarrow f(a) = f(a)$ from $\vdash f(a) = f(a)$.

The deducibly extensional definition would allow us to make this move since it can weaken. In fact, if we assume that transfinite induction is provable[25], we can prove Mostowski's.

[24] Consider the ways given before to construct arbitrarily many dopplegängers.
[25] And it is not clear that it is provable.

The base case of the inductive proof is that $f(x) = f(y) \vdash x = y$ where x and y are R-minimal. The inductive hypothesis is then any R previous elements to x and y, i.e. some aRx and bRy, $f(a) = f(b) \vdash a = b$.

$$\begin{array}{lll} aRx & \Rightarrow f(a) \in f(x) & \text{weakening and existential generalization} \\ & \Rightarrow f(a) \in f(y) & \text{substitution} \\ & \Rightarrow \exists z(zRy \wedge f(z) = f(a)) & \text{definition of f} \\ & \Rightarrow z = a & \text{inductive hypothesis} \\ & \Rightarrow aRy & \end{array}$$

Then one could use universal generalization on a and the proof is secured.[26]

Though this proof might work, there is another problem. It would be impossible to prove that any non-trivial relation was deducibly extensional. Let's assume we are trying to prove $x = y$ under assumption $\forall z(zRx \Leftrightarrow zRy)$. That proof would begin with choosing an element $z \in x$. Then presumably we derive zRx to infer that zRy. However that inference to zRy uses a naive conditional and thus breaks the relevant implication. The relevant implication is what is needed to prove equality.

None of what is done in this section demonstrates that non-well-founded sets can't be handled in DKQ-NST. In fact, it is quite clear that the theory can create some non-well-founded sets. However, this section demonstrates some of the difficulties DKQ-NST has in allowing broader mathematical arguments and constructions like that found is Mostowski's which provides a very useful characterization of constructions possible in classical set theory.

3.2 Further Work

There are still plenty of questions to ask about the theory. Here are some for further research.

The ordinal structure we have doesn't produce more ordinals than $\emptyset$ and On. The question remains whether there is one that can? And if not, what can we make of the theory of cardinals in this theory?

Can we still get some arithmetic results and get around the limitations of the ordinals in this theory if we focus on an alternate construction of the natural numbers? Another possible construction might be $\mathbf{N} := \{x | x = \emptyset \vee \exists y(y = x^+)\}$.

What do we make of bijections in this system? With injections being split into the regular positive direction and contra-injections, do we need to assume both for bijections?

Can the definable "strong negation" $\perp$ be used to greater effect in the theory?

"Transfinite Cardinals" offers a non-constructive proof of the well-ordering theorem in the system. However, no relation, constructive or non-constructive, is offered which well-orders the universe. Can that proof be expanded to include such a relation, or does the theory make sufficient sense of a well-ordering without such a thing?

[26] This type of proof structure would be a non-starter for the non-well-founded case since we need induction.

3.3 Conclusion

DKQ naive set theory offers us our first glance into the naive universe. For that, it is invaluable. However, it doesn't quite seem to accomplish giving us both a truth predicate and sufficient mathematical ability. The relevant conditional becomes too much of a burden to have around in the axioms and heavily restricts what we can do.

The naive conditional, on the other hand, proves to be useful. Thus we'll now see what we can do if that's our only conditional by turning to L_{BCK} and light linear logic naive set theory.

Chapter 4

Simply Contraction-Free

Another approach to achieving consistency with unrestricted comprehension is starting with a classical sequent calculus and dropping both left and right contraction. This allows us to prove a cut elimination result for the theory, which guarantees consistency. The first proof of this fact is attributed to Grišin [24].

A strong feature of this approach is that the resultant logics have conditionals that are more natural to work with.[1] However, while the system is consistent with unrestricted comprehension, it becomes inconsistent again if we add in extensionality. This is known as Grišin's paradox.[2] Despite this problem, these theories provide insight into powerful ways to work with unrestricted comprehension.[3]

We'll start by looking at the "simply contraction-free" approach. We'll look at a particular logic which fits this description, compare and contrast it with DKQ and then go through the cut elimination proof with unrestricted comprehension. The way in which this cut elimination proof works will be used to help motivate light linear logics.

4.1 L_{BCK}

The logic of this type we'll consider is L_{BCK} [37]. This logic, presented as a sequent calculus with conjunction written as $\star$, the conditional as $\multimap$ and the disjunction as $+$[4], follows.

$$\frac{}{A \vdash A}\ \text{Axiom} \qquad\qquad \frac{}{\bot \vdash A}\ \bot$$

$$\frac{\Gamma, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\ \text{Weakening} \qquad\qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash C}{\Delta, \Gamma \vdash C}\ \text{Cut}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \text{Exchange}$$

[1] They look a lot like the naive conditional we defined to go with DKQ.

[2] We'll look at this paradox in the chapter on light affine logic set theory, Chapter 5.

[3] Theories with only unrestricted comprehension and not extensionality are sometimes referred to as "naive class theories" rather than naive set theory.

[4] The reason for this notation is for easy comparison to linear logics.

$$\frac{\Gamma \vdash A \qquad B, \Delta \vdash C}{A \multimap B, \Gamma, \Delta \vdash C} \multimap_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \multimap_R$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A + B} +_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A + B} +_{R2}$$

$$\frac{\Gamma, A, \Delta \vdash C \qquad \Gamma, B, \Delta \vdash C}{\Gamma, A + B, \Delta \vdash C} +_L$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \star B, \Delta \vdash C} \star_L \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \star B} \star_R$$

The application of a left or right rule in this system are referred to as logical inferences, whereas the other rules are referred to as structural inferences. A deduction in this logic is any correct application of inferences. The negation in the logic is the one defined as $\sim A := A \multimap \bot$.

While L_{BCK} only allows a single formula to occur on the right side of the turnstile, there is no particular reason to do this for the consistency of naive set theory. The axiom of comprehension with the logic remains consistent if we allow multiple formulas on the right.

4.1.1 L_{BCK} as a Logic for Mathematics

This logic avoids some of the problems associated with DKQ that we discussed above but also has some of its own.

The negation is not paraconsistent. It's somewhere in between an intuitionistic negation and a classical one: even if we allowed multiple formula on the right, we couldn't derive the law of excluded middle as

$$\vdash A + \sim A$$

without right contraction but we could derive

$$\vdash A, \sim A$$

with right weakening. The failure of the first form of the law of excluded middle is good for the contraction-free property since it can be used to reprove conjunction contraction, $A \vdash A \star A$.

Proposition 4.1. *The law of excluded middle, $\vdash A + \sim A$, proves $A \vdash A \star A$.*

Proof. The proof follows.

$$\dfrac{\vdash A + \sim A \qquad \dfrac{\vdash A + \sim A \qquad A, A \vdash A \star A \qquad \dfrac{A \vdash A \qquad \bot \vdash A \star A}{A, \sim A \vdash A \star A} \multimap_L}{A, A + \sim A \vdash A \star A} +_L}{A \vdash A \star A} \text{Cut}$$

□

This negation can derive counter example, $(A \star \sim B) \vdash \sim (A \multimap B)$. We can also recover some information from a negated conditional, $\sim (A \multimap B)$; we can prove $((A \multimap \bot) \multimap \bot) \star (B \multimap \bot)$. We can't make the inference to $A \star (B \multimap \bot)$ without double negation elimination.

Proof by contradiction is, in some sense, present in this system but is a little odd. The inference requires contraction so the most we can prove is $(A \multimap B) \multimap (A \multimap (B \multimap \bot)) \multimap (A \star A \multimap \bot)$.

Finally, while this conditional acts well in sequent calculus, it still doesn't quite validate an ordinary deduction theorem and thus an ordinary style of conditional proof [13]. This is due to the lack of contraction. What it does verify for conditional proof is one in which the number of times an assumption is used is kept track of, and that many appearances of that assumption end up in the final conditional.[5] This understanding is implicit in the sequent calculus since it doesn't have contraction, i.e. each assumption is distinct even from itself. That is $A, A \vdash C$ acts as if it were $A, B \vdash C$ where A is distinct from B. Put yet another way, we consider the collection of assumptions to be a multi-set.

4.1.2 Theorems Still Contract

Contraction-free systems still verify contraction for theorems. We can derive $A \vdash A \star A$ if we already have that $\vdash A$.

Proposition 4.2. *If* $\vdash A$ *then* $A \vdash A \star A$.

Proof. Use of the right wedge rule.

$$\frac{\vdash A \qquad A \vdash A}{A \vdash A \star A} \star_R$$

□

This suggests an interesting perspective for mathematics performed in the contraction-free systems. When we prove theorems in mathematics, they'll have the form $A \multimap B$. What if we have to use A multiple times? Then the theorem amounts to $A^n \multimap B$ where A^n is a representation for a conjunction of n A's. Then if we have actually constructed an object that satisfies the theorem premise, we're still free to derive B. In practice, it would seem we don't actually lose anything that essential.

Yet, this isn't the whole story. The innocuous action of contraction in conjunction with cut in a proof can actually hide a great deal of complexity. The ordinal analysis of contraction-free proof theoretic systems suggests that these systems are in fact much weaker, on the order of ω [52]. Which means that the cut-free proofs aren't really all that large, and thus the absence of contraction means that cut can't bundle up as much complexity. Compare that to the infinite size of cut-free proofs of a system like PA [55] and it seems like this "in practice" idea really does miss something. The presence of contraction allows more than the contraction of finite premises, but allows us to deal with the infinity needed in inductive arguments like we see in PA.

[5] For an example of how the conditional proof would work, see the natural deduction system in [61] which is basically a natural deduction formulation of the sequent calculus we're considering.

4.1.3 Cut Elimination Proof with Unrestricted Comprehension

The removal of contraction in a logic greatly simplifies its cut elimination proof and as it turns out the, the proof is still valid even in the presence of rules for unrestricted comprehension. To represent unrestricted comprehension in a sequent calculus, we add the rules $\in_L$ and $\in_R$.

$$\frac{A[t/x], \Gamma \vdash C}{t \in \{x|A\}, \Gamma \vdash C} \in_L \qquad\qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash t \in \{x|A\}} \in_R$$

Ordinarily a cut elimination proof requires inducting on the complexity of a cut formula, where complexity is measured as the number of logical connectives present in the formula. The proof proceeds from there by showing that we can always either "push a cut upward", i.e. we can replace the current cut in question with a cut on a lower complexity formula higher in the proof, or we're at a step where we can eliminate a cut step entirely. Then it is shown that we can always push a cut upward until it can be eliminated entirely.

Since this approach relies on induction with the degree of complexity of the cut formula, we need to be sure that when we are eliminating cut, we are guaranteed that the resultant cut formula will be of a lower complexity. Then our inductive hypothesis of "every proof with a cut of complexity less than d can be turned into a cut-free proof" can take over and do the rest of the work.

The problem is that unrestricted comprehension makes the complexity of a cut formula unpredictable.[6] Applying the rules of $\in_L$ or $\in_R$ can actually cause the complexity of a formula to decrease. For example, this is seen in Curry's paradox.

$$\frac{\vdash C \in C \multimap p}{\vdash C \in C} \in_R$$

Thus it happens that when we're attempting to eliminate a cut on $C \in C$, we find that the cut we replace it with doesn't have a cut formula of reduced complexity. To make this concrete, let's revisit the sequent calculus proof of Curry's. Let's take the first part, denoted $\mathbf{D}$.

$$\cfrac{\cfrac{\cfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C, C \in C \multimap p \vdash p} \multimap_L}{C \in C, C \in C \vdash p} \in_L}{C \in C \vdash p} \text{Contraction}$$

If we're attempting to cut eliminate, we'll push a cut up to where the contraction takes place. This creates a subproof which looks as follows.

$$\cfrac{\cfrac{\cfrac{\mathbf{D}}{\vdash C \in C \multimap p} \multimap_R}{\vdash C \in C} \in_R \qquad \cfrac{\cfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C, C \in C \multimap p \vdash p} \multimap_L}{C \in C, C \in C \vdash p} \in_L}{C \in C \vdash p} \text{Cut}$$

If attempt to eliminate this cut, we'll push the cut higher to the step right before and cut on $C \in C \multimap p$.

[6]cf. Petersen [38]

$$\dfrac{\dfrac{\mathbf{D}}{\vdash C \in C \multimap p}\multimap_R \qquad \dfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C, C \in C \multimap p \vdash p}\multimap_L}{C \in C \vdash p}\text{Cut}$$

But this is the problem. We're now cutting on a formula of higher complexity than we were before.[7] This means that our usual approach to cut elimination won't work with comprehension in the mix. However, with a contraction-free logic we don't need to rely on the complexity of the cut formula reducing. We can induct on a simpler parameter.

Definition 4.3. The *logical grade* of a deduction is the number of logical inferences occurring in it. The *logical grade of a cut* is the sum of the logical grades of the deductions occurring in its upper sequents.

Unrestricted comprehension behaves the same as any other sequent calculus inference with respect to this parameter. However, this notion of a proof's complexity is unpredictable when contraction is present. To see why we can not use this parameter for induction if we have contraction, we can turn to the proof of Curry's once more. When we attempt to eliminate the cut in the standard proof of Curry's, we replace the contraction with a cut instead and end up with this subproof.

$$\dfrac{\dfrac{\dfrac{\mathbf{D}}{\vdash C \in C \multimap p}\multimap_R}{\vdash C \in C}\in_R \qquad \dfrac{\dfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C, C \in C \multimap p \vdash p}\multimap_L}{C \in C, C \in C \vdash p}\in_L}{C \in C \vdash p}\text{Cut}$$

The subproof now has an extra copy of the derivation of $\vdash C \in C$. The result is that the logical grade of the proof has increased. This sort of thing never occurs if we don't have contraction.[8]

With this measure we'll perform a double induction in conjunction with the second parameter of rank.[9]

Definition 4.4. Let Δ be a set of formulas that contains an occurrence of C. The *cut formula* of a cut is the formula that the cut operates on. In the following it is C,

Definition 4.5. Let Δ be a set of formulas that contains an occurrence of C. Then $\Delta - \{C\}$ denotes Δ with an instance of C removed.

$$\dfrac{\Gamma \vdash C \qquad \Delta \vdash D}{\Gamma, \Delta - \{C\} \vdash D}\text{Cut}$$

Definition 4.6. The *left/right rank of a cut* is the maximum number of consecutive sequents that contain the cut formula, counting up from the upper left/right sequent respectively. The *rank of a cut* is the sum of the left and right ranks.

[7] An additional oddity about Curry's is that if we continue to apply these reduction steps, our proof ends up looping back to where it started before we applied our first elimination. This is our first glimpse at the motivation for the preferred theory presented in this thesis, NNST.

[8] This is a hint about how the seemingly innocuous contraction inflicts severe complexity problems in our elimination procedures.

[9] Rank is a standard parameter used in cut elimination proofs [49].

We now prove a complicated lemma which almost immediately delivers the desired cut elimination result.

Lemma 4.7. *If* **S** *is the lower sequent of a cut in which both upper sequents are end-sequents of cut-free deductions with logical grades* j *and* i*, respectively, then there is a cut-free deduction of* **S** *with a logical grade* $\leq j + i$ *[38].*

Proof. Let C denote the cut formula. Proceed by double induction on the logical grade and the rank of a cut. Our base case for rank will be 2, since that is the lowest possible. The inductive hypothesis for the rank of the cut is that the proposition holds for all logical grades for ranks less than our current rank. Within each rank, we induct on logical grade. The base case in this instance is 0, in which the cut formula is introduced by initial sequents or there are only structural rules which precede the cut. The inductive hypothesis for the logical grade is that the proposition applies to lower logical grades than our current logical grade.

The overall structure of the proof is to establish the proposition for the base case of rank, and then the inductive step. Both cases will be split into further cases. In those cases we "reduce" the cut by reaching the same end sequent without the cut instance or by reducing the rank or logical grade of the cut.

Consider when the rank of the cut is two. This means the cut formula was introduced in the last step before each upper sequents. Either the cut formula is introduced as

1. an initial sequent in the left upper sequent,
2. an initial sequent in the right upper sequent,
3. introduced in the right upper sequent by weakening,
4. or it's introduced by a logical inference in both the left and right.

In the first case, we have an initial sequent in the left upper sequent.

$$\frac{C \vdash C \qquad \Delta \vdash B}{C, \Delta - \{C\} \vdash B}\,\text{Cut}$$

To remove the cut, we take the right upper sequent and perform as many exchanges as needed.

$$\frac{\Delta \vdash B}{C, \Delta - \{C\} \vdash B}\,\text{Ex}$$

In the second case, we have an initial sequent in the right upper sequent.

$$\frac{\Gamma \vdash C \qquad C \vdash C}{\Gamma \vdash C}\,\text{Cut}$$

To remove the cut, we just need to keep the upper left sequent without any further steps.

In the third case, we have the cut formula introduced by left weakening in the right upper sequent.

$$\frac{\Gamma \vdash C \qquad \dfrac{\Delta \vdash B}{\Delta, C \vdash B}\,\text{Weak}}{\Gamma, \Delta \vdash B}\,\text{Cut}$$

To remove the cut, we can start from the right upper sequent and perform as many weakening and exchange steps as needed.

$$\frac{\Delta \vdash B}{\Gamma, \Delta \vdash B}\text{Weak/Ex}$$

These first three cases can also be used to verify the base case for the logical grade induction.

In the fourth case, we have the cut formula introduced by logical inference in both the left and right upper sequents. These reductions are normally referred to as key cases. As opposed to the above cases, these don't remove the cuts but replace them with cuts of a lower logical grade. We can then invoke the inductive hypothesis with the logical grade. I will demonstrate two of the needed reductions here; none of the reductions performed are novel and the rest can be found in [38, 54, 10].

First, the $(\in_R, \in_L)$ key case. Let C in this case be the formula $t \in \{x|A(x)\}$.

$$\frac{\dfrac{\Gamma \vdash A[t/x]}{\Gamma \vdash t \in \{x|A(x)\}}\in_R \qquad \dfrac{A[t/x], \Delta \vdash B}{t \in \{x|A(x)\}, \Delta \vdash B}\in_L}{\Gamma, \Delta \vdash B}\text{Cut}$$

Let m and n be the logical grades of the uppermost left and right sequents respectively. Then the logical grade of this cut is $(m+1)+(n+1)$. The reduction we perform eliminates the $\in_R$ and $\in_L$ steps.

$$\frac{\Gamma \vdash A[t/x] \qquad A[t/x], \Delta \vdash B}{\Gamma, \Delta \vdash B}\text{Cut}$$

This leaves the cut with a logical grade of $m+n$ and thus the logical grade inductive hypothesis applies.

The other case we'll consider here is $(\multimap_R, \multimap_L)$. Let C in this case be the formula $A \multimap B$.

$$\frac{\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\multimap_R \qquad \dfrac{\Delta_1 \vdash A \qquad B, \Delta_2 \vdash D}{A \multimap B, \Delta_1, \Delta_2 \vdash D}\multimap_L}{\Gamma, \Delta_1, \Delta_2 \vdash D}\text{Cut}$$

Let m, n, l be the logical grades of the uppermost left, middle, and right sequents respectively. Then the logical grade of this cut is $(m+1)+(n+l+1)$. To perform this reduction requires removing the $\multimap_R$ and $\multimap_L$ steps by adding two additional cuts of lower logical grade and then also performing some exchanges.

$$\frac{\Delta_1 \vdash A \qquad \dfrac{\Gamma, A \vdash B \qquad B, \Delta_2 \vdash D}{A, \Gamma, \Delta_2 \vdash D}\text{Cut/Ex}}{\Gamma, \Delta_1, \Delta_2 \vdash D}\text{Cut/Ex}$$

The upper cut will have a logical grade of $m+l$. Thus, by the inductive hypothesis, we can replace the subdeduction with a cut-free deduction of logical grade $\leq m+l$. Then after that replacement, the logical grade of the lower cut will be $\leq m+l+n$, which is less than our original logical grade and thus the inductive hypothesis will still apply.

This takes care of all the cases when the rank of the cut is two. The proof reductions will now omit mentions of exchange rules. Consider when the rank of the cut is greater than two. We split again into cases:

1. the left rank is greater than one
2. or the right rank is greater than one.

The goal in this case is to show that we can reduce the rank of the cut and then the inductive hypothesis finishes it off. I'll do two examples from each case; the others are completely straightforward.

First, the left rank is greater than one. This means that the cut formula wasn't just introduced, and so any other left inference could possibly be the one that precedes the cut in question: weakening, exchange, $\multimap_L$, $+_L$, or $\star_L$. We'll look at weakening and $\star_L$ to exhibit the proof strategy which works for all other inferences.

Thus, consider the proof with left rank greater than one and with a weakening step that precedes cut step.

$$\dfrac{\dfrac{\Gamma \vdash C}{\Gamma, A \vdash C}\text{Weak} \qquad \Delta \vdash B}{A, \Gamma, \Delta - \{C\} \vdash B}\text{Cut}$$

We push the cut up as follows to reduce its rank.

$$\dfrac{\dfrac{\Gamma \vdash C \qquad \Delta \vdash B}{\Gamma, \Delta - \{C\} \vdash B}\text{Cut}}{A, \Gamma, \Delta - \{C\} \vdash B}$$

The inductive hypothesis on rank then applies.

Now, consider the proof with left rank greater than one and with a $\star_L$ step that precedes the cut step.

$$\dfrac{\dfrac{A, B, \Gamma \vdash C}{A \star B, \Gamma \vdash C}\star_L \qquad \Delta \vdash B}{A \star B, \Gamma, \Delta - \{C\} \vdash B}\text{Cut}$$

Similar to the weakening case, we push the cut up.

$$\dfrac{\dfrac{A, B, \Gamma \vdash C \qquad \Delta \vdash B}{A, B, \Gamma, \Delta - \{C\} \vdash B}\text{Cut}}{A \star B, \Gamma, \Delta - \{C\} \vdash B}\star_L$$

Thus the inductive hypothesis applies.

Now consider if the right rank is greater than one. This means the cut formula wasn't introduced by a left rule in the upper right sequent prior to the cut formula. There are more cases to consider in this case of what did happen before since we now have to consider left and right variants where they exist: weakening, $\star$, $\multimap$, $+$.

Consider where the last rule applied in the right upper sequent was $+_{R1}$.

$$\dfrac{\Gamma \vdash C \qquad \dfrac{\Delta \vdash B}{\Delta \vdash A + B}+_{R1}}{\Gamma, \Delta - \{C\} \vdash A + B}\text{Cut}$$

We push the cut up.

$$\dfrac{\dfrac{\Gamma \vdash C \qquad \Delta \vdash B}{\Gamma, \Delta - \{C\} \vdash B}\text{Cut}}{\Gamma, \Delta - \{C\} \vdash A + B}+_{R1}$$

Finally, we'll consider the case in which the last rule applied in the right upper sequent is $\star_R$.

$$\dfrac{\Gamma \vdash C \qquad \dfrac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \star B}\star_R}{\Gamma, \Delta_1 - \{C\}, \Delta_2 \vdash A \star B}\text{Cut}$$

We reduce the cut.

$$\dfrac{\dfrac{\Gamma \vdash C \qquad \Delta_1 \vdash A}{\Gamma, \Delta_1 \vdash A}\text{Cut} \qquad \Delta_2 \vdash B}{\Gamma, \Delta_1, \Delta_2 \vdash A \star B}\star_R$$

This completes the proof. □

Theorem 4.8. *L_{BCK} with the rules for $\in_L$ and $\in_R$ satisfies cut elimination.*

Proof. Use the previous proposition in conjunction with induction on the number of cuts in a proof.[10] □

Corollary 4.9. *L_{BCK} is consistent with the rules for $\in_L$ and $\in_R$.*

Proof. A proof of

$$\vdash \bot$$

requires the use of cut since no right rule can introduce $\bot$. That is, the proof would have to be of the following form.

$$\dfrac{\vdash C \qquad C \vdash \bot}{\vdash \bot}\text{Cut}$$

That the theory can't prove $\bot$ shows the theory is non-trivial. To further conclude the theory must be consistent observe that a proof of A and $A \multimap \bot$ would imply a cut proof of $\bot$. □

This proof demonstrates that there is a way to prove cut elimination in a theory with unrestricted comprehension if we don't rely on the complexity of the cut formula. However, L_{BCK} and similar logics are weak. We can extend their strength by moving to light linear logics. These logics can be seen as extensions of the simply contraction-free logics. These still won't require induction on the complexity of the cut formula and can be proved consistent.[11]

We'll first look at linear logic in general to give a general sense of those theories. We'll then move to discussing how to move from linear logic to the light variants.

4.2 Linear Logic

Linear logic originated in an analysis of the semantics of system F, a form of λ-calculus [14]. Part of its motivation as a logical system comes from providing a unique perspective on the divide between classical and intuitionistic logic. This perspective turns out to be useful in our quest to understand contraction and will let us build systems with more logical strength than L_{BCK}.

[10] While there are steps which temporarily increase the number of cuts in a proof, they are made cut-free by the end of the reduction and thus don't interfere with using induction on the number of cuts.

[11] This was part of Girard's initial motivation in developing light linear logic [22].

Linear logic relies on differentiating between additive and multiplicative connectives. We've already seen this in our differentiation of extensional conjunction and fusion above. I'll use $\star$, $+$, $\mathbf{0}$ to denote the multiplicative conjunction, disjunction and negation, and $\sqcap$, $\sqcup$ and $\perp$ to denote the additive conjunction, disjunction, and negation.[12]

The only assumed structural rules for linear logic are cut, exchange and reflexivity.[13] The rules for the connectives in sequent calculus are as follows.[14]

$$\frac{\Delta, A, B \vdash \Gamma}{\Delta, A \star B \vdash \Gamma}\star_L \qquad \frac{\Delta_2 \vdash C, \Gamma_2 \qquad \Delta_1 \vdash B, \Gamma_1}{\Delta_1, \Delta_2 \vdash B \star C, \Gamma_1, \Gamma_2}\star_R$$

$$\frac{\Delta, B_i \vdash \Gamma}{\Delta, B_1 \sqcap B_2 \vdash \Gamma}\sqcap_{Li}\ (i = 1, 2) \qquad \frac{\Delta \vdash C, \Gamma \qquad \Delta \vdash B, \Gamma}{\Delta \vdash B \sqcap C, \Gamma}\sqcap_R$$

$$\frac{\Delta_2, C \vdash \Gamma_2 \qquad \Delta_1, B \vdash \Gamma_1}{\Delta_1, \Delta_2, C + B \vdash \Gamma_1, \Gamma_2}+_L \qquad \frac{\Delta \vdash B, C, \Gamma}{\Delta \vdash B + C, \Gamma}+_R$$

$$\frac{\Delta, C \vdash \Gamma \qquad \Delta, B \vdash \Gamma}{\Delta, C \sqcup C \vdash \Gamma}\sqcup_L \qquad \frac{\Delta \vdash B_i, \Gamma}{\Delta \vdash B_1 \sqcup B_2, \Gamma}\sqcup_{Ri}\ (i = 1, 2)$$

$$\frac{\Delta_1 \vdash A, \Gamma_1 \qquad \Delta_2, B \vdash \Gamma_2}{\Delta_1, \Delta_2, A \multimap B \vdash \Gamma_1, \Gamma_2}\multimap_L \qquad \frac{\Delta, A \vdash B, \Gamma}{\Delta \vdash A \multimap B, \Gamma}\multimap_R$$

$$\frac{}{\mathbf{0} \vdash}\mathbf{0}_L \qquad \frac{\Delta \vdash \Gamma}{\Delta \vdash \mathbf{0}, \Gamma}\mathbf{0}_R$$

$$\frac{}{\Delta, \perp \vdash \Gamma}\perp_L \qquad \text{no right rule for } \perp$$

One way to view the distinction between the additive and multiplicative connectives is that the additive rules require the context around formulas to be the same.

Definition 4.10. The *context* in a deduction is the secondary formulas not being acted on by the rule. For example, in

$$\frac{\Delta, A, B \vdash \Gamma}{\Delta, A \star B \vdash \Gamma}\star_L$$

both Δ and Γ are part of the context.

The primary negation in the system is then defined by $\sim A := A \multimap \mathbf{0}$. Negation is also sometimes included as a primitive operation, as with classical sequent calculus.

$$\frac{\Delta \vdash B, \Gamma}{\Delta, \sim B \vdash \Gamma}\sim_L \qquad \frac{\Delta, B \vdash \Gamma}{\Delta \vdash \sim B, \Gamma}\sim_R$$

[12]This is a break from the traditional notation used by Girard in linear logics. This notation comes from [54]. For Girard's notation, $\otimes$ will be multiplicative conjunction while $\oplus$ will be additive disjunction. The additive conjunction will be & and the multiplicative disjunction is an upside down &. I find the notation presented here much easier to remember for those unfamiliar with linear logic.

[13]Uses of exchange are omitted in the formal proofs to follow.

[14]The subscripts occuring in the rule definitions that follow are only used to shorten the representation. A subscript means there are two variants of the rule and each variant can be realized by instantiating the rule with one of the values for i. The subscripts on a set of formulas like Γ_1 denotes a different set of formula from Γ_2. This is used rather than having to use more Greek letters.

Proposition 4.11. $\sim A \dashv\vdash A \multimap \mathbf{0}$

Proof. The forward direction first.

$$\dfrac{\dfrac{\dfrac{B \vdash B}{B, \sim B \vdash}\sim_L}{B, \sim B \vdash \mathbf{0}}\mathbf{0}_R}{\sim B \vdash B \multimap \mathbf{0}}\multimap_R$$

And the other direction.

$$\dfrac{\dfrac{B \vdash B \qquad \bot \vdash}{B, B \multimap \bot \vdash}\multimap_L}{B \multimap \bot \vdash \sim B}\sim_R$$

□

With the primitive negation rules, we could also define linear implication in terms of the multiplicative disjunction as $\sim A + B$.

In a classical sequent calculus, the additive and multiplicative rules would be equivalent. This equivalence relies on right weakening and left and right contraction. This equivalence doesn't arise in linear logic since it drops these structural rules.

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}W_L \qquad\qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}W_R$$

$$\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}C_L \qquad\qquad \dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}C_R$$

Proposition 4.12. *The following equivalences hold with W_R, C_L, and C_R:*

1. $A \sqcap B \dashv\vdash A \star B$,
2. $A \sqcup B \dashv\vdash A + B$.

Proof. We'll first prove $A \sqcap B \vdash A \star B$ which uses C_L.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \star B}\star_R}{A \sqcap B, B \vdash A \star B}\sqcap_{L1}}{A \sqcap B, A \sqcap B \vdash A \star B}\sqcap_{L2}}{A \sqcap B \vdash A \star B}C_L$$

Next we'll prove the other direction, which uses C_L and W_L.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}W_L \qquad \dfrac{B \vdash B}{A, B \vdash B}W_L}{A, A, B, B \vdash A \sqcap B}\sqcap_R}{A, B, B \vdash A \sqcap B}C_L}{A, B \vdash A \sqcap B}C_L}{A \star B \vdash A \sqcap B}\star_L$$

Now on to the disjunctions. The proof that $A \sqcup B \vdash A + B$ uses W_R.

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, B}W_R \qquad \dfrac{B \vdash B}{B \vdash A, B}W_R}{A \sqcup B \vdash A, B}\sqcup_L}{A \sqcup B \vdash A + B}+_R$$

Finally, the proof of the other direction uses C_R.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A + B \vdash A, B} +_L}{A + B \vdash A \sqcup B, B} \sqcup_{L1}}{A + B \vdash A \sqcup B, A \sqcup B} \sqcup_{L2}}{A + B \vdash A \sqcup B} C_R$$

□

The removal of those rules results in a logic that is more "constructive", i.e. it's clear about how particular logical formula are constructed in the application of the logical rules, while also maintaining the symmetry of classical sequent calculus.[15] Further, this new perspective allows us to understand the divide between classical and intuitionistic logic as a difference in connectives.

For example, the law of excluded middle is a point of contention between the two different logics. We use the primitive negation connective to make this point. The linear logic perspective is that the law of excluded middle written as $\sim A \sqcup A$ is not provable without right contraction, but written as $\sim A + A$ it amounts to the provable implication $A \multimap A$. Further, the additive disjunction is the "intuitionistic" one since it satisfies the disjunction property, i.e. a disjunction $A \sqcup B$ implies that either A or B is a theorem. The multiplicative disjunction does not.

The previous rules constitute the Multiplicative-Additive fragment of linear logic (MALL). Full linear logic has contraction and weakening, but their use is more explicit. Linear logic achieves this by adding modalities to track when these rules are allowed to be used. These modalities are called exponentials in linear logic and are denoted as ! and ?. In what follows, $!\Delta$ is the set of formulas such that for each formula A in Δ, A is of the form $!B$ for some formula B. Similar for $?\Delta$.

$$\dfrac{!\Delta, B \vdash ?\Gamma}{!\Delta, ?B \vdash ?\Gamma} ?_L \qquad \dfrac{!\Delta \vdash B, ?\Gamma}{!\Delta \vdash !B, ?\Gamma} !_R$$

$$\dfrac{\Delta \vdash \Gamma}{\Delta, !B \vdash \Gamma} !W \qquad \dfrac{\Delta \vdash \Gamma}{\Delta \vdash ?B, \Gamma} ?W$$

$$\dfrac{\Delta, !B, !B \vdash \Gamma}{\Delta, !B \vdash \Gamma} !C \qquad \dfrac{\Delta \vdash ?B, ?B, \Gamma}{\Delta \vdash ?B, \Gamma} ?C$$

$$\dfrac{\Delta, B \vdash \Gamma}{\Delta, !B \vdash \Gamma} !D \qquad \dfrac{\Delta \vdash B, \Gamma}{\Delta \vdash ?B, \Gamma} ?D$$

Using these exponentials we can code the implications of intuitionistic and classical logic in linear logic. The former is $!A \multimap B$ and the latter is $!A \multimap ?B$ [54]. In this sense, linear logic is a more general system than both intuitionistic and classical logic.

A useful property of the exponentials is that they are duals.

Proposition 4.13. *The following hold:*

1. $\sim !A \dashv\vdash ? \sim A$

[15]Normally we think of classical sequent calculus as allowing multiple formulas on the right, while intuitionism restricts this.

2. $\sim ?A \dashv\vdash ! \sim A$

Proof. First, $\sim !A \vdash ? \sim A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, \sim A}\sim_R}{\vdash A, ? \sim A}?D}{\vdash !A, ? \sim A}!_R}{\sim !A \vdash ? \sim A}\sim_L$$

Now $? \sim A \vdash \sim !A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \sim A \vdash}\sim_L}{!A, \sim A \vdash}!D}{!A, ? \sim A \vdash}?_L}{? \sim A \vdash \sim !A}\sim_R$$

The other two proofs have the same proof structure. First, $\sim ?A \vdash ! \sim A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, \sim A}\sim_R}{\vdash ?A, \sim A}?D}{\vdash ?A, ! \sim A}!_R}{\sim ?A \vdash ! \sim A}\sim_L$$

Finally, $! \sim A \vdash \sim ?A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \sim A \vdash}\sim_L}{A, ! \sim A \vdash}!D}{?A, ! \sim A \vdash}?_L}{! \sim A \vdash \sim ?A}\sim_R$$

□

4.2.1 Intuitionistic Linear Logic is Familiar

If we look at the intuitionistic variant of the linear logic, that is if we restrict ourselves to single formula on the right of the turnstile, we'll see something that is awfully familiar.[16] To make this switch we have to remove the ? exponential, the multiplicative negation **0**, and the multiplicative disjunction + since these can't be made to work without multiple formulas on the right.[17] Then the sequent calculus consisting of the connectives linear implication, multiplicative conjunction and additive disjunction with the structural rules of exchange, cut, is precisely L_{BCK} without weakening.

Taking this perspective with L_{BCK} suggests we can then use the exponentials to be explicit about our uses of contraction. However, if we attempt to use the full exponentials with this logic, we'll recover too much logical strength and trivialize naive set theory. (See 4.14 below.) Light linear logics restrict the strength of the exponentials to allow some limited forms of contraction back while still remaining consistent with unrestricted comprehension. These logics were first introduced by Girard in [22].

[16]This does take us away from the initial motivation of linear logic which maintains symmetrical sequents.
[17]Adjusting the other rules where necessary are done so in the obvious way.

4.3 Light Linear Logic

There are a few different variants of light linear logic. We'll be focusing on the "affine" versions of these logics, which allow unrestricted weakening.[18] The particular logic we'll be focusing on is "Intuitionistic Light Affine Logic" (ILAL) which has been used in constructing a naive set theory by Terui [52] which he calls "Light Affine Set Theory" (LAST).[19]

We'll return to Curry's paradox in the linear logic framework to demostrate a ! inference that leads to triviality. First, note the previously given proof of Curry's is no longer valid in linear logic due to the contraction step on a non-exponentiated formula.

$$\dfrac{\dfrac{\dfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C, C \in C \multimap p \vdash p}\multimap_L}{C \in C, C \in C \vdash p}\in_L}{C \in C \vdash p}\text{Contraction}$$

However, the proof can be adapted by using the exponentials.

Proposition 4.14. *The inference*

$$\dfrac{\Delta, B \vdash \Gamma}{\Delta, !B \vdash \Gamma}\,!D$$

implies triviality with $\in_L$ *and* $\in_R$.

Proof. A more appropriate translation Curry's set into the linear logic framework is $\forall x(x \in C \multimapboth !(x \in x \multimap p))$. Then the subproof of $!(C \in C \multimap p)$ is possible.

$$\dfrac{\dfrac{\dfrac{\dfrac{!(C \in C \multimap p) \vdash !(C \in C \multimap p)}{C \in C \vdash !(C \in C \multimap p)}\in_L \qquad \dfrac{\dfrac{\dfrac{\dfrac{!(C \in C \multimap p) \vdash !(C \in C \multimap p)}{!(C \in C \multimap p) \vdash C \in C}\in_R \qquad p \vdash p}{C \in C \multimap p, !(C \in C \multimap p) \vdash p}\multimap_L}{!(C \in C \multimap p), !(C \in C \multimap p) \vdash p}\,!D}{!(C \in C \multimap p) \vdash p}\,!C}{C \in C \vdash p}\text{Cut}}{\vdash C \in C \multimap p}\multimap_R}{\vdash !(C \in C \multimap p)}\,!_R$$

$$\dfrac{\vdash !(C \in C \multimap p)}{\vdash C \in C}\in_R$$

The upper portion up to the cut can then be reused to get $C \in C \vdash p$, and then cut gives us p. Note the $!D$ inference was essential in making this proof go through. □

The $!D$ inference in the above proof is the primary culprit leading to triviality with the ! exponential. It also sometimes referred to as dereliction and it is expressed axiomatically as $!A \multimap A$.[20] Other principles that are done without in ILAL are *digging*, $!A \multimap !!A$, and

[18] Unrestricted weakening has negligible impacts on the cut elimination procedure.

[19] This light linear logic is, relatively speaking, a very weak variant of light linear logics. Elementary linear logic covered in [15] or the even stronger generalizations in [4], linear logic by levels, extend the logical strength. Terui's primary focus was the theory's relationship with polytime computation which is lost with these stronger logics.

[20] The properties of ! will often look a lot like a provability modality, usually denoted $\Box$, especially in light of our earlier observation that theorems still "contract" in contraction-free systems. Taking this perspective, the problem with dereliction mirrors the problem revealed in Lŏb's theorem: if we have $\Box p \multimap p$ then we can infer p.

monoidalness, $!A\star!B \multimap !(A\star B)$. Monoidalness isn't necessarily problematic for our purposes as it does not lead to inconsistency with naive set theory.[21][22] Digging on its own is consistent in naive set theory but leads to another proof of triviality in the presence of *weak dereliction*, $!A \multimap ?A$ in the classical variant of light affine logic.

Proposition 4.15. *Digging and weak dereliction together imply triviality, with all other modality rules being fixed.*

Proof. The proof of this is a bit more complicated. To make the sequent proof be a reasonable size, we rely on the derived duality between ! and ? and also on double negation elimination. These steps are marked by Derived. Further weak dereliction is used as the inference

$$\dfrac{\Gamma \vdash !B}{\Gamma \vdash ?B}\,\mathrm{WD}$$

while digging is used as

$$\dfrac{\Gamma, !!A \vdash B}{\Gamma, !A \vdash B}\,\mathrm{Dig}$$

We can use a translation of Russell's paradox to derive triviality. Take the set instantiation $\forall x(x \in R \multimapboth !(x \notin x))$. Then we can prove $!(R \notin R)$ as follows. The weaker version of the ! rule is also used, as present below in ILAL, but that inference can be replaced with $!D$ and $!_R$.

$$
\begin{prooftree}
\AxiomC{$!R \notin R \vdash !R \notin R$}
\RightLabel{$\in_R$}
\UnaryInfC{$!R \notin R \vdash R \in R$}
\RightLabel{$!$}
\UnaryInfC{$!!R \notin R \vdash !R \in R$}
\RightLabel{WD}
\UnaryInfC{$!!R \notin R \vdash ?R \in R$}
\RightLabel{Dig}
\UnaryInfC{$!R \notin R \vdash ?R \in R$}
\RightLabel{$\sim_R$}
\UnaryInfC{$\vdash \sim !R \notin R, ?R \in R$}
\RightLabel{Derived}
\UnaryInfC{$\vdash ?R \in R, ?R \in R$}
\RightLabel{?C}
\UnaryInfC{$\vdash ?R \in R$}
\AxiomC{$!R \notin R \vdash !R \notin R$}
\RightLabel{$\in_L$}
\UnaryInfC{$R \in R \vdash !R \notin R$}
\RightLabel{$\sim_L$}
\UnaryInfC{$R \in R, \sim !R \notin R \vdash$}
\RightLabel{Derived}
\UnaryInfC{$R \in R, ?R \in R \vdash$}
\RightLabel{Cut}
\BinaryInfC{$R \in R \vdash$}
\RightLabel{$\sim_R$}
\UnaryInfC{$\vdash R \notin R$}
\RightLabel{$!$}
\UnaryInfC{$\vdash !(R \notin R)$}
\RightLabel{$\in_R$}
\UnaryInfC{$\vdash R \in R$}
\end{prooftree}
$$

Then this proof can be repeated up to $R \notin R$. Cut delivers the result. □

The loss of all of these principles make the remaining operations of ! limited. We can mitigate this loss by adding a new exponential §.[23] This exponential satisfies *monoidalness*, $\S A\star\S B \multimap \S(A\star B)$, and *stratified dereliction*, $!A \multimap \S A$. Finally, here's ILAL with unrestricted comprehension as used to construct LAST [52].

[21] It does complicate the cut elimination procedure and forces the loss of polytime reduction [52].

[22] Monoidalness is present in elementary linear logic. Our below rule for ! is highly restrictive since it only applies to single formula on both ends. Monoidalness allows more freedom by letting the rule infer from $\Gamma \vdash A$ to $!\Gamma \vdash !A$.

[23] This extra exponential is not needed in the elementary linear logic variant or in the stronger linear logic by levels variant.

$$\frac{}{A \vdash A}\text{Ax} \qquad \frac{\Gamma_1 \vdash A \qquad A, \Gamma_2 \vdash C}{\Gamma_1, \Gamma_2 \vdash C}\text{Cut}$$

$$\frac{\Gamma \vdash C}{A, \Gamma \vdash C}\text{W} \qquad \frac{!A, !A, \Gamma \vdash C}{!A, \Gamma \vdash C}\text{C}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, B, A, \Gamma_2 \vdash C}\text{E}$$

$$\frac{\Gamma_1 \vdash A \qquad B, \Gamma_2 \vdash C}{A \multimap B, \Gamma_1, \Gamma_2 \vdash C}\multimap_L \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\multimap_R$$

$$\frac{B \vdash A}{!B \vdash !A}\,! \qquad \frac{\Gamma, \Delta \vdash A}{!\Gamma, \S\Delta \vdash \S A}\,\S$$

$$\frac{A[t/x], \Gamma \vdash C}{\forall x A, \Gamma \vdash C}\forall_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\forall_R,\ x \text{ not free in } \Gamma$$

$$\frac{A[t/x], \Gamma \vdash C}{t \in \{x|A\}, \Gamma \vdash C}\in_L \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash t \in \{x|A\}}\in_R$$

The cut elimination proof in this system inducts on a parameter called *depth*[24] which tracks the number of exponentials in a formula.[25] The crucial change of ILAL that helps control depth is that exponentials are always symmetrically introduced on the left and right. Inferences like dereliction and digging interfere with depth being reliably tracked.

Light linear logic still inherit much of the same problems as L_{BCK}. The main difference is that we are now admitted some degree of contraction and thus slightly more logical strength. This extra strength will prove to be enough to show a form of induction for the natural numbers. This is not possible with L_{BCK} naive set theory. The main downside for this theory remains the same: the axiom of extensionality still induces triviality. The proof for that only needs the logical principles of L_{BCK} and thus anything that extends it will run into the same problem.

However, we have seen that these logics offer another way to provide at least a theory with unrestricted comprehension. These do so by removing all forms of contraction. This allows us to prove they are consistent with the rules for naive comprehension through cut elimination arguments. We also find that the connectives, and in particular the conditional, exhibit behavior more resembling that of classical logic than DKQ's conditional(s) did. We now move on to the work done in naive set theories with these logics.

[24] Different from the notion of depth for depth relevance.

[25] The cut elimination proof itself requires (a lot) more machinery than has been presented in the thesis as it's usually presented either via proof nets or the light affine lambda calculus. The latter version of the proof can be found in [53]. A proof sketch of the former version can be found in [4].

Chapter 5

Contraction-Free Naive Set Theory

This chapter discusses some of the work on contraction-free naive set theories with logics like L_{BCK} and light linear logics [38, 24, 11, 52, 53]. We will focus on the work done on Light Affine Set Theory by Terui. Recall that these theories cannot admit the axiom of extensionality due to Grišin's paradox; we will look at the paradox in this chapter.[1]

Despite the lack of extensionality, the work in LAST provides useful ways to work with naive comprehension. We will prove the fixpoint theorem which essentially allows the use of naive comprehension in its axiomatic form with full impredicative definitions. This is a powerful tool which can essentially perform recursive operations without the need for an ordinal structure. For example, the arithmetic operations of addition and multiplication will be defined by using this theorem. Thus we will be able to do some interesting mathematics in what initially appears to be a fairly weak system. We will also spend some time on Gršin's paradox and I will attempt a paraconsistent solution to the paradox. This will fail due to a "revenge" version of the paradox.

5.1 LAST Basics

This theory is presented as a sequent calculus and has the rules given in Section 4.3. The rules are reproduced here.

$$\frac{}{A \vdash A}\ \mathrm{Ax} \qquad\qquad \frac{\Gamma_1 \vdash A \qquad A, \Gamma_2 \vdash C}{\Gamma_1, \Gamma_2 \vdash C}\ \mathrm{Cut}$$

$$\frac{\Gamma \vdash C}{A, \Gamma \vdash C}\ \mathrm{W} \qquad\qquad \frac{!A, !A, \Gamma \vdash C}{!A, \Gamma \vdash C}\ \mathrm{C}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, B, A, \Gamma_2 \vdash C}\ \mathrm{E}$$

$$\frac{\Gamma_1 \vdash A \qquad B, \Gamma_2 \vdash C}{A \multimap B, \Gamma_1, \Gamma_2 \vdash C}\ \multimap_L \qquad\qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\ \multimap_R$$

[1] These theories are sometimes referred to as naive class theories, rather than naive set theories, in the literature due to their lack of extensionality.

$$\frac{B \vdash A}{!B \vdash !A}\,! \qquad \frac{\Gamma, \Delta \vdash A}{!\Gamma, \S\Delta \vdash \S A}\,\S$$

$$\frac{A[t/x], \Gamma \vdash C}{\forall x A, \Gamma \vdash C}\,\forall_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\,\forall_R,\ x \text{ not free in } \Gamma$$

$$\frac{A[t/x], \Gamma \vdash C}{t \in \{x|A\}, \Gamma \vdash C}\,\in_L \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash t \in \{x|A\}}\,\in_R$$

This theory admits a cut elimination argument and thus is known to be consistent. Note that we have not defined a conjunction, disjunction, negation or existential quantifier as primitive. We can leverage the second-order character of set theory to define these.[2] We will define a multiplicative conjunction $\star$, an additive disjunction $\sqcup$, the additive absurdity constant $\perp$, and the existential quantifier $\exists$. First, fix the closed term $t_0 := \{x|x \in x\}$.[3] Then we define the connectives as follows.

- $A \star B := \forall x((A \multimap B \multimap t_0 \in x) \multimap t_0 \in x)$
- $A \sqcup B := \forall x((A \multimap t_0 \in x) \multimap (B \multimap t_0 \in x) \multimap t_0 \in x)$
- $\perp := \forall x(t_0 \in x)$
- $\neg A := A \multimap \perp$
- $\exists y A := \forall x(\forall y(A \multimap t_0 \in x) \multimap t_0 \in x)$

We can use these definitions to derive the usual properties of the connectives.

Proposition 5.1. *The following hold:*

- $A \multimap B \multimap A \star B$
- $(A \multimap B \multimap C) \multimap (A \star B \multimap C)$
- $A \multimap A \sqcup B$
- $(A \multimap C) \multimap (B \multimap C) \multimap (A \sqcup B \multimap C)$
- $A \multimap \neg A \multimap \perp$
- $\perp \multimap A$
- $A[t/y] \multimap \exists y A$
- $A \multimap C$ *implies* $\exists y A \multimap C$ *if* y *is not a free variable of* C*.*

Proof. Some steps will be abbreviated or skipped in the sequent calculus proofs that follow, like extra applications of $\multimap_R$ where obvious or not including final $\multimap_R$'s. First, $A \multimap B \multimap A \star B$.

[2] In a way that is recognizable to those familiar with type theory.
[3] This could be any closed term.

$$\cfrac{\cfrac{\cfrac{B \vdash B \qquad t_0 \in x \vdash t_0 \in x}{B, B \multimap t_0 \in x \vdash t_0 \in x}\,\multimap_L \qquad A \vdash A}{A, B, A \multimap (B \multimap t_0 \in x) \vdash t_0 \in x}\,\multimap_L}{A, B \vdash (A \multimap t_0 \in x) \multimap t_0 \in x}\,\multimap_R$$

Now for $(A \multimap B \multimap C) \multimap (A \star B \multimap C)$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A \qquad \cfrac{B \vdash B \qquad C \vdash C}{B, B \multimap C \vdash C}\,\multimap_L}{A \multimap (B \multimap C), A, B \vdash C}\,\multimap_L}{A \multimap (B \multimap C), A, B \vdash t_0 \in \{x|C\}}\,\in_R}{A \multimap (B \multimap C) \vdash A \multimap (B \multimap t_0 \in \{x|C\})}\,\multimap_R \qquad \cfrac{C \vdash C}{t_0 \in \{x|C\} \vdash C}\,\in_L}{A \multimap (B \multimap C), (A \multimap (B \multimap t_0 \in \{x|C\})) \multimap t_0 \in \{x|C\} \vdash C}\,\multimap_L}{A \multimap (B \multimap C), \forall x(A \multimap (B \multimap t_0 \in x)) \multimap t_0 \in x \vdash C}\,\forall_L$$

Next is $A \multimap A \sqcup B$.

$$\cfrac{\cfrac{\cfrac{A \vdash A \qquad t_0 \in x \vdash t_0 \in x}{A, A \multimap t_0 \in x \vdash t_0 \in x}\,\multimap_L}{A, A \multimap t_0 \in x, B \multimap t_0 \in x \vdash t_0 \in x}\,\mathrm{W}}{A \vdash (A \multimap t_0 \in x) \multimap (B \multimap t_0 \in x) \multimap t_0 \in x}\,\multimap_R$$

Next we will prove $(A \multimap C) \multimap (B \multimap C) \multimap (A \sqcup B \multimap C)$. For space, we omit the derivation to $B \multimap C \vdash B \multimap t_0 \in \{x|C\}$. This mirrors the left derivation of $A \multimap C \vdash A \multimap t_0 \in \{x|C\}$.

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A \qquad \cfrac{C \vdash C}{C \vdash t_0 \in \{x|C\}}\,\in_R}{A, A \multimap C \vdash t_0 \in \{x|C\}}\,\multimap_L}{A \multimap C \vdash A \multimap t_0 \in \{x|C\}}\,\multimap_R \qquad \cfrac{B \multimap C \vdash B \multimap t_0 \in \{x|C\} \qquad \cfrac{C \vdash C}{t_0 \in \{x|C\} \vdash C}\,\in_L}{B \multimap C, (B \multimap t_0 \in \{x|C\}) \multimap t_0 \in \{x|C\} \vdash C}\,\multimap_L}{A \multimap C, B \multimap C, (A \multimap t_0 \in \{x|C\}) \multimap ((B \multimap t_0 \in \{x|C\}) \multimap t_0 \in \{x|C\}) \vdash C}\,\multimap_L}{A \multimap C, B \multimap C, (A \multimap t_0 \in x) \multimap ((B \multimap t_0 \in x) \multimap t_0 \in x) \vdash C}\,\forall_L$$

Next the negation properties. The first follows immediately from an application of $\multimap_L$ to $A \vdash A$ and $\bot \vdash \bot$. The proof for $\bot \multimap A$ is below, which has a somewhat non-intuitive proof.

$$\cfrac{\cfrac{A \vdash A}{t_0 \in \{x|A\} \vdash A}\,\in_L}{\forall x(t_0 \in \{x|A\} \vdash A}\,\forall_L$$

Finally the existential quantifier properties. First $A[t/y] \multimap \exists y A$. We will omit the substitution operation $[t/y]$ as it does not affect the proof.

$$\cfrac{\cfrac{\cfrac{A \vdash A \qquad t_0 \in x \vdash t_0 \in x}{A, A \multimap t_0 \in x \vdash t_0 \in x}\,\multimap_L}{A, \forall y(A \multimap t_0 \in x) \vdash t_0 \in x}\,\forall_L}{A \vdash \forall x(\forall y(A \multimap t_0 \in x) \multimap t_0 \in x)}\,\multimap_R, \forall_R$$

The last proof is for $A \multimap C \vdash \exists y A \multimap C$ if y is not a free variable of C. The free variable restriction occurs when we use $\forall_R$ in the following proof.

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\AxiomC{$C \vdash C$}
\RightLabel{$\in_R$}
\UnaryInfC{$C \vdash t_0 \in \{x|C\}$}
\RightLabel{$\multimap_L$}
\BinaryInfC{$A, A \multimap C \vdash t_0 \in \{x|C\}$}
\RightLabel{$\multimap_R$}
\UnaryInfC{$A \multimap C \vdash A \multimap t_0 \in \{x|C\}$}
\RightLabel{$\forall_R$}
\UnaryInfC{$A \multimap C \vdash \forall y(A \multimap t_0 \in \{x|C\})$}
\AxiomC{$C \vdash C$}
\RightLabel{$\in_L$}
\UnaryInfC{$t_0 \in \{x|C\}$}
\RightLabel{$\multimap_L$}
\BinaryInfC{$A \multimap C, \forall y(A \multimap t_0 \in \{x|C\}) \multimap t_0 \in \{x|C\} \vdash C$}
\RightLabel{$\forall_L$}
\UnaryInfC{$A \multimap C, \forall x(\forall y(A \multimap t_0 \in x) \multimap t_0 \in x) \vdash C$}
\end{prooftree}
$$

□

We can now consider some other basic set theoretic notions, like equality. The approach to equality will look different than what we saw in the Hilbert-style systems. We will not consider equality as a primitive and instead define it using the conditional and membership.

Definition 5.2. Two sets, x and y, are *extensionally equal* when they have the same members:

$$x =_e y := \forall z(z \in x \multimapboth z \in y).$$

Definition 5.3. Two sets, t and u, are *Leibniz equal* when no property can distinguish them:

$$t =_l u := \forall x(t \in x \multimap u \in x).$$

With these notions of equality, we can write the axiom of extensionality in an equivalent but different form from before.

Definition 5.4. The *axiom of extensionality* states that

$$\vdash x =_e y \multimap x =_l y.$$

As we will see in Section 5.2, the axiom of extensionality will make this theory trivial. The converse to the axiom of extensionality, that $x =_l y \multimap x =_e y$, follows from substitution with Leibniz identity. We will focus on Leibniz identity for now and drop the subscript from $=_l$. Leibniz identity explicitly recovers the properties usually associated with identity, i.e. it's an equivalence relation and allows substitution. As it turns out, we can also derive contraction for it as well.

Proposition 5.5. *The following formulas are provable:*

1. $x = x$
2. $x = y \multimap (A(x) \multimap A(y))$
3. $x = y \multimap y = x$
4. $x = y \star y = z \multimap x = z$
5. $x = y \multimap x = y \star x = y$

Proof. The proof for 1 follows from the application of the rules Ax, $\multimap_R$ and $\forall_R$.

The proof for 2, substitution, is as follows.

$$\dfrac{\dfrac{A(x) \vdash A(x)}{A(x) \vdash x \in \{t|A(t)\}} \in_R \qquad \dfrac{A(y) \vdash A(y)}{y \in \{t|A(t)\} \vdash A(y)} \in_L}{\dfrac{x \in \{t|A(t)\} \multimap y \in \{t|A(t)\}, A(x) \vdash A(y)}{x = y, A(x) \vdash A(y)} \forall_L} \multimap_L$$

The final form as given in the proposition follows from two applications of $\multimap_R$. This general proof form for particular propositions A will be called substitution instances. These can be used to give proofs for 3 and 4. For 3, we use the proof that

$$\vdash x = x$$

and the substitution instance that

$$x = y, x = x \vdash y = x.$$

Cut and $\multimap_R$ deliver the result. As for 4, we use the substitution instance

$$x = y, y = z \vdash x = z.$$

Then the property of conjunction proved in Proposition 5.1 delivers

$$x = y \star y = z \vdash x = z$$

and $\multimap_R$ finishes the result.

The proof for 5, contraction of Leibniz identity, follows. This uses some derived rules: 1 from this proposition, Proposition 5.1 for the properties of conjunction and a substitution instance. Cut is also applied multiple times when needed in a single step.

$$\dfrac{\dfrac{\vdash x = x \qquad x = x, x = x \vdash x = x \star x = x}{\vdash x = x \star x = x} \text{Cut} \qquad x = y, x = x \star x = x \vdash x = y \star x = y}{x = y \vdash x = y \star x = y} \text{Cut}$$

□

That Leibniz identity proves its own contraction is what will get us in trouble with the axiom of extensionality. While Leibniz has the essential properties of an equality, it also turns out that it is rarely provable. In fact, it only amounts to syntactic identity.

Definition 5.6. Two terms x and y are syntactically identical if they are composed of the same symbols of the language.

Proposition 5.7. *$x = y$ is provable iff x and y are syntactically identical.*

Proof. For the forward direction, let $x = y$ be provable. Then from the substitution instance

$$x = y, x \in z \vdash y \in z$$

and Cut, we have that

$$x \in z \vdash y \in z$$

must be provable. However, since we have a cut elimination result for this theory,

$$x \in z \vdash y \in z$$

must be provable. Such a cut-free proof can only be generated by an instance of Ax. Thus we must have syntactically identical formulas for x and y. The other direction is by 1 from Proposition 5.5. □

We define some abbreviated notation for the empty set and the set constructions of singleton, pairing, and ordered pairs.

- $\emptyset := \{x|\bot\}$.
- $\{t\} := \{x|x = t\}$
- $\{t, u\} := \{x|x = t \sqcup x = u\}$
- $\langle t, u\rangle := \{\{t\}, \{t, u\}\}$

Proposition 5.8. $\vdash x \notin \emptyset$.

Proof. Immediate from Ax, $\in_L$ and $\multimap_R$. □

Proposition 5.9 ([47])**.** $\langle t, u\rangle = \langle r, s\rangle \multimapboth t = r \star u = s$.

Proof. Note that since we have contraction on Leibniz identity, we do not need to worry about using formula multiple times in this proof. We can use our derived contraction to "Cut out" multiple occurrences.

The right to left direction follows from repeated substitution. The left to right is more complex.

First we prove that

$$\langle t, u\rangle = \langle r, s\rangle \vdash t = r.$$

Since $\{t\} = \{t\}$ we have that $\{t\} \in \langle t, u\rangle$. Thus by substitution, we have that $\{t\} \in \langle r, s\rangle$. Thus we have that $\{t\} = \{r\}$ or $\{t\} = \{r, s\}$. In either case, $t = r$.

Next we prove that

$$\langle t, u\rangle = \langle r, s\rangle \vdash u = s.$$

Since $\{t, u\} = \{t, u\}$ and $\{r, s\} = \{r, s\}$ we have that $\{t, u\} \in \langle t, u\rangle$ and that $\{r, s\} \in \langle r, s\rangle$. Substitution with these gives

$$\langle t, u\rangle = \langle r, s\rangle \vdash \{t, u\} = \{r\} \sqcup \{t, u\} = \{r, s\}$$

and

$$\langle t, u\rangle = \langle r, s\rangle \vdash \{r, s\} = \{t\} \sqcup \{r, s\} = \{t, u\}.$$

If we have $\{t, u\} = \{r\}$ then we can derive $u = r$ and then introduce with $u = r$ to get $u = r \sqcup u = s$, and if we have $\{t, u\} = \{r, s\}$ then we can derive $u = r \sqcup u = s$. Thus in either case we have $u = r \sqcup u = s$. Taken together, we have

$$\{t, u\} = \{r\} \sqcup \{t, u\} = \{r, s\} \vdash u = r \sqcup u = s.$$

Similarly, we can derive

$$\{r, s\} = \{t\} \sqcup \{r, s\} = \{t, u\} \vdash s = t \sqcup s = u.$$

The rest of the proof is another case by case argument. In the cases in which $u = s$ or $s = u$, we are done. In the latter case, we use the derived symmetry of identity. The only remaining case is then $u = r$ and $s = t$. In this case, we use the previous proof that $t = r$ and transitivity of identity to get that $u = s$.

The final desired result comes from two uses of $\langle t, u\rangle = \langle r, s\rangle$ but as noted in the beginning, this can be replaced with one instance by the proof of contraction for Leibniz identity. □

5.1.1 Fixpoint Theorem

With ordered pairs, we can now derive a fixpoint[4] theorem for the theory. This is what allows the construction of the highly impredicative sets often associated with unrestricted comprehension. The proof as given here is from [47]. The most difficult part is choosing the right set terms, the proof is straightforward manipulations after that.

Theorem 5.10 (Fixpoint Theorem)**.** *For any formula A, there exists a term f such that $t \in f \multimapboth A[f/y, t/x]$ is provable for any t.*

Proof. Define

$$s := \{z | \exists u \exists v(z = \langle u, v\rangle \star A[\{w|\langle w, v\rangle \in v\}/y, u/x])\}$$

and

$$f := \{w|\langle w, s\rangle \in s\}.$$

The proof for left to right follows. Derived connective rules from Proposition 5.1 are freely used.

$$\cfrac{\cfrac{\cfrac{\cfrac{\langle t, s\rangle = \langle t, s\rangle, A[f/y, t/x] \vdash A[f/y, t/x]}{\langle t, s\rangle = \langle t, s\rangle \star A[f/y, t/x] \vdash A[f/y, t/x]}\ \star_L}{\exists u \exists v(\langle t, s\rangle = \langle u, v\rangle \star A[\{w|\langle w, v\rangle \in v\}/y, u/x]) \vdash A[f/y, t/x]}\ \exists_L}{\langle t, s\rangle \in s \vdash A[f/y, t/x]}\ \in_L}{t \in f \vdash A[f/y, t/x]}\ \in_L$$

The right to left direction follows.

[4]Fixpoint is often used as an abbreviation for fixed point. My personal preference is to use the abbreviation "fixpoint" when referring to the theorem or the general set of objects that may be fixed points, but using the full "fixed points" when referring to a particular.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \langle t,s\rangle = \langle t,s\rangle \qquad A[f/y,t/x] \vdash A[f/y,t/x]}{A[f/y,t/x] \vdash \langle t,s\rangle = \langle t,s\rangle \star A[f/y,t/x]}\star_R}{A[f/y,t/x] \vdash \exists u \exists v(\langle t,s\rangle = \langle u,v\rangle \star A[\{w|\langle w,v\rangle \in v\}/y,u/x])}\exists_R}{A[f/y,t/x] \vdash \langle t,s\rangle \in s}\in_R}{A[f/y,t/x] \vdash t \in f}\in_R$$

□

The reason for calling this a fixpoint theorem is a bit more obvious if we rewrite it as its seen in [11].

Corollary 5.11. *Let* $fa := \{x|\langle x,a\rangle \in f\}$.[5]. *Then there exists a term* I_f *such that*

$$I_f =_e fI_f.$$

Proof. We need to prove that

$$x \in I_f \multimapboth x \in fI_f \multimap x \in \{z|\langle z, I_f\rangle \in f\}.$$

To accomplish this, we take

$$D_f := \{z|\exists x \exists g(z = \langle x,g\rangle \star x \in f(gg)\}$$

where g is a potential collection of ordered pairs[6] and gg is those elements of g's domain that map again to the *set* that g is. Then $f(gg)$ is those elements of f's domain which maps to the set which has those elements of g's domain which map to g. And we also take

$$I_f := D_f D_f.$$

The proof is now similar to the previous when all definitions are unfolded. □

A useful way to think of this theorem is as finding sets that are stable after the application of some defined operation. Let x be some set, and A some definable operation on the universe of sets, i.e. $x \mapsto A(x)$ where $A(x) := \{z|z \in x \wedge A(z)\}$. The fixpoint theorem says that there is some set in the universe I which remains unchanged after applying A.

To see this, we can take

$$f_A := \{u|\exists v \exists z(u = \langle v,z\rangle \star v \in A(z))\}.$$

The fixpoint thereom says that there is some I such that $I =_e f_A I$. Let's unpack the latter part of the term, $f_A I$. We have that

$$t \in f_A I \multimapboth \langle t, I\rangle \in f_A \multimapboth \exists v \exists z(\langle t,I\rangle = \langle v,z\rangle \star t \in A(z)).$$

[5] fa is then the subset of f's domain that maps to a

[6] gg would be empty if there were no ordered pairs in g

The fixpoint theorem further gives that

$$t \in I \multimapboth t \in f_A I$$

and so we have finally that

$$t \in I \multimapboth t \in A(I).$$

Note that this does not necessarily say that the operation has no effect on the elements of the fixpoint term. It only implies that we get back the same set after applying the operation. So, for example, if we took a successor operation, there would be some subset of the universe which was not altered, but it is not necessarily implied that it means that there must be some element which is its own successor, i.e. a number not "affected" by the successor operation, nor does it imply those elements which behave oddly with regard to the successor function are actually interpretable as natural numbers.

Remark 5.12. One "naive" but interesting fixpoint for successor in a naive set theory with both a fixpoint theorem and the axiom of extensionality would be the singleton containing the universe. The universe, V, would be such that it was extensionally equal to its own successor, $S(V) := V \cup \{V\}$. Thus the singleton $\{V\}$ would be equal to the singleton $\{S(V)\}$.

As an example of an application of the fixpoint thereom, it could be used to define the natural numbers. We will not use this approach for the natural numbers below, but the definition as a fixpoint would be:

$$x \in \mathbf{N} \multimapboth x = \emptyset \vee \exists z(z \in \mathbf{N} \wedge x = S(z).$$

We will use the fixpoint theorem to bypass the recursive machinery needed for defining the arithmetic operations of addition and multiplication.

5.1.2 Natural Numbers

We can define the natural numbers in this theory and derive the Peano axioms with an alteration to induction. The succcessor operation is not defined as the usual $S(t) := t \cup \{t\}$ as the logic is not strong enough to derive $S(x) = S(y) \multimap x = y$ with that definition.

Definition 5.13. $0 := \emptyset$.

Definition 5.14. $S(t) := \langle \emptyset, t \rangle$.

Definition 5.15. $n := S^n(0)$.

Definition 5.16. The set of natural numbers:

$$\mathbf{N} := \{x | \forall \alpha ! \forall y(y \in \alpha \multimap S(y) \in \alpha) \multimap \S(0 \in \alpha \multimap x \in \alpha)\}.$$

Proposition 5.17. *The following are provable:*

1. $0 \in \mathbf{N}$

2. $t \in \mathbf{N} \multimap S(t) \in \mathbf{N}$

3. $S(t) \neq 0$

4. $S(t) = S(u) \multimap t = u$

Proof. The proof for 1 follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{0 \in \alpha \vdash 0 \in \alpha}{\vdash 0 \in \alpha \multimap 0 \in \alpha}\multimap_R}{\vdash \S(0 \in \alpha \multimap 0 \in \alpha)}\S}{!\forall y(y \in \alpha \multimap S(y) \in \alpha) \vdash \S(0 \in \alpha \multimap 0 \in \alpha)}\text{W}}{\vdash !\forall y(y \in \alpha \multimap S(y) \in \alpha) \multimap \S(0 \in \alpha \multimap 0 \in \alpha)}\multimap_R}{\vdash \forall \alpha(!\forall y(y \in \alpha \multimap S(y) \in \alpha) \multimap \S(0 \in \alpha \multimap 0 \in \alpha))}\forall_R}{\vdash 0 \in N}\text{Def}$$

The proof for 2 follows. Denote this first subproof as **D**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{t \in \alpha \vdash t \in \alpha \qquad S(t) \in \alpha \vdash S(t) \in \alpha}{t \in \alpha \multimap S(t) \in \alpha, t \in \alpha \vdash S(t) \in \alpha}\multimap_L}{\forall y(y \in \alpha \multimap S(y) \in \alpha), t \in \alpha \vdash S(t) \in \alpha}\forall_L \qquad 0 \in \alpha \vdash 0 \in \alpha}{0 \in \alpha, \forall y(y \in \alpha \multimap S(y) \in \alpha), 0 \in \alpha \multimap t \in \alpha \vdash S(t) \in \alpha}\multimap_L}{\forall y(y \in \alpha \multimap S(y) \in \alpha), 0 \in \alpha \multimap t \in \alpha \vdash 0 \in \alpha \multimap S(t) \in \alpha}\multimap_R}{!\forall y(y \in \alpha \multimap S(y) \in \alpha), \S(0 \in \alpha \multimap t \in \alpha) \vdash \S(0 \in \alpha \multimap S(t) \in \alpha)}\S$$

Then we finish the proof for 2 using **D** as defined. Denote multiple occurrences of a formula A as A^n.

$$\dfrac{\dfrac{\dfrac{\mathbf{D} \qquad !\forall y(y \in \alpha \multimap S(y) \in \alpha) \vdash !\forall y(y \in \alpha \multimap S(y) \in \alpha)}{!\forall y(y \in \alpha \multimap S(y) \in \alpha)^2, !\forall y(y \in \alpha \multimap S(y) \in \alpha) \multimap \S(0 \in \alpha \multimap t \in \alpha) \vdash \S(0 \in \alpha \multimap S(t) \in \alpha)}\multimap_L}{!\forall y(y \in \alpha \multimap S(y) \in \alpha) \multimap \S(0 \in \alpha \multimap t \in \alpha) \vdash !\forall y(y \in \alpha \multimap S(y) \in \alpha) \multimap \S(0 \in \alpha \multimap S(t) \in \alpha)}\text{C}, \multimap_R}{t \in \mathbf{N} \vdash S(t) \in \mathbf{N}}\forall_L, \multimap_R, \text{Def}$$

The proof for 3 follows by construction of $S(t)$. Suppose $S(t) = 0$. Then since $S(t)$ is non-empty since it must contain $\{\emptyset\}$ since $\{\emptyset\} \in \langle \emptyset, t \rangle$, it implies that $\emptyset$ has a member. This is a contradiction. The proof for 4 follows directly from Proposition 5.9. □

We now turn to the restricted version of induction that's recoverable, called "Light Induction". The primary restriction that's introduced is on the single formula assumption allowed in the inductive step. There's also generally a lot less contraction allowed.

Proposition 5.18. $\dfrac{\Gamma \vdash A[0/x] \qquad B, A[y/x] \vdash A[S(y)/x]}{\S\Gamma, !B, t \in \mathbf{N} \vdash \S A[t/x]}$

Proof. The proof follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B, A[y/x] \vdash A[S(y)/x]}{B, y \in \{x|A\} \vdash S(y) \in \{x|A\}}\in_L, \in_R}{B \vdash \forall y(y \in \{x|A\} \multimap S(y) \in \{x|A\})}\multimap_R, \forall_R}{!B \vdash !\forall y(y \in \{x|A\} \multimap S(y) \in \{x|A\})}! \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash A[0/x]}{\Gamma \vdash 0 \in \{x|A\}}\in_R \qquad \dfrac{A[t/x] \vdash A[t/x]}{t \in \{x|A\} \vdash A[t/x]}\in_L}{\Gamma, 0 \in \{x|A\} \multimap t \in \{x|A\} \vdash A[t/x]}\multimap_L}{\S\Gamma, \S(0 \in \{x|A\} \multimap t \in \{x|A\}) \vdash \S A[t/x]}\S}{\S\Gamma, !B, !\forall y(y \in \{x|A\} \multimap S(y) \in \{x|A\}) \multimap \S(0 \in \{x|A\} \multimap t \in \{x|A\}) \vdash \S A[t/x]}\multimap_L}{\S\Gamma, !B, \forall \alpha(!\forall y(y \in \alpha \multimap S(y) \in \alpha) \multimap \S(0 \in \alpha \multimap t \in \alpha) \vdash \S A[t/x]}\forall_L}{\S\Gamma, !B, t \in \mathbf{N} \vdash \S A[t/x]}\text{Def}$$

$\square$

Finally, we can define the operations of arithmetic as fixpoints using the fixpoint theorem. This is a good demonstration of how useful the fixpoint constructions are. We would normally need a recursion theorem to recover these operations, but the fixpoint theorem can be used instead.

Definition 5.19. Define $+$ as the term

$$\begin{aligned}\langle x, y, z\rangle \in + \multimap\!\circ (y = 0 \star x = z) \sqcup \\ \exists y' \exists z' (y = S(y') \star z = S(z') \star \langle x, y', z'\rangle \in +)\end{aligned}$$

Definition 5.20. Define $\times$ as the term

$$\begin{aligned}\langle x, y, z\rangle \in \times \multimap\!\circ (y = 0 \star z = 0) \sqcup \\ \exists y' \exists z' (y = S(y') \star \langle z', x, z\rangle \in + \star \langle x, y', z'\rangle \in \times\end{aligned}$$

Proposition 5.21. *The following are provable:*

1. $\langle x, 0, z\rangle \in + \multimap\!\circ\ x = z$,
2. $\langle x, S(y), z\rangle \in + \multimap\!\circ\ \exists z'(z = S(z') \star \langle x, y, z'\rangle \in +)$,
3. $\langle x, 0, z\rangle \in \times \multimap\!\circ\ z = 0$,
4. $\langle x, S(y), z\rangle \in \times \multimap\!\circ\ \exists z'(\langle z', x, z\rangle \in + \star \langle x, y, z'\rangle \in \times)$

5.2 Grišin's Paradox

We now look at the paradoxes that result when the axiom of extensionality is added.[7] That the axiom of extensionality leads to triviality in these theories was first recognized by Grišin [24].

We will look at two different derivations for this triviality. A primary feature that allows these derivations is the provable contraction of Leibniz identity. The first derivation will use this contractive property of Leibniz identity with the axiom of extensionality to show that contraction is provable for arbitrary formula. This then allows the proof for Curry's paradox. The second derivation of triviality is a stronger result, and will show that even with the axiom of extensionality only applied to empty sets, triviality still follows. In this section, we will again use subscripted equality to denote each type of equality, $=_e$ for extensional identity and $=_l$ for Leibniz identity. These proofs are from [11].

Lemma 5.22. *A simply contraction-free theory with naive comprehension and the axiom of extensionality implies that* $x =_e y \vdash x =_e y \star x =_e y$.

[7] Note that these proofs will hold even for the weaker contraction-free systems without exponentials.

Proof. We use contraction of Leibniz identity as proved in Proposition 5.5 as a sequent calculus step $=_l C$. The proof follows.

$$
\begin{prooftree}
\AxiomC{$x =_l y \vdash x =_e y$}
\AxiomC{$x =_l y \vdash x =_e y$}
\RightLabel{$\star_R$}
\BinaryInfC{$x =_l y \star x =_l y \vdash x =_e y \star x =_e y$}
\RightLabel{$=_l$ C}
\UnaryInfC{$x =_l y \vdash x =_e y \star x =_e y$}
\AxiomC{}
\RightLabel{Ext}
\UnaryInfC{$x =_e y \vdash x =_l y$}
\RightLabel{Cut}
\BinaryInfC{$x =_e y \vdash x =_e y \star x =_e y$}
\end{prooftree}
$$

□

Proposition 5.23. *A simply contraction-free theory with naive comprehension and the axiom of extensionality derives contraction on arbitrary atomic formula, that is for any t and s,*

$$t \in s \vdash t \in s \star t \in s.$$

Proof. The idea of the proof is to deduce from $t \in s$ some appropriate statement about identity which then allows us to use the derived contraction for identity to derive contraction for all membership formula. That is, we derive

$$t \in s \vdash a =_l b$$

for some particular choice of a and b for which

$$a =_l b \vdash t \in s.$$

The choice for a will be $\{t\}$ and for b it will be $\{t\} \cap s := \{x | x \in \{t\} \star x \in s\}$.

We first prove that $\{t\} =_e \{t\} \cap s \vdash t \in s$.

$$
\begin{prooftree}
\AxiomC{$\vdash t \in \{t\}$}
\AxiomC{$t \in \{t\}, t \in s \vdash t \in s$}
\RightLabel{$\star_L$}
\UnaryInfC{$t \in \{t\} \star t \in s \vdash t \in s$}
\RightLabel{$\in_L$}
\UnaryInfC{$t \in \{t\} \cap s \vdash t \in s$}
\RightLabel{$\multimap_L$}
\BinaryInfC{$t \in \{t\} \multimap t \in \{t\} \cap s \vdash t \in s$}
\RightLabel{W}
\UnaryInfC{$t \in \{t\} \cap s \multimap t \in \{t\}, t \in \{t\} \multimap t \in \{t\} \cap s \vdash t \in s$}
\RightLabel{$\star_L, \forall_L$}
\UnaryInfC{$\forall z (z \in \{t\} \cap s \multimap z \in \{t\}) \star (z \in \{t\} \multimap z \in \{t\} \cap s) \vdash t \in s$}
\RightLabel{Def}
\UnaryInfC{$\{t\} =_e \{t\} \cap s \vdash t \in s$}
\end{prooftree}
$$

Next we prove that $z \in s \cap \{t\} \multimap z \in \{t\}$.

$$
\begin{prooftree}
\AxiomC{$z \in \{t\}, z \in s \vdash z \in \{t\}$}
\RightLabel{$\star_L, \in_L$}
\UnaryInfC{$z \in s \cap \{t\} \vdash z \in \{t\}$}
\RightLabel{$\multimap_R$}
\UnaryInfC{$\vdash z \in s \cap \{t\} \multimap z \in \{t\}$}
\RightLabel{$\forall_R$, Def}
\UnaryInfC{$\vdash s \cap \{t\} \subseteq \{t\}$}
\end{prooftree}
$$

Next we prove that under the assumption $t \in s$, $z \in \{t\} \multimap z \in s \cap \{t\}$. We use the derived contraction on Leibniz identity as $z =_l t \vdash z =_l t \star z =_l t$.

$$\dfrac{\dfrac{\dfrac{\dfrac{z=t,t\in s\vdash z\in s \qquad \dfrac{z=t\vdash z=t}{z=t\vdash z\in\{t\}}\in_R}{z=t,z=t,t\in s\vdash z\in s\star z\in\{t\}}\star_R}{z=t,t\in s\vdash z\in s\star z\in\{t\}}=_l \mathrm{C}}{z\in\{t\},t\in s\vdash z\in s\cap\{t\}}\in_L,\in_R}{t\in s\vdash z\in\{t\}\multimap z\in s\cap\{t\}}\multimap_R$$

Now we can use $\star_R$ and $\forall_R$ with these two previous proofs to derive that

$$t\in s\vdash\{t\}=_e\{t\}\cap s.$$

We now complete the result.

$$\dfrac{\dfrac{\dfrac{\{t\}=_e\{t\}\cap s\vdash t\in s \qquad \{t\}=_e\{t\}\cap s\vdash t\in s}{\{t\}=_e\{t\}\cap s,\{t\}=_e\{t\}\cap s\vdash t\in s\star t\in s}\star_R}{\{t\}=_e\{t\}\cap s\vdash t\in s\star t\in s}=_e \mathrm{C} \qquad t\in s\vdash\{t\}=_e\{t\}\cap s}{t\in s\vdash t\in s\star t\in s}\mathrm{Cut}$$

□

Corollary 5.24. *A simply contraction-free theory with naive comprehension and the axiom of extensionality is trivial.*

The other derivation of triviality uses the fixpoint construction of a set g such that

$$x\in g\multimapboth x=g\star x=\emptyset.$$

We will find that membership in g contracts, and that will be enough to derive triviality. This derivation exclusively uses extensionality on the empty set.

Proposition 5.25. *A simply contraction-free theory with naive comprehension and the axiom of extensionality restricted to the empty set,*

$$t=_e\emptyset\vdash t=_l\emptyset,$$

is trivial.

Proof. We use the fixpoint theorem to get the construction

$$x\in g\multimapboth x=_l g\star x=_l\emptyset.$$

We first prove that membership contracts for g.

$$\dfrac{\dfrac{\dfrac{\dfrac{x=_l g,x=_l\emptyset\vdash x\in g \qquad x=_l g,x=_l\emptyset\vdash x\in g}{x=_l g,x=_l\emptyset,x=_l g,x=_l\emptyset\vdash x\in g\star x\in g}\star_R}{x=_l g,x=_l\emptyset\vdash x\in g\star x\in g}=_l \mathrm{C}}{x=_l g\star x=_l\emptyset\vdash x\in g\star x\in g}\star_L}{x\in g\vdash x\in g\star x\in g}\star_R$$

Next we prove that $g =_e \emptyset$. That $x \in \emptyset \multimap x \in g$ follows from definition of $\emptyset$. The proof for the other direction follows. First, by substitution, we have that

$$g =_l \emptyset, x \in g \vdash x \in \emptyset.$$

We can then use the definition of g and transitivity of identity to finish this direction of extensionality.

$$\dfrac{\dfrac{\dfrac{\dfrac{x \in g \vdash x =_l g \star x =_l \emptyset \qquad x =_l g \star x =_l \emptyset \vdash g =_l \emptyset}{x \in g \vdash g =_l \emptyset}\,\text{Cut} \qquad g =_l \emptyset, x \in g \vdash x \in \emptyset}{x \in g, x \in g \vdash x \in \emptyset}\,\text{Cut}}{x \in g \star x \in g \vdash x \in \emptyset}\,\star_L}{x \in g \vdash x \in \emptyset}\,x \in g\ \text{C}$$

The assumed restricted form of extensionality allows us to conclude that $g =_l \emptyset$. This then allows us to conclude that $g \in g$.

$$\dfrac{\dfrac{\vdash g =_l g \qquad \vdash g =_l \emptyset}{\vdash g =_l g \star g =_l \emptyset}\,\star_R}{\vdash g \in g}\,\in_R$$

Since we also have that $g = \emptyset$, we get that $g \in \emptyset$. Any provable member of the empty set proves triviality. □

5.2.1 Paraconsistent Extension

The success of relevant logic with a paraconsistent negation in maintaining non-triviality with full naive set theory suggests we could try something similar here. The paraconsistent approach also suggests that the extension of a set should split from its "anti-extension", the things which are not a member of that set [34]. Since being a member no longer excludes the possibility of not being member, it opens the possibility that sets could be identical in extension but not identical in anti-extension. Thus, we attempt to define extensionality with a paraconsistent negation which respects this observation. This approach was attempted for this thesis and the results in this subsection are new contributions.

We will use $\neg$ to denote an "arbitrary" paraconsistent negation which has properties as specified in the following propositions. Let $x \notin y$ be shorthand for $\neg(x \in y)$. We will redefine extensional identity to include reference to this assume paraconsistent negation. The axiom of extensionality remains as before,

$$x =_e y \vdash x =_l y,$$

but with the redefined extensional identity,

Definition 5.26. Two sets x and y are (paraconsistently) extensionally equal, $x =_e y$ if $\forall z((z \in x \Leftrightarrow z \in y) \wedge (z \notin x \Leftrightarrow z \notin y))$.

If we used this negation in a theory in such a way that it could never be proved, then the naive set theory with this paraconsistently stated extensionality with a contraction-free logic would be non-trivial. This obviously misses the intended goal of changing the axiom of extensionality. Sadly, the results below show that the negation can have very little proof theoretic ability before we find ourselves in triviality again. Thus we conclude this added paraconsistent negation would have to be rather weak and what would be left of the negation connective would not be a useful form of negation to have in the theory. Thus this does not seem like a viable way of recovering a naive set theory with extensionality with these logics.

To prove these results, we extend the Grišin paradoxes to apply to the paraconsistent notion of extensionality. The first paradox is extended by first showing that it allows us to recover some contraction, namely that any formula can be used as many times as needed after we have assumed it twice.

Proposition 5.27. *The simply contraction-free theory with naive comprehension and the paraconsistent axiom of extensionality in which the paraconsistent negation* $\neg$ *satisfies the following*

- $A, \neg B \vdash \neg(A \multimap B)$,
- $\exists z \neg A \vdash \neg \forall z A$,
- $\neg A \sqcup \neg B \vdash \neg(A \star B)$,

we can derive that

$$t \in s \wedge t \in s \multimap t \in s \wedge t \in s \wedge t \in s.$$

Proof. The proof proceeds as the proof of Proposition 5.23 with additional steps to cover the negated steps needed. First, note that $\{t\} =_e \{t\} \cap s \vdash t \in s$ still holds as before since we have only strengthened $=_e$. Then we prove each of the four clauses needed for equality.

First,

$$\vdash x \in s \cap \{t\} \multimap x \in \{t\}$$

holds by definition as before. The contraposition of this,

$$\vdash x \notin \{t\} \multimap x \notin s \cap \{t\},$$

holds by the assumed property of the negation that $\neg A \sqcup \neg B \multimap \neg(A \star B)$. That

$$t \in s \vdash x \in \{t\} \multimap x \in s \cap \{t\}$$

holds as proved previously.

The remaining clause to prove is

$$t \in s \vdash x \notin s \cap \{t\} \multimap x \notin \{t\}.$$

This proceeds by cases. From $x \notin s \cap \{t\}$ we have that $x \notin s$ or $x \notin \{t\}$. The latter case is trivial. In the former case, we have $t \in s$ and $x \notin s$ which gives $\neg(t \in s \multimap x \in s)$ by counterexample. Then we have that $\exists z \neg(t \in z \multimap x \in z)$ which gives $\neg \forall z(t \in z \multimap x \in z)$.

Thus we have that $x \notin \{t\}$ in this case as well. Thus we have that

$$t \in s \vdash x \notin s \cap \{t\} \multimap x \notin \{t\}.$$

Therefore we have that

$$t \in s \star t \in s \vdash \{t\} =_e s \cap \{t\}.$$

As before, we can "multiply" this conclusion and get the desired result,

$$t \in s \star t \in s \vdash t \in s \star t \in s \star t \in s.$$

$\square$

Corollary 5.28. *The theory described in Proposition 5.27 is trivial.*

Proof. The proof follows from a "revenge" Curry's set $\{x | (x \in x \star x \in x) \multimap A\}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \in C \star C \in C \vdash C \in C \star C \in C \qquad A \vdash A}{C \in C \star C \in C, C \in C \star C \in C \multimap A \vdash A}\to_L}{C \in C \star C \in C, C \in C \vdash A}\in_L}{C \in C \star C \in C \star C \in C \vdash A}\star_L}{C \in C \star C \in C \vdash A}\text{Prop 5.27}}{\vdash C \in C \star C \in C \multimap A}\multimap_R}{\vdash C \in C}\in_R$$

Then the proof is reused up to $C \in C \star C \in C \vdash A$ and Cut completes the derivation. $\square$

To avoid the proof of triviality, the paraconsistent negation has to abandon one of the three assumed properties and anything that might allow us to prove those. These are difficult principles to do without and rather minimal assumptions on a negation. We now adapt the other paradox to show what else the negation can not do.

Proposition 5.29. *The simply contraction-free theory with naive comprehension and the paraconsistent axiom of extensionality restricted to empty sets in which the paraconsistent negation $\neg$ satisfies the law of excluded middle,*

$$\vdash A \vee \neg A$$

for any A, we can derive triviality.

Proof. The proof proceeds as the proof of Proposition 5.25 with additional steps to cover the negated steps needed. That

$$\vdash x \in \emptyset \multimap x \in g$$

and

$$\vdash x \in g \multimap x \in \emptyset$$

hold as before. That

$$\vdash x \notin g \multimap x \notin \emptyset$$

follows from the fact that $x \notin \emptyset$ is provable and weakening.

The final derivation is

$$\vdash x \notin \emptyset \vdash x \notin g.$$

We proceed by cases from the law of excluded middle as $x \in g \vee x \notin g$. In the case that $x \notin g$ we are done. In the case that $x \in g$, we have that

$$x \in g \vdash x =_l g \wedge x =_l \emptyset.$$

Thus we have that

$$x \in g \vdash g =_l \emptyset.$$

As in Proposition 5.25, we can derive that

$$g =_l \emptyset \vdash \bot.$$

and thus by Cut and the property of $\bot$ that $\bot \vdash A$ for any A, we have

$$g =_l \emptyset \vdash x \notin g.$$

Again by Cut, we thus have

$$x \in g \vdash x \notin g.$$

Thus in either case we have that

$$\vdash x \notin \emptyset \multimap x \notin g.$$

Thus we've derived that $g =_e \emptyset$ and thus $g =_l \emptyset$. The proof proceeds as in Proposition 5.25 to derive triviality. □

This result shows that the assumed paraconsistent negation absolutely can not satisfy the law of excluded middle. Given the general weakness of the paraconsistent negation, the loss of LEM for it would make it very difficult to use in the theory.

5.3 Further Work

The main problem in using LAST as a formal system for mathematics is its weakness. Terui notes that they had trouble in proving even the totality of division [52]. Thus, some further work in this area would investigate ways to increase the strength of the system while still maintaining non-triviality.

One way this could be done is by finding a restricted version of extensionality that did not cause triviality. As was initially noted by foundational mathematicians at the beginning of the twentieth century, perhaps the problem is being too loose with non-well-founded sets. It might be interesting to explore extensionality in LAST if it was restricted to only well-founded sets. However, this wouldn't escape triviality as the $\emptyset$ is well-founded itself and we see that extensionality restricted to the $\emptyset$ is problematic.

This suggests trying LAST with extensionality restricted to non-empty and well-founded sets. This would still likely result in membership contraction being provable for such sets; an inspection of the proof of Proposition 5.23 shows that nothing would change except the additional restriction on what sets proved the proposition.

Another way the strength of these systems could be expanded is by turning to stronger variants of light linear logics. The logical strength of the logic LAST uses is merely ω [4]. However, recently a system with strength of ω^ω that still proves cut elimination with unrestricted $\in$ rules was detailed in [4]. It's an open question if the extra strength also allows more mathematical theorems to be proved.

5.4 Conclusion

LAST provides a glimpse of what it looks like to work with naive set theory in a more natural logical environment. We only have to do without contraction, which is not too difficult once practiced. The fixpoint theorem also provides the theory with a lot of expressive power.

However, it is held back in what it can do with the severe restriction of its logical strength. Further, a theory that could contain full extensionality as well would be a nice bonus. In such a case that no better theory could be found, then this approach would appear to be the best available.

We now move to the main theory of this thesis, NNST. It is conjectured that under a suitable translation, NNST would recover all of what can be done in LAST and all similar light linear naive set theories. Such a result is highly desired to connect up the work in these systems with NNST.

Chapter 6

Normalized Naive Set Theory

This chapter discusses the naive set theory called NNST for "Normalized" Naive Set Theory. One of the earliest suggestions for this approach comes from Prawitz in 1965 [39]. Some more recent work can also be found in [18, 17]. A more general study of normalization of proofs in naive set theory was performed in [25]. More generally, this is a non-transitive approach which has been looked at in [60, 45, 51]. However, most of the work done in this chapter is original. The works cited and so far discovered appear to be mostly exploratory. This is likely due to this approaches' computationally difficult aspects, proof theoretic difficulties and quasi-paraconsistent nature. This thesis takes the position that the potential benefits of understanding this theory outweighs its difficulties.

We will first discuss the natural deduction framework for logics and the concept of normalization. Normalization will allow us to define NNST and guarantees its non-triviality. We then find a more "minimal" set of connectives which defines full NNST. Finally we develop some mathematics concluding with the recovery of Heyting Arithmetic[1] in the follow chapter. Of potentially equal importance, but an aside from the purpose for a mathematics thesis, Appendix A lays out the beginnings of a philosophical justification and understanding of NNST.

The chapter ends with far more questions than answers, and it is hoped the questions spur further development of this theory.

6.1 Logical Rules

The presentation for NST, and later NNST, is in a natural deduction system.[2] It will be more straightforward to build these theories on intuitionistic logic but all results here will translate to a classical setting.[3] The intuitionistic variant will suffice for the immediate goal of showing the theory to be of mathematical interest.

[1]The obvious classical extension for NNST should also recover PA, but a proof is not yet known.

[2]Part of the reason for this is that natural deduction mirrors informal reasoning.

[3]Sequent calculi are generally more amenable to classical logic while natural deduction favors intuitionistic. Briefly, there is more difficulty in handling classical negation in normalization in natural deduction systems while the asymmetry required for intuitionistic logic in sequent calculus makes the sequent calculus a bit more difficult to use.

Let Γ, Δ, and Σ stand for sets of formulas. All collections of assumptions are assumed to be sets.[4] We first give the logical rules for NNST in the form of a sequent presentation of a natural deduction system.[5]

$$\frac{}{\Gamma, A \vdash A}\,\text{Ax}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\wedge E_0 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\wedge E_1$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\vee I_0 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\vee I_1 \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\,\vee E$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to I \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B} \to E$$

$$\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A}\,\forall I \qquad \frac{\Gamma \vdash \forall x A}{\Gamma \vdash A[y/x]}\,\forall E$$

$$\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \exists x A}\,\exists I \qquad \frac{\Gamma \vdash \exists x A \qquad \Gamma, A[y/x] \vdash B}{\Gamma \vdash B}\,\exists E$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\bot$$

For $\forall I$, a must not occur free in any formula in Γ.

For $\exists E$, a can not occur free in any formula in Γ nor free in B.

For $\bot$, A is to be different from $\bot$.

$\neg A$ will be an abbreviation for $A \to \bot$.

We extend the logical system to naive set theory by adding the following two sets of introduction and elimination rules, one for equality and one for membership.

$$\frac{\Gamma, x \in s \vdash x \in r \qquad \Gamma, x \in r \vdash x \in s}{\Gamma \vdash s = r} = I \qquad \frac{\Gamma \vdash A(r) \qquad \Gamma \vdash s = r}{\Gamma \vdash A[s/r]} = E_0$$

$$\frac{\Gamma \vdash A(s) \qquad \Gamma \vdash s = r}{\Gamma \vdash A[r/s]} = E_1$$

$$\frac{\Gamma \vdash A[r/x]}{\Gamma \vdash r \in \{x|A(x)\}} \in I \qquad \frac{\Gamma \vdash r \in \{x|A(x)\}}{\Gamma \vdash A[r/x]} \in E$$

For $= I$, x must not occur free in any open assumptions other than $x \in s$, $x \in r$.

$s \notin r$ will be an abbreviation for $s \in r \to \bot$.

With $= E$ rules, we can substitute between any combination of set terms and variables. For $= E_0$, the substitution of s may be performed on any number of free occurrences of r if it is a variable, or any number of occurrences of r so long as s is free for r if r is term.

[4]This automatically implies the natural deduction analog of "exchange" holds, i.e. the order of assumptions does not matter. Contraction also implicitly holds; in natural deduction contraction translates as allowing multiple uses of a formula.

[5]Note that these rules are an intuitionistic natural deduction system extended with rules governing set equality and membership.

Definition 6.1. A set term $\{x|A\}$ is free for a variable or term, t, in a formula B if replacing t would not cause any variable of $\{x|A\}$ to become bound by a quantifier.

Definition 6.2. Any formula which occurs to the left of a turnstile is called an **assumption**.

Definition 6.3. A formula is said to be **discharged** when a rule removes it from the set of assumptions.[6]

Definition 6.4. A formula is an **open assumption** if it is a member of the set of assumptions on a line of the deduction. It is **closed** otherwise.

Definition 6.5. A deduction is **open** if it has an open assumption in the last line of the deduction. It is **closed** otherwise.

For instantiation or substitution, it will usually be clear from context which term is being substituted for which. Thus, instead of writing the substitution of s for r in $A(r)$ as $A[s/r]$ we will sometimes simply write $A(s)$, as in the following.

$$\frac{\Gamma \vdash A(r) \qquad \Gamma \vdash s = r}{\Gamma \vdash A(s)} = E_0$$

In longer proofs, we will not always carry the same Γ throughout the proof. Any such proof that is missing the assumptions on applications of rules can be fixed by weakening the assumptions in using the Axiom rule.[7]

We now define further terminology and build to the ideas of a deduction and a proof. These two terms will come apart for NNST: we call the tree structure that we build in applying rules of NST a deduction, while some subset of deductions are proofs if they satisfy some pre-specified criteria.

Definition 6.6. Each application of a rule is composed of **top formulas** and a **bottom formula**.[8] A formula is called a **premise**[9] to a rule if it occurs to the right of a turnstile in a top formula, it is called a **conclusion**[10] if it occurs to the right of a turnstile in a bottom formula.

Definition 6.7. The **major premise** of an elimination (introduction) rule is the premise which contains the connective being eliminated (introduced) as the primary connective. All other premises are **minor**.

Definition 6.8. Let T be an n-tuple composed of $n-1$ formulas, $A_1, A_2, \ldots A_{n-1}$ and a rule, R_0. Further, the formulas $A_1, A_2, \ldots A_{n-2}$ are the top formulas and A_{n-1} the bottom formula of the application of the rule R_0. T is a **valid** if the top formulas and the bottom formula match the proper shape of formulas for any defined rule of theory.

[6]The only such rules in this system that can discharge are $\to I$, $\exists E$, $= I$, $\vee E$.

[7]We also know this is possible because we are dealing with finite proofs.

[8]Defined in the obvious way.

[9]The same way in which the formulas that precede a rule in a non-sequent presentation of natural deduction are premises.

[10]Again, in the same way as we might call the formula in a non-sequent natural deduction.

Definition 6.9. A finite sequence of n-tuples, $T_0, T_1, T_2 \ldots, T_m$ each composed of $n-1$ formulas, $A_1, A_2, \ldots A_{n-1}$ and a rule, R_0 is a **deduction** if each tuple is valid and the bottom formula of each tuple except for T_m occurs as a top formula in a valid tuple later in the sequence.

Definition 6.10. A deduction in a natural deduction theory is a **proof** if it satisfies a, possibly empty, set of conditions.

6.2 Normal Proofs

In the context of natural deduction formalizations, there is a concept of normal proofs. This concept turns out to be analogous to cut-free proofs in a sequent calculus. The following exposition follows Prawitz's development in [39].

Definition 6.11. A **subformula** of a formula A is defined[11] inductively:

1. A is a subformula of A,
2. if $B \wedge C$, $B \vee C$, or $B \rightarrow C$ are subformulas of A then B and C are as well,
3. if $\forall x B$ or $\exists x B$ is a subformula of A then so is $B[t/x]$,
4. if $x \in \{z|B(z)\}$ is a subformula of A then $B[x/z]$ is a subformula of A,
5. if $x = y$ is a subformula of A then $z \in x$ and $z \in y$ are subformulas of A,
6. if x is a term that is free in A, then $A[y/x]$ is a subformula of A.

Definition 6.12. A sequence of formula occurrences $A_1, A_2, \ldots, A_n$ in a deduction Π is a **thread** if

1. A_1 is the result of an axiom rule in Π
2. A_i stands immediately above A_{i+1} in Π for each $i < n$
3. A_n is the end-formula of Π

Definition 6.13. A **segment** in a deduction Π is a sequence $A_1, A_2, \ldots, A_n$ of consecutive formula occurrences in a thread in Π such that

1. A_1 is not the conclusion of an application of $\vee E$, $\exists E$, or $= E$
2. A_i, for each $i < n$, is a minor premise of an application of $\vee E$, $\exists E$, or $= E$
3. A_n is not the minor premise of an application of $\vee E$, $\exists E$, or $= E$

A single formula that doesn't take place as a minor premise or consequence of $\vee E$ and $\exists E$ qualifies as a segment.

[11] This definition has been extended for the purpose of proving a subformula property.

Definition 6.14. The **formula of a segment** refers to the formula repeated in the sequence that composes the segment.

Definition 6.15. A **maximum segment** is a segment that begins with a consequence of an application of an I rule or the $\perp$ rule and ends with a major premise of an E rule.

Definition 6.16. A **normal deduction** is a deduction that

- Does not contain a maximum segment
- Contains no applications of $\vee E$ or $\exists E$ in which a minor premise does not discharge an assumption.

6.2.1 Rules of Reduction

One way to investigate whether a deduction reduces to a normal form is by rules of reduction. A non-normal proof takes detours, by introducing a connective and then discharging the connective just introduced. These rules demonstrate how to remove these detours of a proof for specific instances of maximum segments. We use a new notation which has the obvious implied definition, $\pi_0[a/x]$, meaning replace all free instances of x in the proof with a. Note that in the case of $\vee E$ and $\vee I_0$, there are technically two reductions to be demonstrated, the other with $\vee I_1$, but this is similar and thus omitted.

$$\dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A\end{array} \quad \begin{array}{c}\pi_1\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}\wedge I}{\Gamma \vdash B}\wedge E \qquad \rightsquigarrow \qquad \begin{array}{c}\pi_1\\ \Gamma \vdash B\end{array}$$

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A\end{array} \quad \dfrac{\begin{array}{c}\pi_1\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \to B}\to I}{\Gamma \vdash B}\to E \qquad \rightsquigarrow \qquad \begin{array}{c}\pi_0\\ \Gamma \vdash A\\ \pi_1\\ \Gamma \vdash B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A[y/x]\end{array}}{\Gamma \vdash \forall x A(x)}\forall I}{\Gamma \vdash A[b/x]}\forall E \qquad \rightsquigarrow \qquad \begin{array}{c}\pi_0[b/y]\\ \Gamma \vdash A[b/x]\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A[a/x]\end{array}}{\Gamma \vdash a \in \{x|A(x)\}}\in I}{\Gamma \vdash A[a/x]}\in E \qquad \rightsquigarrow \qquad \begin{array}{c}\pi_0\\ \Gamma \vdash A[a/x]\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \vee B}\vee I_0 \quad \begin{array}{c}\pi_1\\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\vee E \qquad \rightsquigarrow \qquad \begin{array}{c}\pi_0\\ \Gamma \vdash A\\ \pi_1\\ \Gamma \vdash C\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A[a/x]\end{array}}{\Gamma \vdash \exists x A(x)}\exists I \qquad \begin{array}{c}\pi_1\\ \Gamma, A(x) \vdash B\end{array}}{\Gamma \vdash B}\exists E \quad \rightsquigarrow \quad \begin{array}{c}\pi_0\\ \Gamma \vdash A[a/x]\\ \pi_1[a/x]\\ \Gamma \vdash B\end{array}$$

There are also reduction rules for $= I$ and $= E$.[12] for These are a bit different as they do not always remove the non-normal step and will vary on the form of the formula that substitution is operating on. Some reduction steps will not remove the non-normal step but will reduce the complexity of the major premise, in a similar way that cut elimination often relies on the decrease in complexity [38]. For the sake of brevity, refer to the following proof as **D**.

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma, x \in s \vdash x \in r\end{array} \qquad \begin{array}{c}\pi_2\\ \Gamma, x \in r \vdash x \in s\end{array}}{\Gamma \vdash s = r} = I$$

Now for the reduction rules. The rules are given in regards to reducing $= E_1$, the rules for $= E_0$ are symmetric: where there is an $= E_1$ replace with an $= E_0$ and where there is an $= E_0$ replace with an $= E_1$. We first state the form of the formula $A(x)$ for each and then give the rule.

x is not free in $A(x)$.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A(s)\end{array} \qquad \mathbf{D}}{\Gamma \vdash A[r/s]} = E_1 \quad \rightsquigarrow \quad \begin{array}{c}\pi_0\\ \Gamma \vdash A(s)\end{array}$$

$A(x)$ is of the form $A_1(x) \wedge A_2(x)$.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A_0(s) \wedge A_1(s)\end{array} \qquad \mathbf{D}}{\Gamma \vdash A_0[r/s] \wedge A_1[r/s]} = E_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A_0(s) \wedge A_1(s)\end{array}}{\Gamma \vdash A_0(s)}\wedge E_0 \qquad \mathbf{D}}{\Gamma \vdash A_0[r/s]} = E_1 \qquad \dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A_0(s) \wedge A_1(s)\end{array}}{\Gamma \vdash A_1(s)}\wedge E_1 \qquad \mathbf{D}}{\Gamma \vdash A_1[r/s]} = E_1}{\Gamma \vdash A_0[r/s] \wedge A_1[r/s]} \wedge_I$$

$A(x)$ is of the form $A_0(x) \to A_1(x)$.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash A_0(s) \to A_1(s)\end{array} \qquad \mathbf{D}}{\Gamma \vdash A_0[r/s] \to A_1[r/s]} = E_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A_0(r) \vdash A_0(r) \qquad \mathbf{D}}{\Gamma, A_0(r) \vdash A_0[s/r]} = E_0 \qquad \begin{array}{c}\pi_0\\ \Gamma \vdash A_0(s) \to A_1(s)\end{array}}{\Gamma, A_0(r) \vdash A_1[s/r]} \to_E \qquad \mathbf{D}}{\Gamma, A_0(r) \vdash A_1[r/s]} = E_1}{\Gamma \vdash A_0(r) \to A_1[r/s]} \to_I$$

$A(x)$ is of the form $\forall z B(z, x)$.

[12]Gratitude to Hallnäs for the work of finding these in [25].

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash \forall z B(z,s)\end{array} \quad \mathbf{D}}{\Gamma \vdash \forall z B(z,r)} = E_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash \forall z B(z,s)\end{array}}{\Gamma \vdash B(z,s)}\,\forall E \quad \mathbf{D}}{\dfrac{\Gamma \vdash B(z,r)}{\Gamma \vdash \forall z B(z,r)}\,\forall I} = E_1$$

$A(x)$ is of the form $t(x) \in \{z|B(z,x)\}$.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash t(s) \in \{z|B(z,s)\}\end{array} \quad \mathbf{D}}{\Gamma \vdash t(r) \in \{z|B(z,r)\}} = E_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash t(s) \in \{z|B(z,s)\}\end{array}}{\Gamma \vdash B(t(s),s)}\in E \quad \mathbf{D}}{\dfrac{\Gamma \vdash B(t(r),r)}{\Gamma \vdash t(r) \in \{z|B(z,r)\}}\in I} = E_1$$

$A(x)$ is of the form $t(x) \in x$ with x not free in t.

$$\dfrac{\begin{array}{c}\pi_0\\ t \in s\end{array} \quad \mathbf{D}}{\Gamma \vdash t \in r} = E_1 \quad\rightsquigarrow\quad \begin{array}{c}\pi_0\\ \Gamma \vdash t \in s\\ \pi_1\\ \Gamma \vdash t \in r\end{array}$$

$A(x)$ is of the form $t(x) \in x$ where x is free in t.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash t(s) \in s\end{array} \quad \mathbf{D}}{\Gamma \vdash t(r) \in r} = E_1 \quad\rightsquigarrow\quad \begin{array}{c}\pi_0\\ \Gamma \vdash t(s) \in s\\ \pi_1\\ \dfrac{\Gamma \vdash t(s) \in r \quad \mathbf{D}}{\Gamma \vdash t(r) \in r} = E_1\end{array}$$

$A(x)$ is of the form $t(x) \in z$ where z is atomic.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash t(s) \in z\end{array} \quad \mathbf{D}}{\Gamma \vdash t(r) \in z} = E_1$$

This non-normal step is a strange one. There is no obvious way to reduce the minor premise to something of less complexity due to z being atomic. Luckily, there are not many ways to end up with something like this in a proof. And if we do, from say the use of an axiom rule, it is possible to replace the axiom rule with the axiom rule of the substituted version instead.

$A(x)$ is of the form $t(x) = u(x)$.

$$\dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash t(s) = u(s)\end{array} \quad \mathbf{D}}{\Gamma \vdash t(r) = u(r)} = E_1 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{z \in t(r) \vdash z \in t(r) \quad \mathbf{D}}{\Gamma, z \in t(r) \vdash z \in t(s)} = E_0 \quad \begin{array}{c}\pi_0\\ \Gamma \vdash t(s) = u(s)\end{array}}{\Gamma, z \in t(r) \vdash z \in u(s)} = E_1 \quad \mathbf{D}}{\Gamma, z \in t(r) \vdash z \in u(r)} = E_1 \quad \dfrac{\dfrac{\dfrac{z \in u(r) \vdash z \in u(r) \quad \mathbf{D}}{\Gamma, z \in u(r) \vdash z \in u(s)} = E_0 \quad \begin{array}{c}\pi_0\\ \Gamma \vdash t(s) = u(s)\end{array}}{\Gamma, z \in u(r) \vdash z \in t(s)} = E_1 \quad \mathbf{D}}{\Gamma, z \in u(r) \vdash z \in t(r)} = E_1}{\Gamma \vdash t(r) = u(r)} = I$$

$A(x)$ is of the form $A_0(x) \vee A_1(x)$.

$$\dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash A_0(s) \vee A_1(s)\end{matrix} \qquad \mathbf{D}}{\Gamma \vdash A_0(r) \vee A_1(r)} = E_1 \qquad \rightsquigarrow$$

$$\dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash A_0(s) \vee A_1(s)\end{matrix} \quad \dfrac{\dfrac{\Gamma, A_0(s) \vdash A_0(s) \qquad \mathbf{D}}{\Gamma, A_0(s) \vdash A_0(r)} = E_1}{\Gamma, A_0(s) \vdash A_0(r) \vee A_1(r)} \vee I_0 \quad \dfrac{\dfrac{\Gamma, A_1(s) \vdash A_1(s) \qquad \mathbf{D}}{\Gamma, A_1(s) \vdash A_1(r)} = E_1}{\Gamma, A_1(s) \vdash A_0(r) \vee A_1(r)} \vee I_1}{\Gamma \vdash A_0(r) \vee A_1(r)}$$

$A(x)$ is of the form $\exists z B(z, x)$.

$$\dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash \exists z B(z, s)\end{matrix} \qquad \mathbf{D}}{\Gamma \vdash \exists z B(z, r)} = E_1 \qquad \rightsquigarrow \qquad \dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash \exists z B(z, s)\end{matrix} \quad \dfrac{\dfrac{B(z, s) \vdash B(z, s) \qquad \mathbf{D}}{B(z, s) \vdash B(z, r)} = E_1}{B(z, s) \vdash \exists z B(z, r)} \exists I}{\Gamma \vdash \exists z B(z, r)} \exists E$$

The presence of $\vee$ and $\exists$ introduce another kind of non-normal step. It can happen that non-normality occurs over minor premises in applications of these rules. We can reduce such steps by pushing the non-normal steps up, similar in style to what is done in cut elimination [38].

For $\vee$ this sort of non-normality is as follows where C is introduced by a connective's I rule in the minor premises, and then eliminated as the major premise in the E rule for that same connective after $\vee E$.

$$\dfrac{\dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash A \vee B\end{matrix} \quad \begin{matrix}\pi_1\\ \Gamma \vdash C\end{matrix} \quad \begin{matrix}\pi_2\\ \Gamma \vdash C\end{matrix}}{\Gamma \vdash C} \vee E \qquad \pi_3}{\Gamma \vdash D}$$

We push the maximum segment up as follows.

$$\dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash A \vee B\end{matrix} \quad \dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash C\end{matrix} \quad \pi_3}{\Gamma \vdash D} \quad \dfrac{\begin{matrix}\pi_2\\ \Gamma \vdash C\end{matrix} \quad \pi_3}{\Gamma \vdash D}}{\Gamma \vdash D} \vee E$$

For $\exists$, we have the following form where C is introduced as an I rule for a particular connective and then eliminated by the E rule for that same connective after $\exists E$.

$$\dfrac{\dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash \exists x B\end{matrix} \quad \begin{matrix}\pi_1\\ \Gamma \vdash C\end{matrix}}{\Gamma \vdash C} \exists E \qquad \pi_2}{\Gamma \vdash D}$$

We move the maximum segment as follows.

$$\dfrac{\begin{matrix}\pi_0\\ \Gamma \vdash \exists x B\end{matrix} \quad \dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash C\end{matrix} \quad \pi_2}{\Gamma \vdash D}}{\Gamma \vdash D} \exists E$$

Definition 6.17. A proof is said to **normalize** if it can be transformed into a normal proof by applying the rules of reduction.

6.3 Motivation for NNST

We can now motivate the definition for NNST. First, let's revisit the proof of Russell's Paradox.

Theorem 6.18. *Russell's set deduces $\bot$ in the NST natural deduction system.*

Proof. Define $R := \{x|x \notin x\}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{R\in R\vdash R\in R \qquad \dfrac{R\in R\vdash R\in R}{R\in R\vdash R\notin R}\in E}{R\in R\vdash\bot}\to E}{\vdash R\notin R}\to I}{\vdash R\in R}\in I \qquad \dfrac{\dfrac{R\in R\vdash R\in R \qquad \dfrac{R\in R\vdash R\in R}{R\in R\vdash R\notin R}\in E}{R\in R\vdash\bot}\to E}{\vdash R\notin R}\to I}{\vdash\bot}\to E \qquad \square$$

Remark 6.19. There is nothing fundamentally different about Russell's paradox and Curry's paradox in NST. We can in fact turn a proof of Russell's paradox in NST to a proof of Curry's by a line for line substitution of Russell's set for a Curry set, with such a set being defined as $C := \{x|x \in x \to A\}$ for some formula A. Compare with the situation in DKQ naive set theory where the two paradoxes were inherently different due to the presence of the paraconsistent negation.

The final step in the proof is non-normal. The major premise of $\to E$ is introduced immediately prior to its use. We've previously diagnosed the problem with Russell's as a problem with the rules of the logic, i.e. contraction with LAST or too strong a negation in DKQ naive set theory[13]. In the present context, we also note that the proof of Russell's paradox does not normalize.

Proposition 6.20. *The proof for Russell's paradox does not normalize.*

Proof. Let $\mathbf{D}$ denote the subdeduction

$$\dfrac{R\in R\vdash R\in R \qquad \dfrac{R\in R\vdash R\in R}{R\in R\vdash R\notin R}\in E}{R\in R\vdash\bot}\to E$$

Then the proof for Russell's paradox can be abbreviated as

$$\dfrac{\dfrac{\dfrac{\mathbf{D}}{R\notin R}\to I}{R\in R}\in I \qquad \dfrac{\mathbf{D}}{R\notin R}\to I}{\bot}\to E$$

The only non-normal step in this proof is at the end: an arrow introduction used as a major premise by an arrow elimination. If we attempt to reduce this step by the reduction rule the following results.

$$\dfrac{\dfrac{\dfrac{\mathbf{D}}{R\notin R}\to I}{R\in R}\in I \qquad \dfrac{\dfrac{\dfrac{\mathbf{D}}{R\notin R}\to I}{R\in R}\in I}{R\notin R}\in E}{\bot}\to E$$

[13]And in DKQ there is still the separate issue of contraction and Curry's set.

Now our only non-normal step is the $\in$ introduction and elimination step. If we reduce this step, we're back at where we started. Since there were no other choices for reduction at any step of normalization, reducing this proof can only cycle between these two non-normal proofs. $\square$

Russell's paradox is then an example of a non-normalizable proof in NST. Thus, the idea for NNST is simple: **we restrict ourselves to normal proofs to avoid these problematic inferences**. Further, we will show in Theorem 6.31 that there is no normal deduction of $\bot$.[14] Thus, if we restrict ourselves to only considering theorems with normal proofs, we'll have a provably non-trivial theory.

6.4 "Normalized" Naive Set Theory

The restriction for NNST can be stated in two different and equivalent ways, a global top-down approach, or an inductive bottom-up approach. For the global we have the following definition.

Definition 6.21. A formula A is a theorem of NNST_1 iff it has a normal deduction in NST.

For the inductive approach, we do not need to be concerned about proofs which have rules from the intuitionistic logic fragment.[15] Our concern is with proofs that include $\in$ or $=$ rules. Non-normality is always introduced by the addition of an elimination step, and once a non-normal step is in the proof, the proof can not be made normal by adding further steps.

Definition 6.22. The theory of NNST_2 has the rules of NST with the additional restrictions applied to $\in$ and $=$ rules:

1. An application of a $\in$ or $=$ rule is not valid if the deduction is non-normal.
2. If an $\in$ or $=$ rule occurs in a deduction, then an application of an elimination rule is not valid if it would make the deduction non-normal.

Remark 6.23. It is decidable whether a given deduction is normal by a finite search. Such a search would scan the tree to determine that no major premise of an elimination rule was the result of an introduction rule and would also confirm that no applications of $\vee E$ or $\exists E$ did not discharge an assumption. However, it is *not* decidable if any given deduction is normalizable due to the Halting Problem.

Proposition 6.24. *The proofs of $NNST_1$ are the same as the proofs of $NNST_2$. That is, for any formula A and set of assumptions Γ, $\Gamma \vdash A$ can be deduced in $NNST_1$ iff $\Gamma \vdash A$ can be deduced in $NNST_2$.*

[14] Similar as to why cut elimination implies consistency.
[15] All proofs of intuitionistic logic can be normalized [39].

Proof. If we have a proof from NNST_1, then there is a normal proof of it in NST. Precisely this proof will be allowed in NNST_2 since it has no non-normal steps. On the other hand, if we have a proof in NNST_2, then the proof contains $=$ or $\in$ or not. If not, then it is a proof of intuitionistic logic and can be normalized and thus has a normal proof. If it does contain an instance of a $\in$ or $=$ rule then the proof can only be constructed so that it is normal. Thus the deduction is normal and the proof is in NNST_1. □

As these theories are equivalent, we will now exclusively refer to the theory as NNST. Note that NNST_2 defines an alternate natural deduction theory entirely, while NNST_1 is a restriction on the set of deductions of NST. We will however usually refer to NNST from the perspective of NNST_1, i.e. as a paring down on the deductions of NST.

Remark 6.25 (Working in NNST)**.** It worth mentioning now that neither conditions defining NNST_1 or NNST_2 explicate how we actually go about the work of finding proofs for NNST. We will usually first find deductions of NST. If its normal, then we're done. If not, we will attempt to normalize the proof to find a normal proof.

It is the case that if normalization will succeed, we can determine so in finite time.[16] However, the amount of time and resources needed may very well be impractical. But, due to the Halting Problem, we can not know whether searching for a normal proof will actually complete or not.

6.5 Properties of Normal Proofs

We'll now derive some properties of normal proofs in NST. We will use these to show the non-triviality of NNST.

Definition 6.26. A **path** is a sequence of formulas $A_1, A_2, \ldots, A_n$ in a deduction Π if

1. A_1 is a top formula in Π that is not discharged by an application of $\vee E$ or $\exists E$,
2. A_i, for each $i < n$, is not the minor premise of an application of $\rightarrow E$ or $= E$ and either
   - A_i is not the major premise of $\vee E$ or $\exists E$ and A_{i+1} is the formula occurrence immediately below A_i,
   - A_i is the major premise of an application of $\vee E$ or $\exists E$ and A_{i+1} is an assumption discharged in Π by this application,
3. and A_n is either a minor premise of $\rightarrow E$, a minor premise of $= E$, the end formula of Π, or a major premise of an application of $\vee E$ or $\exists E$ that does not discharge any assumptions.

[16] That this is true can be seen to follow from Corollary 7.22 and the left-most reduction strategy of λ-calculus. The latter is guaranteed to find a normal type if it exists [36].

A path in a normal deduction will always end at a $\to$ or $=$ minor premise or at the end-formula of the deduction. We can now prove a theorem about the shape of all normal proofs. This is adapted from Prawitz's proof in [39] for the proof of the normal form theorem for intuitionistic logic proofs.

Theorem 6.27. *Let Π be a normal deduction in NST, let π be a path in Π, and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the sequence of segments in π. Then there is a segment σ_i, called the minimum segment in π, which separates two (possibly empty) parts of π, called the E-part and I-part of π, with the properties:*

1. *For each σ_j in the E-part (i.e. $j < i$) it holds that σ_j is a major premise of an E-rule and that the formula occurring in σ_{j+1} is a subformula of the one occurring in σ_j.*
2. *σ_i, provided that $i \neq n$, is a premise of an I-rule or of the $\perp$ rule.*
3. *For each σ_j in the I-part, except the last one, (i.e. $i < j < n$) it holds that σ_j is a premise of an I-rule and that the formula occurring in σ_j is a subformula of the one occurring in σ_{j+1}.*

Proof. Let π be a path in Π, and $\sigma_1, \sigma_2, \ldots, \sigma_n$ the sequence of segments in π.

Now, either there are applications of introduction rules or the $\perp$ rule in the path or not. If there are no such application, let σ_i be the last segment in the path and we are done. The rest of the proof covers the case in which there are such applications.

We first prove that all segments which are major premises of E rules precede all segments that are premises of I rules or the $\perp$ rule. Assume to the contrary that there is some segment σ_k that is a major premise of an E rule that succeeds an I rule or $\perp$ rule. Then the formula of segment σ_k is introduced and eliminated as a major premise, making σ_k a maximum segment. This contradicts our assumption that the proof is normal.

Let σ_i be the first segment that is the premise of an I rule or $\perp$ rule. Thus σ_i satisfies condition 1 by what was proved in the paragraph above; σ_i is a segment that occurs after all elimination rules in the path. And by selection it satisfies condition 2, it is a premise of an I rule or $\perp$ rule.

Consider the segments σ_j such that $i < j < n$. By definition of segments, none of these can be minor premises of $\vee$ or $\exists$ elimination (minor premises of these are "inside" the segments). By what was proved before, none of these occur as major premises of elimination rules since they occur after σ_i which is a premise of an I rule or the $\perp$ rule. Further, since they are not the last segment in the path, they can't be minor premises of $\to E$ or $= E$.

It remains that they must be premises of introduction rules or $\perp$ rules. But in the latter case, to be a premise of the $\perp$ rule is to be $\perp$, which can only be derived as the result of an elimination rule. None of the σ_j can occur as conclusions of elimination rules, since this would imply σ_{j-1} was the premise of an elimination rule. Thus none of the σ_j segments can be $\perp$. Therefore they can only be premises of introduction rules. Thus σ_i satisfies condition 3.

That the subformula clauses hold is by inspection of the rules of the system. An elimination rule always produces a subformula of one of the formulas of the premises, and an introduction rule is always composed of subformulas in the premises.[17] □

Definition 6.28. A **main path** in a deduction Π is one that contains the end-formula.

Corollary 6.29. *Any closed normal deduction in NST ends on an I rule.*

Proof. Let Π be a closed normal deduction and assume that Π ends on an E rule. Let π be a main path in Π. There must be at least one open assumption in the path of π since any path starts with a top-formula, which can only be an Ax rule. By Theorem 6.27, there can be no I-rules in π. And by definition of path, the first formula in π is not discharged by an application of $\vee$ or $\exists$ elimination. Thus the open assumption(s) is never discharged, which contradicts our assumption that the deduction is closed. □

Corollary 6.30 (Subformula Principle)**.** *Every formula occurring in a normal deduction of A from Γ is a subformula of A or of some formula of Γ.*

Proof. Let Π be a normal deduction of A with assumptions Γ. We first show that the subformula principle holds for each path in the deduction. Let π be a path in that deduction. Let A_0 be the last formula in the path. By the first clause of Theorem 6.27, we know that each step in the E-part is a subformula of the previous and since the E-part will start with axiom rules, these must in turn be subformulas of our assumptions Γ. In the I-part, each step has the previous formula as a subformula. Thus in particular, each formula occurring prior to A_0 in the I-part is a subformula of A_0. Therefore each formula in the path is a subformula of A_0 or Γ.

Now consider any arbitrary main path and the occurrences of $\to E$ and $= E$ in the E-part. For the former, we have that A_0 must be a subformula of the the major premise which is of the form $A_0 \to B$ and this latter formula must also be a subformula of Γ. Thus A_0 is a subformula of Γ. For the latter, we have that the minor premise of $A_0(s)$ must be a subformula of the conclusion $A_0(r)$ by definition and the major premise will be a subformula of Γ. Thus we have that anything in the I-part of must be a subformula of A and that any formula occurring in the E-part or in subpaths that end at minor premises in the E-part of the main path are subformulas of our assumptions. There are no other formulas in the proof to consider. □

6.5.1 Non-triviality for NNST

There are a few different ways to use the properties proved to derive that NNST is non-trivial. We'll consider two different proofs.

Theorem 6.31. *NNST is non-trivial. That is, there is no closed proof of $\bot$ in NNST.*

Proof. In order for NNST to derive $\vdash \bot$, there must be a normal proof in NST of $\vdash \bot$. There are multiple ways to show this is impossible.

[17] This fact is what motivates the definition of being a subformula.

Our first proof follows from Proposition 6.29. Since all normal deductions must end on an I rule, and there is no I rule that introduces $\bot$, there is no normal deduction of $\bot$.

The second follows from the subformula principle, Proposition 6.30. There are no subformulas of $\bot$ except for $\bot$ itself and since a normal deduction can only be composed of subformulas of the conclusion, there is no normal deduction of $\bot$. $\square$

6.5.2 A Trivial Variant of NNST

In comparison with the Hilbert-style axiom of comprehension for naive set theory, the $\in I$ and $\in E$ rules are a little different. Recall this allows us to assume anything of the form

$$\forall x(x \in y \leftrightarrow A(x)).$$

We call the rule that allows instantiations of this axiom in the natural deduction theory **Comp**. While $\in I$ and $\in E$ look like they might be shortening the uses of modus ponens on that axiom, this is not quite right. As is shown in the following proof, if we were to allow instances of comprehension as initial sequents in this theory, we would find normal proofs of $\bot$.[18]

Proposition 6.32. *NNST with Comp as a rule is trivial.*

Proof. We will use Russell's set to derive triviality, $R := \{x | x \notin x\}$. Let **D** denote the following subproof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R \in R \vdash R \in R \quad \dfrac{\dfrac{\vdash \forall x(x \in R \leftrightarrow x \notin x)}{\vdash R \in R \leftrightarrow R \notin R}\forall E}{\vdash R \in R \to R \notin R}\wedge E}{R \in R \vdash R \notin R}\to E \quad R \in R \vdash R \in R}{R \in R \vdash \bot}\to E}{\vdash R \notin R}\to I \quad \dfrac{\dfrac{\vdash \forall x(x \in R \leftrightarrow x \notin x)}{\vdash R \in R \leftrightarrow R \notin R}\forall E}{\vdash R \notin R \to R \in R}\wedge E}{\vdash R \in R}\to E \quad \dfrac{\dfrac{\vdash \forall x(x \in R \leftrightarrow x \notin x)}{\vdash R \in R \leftrightarrow R \notin R}\forall E}{\vdash R \in R \to R \notin R}\wedge E}{\vdash R \notin R}\to E$$

It is the last two uses of the apparently redundant $\to E$ that allow us to hide the non-normality from $\to I$. This is in effect a $\in I$ followed by a $\in E$ rule but replaced by $\to E$. This allows the normal proof of $\bot$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R \in R \vdash R \in R \quad \dfrac{\dfrac{\vdash \forall x(x \in R \leftrightarrow x \notin x)}{\vdash R \in R \leftrightarrow R \notin R}\forall E}{\vdash R \in R \to R \notin R}\wedge E}{R \in R \vdash R \notin R}\to E \quad R \in R \vdash R \in R}{R \in R \vdash \bot}\to E}{\vdash R \notin R}\to I \quad \dfrac{\dfrac{\vdash \forall x(x \in R \leftrightarrow x \notin x)}{\vdash R \in R \leftrightarrow R \notin R}\forall E}{\vdash R \notin R \to R \in R}\wedge E}{\vdash R \in R}\to E \quad \mathbf{D}}{\vdash \bot}\to E$$

$\square$

Representing comprehension as the $\in$ rules does not seem to give us the full power of comprehension. The $\in$ rules have no way to immediately introduce a fixpoint while the axiom of comprehension does. Consider $\Omega = \{x | x = \Omega\}$ and attempting to derive that $x \in \Omega \to x = \Omega$. It would seem we could do something like the following.

[18]With our current definition of normality at least.

$$\dfrac{\dfrac{x \in \Omega \vdash x \in \Omega}{x \in \Omega \vdash x = \Omega} \in E}{\vdash x \in \Omega \to x = \Omega} \to I$$

But this is technically not allowed in the theory. For names of sets are only convenient shorthand and the only thing that is valid is the set term $\{x|A(x)\}$ itself. We can not write down the set term for Ω; $x \in \Omega$ means $x \in \{x|x = \Omega\}$ which means $x \in \{x|x = \{x|x = \Omega\}\}$ which means ...

What can be done instead is proving a Fixpoint Theorem[19] which provides a construction for these sorts of sets. That these need to be constructed is a good thing; we show now that taking fixpoints as axioms will allow normal proofs of $\perp$.

For the sake of space, we'll have to denote a few different subdeductions instead of typing this proof up as a single deduction. We will use $Prf(A \vdash B)$ to denote the subdeduction with last line $A \vdash B$. When these subdeductions are reused, we will recall the last step in the subdeduction to make sure that normalization restrictions are met.

Proposition 6.33. *NNST with Comp as a rule is trivial via a fixed point.*[20]

Proof. We derive a normal proof of $\perp$ through the use of the set $g := \{x|x = g \wedge x = \emptyset\}$. Our first step and main part of the proof is to derive that $g \in g$ without assumption. We will do this by showing that $g \in \emptyset$ without assumption.

First we prove the general fact that $g = g$,

$$\dfrac{z \in g \vdash z \in g \qquad z \in g \vdash z \in g}{\vdash g = g} =_I$$

This subdeduction is denoted $Prf(\vdash g = g)$.

Next we prove that under assumption $z \in g$ that $g = \emptyset$.

$$\dfrac{\dfrac{\dfrac{z \in g \vdash z \in g \quad \dfrac{\dfrac{\vdash \forall x(x \in g \leftrightarrow x = g \wedge x = \emptyset)}{\vdash z \in g \leftrightarrow z = g \wedge z = \emptyset} \forall_E}{\vdash z \in g \to z = g \wedge z = \emptyset} \wedge_E}{z \in g \vdash z = g \wedge z = \emptyset} \to_E}{z \in g \vdash z = g} \wedge_E \quad \dfrac{\dfrac{z \in g \vdash z \in g \quad \dfrac{\dfrac{\vdash \forall x(x \in g \leftrightarrow x = g \wedge x = \emptyset)}{\vdash z \in g \leftrightarrow z = g \wedge z = \emptyset} \forall_E}{\vdash z \in g \to z = g \wedge z = \emptyset} \wedge_E}{z \in g \vdash z = g \wedge z = \emptyset} \to_E}{z \in g \vdash z = \emptyset} \wedge_E}{z \in g \vdash g = \emptyset} =_E$$

This proof is denoted $Prf(z \in g \vdash g = \emptyset)$.

We next use these two subdeductions to get to derivation that $g \in g$. This conclusion is still under assumption that $z \in g$.

$$\dfrac{\dfrac{\dfrac{\vdash \forall x(x \in g \leftrightarrow x = g \wedge x = \emptyset)}{\vdash g \in g \leftrightarrow g = g \wedge g = \emptyset} \forall_E}{\vdash g = g \wedge g = \emptyset \to g \in g} \wedge_E \quad \dfrac{\dfrac{Prf(\vdash g = g)}{\vdash g = g} =_I \quad \dfrac{Prf(z \in g \vdash g = \emptyset)}{z \in g \vdash g = \emptyset} =_E}{z \in g \vdash g = g \wedge g = \emptyset} \wedge_I}{z \in g \vdash g \in g} \to_E$$

[19]cf. Theorem 4.4 and [52, 47]

[20]This proof is adapted from [11].

We denote this deduction as $Prf(z \in g \vdash g \in g)$.

Now we can show that the two subdeductions $Prf(z \in g \vdash g = \emptyset)$ and $Prf(z \in g \vdash g \in g)$ derives that $z \in \emptyset$. This in conjunction with $z \in \emptyset$ leading to $z \in g$ yields that $g = \emptyset$ with discharged assumptions $z \in g$ and $z \in \emptyset$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Prf(z \in g \vdash g = \emptyset)}{z \in g \vdash g = \emptyset}\,{=_E} \quad \dfrac{Prf(z \in g \vdash g \in g)}{z \in g \vdash g \in g}\,{\to_E}}{z \in g \vdash g \in \emptyset}\,{=_E} \quad \dfrac{\dfrac{\vdash \forall x(x \in \emptyset \leftrightarrow \bot)}{\vdash g \in \emptyset \leftrightarrow \bot}\,{\forall_E}}{\vdash g \in \emptyset \to \bot}\,{\wedge_E}}{z \in g \vdash \bot}\,{\to_E}}{z \in g \vdash z \in \emptyset}\,{\bot} \quad \dfrac{\dfrac{z \in \emptyset \vdash z \in \emptyset \quad \dfrac{\dfrac{\vdash \forall x(x \in \emptyset \leftrightarrow \bot)}{\vdash z \in \emptyset \leftrightarrow \bot}\,{\forall_E}}{\vdash z \in \emptyset \to \bot}\,{\wedge_E}}{z \in \emptyset \vdash \bot}\,{\to_E}}{z \in \emptyset \vdash z \in g}\,{\bot}}{\vdash g = \emptyset}\,{=_I}\ (1,2)$$

This proof will be denoted $Prf(\vdash g = \emptyset)_1$. Note that this proof concludes on $=_I$. We will use Comp to later derive a conclusion of $g = \emptyset$ with our last rule being $\wedge_E$.

Now we can use the subdeductions that $Prf(\vdash g = g)$ and $Prf(\vdash g = \emptyset)_1$ gives that $g \in g$ under no new assumptions.

$$\dfrac{\dfrac{\dfrac{Prf(\vdash g = g)}{\vdash g = g}\,{=_I} \quad \dfrac{Prf(\vdash g = \emptyset)_1}{\vdash g = \emptyset}\,{=_I}\ (1,2)}{\vdash g = g \wedge g = \emptyset}\,{\wedge_I} \quad \dfrac{\dfrac{\vdash \forall x(x \in g \leftrightarrow x = g \wedge x = \emptyset)}{\vdash g \in g \leftrightarrow g = g \wedge g = \emptyset}\,{\forall_E}}{\vdash g = g \wedge g = \emptyset \to g \in g}\,{\wedge_E}}{\vdash g \in g}\,{\to_E}$$

This proof will be denoted $Prf(\vdash g \in g)$.

Now that we have $Prf(\vdash g \in g)$ we can derive $g = \emptyset$ via an instance of Comp. This is essentially what allows us to derive a normal proof of bottom since we don't have to use the proof above that introduces $g = \emptyset$ via $=_I$. The insertion of going through Comp again allows us to close off the branch that introduces $g = \emptyset$ via $=_I$ by using all of this information as a minor premise.

$$\dfrac{\dfrac{\dfrac{Prf(\vdash g \in g)}{\vdash g \in g}\,{\to_E} \quad \dfrac{\dfrac{\vdash \forall x(x \in g \leftrightarrow x = g \wedge x = \emptyset)}{\vdash g \in g \leftrightarrow g = g \wedge g = \emptyset}\,{\forall_E}}{\vdash g \in g \to g = g \wedge g = \emptyset}\,{\wedge_E}}{\vdash g = \emptyset \wedge g = g}\,{\to_E}}{\vdash g = \emptyset}\,{\wedge_E}$$

Denote this proof $Prf(\vdash g = \emptyset)_2$.

Now we can derive bottom.

$$\dfrac{\dfrac{\dfrac{Prf(\vdash g = \emptyset)_2}{\vdash g = \emptyset}\,{\wedge_E} \quad \dfrac{Prf(\vdash g \in g)}{\vdash g \in g}\,{\to_E}}{\vdash g \in \emptyset}\,{=_E} \quad \dfrac{\dfrac{\vdash \forall x(x \in \emptyset \leftrightarrow \bot)}{\vdash g \in \emptyset \leftrightarrow \bot}\,{\forall_E}}{\vdash g \in \emptyset \to \bot}\,{\wedge_E}}{\vdash \bot}\,{\to_E}$$

$\square$

These results suggest something about what our naive set theory is and isn't. First, it is still the case that the unrestricted **axiom** of comprehension is not tenable. Including the axiomatic form reduces us back to triviality even in the present context of normalized proofs. Second, the fixpoint theorem will still allow the constructions that the axiom of comprehension affords. What the fixpoint theorem will not recover is certain implications of those fixpoint constructions. For instance, we certainly "lose" out on the fixpoint construction above deriving bottom since it will not be able to do it in a normal proof.

A salient difference for this theory is that we know the fixpoint theorem provides actual proof constructions. If we take a constructive reading of the axiom of comprehension[21], then the axiom is asserting constructions that we do not have. This is a problematic assertion as opposed to the "actual" constructions afforded by using the rules $\in I$ and $\in E$.

Remark 6.34 ("True" Naive Set Theory?)**.** This section raises the possible objection that whatever NNST is, it is in some sense not "true" naive set theory; such a theory might necessarily require the axiomatic form of unrestricted comprehension. There is no counterpoint here, except that whatever NNST is, it is a whole lot closer to such a theory than many traditional set theories. In this case, we have a Fixpoint theorem that gives a true biconditional in the same way as the axiomatic form, but also necessarily stops short from deriving all the consequences that such a "true" naive set theory might. Further, if the sketch of the ideas in Appendix A are indicating some kind of truth, anything more "naive set theory" than NNST necessarily has deductions which have forever hidden steps, which appear to be suspicious entities to call proofs.

6.6 $\mathbf{NNST}_{\to\in\forall}$

For this section, we'll consider the fragment of NST without the equality rules. We will now show that NST and NNST assume more deductive machinery than necessary. If we only assume the rules for $\to$, $\in$ and $\forall$, we can still deduce the same set of theorems as before.

Definition 6.35. $NST_{\to\in\forall}$ is the natural deduction theory that contains only the *primitive connectives* $\to$, $\in$, and $\forall$ and their accompanying rules.

Definition 6.36 ([52])**.** Fix a closed term $t_0 := \{x|x \in x\}$. The *primitive translation* of each following *composite connectives* is defined as follows:

- $A \wedge B := \forall x((A \to (B \to t_0 \in x)) \to t_0 \in x)$
- $A \vee B := \forall x((A \to t_0 \in x) \to (B \to t_0 \in x) \to t_0 \in x)$
- $\bot := \forall x(t_0 \in x)$
- $\exists y A := \forall x(\forall y(A \to t_0 \in x) \to t_0 \in x)$

[21] The appendix for the justification of NNST strongly advocates a kind of constructive/computational understanding of proofs of NNST. Perhaps counter-intuitively, this understanding of the proofs do not need to necessarily lead to computational objects; classical logic will also sidestep triviality with the normalization restriction.

Definition 6.37. A connective's operational rules are those introduction and elimination rules which were defined in NST for that connective.

Proposition 6.38. *Each primitive translation of the composite connectives derives the equivalent operational rules. That is, the introduction and elimination rules are derivable for the primitive translations of $\wedge$, $\vee$ and $\exists$ and the $\bot$ rule is derivable for the primitive translation of $\bot$.*

Proof. We start with $\wedge$. For the introduction rule, we need that from $\Gamma \vdash A$ and $\Gamma \vdash B$ that we can derive $\Gamma \vdash \forall x((A \to B \to t_0 \in x) \to t_0 \in x)$.

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash B \qquad \dfrac{\Gamma \vdash A \qquad A \to B \to t_0 \in x \vdash A \to B \to t_0 \in x}{A \to B \to t_0 \in x \vdash B \to t_0 \in x} \to E}{\Gamma, A \to B \to t_0 \in x \vdash t_0 \in x} \to E}{\Gamma \vdash (A \to B \to t_0 \in x) \to t_0 \in x} \to I}{\Gamma \vdash \forall x((A \to B \to t_0 \in x) \to t_0 \in x)} \forall I$$

For the elimination rule, we need that from $\Gamma \vdash \forall x((A \to B \to t_0 \in x) \to t_0 \in x)$, we can derive either $\Gamma \vdash A$ or $\Gamma \vdash B$. We'll show $\Gamma \vdash A$, the proof for the other elimination rule derivation is almost identical.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B \vdash A}{A, B \vdash t_0 \in \{x|A\}} \in I}{A \vdash B \to t_0 \in \{x|A\}} \to I}{\vdash A \to B \to t_0 \in \{x|A\}} \to I \qquad \dfrac{\Gamma \vdash \forall x((A \to B \to t_0 \in x) \to t_0 \in x)}{\Gamma \vdash (A \to B \to t_0 \in \{x|A\}) \to t_0 \in \{x|A\}} \forall E}{\Gamma \vdash t_0 \in \{x|A\}} \to E}{\Gamma \vdash A} \in E$$

Now for $\vee$. For the introduction rule, we need to show that from $\Gamma \vdash A$ or $\Gamma \vdash B$ that $\Gamma \vdash \forall x((A \to t_0 \in x) \to (B \to t_0 \in x) \to t_0 \in x)$. We will do $\Gamma \vdash A$ here, the proof for the other introduction rule is almost identical.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash A \qquad A \to t_0 \in x, B \to t_0 \in x \vdash A \to t_0 \in x}{\Gamma, A \to t_0 \in x, B \to t_0 \in x \vdash t_0 \in x} \to E}{\Gamma, A \to t_0 \in x \vdash (B \to t_0 \in x) \to t_0 \in x} \to I}{\Gamma \vdash (A \to t_0 \in x) \to (B \to t_0 \in x) \to t_0 \in x} \to I}{\Gamma \vdash \forall x((A \to t_0 \in x) \to (B \to t_0 \in x) \to t_0 \in x)} \forall I$$

Next is the elimination rule for $\vee$. We need to show that from $\Gamma \vdash \forall x((A \to t_0 \in x) \to (B \to t_0 \in x) \to t_0 \in x)$, $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$ that $\Gamma \vdash C$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, B \vdash C}{\Gamma, B \vdash t_0 \in \{x|C\}} \in I}{\Gamma \vdash B \to t_0 \in \{x|C\}} \to I \qquad \dfrac{\dfrac{\dfrac{\Gamma, A \vdash C}{\Gamma, A \vdash t_0 \in \{x|C\}} \in I}{\Gamma \vdash A \to t_0 \in \{x|C\}} \to I \qquad \dfrac{\Gamma \vdash \forall x((A \to t_0 \in x) \to (B \to t_0 \in x) \to t_0 \in x)}{\Gamma \vdash (A \to t_0 \in \{x|C\}) \to (B \to t_0 \in \{x|C\}) \to t_0 \in \{x|C\}} \forall E}{\Gamma \vdash (B \to t_0 \in \{x|C\}) \to t_0 \in \{x|C\}} \to E}{\Gamma \vdash t_0 \in \{x|C\}} \to E}{\Gamma \vdash C} \in E$$

Now for $\exists$. We'll first show the introduction rule, that from $\Gamma \vdash A[a/y]$ we can derive that $\Gamma \vdash \exists y A$.

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A \qquad \dfrac{\forall y(A \to t_0 \in x) \vdash \forall y(A \to t_0 \in x)}{\forall y(A \to t_0 \in x) \vdash A \to t_0 \in x}\,\forall E}{\Gamma, \forall y(A \to t_0 \in x) \vdash t_0 \in x}\,\to E}{\Gamma \vdash \forall y(A \to t_0 \in x) \to t_0 \in x}\,\to I}{\Gamma \vdash \forall x(\forall y(A \to t_0 \in x) \to t_0 \in x)}\,\forall I$$

Next is the elimination rule. We need to show that from $\Gamma \vdash \exists y A$ and $\Gamma, A[a/y] \vdash B$ that we can derive $\Gamma \vdash B$. The usual restriction of $\exists E$, that y can not be free in Γ or B, comes from the application of $\forall I$.

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \forall x(\forall y(A \to t_0 \in x) \to t_0 \in x)}{\Gamma \vdash \forall y(A \to t_0 \in \{x|B\}) \to t_0 \in \{x|B\}}\,\forall E \qquad \dfrac{\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma, A \vdash t_0 \in \{x|B\}}\,\in I}{\Gamma \vdash A \to t_0 \in \{x|B\}}\,\to I}{\Gamma \vdash \forall y(A \to t_0 \in \{x|B\})}\,\forall I}{\Gamma \vdash t_0 \in \{x|B\}}\,\to E}{\Gamma \vdash B}\,\in E$$

Finally for the $\perp$ rule, we need to start from $\Gamma \vdash \forall x(t_0 \in x)$ and derive that $\Gamma \vdash A$.

$$\dfrac{\dfrac{\Gamma \vdash \forall x(t_0 \in x)}{\Gamma \vdash t_0 \in \{x|A\}}\,\forall E}{\Gamma \vdash A}\,\in E$$

$\square$

Definition 6.39. The *primitive translation of a rule* replaces an operational rule for a composite connective with the *derived rule* for the primitive translation of that connective as found in the proof of Proposition 6.38.

We can use the derivations of Proposition 6.38 to look at translation as a process of applying translation rules to a proof in NST. For example, where a $\wedge I$ joins two propositions A and B, we can replace $\wedge I$ with the derivation of the introduction rule for the primitive translation of $\wedge$. However, some proofs do not allow "partial translation" in which some instances of a connective are translated and others are not. Consider an instance in which a $\wedge I$ precedes a $\wedge E$. To translate only the $\wedge I$ would mean the proof was not validly formed as $\wedge E$ would be applied to formula with a primary connective of $\forall$.

Definition 6.40. The *primitive translation of a NST deduction* replaces composite connectives and instances of their operational rules with their primitive translations.

Proposition 6.41. *The normalization of the primitive translations of the reduction rules for the composite connectives $\wedge$, $\vee$ and $\exists$ normalizes to the same deduction as the reduction rules defined for those composite connectives.*

Proof. We need to compute the normalization of deductions which have a derived introduction rule that preceds a derived elimination rule using the reduction rules for $\to$, $\in$, and $\forall$. This is long, but straightforward.

We start with the $\wedge$ case. We need to normalize the following deduction. For space, we admit the contextual assumptions of Γ which do not influence the computation in any way, and we only perform the computation for normalizing to A. The related reduction rule to B is similar.

$$\dfrac{\dfrac{\vdash B \qquad \dfrac{\vdash A \qquad A \to B \to t_0 \in \{x|A\} \vdash A \to B \to t_0 \in \{x|A\}}{A \to B \to t_0 \in \{x|A\} \vdash B \to t_0 \in \{x|A\}}\to E}{\dfrac{\dfrac{\dfrac{A \to B \to t_0 \in \{x|A\} \vdash t_0 \in \{x|A\}}{\vdash (A \to B \to t_0 \in \{x|A\}) \to t_0 \in \{x|A\}}\to I}{\vdash \forall x((A \to B \to t_0 \in x) \to t_0 \in x)}\forall I}{\vdash (A \to B \to t_0 \in \{x|A\}) \to t_0 \in \{x|A\}}\forall E}\to E \qquad \dfrac{\dfrac{\dfrac{A, B \vdash A}{A, B \vdash t_0 \in \{x|A\}}\in I}{A \vdash B \to t_0 \in \{x|A\}}\to I}{\vdash (A \to B \to t_0 \in \{x|A\}) \to t_0 \in \{x|A\}}\to I}{\dfrac{\vdash t_0 \in \{x|A\}}{\vdash A}\in E}\to E$$

The first reduction to apply is to the $\forall I$ followed by the $\forall E$. After this is applied, a normalization needs to be done on the $\to I$ and $\to E$ with major premise $(A \to B \to t_0 \in \{x|A\}) \to t_0 \in \{x|A\}$. The result of these two reduction steps follows.

$$\dfrac{\vdash B \qquad \dfrac{\vdash A \qquad \dfrac{\dfrac{\dfrac{A, B \vdash A}{A, B \vdash t_0 \in \{x|A\}}\in I}{A \vdash B \to t_0 \in \{x|A\}}\to I}{\vdash A \to B \to t_0 \in \{x|A\}}\to I}{\vdash B \to t_0 \in \{x|A\}}\to E}{\dfrac{\vdash t_0 \in \{x|A\}}{\vdash A}\in E}\to E$$

The next reduction step to apply is $\to I$ and $\to E$ for major premise $A \to B \to t_0\{x|A\}$.

$$\dfrac{\vdash B \qquad \dfrac{\dfrac{B \vdash A}{B \vdash t_0 \in \{x|A\}}\in I}{\vdash B \to t_0 \in \{x|A\}}\to I}{\dfrac{\vdash t_0 \in \{x|A\}}{\vdash A}\in E}\to E$$

The next reduction step is again $\to I$ and $\to E$ for major premise $B \to t_0 \in \{x|A\}$. In this case, the use of B is vacuous and so the reduction leaves us only with the non-normal step of $\in I$ applied to A and and $\in E$ applied to $\{x|A\}$. The reduction of this leaves us with $\vdash A$ as desired.

Next we'll look at $\vee$. As with $\wedge$ we would have an initial reduction step to apply for $\forall I$ and $\forall E$. We will skip writing this step down and produce the proof as it stands after applying that reduction.

$$\dfrac{\dfrac{\dfrac{B \vdash C}{B \vdash t_0 \in \{x|C\}}\in I}{\vdash B \to t_0 \in \{x|C\}}\to I \qquad \dfrac{\dfrac{\dfrac{A \vdash C}{A \vdash t_0 \in \{x|C\}}\in I}{\vdash A \to t_0 \in \{x|C\}}\to I \qquad \dfrac{\dfrac{\dfrac{\vdash A \qquad A \to t_0 \in \{x|C\}, B \to t_0 \in \{x|C\} \vdash A \to t_0 \in \{x|C\}}{A \to t_0 \in \{x|C\}, B \to t_0 \in \{x|C\} \vdash t_0 \in \{x|C\}}\to E}{A \to t_0 \in \{x|C\} \vdash (B \to t_0 \in \{x|C\}) \to t_0 \in \{x|C\}}\to I}{\vdash (A \to t_0 \in \{x|C\}) \to (B \to t_0 \in \{x|C\}) \to t_0 \in \{x|C\}}\to I}{\vdash (B \to t_0 \in \{x|C\}) \to t_0 \in \{x|C\}}\to E}{\dfrac{\vdash t_0 \in \{x|C\}}{\vdash C}\in E}\to E$$

We apply the reduction to $\to$ for the major premise $(A \to t_0 \in \{x|C\}) \to (B \to t_0 \in \{x|C\}) \to t_0 \in \{x|C\}$ first.

$$
\dfrac{\dfrac{\dfrac{\dfrac{B \vdash C}{B \vdash t_0 \in \{x|C\}}\in I}{\vdash B \to t_0 \in \{x|C\}}\to I \qquad \dfrac{\vdash A \qquad \dfrac{\dfrac{B \to t_0 \in \{x|C\}, A \vdash C}{B \to t_0 \in \{x|C\}, A \vdash t_0 \in \{x|C\}}\in I}{\vdash A \to t_0 \in \{x|C\}}\to I}{B \to t_0 \in \{x|C\} \vdash t_0 \in \{x|C\}}\to E}{\vdash t_0 \in \{x|C\}}\to E}{\vdash C}\in E
$$

The next reduction step we apply is for $\to$ and major premise $(B \to t_0 \in \{x|C\}) \to t_0 \in \{x|C\}$. The minor premise, $(B \to t_0 \in \{x|C\}$, is used vacuously and so disappears.

$$
\dfrac{\dfrac{\vdash A \qquad \dfrac{\dfrac{A \vdash C}{A \vdash t_0 \in \{x|C\}}\in I}{\vdash A \to t_0 \in \{x|C\}}\to I}{\vdash t_0 \in \{x|C\}}\to E}{\vdash C}\in E
$$

All that remains is to apply a $\to$ reduction on major premise $A \to t_0 \in \{x|C\}$ and then a $\in$ reduction on the resulting proof. This completes the computation for $\vee$.

Last to look at is $\exists$. Similar to the previous cases, there's an immediate $\forall$ reduction to apply. After that we have the following deduction. We do not explicitly denote what is instantiated when $\forall E$ is applied for $\forall y$ but it only needs to match whatever A has been provided.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B}{A \vdash t_0 \in \{x|B\}}\in I}{\vdash A \to t_0 \in \{x|B\}}\forall I}{\vdash \forall y(A \to t_0 \in \{x|B\})}\forall I \qquad \dfrac{\dfrac{\vdash A \qquad \dfrac{\forall y(A \to t_0 \in \{x|B\}) \vdash \forall y(A \to t_0 \in \{x|B\})}{\forall y(A \to t_0 \in \{x|B\} \vdash A \to t_0 \in \{x|B\}}\forall E}{\forall y(A \to t_0 \in x) \vdash t_0 \in \{x|B\}}\to E}{\vdash \forall y((A \to t_0 \in \{x|B\}) \to t_0 \in \{x|B\})}\to I}{\vdash t_0 \in \{x|B\}}\to E}{\vdash B}\in E
$$

At this point we need to apply a $\to$ reduction for instance with the major premise of $\forall y(A \to t_0 \in \{x|B\}) \to t_0 \in \{x|B\}$. After applying this reduction, there would be reduction to be applied for $\forall$. After these steps we have the following.

$$
\dfrac{\dfrac{\vdash A \qquad \dfrac{\dfrac{A \vdash B}{A \vdash t_0 \in \{x|B\}}\in I}{\vdash A \to t_0 \in \{x|B\}}\to I}{\vdash t_0 \in \{x|B\}}}{\vdash B}\in E
$$

The final two reductions will be another $\to$ reduction followed by the $\in$ reduction as we've seen in the other instances. This computation completes the proposition. $\square$

Theorem 6.42. *A deduction in NST is normal iff the primitive translation of the deduction is normal in* $NST_{\to\in\forall}$.

Proof. Since any proof in $\mathrm{NST}_{\to\in\forall}$ is also a proof of NST, we have the right to left direction. For left to right, let Π be a normal deduction in NST and π a path in that deduction. Then π is of the form specified in Theorem 6.27. We will consider each part of the form in turn.

In the E-part there can not be any $\perp$ rules or introduction rules, so we will only need to translate $\wedge E$, $\vee E$, or $\exists E$. The general proof here is that the translations as given in Proposition

6.38 are such that the string of elimination steps will not be broken in π after translation and those introduction rules which are introduced are closed off from π being used as minor premises in $\to E$.

We'll look at $\wedge E$ in particular, the other two elimination rules satisfy the same necessary form. Let π_n denote arbitrary subpaths of π that precede the part in question and $\Box$ denote an arbitrary connective. The original deduction will look like

$$\dfrac{\dfrac{\pi_0}{\Gamma \vdash A \wedge B}\,\Box E}{\Gamma \vdash A}\,\wedge E$$

which is then translated to the following.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B \vdash A}{A, B \vdash t_0 \in \{x|A\}}\in I}{A \vdash B \to t_0 \in \{x|A\}}\to I}{\vdash A \to B \to t_0 \in \{x|A\}}\to I \qquad \dfrac{\dfrac{\pi_0}{\Gamma \vdash \forall x((A \to B \to t_0 \in x) \to t_0 \in x)}\,\Box E}{\Gamma \vdash (A \to B \to t_0 \in \{x|A\}) \to t_0 \in \{x|A\}}\,\forall E}{\Gamma \vdash t_0 \in \{x|A\}}\to E}{\Gamma \vdash A}\in E$$

The path π can stay the same except by inserting the new elimination steps between the application of $\Box E$ and the derivation of $\Gamma \vdash A$. Either this translation will introduce a maximum segment inside the steps themselves, or by coming into conflict with an introduction rule that precedes the newly inserted elimination steps. We first consider the inserted steps themselves. All introduction steps that are added are in the minor premise of $\to E$ and that is the entire subdeduction on that side, and so no new non-normal steps are introduced there or are in conflict with elimination rules later on. Further, a sequence of elimination steps are never non-normal. We now consider whether a non-normal step was introduced outside of the translation. However, by our original assumption that we're considering the E-part of π, it is also not possible that an introduction or $\perp$ rule precedes the newly added steps. This proof also applies equally to the translations of $\vee E$ and $\exists E$.

Next, we consider the translation of the minimum segment. In the instance that there is a $\perp$ rule, the translation replaces the $\perp$ rule only with elimination rules which only extend the E-part and can't introduce a maximum segment. In the other case, our last elimination step is the premise of an introduction step. This will not introduce a maximum segment for essentially the same reason that that translation does not introduce maximum segments in the I-part of π, with the only difference being that the last step prior to the introduction rules will be an elimination. We thus move to giving the proof for that the translations in the I-parts not introducing non-normal steps.

We consider the translations of $\wedge I$, $\vee I$ and $\exists I$ in the I-part. We consider $\wedge I$ in detail, and the proofs for $\vee I$ and $\exists I$ are nearly identical. The general idea is similar to the observation in the E-part, except here the translation will considerably shorten the path π and start new paths. The original deduction will be of the form

$$\dfrac{\dfrac{\pi_0}{\Gamma \vdash A}\,\Box I \qquad \dfrac{\pi_1}{\Gamma \vdash B}\,\Box I}{\Gamma \vdash A \wedge B}\,\wedge I$$

while the translation looks like the following.

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash B}\,\Box I \qquad \dfrac{\dfrac{\pi_0}{\Gamma \vdash A}\,\Box I \qquad A \to B \to t_0 \in x \vdash A \to B \to t_0 \in x}{A \to B \to t_0 \in x \vdash B \to t_0 \in x} \to E}{\dfrac{\dfrac{\Gamma, A \to B \to t_0 \in x \vdash t_0 \in x}{\Gamma \vdash (A \to B \to t_0 \in x) \to t_0 \in x)} \to I}{\Gamma \vdash \forall x((A \to B \to t_0 \in x) \to t_0 \in x)} \forall I} \to E$$

Again, we consider whether the translation itself contains maximum segments or whether it could introduce maximum segments on the edges of the proof. In the former case, this is not a concern as the translation itself is normal. In the latter case, the rules that precede $\Gamma \vdash A$ and $\Gamma \vdash B$ are closed off inside their own path and so can not lead to the creation of a new maximum segment.

We have considered every part of the path π and as π was arbitrary, we are nearly done. Every path in the resulting proof in $\text{NST}_{\to\in\forall}$ will either end at the end-formula or as a minor premise in $\to E$ rule. Neither of these endings could introduce maximum segments between paths, and so we are done. $\Box$

Corollary 6.43. *A deduction in NST normalizes iff the primitive translation of the deduction normalizes in $NST_{\to\in\forall}$.*

Definition 6.44. A formula A is a theorem of $\text{NNST}_{\to\in\forall}$ iff it has a normal deduction in $\text{NST}_{\to\in\forall}$.

Corollary 6.45. $\Gamma \vdash A$ *in NNST iff* $\Gamma \vdash A$ *in* $NNST_{\to\in\forall}$.

Thus, at least with equality aside, we have the same theory with a much smaller set of connectives. The advantage of this lies in being able to prove theorems about the system, moreso than working with the system. We will use this system, equivalent to NNST, to prove the primary result of this thesis, that NNST recovers all of Heyting Arithmetic.

Chapter 7

Recovering Heyting Arithmetic in NNST

This chapter contains the major result of the thesis, the recovery of second-order Heyting Arithmetic.[1] We first introduce λ-calculus and then define a type system for $\text{NST}_{\to\in\forall}$ that appropriately corresponds with normalization of proofs in the system. We'll prove a subject reduction theorem [53] to show that this correspondence behaves in the way expected. Then we show that the requisite equality properties for HA can be expressed with Leibniz identity, which can be defined in our reduced system and that we have normal proofs of Peano's Axioms. We will be able to translate the proofs of Peano's axioms as well as any proofs in that fragment of NST into an extended System Fω; we essentially take a fragment of the (strongly normalizing) Extended Calculus of Constructions [32]. This translation will work in such a way that the normalization of the λ-terms of the HA fragment of NNST corresponds to the normalization of the λ-terms of the translated fragment of System Fω. As System Fω is strongly normalizing, this implies that this fragment of NNST must normalize as well.[2]

7.1 λ-Calculus

We now introduce the computational model of the λ-calculus. The untyped lambda calculus is a syntax used to streamline the processes of function application and the generation of higher-order functions [29, 36, 5]. As it turns out, it provides enough flexibility to represent all recursive functions and thus it can be viewed as another way of expressing computational procedures. This machinery is what will make our final proof of the recovery of HA work. We use the λ-calculus to first capture the algorithmic properties of our proofs and their normalization using the ideas of the Curry-Howard correspondence, and then use a correspondence with a type

[1] Before we begin believing that NNST is a purely constructive idea in the sense of constructive mathematics, while it is constructive in the sense that all proofs must be constructed, it should be noted that it is believed that PA would be recoverable in a system of NNST constructed on a "classical" naive set theory, i.e. a naive set theory with a classical negation. These initial results were easier to approach in a natural deduction setting. It is also believed that any order HA or PA would also be recoverable, not only up to second order.

[2] The original idea comes from the normalization of HA in System F as seen in [30].

theory to reach our result; before we can get to type theories, we need to start from untyped λ-calculus. To begin, we first need the notion of a lambda term.

Definition 7.1 (Lambda Term)**.** Let $v_0, v_1, v_2, \ldots$ be a given infinite sequence of variables. Optionally, we may also have a finite or infinite sequence of atomic constants, which are different from the variables. (If we have no atomic constants, the lambda calculus is called pure.) The set of λ-terms is defined inductively:

1. all variables and atomic constants are λ-terms (called atoms);
2. if M and N are λ-terms, then (MN) is a λ-term (called an application);
3. if M is any λ-term and x is any variable, then $(\lambda x.M)$ is a λ-term (called an abstraction).

Notation 7.2. Syntactic identity will be denoted by $\equiv$.

Definition 7.3 (Free and Bound Variables)**.** An occurrence of a variable x in a term P is called

- bound if it is in the scope of a λx in P,
- bound and binding if it is the x in λx,
- free otherwise.

Denote the set of free variables of a term P by $FV(P)$.

Definition 7.4 (Substitution)**.** Define $M[N/x]$ to be the result of substituting N for every free occurrence of x in M, and also changing bound variables to avoid clashes. The definition is by induction:

1. $x[N/x] \equiv N$;
2. $a[N/x] \equiv a$ for all atoms $a \not\equiv x$;
3. $(PQ)[N/x] \equiv (P[N/x]Q[N/x])$;
4. $(\lambda x.P)[N/x] \equiv \lambda x.P$;
5. $(\lambda y.P)[N/x] \equiv \lambda y.P$ if $x \notin FV(P)$;
6. $(\lambda y.P)[N/x] \equiv \lambda y.P[N/x]$ if $x \in FV(P)$ and $y \notin FV(N)$
7. $(\lambda y.P)[N/x] \equiv \lambda z.P[z/y][N/x]$ if $x \in FV(P)$ and $y \in FV(N)$.

Definition 7.5 (α-conversion)**.** Let a term P contain an occurrence of $\lambda x.M$, and let $y \notin FV(M)$. Replacing $\lambda x.M$ by

$$\lambda y.M[y/x]$$

is called a change of bound variable or an α-conversion in P. If P can be changed to Q by a finite series of α-conversions, we say that P is congruent to Q or P α-converts to Q, or

$$P \equiv_\alpha Q$$

.

Definition 7.6. Let M, N, and Z be λ-terms. Define one-step β-reduction, $A \triangleright_{1\beta} B$ inductively as follows.

1. $(\lambda x.M)N \triangleright_{1\beta} M[N/x]$.
2. If $M \triangleright_{1\beta} N$ then $ZM \triangleright_{1\beta} ZN$, $MZ \triangleright_{1\beta} NZ$, and $\lambda x.M \triangleright_{1\beta} \lambda x.N$.

We define β-reduction, $A \triangleright_\beta B$ inductively as follows.

1. $M \triangleright_\beta M$.
2. If $M \triangleright_{1\beta} N$ then $M \triangleright_\beta N$.
3. If $M \triangleright_\beta N$ and $N \triangleright_\beta Z$ then $M \triangleright_\beta Z$.

We denote $\triangleright_\beta$ simply as $\triangleright$ from now on when differentiation between the two is not required.

Definition 7.7. A λ-term is normal/reduced if there are no more β-reductions of the form $(\lambda x.M)N \triangleright M[N/x]$ to be performed.

Definition 7.8. A λ-term normalizes if there is some (finite) sequence of λ-terms related by β-reduction such that the final λ-term in the sequence is normal.

Notation 7.9. Let $\lambda x_1 x_2 \ldots x_n.M$ be shorthand for $\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.M)))$.

Definition 7.10. We define β-equality as $P =_\beta Q$ iff there exist $P_0, \ldots, P_n (n \geq 0)$ with $P_0 \equiv P$ and $P_n \equiv Q$ such that

$$(\forall i \leq n-1)(P_i \triangleright P_{i+1} \vee P_{i+1} \triangleright P_i \vee P_i \equiv_\alpha P_i + 1).$$

That is, there is some finite sequence of β-reductions or α-conversions that takes P to Q.

Finally, we can formalize these notions of β-reduction and β-equality in a Hilbert style system as follows.

Definition 7.11 ($\lambda\beta$)**.** The Hilbert system $\lambda\beta$ has as formulas $M = N$, where $=$ is β-equality, and $M \triangleright N$, where $\triangleright$ is β-reduction. For all λ-terms M and N, and variables x and y we have the following axiom schema and rules of inference for theory of β-equality.

Axiom Schema

A1 $\lambda x.M = \lambda y.[y/x]M$ if $y \notin FV(M)$;

A2 $(\lambda x.M)N = [N/x]M$;

A3 $M = M$.

Rules of Inference

$$\frac{M = M'}{NM = NM'}\ \text{LComp} \qquad\qquad \frac{M = M'}{MN = M'N}\ \text{RComp}$$

$$\frac{M = M'}{\lambda x.M = \lambda x.M'}\ \text{SubEval}$$

$$\frac{M = N \qquad N = P}{M = P}\ \text{Trans} \qquad\qquad \frac{M = N}{N = M}\ \text{Sym}$$

The theory of β-reduction is given by replacing all instances of $=$ in the above with $\triangleright$, and removing Sym.

7.2 Type System for NNST

The general idea of adding λ-terms to logical systems is known as the Curry-Howard correspondence [36]. The λ-terms are meant to serve as representations of the algorithmic content of the proofs and the "types" of the system are logical formulas.[3] It will be set up in such a way that if the derived λ-term for a proof of NST normalizes in the λ-calculus sense of the term "normalizes", the proof will also normalize in the natural deduction sense. We define now a type system for $\text{NNST}_{\to\in\forall}$ is as follows.[4]

$$\frac{}{\Gamma, x : A \vdash x : A}\ \text{Ax}$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x.u : A \to B} \to I \qquad\qquad \frac{\Gamma \vdash u : A \qquad \Gamma \vdash t : A \to B}{\Gamma \vdash tu : B} \to E$$

$$\frac{\Gamma \vdash u : A[a/x]}{\Gamma \vdash u : \forall x A}\ \forall I \qquad\qquad \frac{\Gamma \vdash u : \forall x A}{\Gamma \vdash u : A[a/x]}\ \forall E$$

$$\frac{\Gamma \vdash u : A[r/x]}{\Gamma \vdash u : r \in \{x | A(x)\}} \in I \qquad\qquad \frac{\Gamma \vdash u : r \in \{x | A(x)\}}{\Gamma \vdash u : A[r/x]} \in E$$

In the axiom rule, we are not allowed to use, or instantiate/declare, a λ-variable more than once. The usual restriction for $\forall I$ applies, x can not be free in Γ. For convenience, we will sometimes use the same variable x as part of a λ-term while it is also part of a logical formula as in $x \in A$. These are different variables when used in this way and there is no harm in doing this as the syntax can never confuse them.

Note that in this type system we only track introduction and elimination steps for $\to$. This will mean that the same λ-variables and λ-terms can take on different types throughout a proof. We will find that it will only be non-normal $\to$ inferences in $\text{NST}_{\to\in\forall}$ which prevent the normalization of a proof. First we recall the primitive translations of $\wedge$, $\vee$, $\exists$ and $\perp$ to find their corresponding λ-terms and derived rules decorated with λ-terms.

Proposition 7.12. *The derived λ-terms for the derived rules are as follows.*

[3] We shall often refer to logical formulas as types when focusing on the properties of the type system in what follows.

[4] This is a Curry-style type system for those familiar.

1. *For $\wedge I$ with $\Gamma \vdash M : A$, $\Gamma \vdash N : B$, and $z : A \to B \to t_0 \in x$, the term is $\lambda z.(zM)N$.*

2. *For $\wedge E_0$ with $\Gamma \vdash M : A \wedge B$, $x : A$, $y : B$, the term is $M(\lambda xy.x)$.*

3. *For $\wedge E_1$ with $\Gamma \vdash M : A \wedge B$, $x : A$, $y : B$, the term is $M(\lambda xy.y)$.*

4. *For $\vee I_0$ with $\Gamma \vdash M : A$, $x : A \to t_0 \in x$, $y : B \to t_0 \in x$, the term is $\lambda xy.(xM)$.*

5. *For $\vee I_1$ with $\Gamma \vdash M : A$, $x : A \to t_0 \in x$, $y : B \to t_0 \in x$, the term is $\lambda xy.(yM)$.*

6. *For $\vee E$ with $\Gamma \vdash M : A \vee B$, $\Gamma, x : A \vdash N_0 : C$, $\Gamma, y : B \vdash N_1 : C$, the term is $(M(\lambda x.N_0))(\lambda y.N_1)$.*

7. *For $\exists I$ with $\Gamma \vdash M : A$, $x : \forall y(A \to t_0 \in x)$, the term is $(\lambda x.xM)$.*

8. *For $\exists E$ with $\Gamma \vdash M : \exists xA$, $\Gamma, x : A \vdash N : B$ the term is $M(\lambda x.N)$.*

9. *For $\perp$ with $\Gamma \vdash M : \perp$, the term is M.*[5]

Proof. These λ-terms follow from decorating the proofs of Proposition 6.38. □

Corollary 7.13. *The following derived rules are valid for the composite connectives.*

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \lambda z.(zM)N : A \wedge B}\ \wedge I$$

$$\frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M(\lambda xy.x) : A}\ \wedge E_0 \qquad\qquad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M(\lambda xy.y) : B}\ \wedge E_1$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \lambda xy.xM : A \vee B}\ \vee I_0 \qquad\qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash \lambda xy.yM : A \vee B}\ \vee I_1$$

$$\frac{\Gamma \vdash M : A \vee B \qquad \Gamma, x : A \vdash N_0 : C \qquad \Gamma, y : B \vdash N_1 : C}{\Gamma \vdash (M(\lambda x.N_0))\lambda y.N_1 : C}\ \vee E$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \lambda x.xM : \exists xA}\ \exists I \qquad\qquad \frac{\Gamma \vdash M : \exists xA \qquad \Gamma, x : A \vdash N : B}{\Gamma \vdash M(\lambda x.N) : B}\ \exists E$$

$$\frac{\Gamma \vdash M : \perp}{\Gamma \vdash M : A}\ \perp$$

We would like to be able to use these derived operational rules with free abandon. However, the need to check normalization means that we can not simply use rules without worrying about the algorithmic consequences. The properties of the type system we verify below, namely the Substitution Lemma (7.18), Proposition 7.21, and the Subject Reduction Theorem (7.19), will verify that we can use these λ-terms for the derived operational rules to track the necessary consequences for our normalization. To prove these theorems, we first need the following defined relation on types.

Definition 7.14. Let Γ be a set of declarations. Then $>_\Gamma$ is a binary relation on the types of the declared λ-terms such that

- $\forall xA >_\Gamma A[a/x]$,
- $A >_\Gamma \forall xA$, if x is not a free variable of Γ,

[5] $\perp$ has no tracked algorithmic content.

- $A[a/x] >_\Gamma a \in \{x|A(x)\}$,
- $a \in \{x|A(x)\} >_\Gamma A[a/x]$.

Let $\geq_\Gamma$ denote the reflexive and transitive closure of $>_\Gamma$. That is,

- For any type A, $A \geq_\Gamma A$,
- if $A \geq_\Gamma B$ and $B \geq_\Gamma C$ then $A \geq_\Gamma C$.

This relation allows identification of types which are related by logical rules but not tracked in the λ-calculus. Similar relations and results are found in [53, 5].

Lemma 7.15. *If $\Gamma \vdash M : A'$ and $A' \geq_\Gamma A$ then $\Gamma \vdash M : A$.*

Proof. Since $A' \geq_\Gamma A$ there is some, possibly empty, sequence of A_i's such that

$$A' \equiv A_1 >_\Gamma A_2 \dots A_n >_\Gamma A_{n+1} \equiv A.$$

Any individual step in the sequence will not change the λ-term by the definition of $>_\Gamma$ and is equivalent to the application of a logical rule. Further, the restriction on $B >_\Gamma \forall x B$ guarantees that this only holds when the logical rule $\forall I$ can be applied. From here, induction on the length of the sequence gives the proof. □

Lemma 7.16. *If $A \to B \geq_\Gamma A' \to B'$ then $A' \to B'$ is of the same form as $A \to B$ but with some, possibly empty, sequence of substitutions.*

Proof. By induction on the construction of the relation $\geq_\Gamma$ and transitivity of the relation. □

The utility of $\geq_\Gamma$ is that it allows a convenient means of stating the Generation Lemma, which shows that all λ-terms and their types are inductively constructed as we expect.[6]

Lemma 7.17 (Generation Lemma [5])**.** *The following hold for $NNST_{\to\in\forall}$:*

1. *If $\Gamma \vdash x : A$ then $x : B \in \Gamma$ for some $B \geq_\Gamma A$,*
2. *If $\Gamma \vdash \lambda x.t : A$ then $x : B, \Gamma \vdash t : C$ for some B and C such that $B \to C \geq_\Gamma A$.*
3. *If $\Gamma \vdash tu : A$ then $\Gamma \vdash t : B \to C$ and $\Gamma \vdash u : B$ for some B and C where $C \geq_\Gamma A$.*

Proof. We proceed by induction on the length of the derivation. The base case is a derivation of length 1. A derivation of length 1 will only contain a single use of the axiom rule. The proof of each possible case is trivial for the base case: the first must have $x : A \vdash x : A$ as it is an axiom rule and thus $x : A \in \Gamma$, and the other two cases are vacuously true.

We assume now that the proposition holds for derivations of length n and the derivation in question is of length $n + 1$. Further, let our conclusion be of the form $\Gamma \vdash M : A$ and let $\Gamma \vdash P_i : B_i$ for $i \in \{0, 1\}$ denote the derivations prior to the last rule application. We proceed

[6]These next few results, the Generation Lemma, Substitution Lemma and Subject Reduction Theorem, are standard results proved for type theories [5, 53]. The names for them are by convention.

by cases on the form of M. M must be one of the following: of the form x for some term variable, $\lambda x.N$ for some λ-term N and term variable x, or NL for some λ-terms N and L.

Assume we are in the first case, that M is of the form x. Then the last rule applied to the derivation has to be an introduction or elimination of $\forall$ or $\in$. In this case we have a prior subderivation of $\Gamma \vdash x : B_0$ which will satisfy the induction hypothesis and thus there must be some $x : C \in \Gamma$ such that $C \geq_\Gamma B$. As A will result from B by application of a rule of $\in$ or $\forall$, we will have $B \geq_\Gamma A$. Then by the transitive closure of $\geq_\Gamma$ we have $C \geq_\Gamma A$. Thus we have a $x : C \in \Gamma$ for which $C \geq_\Gamma A$.

In the second case in which M is of the form $\lambda x.N$, the last rule applied to the derivation may be any of the $\forall$ or $\in$ rules or $\to I$. In the cases that a $\forall$ or $\in$ rule was applied, the proof is similar to the first case. In the case that $\to I$ was applied, the prior derivation will be of the form $x : C, \Gamma \vdash N : B_0$ and $A \equiv C \to B_0$. By the reflexivity of $\geq_\Gamma$, we have $C \to B_0 \geq_\Gamma A$.

The third and final case is similar to the proofs of the first and second case. This completes the proof. □

Lemma 7.18 (Substitution Lemma [5]). *The following hold:*

1. *If* $\Gamma \vdash M : A$ *then* $\Gamma[C/B] \vdash M : A[C/B]$.
2. *If* $\Gamma, x : B \vdash M : A$ *and* $\Gamma \vdash N : B$ *then* $\Gamma \vdash M[N/x] : A$.

Proof. For 1, we proceed by induction on the length of the derivation of $\Gamma \vdash M : A$. For the base case of length 1, we have that the derivation of the substituted version only requires a change in the axiom rule. Now assume the proposition holds for derivations of length n and assume the derivation in question is $n+1$. We can use the Generation Lemma to find a subderivation(s) of $\Gamma \vdash M : A$ that is of length n. We then apply the inductive hypothesis to these subderivation(s). By inspection any rule of the system applied will maintain the same substituted variables as desired.

For 2, we proceed by induction on the length of the derivation of $\Gamma, x : B \vdash M : A$. In the base case, we have M is of the form y for some λ-variable y. As this derivation is of length 1, it must be the case that y is distinct from x, in which case the substitution of N for x has no effect and thus the desired derivation holds. Now assume it holds for derivations of length n and assume the derivation in question is $n+1$. We then use the Generation Lemma to find a subderivation(s) of length n. The induction hypothesis gives that these smaller subderivation(s) can be used to construct subderivation(s) with the requisite substituted N. Then what rule was originally used to combine the subderivation(s) to the final conclusion in question can be used again to produce the new substituted derivation. □

Theorem 7.19 (Subject Reduction Theorem [5]). *If* $\Gamma \vdash M : A$ *is derivable and* $M \rhd N$ *then* $\Gamma \vdash N : A$ *is derivable.*

Proof. By induction on the definition of β-reduction. The base cases are reflexivity of $\rhd$ and the reduction of $(\lambda x.M)N \rhd M[N/x]$. The former is trivial. For the latter, our derivation is of the form

$$\Gamma \vdash ((\lambda x.P)Q) : A$$

for some λ-terms P and Q. By the Generation Lemma (7.17) we have that there must be derivations

$$\Gamma \vdash \lambda x.P : B \to A'$$

and

$$\Gamma \vdash Q : B$$

such that $A' \geq_\Gamma A$. We can again apply the Generation Lemma to the former derivation to determine that there must be the derivation

$$\Gamma, x : B' \vdash P : A''$$

such that $B' \to A'' \geq_\Gamma B \to A'$.

By Lemma 7.16, there is a sequence of substitutions that converts $B' \to A''$ into $B \to A'$. Apply those substitutions to the derivation

$$\Gamma, x : B' \vdash P : A''$$

to get

$$\Gamma, x : B \vdash P : A'.$$

Then by the Substitution Lemma (7.18), $\Gamma, x : B \vdash P : A'$ and $\Gamma \vdash Q : B$, we can find a derivation

$$\Gamma \vdash P[Q/x] : A'.$$

Then by Lemma 7.15 and $A' \geq_\Gamma A$ we can find the derivation

$$\Gamma \vdash P[Q/x] : A.$$

The rest of the inductive proof is straightforward verification on the inductive clauses defining $\triangleright$. $\square$

Corollary 7.20. *Let M and N be λ-terms such that $M \triangleright N$. The following derived rule is valid.*

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash N : A}\ \mathit{Eval}$$

Proposition 7.21. *If $\Gamma \vdash M : A$ is derivable and M is normal, then the deduction of A is normalizable.*

Proof. First note that if M is normal, then any non-normal steps in the deduction must involve $\in$ or $\forall$. Non-normal steps involving the rules of these connectives can always be reduced by their respective reduction rules. After we apply a reduction, we must either decrease our count of non-normal steps by 1 or our count stays the same by introducing a new non-normal step involving either $\in$ or $\forall$. No non-normal steps involving $\to$ can be introduced after reducing $\in$ or $\forall$ steps, as such steps would result in M being non-normal. If such a non-normal step did result from reducing $\in$ rules or $\forall$ rules, then the non-normality involving the rules of $\to$ would already necessarily appear in the λ-term since we are not tracking $\in$ or $\forall$ rules.

Further, it can not be the case that a reduction of $\in$ or $\forall$ rules never results in a decrease of our count of non-normal steps. For if it never did, then each reduction would imply there was another pair of $\in$ or $\forall$ rules above and below the just-reduced step. This implies the deduction must be infinite. Thus we must eventually eliminate all remaining non-normal steps in the proof. $\square$

Proposition 7.21 and the Subject Reduction Theorem together give that we only need to calculate the reduction of a λ-term involving the basic operations of λ-abstraction and application to determine if a proof will normalize. This makes it clear that full NST does not have any algorithmic machinery not already present in $\mathrm{NST}_{\to\in\forall}$. Even the algorithms that result from computing whether some set is a member of "non-normal sets" like Russell's involve understood computational principles.

Corollary 7.22. *A proof of $NST_{\to\in\forall}$ (and NST) can be normalized if a λ-term that involves only the operations of λ-abstraction and application can be normalized.*

Proposition 7.21 and the Subject Reduction Theorem also verify that the λ-calculus defined for $\mathrm{NST}_{\to\in\forall}$ captures enough of the logic to determine whether a deduction will normalize or not. That is, we've successfully found a correspondence between λ-terms and the deductions of NST in such a way that it captures the process of normalization.[7]

7.3 Recovery of Heyting Arithmetic

We can now proceed to recapturing Heyting Arithmetic in NNST. Our first step is to lay the ground work by proving some basic results of NNST: we will recover the subset operation which we'll need for proving things about successors, we prove the rules for equality we need by defining Leibniz identity, and then we provide normal deductions for the axioms of PA. We also use these as examples of how the type system works and track λ-terms for some of these proofs. We stop providing them as soon as the λ-terms become too complex to be easily readable. Having these will not be necessary for our final goal of recovering HA.

Definition 7.23. $x \subseteq y := \forall z(z \in x \to z \in y)$.

Proposition 7.24. *The subset relation is reflexive and transitive.*

1. $x \subseteq x$
2. $x \subseteq y, y \subseteq z \vdash x \subseteq z$

Proof. The proof for 1 follows.

$$\frac{t : z \in x \vdash t : z \in x}{\vdash \lambda t.t : z \in x \to z \in x}$$

[7]That such correspondences are possible is usually known as the Curry-Howard correspondence.

The proof for 2 follows.

$$\cfrac{\cfrac{x : x \subseteq y \vdash x : t \in x \to t \in y \qquad t : t \in x \vdash t : t \in x}{x : x \subseteq y, t : t \in x \vdash xt : t \in y}\to E \qquad y : y \subseteq z \vdash y : t \in y \to t \in z}{\cfrac{x : x \subseteq y, y : y \subseteq z, t : t \in x \vdash y(xt)t \in z}{x : x \subseteq y, y : y \subseteq z \vdash \lambda t.y(xt) : t \in x \to t \in z}\to I}\to E$$

□

Corollary 7.25. *The following derived rules hold.*

$$\frac{}{\Gamma \vdash \lambda t.t : x \subseteq x}\ \mathit{Refl}$$

$$\frac{\Gamma \vdash x : x \subseteq y \qquad \Gamma \vdash y : y \subseteq z}{\Gamma \vdash \lambda t.y(xt) : x \subseteq z}\ \mathit{Trans}$$

Definition 7.26. Two sets x and y are Leibniz identical if $x = y := \forall z(x \in z \to y \in z)$.

Proposition 7.27. *Leibniz identity satisfies the following:*

1. $\vdash x = x$,
2. $x = y \vdash y = x$,
3. $x = y, y = z \vdash x = z$,
4. $x = y, A(x) \vdash A(y)$.

Proof. The proof for 1 follows.

$$\cfrac{\cfrac{t : x \in z \vdash t : x \in z}{\vdash \lambda t.t : x \in z \to x \in z}\to I}{\vdash \lambda t.t : \forall z(x \in z \to x \in z)}\forall I$$

The closed λ-term for the proof of reflexivity is thus $\lambda t.t$. The proof for 2 follows.

$$\cfrac{\cfrac{\cfrac{t : x = y \vdash t : x = y}{t : x = y \vdash t : x \in \{a : a = x\} \to y \in \{a : a = x\}}\forall E \qquad \cfrac{\vdash \lambda u.u : x = x}{\vdash \lambda u.u : x \in \{a : a = x\}}\in I}{t : x = y \vdash t(\lambda u.u) : y \in \{a : a = x\}}\to E}{t : x = y \vdash t(\lambda u.u) : y = x}\in E$$

The closed λ-term for the proof of symmetry is thus $\lambda t.t(\lambda u.u)$. The proof for 3 follows.

$$\cfrac{\cfrac{v : y = z \vdash v : y \in s \to z \in s \qquad \cfrac{t : x = y \vdash t : x \in s \to y \in s \qquad u : x \in s \vdash u : x \in s}{t : x = y, u : x \in s \vdash tu : y \in s}\to E}{t : x = y, v : y = z, u : x \in s \vdash v(tu) : z \in s}\to E}{t : x = y, v : y = z \vdash \lambda u.(v(tu)) : x \in s \to z \in s}\to I$$

The closed λ-term for the proof of transitivity is thus $\lambda tvu.v(tu)$. Finally the proof for 4 follows.

$$\cfrac{\cfrac{t : \forall z(x \in z \to y \in z) \vdash t : x \in \{a : A(a)\} \to y \in \{a : A(a)\} \qquad u : A(x) \vdash u : x \in \{a : A(a)\}}{t : x = y, u : A(x) \vdash tu : y \in \{a : A(a)\}}\to E}{t : x = y, u : A(x) \vdash tu : A(y)}\in E$$

The closed λ-term for substitution is thus $\lambda tu.tu$. □

Corollary 7.28. *The following derived rules for Leibniz Identity hold.*

$$\frac{}{\Gamma \vdash \lambda t.t : x = x}\ Refl \qquad \frac{\Gamma \vdash t : x = y}{\Gamma \vdash t(\lambda u.u) : y = x}\ Sym$$

$$\frac{\Gamma \vdash t : x = y \qquad \Gamma \vdash v : y = z}{\Gamma \vdash \lambda u.v(tu) : x = z}\ Trans$$

$$\frac{\Gamma \vdash u : A(x) \qquad \Gamma \vdash t : x = y}{\Gamma \vdash tu : A(y)} = E$$

Definition 7.29. Let $x := \{z|z = x\}$ and $x \cup y := \{z|z \in x \vee x \in y\}$.

Definition 7.30. Let $0 := \emptyset$, $Sx := x \cup \{x\}$ and

$$\mathbf{N} := \{x|\forall\alpha(\forall y(y \in \alpha \to Sy \in \alpha) \to (0 \in \alpha \to x \in \alpha))\}.$$

Proposition 7.31. *A set x is a subset of its successor, Sx.*

Proof. The proof follows.

$$\dfrac{\dfrac{\dfrac{t : z \in x \vdash t : z \in x}{t : z \in x \vdash \lambda xy.xt : z \in x \vee z \in \{x\}}\ \vee I}{t : z \in x \vdash z \in Sx}\ \in I}{\vdash \lambda txy.xt : z \in x \to z \in Sx}\ \to I$$

□

Corollary 7.32. *The followed derived rule holds.*

$$\frac{}{\Gamma \vdash \lambda txy.xt : x \subseteq Sx}\ SucSub$$

Proposition 7.33. *The following hold:*

1. $\vdash 0 \in \mathbf{N}$,
2. $\vdash x \in \mathbf{N} \to Sx \in \mathbf{N}$,
3. $\vdash 0 \neq Sx$,
4. $A(0), \forall n(A(n) \to A(Sn)) \vdash \forall n(n \in \mathbf{N} \to A(n))$
5. *Let $T := \{x|A(x)\}$. Then* $0 \in T, \forall n(n \in T \to Sn \in T) \vdash \forall n(n \in \mathbf{N} \to n \in T)$

Proof. The proof for 1 follows.

$$\dfrac{\dfrac{\dfrac{y : \forall y(y \in \alpha \to Sy \in \alpha), x : 0 \in \alpha \vdash x : 0 \in \alpha}{y : \forall y(y \in \alpha \to Sy \in \alpha), \vdash \lambda x.x : 0 \in \alpha \to 0 \in \alpha}\ \to I}{\vdash \lambda yx.x : \forall y(y \in \alpha \to Sy \in \alpha) \to (0 \in \alpha \to 0 \in \alpha)}\ \to I}{\vdash \lambda yx.x : 0 \in \mathbf{N}}\ \forall I$$

The closed λ-term for the proof is thus $\lambda yx.x$. The proof for 2 follows. For the sake of space, we abbreviate $\forall y(y \in \alpha \to Sy \in \alpha)$ as Ind.

$$\dfrac{\dfrac{x : x \in \mathbf{N} \vdash x : \text{Ind} \to (0 \in \alpha \to x \in \alpha) \qquad y : \text{Ind} \vdash y : \text{Ind}}{x : x \in \mathbf{N}, y : \text{Ind} \vdash xy : 0 \in \alpha \to x \in \alpha}\ \to E \qquad z : 0 \in \alpha \vdash z : 0 \in \alpha}{x : x \in \mathbf{N}, y : \text{Ind}, z : 0 \in \alpha \vdash (xy)z : x \in \alpha}\ \to E$$

We continue from here to complete the proof.

$$\dfrac{\dfrac{y:\mathrm{Ind}\vdash y:x\in\alpha\to Sx\in\alpha \qquad x:x\in\mathbf{N},y:\mathrm{Ind},z:0\in\alpha\vdash (xy)z:x\in\alpha}{x:x\in\mathbf{N},y:\mathrm{Ind},z:0\in\alpha\vdash y((xy)z):Sx\in\alpha}\to E}{\vdash \lambda xyz.y((xy)z):x\in\mathbf{N}\to Sx\in\mathbf{N}}\to I$$

The closed λ-term for the proof is thus $\lambda xyz.y((xy)z)$. The proof for 3 follows.

$$\dfrac{\dfrac{\dfrac{t:0=Sx\vdash t:0=Sx \qquad \dfrac{\dfrac{\dfrac{\vdash \lambda u.u:x=x}{\vdash \lambda u.u:x\in\{x\}}\in I}{\vdash \lambda xy.(x(\lambda u.u)):x\in x\vee x\in\{x\}}\vee I}{\vdash \lambda xy.(x(\lambda u.u)):x\in Sx}\in I}{t:0=Sx\vdash t(\lambda xy.(x(\lambda u.u))):x\in 0}=E}{t:0=Sx\vdash t(\lambda xy.(x(\lambda u.u))):\bot}\in E}{\vdash \lambda t.t(\lambda xy.(x(\lambda u.u))):0=Sx\to\bot}\to I$$

The closed λ-term for the proof is thus $\lambda t.t(\lambda xy.(x(\lambda u.u)))$. The proof for the two forms of induction, 4 and 5, follow. We split this proof into two parts due to size.

$$\dfrac{\dfrac{\dfrac{t:n\in\{x|A(x)\}\vdash t:A(n) \qquad y:\forall n(A(n)\to A(Sn))\vdash y:A(n)\to A(Sn)}{y:\forall n(A(n)\to A(Sn)),t:n\in\{x|A(x)\}\vdash yt:A(Sn)}\to E}{y:\forall n(A(n)\to A(Sn)),t:n\in\{x|A(x)\}\vdash yt:Sn\in\{x|A(x)\}}\in I}{y:\forall n(A(n)\to A(Sn))\vdash \lambda t.yt:\forall n(n\in\{x|A(x)\}\to Sn\in\{x|A(x)\})}\to I$$

Let $T:=\{x|A(x)\}$, $D:=\forall y(y\in T\to Sy\in T)\to(0\in T\to n\in T)$. We continue this proof in the following.

$$\dfrac{\dfrac{x:A(0)\vdash x:0\in T \qquad \dfrac{y:\forall n(A(n)\to A(Sn))\vdash \lambda t.yt:\forall n(n\in T\to Sn\in T) \qquad z:n\in\mathbf{N}\vdash z:\mathbf{D}}{y:\forall n(A(n)\to A(Sn)),z:n\in\mathbf{N}\vdash z(\lambda t.yt):0\in T\to n\in T}\to E}{x:A(0),y:\forall n(A(n)\to A(Sn)),z:n\in\mathbf{N}\vdash (z(\lambda t.yt))x:n\in T}\to E}{x:A(0),y:\forall n(A(n)\to A(Sn)),z:n\in\mathbf{N}\vdash (z(\lambda t.yt))x:A(n)}\in E$$

The closed λ-term for the proof is thus $\lambda xyz.(z(\lambda t.yt))x$. The proof for 5 follows, which is nearly identical to the second part of 4. Let D and T continue to denote the same formulas.

$$\dfrac{\dfrac{x:0\in T\vdash x:0\in T \qquad \dfrac{y:\forall n(n\in T\to Sn\in T)\vdash y:\forall n(n\in T\to Sn\in T) \qquad z:n\in\mathbf{N}\vdash z:\mathbf{D}}{y:\forall n(n\in T\to Sn\in T),z:n\in\mathbf{N}\vdash zy:0\in T\to n\in T}\to E}{x:0\in T,y:\forall n(n\in T\to Sn\in T),z:n\in\mathbf{N}\vdash (zy)x:n\in T}\to E}{x:0\in T,y:\forall n(n\in T\to Sn\in T)\vdash \lambda z.(zy)x:n\in\mathbf{N}\to n\in T}\to I$$

□

Corollary 7.34. *The following derived rules hold.*

$$\dfrac{}{\Gamma\vdash \lambda yx.x:0\in\mathbf{N}}\,Zero \qquad\qquad \dfrac{\Gamma\vdash M:n\in\mathbf{N}}{\Gamma\vdash \lambda yz.y((My)z):Sn\in\mathbf{N}}\,Suc$$

$$\dfrac{\Gamma\vdash M:0=Sx}{\Gamma\vdash M(\lambda xy.(x(\lambda u.u))):\bot}\,Min$$

$$\dfrac{\Gamma\vdash N:A(0) \qquad \Gamma\vdash M:\forall n(A(n)\to A(Sn))}{\Gamma\vdash \lambda z.(z(\lambda t.Mt))N:\forall n(n\in\mathbf{N}\to A(n))}\,PropInd$$

$$\dfrac{\Gamma\vdash N:0\in\{x|A(x)\} \qquad \Gamma\vdash M:\forall n(n\in\{x|A(x)\}\to Sn\in\{x|A(x)\})}{\Gamma\vdash \lambda z.(zM)N:\forall n(n\in\mathbf{N}\to n\in\{x|A(x)\})}\,Ind$$

The final Peano Axiom, that if the successors of two numbers are equal so are those numbers, is a little more difficult to prove. This serves as a good example of the computational difficulties in working with the system and shows that working without computer assistance in NNST can be a difficult challenge.[8] We will start the following proof tracking λ-terms to demonstrate how quickly they become unwieldy. We will skip some uses of $\in$ and $\forall$; note these also don't alter the computed λ-terms. We will also use subscripts on λ-term variables to differentiate them between subproofs rather than choosing new letters.

Lemma 7.35. *Let n be any natural number and x any member of n. Then $n \not\subseteq x$.*

Proof. We proceed by induction. Let T be the set of n such that

$$n \in \mathbf{N} \wedge \forall x(x \in n \to (n \not\subseteq x)).$$

Then the base case is as follows.

$$\dfrac{\dfrac{}{\vdash \lambda y_0 x_0.x_0 : 0 \in \mathbf{N}}\,\text{Zero} \qquad \dfrac{\dfrac{\dfrac{\dfrac{z_0 : x \in 0 \vdash z_0 : x \in 0}{z_0 : x \in 0 \vdash z_0 : \bot}\,\in E}{z_0 : x \in 0 \vdash z_0 : n \not\subseteq x}\,\bot}{\vdash \lambda z_0.z_0 : x \in 0 \to n \subseteq x}\,\to I}{\vdash \lambda t_0.(t_0(\lambda y_0 x_0.x_0))(\lambda z_0.z_0) : 0 \in \mathbf{N} \wedge \forall x(0 \in \mathbf{N} \wedge x \in 0 \to n \not\subseteq x)}\,\wedge I$$

We refer to this proof as Base later.

Next assume

$$n \in T := \{n | n \in \mathbf{N} \wedge \forall x(x \in n \to (n \not\subseteq x))\},$$

for the inductive step. That $Sn \in \mathbf{N}$ will follow from the derived rule *Suc*, proved in Proposition 7.33. The second conjunct, that $\forall x(x \in Sn \to (n \not\subseteq x))$ is proved by cases from our assumption of $x \in Sn$. This gives that either $x \in n$ or $x \in \{n\}$. First we give the proof from $x \in n$. Due to the width of the proof tree, we split it up into subproofs. First we call this subproof Left.

$$\dfrac{\dfrac{}{\vdash \lambda t_1 x_1 y_1.x_1 t_1 : n \subseteq Sn}\,\text{SucSub} \qquad u : Sn \subseteq x \vdash u : Sn \subseteq x}{u : Sn \subseteq x \vdash \lambda v_1.u((\lambda t_1 x_1 y_1.x_1 t_1)v_1) : n \subseteq x}\,\text{Trans}$$

This following subproof is Right.

$$\dfrac{\dfrac{q : n \in T \vdash q : n \in T}{q : n \in T \vdash q(\lambda r_1 s_1.s_1) : x \in n \to n \not\subseteq x}\,\wedge E \qquad w_1 : x \in n \vdash w_1 : x \in n}{q : n \in T, w_1 : x \in n \vdash (q(\lambda r_1 s_1.s_1))w_1 : n \not\subseteq x}\,\to E$$

Then we complete the proof from $x \in n$ as follows.

$$\dfrac{\dfrac{\text{Left} \qquad \text{Right}}{q : n \in T, u : Sn \subseteq x, w_1 : x \in n \vdash ((q(\lambda r_1 s_1.s_1))w_1)(\lambda v_1.u((\lambda t_1 x_1 y_1.x_1 t_1)v_1)) : \bot}\,\to E}{q : n \in T, u : Sn \subseteq x, w_1 : x \in n \vdash ((q(\lambda r_1 s_1.s_1))w_1)(\lambda v_1.u(\lambda x_1 y_1.x_1 v_1)) : \bot}\,\text{Eval}$$

We notate this proof Disj-1. As is clear now, the final λ-term for this lemma will be quite complicated. We'll omit its computation for the remainder of the proof. Now the proof from $x \in \{n\}$. We first provide the following subproof to $x \in \{n\}, Sn \subseteq x \vdash n \in n$.

[8] cf. Appendix D

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash n = n}\,\text{Refl}}{\vdash n \in \{n\}}\in I}{\vdash n \in n \vee n \in \{n\}}\vee I_1}{\vdash n \in Sn}\in I
\qquad
\dfrac{Sn \subseteq x \vdash Sn \subseteq x \qquad \dfrac{x \in \{n\} \vdash x \in \{n\}}{x \in \{n\} \vdash x = n}\in E}{x \in \{n\}, Sn \subseteq x \vdash Sn \subseteq n} = E}{x \in \{n\}, Sn \subseteq x \vdash n \in n}\to E
$$

Then complete the proof from $x \in \{n\}$ as follows.

$$
\dfrac{\dfrac{x \in \{n\}, Sn \subseteq x \vdash n \in n \qquad n \in T \vdash n \in T}{n \in T, x \in \{n\}, Sn \subseteq x \vdash n \not\subseteq n}\to E
\qquad
\dfrac{}{\vdash n \subseteq n}\,\text{Refl}}{n \in T, x \in \{n\}, Sn \subseteq x \vdash \bot}\to E
$$

Notate this proof Disj-2. We will first show the final steps to the proof. We first have to apply a disjunction elimination.

$$
\dfrac{\dfrac{\dfrac{\dfrac{x \in Sn \vdash x \in Sn}{x \in Sn \vdash x \in n \vee x \in \{n\}}\in E \qquad \text{Disj-1} \qquad \text{Disj-2}}{x \in Sn, n \in T, Sn \subseteq x \vdash \bot}\vee E}{x \in Sn, n \in T \vdash Sn \subseteq x \to \bot}\to I}{n \in T \vdash x \in Sn \to Sn \not\subseteq x}\to I
$$

We also still need that $Sn \in \mathbf{N}$. We can get this by using the derived rule *Suc* from a conjunct resulting from $n \in T$.

$$
\dfrac{\dfrac{n \in T \vdash n \in \mathbf{N} \wedge \forall x(x \in n \to x \not\subseteq n)}{n \in T \vdash n \in \mathbf{N}}\wedge E}{n \in T \vdash Sn \in \mathbf{N}}\,\text{Suc}
$$

Conjunction introduction with these last two delivers that $Sn \in T$. Note that proof as Ind-Step. We use the derived rule of Ind to finish the proof.

$$
\dfrac{\text{Base} \qquad \text{Ind-Step}}{\vdash \forall n(n \in \mathbf{N} \to n \in T)}\,\text{Ind}
$$

□

Note that despite the apparent simplicity of the lemma, the λ-term quickly became difficult to handle by hand. We continue with the proof of this arithimetic axiom without computing λ-terms, i.e. we are working in normal NST without a type system.

Lemma 7.36. *Let n be any natural number and x any member of n. Then $x \subseteq n$.*

Proof. We will prove this with induction, similar to the previous lemma. Let T be the set of n such that

$$
n \in \mathbf{N} \wedge \forall x(x \in n \to (x \subseteq n)).
$$

The base case is as follows.

$$
\dfrac{\dfrac{}{0 \in \mathbf{N}}\,\text{Zero}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{x \in 0 \vdash x \in 0}{x \in 0 \vdash \bot}\in E}{x \in 0 \vdash x \subseteq n}\bot}{x \in 0 \to x \subseteq n}\to I}{0 \in \mathbf{N} \wedge \forall x(x \in 0 \to x \subseteq n)}
$$

Notate this as Base. Next we assume

$$n \in T := \{n | n \in \mathbf{N} \wedge \forall x(x \in n \to (x \subseteq n))\}$$

for the inductive step. The needed $Sn \in \mathbf{N}$ follows from the derived rule Suc. We will prove the second conjunct by cases that result from the definition of successor, i.e. either $x \in n$ or $x \in \{n\}$. First the proof from $x \in n$.

$$\dfrac{\dfrac{x \in n \vdash x \in n \qquad \dfrac{\dfrac{n \in T \vdash n \in \mathbf{N} \wedge \forall x(x \in n \to x \subseteq n)}{n \in T \vdash x \in n \to x \subseteq} \wedge E}{} }{x \in n \vdash x \subseteq n} \to E \qquad \dfrac{}{\vdash n \subseteq Sn}\,\text{SucSub}}{x \in n \vdash x \subseteq Sn}\,\text{Trans}$$

Notate this proof Disj-1. Next the proof from $x \in \{n\}$.

$$\dfrac{x = n \vdash x = n \qquad \dfrac{}{\vdash n \subseteq Sn}\,\text{SucSub}}{x = n \vdash x \subseteq Sn} = E$$

Notate this proof Disj-2. We now apply disjunction elimination.

$$\dfrac{\dfrac{\dfrac{x \in Sn \vdash x \in Sn}{x \in Sn \vdash x \in n \vee x \in \{n\}} \in E \qquad \text{Disj-1} \qquad \text{Disj-2}}{x \in Sn, n \in T \vdash x \subseteq Sn} \vee E}{n \in T \vdash x \in Sn \to x \subseteq Sn} \to I$$

We can form the full conjunction for $n \in T$ by using the derived rule Suc as noted before. Notate the full proof after forming the conjunction as Ind-Step. Then the proof is completed with Ind.

$$\dfrac{\text{Base} \qquad \text{Ind-Step}}{\vdash \forall n(n \in \mathbf{N} \to n \in T)}\,\text{Ind}$$

□

Corollary 7.37. *The following derived rules hold.*

Let $T := \{n | n \in \mathbf{N} \wedge \forall x(x \in n \to n \not\subseteq x)\}$.

$$\dfrac{}{\vdash \forall n(n \in \mathbf{N} \to n \in T)}\,\textit{Not-Subset}$$

Let $T := \{n | n \in \mathbf{N} \wedge \forall x(x \in n \to x \subseteq n)\}$.

$$\dfrac{}{\vdash \forall n(n \in \mathbf{N} \to n \in T)}\,\textit{NumTrans}$$

Proposition 7.38. $\vdash Sx = Sy \to x = y$.

Proof. This proof proceeds by cases. We first derive that from $Sx = Sy$ both $x \in Sy$ and $y \in Sx$. The former gives that we have either $x \in \{y\}$ or $x \in y$, and the latter gives either $y \in \{x\}$ or $y \in x$. This gives four cases of conjunctions. In three of those, we have either $x \in \{y\}$ or $y \in \{x\}$ which quickly imply $x = y$. The most problematic case is when we have $x \in y$ and $y \in x$. We use the previous lemmas to resolve this case.

We first demonstrate that $Sx = Sy \vdash x \in y \vee x = y$.

$$\dfrac{\dfrac{Sx = Sy \vdash Sx = Sy \qquad \dfrac{\dfrac{\dfrac{\vdash x = x}{\vdash x \in \{x\}}\in I}{\vdash x \in x \lor x \in \{x\}}\lor I_1}{\vdash x \in Sx}\in I}{\vdash x \in Sy}= E}{\vdash x \in y \lor x = y}\in E$$

Next we demonstrate $x \in y, y \in x \vdash x = y$. We first provide a subproof to $x \in y, y \in x \vdash y \not\subseteq x$.

$$\dfrac{\dfrac{\dfrac{y \in x \vdash y \in x \qquad \dfrac{}{\vdash y \in x \to y \in \{n|n \in \mathbf{N} \land \forall z(z \in n \to n \not\subseteq z)}\text{Not-Subset}}{\vdash y \in \{n|n \in \mathbf{N} \land \forall z(z \in n \to n \not\subseteq z)}}{\vdash x \in y \to y \not\subseteq x}\land E \qquad x \in y, y \in x \vdash x \in y}{x \in y, y \in x \vdash y \not\subseteq x}\to E$$

Then complete the proof to $x \in y, y \in x \vdash x = y$ as follows.

$$\dfrac{\dfrac{\dfrac{x \in y, y \in x \vdash y \in x}{\vdash y \subseteq x}\text{NumTrans} \qquad x \in y, y \in x \vdash y \not\subseteq x}{x \in y, y \in x \vdash \bot}\to E}{x \in y, y \in x \vdash x = y}\bot$$

The proof is then finished as follows.

$$\dfrac{Sx = Sy \vdash x \in y \lor x = y \qquad \dfrac{Sx = Sy \vdash y \in x \lor y = x \qquad x \in y, y \in x \vdash x = y \qquad y = x \vdash x = y}{x \in y \vdash x = y}\lor E \qquad x = y \vdash x = y}{Sx = Sy \vdash x = y}\lor E$$

□

Corollary 7.39. *The following derived rule holds.*

$$\dfrac{\Gamma \vdash Sx = Sy}{\Gamma \vdash x = y}\,SucInj$$

Thus we see that even the apparently simple statement can generate a lot of complexity in attempting to prove it. As we will soon be looking at these proofs as algorithms in a type theory, it may be beneficial to get away with a simpler notion of successor that can perform the same work. So we turn now to Zermelo's definition of successor and derive the necessary properties for it.

Definition 7.40. Define $Sx := \{x\} = \{z|z = x\}$.

Proposition 7.41. $\vdash 0 \neq Sx$

Proof. The proof follows.

$$\dfrac{\dfrac{\dfrac{t : 0 = Sx \vdash t : 0 = Sx \qquad \dfrac{\vdash \lambda u.u : x = x}{\vdash \lambda u.u : x \in Sx}\in I}{t : 0 = Sx \vdash t(\lambda u.u) : x \in 0}= E}{t : 0 = Sx \vdash t(\lambda u.u) : \bot}\in E}{\vdash \lambda t.t(\lambda u.u) : 0 = Sx \to \bot}\to I$$

□

Proposition 7.42. $\vdash Sx = Sy \to x = y$

Proof. The proof follows.

$$\cfrac{\cfrac{\vdash Sx = Sy \qquad \vdash x \in Sx}{\vdash x \in Sy} = E}{\vdash x = y} \in E$$

□

We have so far derived the axioms for a fragment of first-order arithmetic; we're missing addition and multiplication rules. We can get addition and multiplication for free if we can recover second-order arithmetic as that theory can define those operations internally [48, 30]. To recover second-order arithmetic we now need something equivalent to the comprehension schema used for those theories. One way to write the comprehension schema in these theories is the following: for a given predicate C where Z is not free in C,

$$\exists Z \forall n (n \in Z \leftrightarrow C).$$

Note there are the implicit restrictions that the $\exists$ on Z is some second-order quantifier, Z is a set and that the $\forall$ on n is a first-order quantifier and n is a number. This means that we can never write some $x \in n$, for n is not treated as a set.[9] Importantly this means we do not need to be concerned with certain comprehension instances that include circular naming and other fixpoint constructions.

Proposition 7.43. *Given some predicate C valid in second-order HA,*

$$\vdash \exists Z \forall n (n \in Z \leftrightarrow C).$$

Proof. The restrictions on the predicate C allow us to prove this with straightforward uses of $\in I$ and $\in E$. □

Our goal now is to show that we can normalize the deductions that are sequences of well-formed applications of our derived rules in NST so that we know NNST contains Heyting Arithmetic. Viewed from another perspective, we hope to take any proof in formal second-order arithmetic, convert the applications of rules in that system to ones of NST with all intermediate steps, and show that these proofs normalize.

Definition 7.44. The *representation of a derivation of HA in NST* takes a valid HA derivation and replaces all formulas of HA into their equivalent forms in NST, and replaces all rules in the derivation with the equivalent derived form in NST.

To prove this, we will use a translation into a strongly normalizing type theory, an extended System Fω. We prepare for this by first proving some simpler translation results with System F.

[9] An alternative comprehension schema directly uses predicates instead of sets. In this case it does not make sense to say the number n is a predicate for number x.

7.3.1 F-translation

The goal of F-translation is to find proofs of NST that translate into derivations in the second-order λ-calculus, System F ($\lambda 2$) [30, 36]. We first lay out the natural deduction system for Curry-style System F. To foreshadow the correspondence to come, I write Π-types as $\forall$ rather than the usual Π.

$$\frac{}{\Gamma, x : A \vdash x : A}\,\text{Var}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\,\text{Abst} \qquad \frac{\Gamma \vdash N : A \qquad \Gamma \vdash M : A \to B}{\Gamma \vdash MN : B}\,\text{Appl}$$

$$\frac{\Gamma \vdash M : A[B/X]}{\Gamma \vdash M : \forall X A}\,\text{Abst}_2 \qquad \frac{\Gamma \vdash M : \forall X A}{\Gamma \vdash M : A[B/X]}\,\text{Appl}_2$$

Abst_2 can only be applied if X is not a free type variable of Γ.

Inspection suggests that NST is "merely" some kind of an extension of System F. The connectives and types have the obvious correspondence and the rules of Abst/Appl and $\text{Abst}_2/\text{Appl}_2$ correspond precisely to $\to I/\!\to E$ and $\forall I/\forall E$. The only difference between the system is thus the $\in I$ and $\in E$ rules. Thus any NST deduction which doesn't use an $\in$ rule could be easily translated over to a System F deduction. The question is then what can we do, if anything, with proofs that do contain $\in$ rules?

First, we define the following translation of deductions of NST into types of System F.

Definition 7.45. The *F-translation of a formula* of NST is defined inductively by the following rules:

- $[\![a \in X]\!] = X$,
- $[\![a \in \{z | A(z)\}]\!] = [\![A]\!]$,
- $[\![\forall X.A]\!] = \forall X.[\![A]\!]$,
- $[\![A \to B]\!] = [\![A]\!] \to [\![B]\!]$.

Definition 7.46. The *F-translation of a rule* of NST is defined by the following rules:

- $[\![Ax]\!] = \text{Var}$,
- $[\![\to I]\!] = \text{Abst}$,
- $[\![\to E]\!] = \text{Appl}$,
- $[\![\forall I]\!] = \text{Abst}_2$,
- $[\![\forall E]\!] = \text{Appl}_2$,
- Both $\in I$ and $\in E$ are erased.

Definition 7.47. The *F-translation* of a deduction of NST first translates all formulas to types by the F-translation for formulas, then all rules of NST to rules of System F by F-translation for rules. Additionally, whenever $\in I$ or $\in E$ is erased, the conclusion of the rule is as well.

Note that the λ-terms formed during NST proofs will be equivalent to the λ-terms of the F-translated deduction. NST only differs in its $\in$ rules and these are not tracked in the λ-terms. We thus have a translation between NST deductions and System F deductions which share the same λ-terms. By the Subject Reduction Theorem (7.19) for NST and the strong normalization of System F, we can thus be guaranteed that any deduction that translates into a valid System F deduction will also normalize.

However, not all NST proofs will translate into valid System F deductions. Some uses of $\in$ rules are harmless but some will result in the F-translation losing necessary type information that makes any subsequent rules applicable.

Example 7.48. The use of $\in I$ on the formula $x = x$ can be used to get the formula $x \in \{z|z = x\}$. Despite the different formulas, the F-translation of both $x = x$ and $x \in \{y|y = x\}$ result in $\forall Z(Z \to Z)$.

Example 7.49. The use of $\in E$ on the formula $\{\emptyset\} \in \{x|\emptyset \in x\}$ will result in $\emptyset \in \{\emptyset\}$ where $\{\emptyset\} := \{x|x = \emptyset\}$. The F-translation of the former formula is simply a type variable X while the latter is $\forall Z(Z \to Z)$.

Thus there are some formulas where applying a $\in$ rule does not result in a change of the F-translation and some formulas where applying a $\in$ does.

Definition 7.50. A use of a $\in$ rule is said to be *invariant under F-translation* if the F-translation of the premise and the F-translation of the conclusion of the $\in$ rule are equivalent.

It turns out there is a nice sufficient condition for which formulas are invariant under F-translation.[10]

Definition 7.51. Let A and B be terms and consider the formula $B \in A$. We say that B or any subformula of B is in the *element position* in the formula $B \in A$ and A or any subformula of A is in the *set position* in the formula $B \in A$.

Proposition 7.52. *Let S and T be arbitrary terms, $B(z)$ be a term which contains z free and $A(z)$ a formula which contains no subformula of the form $S \in B(z)$, i.e. z does not occur in set position in the formula $A(z)$. Then a use of $\in I$ ($\in E$) which results in a conclusion (follows from a premise) of the form $T \in \{z|A(z)\}$ is invariant under F-translation.*

Proof. The F-translation of $T \in \{z|A(z)\}$ is the F-translation of $A(z)$. The formula that was the premise (or conclusion) would be $A[T/z]$. Thus, we need to determine whether $[\![A[T/z]]\!]$ is equivalent to $A(z)$.

By assumption, there are no instances of z which occur in set position. Thus any substitutions of T for z can only occur in the element position. By definition of F-translation, any terms in element position will not affect the resulting translation. $\square$

[10] It is not proved at present whether it is also necessary, that is, if we have some invariant step, it must be of the form specified in the proposition. It does however seem as if it would be provable; cooking up a counterexample has proven to be a challenge. Such a counterexample would require a z in set position which somehow did not change upon $\in E$ or $\in I$.

Thus, we can be sure that the F-translations of any deductions which fit this condition will result in valid System F deductions and thus that any NST deductions that satisfy this condition will also be proofs of NNST.

7.3.2 ω-translation

System F suffices to show the normalization of natural deduction systems of second-order Heyting Arithmetic [30] through a translation argument. However, there is some difficulty in using it to show that second-order Heyting Arithmetic is a normalizing fragment of NST. The proofs of some of the HA rules require $\in$ rule use on formulas which are not invariant under F-translation; the "algorithmic machinery" used to derive Peano's axioms and Leibniz identity in NST is more complex than can be captured in System F. Thus, we need to move to a stronger normalizing type theory to show the recapture of second-order Heyting Arithmetic; we use an extended System Fω, or Fω^+.[11] The extension is not that large in type theoretic terms: we take Fω and add one super-kind to allow a higher-order type which will admit induction.

We start with the natural deduction system for Fω below and then later extend to Fω^+ [36]. Note that we do use Π types here; the Π type will be used to translate both $\to$ and $\forall$. In the rules for Fω, we have $*$ which represents the type of types, and kinds which are types built on the type of types, e.g. $\Pi x : *.*$. These kinds are themselves of type $\Box$. The general layout is thus λ-terms which have types, these types in turn are expressions of λ-variables, which themselves may have types of expressions of $*$'s called kinds, which themselves have type $\Box$. These extra layers are essentially used to control what types can be constructed as well as to generate "type constructors" which generate types on input of other well-formed and appropriately typed expressions of the calculus.

$$\frac{}{\vdash * : \Box}\ \text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\ \text{Var}$$

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\ \text{Weak} \qquad \frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.B : s_2}\ \text{Form}$$

$$\frac{\Gamma \vdash M : \Pi x : A.B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]}\ \text{Appl}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.B : s}{\Gamma \vdash \lambda x : A.M : \Pi x : A.B}\ \text{Abst}$$

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\ \text{Conv}$$

Conv has the additional restriction that $B =_\beta B'$. In the above rules, s is either $*$ or $\Box$. The allowable pairs of s_1 and s_2 in the Form rule are $\langle *, * \rangle$, $\langle \Box, * \rangle$ and $\langle \Box, \Box \rangle$. We now specify a loose idea of the translation we use in what follows. This definition is primarily a guide post as we do often have to use a specified ω translation which breaks the regularity of the pattern.

[11] It may be the case that we can get away with a weaker type system, or maybe even System F. However, the method for this remains undiscovered.

Definition 7.53. The *unspecified ω-translation of a formula* of NST is defined inductively by the following rules. The types assigned to the variables are to be taken as the "default" specification unless otherwise specified:

- $[\![x \in z]\!] = ZX$,
- $[\![\{z|A(z)\}]\!] = \lambda Z.[\![A(z)]\!]$,
- $[\![B \in \{z|A(z)\}]\!] = ([\![\{z|A(z)\}]\!])[\![B]\!]$,
- $[\![\forall X.A]\!] = \Pi X.[\![A]\!]$,
- $[\![A \to B]\!] = \Pi X : [\![A]\!].[\![B]\!]$ where X is a fresh variable and $X : A$.

A *specified ω translation of a formula* of NST supplies the types of the variables and the full type expressions which result in the translation. Additionally, some formulas may be given *special translations* which do not exactly conform to these rules.[12]

Example 7.54. The ω-translation of an atomic membership statement, as in $[\![x \in z]\!] = ZX$ is an unspecified ω-translation. If we include that $Z : (\Pi Y : *.*)$ and $X : *$, then we have a specified ω-translation.

Remark 7.55. The ability to mark things as special translations merely because they do not translate to the type we might like may appear to make this definition worthless. However, the point of these definitions is not to provide a later theorem which states that all translations satisfy some sort of property. It is to show that we can provide translations of the derivations of NST operating with only the derived rules of HA which are valid derivations in Fω^+ so that we can use the normalization result. To that end, the translations are handy inductive definitions to have around to prove some things about, but the inductive definitions themselves will not suffice for the result and will absolutely need to be extended by special cases.

Definition 7.56. For $\in I$ and $\in E$, consider premises of the form $A(B)$ and $B \in \{z|A(z)\}$ respectively. The *ω-translation of a rule* of NST is defined by the following rules:

- $[\![Ax]\!] =$ Var,
- $[\![\to I]\!] =$ Abst,
- $[\![\to E]\!] =$ Appl,
- $[\![\forall I]\!] =$ Abst where the abstracted λ-variable corresponds to the NST formula variable that is being generalized,
- $[\![\forall E]\!] =$ Appl where the additional applied type corresponds to the NST formula variable being instantiated,
- $[\![\in I]\!] =$ Conv where an abstraction is applied so that the resulting type is $[\![B \in \{z|A(z)\}]\!]$,
- $[\![\in E]\!] =$ Conv where a reduction is applied so that the resulting type is $[\![A[B/z]]\!]$.

[12]These special translations can be seen as additional cases added to the inductive definition.

Contrasted with the ω-translation of a formula, the rule translation given must exactly correspond to what is defined here. This guarantees that the translated derivations of System Fω^+ will have the necessary corresponding normalization with the derivation it was translated from.

Definition 7.57. The *ω-translation* of a deduction of NST first translates all formulas to types by a specified ω-translation for formulas, then all rules of NST to rules of System Fω^+ by ω-translation for rules. The ω-translation is further extended by the necessary rules to show the types used in the resultant System Fω translation are well-formed, if such an extension is possible. An ω-translation may also be *special* if it does not conform to the prescribed inductive rules.

We will simply say translation from now on in this section. We always refer to an ω-translation unless otherwise stated and it shall be clear from context whether it is unspecified or specified.

By using ω-translation we gain a significant amount of algorithmic expressivity not accessible by System F. Most notably, the terms in element position are not lost in translation. This will allow us to represent the content of Leibniz identity and the proofs of Peano's axioms as they are in NST. That this is now possible arises from the technical fact that we can now make sense of some applications of $\in I$ and $\in E$ as particular applications of the Conv rule. This increased expressivity comes with the cost that we have no guarantee that the the translated deductions are valid deductions of Fω^+. In fact, they almost certainly are not as Fω^+ has rules devoted to constructing well-formed and valid types, also called *legal* types, whereas in NST there is no such concern; the rules of NST act on formulas that are implicitly assumed to be well-formed.

Definition 7.58. An Fω^+ expression M is *legal* if there exists a valid deduction such that $\Gamma \vdash M : A$ for some type/kind A.

Nearly every translation we look at will need to first be extended by subproofs which show the translated types to be legal. No general method or sufficient condition has been determined that shows which NST deductions ultimately translate to valid Fω^+ trees which are extensible to full Fω^+ derivations. That is, thus far I know of no condition which guarantees that the resultant translated Fω^+ trees are capable of being extended with the addition of the rules that take place prior to Var and other rules like Form which need preceding trees to show the types used are well-formed.[13] Without such a method, the only way we can proceed is to use brute-force and case-by-case constructions of the types and derivations needed in our translations.

However, aside from lacking such a general method, the overall strategy for the proof of this chapter is essentially a more involved version of the strategy exhibited in the previous section on F-translation. The short version is that we show every logical formula of NST that can be constructed out of the HA fragment translates into legal types and kinds of a fragment of Fω^+. This is the "brute force" in what follows. This translation works such that the normalization

[13]Such a method seems possible but is not known.

of the λ-terms of those types and kinds guarantees the normalization of the λ-terms for our proofs in NST. As $\mathrm{F}\omega^+$ is itself strongly normalizing [32], and we have our Subject Reduction Theorem for NST, we have that the proofs in this fragment of NST will normalize, and thus suffice to also be proofs of NNST.

While the general strategy is simple enough, we must take care to construct deductions that show our λ-expressions are legal. In the proofs that follow which are concerned with proving legality, we are primarily doing this to either establish that the specified *translation* of an NST logical formula into a type expression of $\mathrm{F}\omega^+$[14] is legal and can be handled in $\mathrm{F}\omega^+$ or it may be the case that we need to prove that the *type* of a type expression is legal before we can show the legality of the desired type expression which is itself translating an NST formula. Again, note there are no analog proofs of legality in NST essentially due to the fact that we assume in logic that all of formulas are well-formed according to some (easily) specified inductive construction. Now on to the proofs.

Proposition 7.59. $\Pi x : *.*$ *is a legal type.*

Proof. The $\mathrm{F}\omega^+$ derivation follows.

$$\dfrac{\vdash * : \Box \qquad \dfrac{\vdash * : \Box \qquad \vdash * : \Box}{X : * \vdash * : \Box}\,\text{Weak}}{\vdash \Pi X : *.* : \Box}\,\text{Form}$$

□

We now use the shorthand $X \to Y := \Pi Z : X.Y$ when Z does not occur free in Y. The introduction of the fresh variable X in the previous proof by first an application of Var and then Weak is commonly used in constructions of $\to$ types and will usually be omitted for brevity. We simply use a shorthand version of the Form rule in these cases.

Definition 7.60. Let A be some valid type expression using $*$ and $\to$ and let (A) denote A with its outer parenthesis, if it would have them.[15] We define the "digging" function D as follows:

$$D(A) = (A) \to *.$$

Let $D^n(A)$ represent n applications of D to a type expression A.

Example 7.61. The first few outputs of D on $*$ are:

- $D(*) = * \to *$
- $D(* \to *) = (* \to *) \to *$
- $D((* \to *) \to *) = ((* \to *) \to *) \to *$

Also note that $D^{n+1}(*) := D^n(*) \to *$.

Proposition 7.62. *For any natural number n, $D^n(*)$ is a legal type of kind $\Box$.*

[14] Expressions which are composed of λ-variables, $*$ and $\Box$.

[15] This is technically unnecessary since not having outer parenthesis is merely a reading convenience.

Proof. Proved by induction using the Sort rule and the Weak rule with the assumed inductive hypothesis that $D^{n-1}(*)$ is legal. The base case has already been done in Proposition 7.59. The inductive step follows.

$$\dfrac{\vdash * : \square \qquad \dfrac{\vdash * : \square \qquad \vdash D^{n-1}(*) : \square}{X : D^{n-1}(*) \vdash * : \square}\text{Weak}}{\vdash \Pi X : D^{n-1}(*).* : \square}\text{Form}$$

$\square$

Proposition 7.63. *If $Z : * \to *$ and $X : *$ then $ZX : *$ and is legal.*

Proof. One use of Appl gives the derivation. $\square$

Recall that when we defined absurdity and the other logical connectives we used an arbitrary closed term. We will specify this term for the purposes of HA as $U := \{x|x = x\}$. This does not alter the proofs of the properties of those connectives but fixing this set will be necessary for our translation result.

Proposition 7.64. *The translation of the set term $U := \{x|x = x\}$ is a legal type.*

Proof. The translation of this set term is $\lambda X : *.\Pi Z : * \to *.ZX \to ZX$. We omit in the following lines the reproduction of the tree that $ZX : *$ for the first Form rule, that $Z : * \to *$ in the second Form rule, and the tree that shows $* \to *$ is legal in the Abst rule.

$$\dfrac{\dfrac{\dfrac{X : *, Z : * \to *, ZX : * \vdash ZX : *}{X : *, Z : * \to * \vdash ZX \to ZX : *}\text{Form}}{X : * \vdash \Pi Z : * \to *.ZX \to ZX : *}\text{Form}}{\vdash \lambda X : *.\Pi Z : * \to *.ZX \to ZX : * \to *}\text{Abst}$$

$\square$

Proposition 7.65. *The translation of the set term for 0 is a legal type and its kind can be legally specified as $D^n(*)$ for any $n \geq 1$.*

Proof. We proceed by induction. First we show that the translation of 0 is legal. The translation of this term is $\lambda Z : *.\Pi X : * \to *.X(\lambda B : *.\Pi C : * \to *.CB \to CB) : * \to *$.

$$\dfrac{\dfrac{\dfrac{Z : *, X : (* \to *) \to * \vdash X : (* \to *) \to * \qquad \vdash \lambda B : *.\Pi C : * \to *.CB \to CB : * \to *}{Z : *, X : (* \to *) \to * \vdash X(\lambda B : *.\Pi C : * \to *.CB \to CB) : *}\text{Appl}}{Z : * \vdash \Pi X : (* \to *) \to *.X(\lambda B : *.\Pi C : * \to *.CB \to CB) : *}\text{Form}}{\vdash \lambda Z : *.\Pi X : (* \to *) \to *.X(\lambda B : *.\Pi C : * \to *.CB \to CB) : * \to *}\text{Abst}$$

Note this proof requires no particular kind for Z as long as its legal. Thus, this construction works for any arbitrary $D^n(*)$ with $n \geq 1$ as desired. $\square$

Proposition 7.66. *Let m be a natural number ≥ 1. Then the set term that results from n applications of the successor function to 0, $S^n(0)$, translates to a legal type and its kind can be legally specified as $D^m(*)$ for any $m \geq n + 1$.*

Proof. We proceed by induction. Our base case is the previous proposition, 7.65.

Assume that $[\![S^n(0)]\!]$ is a legal type and can be specified as any kind $D^m(*)$ such that $m \geq n+1$. We wish to show that $[\![S^{n+1}(0)]\!]$ is legal. Fix an $m \geq n+2$. We resolve the translation one step to find that

$$[\![S^{n+1}(0)]\!] := \lambda Y.\Pi Z.ZY \to Z[\![S^n(0)]\!].$$

This suggests that we simply need that $Z : D^{m+1}(*)$ and $Y : D^m(*)$. The shorthand derivation follows.

$$\dfrac{\dfrac{\dfrac{Y : D^m(*), Z : D^{m+1}(*) \vdash ZY : * \qquad x : ZY \vdash Z[\![S^m(0)]\!] : *}{Y : D^m(*), Z : D^{m+1}(*) \vdash ZY \to Z[\![S^m(0)]\!] : *}\text{Form}}{Y : D^m(*) \vdash \Pi Z : D^{m+1}(*).ZY \to Z[\![S^m(0)]\!] : *}\text{Form}}{\vdash \lambda Y : D^m(*).\Pi Z : D^{m+1}(*).ZY \to Z[\![S^m(0)]\!] : D^{m+2}(*)}\text{Abst}$$

Thus for any given valid n we can construct the kind of $[\![S^{n+1}(0)]\!]$ as desired, also showing it is a legal type. $\square$

We have shown that the representation of natural numbers in our set theory translates to an entire sequence of legal types. We now add the notation that $[\![n]\!]_m$ for any $m \geq n+1$ is denoting the the specified translation of n which has kind $D^m(*)$.

To continue, we need to introduce the machinery that changes our type theory from Fω to Fω^+. This can be seen as a small fragment of the Extended Calculus of Constructions [36, 32]. We introduce a super kind, $\square_S$ which is such that $\square : \square_S$. This will allow us to define variables of $\square$ type. We first add this typing relationship as another rule.

$$\dfrac{}{\vdash \square : \square_S}\text{Sort}$$

We then add two rules which define a sort of cumulativity relationship between $*$, $\square$ and $\square_S$.

$$\dfrac{\Gamma \vdash A : *}{\Gamma \vdash A : \square}\text{C} \qquad\qquad \dfrac{\Gamma \vdash A : \square}{\Gamma \vdash A : \square_S}\text{C}$$

We also slightly extend the Form rule.

$$\dfrac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.B : s_2}\text{Form}$$

The following pairs of s_1 and s_2 are allowed: $\langle *, * \rangle$, $\langle \square, * \rangle$, $\langle \square, \square \rangle$, $\langle \square_S, \square \rangle$, $\langle \square_S, \square_S \rangle$. As this system is a fragment of the Extended Calculus of Constructions, it strongly normalizes [32]. This is needed to translate $\mathbf{N}$ and induction. We now specify and prove legal a sequence of specified types for $[\![\mathbf{N}]\!]$.

Proposition 7.67. *Fix $m \geq 1$. Let $Y : D^m(*)$, $X : D^m(*)$, and $\alpha : (\Pi T : \square.T \to *)$. Define $[\![N]\!]_m$ as*

$$\lambda X.\Pi\alpha.(\Pi Y.((\alpha D^m(*))Y \to (\alpha D^{m+1}(*))[\![Sy]\!])) \to ((\alpha D^m(*))[\![0]\!]_m \to (\alpha D^m(*))X).$$

This is a legal type.

Proof. We first derive that the type of α is legal.

$$\dfrac{\vdash \Box : \Box_S \qquad \dfrac{\dfrac{\vdash \Box : \Box_S}{T : \Box \vdash T : \Box}\,\text{Var} \qquad \dfrac{\vdash * : \Box}{\vdash Z : T \vdash * : \Box}\,\text{Weak}}{T : \Box \vdash T \to * : \Box}\,\text{Form}}{\vdash \Pi T : \Box . T \to * : \Box}\,\text{Form}$$

Given the legality of this type, we can form the variable α of this type. Now note that $(\alpha D^m(*)) = D^{m+1}(*)$ for any $m \geq 0$. Then the expressions $(\alpha D^m(*))Y$, $(\alpha D^{m+1}(*))[\![Sy]\!])$, $(\alpha D^{m+1}(*))0$, and $(\alpha D^{m+1}(*))X$ can be seen to be legal by an application of Appl. Then the construction of the needed $\to$ types follow from a few uses of Weak to introduce fresh variables and accompanying uses of Form. $\Box$

This construction fits nicely with Proposition 7.66 which shows that any number n has a "base" translation type of at least $D^{n+1}(*)$ but can also be given a type for any m past that. Applying this to our translation of $\mathbf{N}$, we have a sequence of $[\![\mathbf{N}]\!]_m$'s, each of which can legally express "membership" of the translation of all numbers up to but not including m. We can take the numbers n up to but not including m with their special translation of $[\![n]\!]_m$. For example, we can find that the translations of both 0 and $S0$ can be legally applied to $[\![\mathbf{N}]\!]_2$ constructed where the type of X is $D^2(*)$ by taking $[\![0]\!]_2 : D^2(*)$ and $[\![S0]\!]_2 : D^2(*)$.

We can do better than this translation for the natural numbers and give a $[\![\mathbf{N}]\!]$ a sort of master type. To demonstrate the idea behind this, we first provide an updated specified type for the set term $\{x|x = x\}$ which utilizes the extended machinery.

Proposition 7.68. *Let $T : \Box$, $X : T$, and $Z : T \to *$. Define a special translation of $U := \{x|x = x\}$ as*

$$\lambda T . \lambda X . \Pi Z . ZX \to ZX.$$

This is legal and of kind $\Pi T : \Box . T \to *$.

Proof. We first derive that $T \to * : \Box$.

$$\dfrac{\dfrac{\vdash \Box : \Box_S}{T : \Box \vdash T : \Box}\,\text{Var} \qquad \dfrac{\vdash * : \Box \qquad \dfrac{\vdash \Box : \Box_S}{T : \Box \vdash T : \Box}\,\text{Var}}{y : T \vdash * : \Box}\,\text{Weak}}{T : \Box \vdash T \to * : \Box}\,\text{Form}$$

We next derive that $ZX \to ZX : *$.

$$\dfrac{\vdash ZX : * \qquad \dfrac{\dfrac{\vdash T \to * : \Box}{Z : T \to * \vdash Z : T \to *}\,\text{Var} \qquad \dfrac{T : \Box \vdash T : \Box}{T : \Box, X : T \vdash X : T}\,\text{Var}}{y : ZX \vdash ZX : *}\,\text{Appl}}{T : \Box, Z : T \to *, X : T \vdash ZX \to ZX : *}\,\text{Form}$$

We finally finish with the derivation of the desired type.

$$\dfrac{\dfrac{\dfrac{T : \Box \vdash T \to * : \Box \qquad T : \Box, Z : T \to *, X : T \vdash ZX \to ZX : *}{T : \Box, X : T \vdash \Pi Z : T \to * . ZX \to ZX : *}\,\text{Form}}{T : \Box \vdash \lambda X : T . \Pi Z : T \to * . ZX \to ZX : *}\,\text{Abst}}{\vdash \lambda T : \Box . \lambda X : T . \Pi Z : T \to * . ZX \to ZX : \Pi T : \Box . T \to *}\,\text{Abst}$$

$\Box$

From here, we can get the old translation of U back by using Appl with $*$.[16] We now also generalize on $\mathbf{N}$ in the same way.

Proposition 7.69. *Let $R : \Box$, $Y : R$, $X : R$ and $\alpha : (\Pi T : \Box . T \to *)$. Define $[\![\mathbf{N}]\!]$ as*

$$\lambda R.\lambda X.\Pi\alpha.(\Pi Y.((\alpha R)Y \to (\alpha(R \to *))[\![Sy]\!])) \to ((\alpha R)[\![0]\!] \to (\alpha R)X).$$

This is a legal type.

Proof. The proof is similar to before as in Proposition 7.67. The only difference is that (αR) now results in a type of $R \to *$. Now that Y's type is set to R as well, this means that $(\alpha R)Y$ is also legal. The rest of the proof is extended in the same way. $\Box$

We can now begin ensuring that the proofs we have derived in NST for HA also translate into Fω^+.

Proposition 7.70. *All types that result from the translations of the proofs of the reflexivity, symmetry and transitivity of Leibniz identity are legal and the translations of the derivation themselves are legal.*

Proof. The translations only require extending the proofs so that we include the derivation to show the types involved are legal. These are all straightforward derivations. We provide the translated derivation of the proof of reflexivity for Leibniz identity as an example.

$$\dfrac{\dfrac{\dfrac{Z : * \to *, X : * \vdash ZX : *}{Z : * \to *, X : *, t : ZX \vdash t : ZX}\text{Var}}{Z : * \to *, X : * \vdash \lambda t : ZX.t : ZX \to ZX}\text{Abst}}{X : * \vdash \lambda Z : * \to *\lambda t : ZX.t : \Pi Z : * \to *.ZX \to ZX}\text{Abst}$$

$\Box$

Lemma 7.71. *All formulas of first-order HA as expressed in NST can be translated into legal types of Fω^+.*

Proof. We prove this by induction on the construction of logical formula. Our base case requires us to show that we can translate the atomic formulas of first-order HA: $a = b$ for arbitrary natural numbers a and b. We further specify that the translation must be of kind $*$. The translation of $a = b$ is of the form

$$\Pi Z.ZA \to ZB.$$

This type is legal as long as the kinds of A and B are equal. By Proposition 7.66, we know that we can choose types for each A and B such that they are equal. To do so, we take the larger of the two numbers, say it's a and specify its kind as the smallest possible iteration of the digging function, i.e. $D^{a+1}(*)$. Then since b is smaller, and by Proposition 7.66, we can specify its kind as $D^{a+1}(*)$ as well. That the translation will be of kind $*$ is shown the same as

[16] Despite this construction, the original translation of U is still used to define 0. Also, since we never treat numbers as sets in the context of HA, there is no need to extend their translation, though it is possible to do so.

any other translation of an equality formula, as in the proof of Proposition 7.64. Thus $a = b$ for any a and b translates to a legal type of kind $*$.

For the inductive step we consider the construction of formulas of first-order HA as represented in NST using $\to$ and $\forall$. In the former case, we assume we have A and B that translate to legal types of kind $*$ and we want to derive that the type $A \to B$ is legal. For the latter case, we have the translation of an A is a legal type of kind $*$ and we want to derive the type $\forall n A$ is legal.

For the translation of $A \to B$, we already have by the inductive hypothesis that A is of kind $*$. We can thus immediately use Weak with $[\![A]\!]$ and then Form to show the desired type is legal and, since B is also of kind $*$, the kind of the translation will be $*$.

For $[\![\forall X A]\!] := \Pi X.[\![A]\!]$, either X occurs in the context of the derivation of $[\![A]\!]$'s legality or not. If it does, then we have something of the form $\Gamma, X : K \vdash [\![A]\!] : *$ and we can use Form to get to $[\![\forall X A]\!] : *$ as desired. If X does not occur in the context of the derivation of $[\![A]\!]$, then we first use Weak followed by Form.

The representation of all formulas of first-order HA in NST can be composed out of these basic constructions and thus we are done. $\square$

Lemma 7.72. *Consider the theory that has as atomic formulas $a = b$ where a and b are natural numbers and $n \in Z$ where n is a natural number and Z is a set variable, and is closed under the well-formed constructions of the logical connectives $\forall, \to, \wedge, \vee, \exists$. The translation of these formulas as they are represented in NST are legal types of kind $*$.*

Proof. We seek to extend our proof of Lemma 7.71. We proceed by induction as before, to show that the translations are legal and of kind $*$. Our base cases are now $a = b$ and $n \in Z$. The former is the same. As for the latter, we know that $[\![n]\!] : D^m(*)$ by Proposition 7.66. Then we simply take Z as $D^{m+1}(*)$ and use Appl for the desired proof of legality and to show the kind is $*$.

The rest of the inductive steps of Lemma 7.71 made no assumptions beyond assuming the base case was of kind $*$ and so the rest of the proof works as before. $\square$

Lemma 7.73. *Let n be a number and $A(n)$ be a formula of the form previously specified in Lemma 7.72.*[17] *Then the special translations of the set term $[\![\{n|A(n)\}]\!] := \lambda T.\lambda n.[\![A(n)]\!]$ and the membership formula $m \in \{n|A(n)\} := ([\![\{n|A(n)\}]\!]D^l(*))[\![m]\!]_l$ are legal for $l \geq m+1$. The former is of kind $\Pi T : \Box.\Pi n : T.*$ and the latter is of kind $*$.*

Proof. We first show that the translations of the set terms $\{n|A(n)\}$ are legal, where $A(n)$ is as specified, n is number and n is either free or does not occur in $A(n)$. By definition,

$$[\![\{n|A(n)\}]\!] = \lambda n.[\![A(n)]\!].$$

To show this is legal, we first need to determine that the kind of the translation is legal. This kind will be of the form $\Pi n.B$. We know that $[\![A(n)]\!]$ will be of kind $*$, and since $[\![n]\!]$ is

[17] The translation of $A(n)$ is thus legal and of kind $*$.

assumed to be a number, we know that n will be of kind $D^m(*)$ for some m which is of kind $\Box$. Thus the kind of this term could be given as $\Pi T.\Pi n : T.* = \Pi T.T \to *$. This kind is legal by Proposition 7.68. The derivation which yields this is as follows, from the assumed legality of $[\![A(n)]\!]$.

$$\dfrac{\dfrac{T : \Box, n : T \vdash [\![A(n)]\!] : *}{T : \Box \vdash \lambda n.[\![A(n)]\!] : T \to *}\text{Abst}}{\vdash \lambda T.\lambda n.[\![A(n)]\!] : \Pi T.T \to *}\text{Abst}$$

The derivation of $[\![m \in \{n|A(n)\}]\!]$ then results from two uses of Appl, first applying $D^l(*) : \Box$ and then applying $[\![m]\!]_l : D^l(*)$. The resulting kind will be $*$ as desired. $\Box$

Theorem 7.74. *The representation of all formulas of second-order HA in NST translate to legal types of kind $*$.*

Proof. We first observe that since the membership statements that are of the form specified in Lemma 7.73 are of kind $*$, these too can be composed with logical connectives along with first-order HA formulas as in Lemma 7.71. And further, since the resulting kind will be $*$, these can again be abstracted in to set terms in the same way as the proof of Lemma 7.73.

This gets us close to all of the formulas of HA, but not quite. What we now have is that any formula A which involves any formula of first-order HA, any atomic membership formulas of the form $n \in Z$, and any membership formulas with set terms that are constructed out of these former two formulas are valid. What we need is "two more levels of abstractions": that we can form set terms and then membership formula out of any other already well-constructed formula of HA, at least as long as we are not violating our restrictions for second-order HA that natural numbers only occur in element position and sets only occur in set position, and we need the legality of any well-formed formula involving any of these constructions composed with $\to$ or $\forall$. But these next level of abstractions can proceed in the exact same way since we still have that the required formulas will be of kind $*$, the only restrictions needed for our methods used in our proofs of our Lemmas to work. $\Box$

This theorem is necessary so that we know we can legally form any set term we might want to use to perform intersubstitution on two identical numbers, and so that we know that every instance of Induction that would be well-formed in second-order HA will still translate to a legal type. We also need this for conclusions from an instance of a $\perp$ rule.

Proposition 7.75. *The translation of the formulas involved in proving Leibniz identity substitution are legal types and the derivation itself is valid.*

Proof. Given Theorem 7.74, we know that the arbitrary formula A used in intersubstitution will be legal. The rest of the formulas are known to translate to legal types already and the derivation itself is straightforward to translate. $\Box$

Proposition 7.76. *The following specified translations of the derived rules in NST of Zero and Suc are legal and the translated derivations are valid.*

- *Let $m \geq 0$. Then specify the translation for the formulas of Zero as $[\![0 \in N]\!] := [\![N]\!]_m [\![0]\!]_m$.*
- *Let x be a natural number and $m \geq x+1$. Then specify the translation for the formulas of Suc as $[\![x \in N]\!] := [\![N]\!]_m [\![x]\!]_m$ and $[\![Sx \in N]\!] := [\![N]\!]_m [\![Sx]\!]_m$.*

Proof. For the legality of $[\![0 \in \mathbf{N}]\!]$, we need only note that Proposition 7.66 and Proposition 7.67 show that the individual types are valid, and then a use of Appl will give the legality of the desired type. An abbreviated version of the translated derivation of Zero can be found in the figure on the next page.

For the legality of $[\![x \in N]\!]$ and $[\![Sx \in N]\!]$, these also follow from Propositions 7.66 and 7.67 and a use of Appl. The translation of the derivation of Suc is also similar to the translation of Zero. □

$$\begin{array}{cl}
\alpha : \Pi T : \Box.T \to *, y : \Pi Y.((\alpha D^{m}(*))Y \to (\alpha D^{m+1}(*))[\![SY]\!]), x : (\alpha D^{m+1}(*))[\![0]\!]_m \vdash x : (\alpha D^{m+1}(*))[\![0]\!]_m & \\
\hline
\alpha : \Pi T : \Box.T \to *, y : \Pi Y.((\alpha D^{m}(*))Y \to (\alpha D^{m+1}(*))[\![SY]\!]) \vdash \lambda x.x : (\alpha D^{m+1}(*))[\![0]\!]_m \to (\alpha D^{m+1}(*))[\![0]\!]_m & \text{Abst} \\
\hline
\alpha : \Pi T : \Box.T \to * \vdash \lambda y.\lambda x.x : (\Pi Y.((\alpha D^{m}(*))Y \to (\alpha D^{m+1}(*))[\![SY]\!])) \to (\alpha D^{m+1}(*))[\![0]\!]_m \to (\alpha D^{m+1}(*))[\![0]\!]_m & \text{Abst} \\
\hline
\vdash \lambda\alpha.\lambda y.\lambda x.x : \Pi\alpha.(\Pi Y.((\alpha D^{m}(*))Y \to (\alpha D^{m+1}(*))[\![SY]\!])) \to (\alpha D^{m+1}(*))[\![0]\!]_m \to (\alpha D^{m+1}(*))[\![0]\!]_m & \text{Form} \\
\hline
\vdash \lambda\alpha.\lambda y.\lambda x.x : (\lambda X.\Pi\alpha.(\Pi Y.((\alpha D^{m}(*))Y \to (\alpha D^{m+1}(*))[\![SY]\!])) \to (\alpha D^{m+1}(*))[\![0]\!]_m \to (\alpha D^{m+1}(*))[\![X]\!])[\![0]\!]_m & \text{Conv}
\end{array}$$

Figure 7.1: Translated Derivation of Zero

Proposition 7.77. *The $\perp$ rule translates to a valid rule of $F\omega^+$.*

Proof. Proposition 7.65 already includes a proof of the legality of $[\![\perp]\!]$. The $\perp$ rule itself, which is a use of $\forall E$ and $\in E$ and translates as well.

$$\dfrac{\dfrac{\vdash t : \Pi X.X[\![U]\!]}{\vdash tu : [\![\{y|A(y)\}]\!][\![U]\!]}\text{Appl}}{\vdash tu : [\![A(y)]\!][[\![U]\!]/[\![y]\!]]}\text{Conv}$$

□

Proposition 7.78. *The translations of the derived rules in NST of Min and SucInj are legal.*

Proof. That $[\![0 = Sx \to \perp]\!]$ is legal follows from Propositions 7.66 and 7.65, and a use of Weak and Form. That $[\![Sx = Sy]\!]$ and $[\![x = y]\!]$ are legal is similar.

That their derivations translate is straightforward. □

Proposition 7.79. *Let $T : \Box$, $X : T$, and $l \geq 1$. The following translations of the formulas of the derived rule of NST of Ind are legal:*

- $[\![0 \in \{x|A(x)\}]\!] :=$
$$((\lambda T.\lambda X.[\![A(x)]\!])D^l(*))[\![0]\!]_l,$$
- $[\![\forall n(n \in \{x|A(x)\} \to Sn \in \{x|A(x)\}]\!] :=$
$$\Pi n.((\lambda T.\lambda X.[\![A(x)]\!])D^l(*))n \to ((\lambda T.\lambda X.[\![A(x)]\!])D^{l+1}(*))[\![Sn]\!],$$
- $[\![\forall n(n \in \mathbf{N} \to n \in \{x|A(x)\})]\!] :=$
$$\Pi n.(([\![N]\!]D^l(*))n \to ((\lambda T.\lambda X.[\![A(x)]\!])D^l(*))n.$$

Additionally, the translated derivation of Ind is valid.

Proof. We already know all of the necessary types to be legal. We also need to check the translation of the derivation of Ind. The only remaining difficult part of the derivation that has not been shown is that the $\forall E$ on our assumption that $n \in \mathbf{N}$ translates. That is from $n \in \mathbf{N}$ in the proof of Ind we can use $\forall E$ to instantiate α to get $\forall y(y \in T \to Sy \in T) \to (0 \in T \to n \in T)$.

By Theorem 7.74, we know that set terms will translate such that they are of kind $\Pi T : \Box.T \to *$. We also have our translation $[\![\mathbf{N}]\!]$ as specified in Proposition 7.69 which specifies the translation of α as type $\Pi T : \Box.T \to *$. Thus it can accept any set term as is needed. □

Proposition 7.80. *Any valid derivation of HA represented in NST can be translated into $F\omega^+$.*

Proof. We have that every derived rule of HA in NST can be translated as desired. The remaining detail is that sequences of derived rules of HA in NST can be translated into valid deductions. This follows immediately from noting that the translations of the conclusions of the derived rules does not deviate from the necessary translated form so that these derivations can be chained together as would be valid in a regular HA proof. □

Lemma 7.81. *If a λ-term normalizes that results from the translated derivation of an HA proof of NST, then the derivation of NST itself normalizes.*

Proof. We describe the normalization algorithm that will resolve the normalization of the NST proofs. First, we can be guaranteed by the translations of the rules that any non-normal step in the translation is the result of a non-normal step in the original NST proof. We first take any instances in the NST proof of non-normal $\in$ steps, an $\in I$ rule followed by an $\in E$ rule. We normalize all such instances. We then find the translation of this new proof, which will be as before but with fewer Conv instances. Then we resolve the non-normality of the translated derivation that results from translations of $\forall I$ and $\forall E$ steps. We can calculate these reductions directly in the λ-term and then recover the altered Fω^+ deduction which contains fewer instances of these Abst and Appl. At each step of these reductions, we normalize the corresponding $\forall I$ and $\forall E$ rules where these non-normalities originated. Then we move on to normalizing any Abst and Appl instances in our translated derivation that resulted from $\rightarrow I$ followed by $\rightarrow E$. As we reduce these steps, we also reduce the steps in the NST λ-term and recover the new NST deduction that results.

At the end of these three steps, we may have introduced new non-normal steps of any type, and thus may have more reduction that needs to be performed. We thus repeat this process as often as it needs to be done. We know that this process must be finite, as we are also reducing the translated derivation of Fω^+ as well and we know that Fω^+ is strongly normalizing: any sequence of reductions must eventually reach a normal derivation. Further, there can be no remaining non-normal steps in the NST deduction, for if there were, there would be a corresponding non-normal step in the translated derivation. Thus, once the Fω^+ derivation is normal we must also have a normal NST derivation. $\square$

Thus we finally have the desired result of this chapter.

Theorem 7.82. *NNST contains all theorems of second-order HA as translated into NST.*

Proof. The Subject Reduction theorem of our type system for NST in conjunction with Lemma 7.81 and the results of this section showing that the formulas translate complete the result. $\square$

7.4 Summary of the Proof

We now recap the steps of the proof. Our first goal is showing that derivations of Peano's axioms translate into valid derivation of Fω^+. To do this, we must know that the formulas involved in Peano's axioms translate to valid types of Fω^+. Thus we prove the legality of the specified and unspecified translations of Peano's axioms. To accomplish this, we must prove that every possible formula of second-order HA translates to a legal type, since we have two rules, substitution and induction, which can operate on arbitrary formulas. This culminates in Theorem 7.74.

Now that we know the translated types are legal, we can verify that the deductions involving these types, i.e. the rules of arithmetic and Leibniz identity, themselves translate to valid deductions. This part is relatively less difficult, as we have no flexibility in how the rules of our derivations in NST are translated. And we can have no such flexibility for otherwise the crucial Lemma 7.81 would not hold. This Lemma is so crucial because up to this point, we only have the legality of the types of the translated formulas and the correctness of the resulting translated deductions of our rules of HA but we do not have a guarantee that the normalization of these derivations in Fω^+ properly correspond to the normalization of our NST proofs. This Lemma shows that we set up the translation from the beginning such that this is the case. And thus we have means to know any of our NST proofs about HA will themselves normalize, and thus we have theorems of NNST.

Chapter 8

Conclusions and Open Questions

The development in the previous chapters show the viability of NNST as a foundational system for mathematics. What remains to be done is to explore the myriad of questions and implications brought forth by this system. Below, I indicate some possible areas of development

8.1 Recovery of PA for Classical NST

There is nothing special about the normalization restriction for constructive logics; it will still grant non-triviality even in a classical system. It is further likely that we could also use a similar argument as was used here to recover HA to recover second-order PA in such a classical NNST. The only complication will be the normalization characteristics of classical negation. This makes it likely that this project is better carried out with a sequent calculus and a corresponding cut elimination formulation of NNST.

8.2 Is NST Turing-Complete?

It is believed that NST "is Turing-complete", though this is a misleading way to say it. More precisely, it is believed that the provided $\mathrm{NST}_{\to\in\forall}$ type system can be used to type any untyped λ-term and thus any program we can generate has a corresponding proof in NST.

Conjecture 8.1. *The type system of $NST_{\to\in\forall}$ can be used to provide a type for any untyped λ term.*

It does however seem unlikely that the semantics of the proof given for some arbitrary untyped λ-term will always be all that enlightening. Some early attempts at a solution make free use of Curry-like sentences to force proofs to contort themselves to the needed λ-term. However, it is another question whether the typing of untyped λ-terms could be made in such a way that the λ-terms usually associated with things like numbers and arithmetic are still given sensible set representations in NST.

If this conjecture were to hold, the following would hold.

Corollary 8.2. *The set of NNST proofs is not decidable.*

Proof. If it were, we would have a solution to the Halting problem. NNST would correspond exactly to those programs which halt since NST is Turing-complete. □

We thus have that NNST is still subject to a fundamental limitative result. And in fact it is an assertion of this "computational foundation for mathematics" that the Halting Problem is **the** limitative result of mathematics.

8.3 Classical Limitative Results

A little bit more will be said about this in Appendix A, but we place it here as well as an open question. NNST appears to give a more nuanced view of Godel's Incompleteness Theorems and Lob's Theorem. The truth predicate, $\vdash A \leftrightarrow \{z|A\} \in Tr$, resulting from the truth-witness set mentioned in the introduction, $\forall x(x \in \mathrm{Tr} \leftrightarrow \exists z(z \in x))$, ostensibly also functions as some sort of provability predicate for NNST.

Proposition 8.3. *The truth predicate is also a provability predicate for NNST. There is a normal deduction in NST that A if and only if there is a normal deduction in NST that* $\{z|A\} \in Tr$.

Proof. Assuming a normal deduction of NST that A, we can extend this deduction by a single use of $\in I$ to also have a normal deduction that $\{z|A\} \in Tr$. The other direction is a bit trickier. Assume we have a normal proof that $\{z|A\} \in Tr$ in NST. Then we have a normal proof of this in $\mathrm{NST}_{\rightarrow\in\forall}$. This had to result from either $\in I$, or an elimination rule. In the latter case, we can append a use of $\in E$ without changing whether the proof is normal. In the former case, the deduction will become non-normal when we add a $\in E$ rule, however, an instance of $\in I$ was used, we know that we already had a deduction of A and that it was normal. □

Of course, the answer to why we can get away with this might be the obvious answer that the system is apparently inconsistent. For example, we do have a proof that both $R \in R$ and $R \notin R$. Despite this, as we know, the system remains non-trivial as inconsistency has separated from triviality. And what happens with Russell's set and Curry sets suggests the outcome of the odd self-referential sentences that prove these limitative results will "simply" be that both the negated and positive will be true.

This is however, more confusing, as strictly speaking we do not have "negation" in $\mathrm{NST}_{\rightarrow\in\forall}$ but a coding of it that works for mathematical purposes. And further $\mathrm{NST}_{\rightarrow\in\forall}$ shows that there can be no hard delineation of the positive sentences from the false. This is a simple consequence of Curry sets. For as soon as we think we have some proposition p which acts such that only one of $A \rightarrow p$ or A hold, a Curry set will show that this does not happen.

A further primary worry that the limitative results generated was that we could not have simultaneously strong systems and complete systems. However, if the proof that second-order HA is recoverable in NNST is correct, this is shown to be false. We can have systems capable of doing a lot of the mathematics we want, while potentially remaining complete, and still being non-trivial. This is something that is desparately desired to be explored and understood. Taking this understanding, Godel's Theorems was a result because we believed all systems had fully transitive conditionals. Now it seems that these theorems were really saying that we can not have a **transitive** mathematical system which is strong enough to represent PA while also being non-trivial and complete. But if we find that transitive mathematical systems were an incorrect approach in the first place, an initial argument for which is offered in Appendix A, then this is not a loss at all. Godel's Theorems were more an indication that we were off track in our formalizations of mathematical reasoning.

8.3.1 Changing Our "Metatheory"

The success of this project might ultimately suggest the delineation between metatheory and theory is not necessary for the purposes of formalization. There is only the theory which contains its own notion of truth. There is no hierarchy of languages, and no excess philosophical complication with our formalizing. But, then, this seems to run afoul of the above assertions that the Halting Problem is an ultimate limitation. If this foundation is in some sense inconsistent, could we just "decide" or "find" that the Halting Problem is both a limit and not?

I think the ultimate answer to this will be no. And the reason will be somewhat simple. We know that the logical systems we use do in fact normalize. It should further follow that reasoning **about** something will also always normalize. That is, we have formal justification for classical, intuitionistic and many non-classical logics that they all normalize or satisfy cut elimination. Thus these theories can be used to reason about some external system of "facts".

It turns out however that reasoning inside a theory does not always normalize. If we try to reason about something using set theory in addition to logic, then we must be wary of normalization. But these very questions will in principle be answered in the formalization itself, so we do not need to worry about our informal questions running afoul of our new requirements on normalization.

Thus, as an example, we have the fact of the Halting Problem: no general algorithm exists which can decide whether any algorithm will terminate. When we reason about this fact and its influences on other sets of facts resulting from foundational systems, we can reason as we always have, as we know that this reasoning itself will normalize since we have proved that the system we're using, whether intuitionistic or classical, satisfies a normalization result.

8.3.2 Generalizing Normalizing Results

The method of translating deductions of NST into a type theory with a known normalization result is likely to be a way forward for us to carve out universes of NST which fall in NNST. Generalizing on this method seems like a good idea, but a way forward is not clear at this time. Also, we can never hope to find a perfect translation of NNST into a type theory, due to being limited by the Halting Problem. If a type theory existed which corresponded precisely to all normalizing algorithms, we will be in conflict with the Halting Problem.

Appendices

Appendix A

Justification and Implications of NNST

A.1 Why NNST?

Using NNST comes with significant computational overhead. Normal proofs are longer, sometimes much longer, and whether a proof normalizes is only semi-decidable. Solving this computational problem is no small feat but this thesis has provided some partial solutions to suggest it is not as bad as it looks. A problem of a more philosophical nature exists if we suggest that NNST is a foundation for mathematics. This necessarily implies that the transitivity of deducibility does not hold in general. A particular manifestation of this is demonstrated by Curry's paradox.

Proposition 1.1. *Let A be a formula and $C := \{x | x \in x \to A\}$. Both $C \in C$ and $C \in C \to A$ are provable.*

Proof. We provide a normal proof for $\vdash C \in C \to A$.

$$\dfrac{\dfrac{C \in C \vdash C \in C \qquad \dfrac{C \in C \vdash C \in C}{C \in C \vdash C \in C \to A} \in E}{\vdash C \in C \to A} \to E}{\vdash C \in C \to A} \to I$$

The normal proof for $\vdash C \in C$ extends this proof of $\vdash C \in C \to A$ by a single application of $\in I$. □

Despite Curry's, we can conclude that there is no way to derive $\vdash A$ in this way.[1] For a normal proof would satisfy the subformula principle and thus would require that the formulas occurring in the proof are subformulas of A. That A fails to be derived despite having both a conditional and its hypothesis is even more confusing if we observe that modus ponens is itself a derivable theorem in the form of $A, A \to B \vdash B$.

$$\dfrac{A \vdash A \qquad A \to B \vdash A \to B}{A, A \to B \vdash B} \to E$$

[1]Except for some trivial cases in which $C \in C$ is a subformula of A.

In this case, since A and $A \to B$ are assumptions, there are no normalization steps to be done. This is a normal proof. How do we justify this behavior? To do so requires being explicit about the turnstile. As discussed at length in Chapter 3, the turnstile asserts the existence of a deduction from the assumptions to the conclusion.[2]

Our understanding of the turnstile extends from thinking about what it means that "from an assumption of A I can deduce B". Our intuitive understanding of deduction is then used to derive the three fundamental properties of the turnstile: reflexivity, monotonicity and transitivity. As shown in Chapter 3 these are not assumed facts, but derivable constructions based on our understanding of deduction. Then, from there, we may derive some axiomatic system which contains a conditional that mirrors exactly the actions of the turnstile. And we then verify that such a relationship holds by the deduction theorem.

A sufficient axiomatic system for the deduction theorem can be summed up in two axiom schemas and the rule of modus ponens. We only need

1. $B \to (A \to B)$ and
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

Note that this only requires positive axioms and thus does not need to make claims about our negation.

This thesis explored some alternate systems which attempt to capture NST, DKQ naive set theory and LAST. However, assuming that our notion of deducibility is already accurate about mathematical practice, and that the deduction theorem is how we get to that notion of deducibility, then we can reach the conclusion that both of these approaches will fail to capture mathematical reasoning. For, in the provided axiomatic system, the first axiom is weakening. This immediately disqualifies any relevant logic as being capable of mirroring what we mean by mathematical reasoning in general. And the second axiom is self-distribution, which implies contraction. And thus a contraction-free system is thrown out as a possibility. To the mathematician who is convinced that how he works informally is the correct way, and for whom the deduction theorem merely assures the coherence his informal reasoning, these axioms are necessities. To move away from these axioms requires justifying a shift in our informal reasoning methods; the formal gives way to the informal.

But then, why does this perspective not also disqualify the NNST approach and all non-transitive approaches? Let us reconsider the informal proof that our notion of deducibility is transitive.

Definition 1.2. Deduction is *transitive* if given that some set of assumptions Γ deduces A and A in conjunction with some set of formulas Δ deduces B then Γ and Δ deduce B.

Proposition 1.3. *Deduction is transitive.*

[2]The following is likely to only be convincing to mathematicians or people who believe the point of logic is to clarify mathematics and not to replace it. I do not intend to put to rest the philosophical exploration, but to start it.

Proof. Assume that we have a proof of A from assumptions Γ and a proof of B from A and assumptions Δ. Then we can construct our new proof as follows: assume the formulas of Γ and Δ. Use our proof of A from Γ to derive A. Then append to this the proof of B from A and Δ. Thus we've deduced B from the assumptions Γ and Δ. $\square$

This informal proof does seem intuitively correct for any theory. However, how do we formally justify that this informal proof of transitive deduction is coherent? We would do this through proving our deductions normalize. That is, normalization in a natural deduction system is the process by which we construct the proof that transitivity is implying exists. That this is the case is even more apparent in a sequent calculus. The cut rule plainly asserts that transitivity is valid.

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\mathrm{Cut}$$

Then cut elimination is the process by which we verify we can construct the proof that cut asserts exists. But then, this also plainly shows why cut is a bizzare thing to assume. For cut asserts that from two proofs, we can find we have a third proof. But it does not provide a method of constructing that proof.

Is there any sense to be made of a proof that exists but which can not be observed? This is not simply a question over having a non-constructive proof. That is still a proof. Cut literally asserts a proof exists without construction; it is the non-construction of any sort of proof. A proof is only a proof in its ability to convince someone else of the truth of what it asserts. To have nothing to show can not be called a proof at all. Proving cut elimination for a theory verifies that we are okay in using cut to take our shortcuts. It verifies that we can construct the proof that cut asserts exists. Transitioning back to our natural deduction systems, then we see that a normalized proof is the constructed proof asserted by transitivity, and a non-normal piece is an instance of using transitivity.

It then may make sense to desire that we only endorse theories which are in fact provably transitive. If a transitive proof relation is what we mean by mathematical reasoning, then anything which fails to verify transitivity can not be held up as a valid mathematical theory. Further, we know that second-order Peano Arithmetic normalizes and thus verifies transitivity. This theory is widely accepted to be enough to do most "every-day" mathematics. However, if we accept that we must have truly transitive theories, then it turns out we must not accept a widely accepted foundation of mathematics: ZF itself does not normalize, nor does the intuitionistic variant.[3]

Proposition 1.4. *There is a non-normalizable proof in a natural deduction formalization of ZF set theory[25].*

[3]There is a recent paper which shows a kind of normalization for IZF with replacement[35]. However, the normalization notion used does not amount to a construction of the transitive proof, but to the construction of a proof which ends on I-form. Some instances of non-normal steps, according to our current definition, are still allowed in a lazy-type fashion. The result is strong enough to show consistency.

Proof. Consider a natural deduction formalization in which the separation axiom is formalized by only allowing $\in$ rules to apply to formulas of the form $x \in u \wedge A(x)$. Then we can prove what is known as Crabbe's counter-example. Take the set term $\{x | x \in u \wedge x \notin x\}$.

Consider the subproof denoted **D**.

$$\dfrac{\dfrac{t \in t \vdash t \in t \qquad \dfrac{\dfrac{t \in t \vdash t \in t}{t \in t \vdash t \in u \wedge t \notin t}\in E}{t \in t \vdash t \notin t}\wedge E}{t \in t \vdash \bot}\rightarrow E}{\vdash t \notin t}\rightarrow I$$

Then we construct the proof that concludes $t \notin u$, which is the non-normal proof desired.

$$\dfrac{\dfrac{\dfrac{\dfrac{t \in u \vdash t \in u \qquad \mathbf{D}}{t \in u \vdash t \in u \wedge t \notin t}\wedge I}{t \in u \vdash t \in t}\in I \qquad \mathbf{D}}{t \in u \vdash \bot}\rightarrow E}{\vdash t \notin u}\rightarrow I$$

□

This is recognizable as the proof in informal ZF that there can not be universal set. The reasons for its non-normality are the same as Russell's. If we attempt to normalize this proof, we will end up in a loop and always end up back at this proof. Now, the case may be made that the difference with assuming transitivity with ZF and NST is that we know assuming transitivity in NST leads to triviality and for ZF is does not. Again though, this still does not answer the question of whether it is epistemically valid to assert a proof exists without being able to actually construct the proof itself. Perhaps the problem with NNST may also be with the fact that we can prove $R \in R$ and $R \notin R$.

Proposition 1.5. *In NNST, $R \in R$ and $R \notin R$ have normal proofs.*

Proof. Precisely the same as the proof for any Curry set that $C \in C$ and $C \in C \rightarrow A$. □

But as we see with $\mathrm{NST}_{\rightarrow \in \forall}$, negation itself is simply a semantic label for something in our foundational system which does what we think negation does. Negation to NST is an illusion and a distraction from what is really going on at the foundational level. There is no salient difference between Curry sets and Russell's set in NST; Russell's set is merely a Curry set implying a sort of A which we call negation. This fact is the same for any inconsistency in the theory. Inconsistencies are necessarily implications which invalidate non-transitivity in NST and this does make them weird. But then the intuitive difficulty with inconsistences is not with the inconsistency but with non-transitive implications.

We have thus far believed that transitivity is necessary feature of (most) logics, even though the widely accepted set theory of ZF itself is not transitive; it has a proof which can not normalize and thus it fails actually construct the proofs which its assumed transitivity assert exist. But non-transitive implications are not at all weird when we understand them in a computational paradigm as non-terminating algorithms. For it makes intuitive sense that a proof which can never be properly "computed" should not imply anything at all.

Further, in NNST we can never actually write a proof that "non-trivially" uses both proved halves of an inconsistency. For an actual non-trivial use would mean an actual detachment of the implication to $\perp$. In this way, the inconsistency is itself inert beyond being able to be conjoined and other random operations that do not get the "meat" of the matter. We may then be able to consider NNST as containing fragments of mutually "incompatible" theories. For example, there's the NNST proofs produced using $R \in R$ and those produced using $R \notin R$.

We may be pushed to say that we should accept only theories where we can construct the proofs asserted to exist by our assumptions of transitivity. There may be a particular subset of conditions that a theory must meet before it can produce a non-normal proof, in a similar way to how we characterize how strong a theory can be before it is inconsistent with a truth predicate. Such requirements would have to lie somewhere in between the expressive ability and strength of the strongly normalizing type theories and IZF.

If we do not take that path, then NNST can be seen to be straightforward from our understanding of informal mathematical reasoning. All that the normalization restriction is thus asserting is that when we invoke transitivity, we must be able to construct the actual proof that transitivity is asserting exists. We only wish to take normal deductions because those are the only things that are actually *proofs*.

A.2 Non-transitivity and the Church-Turing Thesis

The Church-Turing thesis asserts that any effective computation can be computed by a Turing machine. This assertion can never be formally proved for it attempts to bridge the gap between our informal notion of computation and the formal representation of Turing machines. Despite this, the evidence so far is in favor if it being true for various reasons: many independent formal systems of computability were developed which all turned out to be equivalent, and we have failed for a century to provide anything that plausibly can not be computed by any such system. The untyped λ-calculus is one such equivalent system.

We have also seen in the past few decades the discovery of the Curry-Howard correspondence between logic and type theory. We have used this very idea in providing a type theory for the natural deduction system of NST. A Curry-Howard correspondence is formally provable: given a formal system of logic, can we represent its proof theory as the computational actions of the λ-calculus? That is, can we always represent a sequence of actions on a logical formula in a proof as a sequence of computational actions.

Finally, there is a much more disputed equivalence to discuss: "any proof is an effective computation". This is the informal analog of the Curry-Howard correspondence which precisely asserts "any formal proof corresponds to a λ-term". It is hard to dispute that any effective computation is a proof. If we have performed some effective computation then it is in principle deterministic, can be followed by another person, and can be checked for correctness. The computation that has been performed is then a proof that the particular inputs provided lead to these particular outputs.

It is more difficult and suspicious to assert that any proof is an effective computation. If we do assert this, then we can invoke the Church-Turing thesis and say that any informal proof must then be computable by a Turing machine and is thus in principle formalizable. In particular, this seems to assert that the "informal metatheory" from which Gödel's Incompleteness Theorems are then provable are then formalizable. But then there is no escape from this informal metatheory which appears to include some notion of proof, a notion of proof which must be formalizable under our assumption that any proof is an effective computation, and so these Gödel sentences would exist in our informal metatheory and must also be provably true and not true. That is, our informal metatheory is trivial.

It is harder to deny the intuitive appeal that any proof is an effective computation after the discovery of the Curry-Howard correspondence which demonstrates this plainly works in the formal setting. Further, it seems convincing that a proof should be an effective computation, for if it was not, then it would seem to suggest that proofs were somehow not deterministic, could not be followed by another person, and could not be checked for correctness. But Gödel's Incompleteness results seems to force on us the understanding that we must have gone wrong somewhere.

Unless, our informal reasoning is actually non-transitive. Then we are forced to make no such concession for we can have it be the case that a Gödel sentence is both true and false. And in this case we could argue that our informal method of proof is in principle formalizable in some particular system. And I assert that NST is a good candidate for such a system. For, if the conjecture that it is Turing-complete is true, then there is some proof in NST which follows the actions of any computation we can think of. There is some concern that the semantics of the sets that correspond to some computations might not be all that useful, but if we take an entirely computational perspective of mathematical objects then as long as the set "does the right things" then it is the right thing. For example, there's a myriad of ways to model the numbers, but we still call them the numbers if they act in the way we want.

This also offers a nice perspective on the actual meaning of an "inconsistency". Assume that in fact NST is the formalization of informal reasoning. We are perfectly comfortable in informal reasoning with the idea that we can choose for a theory to have it be the case either "the continuum hypothesis holds" or "the continuum hypothesis does not hold". I assert this is no different from what it would mean to have that $R \in R$ and $R \notin R$ in NST. Both of those facts are floating around in NST as viable assumptions that we can make. We further can not pull them both in to the same theory "fragments" in a non-trivial way. These formulas do not interact as other implications do, but are repelling each other. We thus have some fragment of NST where we work with $R \in R$ and some fragment where we work with $R \notin R$ in the very same way we have theories where we assume the continuum hypothesis and those where we do not.

A way to test this hypothesis would be to discover what NST did with formulas known to be independent of ZF. Given that NST can express a provability predicate, it does seem any usual Gödel sentence would both hold and not hold. We could take it or its negation if we want. Would it actually be the case that the continuum hypothesis did hold and did not hold? Inconsistencies are then not at all bizarre but places where we are allowed to make assumptions. Perhaps these are places where our theories are in fact overdetermined due to the innate complexity at those points.

A.3 Conclusion

I only hope to offer in this Appendix some new ways to think about the theory of NST and to offer some initial thoughts about why it may make sense philosophically. The thesis itself offers a case for its mathematical interest, it escapes the worst of the limitative results and retains usual mathematical strength and reasoning. This section hopes to have at least indicated that: first, non-transitivity is not actually all that weird and may make more sense and two, inconsistency in NST has an interesting way to be understood if NST really does formalize informal proof as Church-Turing/Curry-Howard suggests it could.

Appendix B

Key Cases Reduction

$(\star_R, \star_L)$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{\Gamma \vdash A, \Delta} \qquad \overset{\vdots\,\pi_2}{\Gamma' \vdash B, \Delta'}}{\Gamma, \Gamma' \vdash A \star B, \Delta, \Delta'} \qquad \dfrac{\overset{\vdots\,\pi_3}{\Gamma'', A, B \vdash \Delta''}}{\Gamma'', A \star B \vdash \Delta''}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta, \Delta'}$$

$$\dfrac{\overset{\vdots\,\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\overset{\vdots\,\pi_2}{\Gamma' \vdash B, \Delta'} \qquad \overset{\vdots\,\pi_3}{\Gamma'', A, B \vdash \Delta''}}{\Gamma'', A, \Gamma' \vdash \Delta'', \Delta'}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta', \Delta}$$

$(\multimap_R, \multimap_L)$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{\Gamma, A \vdash B, \Delta}}{\Gamma \vdash A \multimap B, \Delta} \qquad \dfrac{\overset{\vdots\,\pi_2}{\Gamma' \vdash A, \Delta'} \qquad \overset{\vdots\,\pi_3}{\Gamma'', B \vdash \Delta''}}{\Gamma', \Gamma'', A \multimap B \vdash \Delta'', \Delta'}}{\Gamma', \Gamma'', \Gamma \vdash \Delta'', \Delta', \Delta}$$

$$\dfrac{\overset{\vdots\,\pi_2}{\Gamma' \vdash A, \Delta'} \qquad \dfrac{\overset{\vdots\,\pi_1}{\Gamma, A \vdash B, \Delta} \qquad \overset{\vdots\,\pi_3}{\Gamma'', B \vdash \Delta''}}{\Gamma'', \Gamma, A \vdash \Delta'', \Delta}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta, \Delta'}$$

$(!_R, !_L)$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\pi_1\\ !\Gamma\vdash A,?\Delta\end{array}}{!\Gamma\vdash !A,?\Delta} \qquad \dfrac{\begin{array}{c}\vdots\pi_2\\ \Gamma',A\vdash\Delta'\end{array}}{\Gamma',!A\vdash\Delta'}}{\Gamma',!\Gamma\vdash\Delta',?\Delta}$$

$$\dfrac{\begin{array}{c}\vdots\pi_1\\ !\Gamma\vdash A,?\Delta\end{array} \qquad \begin{array}{c}\vdots\pi_2\\ \Gamma',A\vdash\Delta'\end{array}}{\Gamma',!\Gamma\vdash\Delta',?\Delta}$$

$(!_R, W_L)$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\pi_1\\ !\Gamma\vdash A,?\Delta\end{array}}{!\Gamma\vdash !A,?\Delta} \qquad \dfrac{\begin{array}{c}\vdots\pi_2\\ \Gamma'\vdash\Delta'\end{array}}{\Gamma',!A\vdash\Delta'}}{\Gamma',!\Gamma\vdash\Delta',?\Delta}$$

$$\begin{array}{c}\vdots\pi_2\\ \Gamma'\vdash\Delta'\\ \hline\hline \Gamma',!\Gamma\vdash\Delta',?\Delta\end{array}$$

$(!_R, C_L)$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\pi_1\\ !\Gamma\vdash A,?\Delta\end{array}}{!\Gamma\vdash !A,?\Delta} \qquad \dfrac{\begin{array}{c}\vdots\pi_2\\ \Gamma',!A,!A\vdash\Delta'\end{array}}{\Gamma',!A\vdash\Delta'}}{\Gamma',!\Gamma\vdash\Delta',?\Delta}$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\pi_1\\ !\Gamma\vdash A,?\Delta\end{array}}{!\Gamma\vdash !A,?\Delta} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\pi_1\\ !\Gamma\vdash A,?\Delta\end{array}}{!\Gamma\vdash !A,?\Delta} \qquad \begin{array}{c}\vdots\pi_2\\ \Gamma',!A,!A\vdash\Delta'\end{array}}{\Gamma',!\Gamma,!A\vdash\Delta',?\Delta}}{\Gamma',!\Gamma,!\Gamma\vdash\Delta',?\Delta,?\Delta}\\ \hline\hline \Gamma',!\Gamma\vdash\Delta',?\Delta\end{array}$$

Appendix C

NST Derived Rules

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \lambda z.(zM)N : A \wedge B} \wedge I$$

$$\frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M(\lambda xy.x) : A} \wedge E_0 \qquad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M(\lambda xy.y) : B} \wedge E_1$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \lambda xy.xM : A \vee B} \vee I_0 \qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash \lambda xy.yM : A \vee B} \vee I_1$$

$$\frac{\Gamma \vdash M : A \vee B \qquad \Gamma, x : A \vdash N_0 : C \qquad \Gamma, y : B \vdash N_1 : C}{\Gamma \vdash (M(\lambda x.N_0))\lambda y.N_1 : C} \vee E$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \lambda x.xM : \exists x A} \exists I \qquad \frac{\Gamma \vdash M : \exists x A \qquad \Gamma, x : A \vdash N : B}{\Gamma \vdash M(\lambda x.N) : B} \exists E$$

$$\frac{\Gamma \vdash M : \bot}{\Gamma \vdash M : A} \bot$$

$$\frac{}{\Gamma \vdash \lambda t.t : x \subseteq x} \text{Refl}$$

$$\frac{\Gamma \vdash x : x \subseteq y \qquad \Gamma \vdash y : y \subseteq z}{\Gamma \vdash \lambda t.y(xt) : x \subseteq z} \text{Trans}$$

$$\frac{}{\Gamma \vdash \lambda t.t : x = x} \text{Refl} \qquad \frac{\Gamma \vdash t : x = y}{\Gamma \vdash t(\lambda u.u) : y = x} \text{Sym}$$

$$\frac{\Gamma \vdash t : x = y \qquad \Gamma \vdash v : y = z}{\Gamma \vdash \lambda u.v(tu) : x = z} \text{Trans}$$

$$\frac{\Gamma \vdash u : A(x) \qquad \Gamma \vdash t : x = y}{\Gamma \vdash tu : A(y)} = E$$

$$\frac{}{\Gamma \vdash \lambda yx.x : 0 \in \mathbf{N}} \text{Zero} \qquad \frac{\Gamma \vdash M : n \in \mathbf{N}}{\Gamma \vdash \lambda yz.y((My)z) : Sn \in \mathbf{N}} \text{Suc}$$

$$\frac{}{\Gamma \vdash \lambda t.t(\lambda xy.(x(\lambda u.u))) : 0 \neq Sx} \text{Min}$$

$$\frac{\Gamma \vdash N : \phi(0) \qquad \Gamma \vdash M : \forall n(\phi(n) \to \phi(Sn))}{\Gamma \vdash \lambda z.(z(\lambda t.Mt))N : \forall n(n \in \mathbf{N} \to \phi(n))} \text{PropInd}$$

$$\frac{\Gamma \vdash N : 0 \in \{x|\phi(x)\} \qquad \Gamma \vdash M : \forall n(n \in \{x|\phi(x)\} \to Sn \in \{x|\phi(x)\}}{\Gamma \vdash \lambda z.(zM)N : \forall n(n \in \mathbf{N} \to n \in T)} \text{Ind}$$

$$\frac{}{\Gamma \vdash \lambda txy.xt : x \subseteq Sx}\text{SucSub}$$

Let $T := \{n | n \in \mathbf{N} \land \forall x(x \in n \to n \not\subseteq x)\}$.

$$\frac{}{\vdash \forall n(n \in \mathbf{N} \to n \in T)}\text{Not-Subset}$$

Let $T := \{n | n \in \mathbf{N} \land \forall x(x \in n \to x \subseteq n)\}$.

$$\frac{}{\vdash \forall n(n \in \mathbf{N} \to n \in T)}\text{NumTrans}$$

$$\frac{\Gamma \vdash Sx = Sy}{\Gamma \vdash x = y}\text{SucInj}$$

Appendix D

Additional Normal Proofs From a Computer

In the writing of this thesis, a Python computer program was written that attempts to compute by brute force normal forms of λ-terms for non-normal proofs in NST. This program is not featured front and center in the thesis for several reasons: it was based on a first iteration of the type system implemented for NST and it needs some significant improvements. While I believe the program to be correct, it is hard to verify exactly that it is correct given its complexity and it thus does not seem sensible to place the results derived from the program in the main body of the thesis. The code for the program is hosted on Github at `https://github.com/eistre91/NST_normalizer`.

However, it is still worth discussion as it demonstrates another way we can overcome the computationally difficult problem of finding normal proofs. The type system this program was based on is reproduced here.

Definition 4.1. Let i be 0 or 1. The grammar for the lambda terms is:

$$\begin{aligned} M, N :=& x \mid \lambda x.M \mid U\alpha.M \mid MN \mid \iota_i(x) \mid \pi_0 \mid [x.w_1, y.w_2] \mid (a, x) \mid [(a, x).t] \mid \\ & false^A(x) \mid eq(t1.t2, w1.w2) \mid sub_i(t : A(s), x : s = r) \mid in(x) \mid out(x). \end{aligned}$$

$$\frac{}{\Gamma, x : A \vdash x : A}\text{ Ax}$$

$$\frac{\Gamma \vdash u : A \qquad \Gamma \vdash t : B}{\Gamma \vdash < u, t >: A \wedge B}\wedge I$$

$$\frac{\Gamma \vdash u : A \wedge B}{\Gamma \vdash u\pi_0 : A}\wedge E \qquad\qquad \frac{\Gamma \vdash u : A \wedge B}{\Gamma u\pi_1 \vdash B}\wedge E$$

$$\frac{\Gamma \vdash u : A}{\Gamma \vdash \iota_0(u) : A \vee B}\vee I_0 \qquad\qquad \frac{\Gamma \vdash u : B}{\Gamma \vdash \iota_1(u) : A \vee B}\vee I_1$$

$$\frac{\Gamma \vdash u : A \vee B \qquad \Gamma, x : A \vdash w_1 : C \qquad \Gamma, y : B \vdash w_2 : C}{\Gamma \vdash u[x.w_1, y.w_2] : C}\vee E$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x.u : A \to B} \to I \qquad \frac{\Gamma \vdash u : A \qquad \Gamma \vdash t : A \to B}{\Gamma \vdash tu : B} \to E$$

$$\frac{\Gamma \vdash u : A[a/x]}{\Gamma \vdash Ux.u : \forall x A} \forall I \qquad \frac{\Gamma \vdash u : \forall x A}{\Gamma \vdash ua : A[a/x]} \forall E$$

$$\frac{\Gamma \vdash u : A[a/x]}{\Gamma \vdash (a, u) : \exists x A} \exists I \qquad \frac{\Gamma \vdash u : \exists x A \qquad \Gamma, s : A[a/x] \vdash t : B}{\Gamma \vdash u[(x, s).t] : B} \exists E$$

$$\frac{\Gamma \vdash w : \bot}{\Gamma \vdash \mathrm{false}^A(w) : A} \bot$$

$$\frac{\Gamma, t_1 : x \in s \vdash t_2 : x \in r \qquad \Gamma, w_1 : x \in r \vdash w_2 : x \in s}{\Gamma \vdash \mathrm{eq}(t_1.t_2, w_1.w_2)s = r} = I$$

$$\frac{\Gamma \vdash t : A(r) \qquad \Gamma \vdash u : s = r}{\Gamma \vdash \mathrm{sub}_0(t : A(r), u : s = r) : A[s/r]} = E_0$$

$$\frac{\Gamma \vdash t : A(s) \qquad \Gamma \vdash u : s = r}{\Gamma \vdash \mathrm{sub}_1(t : A(s), u : s = r) : A[r/s]} = E_1$$

$$\frac{\Gamma \vdash u : A[r/x]}{\Gamma \vdash \mathrm{in}(u) : r \in \{x|A(x)\}} \in I \qquad \frac{\Gamma \vdash u : r \in \{x|A(x)\}}{\Gamma \vdash \mathrm{out}(u) : A[r/x]} \in E$$

Note the dramatic increase in complexity in the language of the λ-terms. This language was used to derive λ-terms for the proofs of the two important propositions, the ordered pair property and the fixpoint theorem.

Definition 4.2. $\langle a, b\rangle := \{a, \{a, b\}\}$

Proposition 4.3. $\vdash \langle a, b\rangle = \langle c, d\rangle \leftrightarrow a = c \wedge b = d$

Theorem 4.4. *For any formula A, there exists a term f such that $t \in f \leftrightarrow A[f/y, t/x]$ is provable for any t.*

The standard proofs for these propositions are non-normal and are reproducible in NST. These standard proofs are quite long and its a difficult task to normalize these proofs by hand. However, the mentioned program produces normal λ-terms for these proofs. With the Subject Reduction theorem and the assumed correctness of the computer program, this demonstrates that these propositions are theorems of NNST.

These normal λ-terms are themselves quite long and consume quite a few lines of text to write out. There's nothing particularly enlightening about these proofs beyond the proof of concept that this is possible. A stronger program which could parse a standard proof syntax, like LaTeX Bussproofs, compute λ-terms, normalize, and reproduce the syntax for the LaTeX equivalent proof is desirable. This would be much more user-friendly and easier to verify.

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