Constructive aspects of Riemann's permutation theorem for series

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Abstract

The notions of permutable and weak-permutable convergence of a series $\sum_{n=1}^{\infty} a_n$ of real numbers are introduced. Classically, these two notions are equivalent, and, by Riemann's two main theorems on the convergence of series, a convergent series is permutably convergent if and only if it is absolutely convergent. Working within Bishop-style constructive mathematics, we prove that Ishihara's principle **BD**-$\mathbf{N}$ implies that every permutably convergent series is absolutely convergent. Since there are models of constructive mathematics in which the Riemann permutation theorem for series holds but **BD**-$\mathbf{N}$ does not, the best we can hope for as a partial converse to our first theorem is that the absolute convergence of series with a permutability property classically equivalent to that of Riemann implies **BD**-$\mathbf{N}$. We show that this is the case when the property is weak-permutable convergence.

1 Introduction

This paper follows on from [2], in which the first two authors gave proofs, within the framework of Bishop-style constructive analysis **(BISH)**,[1] of the two famous series theorems of Riemann [17]:[2]

$\mathbf{RST}_1$ *If a series* $\sum a_n$ *of real numbers is absolutely convergent, then for each permutation* σ *of the set* $\mathbf{N}^+$ *of positive integers, the series* $\sum a_{\sigma(n)}$ *converges to the same sum as* $\sum a_n$.

$\mathbf{RST}_2$ *If a series* $\sum a_n$ *of real numbers is conditionally convergent, then for each real number* x *there exists a permutation* σ *of* $\mathbf{N}^+$ *such that* $\sum a_{\sigma(n)}$ *converges to* x.

It is not hard to extend the conclusion of $\mathbf{RST}_2$ to what we call its *full, extended version*, which includes the existence of permutations of the series $\sum a_n$ that diverge to ∞ and to $-\infty$. In consequence, a simple reductio ad absurdum argument

[1] That is, analysis using intuitionistic logic, a related set theory such as that of Aczel and Rathjen [1], and dependent choice. For more on **BISH**, see [3, 4, 7].

[2] We use shorthand like $\sum a_n$ and $\sum a_{\sigma(n)}$ for series when it is clear what the index of summation is.

proves classically that if a real series $\sum a_n$ is **permutably convergent**—that is, every permutation of $\sum a_n$ converges in $\mathbf{R}$—then it is absolutely convergent. An intuitionistic proof of this last result was provided by Troelstra ([19], pages 95 ff.), using Brouwer's continuity principle for choice sequences. That result actually has one serious intuitionistic application: Spitters ([18], pages 2101–2) uses it to give an intuitionistic proof of the characterisation of normal linear functionals on the space of bounded operators on a Hilbert space; he also asks whether there is a proof of the Riemann-Troelstra result within **BISH** alone. In Section 3 below, we give a proof, within **BISH** *supplemented* by the constructive-foundationally important principle **BD**-$\mathbf{N}$, that permutable convergence implies absolute convergence. While this proof steps outside unadorned **BISH**, it is valid in both intuitionistic and constructive recursive mathematics, in which **BD**-$\mathbf{N}$ is derivable.

This raises the question: over **BISH**, does the absolute convergence of every permutably convergent series imply **BD**-$\mathbf{N}$? Thanks to Diener and Lubarsky [8], we now know that in certain formal systems of **BISH**, the answer is negative; in other words, the result about permutably convergent series is weaker than **BD**-$\mathbf{N}$. In turn, this raises another question: is there a proposition that is *classically* equivalent to, and clearly cognate with, the absolute convergence of permutably convergent series and that, added to **BISH**, implies **BD**-$\mathbf{N}$? In order to answer this question affirmatively, we introduce in Section 2 the notion of *weak-permutable convergence* and then derive some of its fundamental properties, including its classical equivalence to permutable convergence. In Section 4 we show that the absolute convergence of weak-permutably convergent series implies **BD**-$\mathbf{N}$. Thus, in **BISH**, we have the implications

Every weak-permutably convergent series is absolutely convergent

$\Longrightarrow$ **BD**-$\mathbf{N}$

$\Longrightarrow$ Every permutably convergent series is absolutely convergent.

In view of the Diener-Lubarsky results in [8], neither of these implications can be reversed.

2 Weak-permutably convergent series in BISH

By a **bracketing** of a real series $\sum a_n$ we mean a pair comprising

- a strictly increasing mapping $f : \mathbf{N}^+ \to \mathbf{N}^+$ with $f(1) = 1$, and
- the sequence $\mathbf{b}$ defined by

$$b_k \equiv \sum_{i=f(k)}^{f(k+1)-1} a_i \quad (k \geqslant 1).$$

We also refer, loosely, to the series $\sum b_k$ as a bracketing of $\sum a_n$.

We say that $\sum a_n$ is **weak-permutably convergent** if it is convergent and if for each permutation σ of $\mathbf{N}^+$, there exists a convergent bracketing of $\sum a_{\sigma(n)}$. Clearly, permutable convergence implies weak-permutable convergence. As we shall see in this section, the converse holds classically; later we shall show that it does not hold constructively. As a first step towards this, we have:

Proposition 1. *Let $\sum a_n$ be a weak-permutably convergent series of real numbers, with sum s, and let σ be a permutation of $\mathbf{N}^+$. Then every convergent bracketing of $\sum a_{\sigma(n)}$ converges to s.*

The proof of this proposition will depend on some lemmas.

Lemma 2. *Let $\sum a_n$ be a convergent series of real numbers, with sum s, and let σ be a permutation of $\mathbf{N}^+$. If there exists a bracketing $(f, \mathbf{b})$ of $\sum a_{\sigma(n)}$ that converges to a sum $t \neq s$, then there exist a permutation τ of $\mathbf{N}^+$ and a strictly increasing sequence $(k_i)_{i \geqslant 1}$ of positive integers such*

$$\left| \sum_{n=f(k_i)+1}^{f(k_{i+1})} a_{\tau(n)} \right| > \frac{1}{3} |s - t| \tag{1}$$

for all i.

Proof. Consider, to illustrate, the case where $s < t$. For convenience, let $\varepsilon \equiv \frac{1}{3}(t - s)$. Pick k_1 such that

$$\left| \sum_{n=j}^{k} a_n \right| < \frac{\varepsilon}{2} \quad (k > j > f(k_1)).$$

Then $\sum_{n=1}^{f(k_1)} a_n < s + \varepsilon$. Set $\tau(k) \equiv k$ for $1 \leqslant k \leqslant f(k_1)$. Next pick $k_2 > k_1$ such that

- $\{1, \ldots, f(k_1)\} \subset \{\sigma(n) : 1 \leqslant n \leqslant f(k_2)\}$ and
- $\left| \sum_{n=f(j)}^{f(k)} a_{\sigma(n)} \right| < \varepsilon/2$ whenever $k > j > f(k_2)$.

Define $\tau(n)$ for $f(k_1) < n \leqslant f(k_2)$ so that

$$\{\sigma(n) \mid 1 \leqslant n \leqslant f(k_2),\ \sigma(n) > f(k_1)\} = \{\tau(f(k_1) + 1), \ldots, \tau(f(k_2))\}$$

Note that

$$\sum_{n=1}^{f(k_2)} a_{\tau(n)} = \sum_{n=1}^{f(k_2)} a_{\sigma(n)} > t - \varepsilon.$$

Next, pick $k_3 > k_2$ such that

$$\{\tau(1), \ldots, \tau(f(k_2))\} \subset \{1, \ldots, f(k_3)\}$$

Define $\tau(n)$ for $f(k_2) < n \leqslant f(k_3)$ so that

$$\{n : 1 \leqslant n \leqslant f(k_3),\ n > \tau(f(k_2))\} = \{\tau(f(k_2) + 1), \ldots, \tau(f(k_3))\}.$$

Then

$$\sum_{n=1}^{f(k_3)} a_{\tau(n)} = \sum_{n=1}^{f(k_3)} a_n < s + \varepsilon.$$

Carrying on in this way, we construct, inductively, a strictly increasing sequence $(k_i)_{i\geqslant 1}$ of positive integers, and a permutation τ of $\mathbf{N}^+$, such that for each j,

$$\sum_{n=1}^{f(k_{2j-1})} a_{\tau(n)} < s+\varepsilon \quad \text{and} \quad \sum_{n=1}^{f(k_{2j})} a_{\tau(n)} > t-\varepsilon.$$

When $i \in \mathbf{N}^+$ is even, we obtain

$$\left|\sum_{n=f(k_i)+1}^{f(k_{i+1})} a_{\tau(n)}\right| \geqslant \sum_{n=1}^{f(k_i)} a_{\tau(n)} - \sum_{n=1}^{f(k_{i+1})} a_{\tau(n)} > t-s-2\varepsilon > \frac{1}{3}(t-s).$$

A similar argument gives (1) when i is odd. $\square$

Lemma 3. *Under the hypotheses of* Lemma 2*, the series* $\sum |a_n|$ *diverges.*

Proof. Construct the permutation τ and the sequence $(k_i)_{i\geqslant 1}$ as in Lemma 2. Given $C>0$, compute j such that $(j-1)\,|s-t| > 3C$. Then

$$\sum_{n=1}^{f(k_j)} \left|a_{\tau(n)}\right| \geqslant \sum_{i=1}^{j-1} \left|\sum_{n=f(k_i)+1}^{f(k_{i+1})} a_{\tau(n)}\right| > \frac{j-1}{3}\,|s-t| > C.$$

Then compute M such that

$$\left\{a_{\tau(1)},\ldots,a_{\tau(f(k_j))}\right\} \subset \{a_1,\ldots,a_M\}.$$

Then

$$\sum_{n=1}^{M} |a_n| \geqslant \sum_{n=1}^{f(k_j)} \left|a_{\tau(n)}\right| > C.$$

Since $C>0$ is arbitrary, the conclusion follows. $\square$

Lemma 4. *Let* $\sum a_n$ *be a convergent series of real numbers, and* τ *a permutation of* $\mathbf{N}^+$ *such that* $\sum a_{\tau(n)}$ *diverges to infinity. Then it is impossible that* $\sum a_{\tau(n)}$ *have a convergent bracketing.*

Proof. Suppose there exists a bracketing $(f,\mathbf{b})$ of $\sum a_{\tau(n)}$ that converges to a sum s. Compute $N>1$ such that

$$\sum_{n=1}^{\nu} a_{\tau(n)} > s+1 \quad (\nu \geqslant N). \tag{2}$$

There exists $N_1 > N$ such that

$$\left|\sum_{i=1}^{N_1} \sum_{n=f(i)}^{f(i+1)-1} a_{\tau(n)} - s\right| < 1$$

and therefore

$$\left|\sum_{n=1}^{f(N_1+1)-1} a_{\tau(n)}\right| < s+1.$$

Since $f(N_1+1) > N$, this contradicts (2). $\square$

Lemma 5. *Let $\sum a_n$ be a weak-permutably convergent series of real numbers, and σ a permutation of $\mathbf{N}^+$. Then it is impossible that $\sum \left|a_{\sigma(n)}\right|$ diverge.*

Proof. Suppose that $\sum \left|a_{\sigma(n)}\right|$ does diverge. Then, by the full, extended version of $\mathbf{RST}_2$, there is a permutation τ of $\mathbf{N}^+$ such that $\sum a_{\tau(n)}$ diverges to infinity. Since $\sum a_n$ is weak-permutably convergent, there exists a bracketing of $\sum a_{\tau(n)}$ that converges. This is impossible, in view of Lemma 4. □

Arguing with classical logic, we see that if $\sum a_n$ is weak-permutably convergent, then, by Lemma 5, $\sum |a_n|$ must converge; whence $\sum a_n$ is permutably convergent, by $\mathbf{RST}_1$.

Returning to intuitionistic logic, we have reached the **proof of Proposition 1**:

Proof. Suppose that there exists a bracketing of $\sum a_{\sigma(n)}$ that converges to a sum distinct from s. Then, by Lemma 3, $\sum |a_n|$ diverges. Lemma 5 shows that this is impossible. It follows from the tightness of the inequality on $\mathbf{R}$ that every convergent bracketing of $\sum a_{\sigma(n)}$ converges to s. □

Since permutable convergence implies convergence and is a special case of weak-permutable convergence, we also have:

Corollary 6. *Let $\sum a_n$ be a permutably convergent series of real numbers, and let σ be a permutation of $\mathbf{N}$. Then $\sum a_{\sigma(n)} = \sum a_n$.*

3 BD-N and permutable convergence

A subset S of $\mathbf{N}^+$ is said to be **pseudobounded** if for each sequence $(s_n)_{n\geqslant 1}$ in S, there exists N such that $s_n/n < 1$ for all $n \geqslant N$—or, equivalently, if $s_n/n \to 0$ as $n \to \infty$. Every bounded subset of $\mathbf{N}^+$ is pseudobounded; the converse holds classically, intuitionistically, and in recursive constructive mathematics, but Lietz [14] and Lubarsky [15] have produced models of **BISH** in which it fails to hold for inhabited, countable, pseudobounded sets. Thus the principle

BD-N *Every inhabited, countable, pseudobounded subset of $\mathbf{N}^+$ is bounded*[3]

is independent of **BISH.** It is a serious problem of constructive reverse mathematics [5, 12, 13] to determine which classical theorems are equivalent to **BISH** + **BD**-N. For example, it is known that the full form of Banach's inverse mapping theorem in functional analysis is equivalent, over **BISH**, to **BD**-N; see [11].

This section is devoted to our version of the **Riemann permutability theorem**:

Theorem 7. *In* **BISH** + **BD-N**, *every permutably convergent series of real numbers is absolutely convergent.*

[3] **BD-N** was introduced by Ishihara in [10] (see also [16]).

Proof. Let $\sum_{i=1}^{\infty} a_i$ be a permutably convergent series of real numbers. To begin with, assume that each a_i is rational. Write

$$a_n^+ = \max\{a_n, 0\}, \ a_n^- = \max\{-a_n, 0\}.$$

Given a positive rational number ε, define a binary mapping ϕ on $\mathbf{N}^+ \times \mathbf{N}^+$ such that

$$\phi(m,n) = 0 \Rightarrow m > n \wedge \sum_{i=n+1}^{m} a_i^+ \geqslant \varepsilon,$$

$$\phi(m,n) = 1 \Rightarrow m \leqslant n \vee \sum_{i=n+1}^{m} a_i^+ < \varepsilon.$$

We may assume that $\phi(2,1) = 0$. Let

$$S \equiv \{n : \exists_m (\phi(m,n) = 0)\}.$$

Then S is countable and downward closed. In order to prove that S is pseudobounded, let $(s_n)_{n\geqslant 1}$ be an increasing sequence in S. We may assume that $s_1 = 1$. Define a map $\kappa : S \to \mathbf{N}^+$ by

$$\kappa(n) \equiv \min\left\{m : m > n \wedge \sum_{i=n+1}^{m} a_i^+ \geqslant \varepsilon\right\}.$$

Setting $\lambda_1 = 0$, we construct inductively a binary sequence $\lambda \equiv (\lambda_n)_{n\geqslant 1}$ with the following properties:

$$\forall_n ((\lambda_n = 0 \wedge \lambda_{n+1} = 1) \Rightarrow n+1 \in S) \tag{3}$$

$$\forall_n \exists_m (\lambda_n = 1 \Rightarrow \lambda_{n+m} = 0) \tag{4}$$

$$\forall_n ((\lambda_n = 0 \wedge \lambda_{n+1} = 0) \Rightarrow s_{n+1} \leqslant n+1) \tag{5}$$

Suppose that $\lambda_1, \ldots, \lambda_n$ have been defined such that

$$\forall_{k<n} ((\lambda_k = 0 \wedge \lambda_{k+1} = 1) \Rightarrow k+1 \in S). \tag{6}$$

In the case $\lambda_n = 0$, if $s_{n+1} \leqslant n+1$, we set $\lambda_{n+1} = 0$; and if $s_{n+1} > n+1$, we set $\lambda_{n+1} = 1$, noting that $n+1 \in S$ since S is downward closed. In the case $\lambda_n = 1$, we define

$$n' \equiv \min\{i \leqslant n : \forall_j (i \leqslant j \leqslant n \Rightarrow \lambda_j = 1)\}.$$

Then the hypothesis (6) ensures that $n' \in S$. If $\kappa(n') = n$, then $\sum_{i=n'+1}^{n} a_i^+ \geqslant \varepsilon$ and we set $\lambda_{n+1} = 0$; otherwise, we set $\lambda_{n+1} = 1$. This concludes the inductive construction of the sequence λ. Note that in the case $\lambda_n = \lambda_{n+1} = 1$, this construction will eventually give $\lambda_{n+1+m} = 0$ for some m, since

$$\kappa(n') \geqslant n+1, \sum_{i=n'+1}^{\kappa(n')-1} a_i^+ < \varepsilon, \text{ and } \sum_{i=n'+1}^{\kappa(n')} a_i^+ \geqslant \varepsilon.$$

Hence the sequence λ has all three properties (3)–(5).

For convenience, if $n \leqslant m$ and the following hold, we call the interval $I = [n, m]$ of $\mathbf{N}^+$ a *bad interval*:

- if $n > 1$ then $\lambda_{n-1} = 0$,
- $\lambda_{m+1} = 0$, and
- $\lambda_i = 1$ for all $i \in I$.

Define a permutation σ of $\mathbf{N}^+$ as follows. If $\lambda_n = 0$, then $\sigma(n) \equiv n$. If $[n, m]$ is a bad interval, then the construction of the sequence λ ensures that $\kappa(n) = m$, so $\sum_{i=n+1}^{m} a_i^+ \geqslant \varepsilon$. Let σ map an initial segment $[n, n+k-1]$ of $[n, m]$ onto

$$\left\{i : n \leqslant i \leqslant m \wedge a_i^+ > 0\right\},$$

and map the remaining elements of $[n, m]$ onto

$$\left\{i : n \leqslant i \leqslant m \wedge a_i^+ = 0\right\}.$$

Note that for all $n \geqslant 1$,

$$(\lambda_{n-1} = 0 \wedge \lambda_n = 1) \Rightarrow \exists_{j,k} \left(n \leqslant j < k \wedge \sum_{i=j+1}^{k} a_{\sigma(i)} \geqslant \varepsilon \right). \tag{7}$$

Since $\sum_{i=1}^{\infty} a_{\sigma(i)}$ is convergent, there exists J such that $\sum_{i=j+1}^{k} a_{\sigma(i)} < \varepsilon$ whenever $J \leqslant j < k$. In view of (4), we can assume that $\lambda_J = 0$. If $n \geqslant J$ and $\lambda_J = 1$, then there exists ν such that $J \leqslant \nu < n$, $\lambda_\nu = 0$, and $\lambda_{\nu+1} = 1$; whence there exist j, k such that $J \leqslant \nu \leqslant j < k$ and $\sum_{i=j+1}^{k} a_{\sigma(i)} \geqslant \varepsilon$, a contradiction. Thus $\lambda_n = 0$ for all $n \geqslant J$, and therefore, by (5), $s_n \leqslant n$ for all $n > J$. This concludes the proof that S is pseudobounded.

Applying **BD-N**, we obtain a positive integer N such that $n < N$ for all $n \in S$. If $m > n \geqslant N$ and $\sum_{i=n+1}^{m} a_i^+ > \varepsilon$, then $\phi(m, n) \neq 1$, so $\phi(m, n) = 0$ and therefore $n \in S$, a contradiction. Hence $\sum_{i=n+1}^{m} a_i^+ \leqslant \varepsilon$ whenever $m > n \geqslant N$. Likewise, there exists N' such that $\sum_{i=n+1}^{m} a_i^- \leqslant \varepsilon$ whenever $m > n \geqslant N'$. Thus if $m > n \geqslant \max\{N, N'\}$, then

$$\sum_{i=n+1}^{m} |a_i| = \sum_{i=n+1}^{m} a_i^+ + \sum_{i=n+1}^{m} a_i^- \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the partial sums of the series $\sum |a_n|$ form a Cauchy sequence, and hence that the series converges.

It remains to remove the restriction that the terms a_i be rational. In the general case, for each i pick b_i such that $a_i + b_i$ is rational and $0 < b_i < 2^{-i}$. Note that the series $\sum_{i=1}^{\infty} b_i$ converges absolutely and so, by $\mathbf{RST}_1$, is permutably convergent. Hence $\sum_{i=1}^{\infty} (a_i + b_i)$ is permutably convergent. By the first part of the proof, $\sum_{i=1}^{\infty} |a_i + b_i|$ is convergent, as therefore is $\sum_{i=1}^{\infty} |a_i|$. □

4 Weak-permutable convergence and BD-N

Diener and Lubarsky [8] have recently constructed topological models showing that the absolute convergence of every permutably convergent series in $\mathbf{R}$ neither implies **BD-N** nor is provable within the Aczel-Rathjen set-theoretic formulation of **BISH** [1], and may therefore be of constructive reverse-mathematical significance in its own right. Their models lead us to ask: is there a variant of the Riemann permutability theorem that is *classically equivalent* to the original form and that implies **BD**-$\mathbf{N}$? Since weak-permutable and permutable convergence are classically equivalent, the main result of this section provides an affirmative answer:

Theorem 8. *The statement*

(*) *Every weak-permutably convergent series in* $\mathbf{R}$ *is absolutely convergent*

implies ***BD***-$\mathbf{N}$.

The hard part of the proof is isolated in the complicated construction in the following lemma.

Lemma 9. *Let* $S \equiv \{s_1, s_2, \ldots\}$ *be an inhabited, countable, pseudobounded subset of* $\mathbf{N}$. *Then there exists a sequence* $(a_n)_{n\geqslant 1}$ *of nonnegative rational numbers with the following properties.*

(i) $\sum (-1)^n a_n$ *is convergent and weak-permutably convergent.*

(ii) *If* $\sum a_n$ *converges, then* S *is bounded.*

Proof. To perform this construction, we first replace each s_n by $\max\{s_k : k \leqslant n\}$, thereby obtaining $s_1 \leqslant s_2 \leqslant \cdots$. Now construct a binary sequence $(\lambda_k)_{k\geqslant 1}$ such that

$$\lambda_k = 0 \Rightarrow s_{2^{k+1}} = s_{2^k},$$
$$\lambda_k = 1 \Rightarrow s_{2^{k+1}} > s_{2^k}.$$

For $2^k + 1 \leqslant n + 1 < 2^{k+1}$, set $a_n = \lambda_k / (n+1)$. Note that if $\lambda_k = 1$, then $\sum_{n=2^k+1}^{2^{k+1}} a_n > \frac{1}{2}$. In order to show that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges in $\mathbf{R}$, first observe that if $\lambda_k = 1$ and $2^k < m_1 \leqslant m_2 \leqslant 2^{k+1}$, then

$$\left| \sum_{n=m_1}^{m_2} (-1)^n a_n \right| = \left| \sum_{n=m_1}^{m_2} \frac{(-1)^n}{n+1} \right| < \frac{1}{2^k}. \tag{8}$$

If j, k, m_1, m_2 are positive integers with $2^k < m_1 \leqslant 2^{k+1} \leqslant 2^j < m_2 \leqslant 2^{j+1}$, then

$$\begin{aligned}
&\left| \sum_{n=m_1}^{m_2} (-1)^n a_n \right| \\
&\leqslant \left| \sum_{n=m_1}^{2^{k+1}} (-1)^n a_n \right| + \sum_{\substack{k<\nu<j,\\ \lambda_\nu=1}} \left| \sum_{n=2^\nu+1}^{2^{\nu+1}} (-1)^n a_n \right| + \left| \sum_{n=2^j+1}^{m_2} (-1)^n a_n \right| \\
&\leqslant \frac{1}{2^k} + \sum_{\substack{k<\nu<j,\\ \lambda_\nu=1}} \frac{1}{2^\nu} + \frac{1}{2^j} \\
&\leqslant \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}.
\end{aligned}$$

Hence the partial sums of $\sum_{n=1}^{\infty} (-1)^n a_n$ form a Cauchy sequence, and so the series converges to a sum $s \in \mathbf{R}$.

Consider any permutation σ of $\mathbf{N}^+$. In order to show that $\sum_{n=1}^{\infty} (-1)^{\sigma(n)} a_{\sigma(n)}$ converges, we construct strictly increasing sequences $(j_k)_{k \geqslant 1}$ and $(n_k)_{k \geqslant 1}$ of positive integers such that for each k,

(a) $2^{j_k} < n_k < 2^{j_{k+1}}$,

(b) $\left\{n : n+1 < 2^{j_k}\right\} \subset \{\sigma(n) : n+1 < n_k\} \subset \left\{1, 2, \ldots 2^{j_{k+1}} - 1\right\}$, and

(c) $\left|\sum_{n=2^{j_k}}^{i} (-1)^n a_n\right| < 2^{-k+1}$ for all $k \geqslant 1$ and $i \geqslant 2^{j_k}$.

Setting $j_1 = 2$, pick $n_1 > 4$ such that

$$\{1, 2\} \subset \{\sigma(n) : n+1 < n_1\}.$$

Then pick $j_2 > j_1$ such that $n_1 < 2^{j_2}$,

$$\{\sigma(n) : n+1 < n_1\} \subset \left\{n : n+1 < 2^{j_2}\right\},$$

and $\left|\sum_{n=2^{j_2}}^{i} (-1)^n a_n\right| < 2^{-1}$ for all $i \geqslant 2^{j_2}$. Next pick, in turn, $n_2 > 2^{j_2}$ and $j_3 > j_2$ such that

$$\left\{n : n+1 < 2^{j_2}\right\} \subset \{\sigma(n) : n+1 < n_2\} \subset \left\{n : n+1 < 2^{j_3}\right\}$$

and $\left|\sum_{n=2^{j_3}}^{i} (-1)^n a_n\right| < 2^{-2}$ for all $i \geqslant 2^{j_3}$. Carrying on in this way, we complete the construction of our sequences $(j_k)_{k \geqslant 1}, (n_k)_{k \geqslant 1}$ with properties (a)–(c).

Now consider the sequence $(s_{2^{j_k+1}})_{k \geqslant 1}$. Since S is pseudobounded, there exists a positive integer K_1 such that $s_{2^{j_{k+1}}} < k$ for all $k \geqslant K_1$. Suppose that for each positive integer $k \leqslant K_1$, there exists i_k such that $j_k \leqslant i_k < j_{k+1}$ and $\lambda_{i_k} = 1$. Then

$$s_{2^{i_1}} < s_{2^{i_2}} < \cdots < s_{2^{i_{K_1}}} < s_{2^{j_{K_1+1}}},$$

so $K_1 \leqslant s_{2^{j_{K_1+1}}} < K_1$, a contradiction. Hence there exists $k_1 \leqslant K_1$ such that for each i with $j_{k_1} \leqslant i < j_{k_1+1}$, we have $\lambda_i = 0$, and therefore $a_n = 0$ whenever $2^i \leqslant n+1 < 2^{i+1}$. Thus $a_n = 0$ whenever $2^{j_{k_1}} \leqslant n+1 < 2^{j_{k_1+1}}$. It follows that

$$\begin{aligned}
\left\{a_n : n+1 < 2^{j_{k_1}}\right\} &\subset \left\{a_{\sigma(n)} : n+1 < n_{k_1}\right\} \\
&\subset \left\{a_n : n+1 < 2^{j_{k_1+1}}\right\} \\
&= \left\{a_n : n+1 < 2^{j_{k_1}}\right\} \cup \left\{a_n : 2^{j_{k_1}} \leqslant n+1 < 2^{j_{k_1+1}}\right\} \\
&= \left\{a_n : n+1 < 2^{j_{k_1}}\right\} \cup \{0\}.
\end{aligned}$$

Without loss of generality, we may assume that $a_1 = 0$. Then

$$\left\{a_n : n+1 < 2^{j_{k_1}}\right\} = \left\{a_{\sigma(n)} : n+1 < n_{k_1}\right\}.$$

Next consider the sequence $\left(s_{2^{j_{k_1}+k+1}}\right)_{k \geqslant 1}$. Since S is pseudobounded, there exists a positive integer K_2 such that $s_{2^{j_{k_1}+k+1}} < k$ for all $k \geqslant K_2$. Suppose that for each positive integer $k \leqslant K_2$, there exists i_k such that $j_{k_1+k} \leqslant i_k < j_{k_1+k+1}$ and $\lambda_{i_k} = 1$. Then

$$s_{2^{i_1}} < s_{2^{i_2}} < \cdots < s_{2^{i_{K_2}}} < s_{2^{j_{k_1+K_2+1}}},$$

so $K_2 \leqslant s_{2^{j_{k_1+K_2+1}}} < K_2$, which is absurd. Hence there exists $\kappa \leqslant K_2$ such that for each i with $j_{k_1+\kappa} \leqslant i < j_{k_1+\kappa+1}$, we have $\lambda_i = 0$, and therefore $a_n = 0$ whenever $2^i \leqslant n+1 < 2^{i+1}$. Setting $k_2 \equiv k_1 + \kappa$, we have $a_n = 0$ for all n with $2^{j_{k_2}} \leqslant n+1 < 2^{j_{k_2+1}}$. Hence

$$\begin{aligned}
\left\{a_n : n+1 < 2^{j_{k_2}}\right\} &\subset \left\{a_{\sigma(n)} : n+1 < n_{k_2}\right\} \\
&\subset \left\{a_n : n+1 < 2^{j_{k_2+1}}\right\} \\
&= \left\{a_n : n+1 < 2^{j_{k_2}}\right\} \cup \left\{a_n : 2^{j_{k_2}} \leqslant n+1 < 2^{j_{k_2+1}}\right\} \\
&= \left\{a_n : n+1 < 2^{j_{k_2}}\right\} \cup \{0\}.
\end{aligned}$$

Thus, since $a_1 = 0$,

$$\left\{a_n : n+1 < 2^{j_{k_2}}\right\} = \left\{a_{\sigma(n)} : n+1 < n_{k_2}\right\}.$$

Carrying on in this way, we construct positive integers $k_1 < k_2 < k_3 < \cdots$ such that for each i,

$$\left\{a_n : n+1 < 2^{j_{k_i}}\right\} = \left\{a_{\sigma(n)} : n+1 < n_{k_i}\right\}.$$

Since both σ and σ^{-1} are injective, it readily follows that for each i,

$$\left\{\sigma(n) : n_{k_i} \leqslant n+1 < n_{k_{i+1}}\right\} = \left\{m : 2^{j_{k_i}} \leqslant m < 2^{j_{k_{i+1}}}\right\}$$

and therefore

$$\left|\sum_{n=n_{k_i}}^{n_{k_{i+1}}-1} (-1)^{\sigma(n)}\, a_{\sigma(n)}\right| = \left|\sum_{m=2^{j_{k_i}}}^{2^{j_{k_i+1}}-1} (-1)^m\, a_m\right| < \frac{1}{2^{k_i}}.$$

We now see that

$$\sum_{i=1}^{\infty} \sum_{n=n_{k_i}}^{n_{k_{i+1}}-1} (-1)^{\sigma(n)} a_{\sigma(n)}$$

converges, by comparison with $\sum_{i=1}^{\infty} 2^{-k_i}$. Thus $\sum_{n=1}^{\infty} a_n$ is weak-permutably convergent.

Finally, suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then there exists N such that $\sum_{n=N+1}^{\infty} a_n < 1/2$. It follows that $\lambda_n = 0$, and therefore that $s_n = s_{2^N}$, for all $n \geqslant N$; whence $s_n \leqslant s_{2^N}$ for all n, and therefore S is a bounded set. □

The proof of Theorem 8 is now straightforward:

Proof. Given an inhabited, countable, pseudobounded subset S of $\mathbf{N}$, construct a sequence $(a_n)_{n\geqslant 1}$ of nonnegative rational numbers with properties (i) and (ii) in Lemma 9. Assuming (*), we see that $\sum a_n$ converges; whence, by property (ii), S is a bounded set. □

5 Concluding remarks

We have shown that, over **BISH**,

- with **BD**-$\mathbf{N}$, every permutably convergent series is absolutely convergent;
- the absolute convergence of every weak-permutably convergent series implies **BD**-$\mathbf{N}$.

It follows from the latter result that if weak-permutable convergence constructively implies, and is therefore equivalent to, permutable convergence, then the absolute convergence of every permutably convergent series implies, and is therefore equivalent to, **BD**-$\mathbf{N}$. Since the topological models in [8] show that this is not the case, we see that, relative to **BISH**, weak-permutable convergence is a strictly weaker notion than permutable convergence. In fact, the Diener-Lubarsky result shows that there is no algorithm which, applied to any inhabited, countable, pseudobounded subset S of $\mathbf{N}$ and the corresponding weak-permutably convergent series $\sum a_n$ constructed in the proof of Lemma 9, proves that that series is permutably convergent. Nevertheless, weak-permutable convergence and permutable convergence are classically equivalent notions; the constructive distinction between them is that the former implies, but is not implied by, **BD**-$\mathbf{N}$, which in turn implies, but is not implied by, the latter.

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