A Survey of Confidence Interval Formulae for Coverage Analysis *

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Abstract

Confidence interval estimators for proportions using normal approximation have been commonly used for coverage analysis of simulation output even though alternative approximate estimators of confidence intervals for proportions were proposed. This is because the normal approximation was easier to use in practice than the other approximate estimators. Computing technology has no problem with dealing these alternative estimators. Recently, one of the approximation methods for coverage analysis which is based on *arcsin* transformation has been used for estimating proportion and for controlling the required precision in [Raa95].

In this report, we compare three approximate interval estimators, based on a normal distribution approximation, an *arcsin* transformation and an F-distribution approximation, of a single proportion. These have application in sequential coverage analysis, conducted for assessing the quality of methods used in simulation output data analysis.

Keywords : sequential simulation, coverage analysis, confidence interval, proportion

1 Introduction

In many simulation studies, the analyst is interested not only in the point and interval estimates but also in other characteristics of the simulation output. These characteristics include the variance, quantiles or percentiles, and proportions or percentages. In this report, we discuss confidence interval formulae for the estimation of a single proportion. These have application in coverage analysis of simulation output, conducted for assessing the quality of methods used in simulation output data analysis.

Sequential simulation is generally accepted as the way of producing the results which is statistically reliable. Coverage analysis should be applied to a statistically sufficient number of repeated simulation experiments to determine the proportion of experiments in which the final confidence intervals cover the true value of the estimated parameter. Sequential coverage analysis, however, has a problem that some of the simulation experiments may stop after an

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abnormally short run, because the stopping criterion for the sequential simulation has been temporarily reached.

Considering of these matters, some rules for the proper experimental analysis of coverage in sequential simulation have been formulated and applied for the (non overlapping) batch means method and SA/HW method (Spectral Analysis in its version proposed by Heidelberger and Welch [HW81]) using the $M/M/1/\infty$ and $M/D/1/\infty$ queueing systems in [MPE96] and [PME98].

While some interesting results have been achieved in theoretical and experimental studies of coverage analysis (see for example [Gly82], [HW81], [MPE96], [PME98], [Sch80]), but experimental analysis of coverage is still required for assessing the quality of practical implementations of methods used for determining confidence intervals in steady-state simulation.

Estimators based on normal approximations to estimate the exact values have been widely used for coverage analysis (see for example [JBC97], [LK91], [MPE96], [PME98], [Sch80]). Alternative approximate estimators of confidence intervals for proportions are proposed, but the normal approximation has been commonly used for estimating proportions because the normal distribution is easy to use in practice. Recently, one of the approximation methods for coverage analysis which is based on an *arcsin* transformation has been used for estimating proportions and for controlling the required precision in [Raa95], but we know of no paper which compares these approximate estimators for coverage analysis.

Our motivation is finding the best interval estimator for coverage analysis in sequential steady-state simulation. Thus, we compare three candidate interval estimators for coverage analysis in this report. In Section 2 three interval estimators for coverage analysis are discussed. In Section 3 the numerical results of simulation are presented and conclusions are in Section 4.

2 Interval Estimators for Coverage Analysis

In any performance evaluation studies of dynamic systems by means of stochastic discrete-event simulation, the final estimators should be determined together with their statistical errors, which are usually measured by the half-width of the final confidence intervals. Restricting our attention to interval estimators of proportions, the point estimator of the proportion p in a binomial experiment is simply given by the statistic

$$\hat{p} = \frac{count\ \ of\ \ successes\ \ in\ \ sample}{size\ \ of\ \ sample} = \frac{x}{n}$$

which will be used as the point estimate for the parameter p.

The accuracy with which it estimates an unknown parameter proportion p can be assessed by the probability

$$P(|\hat{p} - p| < \Delta) = 1 - \alpha$$

or

$$P(\hat{p} - \Delta \leq p \leq \hat{p} + \Delta) = 1 - \alpha$$

where Δ is the half-width of the confidence interval for the estimator and $(1 - \alpha)$ is the confidence level, $0 < \alpha < 1$. Thus, if the total width (2Δ) of the confidence interval is found for an assumed confidence level of $(1-\alpha)$ and the simulation experiment were repeated a number of times, the confidence interval $(\hat{p}-\Delta, \hat{p}+\Delta)$ would contain the parameter p in $100(1-\alpha)\%$ of cases. It is well known that if observations $x_1, x_2, \ldots, x_n$ can be regarded as realizations of independent and identically normally distributed random variables $X_1, X_2, \ldots, X_n$, then Δ is estimated by

$$\hat{\Delta} = t_{n-1,1-\alpha/2}\hat{\sigma}[\hat{p}],$$

where $\hat{\sigma}^2[\hat{p}]$ is an estimator of the variance of $\hat{p}$ and $t_{n-1,1-\alpha/2}$ is the $(1-\alpha/2)$ quantile of the Student t-distribution with $n-1$ degrees of freedom. For $n > 30$, the t-distribution can be approximated by the standard normal distribution. Problems and solutions related with estimating variance σ^2 in steady-state simulations are well surveyed in [Paw90].

The robustness of any method is usually measured by the coverage of confidence intervals, defined as the proportion $\hat{p}$ with which the number of the final confidence interval $(\hat{p}-\Delta, \hat{p}+\Delta)$ contains the true value p. An estimator of variance $\hat{\sigma}^2$ used for determining the confidence interval of the point estimate is considered as valid, i.e. producing valid $100(1-\alpha)\%$ confidence intervals of the point estimate, if the upper bound of the confidence interval of the point estimate $\hat{p}$ equals at least $(1-\alpha)$ [Sau79]. Coverage analysis, however, is limited to analytically tractable systems, since the theoretical value of the parameter of interest has to be known.

2.1 Approximated Interval Estimators for Coverage Analysis

Three approximate estimators of the confidence interval for the proportion $\hat{p}$ are described in following sections in detail.

2.1.1 Interval Estimator for Coverage Analysis : I

The inference procedure for finding a confidence interval for the binomial parameter p involves two approximations:

- the normal approximation to the binomial distribution and
- the approximation of p by $\hat{p}$ in the standard deviation.

If the unknown proportion p is not expected to be too close to 0 or 1, we can establish a confidence interval for p by considering the sampling distribution of $\hat{p}$. If each experiment for coverage analysis is independent and identically distributed, with mean $\mu = p$ and variance $\sigma^2 = p(1-p)$, then an exact confidence interval for the estimated proportion $\hat{p}$ is obtained using the binomial distribution. However, by Theorem I (in *Appendix IV*), the sampling distribution of $\hat{p}$ is approximately normally distributed with mean $\mu_{\hat{p}} = p$ and variance $\sigma^2_{\hat{p}} = p(1-p)/n$ when n is large.

To find a confidence interval for p, we can assert that

$$P(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) \doteqdot 1-\alpha,$$

where

$$Z = \frac{\hat{p}-p}{\sqrt{p(1-p)/n}}$$

and $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ quantile of the standard normal distribution. Substituting for Z, we write

$$P(-z_{1-\alpha/2} < \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} < z_{1-\alpha/2}) = 1-\alpha$$

or

$$P(\hat{p} - z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}} < p < \hat{p} + z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}) = 1-\alpha.$$

When n is large, very little error is introduced by substituting the point estimate $\hat{p}$ for the p under the radical sign. Then we can write

$$P(\hat{p} - z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}) \simeq 1-\alpha.$$

Therefore, if the sample size n is sufficiently large($n \geq 30$), an approximate 100(1 - α)% confidence interval for the binomial parameter p is given by

$$(\hat{p} \pm z_{1-\alpha/2} * \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}) \tag{1}$$

where $\hat{p}$ is the proportion of successes in a random sample of size n [LK91].

The accuracy of the normal approximations improves as the sample size n increases. They are most accurate for any fixed n when p is close to 1/2, and least accurate when p is near 0 or 1. Therefore, although $\hat{p}$ has a normal limiting distribution, the confidence interval based on the normal approximation is not appropriate if p is close to 0 or 1[Raa95].

2.1.2 Interval Estimator for Coverage Analysis : II

Another estimator of the confidence interval for the proportion is based on the *arcsin* transformation which has been given by R. A. Fisher. On the basis of the relative between the mean p and the variance $p(1-p)/n$, for the sample proportion $\hat{p} = x/n$ we may determine a function $\hat{y} = g(\hat{p})$ in such a manner that the variance of the transformed variable $\hat{y}$ is independent of p. Here, we can use the *arcsin* transform,

$$\hat{y} = 2\ arcsin\sqrt{\hat{p}}$$

with variance

$$Var\{\hat{y}\} \simeq \frac{1}{n},$$

to construct an approximate confidence interval for the proportion being estimated because the *arcsin* transformation can produce the approximate values of the cumulative distribution function(CDF) of the normal distribution.

If $\hat{p}$ is approximately normally distributed with mean p and variance $p(1-p)/n$, then $\hat{y} = 2\ arcsin\sqrt{\hat{p}}$ is also approximately normally distributed with mean $= 2\ arcsin\sqrt{p}$ and variance $= \frac{1}{n}$, so that

$$Z = (2\ arcsin\sqrt{\hat{p}} - 2\ arcsin\sqrt{p}) * \sqrt{n} \tag{2}$$

is approximately normally distributed with parameters (0,1).

By analogy with Equation(28), Equation(29) and Equation(30) in *Appendix V*, we obtain that the CDF of the sample proportion $\hat{p}$ is defined by

$$P\{\hat{p}\} \simeq \Phi((2\,arcsin\sqrt{\hat{p}+\frac{1}{2n}} - 2\,arcsin\sqrt{p}) * \sqrt{n}), \tag{3}$$

where Φ is the CDF of a normal random variable and by the relation Equation(2) and Equation(29) we can write

$$2\,arcsin\sqrt{\hat{p}+\frac{1}{2n}} \simeq 2\,arcsin\sqrt{p} + \frac{z_{1-\alpha/2}}{\sqrt{n}}.$$

Therefore, we obtain by applying Equation(3) the confidence interval for the proportion $\hat{p}$ which is based on the *arcsin* transform.

A $100(1-\alpha)\%$ confidence interval for the proportion $\hat{p}$ which is based on the *arcsin* transform is $(\hat{p}_L, \hat{p}_U)$:

$$\hat{p}_L = sin(LC/2)^2 \tag{4}$$

and

$$\hat{p}_U = sin(UC/2)^2 \tag{5}$$

where

$$LC = 2\,arcsin\sqrt{\hat{p} - 1/(2n)} - z_{1-\alpha/2}/\sqrt{n}$$

and

$$UC = 2\,arcsin\sqrt{\hat{p} + 1/(2n)} + z_{1-\alpha/2}/\sqrt{n}$$

where $\hat{p}$ is the proportion, $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ quantile of the standard normal distribution [Hal52].

2.1.3 Interval Estimator for Coverage Analysis : III

Confidence intervals for the proportion can also be formulated from the binomial distribution itself as follows:

Let the probability of the event U be $P\{U\} = p$. If the probability of the event U is p at each observation, irrespective of the outcome of previous observations, then the probability that the event will occur exactly x times in n observations is

$$p\{x\} = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n. \tag{6}$$

Let the ratio of two successive terms of Equation(6) be

$$f(x) = \frac{p\{x+1\}}{p\{x\}} = \frac{n-x}{x+1}\frac{p}{1-p}, \quad x = 0, 1, \ldots, n-1. \tag{7}$$

Hence $f(x)$ is a decreasing function,

$$f(0) > f(1) > \cdots > f(n-1),$$

so the course of the probability density function(PDF) may be characterized as follows:

1. The PDF is steadily decreasing if $f(x) < 1$ for all x, which leads to

$$f(0) = \frac{np}{1-p} < 1 \ \ or \ \ p < \frac{1}{n+1}.$$

2. The PDF is steadily increasing if $f(x) > 1$ for all x, i.e.,

$$f(n-1) = \frac{p}{n(1-p)} > 1 \ \ or \ \ p > \frac{n}{n+1}.$$

3. The PDF first increases and then decreases if $f(0) > 1 > f(n-1)$, i.e.,

$$\frac{1}{n+1} < p < \frac{n}{n+1}.$$

In the third case, the mode($x = r$) of the PDF may be found from the inequality

$$f(r-1) \geq 1 > f(r), \tag{8}$$

which is identical with the inequalities $p\{r\} \geq p\{r-1\}$ and $p\{r\} > p\{r+1\}$. Substitution of Equation(7) into Equation(8) leads to

$$\frac{n-r+1}{r}\frac{p}{1-p} \geq 1 > \frac{n-r}{r+1}\frac{p}{1-p}, \tag{9}$$

which gives

$$(n+1)p - 1 < r \leq (n+1)p.$$

The CDF $P\{x\}$ is discontinuous. So the equation $P\{x\} = P$, or $x = x_P$, can be solved only for $P = P\{0\}, P\{1\}, ..., P\{n\}$, and for these values of P the solutions are $x_P = 0, 1, ..., n$.

The quantiles x_P of the binomial distribution are approximated by the quantiles of the F-distribution(see the *Appendix I* for proof), as

$$P\{x\} \simeq 1 - P\{F(r_1, r_2) < \frac{n-x}{x+1}\frac{p}{1-p}\} =$$

$$1 - P\{\frac{(x+1)F(r_1, r_2)}{(n-x) + (x+1)F(r_1, r_2)} < p\}, \tag{10}$$

where the degrees of freedom for F-distribution are $r_1 = 2(x+1)$ and $r_2 = 2(n-x)$. Thus, the equation $P\{x_P\} = P$ is identical with

$$P\{F(r_1, r_2) < \frac{n-x_P}{x_P+1}\frac{p}{1-p}\} = 1 - P. \tag{11}$$

The confidence intervals are functions of x, n, and P. The upper confidence interval $\bar{p}$ is determined so that the probability of frequencies, h, smaller than or equal to the one observed frequency, $h_0 = \frac{x}{n}$, is P for $p = \bar{p}$, i.e., $\bar{p}$ is determined from the equation

$$P\{h \leq h_0; p = \bar{p}\} = \sum_{v=0}^{x} \binom{n}{v} \bar{p}^v (1-\bar{p})^{n-v} = P, \tag{12}$$

and similarly the lower confidence interval $\underline{p}$ is determined from the equation

$$P\{h \geq h_0; p = \underline{p}\} = \sum_{v=x}^{n} \left(\begin{array}{c} n \\ v \end{array} \right) \underline{p}^v (1-\underline{p})^{n-v} = 1 - P. \tag{13}$$

These equations may be solved directly means of Equation(10) from which we obtain the confidence interval for proportion.

From Equation(9), Equation(10), and Equation(12), a $100(1-\alpha)\%$ upper confidence interval for the proportion is

$$\hat{p}_U = \frac{(x+1)F_{1-\alpha/2}(r_1, r_2)}{(n-x) + (x+1)F_{1-\alpha/2}(r_1, r_2)} \tag{14}$$

where n is the sample size, x is the number of events of interest occurring in the n observations and $F_{1-\alpha/2}(r_1, r_2)$ is the $(1-\alpha/2)$ quantile of the F distribution with $r_1 = 2*(x+1)$ degrees of freedom for the numerator and $r_2 = 2*(n-x)$ degrees of freedom for the denominator.

Similarly, from Equation(9), Equation(10), and Equation(13), a $100(1-\alpha)\%$ lower confidence interval for the proportion is

$$\hat{p}_L = \frac{x}{x + (n-x+1)F_{1-\alpha/2}(r_3, r_4)} \tag{15}$$

where $F_{1-\alpha/2}(r_3, r_4)$ is the $(1-\alpha/2)$ quantile of the F distribution with $r_3 = 2*(n-x+1)$ degrees of freedom for the numerator and $r_4 = 2*x$ degrees of freedom for the denominator [Hal52], [PW93].

3 Numerical Results

Implementing the interval estimators of sequential coverage analysis has been discussed in the previous section. Our other implementation rules for sequential coverage analysis on a single processor and multiple processors under MRIP (Multiple Replications In Parallel) scenario of AKAROA ([PYM94] and [EPM98]) are adopted from [MPE96] and [PME98].

For finding a robust interval estimator for coverage analysis, we used the SA/HW method applied to estimating the mean response times of M/M/1/∞ and M/D/1/∞ queueing systems. All numerical results in this paper were obtained by stopping the simulation experiments when the final steady-state results reached a required precision of 5% or less, at the 0.95 confidence level, and 200 or more bad confidence intervals (to secure representativeness in the analysed data) had been collected. All results were also filtered of strangely short simulation runs after 200 bad confidence intervals are collected. As argued in [MPE96] and [PME98], the filtering of short simulation runs guess a more realistic estimate of actual coverage likely to be achieved under experimental conditions.

The results of sequential coverage analysis for SA/HW using three interval estimators on a single processor are presented in Figure 1 for M/M/1/∞ and Figure 4 for M/D/1/∞ queueing systems. In both simulation models, the half-width of confidence interval of proportions using the normal distribution, the *arcsin* transformation and the F distribution is almost the same.

The simulation results of three estimators of confidence interval for coverage using 2 and 4 processors under MRIP scenario of AKAROA are also presented in Figure 2 and Figure 3 for

M/M/1/∞ and Figure 5 and Figure 6 for M/D/1/∞. As we can see from these Figures, in both M/M/1/∞ and M/D/1/∞ queueing systems, there is no significant difference between these three estimators in this case, as well.

Figure 7 to Figure 9 for M/M/1/∞ and Figure 10 to Figure 12 for M/D/1/∞ depict the results obtained from sequential coverage analysis of SA/HW for the three interval estimators when the simulations were run using $P = 1$, 2 and 4 processors under MRIP scenario of AKAROA. Using more processors usually produces narrow confidence intervals for heavier loaded systems whatever interval estimator is used. For lighter loaded systems, however, using more processors sometimes produces narrower confidence intervals and sometimes produces wider confidence intervals because the required coverage has actually been reached by the actual coverage with sufficiently small confidence intervals using single processor.

Confidence intervals of proportions using the normal, the *arcsin*, and the F distribution at $\alpha = 0.001$ and sample size $n = 50$ are depicted in Figure 13 and the upper limits of confidence intervals of proportions from 0.5 to 1 using the normal, the *arcsin*, and the F distribution are in Table 1. The figures confirm the theoretical claims, that interval estimators of proportion based on the *arcsin* transformation and the F distribution never exceed the practical lower and upper limits of confidence intervals. On the other hand, it shows that the upper limit of the interval estimator of proportion based on the normal distribution can exceed 1.0, making it inappropriate in simulation coverage analysis.

4 Conclusions

While some interesting results have been achieved in theoretical and experimental studies of coverage analysis, experimental analysis of coverage is still required for assessing the quality of practical implementations of methods used for determining confidence intervals in steady-state simulation.

We have experimented with three interval estimators, based on the normal distribution approximation, the *arcsin* transformation and the F distribution, for sequential coverage analysis for the SA/HW method of M/M/1/∞ and M/D/1/∞ queueing systems on a single and multiple processors. Although the results obtained show that they are basically equivalent, being concerned about their validity, we would point at the estimators based on the *arcsin* transformation and the F distribution as more appropriate one in coverage studies, especially if higher value of confidence level is assumed.

Confidence interval estimators for proportions using the (symmetric) normal approximation have been commonly used for coverage analysis of simulation output even though alternative estimators of (asymmetric) confidence intervals for proportions have been proposed in the past. This is probably because the normal approximation had been easier to use in practice than the other estimators. But, the current computing technology has no longer a problem with dealing with alternative estimators. Even confidence intervals for coverage analysis based on the F distribution can be calculated easily by a standard computer. They also guarantee that the upper limits of confidence intervals for proportions do not exceed 1.0.

References

[EPM98] G. C. Ewing, K. Pawlikowski, and D. McNickle. Akaroa 2.5 User's Manual. Technical Report TR-COSC 07/98, Department of Computer Science, University of Canterbury, Christchurch, New Zealand, 1998.

[Gly82] P. W. Glynn. Coverage Error for Confidence Intervals Arising in Simulation Output Analysis. In *Proceedings of the 1982 Winter Simulation Conference*, pages 369–375, 1982.

[Hal52] A. Hald. *Statistical Theory with Engineering Applications*. John Wiley and Sons, Inc., 1952.

[HW81] P. Heidelberger and P. D. Welch. A Spectral Method for Confidence Interval Generation and Run Length Control in Simulations. *Communications of the ACM*, 25:233–245, 1981.

[JBC97] J. Judge, H. W. P. Beadle, and J. Chicharo. Correlation of HTTP Response Packet Size and Estimating Confidence intervals for for Mean Packet Size and WWW Traffic Volume. In *Proceedings of APCC 1997, Sydney, Australia*, pages 382–386, 1997.

[LK91] A. M. Law and W. D. Kelton. *Simulation Modeling and Analysis*. McGraw-Hill, Inc., 2nd Ed., 1991.

[MPE96] D. C. McNickle, K. Pawlikowski, and G. Ewing. Experimental Evaluation of Confidence Interval Procedures in Sequential Steady-State Simulation. In *Proceedings of the 1996 Winter Simulation Conference*, pages 382–389, 1996.

[Paw90] K. Pawlikowski. Steady-State Simulation of Queueing Processes: A Survey of Problems and Solutions. *ACM Computing Surveys*, 22(2):122–170, 1990.

[PME98] K. Pawlikowski, D. C. McNickle, and G. Ewing. Coverage of Confidence Intervals in Sequential Steady-State Simulation. *Simulation Practice and Theory*, 6:255–267, 1998.

[PW93] U. W. Pooch and J. A. Wall. *Discrete Event Simulation: A Practical Approach*. CRC Press, 1993.

[PYM94] K. Pawlikowski, V. Yau, and D. C. McNickle. Distributed and Stochastic Discrete-event Simulation in Parallel Time Streams. *Proceedings of the 1994 Winter Simulation Conference*, pages 723–730, 1994.

[Raa95] K. E. E. Raatikainen. Simulation-Based Estimation of Proportions. *Management Science*, 41(7):1202–1223, 1995.

[Sau79] C. H. Sauer. Confidence Intervals for Queueing Simulations of Computer Systems. *ACM Performance Evaluation Review*, 8(1-2):46–55, 1979.

[Sch80] L. W. Schruben. A Coverage Function for Interval Estimators of Simulation Response. *Management Science*, 26:18–27, 1980.

Appendix I : *Relationship between Binomial and F-Distributions*

When deriving Equation(10) we consider the integral

$$\int_p^1 y^x(1-y)^{n-x-1}dy, \;\; x = 0, 1, \ldots, n$$

which by integration by parts gives

$$\int_p^1 y^x(1-y)^{n-x-1}dy = \frac{1}{n-x}p^x(1-p)^{n-x} + \frac{x}{n-x}\int_p^1 y^{x-1}(1-y)^{n-x}dy$$

If reduction of the integral is continued by successive applications of this formula, we obtain, after multiplication by $n\begin{pmatrix} n-1 \\ x \end{pmatrix}$,

$$n\begin{pmatrix} n-1 \\ x \end{pmatrix}\int_p^1 y^x(1-y)^{n-x-1}dy = \sum_{v=0}^{x}\begin{pmatrix} n \\ v \end{pmatrix}p^v(1-p)^{n-v} = P\{x\}. \tag{16}$$

This result may be expressed by the incomplete Beta-function(see the *Appendix III*), which according to Equation(24) may be written as

$$B_p(x+1, n-x) = \int_0^p y^x(1-y)^{n-x-1}dy.$$

Substitution into Equation(16) leads to

$$P\{x\} = \frac{B(x+1, n-x) - B_p(x+1, n-x)}{B(x+1, n-x)} = 1 - I_p(x+1, n-x), \tag{17}$$

see Equation(23) and Equation(25).

Furthermore, according to Equation(26) the CDF of F distribution for the variable (see the Equation(20) in *Appendix II*)

$$y = \frac{r_1F(r_1, r_2)}{r_2 + r_1F(r_1, r_2)}$$

is

$$P\{y\} = I_y(\frac{r_1}{2}, \frac{r_2}{2})$$

from which we - after comparison with Equation(17) - see that

$$P\{x\} = 1 - P\{y\}$$

for

$$y = \frac{r_1F(r_1, r_2)}{r_2 + r_1F(r_1, r_2)} = p$$

and

$$r_1 = 2(x+1), \;\; r_2 = 2(n-x).$$

This result may also be written

$$P\{x\} \simeq 1 - P\{\frac{(x+1)F(r_1, r_2)}{(n-x) + (x+1)F(r_1, r_2)} < p\} \tag{18}$$

or as

$$P\{x\} \simeq 1 - P\{F(r_1, r_2) < \frac{(n-x)}{(x+1)}\frac{p}{1-p}\}\text{[Hal52]}.$$

Appendix II : *Relationship between Beta and F-Distributions*

The F-distribution may be expressed by the well-known Beta function in the following way(For detail, see the pp.381-384 in [Hal52]): The probability density function of F-distribution is

$$p\{z\} = \frac{\Gamma(\frac{r_1+r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} r_1^{\frac{r_1}{2}} r_2^{\frac{r_2}{2}} \frac{z^{\frac{r_1}{2}-1}}{(r_2 + r_1 z)^{\frac{r_1+r_2}{2}}}, \quad (0 \leq z < \infty), \tag{19}$$

for $z = F(r_1, r_2)$.

Introducing the variable y by the transformation

$$F(r_1, r_2) = \frac{r_2}{r_1}\frac{y}{1-y} \quad (0 \leq y \leq 1),$$

i.e.,

$$y = \frac{r_1 F(r_1, r_2)}{r_2 + r_1 F(r_1, r_2)} \tag{20}$$

and

$$1 - y = \frac{r_2}{r_2 + r_1 F(r_1, r_2)} \tag{21}$$

we obtain from Equation(19) the following equation

$$p\{y\} = \frac{\Gamma(\frac{r_1+r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} y^{\frac{r_1}{2}-1}(1-y)^{\frac{r_2}{2}-1} \quad (0 \leq y \leq 1), \tag{22}$$

which is called the Beta-distribution.

Appendix III : *Beta Function and Incomplete Beta Function*

Beta function is

$$B(s,t) = \int_0^1 y^{s-1}(1-y)^{t-1}dy = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \tag{23}$$

Incomplete Beta function is the integral

$$B_y(s,t) = \int_0^y y^{s-1}(1-y)^{t-1}dy, (0 \le y \le 1) \tag{24}$$

The ratio between the complete and the incomplete Beta function is

$$I_y(s,t) = \frac{B_y(s,t)}{B(s,t)}. \tag{25}$$

From Equation(22)-(25) it follows that

$$P\{y\} = I_y(\frac{r_1}{2}, \frac{r_2}{2}) \tag{26}$$

so that the quantiles of the y-distribution - and therefore also of the F-distribution - may be computed from complete and incomplete Beta function.

Appendix IV : *Theorem I*

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$, then the standardized variable

$$Z = \frac{X - np}{\sqrt{np(1-p)}}, \tag{27}$$

as $n \to \infty$, is approximately the standardized normal distribution $n(0,1)$.

Appendix V : *Relationship between Binomial and Normal Distribution*

Dividing both numerator and denominator of Equation(27) of *Appendix IV* by n, we obtain

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}},$$

i.e., the proportion $\hat{p} = x/n$ is approximately normally distributed with mean p and variance $p(1-p)/n$.

In most applications of the binomial distribution it is not a single value of $p\{x\}$ which is required, but sums of $p\{x\}$ of the form

$$P\{x_1 \le x \le x_2\} = \sum_{x=x_1}^{x_2} p\{x\}.$$

The areas of the corresponding to this sum may be approximated by the area of the normal distribution from $x_1 - \frac{1}{2}$ to $x_2 + \frac{1}{2}$, so that

$$P\{x_1 \leq x \leq x_2\} = \sum_{x=x_1}^{x_2} p\{x\} \simeq \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1-\frac{1}{2}}^{x_2+\frac{1}{2}} e^{-\frac{(x-\xi)^2}{2\sigma^2}} dx =$$

$$\Phi(\frac{x_2 + \frac{1}{2} - \xi}{\sigma}) - \Phi(\frac{x_1 - \frac{1}{2} - \xi}{\sigma}),$$

where Φ is the CDF of a Gaussian random variable with $\xi = np$ and $\sigma = \sqrt{np(1-p)}$. For $x = x_1 = x_2$, we obtain an approximation to $p\{x\}$, namely,

$$p\{x\} \simeq \Phi(\frac{x + \frac{1}{2} - \xi}{\sigma}) - \Phi(\frac{x - \frac{1}{2} - \xi}{\sigma}).$$

For the cumulative distribution function $P\{x\}$ we have

$$P\{x\} \simeq \Phi(\frac{x + \frac{1}{2} - np}{\sqrt{np(1-p)}}), \tag{28}$$

which leads to

$$\frac{x_{1-\alpha/2} + \frac{1}{2} - np}{\sqrt{np(1-p)}} \simeq Z_{1-\alpha/2},$$

or

$$x_{1-\alpha/2} \simeq np - \frac{1}{2} + Z_{1-\alpha/2}\sqrt{np(1-p)},$$

and by division by n

$$\hat{p}_{1-\alpha/2} \simeq p - \frac{1}{2n} + Z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}},$$

or

$$\hat{p}_{1-\alpha/2} + \frac{1}{2n} \simeq p + Z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}, \tag{29}$$

or

$$\frac{\hat{p}_{1-\alpha/2} + \frac{1}{2n} - p}{\sqrt{\frac{p(1-p)}{n}}} \simeq Z_{1-\alpha/2}. \tag{30}$$

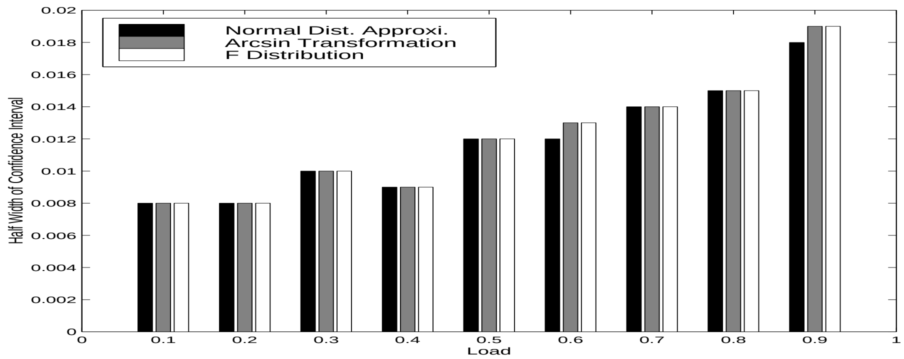

Figure 1: Half-Width of Confidence Interval of Coverage Using Normal Distribution Approximation, *Arcsin* Transformation and F-Distribution Approximation in M/M/1/∞ Queueing System ($P = 1$)

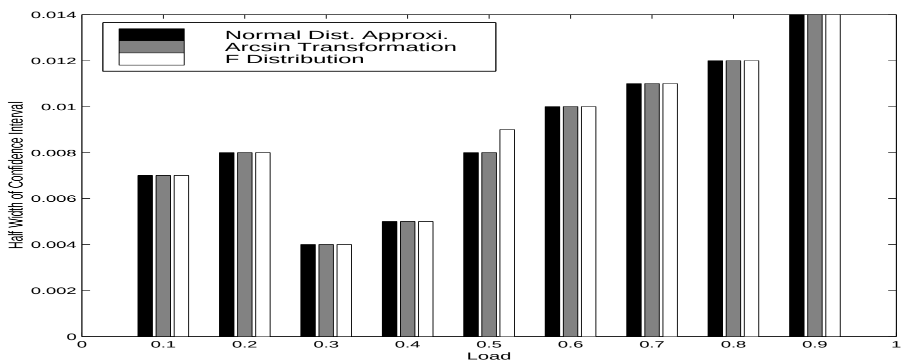

Figure 2: Half-Width of Confidence Interval of Coverage Using Normal Distribution Approximation, *Arcsin* Transformation and F-Distribution Approximation in M/M/1/∞ Queueing System ($P = 2$)

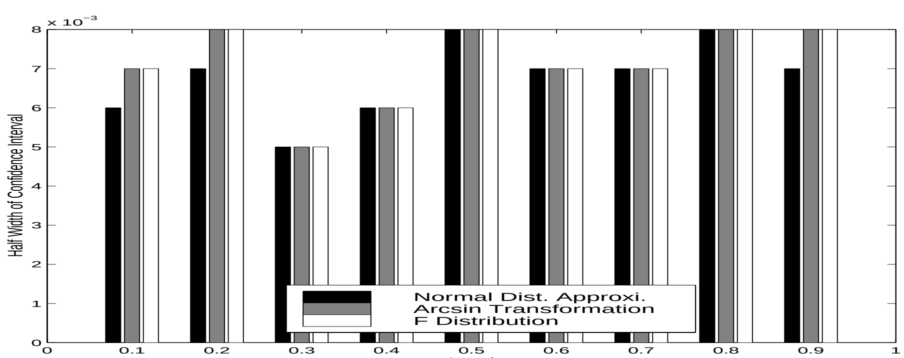

Figure 3: Half-Width of Confidence Interval of Coverage Using Normal Distribution Approximation, *Arcsin* Transformation and F-Distribution Approximation in M/M/1/∞ Queueing System ($P = 4$)

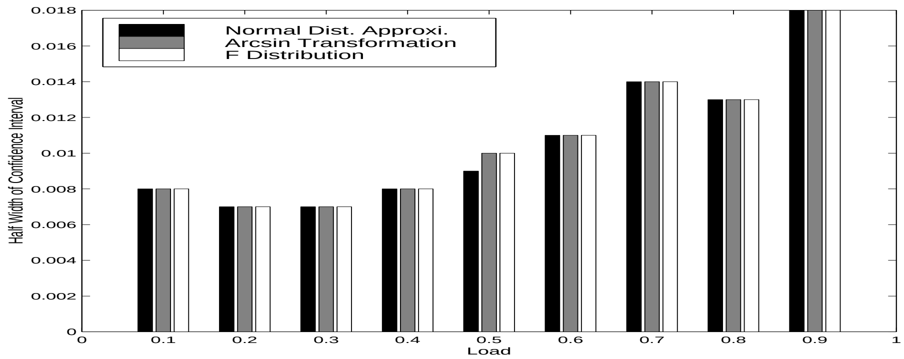

Figure 4: Half-Width of Confidence Interval of Coverage Using Normal Distribution Approximation, *Arcsin* Transformation and F-Distribution Approximation in M/D/1/∞ Queueing System ($P = 1$)

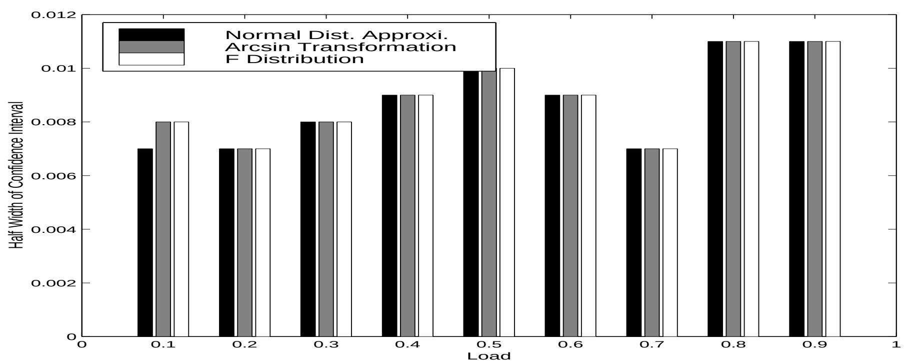

Figure 5: Half-Width of Confidence Interval of Coverage Using Normal Distribution Approximation, *Arcsin* Transformation and F-Distribution Approximation in M/D/1/∞ Queueing System ($P = 2$)

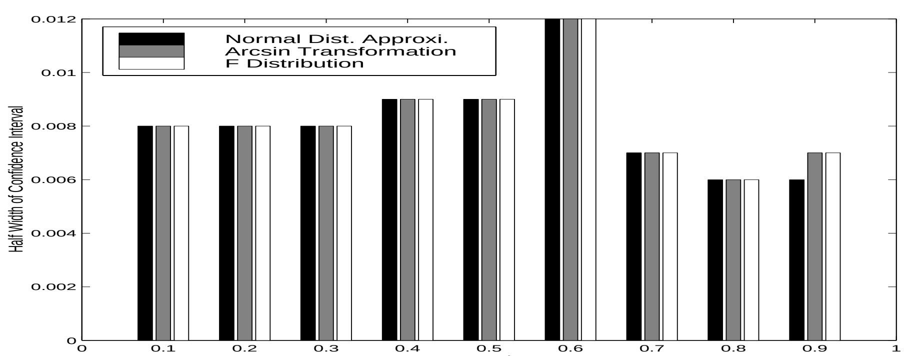

Figure 6: Half-Width of Confidence Interval of Coverage Using Normal Distribution Approximation, *Arcsin* Transformation and F-Distribution Approximation in M/D/1/∞ Queueing System ($P = 4$)

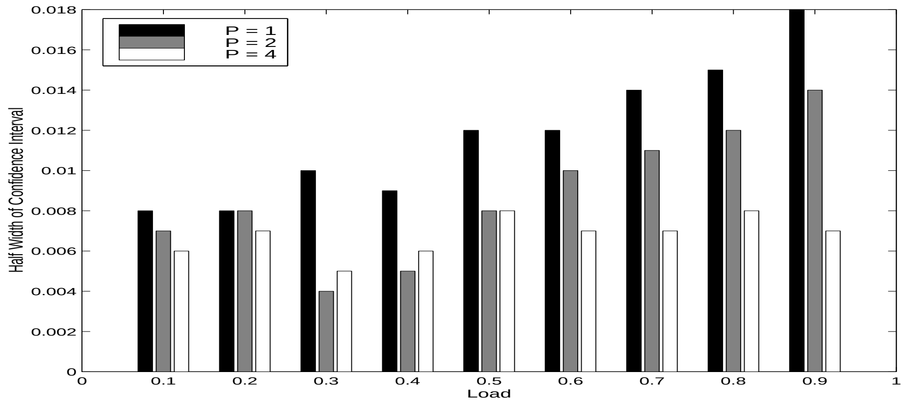

Figure 7: Half-Width of Confidence Interval of Coverage Using 1, 2, and 4 Processors in Normal Distribution Approximation of M/M/1/∞ Queueing System

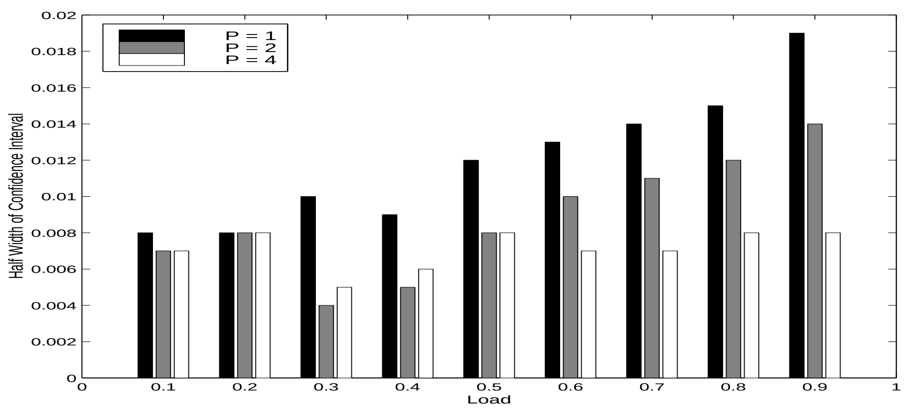

Figure 8: Half-Width of Confidence Interval of Coverage Using 1, 2, and 4 Processors in *Arcsin* Transformation of M/M/1/∞ Queueing System

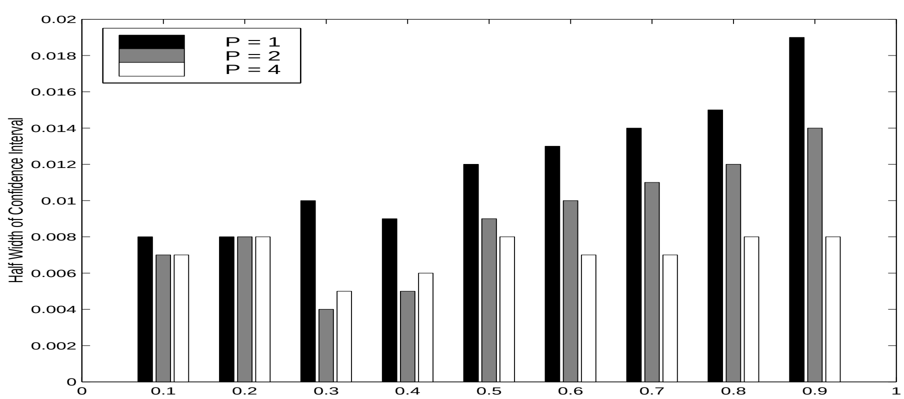

Figure 9: Half-Width of Confidence Interval of Coverage Using 1, 2, and 4 Processors in F-Distribution Approximation of M/M/1/∞ Queueing System

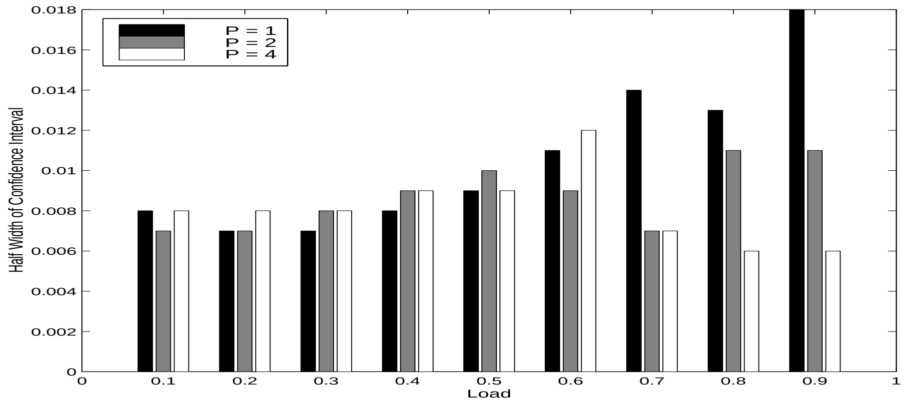

Figure 10: Half-Width of Confidence Interval of Coverage Using 1, 2, and 4 Processors in Normal Distribution Approximation of M/D/1/∞ Queueing System

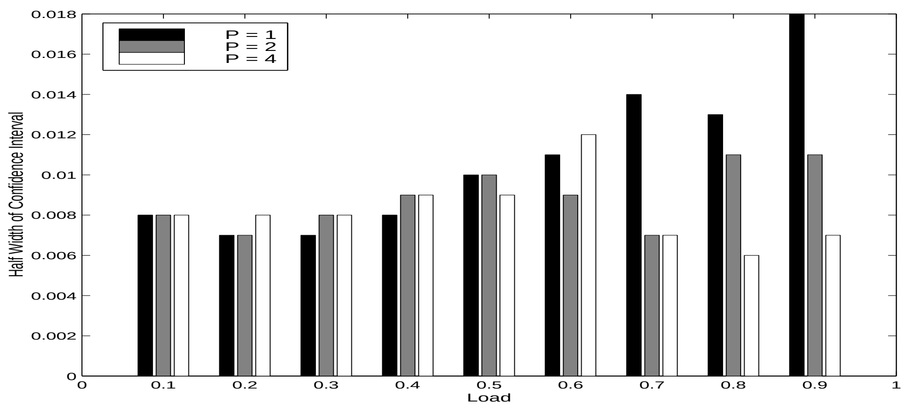

Figure 11: Half-Width of Confidence Interval of Coverage Using 1, 2, and 4 Processors in *Arcsin* Transformation of M/D/1/∞ Queueing System

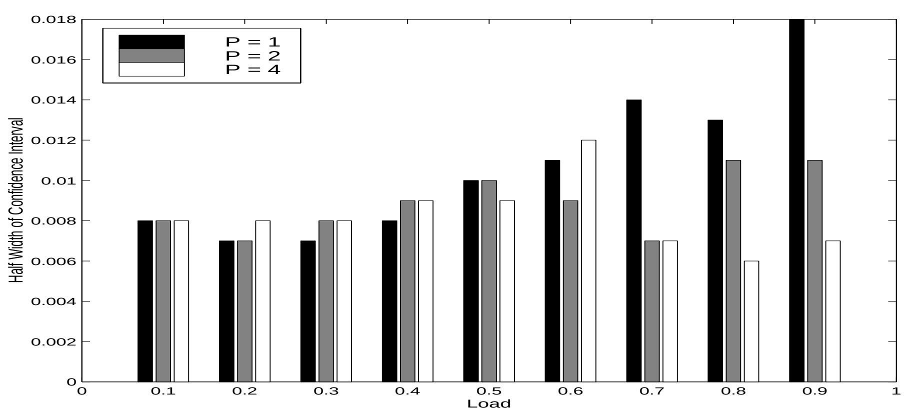

Figure 12: Half-Width of Confidence Interval of Coverage Using 1, 2, and 4 Processors in F-Distribution Approximation of M/D/1/∞ Queueing System

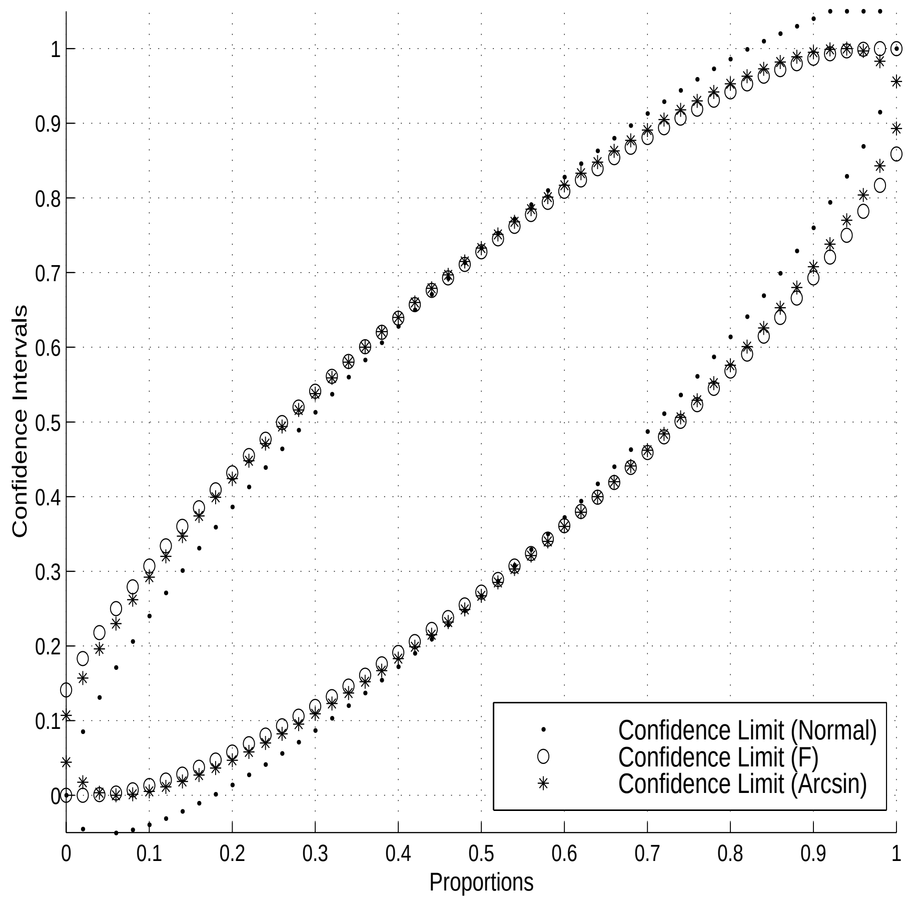

Figure 13: Confidence intervals of proportions using normal, *arcsin*, and F distribution ($\alpha = 0.001$ & $n = 50$)

Table 1: Upper limits of confidence intervals of proportions using normal, *arcsin*, and F distribution ($\alpha = 0.001$ & $n = 50$)

Proportion	*Normal*	*Arcsin*	F
0.5	0.733	0.733	0.728
0.52	0.753	0.751	0.745
0.54	0.772	0.768	0.762
0.56	0.791	0.785	0.778
0.58	0.81	0.801	0.794
0.6	0.828	0.817	0.809
0.62	0.846	0.833	0.824
0.64	0.863	0.848	0.839
0.66	0.88	0.863	0.854
0.68	0.897	0.877	0.868
0.7	0.913	0.891	0.881
0.72	0.929	0.905	0.894
0.74	0.944	0.918	0.907
0.76	0.959	0.93	0.919
0.78	0.973	0.942	0.931
0.8	0.986	0.953	0.942
0.82	0.999	0.963	0.953
0.84	1.01	0.973	0.963
0.86	1.02	0.982	0.972
0.88	1.03	0.989	0.98
0.9	1.04	0.995	0.987
0.92	1.05	0.999	0.993
0.94	1.05	1	0.997
0.96	1.05	0.997	0.999
0.98	1.05	0.983	1
1	1	0.956	1