Group divisible designs with two associate classes

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1 Introduction

The work in this research report was done while visiting The University of Canterbury, and was completed jointly in cooperation with H.L. Fu and D. Sarvate. These results follow upon previous efforts where we were investigating the existence of group divisible designs with first and second associates and with block size 3. Background information concerning this problem will be added to the final version which will incorporate all our work on the topic.

Graph theoretically, we are looking for a partition of the edges of a graph H into copies of K_3 (each K_3 is also called a *triple*). In our case, H is the multigraph with vertex set $V = V_0 \cup V_1 \cup \ldots \cup V_{m-1}$, $|V_i| = n$ for each $i \in \boldsymbol{Z}_m$, in which two vertices are joined by λ_1 edges if they both occur in V_i for some i, and otherwise are joined by λ_2 edges. Edges joining vertices in the same or different groups are called *pure* or *cross* edges respectively. Such a decomposition of H into copies of K_3 is called a *group divisible design* and is denoted by a GDD(n, m) of index (λ_1, λ_2). Formally, such a GDD is represented by the ordered triple $(V, \{V_0, \ldots, V_{m-1}\}, B)$, where B is the collection of triples. If $m = 1$ then the GDD is simply a triple system, so a GDD$(n, 1)$ of index (λ_1, λ_2) is denoted more simply by a TS(n) of index λ_1.

We have already completely solved this problem in the case where $n, m \geq 3$, proving the following result.

Theorem 1.1 *Let $n, m \geq 3$ and $\lambda_1, \lambda_2 \geq 1$. There exists a* GDD$(n, m)$ *of index* (λ_1, λ_2) *if and only if*

(1) *2 divides* $\lambda_1(n-1) + \lambda_2(m-1)n$, *and*

(2) *3 divides* $\lambda_1 n(n-1) + \lambda_2 m(m-1)n^2$.

In this report, the case where $m = 2$ is completely solved. At first sight, this would seem to be quite simple to handle compared to the myriad of cases that have to be considered to prove Theorem 1.1. However, it turns out to be a very interesting case, requiring different solution techniques and another necessary condition.

Lemma 1.2 *If there exists a* $\mathrm{GDD}(n, 2)$ *of index* (λ_1, λ_2) *then*

(1) *2 divides* $\lambda_1(n-1) + \lambda_2 n$,

(2) *3 divides* $\lambda_1 n(n-1) + \lambda_2 n^2$, *and*

(3) $\lambda_1 \geq \lambda_2 n/2(n-1)$.

Proof: (1) and (2) follow because each vertex must have even degree, and the number of edges must be divisible by 3. (3) follows since any cross edge must be contained in a triple that contains another cross edge and a pure edge, so the number of pure edges must be at least half the number of cross edges. □

We will now proceed to show that these three conditions are also sufficient for the existence of a $\mathrm{GDD}(n, 2)$ of index (λ_1, λ_2). The main result is finally stated as Theorem 3.7.

2 Preliminary Results

In this section we obtain several building blocks. In Section 3, these will be put together in various ways to obtain the main result.

Lemma 2.1 *Let* $n \geq 3$. *There exists a* $\mathrm{GDD}(n, 2)$ *of index* $(n, 2n-2)$.

Proof: Define

$$B = \{\{(a,0),(b,0),(c,1)\},\{(a,1),(b,1),(c,0)\} \,|\, 0 \leq a < b \leq n-1, c \in \boldsymbol{Z}_n\}\ .$$

Then $(\boldsymbol{Z}_n \times \boldsymbol{Z}_2,\ \{\boldsymbol{Z}_n \times \{i\} \,|\, i \in \boldsymbol{Z}_2\}, B)$ is a $\mathrm{GDD}(n, 2)$ of index $(n, 2n-2)$. □

The following is a result of Petersen.

Theorem 2.2 *[1] Let H be a regular multigraph of even degree. Then there exists a 2-factorization of H.*

Lemma 2.3 is a special case of a result of Rodger and Stubbs.

Lemma 2.3 *[3] Let $\lambda, n \geq 1$. Suppose that $0 \leq x \leq \lambda(n-1)$, x is even, and 3 divides xn. Then there exists an x-regular multigraph of multiplicity at most λ with n vertices whose edges can be partitioned into triples.*

These two results can be combined to obtain Corollary 2.4. Let $E(H)$ be the set of edges in H.

Corollary 2.4 *Suppose that $\lambda, n \geq 1$, $0 \leq x \leq \lambda(n-1)$, 3 divides xn, and $\lambda(n-1)$ and x are even. Then there exists an x-regular multigraph H of multiplicity at most λ with n vertices whose edges can be partitioned into triples, such that $\lambda K_n - E(H)$ has a 2-factorization.*

Proof: Choose H using Lemma 2.3, then apply Theorem 2.2 to $\lambda K_n - E(H)$. $\square$

We will need a companion result to Corollary 2.4 to cope with the situation where $\lambda(n-1)$ is odd. Obtaining this result will require the following results, the first by Stern and Lenz, the second by Rees, and the third by Simpson. For any $D \subseteq \boldsymbol{Z}_{\lfloor a/2 \rfloor}$, let $H[D]$ be the graph with vertex set $\boldsymbol{Z}_n$ and edge set $\{\{j, j+d\} | d \in D, j \in \boldsymbol{Z}_n\}$, reducing the sum modulo n.

Lemma 2.5 *[5] There exists a 1-factorization of $H[D]$ if and only if there exists a $d \in D$ such that $d/gcd(n,d)$ is even.*

Notice that if $d = n/2 \in D$ then since $d/gcd(n,d)$ is even, $H[D]$ has a 1-factorization.

Theorem 2.6 *[2] For all $\equiv 0 \pmod 6$ and for all even x with $0 \leq x < n$ except for $(n,x) \in \{(12,10),(6,4)\}$, there exists an x-regular simple graph H on n vertices whose edges can be resolvably partitioned into triples, such that $K_n - E(H)$ has a 1-factorization.*

Theorem 2.7 *[4] For any $y \geq 1$ and for some $s \in \{3y, 3y+1\}$, the integers in $\{y+1, y+2, \ldots, 3y+1\} \backslash \{s\}$ can be partitioned into pairs (a_i, b_i) with $b_i > a_i$ such that $\{b_i - a_i | 1 \leq i \leq y\} = \{1, 2, \ldots, y\}$.*

We can now present the companion to Corollary 2.4. It is probably a result that is of interest in its own right.

Theorem 2.8 *Suppose that $\lambda \geq 1, n \geq 3, 0 \leq x \leq \lambda(n-1)$, 3 divides xn and 2 divides n and x. Then*

(i) there exists an x-regular graph H on n vertices and of multiplicity at most λ whose edges can be partitioned into triples, and

(ii) such that $\lambda K_n - E(H)$ has a 1-factorization.

Proof: For each $\lambda \geq 1$ and each even $n \geq 3$, let $S(n, \lambda)$ be the set of integers x for which (i) and (ii) are true. Let $\ell = 2$ if $n \equiv 0$ or 4 (mod 6) and let $\ell = 6$ if $n \equiv 2$ (mod 6).

Since there exists a 1-factorization of K_n, if $x \in S(n, \lambda)$ then $x \in S(n, \lambda')$ for all $\lambda' \geq \lambda$. Also, since there exists a $TS(n)$ of index ℓ, if $x = y\ell(n-1) + x'$ with $0 \leq x' < \lambda(n-1)$ and $\lambda \leq \ell$, and if $x' \in S(n, \lambda)$, then $x \in S(n, \lambda + y\ell)$. Therefore we need only consider the cases where $x < \ell(n-1)$.

Suppose that $n \equiv 0$ (mod 6). We need only consider the cases where $x < 2(n-1)$. If $x < n$ then the result follows from Theorem 2.6 unless $(n, x) \in \{(12, 10), (6, 4)\}$. Fortunately, since we do not require the set of triples to be resolvable, we can obtain solutions in these cases too: for each $m \in \{3, 6\}$ the complement of the edges in the triples of a GDD$(2, m)$ of index $(0, 1)$ is a 1-factor. If $n \leq x \leq 2n-4$ then we can simply combine a solution where $x' = n-2$ and $\lambda' = 1$ with a solution where $x'' = x - (n-2)$ and $\lambda = 1$.

If $n \equiv 2$ or 4 (mod 6) then since x is even and 3 divides xn, we have that $x \equiv 0$ (mod 6), so let $x = 6y$. If $x = n-2$ then $n \equiv 2$ (mod 6); since there exists a GDD$(2, 3y+1)$ of index $(0, 1)$ we have that $n - 2 \in S(n, 1)$. If $x < n-2$ then define s, a_i and b_i as in Theorem 2.7, and let $T = \{\{j, a_i + j, b_i + j\} | j \in \boldsymbol{Z}_n\}$, reducing sums modulo n. Then T is a set of triples that partition $H = H[D']$ where $D' = \{1, 2, \ldots, 3y+1\} \backslash \{s\}$, and $K_n - E(H) = H[D]$ where $d = \{1, 2, \ldots, n/2\} \backslash D$. Since $x < n-2, n/2 \in D$, so $K_n - E(H)$ has a 1-factorization by Lemma 2.5. So it remains to consider $x \geq n$.

If $n \equiv 4$ (mod 6) then $\ell = 2$ so we can assume that $x < 2(n-1)$; so $n+2 \leq x \leq 2n-8$ (since $x \equiv 0$ (mod 6)). We can combine a solution where $x' = n-4$ and $\lambda' = 1$ with a solution where $x'' = x - (n-4) \leq n-4$ and $\lambda'' = 1$.

If $n \equiv 2$ (mod 6) then $\ell = 6$, so we can assume that $x < 6(n-1)$; so $n + 4 \leq x \leq 6n-12$ (since $x \equiv 0$ (mod 6)). Let ℓ' be such that $\ell'(n-2) < x \leq (\ell'+1)(n-2)$. Combine

ℓ' solutions where $x' = n-2$ and $\lambda' = 1$ with a solution where $x'' = x - \ell'(n-2) \leq n-2$ and $\lambda'' = 1$. $\square$

It will be useful to define $[x, y, z]$ to denote the graph with vertex set $\boldsymbol{Z}_n \times \boldsymbol{Z}_2$ in which two vertices (u, i) and (v, j) are joined by x edges if $i = j = 0$, by y edges if $i \neq j$, and by z edges if $i = j = 1$.

The next four results are crucial building blocks in the construction of the GDD's in Section 3.

Lemma 2.9 *For each $i \in \boldsymbol{Z}_2$, let T_i be an xn-regular multigraph on the vertex set $\boldsymbol{Z}_n \times \{i\}$ that has a 1-factorization. Then there exists a set of triples whose edges partition the edges of $[0, x, 0] + T_i$.*

Proof: Partition the xn 1-factors in a 1-factorization of T_i into x sets $S_0, S_1, \ldots, S_{n-1}$, each of size x. For each $a \in \boldsymbol{Z}_n$ and for each edge $\{(u, i), (v, i)\}$ in a 1-factor in S_a, let B contain the triple $\{(u, i), (v, i), (a, i+1)\}$, reducing the sum modulo 2. $\square$

Lemma 2.10 *Let n be odd, and let F be any 1-factor of $[0, 1, 0]$. Then there exists an edge-disjoint decomposition of $[1, 1, 0] - F$ and of $[0, 1, 1] - F$ into copies of K_3.*

Proof: Let $(\boldsymbol{Z}_n, \circ)$ be a symmetric idempotent quasigroup of order n. Let $i \in \boldsymbol{Z}_n$ and let $F' = \{\{(a, 0), (a, 1)\} | a \in \boldsymbol{Z}_n\}$. Let $B'_i = \{\{(a, i), (b, i), (a \circ b, i+1)\} | 0 \leq a < b \leq n-1\}$, reducing $i+1$ modulo 2. Then clearly the triples in B' partition the edges in $[1, 1, 0] - F'$ or $[0, 1, 1] - F'$ if $i = 0$ or 1 respectively. The first coordinate of the symbols in the triples in B'_i whose second coordinate is $i+1$ can easily be renamed to produce a set of triples B_i that partition the edges of $[1, 1, 0] - F$ or $[0, 1, 1] - F$ as required. $\square$

Lemma 2.11 *Let $i \in \boldsymbol{Z}_2$, and let H_i be a $2x$-regular graph on the vertex set $\boldsymbol{Z}_n \times \{i\}$. Then there exists a $2x$-regular multigraph T consisting of $2x$ 1-factors, each being in $[0, 1, 0]$, such that there exists an edge-disjoint decomposition of $H_i + T$ into copies of K_3.*

Proof: By Theorem 2.2, H_i has a 2-factorization into x 2-factors $T_0, T_1, \ldots, T_{x-1}$. For each $j \in \boldsymbol{Z}_x, T_x$ consists of vertex disjoint cycles which we can arbitrarily orient to form directed cycles; call the resulting directed graph T'_j. Let H'_i be the corresponding directed graph. For each directed edge (a, b) in T'_j, let $\{(a, i), (b, i+1)\} \in F_{2j}$ and

$\{(a,i),(a,i+1)\} \in F_{2j+1}$. Let T be the $2x$-regular multigraph formed by the sum of $F_0, \ldots, F_{2x-1}$. Then $B = \{\{(a,i),(b,i),(b,i+1)\} | (a,b) \in E(H_i')\}$ is a set of triples whose edges partition the edges of $H_i + T$. □

Lemma 2.12 *Let $n \geq 4$ be even. Let $\epsilon = 0$ if $n \equiv 0 \pmod 4$, $\epsilon = 1$ if $n \equiv 6 \pmod{12}$, and $\epsilon = 3$ if $n \equiv 2$ or $10 \pmod{12}$. For each $i \in \boldsymbol{Z}_2$ there exists a simple graph H_i on the vertex set $\boldsymbol{Z}_n \times \{i\}$ such that:*

(i) H_0 is $(n/2+\epsilon)$-regular and H_1 is $(n/2-\epsilon)$-regular,

(ii) the edges of $[0,1,0] + H_0 + H_1$, can be partitioned into triples, and

(iii) there exists a 1-factorization of $K_n - E(H_i), i \in \boldsymbol{Z}_2$.

Proof: Let $D = \{2k-1 \,|\, 1 \leq k \leq n/4\}$. Define

$$D_0 = \begin{cases} D & \text{if } \epsilon = 0, \\ D \cup \{2\} & \text{if } \epsilon = 1, \\ D \cup \{2,4\} & \text{if } \epsilon = 3, \end{cases}$$

and define

$$D_1 = \begin{cases} D & \text{if } \epsilon = 0, \\ (D \cup \{2\}) \backslash \{n/2-2\} & \text{if } \epsilon = 1, \\ D \cup \{n/2-4\} & \text{if } \epsilon = 3. \end{cases}$$

In any case, define $H_i = H[D_i]$ on the vertex set $\boldsymbol{Z}_n \times \{i\}$, for each $i \in \boldsymbol{Z}_2$. Then clearly H_i satisfies (i), and since $n/2 \in D_i$ it follows from Lemma 2.5 that (iii) is satisfied.

If $\epsilon = 0$ then let $B = \{\{(j,0),(j+2k-1,0),(j+k+n/4,1)\},\ \{(j,1),(j+2k-1,1),(j+k+n/4-1,0)\} \,|\, j \in \boldsymbol{Z}_n,\ 1 \leq k \leq n/4\}$.

If $\epsilon = 1$ then let $B = \{\{(j,0),(j+2k-1,0),(j+k+(n+2)/4,1)\},\ \{(j,0),(j+2,0),(j+1,1)\} \,|\, j \in \boldsymbol{Z}_n,\ 1 \leq k \leq (n-2)/4\} \cup \{\{(j,1),(j+2k-1,1),\ (j+k+(n+2)/4,0)\},\ \{(j,1),(j+2,1),(j+2,0)\} \,|\, j \in \boldsymbol{Z}_n,\ 1 \leq k \leq (n-6)/4\}$.

If $\epsilon = 3$ then let $B = \{\{(j,0),(j+2k-1,0),(j+k+(n+6)/4,1)\} \,|\, j \in \boldsymbol{Z}_n, 1 \leq k \leq (n-2)/4\} \cup \{\{(j,0),(j+2,0),(j+1,1)\},\{(j,0),(j+4,0),(j+2,1)\},\{(j,1),(j+(n-4)/2,1),(j+(n-4)/2,0) \,|\, j \in \boldsymbol{Z}_n\} \cup \{\{(j,1),(j+2k-1,1),(j+k+(n-2)/4,0)\} \,|\, j \in \boldsymbol{Z}_n, 1 \leq k \leq (n-10)/4\}$.

Then in each case, B is a set of triples which partition the edges of $[0,1,0]+H_0+H_1$. □

The following structure will be needed in Section 3.

Let n be even, and let F be a partition of $\boldsymbol{Z}_n$ into sets of size 2. A *symmetric quasigroup* $(\boldsymbol{Z}_n, \circ)$ with *holes* F and of *order* n is an $n \times n$ array in which: cell (a, b) contains exactly one symbol in $\boldsymbol{Z}_n$ if $\{a, b\} \notin F$ and no symbols if $\{a, b\} \in F$; for each $a \in \boldsymbol{Z}_n$ row and column a contain each symbol in $\boldsymbol{Z}_n$ exactly once except for symbols a and b, where $\{a, b\} \in F$; and cells (a, b) and (b, a) either contain the same symbol or are both empty, for $0 \leq a < b \leq n-1$. The following is well known.

Lemma 2.13 *For all even $n \geq 6$, there exists a symmetric quasigroup with holes F and of order n, where F is a partition of $\boldsymbol{Z}_n$ into sets of size 2.*

Since maximum packings and minimum coverings of triple systems have been completely determined, we have the following result.

Lemma 2.14 *Let $n \equiv 2 \pmod 6$, $n \geq 8$ and let L be a set of 2 independent edges in K_n. Then there exists an edge-disjoint decomposition of $(6y+2)K_n + 2L$ and of $(6y+4)K_n - 2L$ into copies of K_3, for all $y \geq 0$.*

Finally, it will probably help enormously to list the values of n that satisfy conditions (1) and (2) of Lemma 1.2 for all values of λ_1 and λ_2. This is done in Table 1.

λ_1 \ λ_2	0	1	2	3	4	5
0	any	0	$0,3$	even	$0,3$	0
1	$1,3$	–	3	–	$3,5$	–
2	$0,1,3,4$	0	$0,2,3,5$	$0,4$	$0,3$	$0,2$
3	odd	–	3	–	3	–
4	$0,1,3,4$	$0,2$	$0,3$	$0,4$	$0,2,3,5$	0
5	$1,3$	–	$3,5$	–	3	–

Table 1. The values of $n \pmod 6$ for each value of $\lambda_1 \pmod 6$ and $\lambda_2 \pmod 6$ that satisfy conditions (1) and (2) of Lemma 1.2.

3 The Main Results

We begin with a result that helps us deal with condition (3) of Lemma 1.2. It allows us to focus on large values of n, so then this lower bound on λ_1 will no longer be a moving target (that is, a function of n).

Proposition 3.1 *If conditions (1-3) of Lemma 1.2 are sufficient for the existence of a* GDD$(n, 2)$ *of index* (λ_1, λ_2) *whenever* $\lambda_2 \leq 2(n-1)$*, then they are sufficient for all* $\lambda_2 \geq 1$.

Proof: Suppose that n, λ_1 and λ_2 satisfy conditions (1-3) of Lemma 1.2, that $2x(n-1) < \lambda_2 \leq (2x+2)(n-1)$, and that $x \geq 1$. Then by (3),

$$\lambda_1 \geq \begin{cases} \lambda_2/2 + x + 1 & \text{if } \lambda_2 \text{ is odd and } \lambda_2 > (2x+1)(n-1), \\ \lambda_2/2 + x & \text{otherwise.} \end{cases}$$

Let $\epsilon = 1$ if λ_2 is odd and $\lambda_2 > (2x+1)(n-1)$, and $\epsilon = 0$ otherwise. Since $\lambda_2' = \lambda_2 - 2x(n-1) \leq 2(n-1)$, and since $\lambda_1' = \lambda_1 - xn \geq \lambda_2/2 + x + \epsilon - xn = (\lambda_2 - 2x(n-1))/2 + \epsilon = \lambda_{2/2}' + \epsilon$, so $\lambda_1 \geq \lambda_2' n/2(n-1)$, (3) is satisfied by n, λ_1' and λ_2', and (1) and (2) are easily seen to be satisfied too. Therefore, by our assumption there exists a GDD$(n, 2)$ of index $(\lambda_1 - xn, \lambda_2 - 2x(n-1))$. Also, by Lemma 2.1 there exists a GDD$(n, 2)$ of index $(xn, x(2n-2))$ for any $x \geq 1$. So together these two GDD's form a GDD$(n, 2)$ of index (λ_1, λ_2). □

Therefore, it remains to consider the case where $\lambda_2 \leq 2(n-1)$; or $n \geq \lambda_2/2 + 1$. Under this condition, (3) simply becomes $\lambda_1 \geq (\lambda_2 + 1)/2$. So throughout the rest of this paper we will assume that n and λ_1 satisfy these lower bounds imposed by λ_2.

Proposition 3.2 *Suppose that n is odd, $\lambda_1 \geq \lambda_2/2 + 1$ and $n \geq \lambda_2/2 + 1$. Let n, λ_1 and λ_2 satisfy conditions (1) and (2) of Lemma 1.2. Then there exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2).

Proof: Since n is odd, λ_2 is even (see Table 1). Let $\lambda = \lambda_1 - \lambda_2/2$. So $\lambda \geq 1$. The result will follow if we can find an integer t that satisfies the following conditions:

(i) $0 \leq 2t \leq \lambda(n-1)$ and 3 divides $(\lambda(n-1) - 2t)n$, and

(ii) $\lambda_2 - \lambda(n-1) \leq 2t \leq \lambda_2$, and 3 divides $(\lambda(n-1) - \lambda_2 + 2t)n$.

For, once these conditions are met, we can proceed as follows. Condition (i) ensures that the conditions of Corollary 2.4 are met when $x = \lambda(n-1) - 2t$, so there exists a $(\lambda(n-1)-2t)$-regular graph H_0 on the vertex set $\boldsymbol{Z}_n \times \{0\}$ such that there exists a set B_0 of triples which partition the edges of H_0; so $\lambda K_n - E(G_0)$ is a $2t$-regular graph. Similarly, condition (ii) ensures that the conditions of Corollary 2.4 are met with $x = \lambda(n-1) - \lambda_2 + 2t$, so there exists a $(\lambda(n-1) - \lambda_2 + 2t)$-regular graph H_1 on the vertex set $\boldsymbol{Z}_n \times \{1\}$ such that there exists a set B_1 of triples which partition the edges of H_1; so $\lambda K_n - E(H_1)$ is a $(\lambda_2 - 2t)$-regular graph. Since λ_2 is even, by Lemma 2.11 there exists a set F_0 of $2t$ 1-factors and a set F_1 of $\lambda_2 - 2t$ 1-factors, each 1-factor being in $[0,1,0]$, such that for each $i \in \boldsymbol{Z}_2$ there exists a collection B_i' of triples which partition the edges of $\lambda K_n - E(H_i)$ and the edges in the 1-factors in F_i. Finally, if F is the λ_2-regular multigraph consisting of all the edges in F_0 and F_1, then by Lemma 2.10 there exists a collection B of triples that partition the edges of $[\lambda_2/2, \lambda_2, \lambda_2/2] - E(F)$. Then each edge $\{(u,i),(v,i)\}$ with $i \in \boldsymbol{Z}_2$ is contained in λ triples in B_i and B_i', and is in $\lambda_2/2$ triples in B, and clearly each edge $\{(u,0),(v,1)\}$ is in λ_2 triples, so the result will follow. So it remains to find an appropriate integer t. Recall that $\lambda \geq 1$.

If $\lambda_2 = 6x+2$ and $n \equiv 3 \pmod 6$ then $\lambda_1 \geq 3x+2$ (since $\lambda_1 \geq \lambda_2/2+1$) and $n \geq 3x+3$ (since $n \geq \lambda_2/2+1$). Choose $t = \lceil (3x+1)/2 \rceil$. Then $2t \leq n-1$, 3 divides n, and $\lambda_2 - (n-1) \leq 2t$.

If $\lambda_2 = 6x+2$ and $n \equiv 5 \pmod 6$ then $\lambda_1 \equiv 2 \pmod 3$ (see Table 1), so $\lambda \equiv 1 \pmod 3$.
If x is odd then $n \geq 3x+2$, so choose $t = (3x+1)/2$.
If x is even then $n \geq 3x+5$ (since $n \equiv 5 \pmod 6$), so choose $t = (3x+4)/2$.

If $\lambda_2 = 6x+4$ and $n \equiv 3 \pmod 6$ then $n \geq 3x+3$, so choose $t = \lceil (3x+1)/2 \rceil$.

If $\lambda_2 = 6x+4$ and $n \equiv 5 \pmod 6$ then $\lambda_1 \equiv 1 \pmod 3$ (see Table 1). If x is even then $n \geq 3x+5$, so choose $t = (3x+2)/2$. If x is odd then $n \geq 3x+8$, so choose $t = (3x+5)/2$.

If $\lambda_2 = 6x$ and $n \equiv 1 \pmod 6$ then: if x is odd then $n \geq 3x+4$, so choose $t = (3x+3)/2$; if x is even then $n \geq 3x+1$, so choose $t = 3x/2$.

If $\lambda_2 = 6x$ and $n \equiv 3 \pmod 6$ then $n \geq 3x+3$, so choose $t = \lceil 3x/2 \rceil$.

If $\lambda_2 = 6x$ and $n \equiv 5 \pmod 6$ then $\lambda_1 \equiv 0 \pmod 3$ (see Table 1) and so $\lambda \geq 3$, and $n \geq 3x+2$. If x is even then choose $t = 3x/2$, and if x is odd then choose $t = (3x+3)/2$. □

It turns out that if λ_2 is odd then we need to consider the smallest value of λ_1 by itself.

Proposition 3.3 *Suppose that λ_2 is odd and $\lambda_1 = (\lambda_2+1)/2$. Let n, λ_1 and λ_2 satisfy conditions (1-3) of Lemma 1.2. Then there exists a* GDD$(n,2)$ *of index* (λ_1, λ_2).

Proof: By (3) of Lemma 1.2, $n \geq \lambda_2 + 1$. Since λ_2 is odd, n and λ_1 are even (see Table 1), so we can write $\lambda_1 = 6x + 2y$, $\lambda_2 = 12x + 4y - 1$, and $n \geq 12x + 4y$, where $y \in \boldsymbol{Z}_3$. So Table 1 shows that λ_1, λ_2 and n are restricted even more: if $\lambda_1 \equiv 0 \pmod 6$ then $\lambda_2 \equiv 5 \pmod 6$ so $n \equiv 0 \pmod 6$; if $\lambda_1 \equiv 2 \pmod 6$ then $\lambda_2 \equiv 3 \pmod 6$ so $n \equiv 0$ or $4 \pmod 6$; and if $\lambda_1 \equiv 4 \pmod 6$ then $\lambda_2 \equiv 1 \pmod 6$ so $n \equiv 0$ or $2 \pmod 6$. Notice that in every case

(a) either $n \equiv 0 \pmod 6$ or $n/2 - \lambda_1 \equiv 0 \pmod 3$.

It will also be useful later to notice that if $n \equiv 2$ or $10 \pmod{12}$ then $\lambda_1 \equiv 4$ or $2 \pmod 6$ respectively, and so since $n/2 \geq (\lambda_2+1)/2 = \lambda_1$ we have:

(b) if $n \equiv 2$ or $10 \pmod{12}$ then $n/2 \geq \lambda_1 + 3$;

and if $n \equiv 6 \pmod{12}$ then $n/2$ is odd, so we have:

(c) if $n \equiv 6 \pmod{12}$ then $n/2 \geq \lambda_1 + 1$.

Let ϵ be defined as in Lemma 2.12. By Lemma 2.12, for each $i \in \boldsymbol{Z}_2$, there exists a simple graph H_i on the vertex set $\boldsymbol{Z}_n \times \{i\}$ satisfying (i-iii). Let B_0 be a set of triples that partitions the edges of $[0,1,0] + H_0 + H_1$ (see (ii)). By (iii), $K_n - E(H_i)$ can be partitioned into $n - 1 - (n/2 + (-1)^i\epsilon) = n/2 - 1 - (-1)^i\epsilon$ 1-factors.

We want to apply Theorem 2.8 with $x = n/2 - \lambda_1 - (-1)^i\epsilon$ and $\lambda = 1$, so we have some things to check. If $n \equiv 2$ or $4 \pmod 6$ then $\epsilon \in \{0,3\}$, so by (a) we have that 3 divides xn. In each case $n/2 - (-1)^i\epsilon$ is even, so x is even because λ_1 is even. Clearly $x \leq n-1$, and by (b) and (c) we have that $x \geq 0$.

Therefore, by Theorem 2.8, for each $i \in \boldsymbol{Z}_2$ there exists a set of triples B'_i and there exists an $(n/2 - \lambda_1 - (-1)^i\epsilon)$ - regular graph H'_i with vertex set $\boldsymbol{Z}_n \times \{i\}$ whose edges are partitioned by the triples in B'_i such that $K_n - E(H'_i)$ has a 1-factorization into $n - 1 - (n/2 - \lambda_1 - (-1)^i\epsilon) = n/2 + \lambda_1 - 1 + (-1)^i\epsilon$ 1-factors.

Finally, for each $i \in \boldsymbol{Z}_2$, since $\lambda_1 \geq 2$ we can take the $(\lambda_1 - 2)(n-1)$ 1-factors in a 1-factorization of $(\lambda_1 - 2)K_n$ on the vertex set $\boldsymbol{Z}_n \times \{i\}$. So for each $i \in \boldsymbol{Z}_2$, altogether on the vertex set $\boldsymbol{Z}_n \times \{i\}$ we have defined $(n/2 - 1 - (-1)^i\epsilon) + (n/2 + \lambda_1 - 1 + (-1)^i\epsilon) + (\lambda_1 - 2)(n-1) = n(\lambda_1 - 1) = n(\lambda_2 - 1)/2$ 1-factors. By Lemma 2.9, there exists a set B_1 of triples that partition the edges in these 1-factors together with the edges in $[0, \lambda_2 - 1, 0]$.

Then clearly the triples in B_0, B_1, B'_0 and B'_1 form a GDD$(n,2)$ of index (λ_1, λ_2). □

Before presenting our last proposition, we need to deal with two exceptional cases.

Lemma 3.4 *Let $n \equiv 2$ or 4 (mod 6), $\lambda_1 = 6y+6$ $\lambda_2 = 12y+9$ and $n \geq 6y+6$. Then there exists a* GDD$(n,2)$ *of index* (λ_1, λ_2).

Proof: If $n \equiv 2 \pmod 6$ then there exists a TS$(2n)$ of index 2, and by Proposition 3.3 there exists a GDD$(n,2)$ of index $(6y+4,\ 12y+7)$, which together produce a GDD$(n,2)$ of index $(6y+6,\ 12y+9)$.

If $n \equiv 4 \pmod 6$ then define ϵ as in Lemma 2.12. By Lemma 2.12, for each $i \in \boldsymbol{Z}_2$ there exists a simple graph H_i on the vertex set $\boldsymbol{Z}_n \times \{i\}$ that is $(n/2+(-1)^i\epsilon)$-regular, such that there exists a set B of triples that partition the edges of $[0,1,0]+H_0+H_1$, and such that $K_n - E(H_i)$ has a 1-factorization into a set $F_1(i)$ of $n/2-1-(-1)^i\epsilon$ 1-factors. Since 6 divides $x = 3n/2-6y-6-(-1)^i\epsilon$ and $0 \leq x \leq n-1$, by Theorem 2.8, for each $i \in \boldsymbol{Z}_2$ there exists a set B_i of triples and an x-regular graph H_i in $(6y+5)K_n$ defined on the vertex set $\boldsymbol{Z}_n \times \{i\}$ whose edges are partitioned by the triples in B_i, such that $(6y+5)K_n - E(H)$ has a 1-factorization into a set $F_2(i)$ of $(6y+5)(n-1)-x$ 1-factors. In $F_1(i)$ and $F_2(i)$, $i \in \boldsymbol{Z}_2$ there are a total of $(6y+4)n$ 1-factors, which altogether with the edges in $[0,12y+8,0]$ can be partitioned into a set B' of triples (by Lemma 2.9).

Clearly the triples in B, B', B_0 and B_1 together form a GDD$(n,2)$ of index $(6y+6, 12y+9)$.

□

Lemma 3.5 *Let $\lambda_1 \equiv 4$ (mod 6), $\lambda_2 = 1$ and $n \equiv 2$ (mod 6). Let n, λ_1 and λ_2 satisfy conditions (1-3) of Lemma 1.2. Then there exists a* GDD$(n,2)$ *of index* (λ_1, λ_2).

Proof: Let $\lambda_1 = 6y+4$. Let $F = \{\{2a, 2a+1\} \mid a \in \boldsymbol{Z}_{n/2}\}$ and $F_0 = \{\{(a,0),(b,0)\} | \{a,b\} \in F\}$. Let $L = \{\{0,1\},\{2,3\}\}$, and for each $i \in \boldsymbol{Z}_2$ let $L_i = \{\{(a,i),(b,i)\} \mid \{a,b\} \in L\}$.

Let $(\boldsymbol{Z}_n, \circ)$ be a symmetric quasigroup with holes F and of order n (see Lemma 2.13). Define

$$\begin{aligned} B \;=\; & \{\{(a,0),(b,0),(a \circ b,1)\} \mid 0 \leq a < b \leq n-1, \{a,b\} \notin F\} \cup \\ & \{\{(2a,0),(2a+1,0),(2a,1)\},\{(2a,0),(2a+1,0),(2a+1,1)\} \mid 2 \leq a \leq n/2\} \cup \\ & \{\{(2a,0),(2a,1),(2a+1,1)\},\{(2a+1,0),(2a,1),(2a+1,1)\} \mid 0 \leq a \leq 1\} \;. \end{aligned}$$

Then the triples in B contain: each edge $\{(a,0),(b,0)\}$ exactly once if $\{a,b\} \notin F$, exactly twice if $\{a,b\} \in F \backslash L$, and not at all if $\{a,b\} \in L$; each edge $\{(a,0),(b,1)\}$ exactly once; and each edge $\{(a,1),(b,1)\}$ exactly twice if $\{a,b\} \in L$, and otherwise not at all.

Using Lemma 2.14, let B_0 be a collection of triples that partition the edges of $(6y+2)K_n+2L_0$ on the vertex set $\boldsymbol{Z}_n \times \{0\}$, and let B_1 be a collection of triples that partition the edges of $(6y+4)K_n - 2L_1$ on the vertex set $\boldsymbol{Z}_n \times \{1\}$.

Finally, let $(\boldsymbol{Z}_n \times \{0\}, F_0, B')$ be a GDD$(n, 2)$ of index $(0, 1)$.

Then the triples in B, B', B_0 and B_1 together form a GDD$(n, 2)$ of index $(6y+4, 1)$. □

Proposition 3.6 *Suppose that n is even, $\lambda_1 \geq \lambda_2/2+1$ and $n \geq \lambda_2/2+1$. Let n, λ_1 and λ_2 satisfy conditions (1) and (2) of Lemma 1.2. Then there exists a* GDD$(n, 2)$ *of index (λ_1, λ_2).*

Proof: The result will follow if we can find an integer t that satisfies the following conditions:

(i) $0 \leq t$, $nt \leq \lambda_1(n-1)$, and 3 divides $(\lambda_1(n-1) - tn)n$, and

(ii) $t \leq \lambda_2$, $(\lambda_2 - t)n \leq \lambda_1(n-1)$, and 3 divides $(\lambda_1(n-1) - (\lambda_2 - t)n)n$.

For, once these conditions are met, we proceed as follows.

Since n is even λ_1 is even, so $(\lambda_1(n-1) - tn)$ is even. Therefore, by Theorem 2.8 and using (i), there exists a $(\lambda_1(n-1) - tn)$-regular graph H_0 on the vertex set $\boldsymbol{Z}_n \times \{0\}$ of multiplicity at most λ_1 and there exists a set B_0 of triples such that: these triples partition the edges of H_0; and $T_0 = \lambda_1 K_n - E(H_0)$ has a 1-factorization into tn 1-factors. Similarly, by Theorem 2.8 and (ii), there exists a $(\lambda_1(n-1) - (\lambda_2 - t)n)$-regular graph H_1 on the vertex set $\boldsymbol{Z}_n \times \{1\}$ and there exists a set B_1 of triples such that: these triples partition the edges of H_1; and $T_1 = \lambda K_n - E(H_1)$ has a 1-factorization into $(\lambda_2 - t)n$ 1-factors. Finally, by Lemma 2.9, there exists a set B of triples which partition the edges of $[0, \lambda_2, 0] + T_0 + T_1$. Then clearly the triples in B_0, B_1 and B together form a GDD$(n, 2)$ of index (λ_1, λ_2). So it remains to find a suitable value of t in each case.

In the following, to check that $nt \leq \lambda_1(n-1)$ it is easiest to check that $t \leq (\lambda_1 - t)(n-1)$. Also, we will choose t so that $t \geq \lambda_2/2$, in which case $nt \leq \lambda_1(n-1)$ implies that $(\lambda_2 - t)n \leq \lambda_1(n-1)$.

If $\lambda_2 = 6x$ then $\lambda_1 \geq 3x+1$ and $n \geq 3x+1$. Choose $t = 3x$. From Table 1, 3 divides λ_1, n or $n-1$, and since 3 divides t, the divisibility by 3 conditions in (i-ii) are met.

If $\lambda_2 = 6x+1$ and $n \equiv 0 \pmod 6$ then $\lambda_1 \geq 3x+2$ and $n \geq 3x+2$. Choose $t = 3x+1$.

If $\lambda_2 = 6x+1$ and $n \equiv 2 \pmod 6$, then $\lambda_1 \equiv 4 \pmod 6$ (see Table 1), so $\lambda_1 \geq 3x+4$ and $n \geq 3x+2$. Choose $t = 3x+2$. Then all conditions in (i-ii) are met except that if $x = 0$ then $\lambda_2 < t$; but then we seek a GDD$(n, 2)$ of index $(6y+4, 1)$ which was constructed in Lemma 3.5.

If $\lambda_2 = 6x+2$ then $\lambda_1 \geq 3x+2$ and $n \geq 3x+2$. Choose $t = 3x+1$.

If $\lambda_2 = 6x+3$ and $n \equiv 0 \pmod 6$ then $\lambda_1 \geq 3x+3$ and $n \geq 3x+3$. Choose $t = 3x+2$.

If $\lambda_2 = 6x+3$ and $n \equiv 2 \pmod 6$ then $\lambda_1 \equiv 0 \pmod 6$ (see Table 1), so $\lambda_1 \geq 3x+3$ and $n \geq 3x+5$. Choose $t = 3x+3$. Then all conditions in (i-ii) are met except that if $\lambda_1 = 3x+3$ then $nt > \lambda_1(n-1)$. However, if $\lambda_1 = 3x+3$ then we can write $\lambda_1 = 6y+6$, $\lambda_2 = 12y+9$ and $n \equiv 2 \pmod 6$, so we can use Lemma 3.4.

If $\lambda_2 = 6x+3$ and $n \equiv 4 \pmod 6$ then $\lambda_1 \geq 3x+3$ and $n \geq 3x+4$. Choose $t = 3x+3$. Then all conditions in (i-ii) are satisfied unless $\lambda_1 = 3x+3$, for then $nt > \lambda_1(n-1)$. If $\lambda_1 = 3x+3$ then again the GDD can be obtained from Lemma 3.4.

If $\lambda_2 = 6x+4$ then $\lambda_1 \geq 3x+3$ and $n \geq 3x+3$. Choose $t = 3x+2$.

If $\lambda_2 = 6x+5$ and $n \equiv 0 \pmod 6$ then $\lambda_1 \geq 3x+4$ and $n \geq 3x+6$. Choose $t = 3x+3$.

If $\lambda_2 = 6x+5$ and $n \equiv 2 \pmod 6$ then $\lambda_1 \equiv 2 \pmod 6$ (see Table 1), so $\lambda_1 \geq 3x+5$ and $n \geq 3x+5$. Choose $t = 3x+4$. □

Finally, we can present the main result.

Theorem 3.7 *Let $n \geq 3$ and $\lambda_1, \lambda_2 \geq 1$. There exists a* GDD$(n, 2)$ *of index (λ_1, λ_2) if and only if*

(1) 2 divides $\lambda_1(n-1) + \lambda_2 n$,

(2) 3 divides $\lambda_1 n(n-1) + \lambda_2 n^2$, and

(3) $\lambda_1 \geq \lambda_2 n/2(n-1)$.

Proof: By Proposition 3.1, it suffices to consider the case where $\lambda_2 \leq 2(n-1)$, so $n \geq \lambda_2/2 + 1$ and therefore by (3) $\lambda_1 \geq (\lambda_2+1)/2$.

If n is odd (so λ_2 is even) the result follows from Proposition 3.2.

If $\lambda_1 = (\lambda_2 + 1)/2$ then the result follows from Proposition 3.3.

If n is even and $\lambda_1 \geq \lambda_2/2 + 1$ then the result follows from Proposition 3.6. $\square$

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