IMPROVEMENTS IN RANKED SET SAMPLING

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Abstract

The main focus of many agricultural, ecological and environmental studies is to develop well designed, cost-effective and efficient sampling designs. Ranked set sampling (RSS) is one of those sampling methods that can help accomplish such objectives by incorporating prior information and expert knowledge to the design. In this thesis, new RSS schemes are suggested for efficiently estimating the population mean. These sampling schemes can be used as cost-effective alternatives to the traditional simple random sampling (SRS) and RSS schemes. It is shown that the mean estimators under the proposed sampling schemes are at least as efficient as the mean estimator with SRS. We consider the best linear unbiased estimators (BLUEs) and the best linear invariant estimators (BLIEs) for the unknown parameters (location and scale) of a location-scale family of distributions under double RSS (DRSS) scheme. The BLUEs and BLIEs with DRSS are more precise than their counterparts based on SRS and RSS schemes. We also consider the BLUEs based on DRSS and ordered DRSS (ODRSS) schemes for the unknown parameters of a simple linear regression model using replicated observations. It turns out that, in terms of relative efficiencies, the BLUEs under ODRSS are better than the BLUEs with SRS, RSS, ordered RSS (ORSS) and DRSS schemes.

Quality control charts are widely recognized for their potential to be a powerful process monitoring tool of the statistical process control. These control charts are frequently used in many industrial and service organizations to monitor in-control and out-of-control performances of a production or manufacturing process. The RSS schemes have had considerable attention in the construction of quality control charts. We propose new exponentially weighted moving average (EWMA) control charts for monitoring the process mean and the process dispersion based on the BLUEs obtained under ORSS and ODRSS schemes. We also suggest an improved maximum EWMA control chart for simultaneously monitoring the process mean and dispersion based on the BLUEs with ORSS scheme. The proposed EWMA control charts perform substantially better than their counterparts based on SRS and RSS schemes. Finally, some new EWMA charts are also suggested for monitoring the process dispersion using the best linear unbiased absolute estimators of the scale parameter under SRS and RSS schemes.

Dedication

To the memory of

'Al-Sayed Muhiyuddin Abu Muhammad Abdul Qadir Al-Gilani Al-Hasani Wal-Hussaini' (RA),

Ghaus-e-Aazam, Baghdad, Iraq.

Acknowledgments

In The Name of Allah, The Most Gracious, The Most Merciful.

> *"He is Allah, other than whom there is no deity, Knower of the unseen and the witnessed. He is the Entirely Merciful, the Especially Merciful."* — Holy Quran 59:21.

First and foremost, I'm extremely thankful to The Almighty Allah for blessing, protecting and guiding me throughout this period to complete this task. All my respect and love go to The Prophet Muhammad (peace be upon him), who once said

> *"Learning is from cradle to grave—seek knowledge even if it is in China."*

I express my profound sense of reverence to my supervisors, Prof. Jennifer Brown and Dr. Elena Moltchanova, for their constant guidance, support, time, motivation and help during the course of my PhD. I'm very thankful to Prof. Jennifer Brown for giving me freedom during my research, having trust in my abilities, and she's always been very kind and nice to me.

I'm also thankful to the University of Canterbury (UoC) for financially supporting my PhD research through "University of Canterbury International Doctoral Scholarship". Without this support, it'd have been very difficult for me to get the PhD from UoC, Christchurch, New Zealand. Thanks are also extended to all the administration staff at the School of Mathematics and Statistics, UoC.

Last but not the least, I cheerfully express my profound thanks to my parents, grand parents, my brother and sisters, teachers and friends for their support, help and cooperation during my work.

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Preface

This thesis is a collection of research articles on improvements in ranked set sampling (RSS). All of the chapters have been either published or accepted for publication in different international journals, including Environmetrics, Journal of Applied Statistics, Communications in Statistics-Theory and Methods, and Quality and Reliability Engineering International. Since each chapter is an independent research article focusing on RSS, there are some repetitions in the form of RSS methods, literature review and notations.

The outline of the thesis is as follows: In Chapter 1, we propose a cost-effective sampling scheme, named partial RSS (PRSS), for estimating the population mean, median and variance. The PRSS scheme selects samples using both simple random sampling (SRS) and RSS schemes, and thus reduces the cost of ranking. In Chapter 2, we extend the work on PRSS, and propose a mixed RSS (MxRSS) scheme, as a cost-effective alternative to the RSS scheme, for estimating the population mean and median. The MxRSS scheme encompasses both SRS and RSS schemes, and it helps in selecting more representative samples from the parent population. Under MxRSS scheme, there are more possibilities to select a sample than those with the PRSS scheme. It is shown that the MxRSS scheme, generally, provides more efficient mean and median estimators than those with SRS and PRSS schemes. Chapter 3 further extends this work, and suggests a new paired double RSS (PDRSS) scheme, as a cost-effective alternative to the double RSS (DRSS) scheme, for estimating the population mean. The mean estimator under PDRSS scheme is at least as efficient as the mean estimator based on RSS. Note that in Chapter 3 we use the notation "PRSS" for paired RSS scheme, and it should not be confused with the notation of the partial RSS (PRSS) scheme used in Chapters 1 and 2.

In Chapter 4, we derive the best linear unbiased and invariant estimators for the unknown parameters (location and scale) of a location-scale family of distributions under DRSS scheme. Chapter 5 proposes the best linear unbiased estimators (BLUEs) for the unknown parameters of a simple linear regression model with replicated observations using DRSS and ordered DRSS (ODRSS) schemes.

In Chapter 6, we suggest new exponentially weighted moving average (EWMA) control charts for monitoring the process mean and the process dispersion using the BLUEs (location and scale) obtained under ordered RSS (ORSS). In Chapter 7, we extend the work on ORSS scheme, and suggest an improved maximum EWMA control chart for simultaneously monitoring the process mean and dispersion. Chapter 8 extends the work on ODRSS scheme, and proposes new EWMA control charts based on the BLUEs (location and scale) using ODRSS for monitoring the process mean and the process dispersion. In Chapter 9, we suggest new

EWMA control charts based on the best linear unbiased absolute estimators of the scale parameter under SRS and RSS schemes for monitoring the process dispersion.

Chapter 1

Partial Ranked Set Sampling Design

This chapter appeared in:

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In many environmental studies, the main focus is on observational economy, that is, to obtain data on the basis of cost-effective and efficient sampling methods. In this chapter, we propose a partial ranked set sampling (PRSS) method for estimation of population mean, median and variance. On the basis of perfect and imperfect rankings, Monte Carlo simulations from symmetric and asymmetric distributions are used to evaluate the effectiveness of the proposed estimators. It is found that the estimators under PRSS are more efficient than the estimators based on simple random sampling. The procedure is illustrated with a case study using a real data set

1.1 Introduction

In many studies where sampling is used, such as environmental management, ecology, sociology and agriculture, exact measurement of a selected unit is either difficult or costly and time-consuming. However, the ranking of a small set of selected units can be carried out easily either by visual inspection with respect to the study variable or on the basis of auxiliary variable. For example, hazardous waste sites with different levels of contamination can be ranked by a visual inspection of soil discoloration, whereas the actual measurements of toxic chemicals and quantifying their environmental impact is very costly. McIntyre (1952) proposed a method, later called ranked set sampling (RSS), for estimating mean pasture and forage yields when measurement is costly. Takahasi and Wakimoto (1968) derived the statistical theory of the RSS procedure. Dell and Clutter (1972) showed that under imperfect ranking, the sample mean remains an unbiased estimator of the population mean, but ranking should be better than at least a random ordering. As mentioned by

Stokes (1977), the concomitant variables can be used to judge the ranks of the study variable. For a detailed review and bibliography on RSS, see Patil (1995) and Kaur et al. (1995). For some real applications of RSS, see Yu and Lam (1997), Al-Saleh and Al-Shrafat (2001), Al-Saleh and Al-Hadrami (2003), Al-Saleh and Al-Omari (2002), Husby et al. (2005), Chen (2007), Wang et al. (2009), Ozturk (2011), Samawi (2011) and references therein.

Under the RSS scheme, the experimenter selects m random samples, each of size m, from the target population. The units within each sample are ranked visually without actual measurements. This may be difficult when the data arrive in batches of varying sizes or when the ranking is difficult and results in large inaccuracies or is time-consuming. An initial sample of m^2 experimental units under RSS produces the final sample of m units.

In this chapter, we propose a cost-effective sampling method, namely partial ranked set sampling (PRSS) design. This design provides flexibility to the experimenter in selecting the sample when it is either difficult to rank the units within each set with full confidence or due to non-availability of experimental units. Under the PRSS scheme, the experimenter selects A units using simple random sampling (SRS) and B units using the RSS, producing the final sample of size $m = A + B$ units. It thus requires less sampling units and less ranking than the RSS and proves to be more efficient than SRS.

The rest of the chapter is organized as follows: In Section 1.2, the RSS and PRSS methods are described. Estimation of the population mean is considered in Section 1.3. In Section 1.4, the PRSS is considered for median and variance estimation. An application to real data set is given in Section 1.5. Finally, we summarize our results in Section 1.6.

1.2 Sampling methods

In this section, we explain the RSS and PRSS procedures.

1.2.1 Ranked set sampling

The RSS can be described as follows: identify m^2 units from the target population. Randomly allocate these units into m sets, each of size m. The units within each set are ranked visually or by any inexpensive method with respect to the variable of interest. From the first set of m units, the smallest ranked unit is measured; the second smallest ranked unit is measured from the second set of m units. The process continues until the mth smallest ranked unit is measured from the last set. The process can be repeated r number of times to obtain a large sample of size mr.

Let the study variable X has a probability density function (PDF) $f(x)$ and cumulative distribution function $F(x)$, with mean μ and variance σ^2. Let $X_1, X_2, ..., X_m$ be a simple random sample of size m drawn from $f(x)$. The SRS estimator of μ is $\bar{X}_{\text{SRS}} = \frac{1}{m}\sum_{i=1}^{m} X_i$ with variance $\text{Var}(\bar{X}_{\text{SRS}}) = \frac{\sigma^2}{m}$. Consider $X_{11}, X_{12}, ..., X_{1m}, X_{21}, X_{22}, ..., X_{2m}, ..., X_{m1}, X_{m2}, ..., X_{mm}$ be m independent simple random samples each of size m. Let $X_{i(1:m)}, X_{i(2:m)}, ..., X_{i(m:m)}$ represent the order statistics of the ith sample. Using RSS scheme, the measured RSS units are denoted by $X_{1(1:m)}, X_{2(2:m)}, ..., X_{m(m:m)}$. Let $g_{(i:m)}(x)$ be the PDF of the ith order statistic, i.e., $X_{(i:m)}$, $i = 1, 2, ..., m$, from a random sample of size m. It can be shown that

$$g_{(i:m)}(x) = m\binom{m-1}{i-1}\{F(x)\}^{i-1}\{1 - F(x)\}^{m-i}f(x), \quad -\infty < x < +\infty,$$

see David and Nagaraja (2003).

The mean and variance of $X_{(i:m)}$, respectively, are

$$\mu_{(i:m)} = \int_{-\infty}^{\infty} x g_{(i:m)}(x)\mathrm{d}x \quad \text{and} \quad \sigma^2_{(i:m)} = \int_{-\infty}^{\infty} (x - \mu_{(i:m)})^2 g_{(i:m)}(x)\mathrm{d}x.$$

The RSS estimator of the population mean is

$$\bar{X}_{\mathrm{RSS}} = \frac{1}{m}\sum_{i=1}^{m} X_{i(i:m)},$$

with variance

$$\mathrm{Var}(\bar{X}_{\mathrm{RSS}}) = \frac{1}{m^2}\sum_{i=1}^{m}\sigma^2_{(i:m)} = \frac{\sigma^2}{m} - \frac{1}{m^2}\sum_{i=1}^{m}(\mu_{(i:m)} - \mu)^2.$$

1.2.2 Partial ranked set sampling

The PRSS scheme is a mixture of both SRS and RSS designs. We propose this design for use when the experimenter is unable to inspect the number of units that are required for a balanced ranked set sample or when inspection cost per unit is high. The PRSS scheme requires fewer identified units as compared with a ranked set sample, and at the same time, it provides more precise estimates than the commonly used SRS scheme. Thus, PRSS scheme helps in reducing the total cost and expenditure that is involved in sampling.

In order to select a partial ranked set sample of size m, the following steps are carried out:

Step 1: Define a coefficient k such that $k = [\alpha m]$, where $0 \le \alpha < 0.5$. Here, $[t]$ represents the largest integer value less than or equal to t.

Step 2: Select $2k$ simple random samples each of size one from the parent population. In order to select the remaining $m - 2k$ units, select $m - 2k$ sets each of size m from the parent population. Rank the units within each set and select the ith ranked unit of the ith sample, for $i = k + 1, ..., m - k$. This completes one cycle of a partial ranked set sample of size m.

Step 3: The above Steps 1 and 2 can be repeated r times in order to select a partial ranked set sample of size $n = mr$.

The total number of units that were involved in selecting a partial ranked set sample of size m are $m^2 - 2k(m-1)$. Note that with $k = 0$, PRSS is equivalent to RSS.

1.3 Estimation of population mean

Let $X_{11}, X_{12}, ..., X_{1m}, X_{21}, X_{22}, ..., X_{2m}, ..., X_{m1}, X_{m2}, ..., X_{mm}$ be m independent simple random samples each of size m. Apply the PRSS procedure on these m samples, as explained in Section 1.2.2. The PRSS estimator of population mean is defined as

$$\bar{X}_{\text{PRSS}} = \frac{1}{m}\left(\sum_{i=1}^{k} X_i + \sum_{i=k+1}^{m-k} X_{i(i:m)} + \sum_{i=m-k+1}^{m} X_i\right),$$

with variance

$$\text{Var}(\bar{X}_{\text{PRSS}}) = \frac{2k\sigma^2}{m^2} + \frac{1}{m^2}\sum_{i=k+1}^{m-k} \sigma^2_{(i:m)}.$$

For a symmetric distribution, we have $\mu_{(i:m)} + \mu_{(m-i+1:m)} = 2\mu$ and $\sigma^2_{(i:m)} = \sigma^2_{(m-i+1:m)}$, for $i = 1, 2, ..., m$, see David and Nagaraja (2003). Then, it is easy to show $E(\bar{X}_{\text{PRSS}}) = \mu$, $\text{Var}(\bar{X}_{\text{PRSS}}) = \text{Var}(\bar{X}_{\text{RSS}}) + \frac{2}{m^2}\sum_{i=1}^{k}(\sigma^2 - \sigma^2_{(i:m)})$. Also, $\text{Var}(\bar{X}_{\text{PRSS}}) \leq \text{Var}(\bar{X}_{\text{SRS}})$ if and only if $\sigma^2 \geq \frac{1}{m-2k}\sum_{i=k+1}^{m-k}\sigma^2_{(i:m)}$.
For a symmetric population, the relative efficiency (RE) of $\bar{X}_{\text{PRSS}}$ with respect to $\bar{X}_{\text{SRS}}$ is

$$\text{RE}(\bar{X}_{\text{PRSS}}, \bar{X}_{\text{SRS}}) = \frac{\text{Var}(\bar{X}_{\text{SRS}})}{\text{Var}(\bar{X}_{\text{PRSS}})} = \frac{m\sigma^2}{2k\sigma^2 + \sum_{i=k+1}^{m-k}\sigma^2_{(i:m)}}.$$

Similarly, for an asymmetric population, the RE will be

$$\text{RE}(\bar{X}_{\text{PRSS}}, \bar{X}_{\text{SRS}}) = \frac{\text{Var}(\bar{X}_{\text{SRS}})}{\text{MSE}(\bar{X}_{\text{PRSS}})} = \frac{m\sigma^2}{2k\sigma^2 + \sum_{i=k+1}^{m-k}\sigma^2_{(i:m)} + m^2\{E(\bar{X}_{\text{PRSS}} - \mu)\}^2},$$

where MSE is the mean squared error.

1.3.1 Simulation study for mean estimation

In this section, a simulation study is conducted to investigate the efficiency of PRSS for estimating the population mean with $m = 4, 5, 6, 7$. The RE is used as a performance criterion for estimators. We consider symmetric distributions: Normal (0,1), Uniform (0,1), Logistic (0,1) and Beta (6,6) and asymmetric distributions: Exponential (1), Weibull (0.5,1), Lognormal (0,1) and Gamma (0.5,1). The sampling schemes (SRS and PRSS) are based on the same sample size. Under each sampling scheme, for given values of m and k, from each distribution, one million estimates of μ and their MSEs are estimated. The estimated REs are calculated and displayed in Figure 1.1.

(a) Univariate case

For all the distributions considered in this study, the mean estimators based on PRSS are more efficient than the estimators from SRS (RE > 1). It is observed that as the value of m increases, the RE of mean estimator based on PRSS also increases and vice-versa. For symmetric distributions, with fixed m, the RE decreases as the value of k increases. In asymmetric distributions, the RE increases as m increases, but at

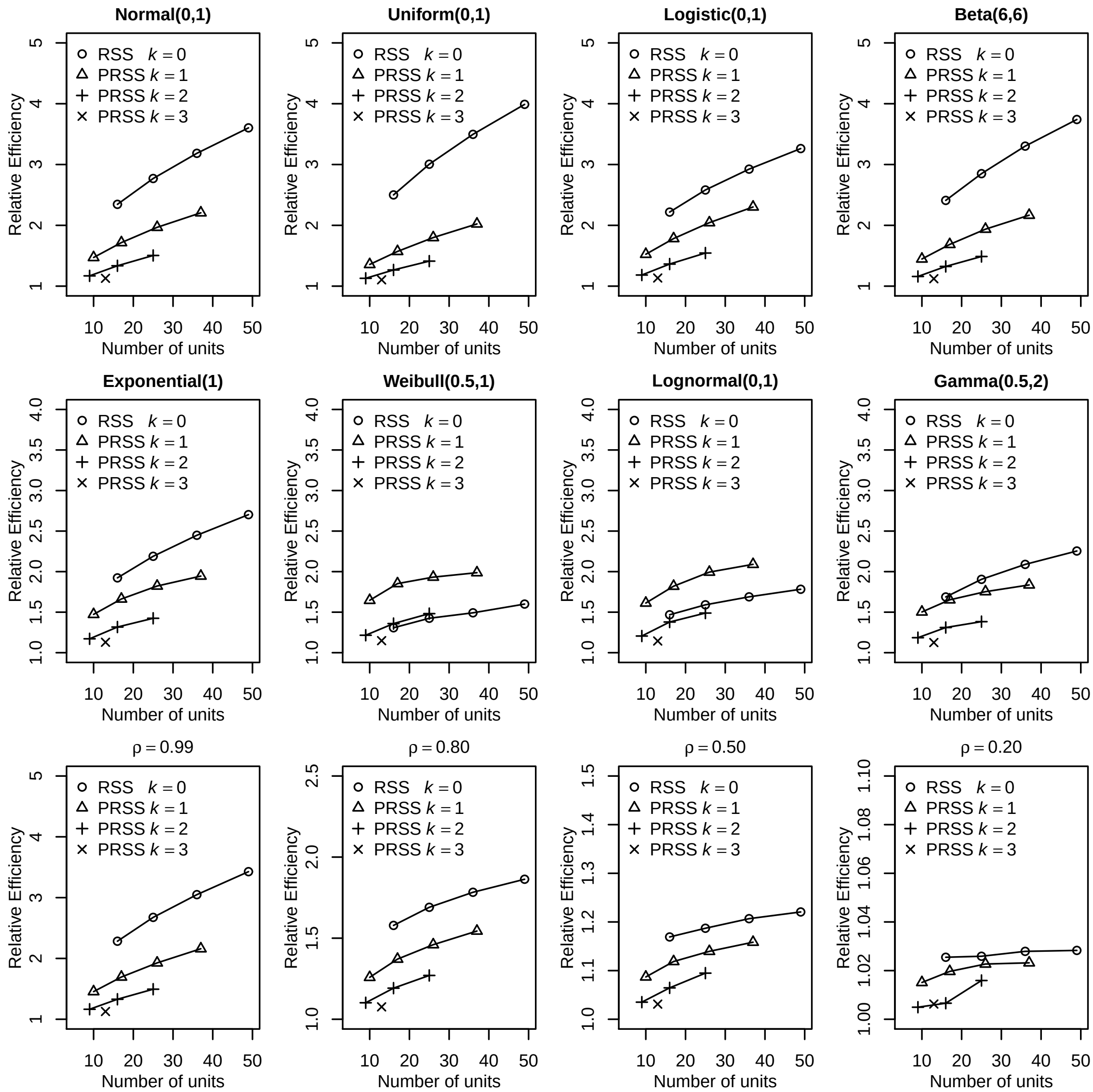

Figure 1.1: REs of PRSS with respect to SRS for estimating population mean under perfect and imperfect rankings

the same time, it also depends on the value k. The RSS mean estimator is more precise than the PRSS mean estimator because it uses more numbers of units. It is of interest to note that when the underlying distribution is asymmetric like Weibull (0.5,1) or Lognormal (0,1), the RE of PRSS mean estimator is higher as compared with the RSS mean estimator. From Section 1.3, we have

$$\text{MSE}(\bar{X}_{\text{PRSS}}) = \text{Var}(\bar{X}_{\text{RSS}}) + \frac{1}{m^2}\{2\sigma^2 - (\sigma^2_{(1:m)} + \sigma^2_{(m:m)})\} + \{\text{Bias}(\bar{X}_{\text{PRSS}})\}^2, \quad \text{for } k = 1.$$

Note that for some highly skewed distributions, $2\sigma^2 < (\sigma^2_{(1:m)} + \sigma^2_{(m:m)})$, as a result of which $\text{MSE}(\bar{X}_{\text{PRSS}}) \leq \text{Var}(\bar{X}_{\text{RSS}})$. Moreover, the PRSS scheme uses fewer units to achieve higher efficiency.

(b) Bivariate case

In most of the real life situations, it is difficult to rank the study variable visually or it is too costly. In such environments, it is beneficial to use any auxiliary variable that is highly correlated with the study variable.

In order to investigate the performances of the estimators under the PRSS design, we assume that both the study and the auxiliary variables follow a standard bivariate normal distribution, having PDF:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}\right\}, \quad -\infty < x, y < +\infty.$$

Different values of the correlation coefficient, $\rho = 0.99, 0.80, 0.50, 0.20$, were considered. Here, we have assumed that the ranking on the auxiliary variable Y is perfect, whereas there are errors in ranking the study variable X. On the basis of extensive Monte Carlo simulations, the REs are calculated and displayed in Figure 1.1.

We can conclude that even when the ranking of the study variable is imperfect, PRSS is more efficient than SRS in estimating the population mean of X. Also, the RE increases with an increase in the value of m, whereas it is a decreasing function of k. The value of ρ plays a key role in the performance of the PRSS mean estimator. As the value of ρ increases, the efficiency of PRSS estimator also increases as compared with the estimator under SRS.

1.4 Estimation of population median and variance

The median is often considered as a more suitable measure of location than the mean when the underlying population is highly skewed such as income, expenditure and production. In this section, we compare the estimators of population median and variance on the basis of SRS, RSS and PRSS methods.

We use Monte Carlo simulations from both symmetric and asymmetric distributions to compare the REs of the median and variance estimators. The standard bivariate normal distribution is also used to study the impact of imperfect ranking on the proposed median estimator under PRSS.

The SRS estimator of the population median is defined as

$$\hat{\theta}_{\text{SRS}} = \text{Median}\{X_1, X_2, ..., X_m\} = \begin{cases} X_{((m+1)/2:m)}, & \text{if } m \text{ is odd,} \\ \{X_{(m/2:m)} + X_{((m+2)/2:m)}\}/2, & \text{if } m \text{ is even.} \end{cases}$$

Similarly, the median estimator under PRSS is defined as

$$\hat{\theta}_{\text{PRSS}} = \text{Median}\{X_1, ..., X_k, X_{k+1(k+1:m)}, ..., X_{m-k(m-k:m)}, X_{m-k+1}, ..., X_m\}.$$

The RE of $\hat{\theta}_{\text{PRSS}}$ with respect to $\hat{\theta}_{\text{SRS}}$ is $\text{RE}(\hat{\theta}_{\text{PRSS}}, \hat{\theta}_{\text{SRS}}) = \frac{\text{MSE}(\hat{\theta}_{\text{SRS}})}{\text{MSE}(\hat{\theta}_{\text{PRSS}})}$. The estimated MSE of any estimator, say $\hat{\theta}_J$, is $\text{MSE}(\hat{\theta}_J) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_{i,J} - \theta_m)^2$, for $J =$ SRS, PRSS. Here, θ_m represents the population median and N is the number of replications (10^6).

The traditional unbiased estimator of the population variance, based on SRS, is $\hat{\sigma}^2_{\text{SRS}} = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \bar{X}_{\text{SRS}})^2$. Stokes (1980) proposed an estimator of population variance on the basis of RSS, $\hat{\sigma}^2_{\text{RSS}} = \frac{1}{m-1}\sum_{i=1}^{m}(X_{i(i:m)} - \bar{X}_{\text{RSS}})^2$. This estimator is biased for small samples, and it is asymptotically unbiased. Suppose for given values of m and k, $X_1^*, X_2^*, , ..., X_m^*$ represent a partial ranked set

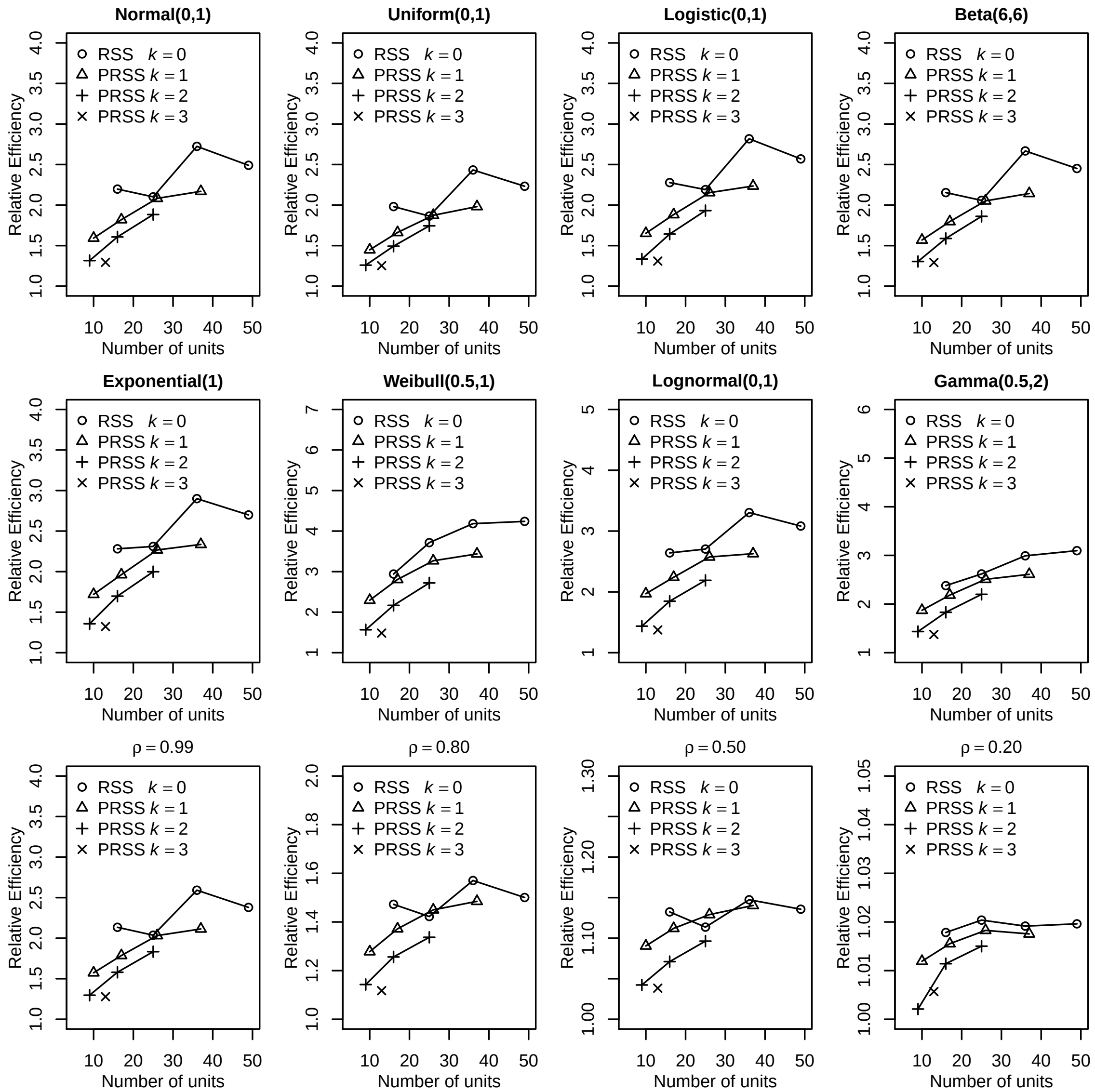

Figure 1.2: REs of PRSS with respect to SRS for estimating population median under perfect and imperfect rankings

sample of size m from the parent population. Then, analogous to $\hat{\sigma}^2_{\text{RSS}}$, the variance estimator under PRSS is $\hat{\sigma}^2_{\text{PRSS}} = \frac{1}{m-1}\sum_{i=1}^{m}(X_i^* - \bar{X}_{\text{PRSS}})^2$. The REs of $\hat{\sigma}^2_{\text{PRSS}}$ and $\hat{\sigma}^2_{\text{RSS}}$ with respect to $\hat{\sigma}^2_{\text{SRS}}$, respectively, are given by

$$\text{RE}(\hat{\sigma}^2_{\text{PRSS}}, \hat{\sigma}^2_{\text{SRS}}) = \frac{\text{MSE}(\hat{\sigma}^2_{\text{SRS}})}{\text{MSE}(\hat{\sigma}^2_{\text{PRSS}})} \quad \text{and} \quad \text{RE}(\hat{\sigma}^2_{\text{RSS}}, \hat{\sigma}^2_{\text{SRS}}) = \frac{\text{MSE}(\hat{\sigma}^2_{\text{SRS}})}{\text{MSE}(\hat{\sigma}^2_{\text{RSS}})}.$$

The estimated MSE of any estimator, say $\hat{\sigma}^2_J$, is $\text{MSE}(\hat{\sigma}^2_J) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\sigma}^2_{i,J} - \sigma^2)^2$, for J = SRS, RSS, PRSS.

On the basis of Figure 1.2, we can conclude that the PRSS median estimator is more efficient than SRS median estimator based on the same sample size. As the value of the coefficient k decreases, the RE under PRSS increases. Under perfect and imperfect rankings, the PRSS estimator still performs better than the SRS estimator for all cases considered here. Furthermore, as the value of ρ decreases, the REs also decrease because of more errors in ranking and vice-versa.

Figure 1.3 shows that, for both symmetric and asymmetric distributions, with small samples, the RE of

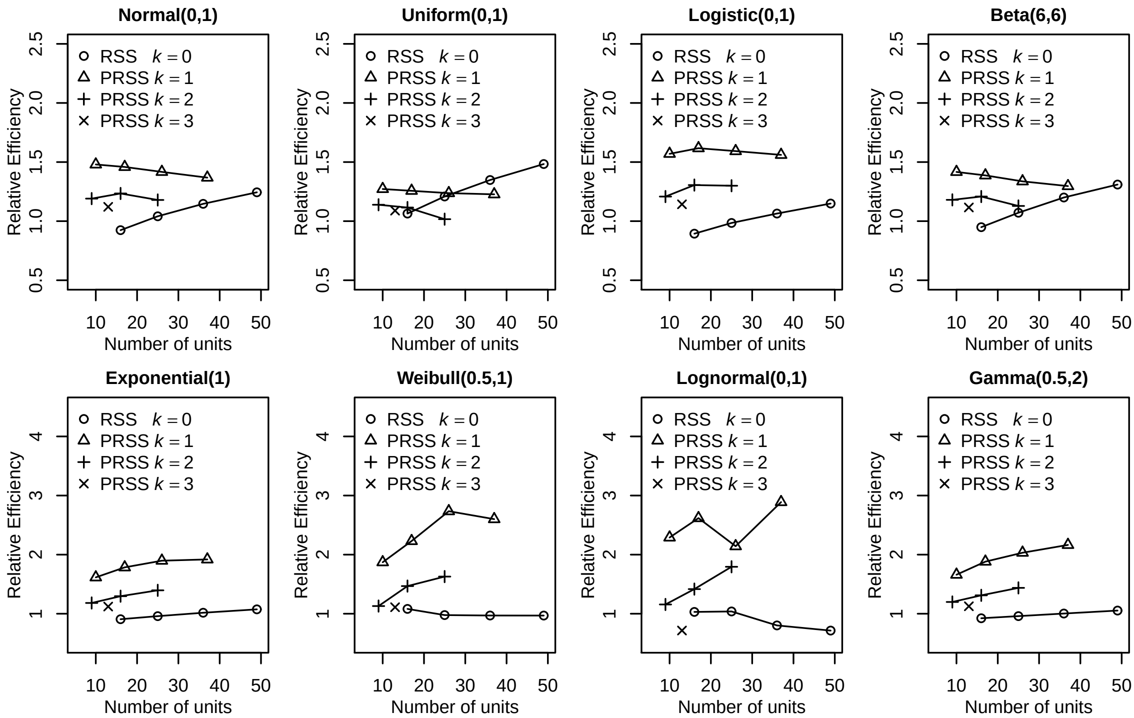

Figure 1.3: REs of PRSS with respect to SRS for estimating population variance under perfect and imperfect rankings

the proposed variance estimator is high as compared with the estimators based on SRS and RSS schemes. For symmetric populations, under RSS, as the number of units increase, this leads to the gain in RE of the estimator. On the other hand, the RE is a decreasing function of the number of units under PRSS for symmetric population. In variance estimation, PRSS is more economical than SRS and RSS. The variance estimator under PRSS uses less number of units and performs better than the other estimators. Therefore, in practice, for small samples, it is preferable to use PRSS variance estimator.

Generally, the optimum choice of k depends on the environment and experimenter. For instance, if the experimenter can rank all of sets with full confidence, then it is better to take $k = 0$. But when there are cost or time constraints or lack of units, then it is preferable to use PRSS with $k > 0$. In case of mean estimation, if the underlying distribution is highly skewed, then it is preferable to apply the PRSS with $k = 1$ instead of using RSS method. Finally, in variance estimation with small samples, $k = 1$ is the optimal choice.

1.5 An application

A real data set is used to study the efficiency of the PRSS design in estimating the population mean, median and variance as compared with the SRS. The data consist of the diameter of conifer tree at breast height, say Y, and the height of the conifer tree, say X. For more details about the data, see Platt et al. (1988). Table 1.1 contains the summary statistics of the trees data. Here, we are interested in estimating the mean, median and variation among the heights of the conifer trees population. The values of the samples sizes are $m = 4, 5, 6, 7$ with different possible values of k. In order to select a ranked set sample of size $m = 7$, the experimenter needs to identify 49 conifer trees; but because of limited time or budget, it is difficult to apply the RSS procedure. Suppose no more than 40 trees can be measured. In such situations, PRSS provides an opportunity to the PRSS(m, k) scheme, i.e., PRSS$(7, 1)$ with 37 units, PRSS$(7, 2)$ with 25 units and PRSS$(7, 3)$ with 13 units. One million replications were used to estimate the MSEs of the estimators under SRS, RSS and PRSS. The coefficient of skewness of the diameter is 0.884 and the skewness of height is 1.619. Therefore, these data are asymmetrically distributed.

Table 1.1: Summary statistics of 399 trees data

Variable	Mean	Median	Variance	Skewness	Kurtosis
Diameter (Y)(cm)	20.84	14.5	310.11	0.884	-0.423
Height (X) (ft)	52.36	29	325.14	1.619	1.776
Correlation coefficient (ρ)	0.908				

It is evident from Table 1.2 that the mean and median estimators based on PRSS are more efficient as compared with their competitors under SRS. As expected, the REs are generally high under perfect ranking as compared with imperfect ranking. For a fixed value of m, as the value of k increases, the REs tend to decrease. It is of interest to note here that, for small samples, the PRSS variance estimator is more efficient than the variance estimators under SRS and RSS. Furthermore, PRSS uses less number of units as compared with the units required in RSS procedure, and at the same time, it provides more efficient estimates than RSS.

1.6 Concluding remarks

In this chapter, we proposed a PRSS design for estimating the population mean, median and variance. PRSS provides an unbiased estimator of the population mean when the underlying population is symmetric. On the basis of extensive Monte Carlo simulations, it was observed that for both perfect and imperfect rankings, the estimators under PRSS are more efficient than the estimators based on SRS. In the variance estimation, especially for small samples, PRSS provides efficient variance estimates than the estimates under SRS and RSS designs. Therefore, it is recommended to use PRSS design as an efficient alternative to SRS design in case of population mean, median and variance estimation.

This work can be extended to develop ratio and regression estimators of the population mean and median under PRSS. Also, the current work can be extended to multistage partial ranked set sampling design.

Table 1.2: Estimation of the population mean, median and variance of the study variable X under perfect and imperfect rankings

	Mean				Median				Variance			
	RSS	PRSS	PRSS	PRSS	RSS	PRSS	PRSS	PRSS	RSS	PRSS	PRSS	PRSS
Ranking	$k=0$	$k=1$	$k=2$	$k=3$	$k=0$	$k=1$	$k=2$	$k=3$	$k=0$	$k=1$	$k=2$	$k=3$
						$m=4$						
X	1.93229	1.40117	——	——	2.35689	1.88827	——	——	0.95619	1.28807	——	——
Y	1.76474	1.32658	——	——	2.02863	1.70739	——	——	0.94762	1.21507	——	——
						$m=5$						
X	2.22719	1.52644	1.16415	——	3.01825	2.40034	1.47281	——	1.07035	1.30545	1.09905	——
Y	1.96547	1.43816	1.14066	——	2.60930	2.16476	1.39176	——	1.02916	1.22735	1.08738	——
						$m=6$						
X	2.52027	1.62581	1.27675	——	3.53694	2.82148	1.96619	——	1.18465	1.29887	1.13812	——
Y	2.14987	1.52776	1.24022	——	2.95560	2.50393	1.80554	——	1.10424	1.22805	1.12365	——
						$m=7$						
X	2.80720	1.71135	1.34462	1.12098	4.00582	3.22178	2.54038	1.43774	1.30096	1.28833	1.13252	1.05779
Y	2.32114	1.60863	1.30656	1.10899	3.38511	2.86422	2.28398	1.38009	1.17128	1.22848	1.12108	1.05496

Chapter 2

Mixed Ranked Set Sampling Design

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The main focus of agricultural, ecological and environmental studies is to develop well-designed, cost-effective and efficient sampling designs. Ranked set sampling (RSS) is one method that leads to accomplish such objectives by incorporating expert knowledge to its advantage. In this chapter, we propose an efficient sampling scheme, named mixed RSS (MxRSS), for estimation of the population mean and median. The MxRSS scheme is a suitable mixture of both simple random sampling (SRS) and RSS schemes. The MxRSS scheme provides an unbiased estimator of the population mean, and its variance is always less than the variance of sample mean based on SRS. For both symmetric and asymmetric populations, the mean and median estimators based on SRS, partial RSS (PRSS) and MxRSS schemes are compared. It turns out that the mean and median estimates under MxRSS scheme are more precise than those based on SRS scheme. Moreover, when estimating the mean of symmetric and some asymmetric populations, the mean estimates under MxRSS scheme are found to be more efficient than the mean estimates with PRSS scheme. An application to real data is also provided to illustrate the implementation of the proposed sampling scheme.

2.1 Introduction

In biomedical, environmental and ecological studies, situations may arise where taking the actual measurement of the sample observation is difficult (e.g., costly, destructive, time-consuming) but ranking a small set of selected units is comparatively easy and reliable. Ranking may be visually with respect to the study variable or by any inexpensive method. For example, if the interest lies in estimating the average height of trees in a forest, then it is easy to rank a small set of selected trees with respect to their heights. As another example, in animal and growth studies, ages of animals may need to be determined but aging an animal is costly and time-consuming. However, variables on the physical size of an animal that are highly correlated with age are cheap and easy to collect. In all such situations, the traditional ranked set sampling (RSS) scheme can be used to achieve observational economy. The RSS scheme incorporates inexpensive auxiliary information related to the variable of interest as a way of gathering additional information in order to rank the selected sampling units. This use of supplementary information helps in selecting more representative samples from the target population.

The RSS scheme was first introduced by McIntyre (1952) for estimating mean pasture and forage yields. The statistical theory of the RSS scheme was developed by Takahasi and Wakimoto (1968). They showed that the sample mean under RSS scheme is an unbiased estimator of the population mean, and it is more efficient than the sample mean based on simple random sampling (SRS). Dell and Clutter (1972) studied the effect of imperfect rankings on the performance of RSS-based mean estimator. For more details and real applications of RSS scheme, see Yu and Lam (1997), Mode et al. (1999), Al-Saleh and Al-Shrafat (2001), Al-Saleh and Al-Omari (2002), Chen and Wang (2004), Husby et al. (2005), Chen (2007), Wang et al. (2009), Ozturk (2011) and references cited therein.

Recently, Haq et al. (2013b) suggested partial RSS (PRSS) scheme for estimation of the population mean, median and variance. They showed that RSS is a special case of PRSS. The PRSS scheme becomes a suitable alternative to the RSS scheme when there is a shortage of experimental units, identification of units is costly and time-consuming, data arrives in different batches, etc. In such situations, it is beneficial to make use of PRSS scheme for efficient estimation of population parameters. The main disadvantage of PRSS scheme is that it lacks flexibility or the options to select partial ranked set samples are limited. Additionally, the PRSS scheme provides biased mean estimates when sampling from asymmetric populations.

In this chapter, we extend the work on PRSS scheme and propose a mixed RSS (MxRSS) scheme for estimation of the population mean and median. The MxRSS scheme provides plenty of options to the experimenter when selecting the sample from population. This helps in keeping the cost at an affordable level. We show that the mean estimates under MxRSS scheme are not only unbiased but also more precise than the mean estimates with SRS scheme. For symmetric and some asymmetric distributions, the mean estimates under MxRSS are found to be more efficient than the mean estimates based on PRSS.

The rest of the chapter is organized as follows: Section 2.2 contains brief details about RSS and PRSS schemes. Section 2.3 introduces MxRSS scheme for estimation of the population mean based on perfect and imperfect rankings. In this section, we also estimate the means of symmetric and asymmetric populations based on SRS, PRSS and MxRSS schemes. Estimation of the population median under the aforesaid sampling schemes is considered in Section 2.4. Section 2.5 provides the numerical results obtained from real data, and Section 2.6 summarizes the main findings.

2.2 Sampling schemes

In this section, we explain RSS and PRSS schemes for estimation of the population mean.

2.2.1 Ranked set sampling

The RSS scheme incorporates cheap quantitative or qualitative auxiliary information in order to obtain a more representative sample from the underlying population before the real and more expensive sampling is done. The RSS procedure is as follows: identify m^2 units from the parent population. These units are then allocated to m sets, each of size m units. Without knowing the actual values, the units within each set are ranked in an increasing order of magnitude with respect to the study variable. The ranking can be done by employing expert knowledge or by using any concomitant variable that is highly correlated with the study variable. After ranking all sets, the smallest ranked unit is quantified from the first set. Similarly, the second smallest ranked unit is quantified from the second set, and the procedure continues until the largest ranked unit is quantified from the last set. This completes one cycle of a ranked set sample of size m. The whole procedure can be repeated r times to obtain r cycles of ranked set sample with total sample size $n = mr$.

Let Y be the study variable with probability density function $f_Y(y)$ and cumulative distribution function $F_Y(y)$, with mean μ_Y and variance σ_Y^2. Let $Y_1, Y_2, ..., Y_n$ represent a simple random sample of size n drawn from $f_Y(y)$. The sample mean $\bar{Y}_{\text{SRS}} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is an unbiased estimator of μ_Y with variance σ_Y^2/n, i.e., $E(\bar{Y}_{\text{SRS}}) = \mu_Y$ and $\text{Var}(\bar{Y}_{\text{SRS}}) = \sigma_Y^2/n$.

Let $(Y_{11j}, Y_{12j}, ..., Y_{1mj}), (Y_{21j}, Y_{22j}, ..., Y_{2mj}), ..., (Y_{m1j}, Y_{m2j}, ..., Y_{mmj})$ be m independent simple random samples each of size m for the jth cycle for $j = 1, 2, ..., r$. Let $(Y_{i(1:m)j}, Y_{i(2:m)j}, ..., Y_{i(m:m)j})$ denote the order statistics of the ith simple random sample $(Y_{i1j}, Y_{i2j}, ..., Y_{imj})$ obtained in the jth cycle. Apply the RSS scheme to m samples, obtained in the jth cycle, to obtain a ranked set sample of size m, denoted by $Y_{i(i:m)j}$, $i = 1, 2, ..., m$ and $j = 1, 2, ..., r$. Let $\bar{Y}_{\text{RSS}} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i(i:m)j}$ be the sample mean based on a ranked set sample of size n. Takahasi and Wakimoto (1968) showed that $\bar{Y}_{\text{RSS}}$ is also an unbiased estimator of μ_Y, and its variance is less than the variance of the $\bar{Y}_{\text{SRS}}$, i.e., $\text{Var}(\bar{Y}_{\text{RSS}}) = \frac{1}{nm}\sum_{i=1}^{m} \sigma_{Y(i:m)}^2 = \text{Var}(\bar{Y}_{\text{SRS}}) - \frac{1}{nm}\sum_{i=1}^{m}(\mu_{Y(i:m)} - \mu_Y)^2$. Here $\mu_{Y(i:m)}$ and $\sigma_{Y(i:m)}^2$ are the population mean and the population variance of $Y_{i(i:m)j}$, respectively.

2.2.2 Partial ranked set sampling

Recently, Haq et al. (2013b) suggested another version of RSS, named PRSS, for estimation of the population mean, median and variance. The PRSS scheme is a mixture of both SRS and RSS schemes. This scheme requires less number of identified units than the RSS scheme when selecting a sample of size n, thus reducing the total cost, time and expenditure that are involved in sampling.

The PRSS procedure is as follows: Define a constant k such that $k = [\alpha m]$ for $0 \leq \alpha < 0.5$. Here, $[\cdot]$ represents the largest integer not greater than αm. Firstly, select a simple random sample of size $2k$ from the target population. Identify $m(m - 2k)$ units from the target population, and allocate them into $m - 2k$ sets, each of size m units. Rank the units within each set with respect to the study variable or as aforementioned. Select the ith smallest ranked unit from the ith set for $i = k+1, ..., m-k$. This completes one cycle of a partial ranked set sample of size m. For a large sample, the whole process can be repeated r times in order to obtain a partial ranked set sample of size n.

Note that the PRSS scheme is equivalent to the RSS scheme when $k = 0$. Given k, the PRSS scheme requires $nm - 2k(n - r)$ identified units from the target population in order to select a sample of size n.

The sample mean based on a partial ranked set sample of size n is given by

$$\bar{Y}_{\text{PRSS}} = \frac{1}{n}\left(\sum_{j=1}^{r}\sum_{i=1}^{k} Y_{ij} + \sum_{j=1}^{r}\sum_{i=k+1}^{m-k} Y_{i(i:m)j} + \sum_{j=1}^{r}\sum_{i=m-k+1}^{m} Y_{ij}\right),$$

with variance

$$\text{Var}(\bar{Y}_{\text{PRSS}}) = \frac{2k\sigma_Y^2}{nm} + \frac{1}{nm}\sum_{i=k+1}^{m-k}\sigma_{Y(i:m)}^2.$$

For symmetric populations, $\bar{Y}_{\text{PRSS}}$ is an unbiased estimator of μ_Y, and it is conditionally better than $\bar{Y}_{\text{SRS}}$. For details see Haq et al. (2013b).

2.3 Proposed sampling scheme

In this section, we propose MxRSS scheme for efficient estimation of the population mean.

In some ecological and environmental field studies, the ranking or identification of the experimental units is costly or time-consuming. Therefore, it is difficult to apply the RSS scheme with full confidence. The PRSS scheme is an alternative option to the RSS scheme that helps in reducing the ranking cost, but it is also restricted to some choices of k for each m. The MxRSS scheme is a suitable mixture of both SRS and RSS schemes that offers more flexibility to the experimenter in selecting more representative samples from the target population in different ways. This not only helps in reducing the ranking cost, time and expenditure, but the estimates based on MxRSS scheme turn out to be more precise than the estimates with SRS and PRSS schemes.

A mixed ranked set sample of size n can be selected based on the following steps:

Step 1: Select $k_1 (0 \leq k_1 \leq m)$ units from the target population based on SRS scheme.

Step 2: Let k_2 be a constant such that $k_2 = [\beta(m - k_1)]$ for $0 \leq \beta \leq 0.5$. Identify $(m - k_1)(m - k_1 - k_2)$ units from the target population, and partition them into $m - k_1 - k_2$ sets, each of size $m - k_1$ units. Without knowing the actual values, rank the units within each set with respect to the study variable or by any inexpensive method.

Step 3: Select the ith smallest ranked unit from the first $m - k_1 - k_2$ sets, for $i = 1, 2, ..., m - k_1 - k_2$. Also select the $(m - k_1 - i + 1)$th smallest ranked unit from the first k_2 sets. This completes one cycle of a mixed ranked set sample of size m.

Step 4: The above Steps 1–3 can be repeated r times in order to obtain a mixed ranked set sample of size n.

It is to be noted that both SRS and RSS methods are special cases of the MxRSS scheme. For $k_1 = m$, MxRSS scheme is equivalent to the traditional SRS scheme. Similarly, for $k_1 = k_2 = 0$, MxRSS scheme is equivalent to the RSS scheme. The total number of units that are required to select a mixed ranked set sample of size n are $k_1 r + r(m - k_1)(m - k_1 - k_2)$.

2.3.1 Estimation of the population mean

The sample mean based on a mixed ranked set sample of size n is defined as

$$\bar{Y}_{\text{MxRSS}} = \frac{1}{n}\left(\sum_{j=1}^{r}\sum_{i=1}^{k_1} Y_{ij} + \sum_{j=1}^{r}\sum_{i=1}^{m-k_1-k_2} Y_{i(i:m-k_1)j} + \sum_{j=1}^{r}\sum_{i=1}^{k_2} Y_{i(m-k_1-i+1:m-k_1)j}\right)$$

with variance

$$\begin{aligned}\text{Var}(\bar{Y}_{\text{MxRSS}}) &= \frac{1}{nm}\left(k_1\sigma_Y^2 + \sum_{i=1}^{m-k_1-k_2}\sigma^2_{Y(i:m-k_1)} + \sum_{i=1}^{k_2}\sigma^2_{Y(m-k_1-i+1:m-k_1)}\right.\\ &\left. +2\sum_{1\le i<m-k_1-i+1}^{k_2}\sigma_{Y(i,m-k_1-i+1:m-k_1)}\right).\end{aligned}$$

Lemma 1:

(i) $\bar{Y}_{\text{MxRSS}}$ is an unbiased estimator of the population mean μ_Y.

(ii) For any population, we have $\text{Var}(\bar{Y}_{\text{MxRSS}}) \le \text{Var}(\bar{Y}_{\text{SRS}})$.

Proof:

(i)

$$\begin{aligned}E(\bar{Y}_{\text{MxRSS}}) &= \frac{r}{n}\left(\sum_{i=1}^{k_1}\mu_Y + \sum_{i=1}^{m-k_1-k_2}\mu_{Y(i:m-k_1)} + \sum_{i=1}^{k_2}\mu_{Y(m-k_1-i+1:m-k_1)}\right),\\ &= \frac{1}{m}\{k_1\mu_Y + (m-k_1)\mu_Y\} = \mu_Y,\end{aligned}$$

using the fact that $\sum_{i=1}^{t}\mu_{Y(i:t)} = t\mu_Y$, see Takahasi and Wakimoto (1968).

(ii)

$$\begin{aligned}\text{Var}(\bar{Y}_{\text{MxRSS}}) &= \frac{1}{nm}\left(k_1\sigma_Y^2 + \sum_{i=1}^{m-k_1-k_2}\sigma^2_{Y(i:m-k_1)} + \sum_{i=1}^{k_2}\sigma^2_{Y(m-k_1-i+1:m-k_1)}\right.\\ &\left. +2\sum_{1\le i<m-k_1-i+1}^{k_2}\sigma_{Y(i,m-k_1-i+1:m-k_1)}\right),\\ &= \frac{1}{nm}\left(k_1\sigma_Y^2 + \sum_{i=1}^{m-k_1}\sigma^2_{Y(i:m-k_1)} + 2\sum_{1\le i<m-k_1-i+1}^{k_2}\sigma_{Y(i,m-k_1-i+1:m-k_1)}\right),\end{aligned}$$

where $\sigma_{Y(i,m-k_1-i+1:m-k_1)} \ge 0$, $i = 1, 2, ..., k_2$, is the positive covariance between $Y_{i(i:m-k_1)j}$ and $Y_{i(m-k_1-i+1:m-k_1)j}$.

Following Al-Saleh and Al-Omari (2002), we can write

$$\sum_{i=1}^{m-k_1} \sigma^2_{Y(i:m-k_1)} = (m-k_1)\sigma^2_Y - 2\sum_{1\leq i<j}^{m-k_1} \sigma_{Y(i,j:m-k_1)}.$$

Then, it follows that

$$\mathrm{Var}(\bar{Y}_{\mathrm{MxRSS}}) = \frac{\sigma^2_Y}{n} - \frac{2}{nm}\left(\sum_{1\leq i<j}^{m-k_1} \sigma_{Y(i,j:m-k_1)} - \sum_{1\leq i<m-k_1-i+1}^{k_2} \sigma_{Y(i,m-k_1-i+1:m-k_1)}\right) \geq 0.$$

Note that the first term, $\sum_{1\leq i<j}^{m-k_1} \sigma_{Y(i,j:m-k_1)} \geq 0$, contains all positive covariance terms including those terms being subtracted from it, i.e., $\sum_{1\leq i<m-k_1-i+1}^{k_2} \sigma_{Y(i,m-k_1-i+1:m-k_1)}$. Thus, overall their difference is a positive quantity, which completes the proof.

For any population, the relative efficiency (RE) of $\bar{Y}_{\mathrm{MxRSS}}$ with respect to $\bar{Y}_{\mathrm{SRS}}$ is given by

$$\begin{aligned}\mathrm{RE}(\bar{Y}_{\mathrm{MxRSS}}, \bar{Y}_{\mathrm{SRS}}) &= \frac{\mathrm{Var}(\bar{Y}_{\mathrm{SRS}})}{\mathrm{Var}(\bar{Y}_{\mathrm{MxRSS}})}, \\ &= \frac{1}{1-\frac{2}{m\sigma^2_Y}\left(\sum_{1\leq i<j}^{m-k_1} \sigma_{Y(i,j:m-k_1)} - \sum_{1\leq i<m-k_1-i+1}^{k_2} \sigma_{Y(i,m-k_1-i+1:m-k_1)}\right)},\end{aligned}$$

which is independent of r.

2.3.2 Imperfect ranking schemes

Sometimes, it is difficult for the experimenter to rank the experimental units with full confidence with respect to the study variable. Dell and Clutter (1972) showed that the sample mean based on imperfect ranking remains an unbiased estimator of population mean as long as the ranking is better than the random ordering of the experimental units. Stokes (1977) showed that it is possible to judge the ranks of the study variable with respect to the ranks of the concomitant variable, say X, that is cheap and correlated with the study variable. The assumptions imposed by Stokes (1977) in developing the model for imperfect ranking are:

(i) the regression of Y on X is linear,

(ii) the underlying distribution of standardized variables, i.e., $\frac{Y-\mu_Y}{\sigma_Y}$ and $\frac{X-\mu_X}{\sigma_X}$, is same.

These conditions can be easily met when both (Y, X) follow a bivariate normal distribution. The mathematical model suggested by Stokes (1977) for imperfect ranking is given by

$$Y_{i[i:m]j} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}\left(X_{i(i:m)j} - \mu_X\right) + \xi_{ij}, \quad i = 1, 2, ..., m,\ j = 1, 2, ..., r,$$

where μ_Y and μ_X are the population means, σ_Y and σ_X are the population standard deviations, of Y and X, respectively, ρ is the correlation coefficient between Y and X. Here, $X_{i(i:m)j}$ is the ith order statistic corresponding to the ith judgment order statistic $Y_{i[i:m]j}$ obtained from the ith sample in the jth cycle. ξ_{ij} is a random error term with zero mean and constant variance, i.e., $E(\xi_{ij}) = 0$ and $\text{Var}(\xi_{ij}) = \sigma_\xi^2 = \sigma_Y^2(1-\rho^2)$. For more details, see Stokes (1977).

Haq et al. (2013b) considered bivariate normal distribution for imperfect RSS (IPRSS) scheme. They estimated the RE of the mean estimator under IRSS using Monte Carlo simulations. Here we calculate the exact RE of the PRSS-based mean estimator under the aforementioned ranking model. The sample mean based on IPRSS scheme is given by

$$\bar{Y}_{\text{IPRSS}} = \frac{1}{n}\left(\sum_{j=1}^{r}\sum_{i=1}^{k} Y_{ij} + \sum_{j=1}^{r}\sum_{i=k+1}^{m-k} Y_{i[i:m]j} + \sum_{j=1}^{r}\sum_{i=m-k+1}^{m} Y_{ij}\right),$$

with variance

$$\text{Var}(\bar{Y}_{\text{IPRSS}}) = \frac{1}{nm}\left[\sigma_Y^2\{2k + (m-2k)(1-\rho^2)\} + \rho^2\frac{\sigma_Y^2}{\sigma_X^2}\sum_{i=k+1}^{m-k}\sigma_{X(i:m)}^2\right].$$

The RE of $\bar{Y}_{\text{IPRSS}}$ with respect to $\bar{Y}_{\text{SRS}}$ is given by

$$\begin{aligned}\text{RE}(\bar{Y}_{\text{IPRSS}}, \bar{Y}_{\text{SRS}}) &= \frac{\text{Var}(\bar{Y}_{\text{SRS}})}{\text{Var}(\bar{Y}_{\text{IPRSS}})}, \\ &= \frac{m\sigma_X^2}{\sigma_X^2\{2k + (m-2k)(1-\rho^2)\} + \rho^2\sum_{i=k+1}^{m-k}\sigma_{X(i:m)}^2},\end{aligned}$$

which is independent of r.

Similarly, the mean estimator based on imperfect MxRSS (IMxRSS) is defined as

$$\bar{Y}_{\text{IMxRSS}} = \frac{1}{n}\left(\sum_{j=1}^{r}\sum_{i=1}^{k_1} Y_{ij} + \sum_{j=1}^{r}\sum_{i=1}^{m-k_1-k_2} Y_{i[i:m-k_1]j} + \sum_{j=1}^{r}\sum_{i=1}^{k_2} Y_{i[m-k_1-i+1:m-k_1]j}\right),$$

with variance

$$\begin{aligned}\text{Var}(\bar{Y}_{\text{IMxRSS}}) &= \frac{1}{nm}\Bigg[\sigma_Y^2\{k_1 + (m-k_1)(1-\rho^2)\} \\ &\quad + \rho^2\frac{\sigma_Y^2}{\sigma_X^2}\left(\sum_{i=1}^{m-k_1}\sigma_{X(i:m-k_1)}^2 + 2\sum_{1\le i<m-k_1-i+1}^{k_2}\sigma_{X(i,m-k_1-i+1:m-k_1)}\right)\Bigg].\end{aligned}$$

The RE of $\bar{Y}_{\text{IMxRSS}}$ with respect to $\bar{Y}_{\text{SRS}}$ is given by

$$\text{RE}(\bar{Y}_{\text{IMxRSS}}, \bar{Y}_{\text{SRS}}) = \frac{\text{Var}(\bar{Y}_{\text{SRS}})}{\text{Var}(\bar{Y}_{\text{IMxRSS}})} = \frac{m\sigma_X^2}{\sigma_X^2\{k_1 + (m-k_2)(1-\rho^2)\} + \rho^2 A},$$

which is also independent of r, where $A = \sum_{i=1}^{m-k_1}\sigma_{X(i:m-k_1)}^2 + 2\sum_{1\le i<m-k_1-i+1}^{k_2}\sigma_{X(i,m-k_1-i+1:m-k_1)}$.

2.3.3 Comparison of mean estimators

In this section, we compare the mean estimators based on SRS, PRSS and MxRSS schemes. The exact REs of the mean estimators under each sampling scheme have been calculated.

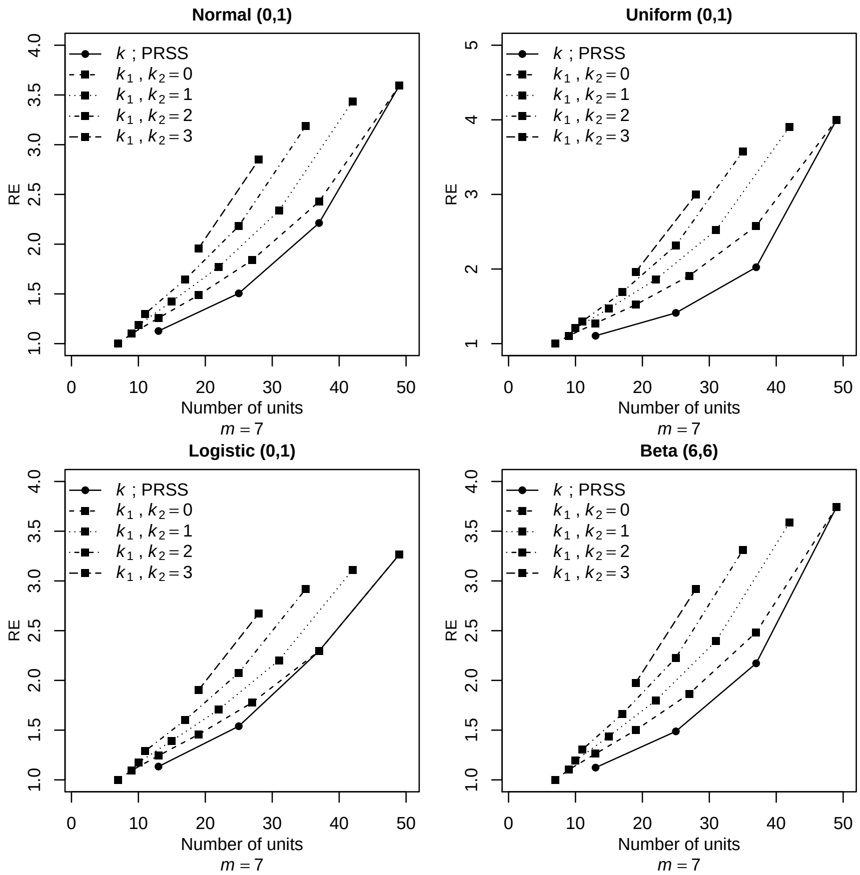

Figure 2.1: REs of mean estimators based on PRSS and MxRSS versus SRS for symmetric distributions

For a fair comparison of mean estimators based on PRSS and MxRSS schemes, we consider both symmetric and asymmetric distributions considered by Haq et al. (2013b). The symmetric and asymmetric distributions considered here are Normal (0,1), Uniform (0,1), Logistic (0,1), Beta (6,6) and Exponential (1), Weibull (0.5,1), Lognormal (0,1), Gamma (0.5,2), respectively. The REs of mean estimators, based on $m = 7$, have been computed from both symmetric and asymmetric distributions and are displayed in Figures 2.1 and 2.2, respectively.

From Figure 2.1, it is clear that the RE of an estimator is an increasing function of the number of units, i.e., with an increase in the value of m or number of units, the RE of the mean estimator also increases and vice-versa. It is observed that all REs are greater than one, which shows that the mean estimators based on PRSS and MXRSS are more precise than the mean estimators based on SRS. An interesting feature of the MxRSS scheme is that a sample can be selected in different possible ways as compared with the PRSS scheme. Under symmetric distributions, the mean estimators under MxRSS outperform the mean estimators based on PRSS when using the same number of experimental units. It is also worth mentioning that, under MxRSS scheme, the mean estimates based on less number of units are more precise than the mean estimates under PRSS scheme. This shows the superiority of the MxRSS scheme over PRSS scheme when estimating the population mean of a symmetric population.

Figure 2 compares the mean estimators based on different asymmetric distributions. It turns out that under MxRSS scheme, the REs are increasing with the number of units. In case when the underlying

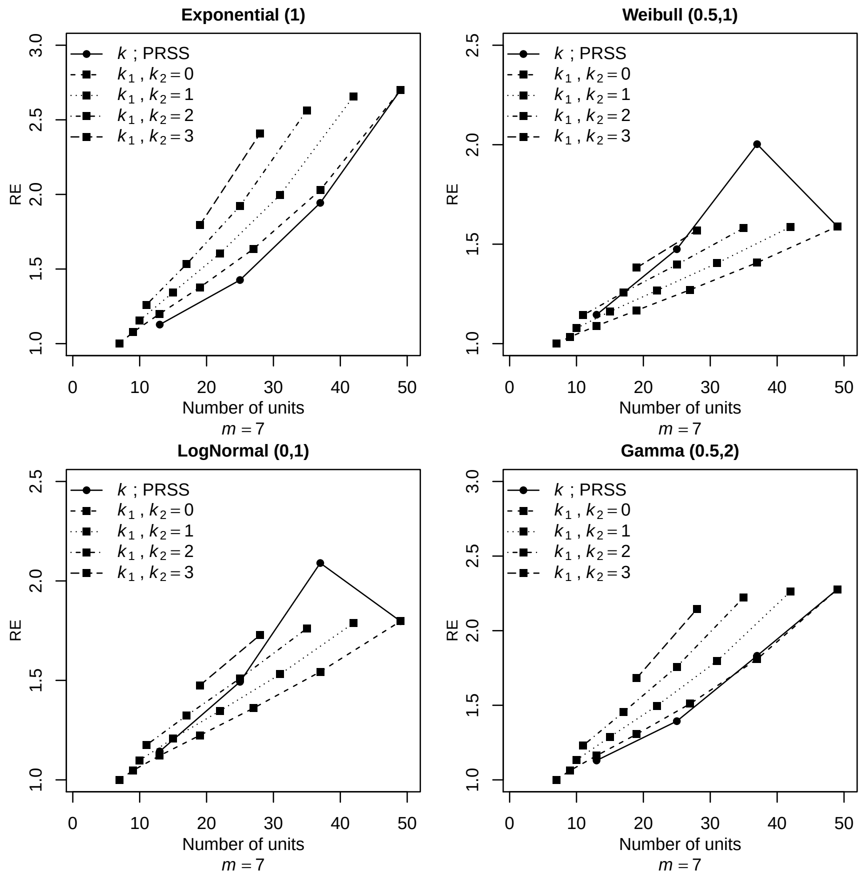

Figure 2.2: REs of mean estimators based on PRSS and MxRSS versus SRS for asymmetric distributions

distribution is Exponential (1) or Gamma (0.5,2), the REs of mean estimators under MxRSS are greater than the REs of their counterparts based on PRSS. However, for some highly skewed distributions, i.e., Weibull (0.5,1) or Lognormal (0,1), under PRSS scheme when $k = 1$, the mean estimates are more precise than the mean estimates with MxRSS. From Figures 2.1 and 2.2, we conclude that MxRSS-based mean estimates are better than those based SRS for all cases considered here. Moreover, these estimates turn out to be more efficient than the mean estimates based on PRSS scheme when estimating the population mean of symmetric and some asymmetric distributions.

In Figure 2.3, we compare the performances of the mean estimators based on imperfect RSS schemes. For a fair comparison, we consider different values of ρ with $m = 7$. The REs of mean estimators under both IPRSS and IMxRSS schemes are calculated and displayed against the number of units in Figure 2.3. From Figure 2.3, it is clear that all REs are greater than one. This shows that the mean estimates based on IPRSS and IMxRSS schemes are better than those based on SRS scheme. Furthermore, in all cases, the mean estimates under IMxRSS scheme are more precise than the mean estimates with IPRSS scheme.

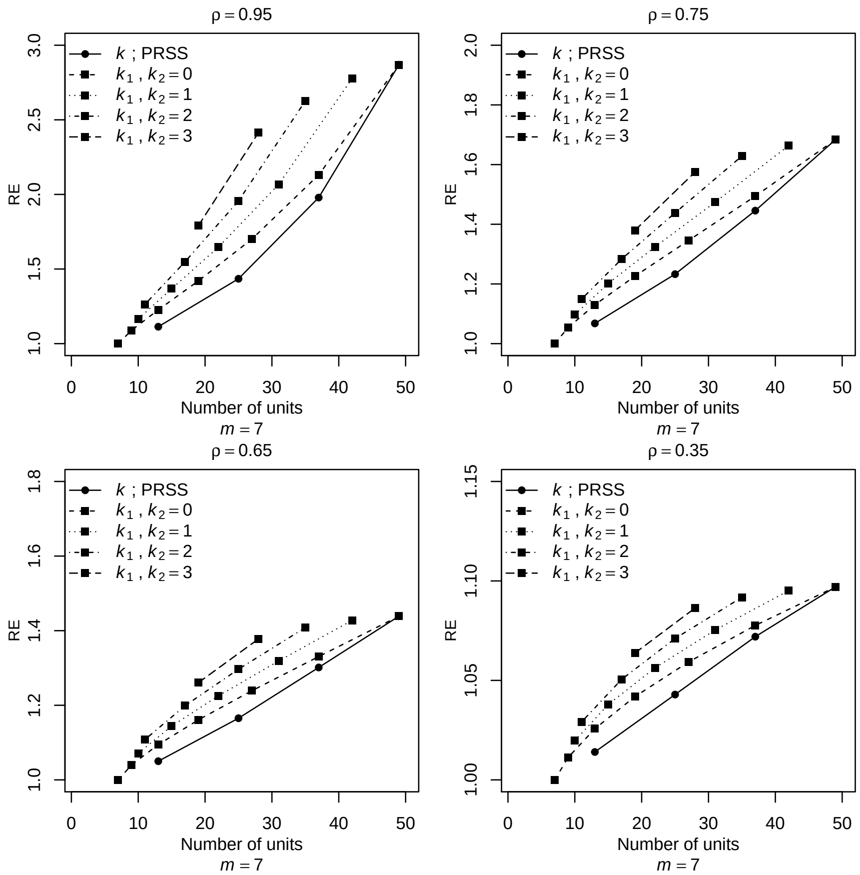

Figure 2.3: REs of mean estimators based on IPRSS and IMxRSS versus SRS for standard bivariate normal distribution

2.4 Estimation of the population median

In this section, we compare the median estimators based on SRS, PRSS and MxRSS schemes.

In survey sampling, we often encounter several variables that follow highly skewed distributions, such as income, expenditure and production. In such situations, the sample median is considered as a more suitable measure of location than the sample mean. Following Haq et al. (2013b), we perform extensive Monte Carlo simulations from both symmetric and asymmetric distributions in order to study the performances of the median estimators based on SRS, PRSS and MxRSS schemes.

The sample median based on a simple random sample of size n is given by

$$\tilde{Y}_{\text{SRS}} = \text{Median}\{Y_1, Y_2, ..., Y_n\} = \begin{cases} Y_{(n/2+1/2:n)}, & \text{if } n \text{ is odd,} \\ \{Y_{(n/2:n)} + X_{(n/2+1:n)}\}/2, & \text{if } n \text{ is even.} \end{cases}$$

Similarly, the median estimators based on partial and mixed ranked set samples of size n, respectively, are given by

$$\tilde{Y}_{\text{PRSS}} = \text{Median}\left\{ \begin{array}{l} \left(Y_{11}, ..., Y_{k1}, Y_{k+1(k+1:m)1}, ..., Y_{m-k(m-k:m)1}, Y_{(m-k+1)1}, ..., Y_{m1}\right), ..., \\ \left(Y_{1r}, ..., Y_{kr}, Y_{k+1(k+1:m)r}, ..., Y_{m-k(m-k:m)r}, Y_{(m-k+1)r}, ..., Y_{mr}\right) \end{array} \right\}$$

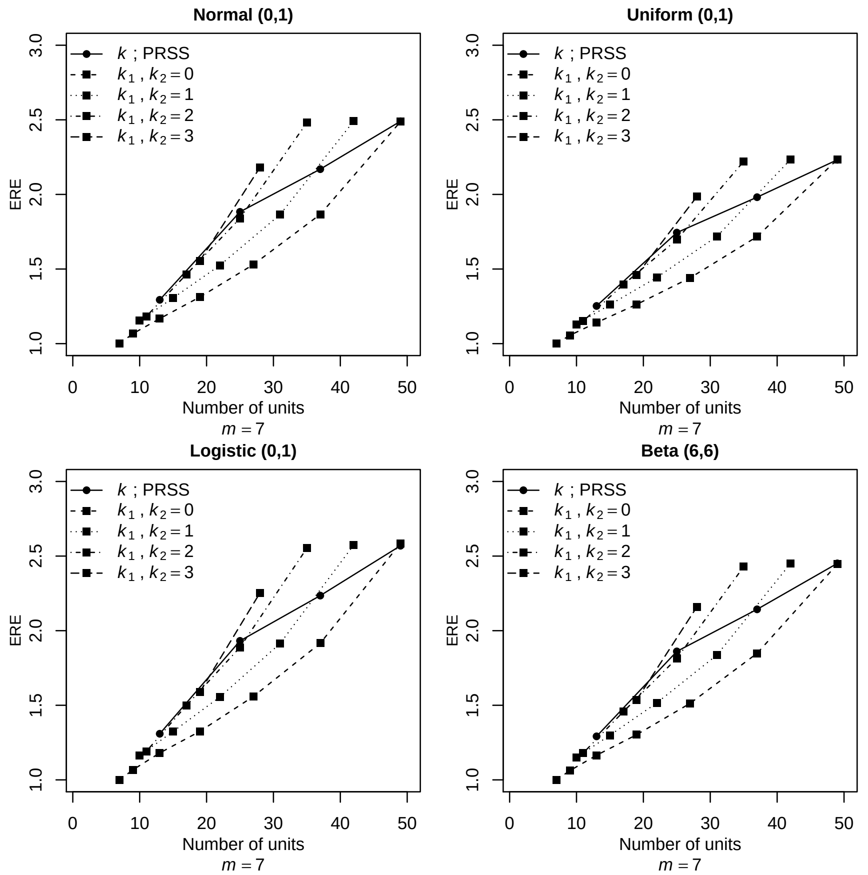

Figure 2.4: EREs of median estimators based on PRSS and MxRSS versus SRS for symmetric distributions

and

$$\tilde{Y}_{\mathrm{MxRSS}} = \text{Median}\left\{\begin{array}{l}\big(Y_{11}, ..., Y_{k_1 1}, Y_{1(1:m-k_1)1}, ..., Y_{m-k_1-k_2(m-k_1-k_2:m-k_1)1}, \\ Y_{1(m-k_1:m-k_1)1}, ..., Y_{k_2(m-k_1-k_2+1:m-k_1)1}\big), ..., \\ \big(Y_{1r}, ..., Y_{k_1 r}, Y_{1(1:m-k_1)r}, ..., Y_{m-k_1-k_2(m-k_1-k_2:m-k_1)r}, \\ Y_{1(m-k_1:m-k_1)r}, ..., Y_{k_2(m-k_1-k_2+1:m-k_1)r}\big)\end{array}\right\}.$$

The REs of $\tilde{Y}_{\mathrm{PRSS}}$ and $\tilde{Y}_{\mathrm{MxRSS}}$ with respect to $\tilde{Y}_{\mathrm{SRS}}$, respectively, are given by

$$\mathrm{RE}(\tilde{Y}_{\mathrm{PRSS}}, \tilde{Y}_{\mathrm{SRS}}) = \frac{\mathrm{MSE}(\tilde{Y}_{\mathrm{SRS}})}{\mathrm{MSE}(\tilde{Y}_{\mathrm{PRSS}})} \quad \text{and} \quad \mathrm{RE}(\tilde{Y}_{\mathrm{MxRSS}}, \tilde{Y}_{\mathrm{SRS}}) = \frac{\mathrm{MSE}(\tilde{Y}_{\mathrm{SRS}})}{\mathrm{MSE}(\tilde{Y}_{\mathrm{MxRSS}})},$$

where MSE is the mean squared error (MSE) of $(\cdot)$. It is difficult to derive the exact mathematical expressions of the MSEs of the median estimators based on PRSS and MxRSS schemes. Therefore, the MSEs are estimated using Monte Carlo simulations, and then estimated REs (EREs) are calculated. The estimated MSE (EMSE) of any median estimator $\tilde{Y}_{\mathrm{H}}$ under H sampling scheme is given by

$$\mathrm{EMSE}(\tilde{Y}_{\mathrm{H}}) = \frac{1}{T}\sum_{i=1}^{T}(\tilde{Y}_{i,\mathrm{H}} - \tilde{Y})^2,$$

where H $=$ SRS, PRSS, MxRSS. Here, $\tilde{Y}$ represents the population median and T is the total number of

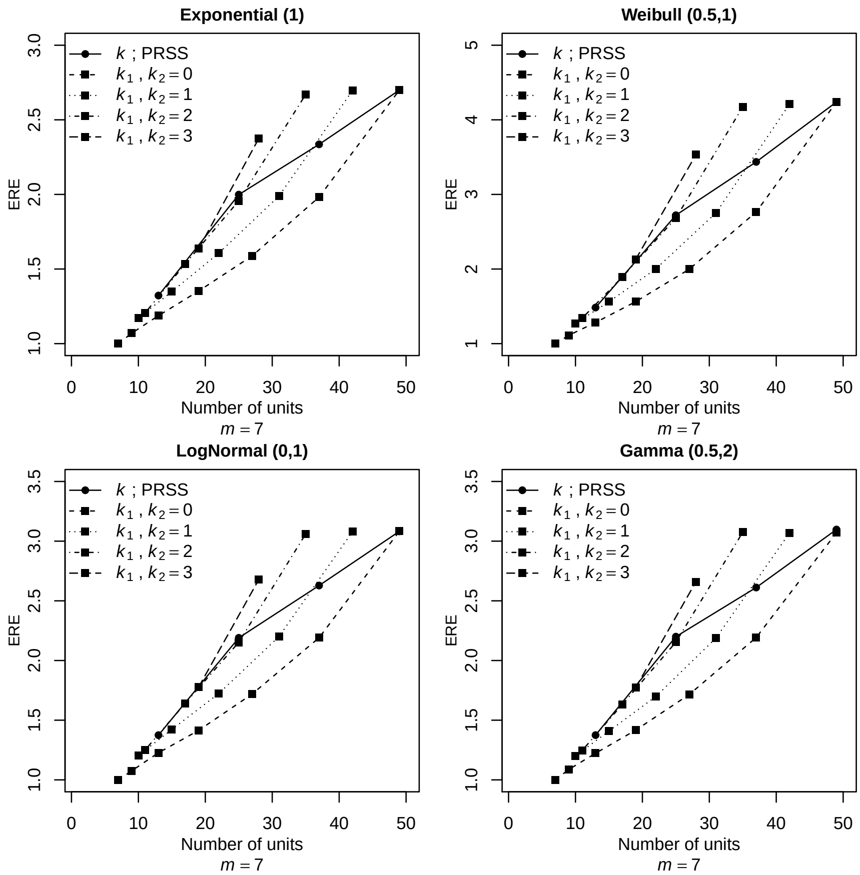

Figure 2.5: EREs of median estimators based on PRSS and MxRSS versus SRS for asymmetric distributions

replications (10^6). The ERE of $\tilde{Y}_{\mathrm{H}}$ with respect to $\tilde{Y}_{\mathrm{SRS}}$ is defined as

$$\mathrm{ERE}(\tilde{Y}_{\mathrm{H}}, \tilde{Y}_{\mathrm{SRS}}) = \frac{\mathrm{EMSE}(\tilde{Y}_{\mathrm{SRS}})}{\mathrm{EMSE}(\tilde{Y}_{\mathrm{H}})}.$$

The EREs of the median estimators based on SRS, PRSS and MxRSS schemes are calculated for both symmetric and asymmetric distributions, and are displayed in Figures 2.4 and 2.5, respectively.

From Figures 2.4 and 2.5, almost a similar trend is observed as was seen in Section 2.3.3. Note that under PRSS scheme, the choices are limited and the experimenter is forced to select certain ranks with full confidence to achieve precision whereas under MxRSS scheme the experimenter has plenty of options to get more efficient estimates. Therefore, under MxRSS the median estimators have high REs than the median estimators with PRSS when both designs use different number of experimental units. The median estimators with the same number of units under PRSS are better than the median estimation with MxRSS.

2.5 An application to real data

In this section, a real data set is taken to study and compare the performances of the mean and median estimators based on SRS, PRSS and MxRSS schemes.

Following Haq et al. (2013b), data on 399 conifer trees are considered for application of the perfect and imperfect RSS schemes considered here. For more details about trees data, see Platt et al. (1988). Let the

Table 2.1: Summary statistics of 399 trees data

Variable	Mean	Median	Variance	Skewness	Kurtosis
Diameter (Y)(cm)	20.84	14.5	310.11	0.884	−0.423
Height (X) (ft)	52.36	29	325.14	1.619	1.776
Correlation coefficient (ρ)	0.908				

study variable Y be the height of the tree, measured in feet, and the auxiliary variable X be the diameter of the tree at breast height, measured in centimeters. Our objective is to estimate the average and median height of the 399 trees using different RSS schemes based on different sample sizes. We consider different values of m, i.e., $m = 4, 5, 6$, with $r = 1$. Consider the case when $n = 5$, the experimenter need to identify 25 conifer trees in order to select a ranked set sample of size $n = 5$. However, under PRSS scheme, the same sample size can be selected by identifying 25, 17 and 9 trees. Suppose that it is difficult to identify these fixed number of trees. In such situations, MxRSS scheme is more economical and flexible than the RSS and PRSS schemes. Under MxRSS scheme, a sample of size $n = 5$ can be selected by identifying 25, 20, 17, 15, 11, 13, 9, 8, 7 and 5 trees. Note that both SRS and RSS are special cases of the MxRSS scheme.

Table 2.2: Comparison of EBs and EREs of the mean and median estimators based on perfect and imperfect PRSS schemes with respect to their counterparts based on SRS for trees data

$m = 4$	PRSS	$(4, k)$	(4,0)	(4,1)	
		No. of units	16	10	
Mean	ERE	ranking on Y	1.93229	1.40117	
	EB		−0.00569	−5.74554	
Mean	ERE	ranking on X	1.76474	1.32658	
	EB		0.00002	−4.85254	
Median	ERE	ranking on Y	2.35689	1.88827	
	EB		8.07091	7.74228	
Median	ERE	ranking on X	2.02863	1.70739	
	EB		8.61632	8.22104	
$m = 5$	PRSS	$(5, k)$	(5,0)	(5,1)	(5,2)
		No. of units	25	17	9
Mean	ERE	ranking on Y	2.22719	1.52644	1.16415
	EB		−0.00213	−6.59463	−3.15085
Mean	ERE	ranking on X	1.96547	1.43816	1.14066
	EB		0.00118	−5.53820	−2.74135
Median	ERE	ranking on Y	3.01825	2.40034	1.47281
	EB		3.37075	3.96855	5.67247
Median	ERE	ranking on X	2.60930	2.16476	1.39176
	EB		3.72485	4.26022	5.91785
$m = 6$	PRSS	$(6, k)$	(6,0)	(6,1)	(6,2)
		No. of units	36	26	16
Mean	ERE	ranking on Y	2.52027	1.62581	1.27675
	EB		−0.00209	−6.94324	−5.24357
Mean	ERE	ranking on X	2.14987	1.52776	1.24022
	EB		−0.00064	−5.80143	−4.58378
Median	ERE	ranking on Y	3.53694	2.82148	1.96619
	EB		3.86064	3.76995	4.62393
Median	ERE	ranking on X	2.95560	2.50393	1.80554
	EB		4.22140	4.07353	4.92899
SRS		$m = 4$	$m = 5$	$m = 6$	
Mean	EB	0.00541	−0.00702	0.00210	
Median	EB	11.89240	7.62705	7.64172	

The summary statistics of the data are given in Table 2.1. Based on one million replications, estimated

Table 2.3: Comparison of EBs and EREs of the mean and median estimators based on perfect and imperfect MxRSS schemes with respect to their counterparts based on SRS for trees data

$m=4$	MxRSS	$(4,k_1,k_2)$	(4,0,0)	(4,1,0)	(4,2,0)	(4,3,0)	(4,0,1)	(4,1,1)	(4,0,2)							
		no. of units	16	10	6	4	12	7	8							
Mean	ERE	ranking on Y	1.93256	1.40841	1.13963	1.00000	1.84463	1.32000	1.58695							
	EB		0.00336	–0.00098	–0.00682	0.00374	–0.00777	0.00385	0.00801							
Mean	ERE	ranking on X	1.76558	1.35749	1.12634	1.00000	1.69890	1.28147	1.47207							
	EB		0.00184	0.00179	0.00518	0.00426	0.01286	0.00301	0.00253							
Median	ERE	ranking on Y	2.35364	1.47922	1.15021	1.00000	2.32047	1.42291	1.61269							
	EB		8.07210	9.84056	11.17660	11.88510	8.09098	9.96757	8.48153							
Median	ERE	ranking on X	2.03443	1.40792	1.13537	1.00000	1.92268	1.35083	1.46902							
	EB		8.61664	10.0854	11.23380	11.86480	8.64852	10.24200	9.05837							
$m=5$	MxRSS	$(5,k_1,k_2)$	(5,0,0)	(5,1,0)	(5,2,0)	(5,3,0)	(5,4,0)	(5,0,1)	(5,1,1)	(5,2,1)	(5,0,2)	(5,1,2)				
		no. of units	25	17	11	7	5	20	13	8	15	9				
Mean	ERE	ranking on Y	2.22659	1.62688	1.30088	1.10762	1.00000	2.16771	1.57850	1.24189	2.00468	1.41964				
	EB		0.00655	–0.00033	0.00765	–0.00010	0.00646	0.00708	–0.00327	–0.00274	0.00111	0.00574				
Mean	ERE	ranking on X	1.96621	1.53073	1.26636	1.09945	1.00000	1.92587	1.49061	1.21217	1.79445	1.34481				
	EB		0.00719	0.00746	–0.00315	0.00295	0.00984	–0.00557	0.00092	0.01067	0.00195	0.01137				
Median	ERE	ranking on Y	3.03046	1.83716	1.37787	1.12453	1.00000	3.01445	1.82599	1.34923	2.65210	1.46753				
	EB		3.36044	4.87692	6.02353	6.97749	7.61286	3.37287	4.91340	6.12513	3.73313	5.82280				
Median	ERE	ranking on X	2.60352	1.70051	1.32847	1.11180	1.00000	2.59615	1.68400	1.29218	2.27652	1.39193				
	EB		3.71869	5.12095	6.16259	7.02714	7.62934	3.74187	5.16080	6.28805	4.14630	5.99082				
$m=6$	MxRSS	$(6,k_1,k_2)$	(6,0,0)	(6,1,0)	(6,2,0)	(6,3,0)	(6,4,0)	(6,5,0)	(6,0,3)	(6,01)	(6,1,1)	(6,2,1)	(6,3,1)	(6,0,2)	(6,1,2)	(6,2,2)
		no. of units	36	26	18	12	8	6	18	30	21	14	9	24	16	10
Mean	ERE	ranking on Y	2.52157	1.84918	1.47285	1.24075	1.08848	1.00000	2.47660	1.81548	1.43803	1.19296	2.35962	1.71531	1.32662	2.15072
	EB		0.00350	–0.00568	0.00104	–0.01149	0.00569	0.00574	–0.00149	0.00725	0.00654	–0.00234	–0.00039	–0.00237	–0.00190	–0.00608
Mean	ERE	ranking on X	2.14993	1.69478	1.40669	1.21466	1.08054	1.00000	2.12322	1.66989	1.37927	1.17146	2.03798	1.58624	1.27032	1.86833
	EB		0.00239	–0.00081	0.00236	–0.01281	0.00006	–0.00045	–0.00388	–0.00458	–0.01002	0.00638	–0.00439	–0.00369	–0.00296	0.00169
Median	ERE	ranking on Y	3.53310	2.20632	1.62323	1.30568	1.11253	1.00000	3.51832	2.19419	1.61237	1.27272	3.36424	2.01926	1.35444	2.18721
	EB		3.86024	4.91433	5.81464	6.55568	7.19659	7.62561	3.86889	4.92811	5.81248	6.63550	3.90960	5.08227	6.31102	4.36739
Median	ERE	ranking on X	2.95616	1.99832	1.53809	1.26998	1.09910	1.00000	2.94801	1.99315	1.52206	1.23517	2.80219	1.82713	1.30350	2.00759
	EB		4.21633	5.17143	5.96829	6.64940	7.24929	7.62204	4.22448	5.17758	5.99389	6.74987	4.29779	5.37721	6.45888	4.72278
SRS		$m=4$	$m=5$	$m=6$												
Mean	EB	0.00742	0.00412	–0.00296												
Median	EB	11.9033	7.63485	7.61689												

biases (EBs) and EREs of both mean and median estimators have been computed for all sampling scheme and reported in Tables 2.2 and 2.3. From Table 2.2, under PRSS scheme, for $m = 5$ and $k = 1$, the EB and ERE of the mean estimator are -6.59463 and 1.52644, respectively, when 17 trees were identified. However, under MxRSS scheme for $m = 5$, $k_1 = 1$ and $k_2 = 0$, the EB and ERE of the mean estimator are -0.00033 and 1.62688, respectively, when 17 trees were identified. Similarly, these quantities are 0.00111 and 2.00468, respective, with $k_1 = 0$ and $k_2 = 2$ when 15 trees were identified. It is clear that with small number of identified trees, the mean estimates under MxRSS scheme are much more precise than the mean estimates with PRSS scheme. Note that the RE under MxRSS scheme with 15 trees is greater than that of PRSS with 17 trees, because in 17 trees one unit is selected using SRS, which reduces the efficiency of the PRSS-based mean estimator. Moreover, the mean estimates under SRS and MxRSS schemes are unbiased as compared with the estimates under PRSS scheme.

With the same number of units, the median estimates under PRSS scheme are better than the MxRSS-based median estimates, but the options under PRSS are limited to achieve higher efficiency which is accomplishable by using MxRSS scheme. Moreover, the median estimates under MxRSS are less biased than the estimates under PRSS, provided both schemes use different number of identified units. Under PRSS scheme, the EB and ERE of the median estimator are 3.96855 and 2.40034, respectively, with $m = 5$, $k_1 = 1$ and $k_2 = 0$ when 17 trees were identified. However, under MxRSS scheme, the EB and ERE of the median estimator are 3.73313 and 2.65210, respectively, with $m = 5$, $k_1 = 0$ and $k_2 = 2$ when 15 trees were identified. Similarly, the EB of the median estimator when $m = 5$ under SRS scheme is 7.62705. In case of imperfect rankings, as expected, the EREs (EBs) of the median estimators have decreases (increased) as compared with perfect rankings.

2.6 Concluding remarks

In this chapter, we have proposed an improved MxRSS design for efficient estimation of the population mean and median. It is shown that the MxRSS scheme provides an unbiased estimator of the population mean and its variance is always less than the variance of the sample mean based on SRS. We also showed that both unbiased sampling schemes, i.e., SRS and RSS, are special cases of the MxRSS scheme. Both symmetric and asymmetric distributions were used to evaluate the performances of the estimators under perfect and imperfect ranking schemes. It is noteworthy that MxRSS provides more flexibility to the experimenter in selecting more representative samples from the target population as compared with RSS and PRSS schemes. It is observed that the mean and median estimates based on MxRSS scheme are more precise than their counterparts based on SRS scheme. Moreover, when estimating the mean of a symmetric population, MxRSS scheme provides more precise mean estimates than the mean estimates under PRSS scheme. Similarly, if the interest lies in estimating the population median, the median estimates under MxRSS scheme are more efficient than those based on PRSS scheme, provided MxRSS scheme uses more number of identified units than the PRSS scheme. Thus, we recommend using MxRSS scheme for estimation of the population mean and median when it is difficult to apply the RSS and PRSS schemes with full confidence.

Chapter 3

Paired Double Ranked Set Sampling

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In environmental monitoring and assessment, the main focus is to achieve observational economy and to collect data with unbiased, efficient and cost-effective sampling methods. Ranked set sampling (RSS) is one traditional method that is mostly used for accomplishing observational economy. In this chapter, we propose an unbiased sampling scheme, named paired double RSS (PDRSS) for estimating the population mean. We study the performance of the mean estimators under PDRSS based on perfect and imperfect rankings. It is shown that, for perfect ranking, the variance of the mean estimator under PDRSS is always less than the variance of mean estimator based on simple random sampling (SRS), paired RSS and RSS. The mean estimators under RSS, median RSS, PDRSS and double RSS are also compared with the regression estimator of the population mean based on SRS. The procedure is also illustrated with a case study using a real data set.

3.1 Introduction

There are many sampling methods that can be used in surveys of natural resources in agriculture, biology, ecology, environmental management, forestry, etc. The main objective of whatever sampling method is used to obtain precise estimates of population parameters with minimum cost and expenditure. One method is ranked set sampling (RSS). RSS becomes an efficient alternative to simple random sampling (SRS) when taking exact measurements of selected units is very costly whereas ranking a small set of selected units is cheap. Ranking may be visually with respect to the study variable or by any inexpensive method. For example, if interest lies in estimating the average height of trees, then it is easy to rank a small set of trees with respect to their heights visually. Similarly, the hazardous waste sites with different levels of contamination can be ranked by a visual inspection of soil staining whereas the actual measurement of toxic chemicals and assessing their environmental impact may be very costly.

McIntyre (1952) was the first to suggest the RSS method for estimation of pasture and forage yields. Takahasi and Wakimoto (1968) developed the theory of RSS procedure under the assumption of perfect ranking. Dell and Clutter (1972) showed that the mean estimator under imperfect RSS remains an unbiased estimator of the population mean. It is also possible to rank the values of the study variable on the basis of a cheap concomitant variable (cf. Stokes, 1977). For more detail and applications of RSS, see Johnson et al. (1993), Patil et al. (1999), Mode et al. (1999), Al-Saleh and Al-Shrafat (2001), Yu and Tam (2002), Al-Saleh and Al-Hadrami (2003), Chen et al. (2004), Buchanan et al. (2005), Haq et al. (2013b) and references cited therein.

Patil et al. (1993) compared the mean estimator based on perfect and imperfect rankings with the SRS-based regression estimator of the population mean. It is shown that, under perfect ranking, RSS mean estimator is considerably more efficient than the regression estimator unless the correlation between the study variable and the auxiliary variable exceeds 0.85. Muttlak (1996) and Muttlak (1997) introduced paired RSS (PRSS) and median RSS (MRSS) schemes for estimation of population mean, respectively. Muttlak (1998) extended the work of Patil et al. (1993) and showed that the mean estimator under MRSS is more efficient than the mean estimators with SRS and RSS. When the correlation between the study variable and the auxiliary variable is high, then SRS-based regression estimator outperforms both RSS and MRSS mean estimators based on perfect rankings. Al-Saleh and Al-Kadiri (2000) introduced the double RSS (DRSS) procedure for efficient estimation of the population mean. They showed that the sample mean based on DRSS is more efficient than the sample mean with RSS. In order to select a double ranked set sample of size m, the experimenter needs to identify m^3 units from the target population. This may be difficult when an epidemic breaks out in some area or in queuing problems when data arrive in batches of varying sizes. Moreover, there may be a shortage of experimental units or ranking is difficult, time-consuming and costly, see Samawi (2011) and Haq et al. (2013b). In all such situations, the balanced DRSS cannot be conducted with full confidence or it is costly.

In this chapter, we introduce a new unbiased sampling scheme, that we call paired double RSS (PDRSS) for estimation of the population mean. PDRSS scheme can be used as an alternative to DRSS scheme when it it difficult to apply DRSS procedure due to non-availability of experimental units or ranking costs cannot be ignored. The main advantage of using PDRSS over DRSS is that it requires less number of identified units as compared with DRSS. This helps in reducing the time and cost that is involved in the ranking process. We show both theoretically and numerically that the mean estimators under PDRSS are more precise than the mean estimators based on SRS, RSS and PRSS schemes. Furthermore, we extend the work of Patil et al. (1993) and Muttlak (1998) to PDRSS. It is observed that with perfect and imperfect rankings, in comparison with the regression estimator, the mean estimates under PDRSS scheme are more precise than their counterparts based on RSS and MRSS methods. The proposed mean estimator under perfect PDRSS is also better in terms of relative precision (RP) than the regression estimator of the population mean based on SRS.

The rest of the chapter is organized as follows: Section 3.2 provides some mathematical results for RSS methods. Section 3.3 introduces PDRSS method with mean estimators. Section 3.4 compares mean estimators in terms of RP when sampling from symmetric and asymmetric distributions. The proposed and existing estimators are also compared with the linear regression estimator of the population mean based on SRS. We present a case study in Section 3.5. Finally, Section 3.6 summarizes the main findings.

3.2 Mathematical setup and RSS methods

Let Y be the study variable with probability density function (PDF) $f(y)$ and cumulative distribution function (CDF) $F(y)$. Let μ_Y and σ_Y^2 be the mean and variance of Y, respectively. Let $Y_1, Y_2, ..., Y_m$ be a simple random sample of size m drawn from $f(y)$. Let $Y_{(1:m)}, Y_{(2:m)}, ..., Y_{(m:m)}$ be the order statistics of this random sample. The PDF and CDF of the ith order statistic $Y_{(i:m)}$, for $i = 1, 2, ..., m$, are respectively given by

$$\begin{aligned} f_{(i:m)}(y) &= \frac{m!}{(i-1)!(m-i)!}\{F(y)\}^{i-1}\{1-F(y)\}^{m-i}f(y), \quad -\infty < y < \infty, \\ F_{(i:m)}(y) &= \sum_{r=i}^{m}\binom{m}{r}\{F(y)\}^{r}\{1-F(y)\}^{m-r}. \end{aligned}$$

The mean and variance of $Y_{(i:m)}$, for $i = 1, 2, ..., m$, respectively, are

$$\mu_{Y(i:m)} = \int y f_{(i:m)}(y)\mathrm{d}y \quad \text{and} \quad \sigma^2_{Y(i:m)} = \int (y - \mu_{Y(i:m)})^2 f_{(i:m)}(y)\mathrm{d}y,$$

see David and Nagaraja (2003).

3.2.1 Ranked set sampling

The RSS procedure is explained as follows: identify m^2 units from the target population. Randomly allocate these units to m sets, each of size m. Now, rank the units within each set visually with respect to the study variable or by any inexpensive method. Select the smallest ranked unit from the first set. The second smallest ranked unit is selected from the second set. The procedure continues until the largest ranked unit is selected from the last set. This completes one cycle of a ranked set sample of size m. This procedure can be repeated r times in order to obtain a ranked set sample of size $n = mr$.

Let $Y_{11j}, Y_{12j}, ..., Y_{1mj}; Y_{21j}, Y_{22j}, ..., Y_{2mj}; ...; Y_{m1j}, Y_{m2j}, ..., Y_{mmj}$ be m independent simple random samples, each of size m, in the jth cycle, for $j = 1, 2, ..., r$. Apply RSS procedure to these samples in order to obtain a ranked set sample of size m for the jth cycle, denoted by $Y_{i(i:m)j}$, for $i = 1, 2, ..., m$. The mean of the ranked set sample is $\bar{Y}_{\text{RSS}} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i(i:m)j}$. Takahasi and Wakimoto (1968) showed that under perfect ranking $\bar{Y}_{\text{RSS}}$ is an unbiased estimator of μ_Y and it is more precise than $\bar{Y}_{\text{SRS}} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, i.e.,

$$E(\bar{Y}_{\text{RSS}}) = \mu_Y \quad \text{and} \quad \text{Var}(\bar{Y}_{\text{RSS}}) = \text{Var}(\bar{Y}_{\text{SRS}}) - \frac{1}{nm}\sum_{i=1}^{m}(\mu_{Y(i:m)} - \mu_Y)^2. \tag{3.1}$$

Sometimes, it is difficult to measure or rank the values of the study variable visually. Stokes (1977) developed a model for imperfect RSS (IRSS) and showed that it is possible to judge the ranks of the study variable with respect to the ranks of any auxiliary variable, say X, that is correlated with the study variable Y. Following Stokes (1977), it is assumed that (Y, X) follows a bivariate normal distribution and the regression of Y on X is linear, i.e.,

$$Y_{i[i:m]j} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X_{i(i:m)j} - \mu_X) + \zeta_{ij}, \quad i = 1, 2, ..., m, \ j = 1, 2, ..., r, \tag{3.2}$$

where μ_X and σ_X^2 are the mean and variance of X, respectively, and ρ is the correlation coefficient between Y and X. Here, ζ_{ij} is the random error term with mean zero and a constant variance, i.e., $E(\zeta_{ij}) = 0$ and $\text{Var}(\zeta_{ij}) = \sigma_\zeta^2 = \sigma_Y^2(1-\rho^2)$. Note that $X_{i(i:m)j}$ is the ith order statistic and the analogous $Y_{i[i:m]j}$ is the corresponding ith judgment order statistic from the ith sample in the jth cycle. Under IRSS, the sample mean $\bar{Y}_{\text{IRSS}}$ and its variance, respectively, are given by

$$\bar{Y}_{\text{IRSS}} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i[i:m]j} \quad \text{and} \quad \text{Var}(\bar{Y}_{\text{IRSS}}) = \frac{1}{nm}\left\{m\sigma_Y^2(1-\rho^2) + \rho^2\frac{\sigma_Y^2}{\sigma_X^2}\sum_{i=1}^{m}\sigma_{X(i:m)}^2\right\}. \tag{3.3}$$

3.2.2 Paired ranked set sampling

Muttlak (1996) introduced PRSS procedure for estimation of population mean. PRSS procedure is as follows: for even sample size m, identify $m/2$ sets each of size m from the target population. For odd sample size m, identify $(m+1)/2$ sets each of size m. Now, rank the units within each set as aforementioned. Select the smallest and largest ranked units from the first set. Similarly, select the second smallest and second largest ranked units from the second set. The procedure continues, in case when m is even, then $(m/2)$th and $\{(m+2)/2\}$th ranked units are selected from the last set, and if m is odd, then $\{(m+1)/2\}$th ranked unit is selected from the last set. This completes one cycle of a paired ranked set sample of size m. This process can be repeated r times to obtain a paired ranked set sample of size $n = mr$. The sample means under PRSS depending on even (E) and odd (O) sample sizes are respectively given by

$$\begin{aligned}
\bar{Y}_{\text{PRSS}}^{\text{E}} &= \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{m/2} Y_{i(i:m)j} + \sum_{i=1}^{m/2} Y_{i(m-i+1:m)j}\right) \quad \text{and} \\
\bar{Y}_{\text{PRSS}}^{\text{O}} &= \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{(m+1)/2} Y_{i(i:m)j} + \sum_{i=1}^{(m-1)/2} Y_{i(m-i+1:m)j}\right).
\end{aligned}$$

Both $\bar{Y}^{\text{E}}_{\text{PRSS}}$ and $\bar{Y}^{\text{O}}_{\text{PRSS}}$ are unbiased estimators of μ_Y. The variances of $\bar{Y}^{\text{E}}_{\text{PRSS}}$ and $\bar{Y}^{\text{O}}_{\text{PRSS}}$, respectively, are

$$\begin{aligned}\text{Var}(\bar{Y}^{\text{E}}_{\text{PRSS}}) &= \text{Var}(\bar{Y}_{\text{RSS}}) + \frac{2}{nm}\sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m/2}\sigma_{Y(i,m-i+1:m)},\\ \text{Var}(\bar{Y}^{\text{O}}_{\text{PRSS}}) &= \text{Var}(\bar{Y}_{\text{RSS}}) + \frac{2}{nm}\sum_{i=1}^{(m-1)/2}\sum_{i<m-i+1}^{(m-1)/2}\sigma_{Y(i,m-i+1:m)},\end{aligned} \tag{3.4}$$

where $\sigma_{Y(i,m-i+1:m)} \geq 0$ represents the covariance between $Y_{i(i:m)j}$ and $Y_{i(m-i+1:m)j}$. For more details, see Muttlak (1996)

3.2.3 Median ranked set sampling

Muttlak (1997) introduced MRSS procedure for estimation of population mean. MRSS procedure is explained as follows: identify m^2 units from the target population and divide them into m sets each of size m. Rank the units within each set as aforesaid. For even sample size m, select the $(m/2)$th ranked unit and $\{(m+2)/2\}$th ranked unit from first and last $m/2$ sets, respectively. For odd sample size m, select $\{(m+1)/2\}$th ranked unit from all sets. This completes one cycle of a median ranked set sample of size m. The whole procedure can be repeated r times to get a median ranked set sample of size $n = mr$. The sample means under MRSS for even and odd m are given below:

$$\bar{Y}^{\text{E}}_{\text{MRSS}} = \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{m/2} Y_{i(m/2:m)j} + \sum_{i=m/2+1}^{m} Y_{i((m+2)/2:m)j}\right) \quad \text{and} \quad \bar{Y}^{\text{O}}_{\text{MRSS}} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i((m+1)/2:m)j}.$$

For symmetric populations, both $\bar{Y}^{\text{E}}_{\text{MRSS}}$ and $\bar{Y}^{\text{O}}_{\text{MRSS}}$ are unbiased estimators of μ_Y. The mean squared errors (MSEs) of $\bar{Y}^{\text{E}}_{\text{MRSS}}$ and $\bar{Y}^{\text{O}}_{\text{MRSS}}$, respectively, are

$$\begin{aligned}\text{MSE}(\bar{Y}^{\text{E}}_{\text{MRSS}}) &= \{E(\bar{Y}^{\text{E}}_{\text{MRSS}} - \mu_Y)\}^2 + \frac{1}{nm}\left(\sum_{i=1}^{m/2}\sigma^2_{Y(m/2:m)} + \sum_{i=m/2+1}^{m}\sigma^2_{Y((m+2)/2:m)}\right),\\ \text{MSE}(\bar{Y}^{\text{O}}_{\text{MRSS}}) &= \{E(\bar{Y}^{\text{O}}_{\text{MRSS}} - \mu_Y)\}^2 + \frac{1}{nm}\sum_{i=1}^{(m+1)/2}\sigma^2_{Y((m+1)/2:m)}.\end{aligned} \tag{3.5}$$

Muttlak (1998) considered imperfect MRSS (IMRSS) for estimation of population mean under bivariate normality. Consider the model given in (3.2), the sample means under IMRSS for even and odd m, respectively, are given by

$$\bar{Y}^{\text{E}}_{\text{IMRSS}} = \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{m/2} Y_{i[m/2:m]j} + \sum_{i=m/2+1}^{m} Y_{i[(m+2)/2:m]j}\right) \quad \text{and} \quad \bar{Y}^{\text{O}}_{\text{IMRSS}} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i[(m+1)/2:m]j}.$$

The variances of $\bar{Y}^{\text{E}}_{\text{IMRSS}}$ and $\bar{Y}^{\text{O}}_{\text{IMRSS}}$, respectively, are

$$\text{Var}(\bar{Y}^{\text{E}}_{\text{IMRSS}}) = \frac{1}{nm}\left\{m\sigma^2_Y(1-\rho^2) + \rho^2\frac{\sigma^2_Y}{\sigma^2_X}\left(\sum_{i=1}^{m/2}\sigma^2_{X(m/2:m)} + \sum_{i=m/2+1}^{m}\sigma^2_{X((m+2):m)}\right)\right\},$$

$$\mathrm{Var}(\bar{Y}^{\mathrm{O}}_{\mathrm{IMRSS}}) = \frac{1}{nm}\left\{m\sigma^2_Y(1-\rho^2)+\rho^2\frac{\sigma^2_Y}{\sigma^2_X}\sum_{i=1}^{m}\sigma^2_{X((m+1)/2:m)}\right\}. \quad (3.6)$$

3.2.4 Double ranked set sampling

Al-Saleh and Al-Kadiri (2000) introduced DRSS procedure for estimation of the population mean. The DRSS scheme can be described as follows: identify m^3 units from the parent population. Randomly allocate these units to m sets each of size m^2 units. Apply RSS procedure to m sets in order to obtain m ranked set samples each of size m. Again apply the RSS procedure to these m ranked sets, each of size m, to obtain a double ranked set sample of size m. This completes one cycle of a double ranked set sample of size m. This process can be repeated r times to obtain a double ranked set sample of size $n = mr$.

Let $Y^{(i)(i:m)}_{i(i:m)j}$, for $i = 1, 2, ..., m$, $j = 1, 2, ..., r$, represent a double ranked set sample of size n. Then, Al-Saleh and Al-Kadiri (2000) showed that the sample mean under DRSS, say $\bar{Y}_{\mathrm{DRSS}}$, is an unbiased estimator of μ_Y and its variances is less than the variance of the sample mean based on RSS, i.e.,

$$E(\bar{Y}_{\mathrm{DRSS}}) = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m}E(Y^{(i)(i:m)}_{i(i:m)j}) = \mu_Y \quad \text{and} \quad \mathrm{Var}(\bar{Y}_{\mathrm{RSS}}) = \mathrm{Var}(\bar{Y}_{\mathrm{DRSS}}) + \frac{1}{nm}\sum_{i\neq l}^{m}\sigma^{(i,l:m)}_{Y(i,l:m)}, \quad (3.7)$$

where $\sigma^{(i,l:m)}_{Y(i,l:m)} \geq 0$, for $i \neq l = 1, 2, ..., m$, represents the covariance between $Y^{(i)(i:m)}_{i(i:m)j}$ and $Y^{(l)(l:m)}_{l(l:m)j}$. For more details, see Al-Saleh and Al-Kadiri (2000).

3.3 Paired double ranked set sampling

In this section, we propose a new unbiased RSS scheme, namely, PDRSS, for estimation of population mean. This scheme can be used as an alternative to DRSS when it is difficult or time-consuming or costly or identify m^3 units from the target population, especially when there is a shortage of experimental units.

The main steps involved in selecting a paired double ranked set sample of size m are as follows: for even sample size m, identify $m^3/2$ elements from the target population. Randomly allocate these units to $m/2$ sets each of size m^2 units. Apply RSS procedure on each set in order to obtain $m/2$ ranked set samples each of size m. Now, apply PRSS procedure on $m/2$ sets to get an even paired double ranked set sample of size m. Similarly, for odd sample size m, identify $m^2(m+1)/2$ elements from the target population. Randomly divide these units to $(m+1)/2$ sets each of size m^2 units. Apply RSS scheme on each set to obtain $(m+1)/2$ ranked sets each of size m. Now, apply PRSS procedure to $(m+1)/2$ ranked sets to get an odd paired double ranked set sample of size m. This completes one cycle of a paired double ranked set sample of size m. This procedure can be repeated r times to obtain a paired double ranked set sample of size $n = mr$.

Examples

1. In order to draw a ranked set sample of size $m = 3$, we identify 9 units from the target population and divide them into 3 sets each of size 3. Let (Y_{11}, Y_{12}, Y_{13}) , (Y_{21}, Y_{22}, Y_{23}) and (Y_{31}, Y_{32}, Y_{33}) be the identified units in the first, second and third set, respectively. Now rank the units within each set to get

$$\begin{bmatrix} \underline{Y_{1(1:3)}} & Y_{2(1:3)} & Y_{3(1:3)} \\ Y_{1(2:3)} & \underline{Y_{2(2:3)}} & Y_{3(2:3)} \\ Y_{1(3:3)} & Y_{2(3:3)} & \underline{Y_{3(3:3)}} \end{bmatrix}.$$

Now select the diagonal of the matrix in order to select a ranked set sample of size 3, i.e., $Y_{i(i:3)}$, for $i = 1, 2, 3$. Similarly, if we select the median of each column in the above matrix, we get

$$\begin{bmatrix} Y_{1(1:3)} & Y_{2(1:3)} & Y_{3(1:3)} \\ \underline{Y_{1(2:3)}} & \underline{Y_{2(2:3)}} & \underline{Y_{3(2:3)}} \\ Y_{1(3:3)} & Y_{2(3:3)} & Y_{3(3:3)} \end{bmatrix}.$$

Here, $Y_{i(2:3)}$, for $i = 1, 2, 3$, represent a median ranked set sample of size $m = 3$.

2. In order to select a paired ranked set sample of size $m = 3$, identify 6 units from the target population and divide them into 2 sets each of size 3. Let (Y_{11}, Y_{12}, Y_{13}) and (Y_{21}, Y_{22}, Y_{23}) be the identified units in the first and second set, respectively. Now rank the units within each set to get

$$\begin{bmatrix} \underline{Y_{1(1:3)}} & Y_{2(1:3)} \\ Y_{1(2:3)} & \underline{Y_{2(2:3)}} \\ \underline{Y_{1(3:3)}} & Y_{2(3:3)} \end{bmatrix}.$$

Now apply the PRSS procedure to select a paired ranked set sample of size $m = 3$, represented by $Y_{1(1:3)}$, $Y_{2(2:3)}$ and $Y_{1(3:3)}$.

3. In order to draw a double ranked set sample of size $m = 3$, we identify 27 units (3 sets each of size 9) from the target population. The identified elements in the first, second and third set are $(Y_{11}^{(1)}, Y_{12}^{(1)}, ..., Y_{33}^{(1)})$, $(Y_{11}^{(2)}, Y_{12}^{(2)}, ..., Y_{33}^{(2)})$ and $(Y_{11}^{(3)}, Y_{12}^{(3)}, ..., Y_{33}^{(3)})$, respectively. Now, rank the units within each set to get

$$\begin{bmatrix} \underline{Y_{1(1:3)}^{(1)}} & Y_{2(1:3)}^{(1)} & Y_{3(1:3)}^{(1)} \\ Y_{1(2:3)}^{(1)} & \underline{Y_{2(2:3)}^{(1)}} & Y_{3(2:3)}^{(1)} \\ Y_{1(3:3)}^{(1)} & Y_{2(3:3)}^{(1)} & \underline{Y_{3(3:3)}^{(1)}} \end{bmatrix}, \begin{bmatrix} \underline{Y_{1(1:3)}^{(2)}} & Y_{2(1:3)}^{(2)} & Y_{3(1:3)}^{(2)} \\ Y_{1(2:3)}^{(2)} & \underline{Y_{2(2:3)}^{(2)}} & Y_{3(2:3)}^{(2)} \\ Y_{1(3:3)}^{(2)} & Y_{2(3:3)}^{(2)} & \underline{Y_{3(3:3)}^{(2)}} \end{bmatrix} \text{ and } \begin{bmatrix} \underline{Y_{1(1:3)}^{(3)}} & Y_{2(1:3)}^{(3)} & Y_{3(1:3)}^{(3)} \\ Y_{1(2:3)}^{(3)} & \underline{Y_{2(2:3)}^{(3)}} & Y_{3(2:3)}^{(3)} \\ Y_{1(3:3)}^{(3)} & Y_{2(3:3)}^{(3)} & \underline{Y_{3(3:3)}^{(3)}} \end{bmatrix}.$$

Now apply the RSS scheme to get

$$\begin{bmatrix} Y_{1(1:3)}^{(1)} & Y_{1(1:3)}^{(2)} & Y_{1(1:3)}^{(3)} \\ Y_{2(2:3)}^{(1)} & Y_{2(2:3)}^{(2)} & Y_{2(2:3)}^{(3)} \\ Y_{3(3:3)}^{(1)} & Y_{3(3:3)}^{(2)} & Y_{3(3:3)}^{(3)} \end{bmatrix}.$$

Again rank the units within each column to get

$$\begin{bmatrix} \underline{Y_{1(1:3)}^{(1)(1:3)}} & Y_{1(1:3)}^{(2)(1:3)} & Y_{1(1:3)}^{(3)(1:3)} \\ Y_{2(2:3)}^{(1)(2:3)} & \underline{Y_{2(2:3)}^{(2)(2:3)}} & Y_{2(2:3)}^{(3)(2:3)} \\ Y_{3(3:3)}^{(1)(3:3)} & Y_{3(3:3)}^{(2)(3:3)} & \underline{Y_{3(3:3)}^{(3)(3:3)}} \end{bmatrix}.$$

Now, apply the RSS scheme to get $Y_{i(i:3)}^{(i)(i:3)}$, for $i = 1, 2, 3$, which is a double ranked set sample of size $m = 3$.

4. In order to draw a paired double ranked set sample of size $m = 3$, we identify 18 elements (2 sets each of size 9) from the target population. The identified elements in the first and the second set are $(Y_{11}^{(1)}, Y_{12}^{(1)}, ..., Y_{33}^{(1)})$ and $(Y_{11}^{(2)}, Y_{12}^{(2)}, ..., Y_{33}^{(2)})$, respectively. Now, rank the units within each set to get

$$\begin{bmatrix} \underline{Y_{1(1:3)}^{(1)}} & Y_{2(1:3)}^{(1)} & Y_{3(1:3)}^{(1)} \\ Y_{1(2:3)}^{(1)} & \underline{Y_{2(2:3)}^{(1)}} & Y_{3(2:3)}^{(1)} \\ Y_{1(3:3)}^{(1)} & Y_{2(3:3)}^{(1)} & \underline{Y_{3(3:3)}^{(1)}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \underline{Y_{1(1:3)}^{(2)}} & Y_{2(1:3)}^{(2)} & Y_{3(1:3)}^{(2)} \\ Y_{1(2:3)}^{(2)} & \underline{Y_{2(2:3)}^{(2)}} & Y_{3(2:3)}^{(2)} \\ Y_{1(3:3)}^{(2)} & Y_{2(3:3)}^{(2)} & \underline{Y_{3(3:3)}^{(2)}} \end{bmatrix}.$$

Now apply the RSS scheme to obtain

$$\begin{bmatrix} Y_{1(1:3)}^{(1)} & Y_{1(1:3)}^{(2)} \\ Y_{2(2:3)}^{(1)} & Y_{2(2:3)}^{(2)} \\ Y_{3(3:3)}^{(1)} & Y_{3(3:3)}^{(2)} \end{bmatrix}.$$

Again rank the units within each column to get

$$\begin{bmatrix} \underline{Y_{1(1:3)}^{(1)(1:3)}} & Y_{1(1:3)}^{(2)(1:3)} \\ Y_{2(2:3)}^{(1)(2:3)} & \underline{Y_{2(2:3)}^{(2)(2:3)}} \\ \underline{Y_{3(3:3)}^{(1)(3:3)}} & Y_{3(3:3)}^{(2)(3:3)} \end{bmatrix}.$$

Now, apply the PRSS scheme to get $Y_{1(1:3)}^{(1)(1:3)}, Y_{3(3:3)}^{(1)(3:3)}, Y_{2(2:3)}^{(2)(2:3)}$, which is an odd paired double ranked set sample of size 3. Note that the random variables $Y_{1(1:3)}^{(1)(1:3)}$ and $Y_{3(3:3)}^{(1)(3:3)}$ are dependent but $Y_{2(2:3)}^{(2)(2:3)}$ is independent from $Y_{1(1:3)}^{(1)(1:3)}$ and $Y_{3(3:3)}^{(1)(3:3)}$.

Similarly, consider a sample of size m, then $\left(Y_{i(i:m)j}^{(i)(i:m)}, Y_{(m-i+1)(m-i+1:m)j}^{(i)(m-i+1:m)}\right)$, $i = 1, 2, ..., m/2$, and $\left\{\left(Y_{i(i:m)j}^{(i)(i:m)}, Y_{(m-i+1)(m-i+1:m)j}^{(i)(m-i+1:m)}\right), Y_{(m/2+1/2)(m/2+1/2:m)j}^{(m/2+1/2)(m/2+1/2:m)}\right\}$, $i = 1, 2, ..., (m-1)/2$, represent even and odd paired double ranked set samples each of size m in the jth cycle for $j = 1, 2, ..., r$, respectively. It is to be noted that, for fixed value of j, if $Y_{i(i:m)j}^{(i)(i:m)}$, for $i = 1, 2, ..., m$, are all independent, then $Y_{i(i:m)j}^{(i)(i:m)}$, for $i = 1, 2, ..., m$, represent a double ranked set sample of size m in the jth cycle.

Let $\bar{Y}_{\text{PDRSS}}^{\text{E}}$ and $\bar{Y}_{\text{PDRSS}}^{\text{O}}$ represent the mean estimators based on PDRSS for even and odd m, respectively, defined as

$$\bar{Y}_{\text{PDRSS}}^{\text{E}} = \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{m/2} Y_{i(i:m)j}^{(i)(i:m)} + \sum_{i=1}^{m/2} Y_{(m-i+1)(m-i+1:m)j}^{(i)(m-i+1:m)}\right) \quad \text{and}$$

$$\bar{Y}^{\mathrm{E}}_{\mathrm{PDRSS}} = \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{(m+1)/2} Y^{(i)(i:m)}_{i(i:m)j} + \sum_{i=1}^{(m-1)/2} Y^{(i)(m-i+1:m)}_{(m-i+1)(m-i+1:m)j}\right).$$

The variances of $\bar{Y}^{\mathrm{E}}_{\mathrm{PDRSS}}$ and $\bar{Y}^{\mathrm{O}}_{\mathrm{PDRSS}}$, respectively, are

$$\begin{aligned}
\mathrm{Var}(\bar{Y}^{\mathrm{E}}_{\mathrm{PDRSS}}) &= \frac{1}{nm}\left(\sum_{i=1}^{m/2}\sigma^{(i,i:m)}_{Y(i,i:m)} + \sum_{i=1}^{m/2}\sigma^{(m-i+1,m-i+1:m)}_{Y(m-i+1,m-i+1:m)}\right) + \frac{2}{nm}\sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m/2}\sigma^{(i,m-i+1:m)}_{Y(i,m-i+1:m)},\\
\mathrm{Var}(\bar{Y}^{\mathrm{O}}_{\mathrm{PDRSS}}) &= \frac{1}{nm}\left(\sum_{i=1}^{(m+1)/2}\sigma^{(i,i:m)}_{Y(i,i:m)} + \sum_{i=1}^{(m-1)/2}\sigma^{(m-i+1,m-i+1:m)}_{Y(m-i+1,m-i+1:m)}\right)\\
&\quad + \frac{2}{nm}\sum_{i=1}^{(m-1)/2}\sum_{i<m-i+1}^{(m-1)/2}\sigma^{(i,m-i+1:m)}_{Y(i,m-i+1:m)},
\end{aligned} \tag{3.8}$$

where $\mathrm{Var}(Y^{(i)(i:m)}_{i(i,i:m)j}) = \sigma^{(i,i:m)}_{Y(i,i:m)}$, $\mathrm{Var}(Y^{(i)(m-i+1:m)}_{i(m-i+1:m)j}) = \sigma^{(m-i+1,m-i+1:m)}_{Y(m-i+1,m-i+1:m)}$, and $\sigma^{(i,m-i+1:m)}_{Y(i,m-i+1:m)} \geq 0$ is the covariance between $Y^{(i)(i:m)}_{i(i:m)j}$ and $Y^{(m-i+1)(m-i+1:m)}_{(m-i+1)(m-i+1:m)j}$. Note that, for a given value of i, $Y^{(i)(i:m)}_{i(i:m)j}$, for $j = 1, 2, ..., r$, are independent and identically distributed (IID) random variables. Therefore, without loss of generality, we consider $Y^{(i)(i:m)}_{i(i:m)j} \equiv Y^{(i)(i:m)}_{i(i:m)}$, for $j = 1, 2, ..., r$. As under DRSS, $Y^{(i)(i:m)}_{i(i:m)j}$, for $i = 1, 2, ..., m$, are independent and non-identically distributed (INID) random variables. Then, the CDF of $Y^{(i)(i:m)}_{i(i:m)j}$ is given by

$$F^{\mathrm{PDRSS}}_{(i:m)}(y) = \sum_{r=i}^{m}\sum_{P_r}\prod_{k=1}^{r} F_{(t_k:m)}(y)\prod_{k=r+1}^{m}\{1 - F_{(t_k:m)}(y)\}, \quad -\infty < y < \infty, \tag{3.9}$$

where $\sum_{P_r}$ denotes the sum over all permutations $(t_1, t_2, ..., t_m)$ of $(1, 2, ..., m)$ for which $t_1 < t_2 < \cdots < t_r$ and $t_{r+1} < t_{r+2} < \cdots < t_m$. For more details about INID random variables, see David and Nagaraja (2003). Following Vaughan and Venables (1972) and Bapat and Beg (1989), another equivalent expression of the CDF of $Y^{(i)(i:m)}_{i(i:m)}$ is

$$F^{\mathrm{PDRSS}}_{(i:m)}(y) = \sum_{r=i}^{m}\frac{1}{r!(m-r)!}\,\mathrm{Per}(\mathbf{\Omega}), \quad -\infty < y < \infty, \tag{3.10}$$

where $\mathbf{\Omega} = \begin{array}{c} \begin{pmatrix} F_{(1:m)}(y) & F_{(2:m)}(y) & \cdots & F_{(m:m)}(y) \\ 1 - F_{(1:m)}(y) & 1 - F_{(2:m)}(y) & \cdots & 1 - F_{(m:m)}(y) \end{pmatrix} \end{array} \begin{array}{l} \}\, r \\ \}\, m-r \end{array}$ and $\mathrm{Per}(\mathbf{\Omega})$ represents the permanent of the matrix $\mathbf{\Omega}$. Here, "$\}\,r$" and "$\}\,m-r$" show that the first and second rows are repeated r and $m-r$ times, respectively. Similarly, the PDF of $Y^{(i)(i:m)}_{i(i:m)}$ is given by

$$f^{\mathrm{PDRSS}}_{(i:m)}(y) = \frac{1}{(i-1)!(m-i)!}\,\mathrm{Per}(\mathbf{\Psi}), \quad -\infty < y < \infty, \tag{3.11}$$

where $\mathbf{\Psi} = \begin{pmatrix} F_{(1:m)}(y) & F_{(2:m)}(y) & \cdots & F_{(m:m)}(y) \\ f_{(1:m)}(y) & f_{(2:m)}(y) & \cdots & f_{(m:m)}(y) \\ 1 - F_{(1:m)}(y) & 1 - F_{(2:m)}(y) & \cdots & 1 - F_{(m:m)}(y) \end{pmatrix} \begin{array}{l} \}\, i-1 \\ \}\, 1 \\ \}\, m-i \end{array}.$

The joint density function of $Y_{i(i:m)}^{(i)(i:m)}$ and $Y_{l(l:m)}^{(l)(l:m)}$, $1 \le i < l \le m$, is

$$f_{(i,l:m)}^{\text{PDRSS}}(y_i, y_l) = \frac{1}{(i-1)!(l-i-1)!(m-l)!}\ \text{Per}(\boldsymbol{\Theta}), \quad -\infty < y_i < y_l < \infty, \tag{3.12}$$

$$\text{where } \boldsymbol{\Theta} = \begin{array}{c} \left(\begin{array}{cccc} F_{(1:m)}(y_i) & F_{(2:m)}(y_i) & \cdots & F_{(m:m)}(y_i) \\ f_{(1:m)}(y_i) & f_{(2:m)}(y_i) & \cdots & f_{(m:m)}(y_i) \\ F_{(1:m)}(y_l) - F_{(1:m)}(y_i) & F_{(2:m)}(y_l) - F_{(2:m)}(y_i) & \cdots & F_{(m:m)}(y_l) - F_{(m:m)}(y_i) \\ f_{(1:m)}(y_l) & f_{(2:m)}(y_l) & \cdots & f_{(m:m)}(y_l) \\ 1 - F_{(1:m)}(y_l) & 1 - F_{(2:m)}(y_l) & \cdots & 1 - F_{(m:m)}(y_l) \end{array}\right) \end{array} \begin{array}{l} \}\, i-1 \\ \}\, 1 \\ \}\, l-i-1 \\ \}\, 1 \\ \}\, m-l \end{array}.$$

Theorem 1:

(i) $\bar{Y}_{\text{PDRSS}}^{\text{J}}$ is an unbiased estimator of the population mean μ_Y for J = E, O.

(ii) $\text{Var}(\bar{Y}_{\text{PDRSS}}^{\text{J}}) \le \text{Var}(\bar{Y}_{\text{RSS}}) \le \text{Var}(\bar{Y}_{\text{PRSS}}^{\text{J}}) \le \text{Var}(\bar{Y}_{\text{SRS}})$.

Proof:

(i) From (3.7), it is easy to show that $\text{E}(\bar{Y}_{\text{PDRSS}}^{\text{J}}) = E(\bar{Y}_{\text{DRSS}}) = \mu_Y$.

(ii) If m is even, then from (3.8), we have

$$\begin{aligned} \text{Var}(\bar{Y}_{\text{PDRSS}}^{\text{E}}) &= \frac{1}{nm}\left(\sum_{i=1}^{m/2} \sigma_{Y(i,i:m)}^{(i,i:m)} + \sum_{i=1}^{m/2} \sigma_{Y(m-i+1,m-i+1:m)}^{(m-i+1,m-i+1:m)}\right) + \frac{2}{nm}\sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m/2} \sigma_{Y(i,m-i+1:m)}^{(i,m-i+1:m)}, \\ &= \text{Var}(\bar{Y}_{\text{DRSS}}) + \frac{2}{nm}\sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m/2} \sigma_{Y(i,m-i+1:m)}^{(i,m-i+1:m)}. \end{aligned} \tag{3.13}$$

Replacing (3.7) in (3.13), we have

$$\text{Var}(\bar{Y}_{\text{PDRSS}}^{\text{E}}) = \text{Var}(\bar{Y}_{\text{RSS}}) - \frac{2}{nm}\left(\sum_{i=1}^{m}\sum_{i<l}^{m} \sigma_{Y(i,l:m)}^{(i,l:m)} - \sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m} \sigma_{Y(i,m-i+1:m)}^{(i,m-i+1:m)}\right).$$

As we know that, all of the covariance terms are positive, i.e., $\sigma_{Y(i,l:m)}^{(i,l:m)} \ge 0$. Therefore, the second term in the above equation on the right side is positive, i.e., $\sum_{i=1}^{m}\sum_{i<l}^{m} \sigma_{Y(i,l:m)}^{(i,l:m)} - \sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m} \sigma_{Y(i,m-i+1:m)}^{(i,m-i+1:m)} \ge 0$, because the first term contains all covariance terms including the covariance terms that are being subtracted from the first term. In what follows $\text{Var}(\bar{Y}_{\text{PDRSS}}^{\text{E}}) \le \text{Var}(\bar{Y}_{\text{RSS}}^{\text{E}})$.

Similarly, for odd m, we have

$$\text{Var}(\bar{Y}_{\text{PDRSS}}^{\text{O}}) = \text{Var}(\bar{Y}_{\text{RSS}}) - \frac{2}{nm}\left(\sum_{i=1}^{m}\sum_{i<l}^{m} \sigma_{Y(i,l:m)}^{(i,l:m)} - \sum_{i=1}^{(m-1)/2}\sum_{i<m-i+1}^{(m-1)/2} \sigma_{Y(i,m-i+1:m)}^{(i,m-i+1:m)}\right).$$

From (3.4), we can write

$$\begin{aligned}
\mathrm{Var}(\bar{Y}^{\mathrm{E}}_{\mathrm{PDRSS}}) &= \mathrm{Var}(\bar{Y}^{\mathrm{E}}_{\mathrm{PRSS}}) - \frac{2}{nm}\left(\sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m/2}\sigma_{Y(i,m-i+1:m)}\right) \\
&\quad - \frac{2}{nm}\left(\sum_{i=1}^{m}\sum_{i<l}^{m}\sigma^{(i,l:m)}_{Y(i,l:m)} - \sum_{i=1}^{m/2}\sum_{i<m=i+1}^{m/2}\sigma^{(i,m-i+1:m)}_{Y(i,m-i+1:m)}\right), \\
\mathrm{Var}(\bar{Y}^{\mathrm{O}}_{\mathrm{PDRSS}}) &= \mathrm{Var}(\bar{Y}^{\mathrm{O}}_{\mathrm{PRSS}}) - \frac{2}{nm}\left(\sum_{i=1}^{(m-1)/2}\sum_{i<m-i+1}^{(m-1)/2}\sigma_{Y(i,m-i+1:m)}\right) \\
&\quad - \frac{2}{nm}\left(\sum_{i=1}^{m}\sum_{i<l}^{m}\sigma^{(i,l:m)}_{Y(i,l:m)} - \sum_{i=1}^{(m-1)/2}\sum_{i<m=i+1}^{(m-1)/2}\sigma^{(i,m-i+1:m)}_{Y(i,m-i+1:m)}\right).
\end{aligned}$$

From (3.1), it is easy to write

$$\mathrm{Var}(\bar{Y}^{\mathrm{E}}_{\mathrm{PDRSS}}) = \mathrm{Var}(\bar{Y}_{\mathrm{SRS}}) - \frac{1}{nm}\sum_{i=1}^{m}(\mu_{Y(i:m)} - \mu_Y)^2 - \frac{2}{nm}\left(\sum_{i=1}^{m}\sum_{i<l}^{m}\sigma^{(i,l:m)}_{Y(i,l:m)} - \sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m/2}\sigma^{(i,m-i+1:m)}_{Y(i,m-i+1:m)}\right),$$

$$\mathrm{Var}(\bar{Y}^{\mathrm{O}}_{\mathrm{PDRSS}}) = \mathrm{Var}(\bar{Y}_{\mathrm{SRS}}) - \frac{1}{nm}\sum_{i=1}^{m}(\mu_{Y(i:m)} - \mu_Y)^2 - \frac{2}{nm}\left(\sum_{i=1}^{m}\sum_{i<l}^{m}\sigma^{(i,l:m)}_{Y(i,l:m)} - \sum_{i=1}^{(m-1)/2}\sum_{i<m-i+1}^{(m-1)/2}\sigma^{(i,m-i+1:m)}_{Y(i,m-i+1:m)}\right),$$

which completes the proof.

Sometimes, it becomes difficult for the experimenter to rank the units with respect to the study variable. Therefore, it is customary to utilize the concomitant variables that are highly correlated with the study variable. Following Stokes (1977), under bivariate normal distribution, we assume the following model for imperfect PDRSS (IPDRSS) scheme:

$$Y^{(i)[i:m]}_{i[i:m]j} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}\left(X^{(i)(i:m)}_{i(i:m)j} - \mu_X\right) + \delta_{ij}, \quad i = 1,2,...,m,\ j = 1,2,...,r, \tag{3.14}$$

where δ_{ij} is the random error term with mean zero and constant variance, $E(\delta_{ij}) = 0$ and $\mathrm{Var}(\delta_{ij}) = \sigma^2_\delta = \sigma^2_Y(1-\rho^2)$. Note that $X^{(i)(i:m)}_{i(i:m)j}$ represents the ith order statistic from the ith set of the ith order statistics, i.e., $X^{(i)}_{i(i:m)j}$, $i = 1,2,...,m$, and $Y^{(i)[i:m]}_{i[i:m]j}$ is the corresponding ith judgment order statistic from the ith set of the ith judgment order statistics, i.e., $Y^{(i)}_{i[i:m]j}$, $i = 1,2,...,m$, in the jth cycle. Based on even and odd set based sample sizes, the mean estimators and their variances under IPDRSS are as follows:

$$\begin{aligned}
\bar{Y}^{\mathrm{E}}_{\mathrm{IPDRSS}} &= \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{m/2}Y^{(i)[i:m]}_{i[i:m]j} + \sum_{i=1}^{m/2}Y^{(i)[m-i+1:m]}_{(m-i+1)[m-i+1:m]j}\right) \quad \text{and} \\
\bar{Y}^{\mathrm{O}}_{\mathrm{IPDRSS}} &= \frac{1}{n}\sum_{j=1}^{r}\left(\sum_{i=1}^{(m+1)/2}Y^{(i)[i:m]}_{i[i:m]j} + \sum_{i=1}^{m-1/2}Y^{(i)[m-i+1:m]}_{(m-i+1)[m-i+1:m]j}\right), \qquad (3.15) \\
\mathrm{Var}(\bar{Y}^{\mathrm{E}}_{\mathrm{IPDRSS}}) &= \frac{1}{nm}\left\{m\sigma^2_Y(1-\rho^2) + \rho^2\frac{\sigma^2_Y}{\sigma^2_X}\left(\sum_{i=1}^{m}\sigma^{(i,i:m)}_{X(i,i:m)} + 2\sum_{i=1}^{m/2}\sum_{i<m-i+1}^{m/2}\sigma^{(i,m-i+1:m)}_{X(i,m-i+1:m)}\right)\right\} \quad \text{and}
\end{aligned}$$

$$\mathrm{Var}(\bar{Y}^{\mathrm{O}}_{\mathrm{IPDRSS}}) = \frac{1}{nm}\left\{m\sigma_Y^2(1-\rho^2)+\rho^2\frac{\sigma_Y^2}{\sigma_X^2}\left(\sum_{i=1}^{m}\sigma_{X(i,i:m)}^{(i,i:m)}+2\sum_{i=1}^{(m-1)/2}\sum_{i<m-i+1}^{(m-1)/2}\sigma_{X(i,m-i+1:m)}^{(i,m-i+1:m)}\right)\right\}. \quad (3.16)$$

Similarly, from (3.15), the mean estimator based on imperfect DRSS (IDRSS) is $\bar{Y}_{\mathrm{IDRSS}} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{i[i:m]j}^{(i)[i:m]}$. The corresponding variance of $\bar{Y}_{\mathrm{IDRSS}}$ is

$$\mathrm{Var}(\bar{Y}_{\mathrm{IDRSS}}) = \frac{1}{nm}\left(m\sigma_Y^2(1-\rho^2)+\rho^2\frac{\sigma_Y^2}{\sigma_X^2}\sum_{i=1}^{m}\sigma_{X(i,i:m)}^{(i,i:m)}\right). \quad (3.17)$$

Remark: It is to be noted that the mean estimators under all IRSS schemes are based on the assumption that the study variable and the auxiliary variable jointly follow a bivariate normal distribution. Therefore, the estimators based on imperfect rankings are not necessarily unbiased.

3.4 Comparison of estimators

In this section, we compare the mean estimators based on different sampling schemes. For a fair comparison of estimators, we have considered both symmetric and asymmetric probability distributions. The RPs of estimators are calculated for some choices of set size m and are given in Table 3.1. Note that the RPs do not depend on the value of r. The estimators are compared based on their RPs. The RP of an estimator, say $\hat{\theta}_1$, with respect to other estimator, say $\hat{\theta}_2$, is defined as $\mathrm{RP}(\hat{\theta}_1,\hat{\theta}_2) = \frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1)}$. Here $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased estimators. In case of a biased estimator, the variance is replaced by the MSE of the estimator.

Table 3.1 shows that the RP is an increasing function of the set size m. In most cases, the proposed mean estimators are more precise than the existing estimators based on PRSS, RSS and MRSS schemes. The gains in precision mean that smaller sample sizes may suffice, thus reducing the total cost of the study. Moreover, MRSS also provides efficient estimates for some populations but for some cases, as the set size increases, the RPs of estimators under MRSS decrease as compared with that of the estimators under PDRSS and DRSS. It is also clear that the DRSS scheme dominates all sampling schemes and provides most precise estimates of the population mean for all distributions.

Sometimes, it is difficult or very costly to rank the values of the study variable, while it is easy to rank the values of the auxiliary variable that is correlated with the study variable. In Table 3.2, we compare the mean estimators based on IRSS, IMRSS, IPDRSS and IDRSS methods. It turns out that under all sampling schemes, the RPs are increasing with ρ and vice-versa. Except for very small values of ρ, the mean estimators under IPDRSS are more efficient than their counterparts based on RSS and MRSS. Generally, when $\rho < 0.70$, the mean estimates under IPDRSS are roughly equivalent to the mean estimates obtained under IDRSS.

3.4.1 Comparison with regression estimator under SRS

In this section, we compare the proposed and existing mean estimators under perfect and imperfect ranking schemes with the regression estimators of the population mean based on SRS.

Patil et al. (1993) compared RSS- and IRSS-based mean estimators with the SRS-based regression of the population mean. They considered two cases, i.e., mean of the auxiliary variable is known or unknown. The linear regression estimator of μ_Y when μ_X is known is given by

$$\bar{Y}_{\text{reg}} = \bar{Y} + \hat{\beta}(\mu_X - \bar{X}), \tag{3.18}$$

where $\bar{Y}$ and $\bar{X}$ are the sample means based on a simple random sample of size n. Here $\hat{\beta}$ is the least-squares estimate of the population regression coefficient β. Sukhatme and Sukhatme (1970) showed that when (Y, X) follows a bivariate normal distribution, then $\bar{Y}_{\text{reg}}$ is an unbiased estimator of μ_Y. The variance of $\bar{Y}_{\text{reg}}$ is given by

$$\text{Var}(\bar{Y}_{\text{reg}}) = \frac{1}{n}\sigma_Y^2(1-\rho^2)\left(1 + \frac{1}{n-3}\right). \tag{3.19}$$

In field investigations, sometimes, the mean of an auxiliary variable μ_X is unknown. In such situations, it is customary to use the method of two-phase sampling or double sampling. In two-phase sampling, in the first-phase a large sample of size nm is drawn from the target population and μ_X is estimated first. Then, in second-phase, a subsample of size n is drawn from nm units to find the sample means, i.e., $\bar{Y}$ and $\bar{X}$. The linear regression estimator of μ_Y under two-phase sampling is given by

$$\bar{Y}_{\text{reg}}^{\text{d}} = \bar{Y} + \hat{\beta}(\bar{X}_{\text{d}} - \bar{X}), \tag{3.20}$$

where $\bar{X}_{\text{d}} = \frac{1}{nm}\sum_{i=1}^{nm} X_i$ is also an unbiased estimator of μ_X. If (Y, X) follows a bivariate normal distribution, then Sukhatme and Sukhatme (1970) showed that $\bar{Y}_{\text{reg}}^{\text{d}}$ is an unbiased estimator of μ_Y, and its variance is given by

$$\text{Var}(\bar{Y}_{\text{reg}}^{\text{d}}) = \sigma_Y^2(1-\rho^2)\left(\frac{1}{n} - \frac{1}{nm}\right)\left(1 + \frac{1}{n-3}\right) + \frac{\sigma_Y^2}{nm}. \tag{3.21}$$

In Table 3.3, we compare the performances of the mean estimators based on perfect RSS schemes with the regression estimator based on SRS. For this comparison, we consider the variances of mean estimators given in (3.1), (3.5), (3.7), (3.8), and compare them with the variances of the regression estimators give in (3.19) and (3.21). The RPs of these estimators (based on perfect RSS schemes) are given in Table 3.3. Similarly, under imperfect RSS schemes, we consider the variances of the mean estimators given in (3.3), (3.6), (3.16), (3.17), and compare them with the variances of the regression estimators given in (3.19) and (3.21). The RPs of these estimators based on imperfect RSS schemes are given in Table 3.4.

From Table 3.3, it is clear that under perfect ranking, the mean estimators under RSS designs are better than the regression estimator based on SRS unless the correlation between the study and the auxiliary variable is greater than 0.85. Similarly, in case of imperfect rankings, from Table 3.4, it is noteworthy that when the correlation is small, the RSS based estimators are preferable to the SRS-based regression estimator. The RPs of the mean estimators tend to decrease under each of the RSS design as the number of cycles r increases and vice-versa. For all cases, the mean estimators under PDRSS are more precise when compared with the mean estimates under RSS and MRSS. It is interesting to note that the RPs of both DRSS- and PDRSS-based estimators remain roughly closer to each other.

Table 3.1: Exact RPs relative to SRS under symmetric and asymmetric distributions

Distribution	m	PRSS	RSS	MRSS	PDRSS	DRSS	Distribution	m	PRSS	RSS	MRSS	PDRSS	DRSS
Uniform (0,1)	3	1.67	2.00	1.67	2.89	3.03	Exponential (1)	3	1.46	1.64	2.25	1.99	2.02
	4	1.67	2.50	2.08	3.41	4.28		4	1.57	1.92	2.44	2.37	2.52
	5	2.33	3.00	2.33	5.28	5.67		5	1.95	2.19	2.23	2.97	3.02
Normal (0,1)	3	1.58	1.91	2.23	2.54	2.63	Gamma (2,1)	3	1.65	1.75	2.23	2.21	2.27
	4	1.68	2.35	2.77	3.10	3.53		4	1.62	2.10	2.56	2.66	2.91
	5	2.22	2.77	3.49	4.29	4.46		5	2.07	2.42	2.64	3.46	3.55
Logistic (0,1)	3	1.53	1.84	2.55	2.34	2.41	Weibull (4,1)	3	1.59	1.93	2.15	2.60	2.69
	4	1.64	2.22	3.16	2.84	3.12		4	1.68	2.38	2.67	3.17	3.65
	5	2.11	2.58	4.17	3.72	3.83		5	2.25	2.82	3.31	4.46	4.65
Beta (3,3)	3	1.62	1.97	1.97	2.72	2.83	Beta (9,2)	3	1.57	1.86	2.08	2.45	2.53
	4	1.69	2.44	2.46	3.31	3.90		4	1.66	2.27	2.48	2.98	3.37
	5	2.29	2.91	2.97	4.81	5.06		5	2.19	2.66	2.73	4.09	4.23

Table 3.2: Exact RPs relative to SRS under imperfect ranking

m	$\rho \rightarrow$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
3	IRSS	1.00	1.02	1.04	1.08	1.14	1.21	1.31	1.44	1.63	1.76	1.88
	IMRSS	1.01	1.02	1.05	1.10	1.16	1.25	1.37	1.55	1.81	1.99	2.18
	IPDRSS	1.01	1.02	1.06	1.11	1.18	1.28	1.42	1.63	1.97	2.21	2.47
	IDRSS	1.01	1.03	1.06	1.11	1.18	1.29	1.44	1.66	2.01	2.27	2.55
4	IRSS	1.01	1.02	1.05	1.10	1.17	1.26	1.39	1.58	1.87	2.07	2.29
	IMRSS	1.01	1.03	1.06	1.11	1.19	1.30	1.46	1.69	2.07	2.37	2.68
	IPDRSS	1.01	1.03	1.06	1.12	1.20	1.32	1.50	1.77	2.22	2.57	2.98
	IDRSS	1.01	1.03	1.07	1.13	1.22	1.35	1.54	1.85	2.38	2.83	3.36
5	IRSS	1.01	1.03	1.06	1.11	1.19	1.30	1.46	1.69	2.07	2.36	2.68
	IMRSS	1.01	1.03	1.07	1.13	1.22	1.35	1.54	1.84	2.37	2.81	3.32
	IPDRSS	1.01	1.03	1.07	1.14	1.24	1.38	1.60	1.96	2.64	3.25	4.03
	IDRSS	1.01	1.03	1.08	1.14	1.24	1.39	1.61	1.99	2.69	3.33	4.17

Table 3.3: Exact RPs of estimators under perfect ranking relative to SRS-based regression estimator

			Auxiliary variable mean known							Auxiliary variable mean unknown					
m	r	$\rho \rightarrow$	$\pm$ 0.10	$\pm$0.30	$\pm$0.50	$\pm$0.70$\pm$	$\pm$0.90	$\pm$0.99	$\rho \rightarrow$	$\pm$0.10	$\pm$0.30	$\pm$0.50	$\pm$0.70	$\pm$0.90	$\pm$0.99
4	1	RSS	4.65	4.27	3.52	2.39	0.89	0.09	RSS	4.07	3.79	3.23	2.38	1.26	0.66
4	1	MRSS	5.49	5.05	4.16	2.83	1.05	0.11	MRSS	4.81	4.48	3.81	2.82	1.48	0.78
4	1	PDRSS	6.14	5.65	4.65	3.16	1.18	0.12	PDRSS	5.38	5.01	4.27	3.15	1.66	0.87
4	1	DRSS	6.98	6.42	5.29	3.60	1.34	0.14	DRSS	6.12	5.70	4.85	3.58	1.89	0.99
4	3	RSS	2.58	2.37	1.96	1.33	0.50	0.05	RSS	2.52	2.37	2.05	1.58	0.96	0.63
4	3	MRSS	3.05	2.81	2.31	1.57	0.59	0.06	MRSS	2.98	2.80	2.43	1.87	1.13	0.74
4	3	PDRSS	3.41	3.14	2.59	1.76	0.65	0.07	PDRSS	3.34	3.13	2.71	2.09	1.27	0.83
4	3	DRSS	3.88	3.57	2.94	2.00	0.74	0.08	DRSS	3.79	3.56	3.09	2.38	1.44	0.94
4	∞	RSS	2.46	2.26	1.86	1.27	0.47	0.05	RSS	2.43	2.28	1.98	1.54	0.94	0.62
4	∞	MRSS	2.91	2.67	2.20	1.50	0.56	0.06	MRSS	2.87	2.70	2.35	1.82	1.11	0.74
4	∞	PDRSS	3.25	2.99	2.46	1.68	0.62	0.07	PDRSS	3.21	3.02	2.62	2.03	1.24	0.82
4	∞	DRSS	3.70	3.40	2.80	1.90	0.71	0.07	DRSS	3.65	3.43	2.98	2.31	1.41	0.94
5	1	RSS	2.32	2.14	1.76	1.20	0.45	0.05	RSS	2.33	2.19	1.91	1.48	0.92	0.62
5	1	MRSS	2.75	2.52	2.08	1.41	0.53	0.06	MRSS	2.75	2.59	2.25	1.75	1.09	0.73
5	1	PDRSS	3.07	2.82	2.33	1.58	0.59	0.06	PDRSS	3.08	2.89	2.52	1.96	1.22	0.82
5	1	DRSS	3.49	3.21	2.64	1.80	0.67	0.07	DRSS	3.50	3.29	2.87	2.23	1.38	0.93
5	3	RSS	4.11	3.78	3.12	2.12	0.79	0.08	RSS	3.85	3.58	3.05	2.25	1.19	0.62
5	3	MRSS	5.18	4.76	3.92	2.67	0.99	0.10	MRSS	4.84	4.50	3.83	2.83	1.49	0.78
5	3	PDRSS	6.37	5.86	4.83	3.28	1.22	0.13	PDRSS	5.96	5.54	4.72	3.48	1.84	0.96
5	3	DRSS	6.62	6.08	5.01	3.41	1.27	0.13	DRSS	6.18	5.76	4.90	3.62	1.91	1.00
5	∞	RSS	2.97	2.73	2.25	1.53	0.57	0.06	RSS	2.93	2.74	2.35	1.78	1.01	0.60
5	∞	MRSS	3.74	3.44	2.83	1.93	0.72	0.08	MRSS	3.69	3.45	2.96	2.24	1.27	0.76
5	∞	PDRSS	4.60	4.23	3.49	2.37	0.88	0.09	PDRSS	4.54	4.24	3.65	2.75	1.56	0.93
5	∞	DRSS	4.78	4.39	3.62	2.46	0.92	0.10	DRSS	4.71	4.41	3.79	2.86	1.62	0.97

Table 3.4: Exact RPs of estimators under imperfect ranking relative to SRS-based regression estimator

			Auxiliary variable mean known							Auxiliary variable mean unknown					
m	r	$\rho \rightarrow$	$\pm$ 0.10	$\pm$0.30	$\pm$0.50	$\pm$0.70$\pm$	$\pm$0.90	$\pm$0.99	$\rho \rightarrow$	$\pm$0.10	$\pm$0.30	$\pm$0.50	$\pm$0.70	$\pm$0.90	$\pm$0.99
4	1	RSS	1.99	1.92	1.75	1.42	0.71	0.09	RSS	1.00	0.98	0.95	0.88	0.73	0.61
4	1	MRSS	1.99	1.93	1.79	1.49	0.79	0.11	MRSS	1.00	0.99	0.97	0.92	0.81	0.71
4	1	PDRSS	1.99	1.94	1.81	1.53	0.84	0.12	PDRSS	1.00	0.99	0.98	0.95	0.87	0.79
4	1	DRSS	1.99	1.95	1.83	1.57	0.91	0.13	DRSS	1.00	1.00	0.99	0.97	0.94	0.89
4	3	RSS	1.11	1.07	0.97	0.79	0.39	0.05	RSS	1.40	1.37	1.31	1.18	0.89	0.60
4	3	MRSS	1.11	1.07	0.99	0.83	0.44	0.06	MRSS	1.40	1.38	1.34	1.25	1.01	0.74
4	3	PDRSS	1.11	1.08	1.00	0.85	0.47	0.07	PDRSS	1.40	1.39	1.36	1.30	1.13	0.90
4	3	DRSS	1.11	1.08	1.02	0.87	0.50	0.07	DRSS	1.40	1.39	1.36	1.31	1.15	0.93
4	∞	RSS	1.05	1.02	0.93	0.75	0.38	0.05	RSS	1.06	1.05	1.01	0.93	0.76	0.58
4	∞	MRSS	1.05	1.02	0.95	0.79	0.42	0.06	MRSS	1.07	1.06	1.03	0.99	0.86	0.72
4	∞	PDRSS	1.06	1.03	0.96	0.81	0.45	0.06	PDRSS	1.07	1.06	1.05	1.03	0.96	0.87
4	∞	DRSS	1.06	1.03	0.97	0.83	0.48	0.07	DRSS	1.07	1.06	1.05	1.04	0.98	0.91
5	1	RSS	1.00	0.96	0.88	0.71	0.36	0.05	RSS	1.03	1.02	0.98	0.91	0.74	0.58
5	1	MRSS	1.00	0.97	0.89	0.74	0.39	0.05	MRSS	1.04	1.03	1.01	0.96	0.85	0.72
5	1	PDRSS	1.00	0.97	0.90	0.76	0.42	0.06	PDRSS	1.04	1.03	1.02	1.00	0.95	0.87
5	1	DRSS	1.00	0.97	0.91	0.79	0.45	0.07	DRSS	1.04	1.03	1.03	1.01	0.97	0.90
5	3	RSS	1.49	1.45	1.34	1.11	0.59	0.08	RSS	1.00	0.98	0.95	0.89	0.73	0.58
5	3	MRSS	1.50	1.46	1.37	1.18	0.67	0.10	MRSS	1.00	0.99	0.97	0.93	0.83	0.72
5	3	PDRSS	1.50	1.47	1.39	1.23	0.75	0.12	PDRSS	1.00	1.00	0.99	0.97	0.93	0.87
5	3	DRSS	1.50	1.47	1.40	1.23	0.77	0.12	DRSS	1.00	1.00	0.99	0.98	0.95	0.90
5	∞	RSS	1.08	1.05	0.97	0.80	0.43	0.06	RSS	1.00	0.98	0.95	0.88	0.73	0.61
5	∞	MRSS	1.08	1.05	0.99	0.85	0.49	0.07	MRSS	1.00	0.99	0.97	0.92	0.81	0.71
5	∞	PDRSS	1.08	1.06	1.01	0.89	0.54	0.09	PDRSS	1.00	0.99	0.98	0.95	0.87	0.79
5	∞	DRSS	1.08	1.06	1.01	0.89	0.55	0.09	DRSS	1.00	1.00	0.99	0.97	0.94	0.89

3.5 An application to real data

In this section, we explain the PDRSS scheme using a real data set that was previously used by Haq et al. (2013b) to illustrate RSS schemes. The data set is taken from Platt et al. (1988) related to 399 conifer (pinus palustris) trees. We consider only two variables out of the seven originally collected: the height of the tree measured in feet (the study variable Y), and the diameter of the tree at the breast height measured in centimeters (the auxiliary variable X). The summary statistics of the data set are presented in Table 3.5.

Table 3.5: Summary statistics of 399 trees data

Variable	Mean	Median	Variance	Skewness	Kurtosis
Diameter (Y)(cm)	20.84	14.5	310.11	0.884	−0.423
Height (X) (ft)	52.36	29	325.14	1.619	1.776
Correlation coefficient (ρ)	0.908				

Both variables are asymmetrically distributed with non-zero skewness and the mean greater than their respective median. We use one million replications to estimate MSEs of the mean estimators under the existing sampling methods. For an unbiased estimator, the MSE of estimator is equivalent to its variance. For brevity of discussion, we consider MSE for each estimator.

Each iteration is performed as follows: a sample of size m is selected under each of the sampling scheme. Here, the assumed values of m are 3, 4 and 5. The ranking is performed by using the study variable Y and the auxiliary variable X. The sample means under both perfect and imperfect rankings are calculated. One million iterations are used to obtain the sampling distributions of sample means under each sampling schemes. Then, based on estimated MSEs (EMSEs) of mean estimators, RPs are calculated and reported in Table 3.6. The estimated RP (ERP) of any estimator, say $\bar{Y}_{\mathrm{H}}$, with respect to $\bar{Y}_{\mathrm{SRS}}$ is defined as $\mathrm{ERP}(\bar{Y}_{\mathrm{H}}, \bar{Y}_{\mathrm{SRS}}) = \frac{\mathrm{EMSE}(\bar{Y}_{\mathrm{SRS}})}{\mathrm{EMSE}(\bar{Y}_{\mathrm{H}})}$, for H = RSS, PRSS, MRSS, PDRSS and DRSS.

Table 3.6: ERPs relative to SRS for trees data

m	Ranking	PRSS	RSS	MRSS	PDRSS	DRSS
3	Perfect	1.4793	1.6328	1.8911	2.0163	2.0447
4	Perfect	1.5869	1.9306	1.9627	2.4975	2.6404
5	Perfect	2.0049	2.2274	1.6607	3.2392	3.2842
3	Imperfect	1.4139	1.5419	1.6698	1.8589	1.8834
4	Imperfect	1.4747	1.7665	1.7679	2.1666	2.2841
5	Imperfect	1.7945	1.9673	1.6194	2.6101	2.6404

It is notable that in order to select a sample of size $m = 5$, RSS or MRSS requires 25 identified units, PRSS requires 15 units, PDRSS requires 75 units, and DRSS requires 125 units. It is clear that although DRSS provides efficient estimates but it requires a large number of identified units. The identification and ranking of 125 trees in different sets may be more difficult than 75 trees. Therefore, PDRSS can be used as an alternative to DRSS because it requires less identification and less ranking of trees. Thus, PDRSS scheme can be more practical and economical than DRSS scheme. Table 3.6 shows that the estimated RPs are increasing with the set size m. It is clear that the estimates obtained under perfect rankings are superior to the estimates based on imperfect rankings. As expected, the mean estimates under PDRSS are more precise than their counterparts based on SRS, RSS, MRSS and PRSS schemes. Generally, PDRSS scheme is best for both perfect and imperfect rankings as compared with these sampling methods. For all cases the mean estimates under DRSS are more precise than its counterparts.

3.6 Conclusion

In this chapter, we proposed an efficient and unbiased sampling scheme for estimation of the population mean. We showed both analytically and numerically that under perfect ranking, the mean estimators under PDRSS are more precise than the mean estimators under SRS, RSS and PRSS for both symmetric and asymmetric distributions. Generally for large samples, the proposed mean estimates are also more precise than the estimates under MRSS. In comparison with the regression estimators of the population mean, PDRSS mean estimates are preferable to SRS, RSS and MRSS estimates for both perfect and imperfect rankings. Based on the results obtained from real data, PDRSS provided efficient estimates than its counterparts. Finally, the use of PDRSS is recommended over the existing RSS schemes. It is also a good alternative to DRSS when ranking or identification cost cannot be ignored. The current work can be extended to develop ratio ratio and regression estimators of population mean based on PDRSS.

Chapter 4

Best Linear Unbiased and Invariant Estimation in Location-Scale Families Based on Double Ranked Set Sampling Scheme

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In this chapter, we propose best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) for the unknown parameters of location-scale family of distributions based on double ranked set sampling (DRSS) using perfect and imperfect rankings. These estimators are then compared with the BLUEs and BLIEs based on ranked set sampling (RSS). It is shown that under perfect ranking, the proposed estimators are uniformly better than the BLUEs and BLIEs obtained via RSS. We also propose best linear unbiased quantile (BLUQ) and best linear invariant quantile (BLIQ) estimators for normal distribution under DRSS. It is observed that the proposed quantile estimators are more efficient than the BLUQ and BLIQ estimators based on RSS for both perfect and imperfect orderings.

4.1 Introduction

The ranked set sampling (RSS) scheme was introduced by McIntyre (1952) for real life situations where the variable of interest is difficult to measure or costly. However, the ranking of a small set of selected units can be carried out easily by visual inspection or by any inexpensive method. The theoretical foundation of RSS was set by Takahasi and Wakimoto (1968). They proved that under perfect ranking, RSS provides an unbiased estimator of the population mean and it is more efficient than the sample mean based on simple random sampling (SRS). Additionally, Dell and Clutter (1972) showed that under imperfect ranking, the sample mean remains an unbiased estimator of the population mean but ranking should be better than at least a random ordering.

Lloyd (1952) derived the best linear unbiased estimators (BLUEs) of the location and scale parameters using general least-squares theory. Mann (1969) obtained the best linear invariant estimators (BLIEs) of the location and scale parameters based on BLUEs. The best linear unbiased quantile (BLUQ) estimator for normal distribution was derived by Hassanein et al. (1986). Under RSS protocol, Stokes (1995) derived the BLUEs of location and scale parameters when one parameter of the location-scale family of distributions is known. Later on Sinha et al. (1996) obtained BLUEs of the location and scale parameters of normal distribution and for a scale parameter of exponential distribution using RSS. The BLUEs of location and scale parameters of generalized geometric distribution were obtained by Bhoj and Ahsanullah (1996) under RSS. Barnett and Moore (1997) suggested BLUEs of the location and scale parameters with particular reference to imperfect RSS (IRSS). By extending the same work, Hossain and Muttlak (2000) obtained minimum variance linear unbiased estimators (MVLUEs) of location-scale family of distributions under RSS. They showed that the MVLUEs of population mean under RSS are more precise than the usual SRS or RSS mean estimators of the population mean. Raqab et al. (2002) extended the work of Mann (1969) and suggested BLIEs of the mean, location and scale parameters of several distributions under RSS. Barnett and Bown (2002) developed an approximate significance testing procedure for normal quantile based on BLUQ estimators under SRS and RSS. Al-Saleh and Al-Kadiri (2000) introduced an extension of RSS, namely, double RSS (DRSS). The DRSS scheme also provides an unbiased estimator of the population mean and it is more efficient than the RSS mean estimator. Balakrishnan and Li (2005) proposed BLUEs based on ordered RSS (ORSS) for unknown location and scale parameters of generalized geometric distribution. It is shown that the BLUEs based on ORSS are uniformly better than the BLUEs based on SRS and RSS. Balakrishnan and Li (2008) obtained BLUEs based on ORSS for some location-scale distributions. Shadid et al. (2011) derived some modified BLUEs and BLIEs of the unknown parameters of location-scale family of distributions under RSS. It is shown that the modified BLUEs and BLIEs are more efficient than their competitors when the underlying distribution is symmetric. For some other application and related work, see Sinha et al. (1996), Bhoj (1997), Chuiv and Sinha (1998), Kim and Arnold (1999), Zheng and Al-Saleh (2003), Balakrishnan and Li (2006), Tiensuwan et al. (2007) and references therein.

In this chapter, we extend the work and propose BLUEs and BLIEs of the location and scale parameters under DRSS scheme using perfect and imperfect rankings. Furthermore, we also propose some BLUQ and best linear invariance quantile (BLIQ) estimators of normal quantiles based on DRSS. It is observed that under perfect DRSS, the proposed BLUEs and BLIEs of location and scale parameters are uniformly better than the BLUEs and BLIEs under RSS. In quantile estimation, the suggested quantile estimators are more precise than their counterparts for perfect and worst rankings.

The rest of the chapter is organized as follows: the RSS and DRSS sampling methods are briefly elucidated in Section 4.2. Section 4.3 focuses on BLUEs and BLIEs under RSS. In Sections 4.4 and 4.5, we present the BLUEs and BLIEs of the location and scale parameters under DRSS using perfect and imperfect rankings, respectively. Section 4.6 provides a numerical comparison of estimators based on several symmetric and asymmetric distributions. In Section 4.7, we consider quantile estimation based on RSS and DRSS schemes. Section 4.8 contains a numerical comparison of quantile estimators. Section 4.9 provides the concluding remarks.

4.2 Sampling methods

In this section, we provide a brief introduction to RSS and DRSS methods.

4.2.1 Ranked set sampling

The RSS scheme can be described as follows:

(i) Identify m^2 units from the target population. Randomly allocate these m^2 units to m sets each of size m units.

(ii) Rank the units within each set without yet knowing the actual measurements of the study variable.

(iii) Select the smallest ranked unit from the first set of m units. Similarly, select the second smallest ranked unit from the second set of m units. The procedure continues until the largest ranked unit is selected from the last set. This completes one cycle of a ranked set sample of size m.

(iv) The above steps (i)-(iii) can be repeated k number of times, if necessary, in order to obtain a ranked set sample of size $n = mk$.

4.2.2 Double ranked set sampling

Al-Saleh and Al-Kadiri (2000) introduced the DRSS procedure for an efficient estimation of population mean. The DRSS scheme can be described as follows:

(i) Identify m^3 units from the target population. Randomly allocate these m^3 units to m sets each containing m^2 units.

(ii) Apply the RSS procedure to these m sets in order to obtain m ranked set samples each of size m.

(iii) Again apply the RSS procedure to these m sets, each of size m, to obtain a double ranked set sample of size m. This completes one cycle of a double ranked set sample of size m.

(iv) This step is similar to the step (iv) of the RSS procedure.

4.3 BLUEs and BLIEs using RSS

Let $Y_1, Y_2, ..., Y_m$ be a simple random sample of size m, drawn from an absolutely continuous distribution having distribution (CDF) $F\left(\frac{y-\mu}{\sigma}\right)$ and probability density function (PDF) $\frac{1}{\sigma}\left(\frac{y-\mu}{\sigma}\right)$, where μ is the location parameter and $\sigma(>0)$ is the scale parameter. For brevity, let $f^*(y) = \frac{1}{\sigma}\left(\frac{y-\mu}{\sigma}\right)$ and $F^*(y) = F\left(\frac{y-\mu}{\sigma}\right)$.

Let $Y_{11t}, Y_{12t}, ..., Y_{1mt}, Y_{21t}, Y_{22t}, ..., Y_{2mt}, ..., Y_{m1t}, Y_{m2t}, ..., Y_{mmt}$ be m sets each of size m in the tth cycle for $t = 1, 2, ..., k$. By applying RSS procedure to these m sets, we obtain a ranked set sample of size m for the tth cycle, i.e., $Y_{(1)t}, Y_{(2)t}, ..., Y_{(m)t}$. Here $Y_{(i)t}$ is the ith independent order statistic obtained from the ith sample of size m in the tth cycle, i.e., $Y_{(i)t} = i\text{th}\ \min(Y_{i1t}, Y_{i2t}, ..., Y_{imt})$. Let $U_{(i)t} = \frac{Y_{(i)t}-\mu}{\sigma}$ be the standardized variate with PDF independent of μ and σ. Let $\boldsymbol{Y}' = (\boldsymbol{Y}'_1, \boldsymbol{Y}'_2, ..., \boldsymbol{Y}'_k)$ be the vector of observed order statistics of a ranked set sample of size mk and let $\boldsymbol{U}' = (\boldsymbol{U}'_1, \boldsymbol{U}'_2, ..., \boldsymbol{U}'_k)$ be the vector of standardized order statistics corresponding to $\boldsymbol{Y}$, where $\boldsymbol{Y}'_t = (Y_{(1)}, Y_{(2)}, ..., Y_{(m)})$ and $\boldsymbol{U}'_t = (U_{(1)}, U_{(2)}, ..., U_{(m)})$ for $t = 1, 2, ..., k$. Let $\boldsymbol{\alpha}' = (\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2, ..., \boldsymbol{\alpha}'_k)$ be the mean vector of $\boldsymbol{U}$, where $\boldsymbol{\alpha}'_t = (\alpha_{(1)}, \alpha_{(2)}, ..., \alpha_{(m)})$ for $t = 1, 2, ..., k$, and let $\boldsymbol{\theta}' = (\mu, \sigma)$ be the vector of unknown parameters. Let $\boldsymbol{\Sigma}$ be a $mk \times mk$ diagonal matrix of $\boldsymbol{U}$, i.e., $\boldsymbol{\Sigma} = \text{diag}(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2, ..., \boldsymbol{\Sigma}_k)$. Here 'diag' indicates a diagonal matrix. As in the ranked set sample, all of the order statistics are independent, thus, here $\boldsymbol{\Sigma}_t$ is also a $m \times m$ diagonal matrix, i.e., $\boldsymbol{\Sigma}_t = \text{diag}(\sigma_{(11)}, \sigma_{(22)}, ..., \sigma_{(mm)})$, for $t = 1, 2, ..., k$, where $\sigma_{(ii)} = \text{Var}(U_{(i)})$, for $i = 1, 2, ..., m$. The expected value of $\boldsymbol{Y}$ is $\text{E}(\boldsymbol{Y}) = \boldsymbol{A\theta}$, where $\boldsymbol{A} = (\boldsymbol{1}, \boldsymbol{\alpha})$, $\boldsymbol{1}' = (\boldsymbol{1}_1, \boldsymbol{1}_2, ..., \boldsymbol{1}_k)$ is a $mk \times 1$ vector of all 1's, i.e., $\boldsymbol{1}'_t = (1, 1, ..., 1)$ is a $m \times 1$ vector, for $t = 1, 2, ..., k$. Similarly, the covariance matrix of $\boldsymbol{Y}$ is $\text{Cov}(\boldsymbol{Y}) = \sigma^2\boldsymbol{\Sigma}$. The PDF and CDF of $U_{(i)}$, for $i = 1, 2, ..., m$, respectively, are

$$
\begin{aligned}
F^*_{(i)}(u) &= \{F^*(u)\}^i \sum_{j=0}^{m-i} \binom{i+j-1}{i-1} \{1 - F^*(u)\}^j, \quad -\infty < u < \infty, \\
f^*_{(i)}(u) &= m\binom{m-1}{i-1}\{F^*(u)\}^{i-1}\{1 - F^*(u)\}^{m-i} f^*(u).
\end{aligned}
$$

Similarly, the mean and variance of $U_{(i)}$, respectively, are

$$
\alpha_{(i)} = \int u f^*_{(i)}(u) \mathrm{d}u \quad \text{and} \quad \sigma_{(ii)} = \int (u - \alpha_{(i)})^2 f^*_{(i)}(u) \mathrm{d}u. \tag{4.1}
$$

From (4.1), it is easy to find $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$, that are required in obtaining the BLUEs of location and scale parameters under RSS.

4.3.1 BLUEs based on RSS

Following the work of Lloyd (1952), Barnett and Moore (1997) obtained the BLUEs of the location and scale parameters under RSS, i.e., $\hat{\boldsymbol{\Theta}}'_{\text{RSS}} = (\hat{\mu}_{\text{RSS}}, \hat{\sigma}_{\text{RSS}})$, given by

$$\hat{\boldsymbol{\Theta}}_{\text{RSS}} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}.$$

The covariance matrix of $\hat{\boldsymbol{\Theta}}_{\text{RSS}}$ is

$$\text{Cov}(\hat{\boldsymbol{\Theta}}_{\text{RSS}}) = \sigma^2(\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}.$$

After some simplification, we can write

$$\hat{\mu}_{\text{RSS}} = -\boldsymbol{\alpha}'\boldsymbol{\Gamma}\boldsymbol{Y} \quad \text{and} \quad \hat{\sigma}_{\text{RSS}} = \boldsymbol{1}'\boldsymbol{\Gamma}\boldsymbol{Y},$$

where $\boldsymbol{\Gamma} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{1}\boldsymbol{\alpha}'-\boldsymbol{\alpha}\boldsymbol{1}')\boldsymbol{\Sigma}^{-1}}{\Delta}$ and $\Delta = (\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1})(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}) - (\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1})$.
Similarly, the variances and covariance of BLUEs under RSS are

$$\text{Var}(\hat{\mu}_{\text{RSS}}) = \sigma^2\left(\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{\Delta}\right), \text{Var}(\hat{\sigma}_{\text{RSS}}) = \sigma^2\left(\frac{\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1}}{\Delta}\right) \text{ and } \text{Cov}(\hat{\mu}_{\text{RSS}}, \hat{\sigma}_{\text{RSS}}) = -\sigma^2\left(\frac{\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{\Delta}\right).$$

Note that if the underlying distribution is symmetric, then due to symmetric $\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha} = -\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha} = 0$. It helps in further simplifying the expressions of the BLUEs, i.e.,

$$\hat{\mu}^{\text{s}}_{\text{RSS}} = \frac{\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}}{\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1}}, \ \hat{\sigma}^{\text{s}}_{\text{RSS}} = \frac{\boldsymbol{\alpha}'\boldsymbol{\sigma}^{-1}\boldsymbol{Y}}{\boldsymbol{\alpha}'\boldsymbol{\sigma}^{-1}\boldsymbol{\alpha}},$$

$$\text{Var}(\hat{\mu}^{\text{s}}_{\text{RSS}}) = \frac{\sigma^2}{\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1}}, \ \text{Var}(\hat{\sigma}^{\text{s}}_{\text{RSS}}) = \frac{\sigma^2}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}} \text{ and } \text{Cov}(\hat{\mu}^{\text{s}}_{\text{RSS}}, \hat{\sigma}^{\text{s}}_{\text{RSS}}) = 0.$$

For more details, see David and Nagaraja (2003).

In order to study the performance of BLUEs under IRSS, Barnett and Moore (1997) considered the following model:

$$Y_{[i]t} = \mu + \rho\left(\frac{\sigma}{\sigma_X}\right)(X_{(i)t} - \mu_X) + \tau_{[i]t}, \quad i = 1, 2, ..., m, \ t = 1, 2, ..., k, \tag{4.2}$$

where $X_{(i)t}$ and $\tau_{[i]t}$ are mutually independent. Under this model, (Y, X) follows a bivariate normal distribution with parameters, μ, μ_X, σ, σ_X and ρ. Here $E(\tau_{[it]}) = E(\tau_{it}) = 0$, $E(Y) = \mu$, $E(X) = \mu_X$, $\text{Var}(Y) = \sigma^2$, $\text{Var}(X) = \sigma_X^2$, $\text{Var}(\tau_{[i]t}) = \text{Var}(\tau_{it}) = \sigma^2(1-\rho^2)$, $\text{Corr}(X, Y) = \rho$ and $\text{Cov}(Y_{[i]t}, Y_{[j]t}) = 0$ for $i \neq j$. Here $\tau_{[i]t}$ is the particular value of τ_{it} associated with $X_{(i)t}$. If a ranked set sample of size m is ordered by X_{it} in the tth cycle, then $Y_{[i]t}$ is the ith concomitant or induced order statistic of the ith order statistic $X_{(i)t}$ for the tth cycle. Let $\alpha_{X_{(i)}} = E\left(\frac{X_{(i)t}-\mu_X}{\sigma_X}\right)$, $\sigma_{X_{(ii)}} = \text{Var}\left(\frac{X_{(i)t}-\mu_X}{\sigma_X}\right)$, $\sigma_{X_{(ij)}} = \text{Cov}(\frac{X_{(i)t}-\mu_X}{\sigma_X}, \frac{X_{(j)t}-\mu_X}{\sigma_X}) = 0$, for $i \neq j$, $i, j = 1, 2, ..., m$, $t = 1, 2, ..., k$.

From (4.2), we can write

$$E(Y_{[i]t}) = \mu + \sigma\alpha^*_{X_{(i)}},\ \mathrm{Var}(Y_{[i]t}) = \sigma^2\sigma^*_{X_{(ii)}},\ \mathrm{Cov}(Y_{[i]t}, Y_{[j]t}) = 0 \text{ for } i \neq j,$$

where $\alpha^*_{X_{(i)}} = \rho\alpha_{X_{(i)}}$ and $\sigma^*_{X_{(ii)}} = 1 - \rho^2(1 - \sigma_{X_{(ii)}})$.

The BLUEs of μ and σ based on IRSS are $\hat{\mu}_{\mathrm{IRSS}} = \sum_{i=1}^m \gamma_i \bar{Y}_{[i]}$ and $\hat{\sigma}_{\mathrm{IRSS}} = \sum_{i=1}^m \eta_i \bar{Y}_{[i]}$, respectively. Here $\gamma_i = \frac{\sum_{j=1}^m (\alpha^2_{X_{(j)}}/\sigma^*_{X_{(jj)}}) - \alpha_{X_{(ii)}} \sum_{j=1}^m (\alpha_{X_{(j)}}/\sigma^*_{X_{(jj)}})}{\Xi\sigma^*_{X_{(ii)}}}$, $\eta_i = \frac{\alpha_{X_{(i)}} \sum_{j=1}^m (1/\sigma^*_{X_{(jj)}}) - \sum_{j=1}^m (\alpha_{X_{(j)}}/\sigma^*_{X_{(jj)}})}{\Xi\sigma^*_{X_{(ii)}}}$, $\bar{Y}_{[i]} = \frac{1}{k}\sum_{t=1}^k Y_{[i]t}$, where $\Xi = \sum_{j=1}^m (\alpha^2_{X_{(j)}}/\sigma^*_{X_{(jj)}}) \sum_{j=1}^m (1/\sigma^*_{X_{(jj)}}) - \{\sum_{j=1}^m (\alpha_{X_{(j)}}/\sigma^*_{X_{(jj)}})\}^2$.

The variances and covariance of $\hat{\mu}_{\mathrm{IRSS}}$ and $\hat{\sigma}_{\mathrm{IRSS}}$ are, respectively, given by

$$\mathrm{Var}(\hat{\mu}_{\mathrm{IRSS}}) = \frac{\sigma^2}{k\Xi}\sum_{i=1}^m \left(\alpha^2_{X_{(i)}}/\sigma^*_{X_{(ii)}}\right),\ \mathrm{Var}(\hat{\sigma}^2_{\mathrm{IRSS}}) = \frac{\sigma^2}{k\Xi}\sum_{i=1}^m \left(1/\sigma^*_{X_{(ii)}}\right) \text{ and}$$
$$\mathrm{Cov}(\hat{\mu}_{\mathrm{IRSS}}, \hat{\sigma}_{\mathrm{IRSS}}) = -\frac{\sigma^2}{k\Xi}\sum_{i=1}^m \left(\alpha_{X_{(i)}}/\sigma^*_{X_{(ii)}}\right).$$

4.3.2 BLIEs based on RSS

Mann (1969) derived the BLIEs of the unknown parameters of location-scale family of distributions. The BLIE possesses the minimum mean squared error (MSE) among all linear invariant estimators.

Following Mann (1969), Raqab et al. (2002) proposed BLIEs of the location, scale and mean of the different distributions under RSS.

Let $\tilde{\mu}_{\mathrm{RSS}}$ and $\tilde{\sigma}_{\mathrm{RSS}}$ be the BLIEs of the μ and σ, respectively, given by

$$\tilde{\mu}_{\mathrm{RSS}} = \hat{\mu}_{\mathrm{RSS}} - \hat{\sigma}_{\mathrm{RSS}}\xi_{12}(1+\xi_{22})^{-1} \quad \text{and} \quad \tilde{\sigma}_{\mathrm{RSS}} = \hat{\sigma}_{\mathrm{RSS}}(1+\xi_{22})^{-1},$$

where $\xi_{11} = \sigma^{-2}\mathrm{Var}(\hat{\mu}_{\mathrm{RSS}})$, $\xi_{12} = \sigma^{-2}\mathrm{Cov}(\hat{\mu}_{\mathrm{RSS}}, \hat{\sigma}_{\mathrm{RSS}})$ and $\xi_{22} = \sigma^{-2}\mathrm{Var}(\hat{\sigma}_{\mathrm{RSS}})$.

The MSEs of $\tilde{\mu}_{\mathrm{RSS}}$ and $\tilde{\sigma}_{\mathrm{RSS}}$ are given by

$$\begin{aligned}
\mathrm{MSE}(\tilde{\mu}_{\mathrm{RSS}}) &= \sigma^2\{\xi_{11} - \xi^2_{12}(1+\xi_{22})^{-1}\},\\
\mathrm{MSE}(\tilde{\sigma}_{\mathrm{RSS}}) &= \sigma^2\xi_{22}(1+\xi_{22})^{-1},\\
E\{(\tilde{\mu}_{\mathrm{RSS}} - \mu)(\tilde{\sigma}_{\mathrm{RSS}} - \sigma)\} &= \sigma^2\xi_{12}(1+\xi_{22})^{-1}.
\end{aligned}$$

For further details see Raqab et al. (2002) and Shadid et al. (2011).

4.3.3 BLIEs based on IRSS

Following the work of Barnett and Moore (1997) and Raqab et al. (2002), we propose BLIEs of location and scale parameters under IRSS scheme.

Let $\tilde{\mu}_{\mathrm{IRSS}}$ and $\tilde{\sigma}_{\mathrm{IRSS}}$ be the BLIEs of μ and σ, respectively. Following Raqab et al. (2002), the BLIEs of μ

and σ are

$$\tilde{\mu}_{\text{IRSS}} = \hat{\mu}_{\text{IRSS}} - \hat{\sigma}_{\text{IRSS}}\xi^*_{12}(1+\xi^*_{22})^{-1} \quad \text{and} \quad \tilde{\sigma}_{\text{IRSS}} = \hat{\sigma}_{\text{IRSS}}(1+\xi^*_{22})^{-1},$$

where $\xi^*_{11} = \sigma^{-2}\text{Var}(\hat{\mu}_{\text{IRSS}})$, $\xi^*_{12} = \sigma^{-2}\text{Cov}(\hat{\mu}_{\text{IRSS}}, \hat{\sigma}_{\text{IRSS}})$ and $\xi^*_{22} = \sigma^{-2}\text{Var}(\hat{\sigma}_{\text{IRSS}})$.
The MSEs of $\tilde{\mu}_{\text{IRSS}}$ and $\tilde{\sigma}_{\text{IRSS}}$ are given by

$$\begin{aligned}
\text{MSE}(\tilde{\mu}_{\text{IRSS}}) &= \sigma^2\{\xi^*_{11} - \xi^{*2}_{12}(1+\xi^*_{22})^{-1}\},\\
\text{MSE}(\tilde{\sigma}_{\text{IRSS}}) &= \sigma^2\xi^*_{22}(1+\xi^*_{22})^{-1},\\
E\{(\tilde{\mu}_{\text{IRSS}} - \mu)(\tilde{\sigma}_{\text{IRSS}} - \sigma)\} &= \sigma^2\xi^*_{12}(1+\xi^*_{22})^{-1}.
\end{aligned}$$

4.4 BLUEs and BLIEs under DRSS based on perfect ranking

Al-Saleh and Al-Kadiri (2000) introduced DRSS procedure for estimation of population mean. It is shown that the DRSS scheme provides an unbiased estimator of population mean and it is more efficient than the mean estimator based on RSS. In this section, we propose some improved BLUEs and BLIEs of the location and scale parameters based on DRSS.

Let $Y_i \sim F^*(y)$, for $i = 1, 2, ..., m$. Suppose $S_{1t}, S_{2t}, ..., S_{mt}$ be m sets each of size m^2 for the tth cycle, for $t = 1, 2, ..., k$. Randomly allocate the m^2 units in the the ith set S_{it} to m subsets s_{ijt} each of size m, i.e., $S_{it} = \{s_{ijt}\} = \{s_{i1t}, s_{i2t}, ..., s_{imt}\}$, for $j = 1, 2, ..., m$. The units of the jth subset s_{ijt} of the ith set S_{it} in the tth cycle are given by $s_{ijt} = \{Y^{(i)}_{j1t}, Y^{(i)}_{j2t}, ..., Y^{(i)}_{jmt}\}$. Apply the RSS procedure to these m sets in order to obtain m ranked set samples each of size m. Suppose the ith set S^*_{it} contains the ith ranked sample, i.e., $S^*_{it} = \{Y^{(i)}_{(1)t}, Y^{(i)}_{(2)t}, ..., Y^{(i)}_{(m)t}\}$. Again apply the RSS procedure to these m sets in order to obtain a double ranked set sample of size m in the tth cycle. Let $Z_{(i)t} = i\text{th} \min\{S^*_{it}\}$, then $\{Z_{(1)t}, Z_{(2)t}, ..., Z_{(m)t}\}$ represents a double ranked set sample of size m for the tth cycle, for $t = 1, 2, ..., k$.

As $Y_1, Y_2, ..., Y_m$ be independent and identically distributed (IID) random variables from a location-scale PDF $f^*(y)$. Then, the CDF and PDF of the ith order statistic $Y_{(i)}$, respectively, are given by

$$\begin{aligned}
F^*_{(i)}(y) &= \{F^*(y)\}^i \sum_{j=0}^{m-i} \binom{i+j-1}{i-1}\{1-F^*(y)\}^j, \quad -\infty < y < \infty,\\
f^*_{(i)}(y) &= m\binom{m-1}{i-1}\{F^*(y)\}^{i-1}\{1-F^*(y)\}^{m-i} f^*(y).
\end{aligned}$$

Consider $\{Z_{(1)t}, Z_{(2)t}, ..., Z_{(m)t}\}$, for $t = 1, 2, ..., k$, be a double ranked set sample of size mk, then it is assumed that the PDF of $Z_{(i)t}$ is $g^*_{(i)}(z)$, i.e., $Z_{(i)t} \sim g^*_{(i)}(z)$, with corresponding CDF $G^*_{(i)}(z)$. Note that for each t, $Z_{(i)t}$ and $Z_{(i)}$ are identically distributed, i.e., $Z_{(i)t} \overset{d}{=} Z_{(i)}$. As explained by Al-Saleh and Al-Kadiri (2000), here $g^*_{(i)}(z)$ is the PDF of the ith order statistic from a ranked set sample, say $Y_{(1)}, Y_{(2)}, ..., Y_{(m)}$, with $Y_{(i)} \sim f^*_{(i)}(y)$, for $i = 1, 2, ..., m$. Obviously $\{Z_{(1)t}, Z_{(2)t}, ..., Z_{(m)t}\}$ are independent but not identically distributed (INID) random variables in the tth cycle.

Following David and Nagaraja (2003), the CDF of the rth order statistic from INID random variables, $Y_{(i)}$, for $i = 1, 2, ..., m$, is given by

$$G^*_{(r)}(z) = \sum_{i=r}^{m} \sum_{\Lambda_i} \prod_{l=1}^{i} F^*_{(j_l)}(z) \prod_{l=i+1}^{m} \{1 - F^*_{(j_l)}(z)\}, \quad -\infty < z < \infty, \ r = 1, 2, ..., m,$$

where the summation Λ_i extends over all permutations $(j_1, j_2, ..., j_m)$ of $1, 2, ..., m$ for which $j_1 < j_2 < \cdots < j_i$ and $j_{i+1} < j_{i+2} < \cdots < j_m$.

An alternative form of $G^*_{(r)}(z)$ is given by

$$G^*_{(r)}(z) = \sum_{i=r}^{m} \frac{1}{i!(m-i)!} \, \mathrm{Per}(\boldsymbol{B}_1),$$

where $\mathrm{Per}(\boldsymbol{B}_1)$ is the permanent of the matrix $\boldsymbol{B}_1$.

Here $\boldsymbol{B}_1$ is defined as

$$\boldsymbol{B}_1 = \begin{array}{c} \begin{pmatrix} F^*_{(1)}(z) & F^*_{(2)}(z) & \cdots & F^*_{(m)}(z) \\ 1 - F^*_{(1)}(z) & 1 - F^*_{(2)}(z) & \cdots & 1 - F^*_{(m)}(z) \end{pmatrix} \end{array} \begin{array}{l} \}\, r \\ \}\, m - r \end{array} .$$

Here "$\}\, i$" shows that the first row is repeated i times and "$\}\, m - i$" shows second row is repeated $m - i$ times.

Similarly, the PDF of the rth order statistic from INID random variables, $Y_{(i)}$, for $i = 1, 2, ..., m$, is given by

$$g^*_{(r)}(z) = \frac{1}{(r-1)!(m-r)!} \, \mathrm{Per}(\boldsymbol{B}_2),$$

where $\boldsymbol{B}_2 = \begin{pmatrix} F^*_{(m)}(z) & F^*_{(m)}(z) & \cdots & F^*_{(m)}(z) \\ f^*_{(m)}(z) & f^*_{(m)}(z) & \cdots & f^*_{(m)}(z) \\ 1 - F^*_{(m)}(z) & 1 - F^*_{(m)}(z) & \cdots & 1 - F^*_{(m)}(z) \end{pmatrix} \begin{array}{l} \}\, r - 1 \\ \}\, 1 \\ \}\, m - r \end{array} .$

Examples

Based on above formulae, we provide the CDFs of the order statistics based on double ranked set samples for different sample sizes when $Z = y$.

Case I: If $m = 2$, then

$G^*_{(1)}(y) = F^*_{(1)}(y)\{1 - F^*_{(2)}(y)\}$, $G^*_{(2)}(y) = F^*_{(1)}(y) * F^*_{(2)}(y)$.

Case II: If $m = 3$, then

$G^*_{(1)}(y) = F^*_{(1)}(y)\{1 - F^*_{(2)}(y)\}\{1 - F^*_{(3)}(y)\} + F^*_{(2)}(y)\{1 - F^*_{(3)}(y)\} + F^*_{(3)}(y)$,

$G^*_{(2)}(y) = F^*_{(2)}(y)F^*_{(3)}(y) + F^*_{(1)}(y)\left[F^*_{(2)}(y)\{1 - 2F^*_{(3)}(y)\} + F^*_{(3)}(y)\right]$,

$G^*_{(3)}(y) = F^*_{(1)}(y)F^*_{(2)}(y)F^*_{(3)}(y)$.

Case III: If $m = 4$, then

$G^*_{(1)}(y) = F^*_{(3)}(y) + F^*_{(1)}(y)\{1 - F^*_{(2)}(y)\}\{1 - F^*_{(3)}(y)\}\{1 - F^*_{(4)}(y)\} + F^*_{(2)}(y)\{1 - F^*_{(3)}(y)\}\{1 - F^*_{(4)}(y)\} + F^*_{(4)}(y) - F^*_{(3)}(y)F^*_{(4)}(y)$,

$G^*_{(2)}(y) = F^*_{(3)}(y)F^*_{(4)}(y) + F^*_{(2)}(y)\{F^*_{(3)}(y) + F^*_{(4)}(y) - 2F^*_{(3)}(y)F^*_{(4)}(y)\} + F^*_{(1)}(y)\big[F^*_{(3)}(y) + F^*_{(4)}(y) - 2F^*_{(3)}(y)F^*_{(4)}(y) + F^*_{(2)}(y)\{1 - 2F^*_{(3)}(y) - (2 - 3F^*_{(3)}(y))F^*_{(3)}(y)\}\big]$,

$G^*_{(3)}(y) = F^*_{(2)}(y)F^*_{(3)}(y)F^*_{(4)}(y) + F^*_{(1)}(y)\big[F^*_{(3)}(y)F^*_{(4)}(y) + F^*_{(2)}(y)\{F^*_{(3)}(y) + F^*_{(4)}(y) - 3F^*_{(3)}(y)F^*_{(3)}(y)\}\big]$,

$G^*_{(4)}(y) = F^*_{(1)}(y)F^*_{(2)}(y)F^*_{(3)}(y)F^*_{(4)}(y)$.

Let $W_{(i)t} = \frac{Z_{(i)t} - \mu}{\sigma}$ be the standardized variate with PDF independent of μ and σ. Let $\boldsymbol{Z}' = (\boldsymbol{Z}'_1, \boldsymbol{Z}'_2, ..., \boldsymbol{Z}'_k)$ be the vector of observed order statistics of a double ranked set sample of size mk and let $\boldsymbol{W}' = (\boldsymbol{W}'_1, \boldsymbol{W}'_2, ..., \boldsymbol{W}'_k)$ be the vector of order statistics corresponding to $\boldsymbol{Z}$ where $\boldsymbol{Z}'_t = (Z_{(1)}, Z_{(2)}, ..., Z_{(m)})$ and $\boldsymbol{W}'_t = (W_{(1)}, W_{(2)}, ..., W_{(m)})$ for $t = 1, 2, ..., k$. Let $\boldsymbol{v}' = (\boldsymbol{v}'_1, \boldsymbol{v}'_2, ..., \boldsymbol{v}'_k)$ be the mean vector of $\boldsymbol{W}$, where $\boldsymbol{v}'_t = (v_{(1)}, v_{(2)}, ..., v_{(m)})$. Let $\boldsymbol{\Omega} = \text{diag}(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, ..., \boldsymbol{\Omega}_k)$ be the covariance matrix of $\boldsymbol{W}$. As all of the order statistics in the double ranked set sample came from independent samples, therefore, here $\boldsymbol{\Omega}$ is a $mk \times mk$ diagonal matrix, i.e., where $\boldsymbol{\Omega}_t = \text{diag}(\omega_{(11)}, \omega_{(22)}, ..., \omega_{(mm)})$, for $t = 1, 2, ..., k$, where $\omega_{(ii)} = \text{Var}(W_{(i)})$, for $i = 1, 2, ..., m$. The expected value of $\text{E}(\boldsymbol{Z})$ is $E(\boldsymbol{Z}) = \boldsymbol{B\theta}$, where $\boldsymbol{B} = (\boldsymbol{1}, \boldsymbol{v})$. The covariance matrix of $\boldsymbol{Z}$ is $\text{Cov}(\boldsymbol{Z}) = \sigma^2\boldsymbol{\Omega}$.

Here, the PDF of $W_{(i)}$, $i = 1, 2, ..., m$, is given by

$$g^*_{(i)}(w) = \frac{1}{(i-1)!(m-i)!} \text{Per}(\boldsymbol{B}_3),$$

where
$$\boldsymbol{B}_3 = \begin{array}{cl} \begin{pmatrix} F^*_{(1)}(w) & F^*_{(2)}(w) & \cdots & F^*_{(m)}(w) \\ f^*_{(1)}(w) & f^*_{(2)}(w) & \cdots & f^*_{(m)}(w) \\ 1 - F^*_{(1)}(w) & 1 - F^*_{(2)}(w) & \cdots & 1 - F^*_{(m)}(w) \end{pmatrix} & \begin{array}{l} \} \, i-1 \\ \} \, 1 \\ \} \, m-i \end{array} \end{array}.$$

Similarly, the mean and variance of $W_{(i)}$, respectively, are

$$v_{(i)} = \int w g^*_{(i)}(w)\text{d}w \quad \text{and} \quad \omega_{(ii)} = \int (w - v_{(i)})^2 g^*_{(i)}(w)\text{d}w. \tag{4.3}$$

From (4.3), it is easy to find $\boldsymbol{v}$ and $\boldsymbol{\Omega}$, that are required in obtaining the BLUEs of location and scale parameters under DRSS.

4.4.1 BLUEs based on DRSS

Following the works of Lloyd (1952) and Barnett and Moore (1997), the BLUEs of the location and scale parameters, i.e., $\hat{\boldsymbol{\Theta}}'_{\text{DRSS}} = (\hat{\mu}_{\text{DRSS}}, \hat{\sigma}_{\text{DRSS}})$, under DRSS are

$$\hat{\boldsymbol{\Theta}}_{\text{DRSS}} = (\boldsymbol{B}'\boldsymbol{\Omega}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{\Omega}^{-1}\boldsymbol{Z}.$$

The covariance matrix of $\hat{\boldsymbol{\Theta}}_{\text{DRSS}}$ is

$$\text{Cov}(\hat{\boldsymbol{\Theta}}_{\text{DRSS}}) = \sigma^2(\boldsymbol{B}'\boldsymbol{\Omega}^{-1}\boldsymbol{B})^{-1}.$$

After some simplification, it is easy to write

$$\hat{\mu}_{\text{DRSS}} = -\boldsymbol{v}'\boldsymbol{\Pi}\boldsymbol{Z} \quad \text{and} \quad \hat{\sigma}_{\text{DRSS}} = \mathbf{1}'\boldsymbol{\Pi}\boldsymbol{Z},$$

where $\boldsymbol{\Pi} = \frac{\boldsymbol{\Omega}^{-1}(\mathbf{1}\boldsymbol{v}'-\boldsymbol{v}\mathbf{1}')\boldsymbol{\Omega}^{-1}}{\Psi}$ and $\Psi = (\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1})(\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}) - (\mathbf{1}'\boldsymbol{\Omega}^{-1}\boldsymbol{v})(\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\mathbf{1})$.
Similarly, the variances and covariance of BLUEs under DRSS, respectively, are

$$\text{Var}(\hat{\mu}_{\text{DRSS}}) = \sigma^2\left(\frac{\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}}{\Psi}\right), \text{Var}(\hat{\sigma}_{\text{DRSS}}) = \sigma^2\left(\frac{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\Psi}\right) \text{ and } \text{Cov}(\hat{\mu}_{\text{DRSS}}, \hat{\sigma}_{\text{DRSS}}) = -\sigma^2\left(\frac{\mathbf{1}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}}{\Psi}\right).$$

The DRSS BLUEs can be written as a linear combination of the independent order statistics obtained from a double ranked set sample, i.e.,

$$\hat{\mu}_{\text{DRSS}} = \sum_{i=1}^{m}\vartheta_i\bar{Z}_{(i)} \quad \text{and} \quad \hat{\sigma}_{\text{DRSS}} = \sum_{i=1}^{m}\varphi_i\bar{Z}_{(i)},$$

where $\vartheta_i = \frac{1}{\omega_{(ii)}\Psi}\{\sum_{i=1}^{m}(v_{(i)}^2/\omega_{(ii)}) - v_{(i)}\sum_{i=1}^{m}(v_{(i)}/\omega_{(ii)})\}$, $\varphi_i = \frac{1}{\omega_{(ii)}\Psi}\{v_{(i)}\sum_{i=1}^{m}(1/\omega_{(ii)}) - \sum_{i=1}^{m}(v_{(i)}/\omega_{(ii)})\}$, $\bar{Z}_{(i)} = \frac{1}{k}\sum_{i=1}^{k}Z_{(i)t}$ and $\Psi = \sum_{i=1}^{m}(v_{(i)}^2/\omega_{(ii)})\sum_{i=1}^{m}(1/\omega_{(ii)}) - \{\sum_{i=1}^{m}(v_{(i)}/\omega_{(ii)})\}^2$.
The variances and covariance of $\hat{\mu}_{\text{DRSS}}$ and $\hat{\sigma}_{\text{DRSS}}$, respectively, are

$$\text{Var}(\hat{\mu}_{\text{DRSS}}) = \frac{\sigma^2}{k\Psi}\sum_{i=1}^{m}\left(v_{(i)}^2/\omega_{(ii)}\right), \ \text{Var}(\hat{\sigma}_{\text{DRSS}}^2) = \frac{\sigma^2}{k\Psi}\sum_{i=1}^{m}\left(1/\omega_{(ii)}\right) \text{ and}$$
$$\text{Cov}(\hat{\mu}_{\text{DRSS}}, \hat{\sigma}_{\text{DRSS}}) = -\frac{\sigma^2}{k\Psi}\sum_{i=1}^{m}\left(v_{(i)}/\omega_{(ii)}\right).$$

Note that if the underlying probability distribution is symmetric, then due to symmetry, we have $\mathbf{1}'\boldsymbol{\Omega}\boldsymbol{v} = -\mathbf{1}'\boldsymbol{\Omega}\boldsymbol{v} = 0$. Then the simplified expressions of the BLUEs are given by

$$\hat{\mu}_{\text{DRSS}}^{\text{s}} = \frac{\mathbf{1}'\boldsymbol{\Omega}^{-1}\boldsymbol{Z}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}, \ \hat{\sigma}_{\text{DRSS}}^{\text{s}} = \frac{\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{Z}}{\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}},$$
$$\text{Var}(\hat{\mu}_{\text{DRSS}}^{\text{s}}) = \frac{\sigma^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}, \ \text{Var}(\hat{\sigma}_{\text{DRSS}}^{\text{s}}) = \frac{\sigma^2}{\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}} \text{ and } \text{Cov}(\hat{\mu}_{\text{DRSS}}^{\text{s}}, \hat{\sigma}_{\text{DRSS}}^{\text{s}}) = 0.$$

For symmetric populations, the expressions of the BLUEs and their variances under DRSS are

$$\hat{\mu}_{\text{DRSS}}^{\text{s}} = \frac{\sum_{i=1}^{m}(\bar{Z}_{(i)}/\omega_{(ii)})}{\sum_{i=1}^{m}(1/\omega_{(ii)})} \quad \text{and} \quad \hat{\sigma}_{\text{DRSS}}^{\text{s}} = \frac{\sum_{i=1}^{m}(v_{(i)}\bar{Z}_{(i)}/\omega_{(ii)})}{\sum_{i=1}^{m}(v_{(i)}^2/\omega_{(ii)})},$$
$$\text{Var}(\hat{\mu}_{\text{DRSS}}^{\text{s}}) = \frac{\sigma^2}{k\sum_{i=1}^{m}(1/\omega_{(ii)})} \quad \text{and} \quad \text{Var}(\hat{\sigma}_{\text{DRSS}}^{\text{s}}) = \frac{\sigma^2}{k\sum_{i=1}^{m}(v_{(i)}^2/\omega_{(ii)})}.$$

4.4.2 BLIEs based on DRSS

Following Mann (1969) and Raqab et al. (2002), in this section, we obtain the BLIEs of location and scale parameters.

Let $\tilde{\mu}_{\text{DRSS}}$ and $\tilde{\sigma}_{\text{DRSS}}$ be the BLIEs of μ and σ under DRSS, respectively, given by

$$\tilde{\mu}_{\text{DRSS}} = \hat{\mu}_{\text{DRSS}} - \hat{\sigma}_{\text{DRSS}}\psi_{12}(1+\psi_{22})^{-1} \quad \text{and} \quad \tilde{\sigma}_{\text{DRSS}} = \hat{\sigma}_{\text{DRSS}}(1+\psi_{22})^{-1},$$

where $\psi_{11} = \sigma^{-2}\text{Var}(\hat{\mu}_{\text{DRSS}})$, $\psi_{12} = \sigma^{-2}\text{Cov}(\hat{\mu}_{\text{DRSS}}, \hat{\sigma}_{\text{DRSS}})$ and $\psi_{22} = \sigma^{-2}\text{Var}(\hat{\sigma}_{\text{DRSS}})$.

The MSEs of $\tilde{\mu}_{\text{DRSS}}$ and $\tilde{\sigma}_{\text{DRSS}}$ are given by

$$\begin{aligned}
\text{MSE}(\tilde{\mu}_{\text{DRSS}}) &= \sigma^2\{\psi_{11} - \psi_{12}^2(1+\psi_{22})^{-1}\}, \\
\text{MSE}(\tilde{\sigma}_{\text{DRSS}}) &= \sigma^2\psi_{22}(1+\psi_{22})^{-1}, \\
E\{(\tilde{\mu}_{\text{DRSS}} - \mu)(\tilde{\sigma}_{\text{DRSS}} - \sigma)\} &= \sigma^2\psi_{12}(1+\psi_{22})^{-1}.
\end{aligned}$$

The simplified expressions of $\tilde{\mu}_{\text{DRSS}}$ and $\tilde{\sigma}_{\text{DRSS}}$ are given by

$$\tilde{\mu}_{\text{DRSS}} = \sum_{i=1}^{m} \varsigma_i \bar{Z}_{(i)} \quad \text{and} \quad \tilde{\sigma}_{\text{DRSS}} = \sum_{i=1}^{m} \zeta_i \bar{Z}_{(i)},$$

where $\varsigma_i = \vartheta_i + \frac{\varphi_i}{\lambda}\sum_{i=1}^{m}(\upsilon_{(i)}/\omega_{(ii)})$, $\zeta_i = \Psi\varphi_i\lambda^{-1}$, $\lambda = k\Psi + \sum_{i=1}^{m}(1/\omega_{(ii)})$.

Similarly, the simplified MSEs of $\tilde{\mu}_{\text{DRSS}}$ and $\tilde{\sigma}_{\text{DRSS}}$ are given by

$$\begin{aligned}
\text{MSE}(\tilde{\mu}_{\text{DRSS}}) &= \frac{\sigma^2}{k\Psi}\left[\sum_{i=1}^{m}(\upsilon_{(i)}^2/\omega_{(ii)}) - \frac{1}{\lambda}\left\{\sum_{i=1}^{m}(\upsilon_{(i)}/\omega_{(ii)})\right\}^2\right], \\
\text{MSE}(\tilde{\sigma}_{\text{DRSS}}) &= \frac{\sigma^2}{\lambda}\sum_{i=1}^{m}(1/\omega_{(ii)}), \\
\text{Cov}(\tilde{\mu}_{\text{DRSS}}, \tilde{\sigma}_{\text{DRSS}}) &= -\frac{k\Psi\sigma^2}{\lambda^2}\sum_{i=1}^{m}(\upsilon_{(i)}/\omega_{(ii)}).
\end{aligned}$$

Since in case of symmetric distribution, we have $\mathbf{1}'\boldsymbol{\Omega}\boldsymbol{\upsilon} = -\mathbf{1}'\boldsymbol{\Omega}\boldsymbol{\upsilon} = \sum_{i=1}^{m}(\upsilon_{(i)}/\omega_{(ii)}) = 0$.

The BLIEs based on DRSS and their MSEs can be further simplified to following forms:

$$\begin{aligned}
&\tilde{\mu}^{\text{s}}_{\text{DRSS}} = \frac{\sum_{i=1}^{m}(\bar{Z}_{(i)}/\omega_{(ii)})}{\sum_{i=1}^{m}(1/\omega_{(ii)})}, \quad \tilde{\sigma}^{\text{s}}_{\text{DRSS}} = \frac{\sum_{i=1}^{m}(\upsilon_{(i)}\bar{Z}_{(i)}/\omega_{(ii)})}{1+\sum_{i=1}^{m}(\upsilon_{(i)}^2/\omega_{(ii)})}, \\
&\text{Var}(\tilde{\mu}^{\text{s}}_{\text{DRSS}}) = \text{Var}(\hat{\mu}^{\text{s}}_{\text{DRSS}}), \quad \text{MSE}(\tilde{\sigma}^{\text{s}}_{\text{DRSS}}) = \frac{\sigma^2}{1+k\sum_{i=1}^{m}(\upsilon_{(i)}^2/\omega_{(ii)})}.
\end{aligned}$$

4.5 BLUEs and BLIEs using DRSS under imperfect ranking

Let Y_1 and Y_2 be the study variable and the auxiliary variable, respectively. Let $E(Y_1) = \mu_{Y_1}$, $E(Y_2) = \mu_{Y_2}$, $\mathrm{Var}(Y_1) = \sigma^2_{Y_1} = \sigma^2$, $\mathrm{Var}(Y_2) = \sigma^2_{Y_2}$ and ρ be the correlation coefficient between Y_1 and Y_2. Consider (Y_{1i}, Y_{2i}), $i = 1, 2, ..., m$, be a random sample from a bivariate normal distribution with location-scale CDF $F^*(y_1, y_2) = F\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}, \frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)$. Suppose $(Z_{1[i]t}, Z_{2(i)t})$, $i = 1, 2, ..., m$, $t = 1, 2, ..., k$, represent a bivariate ranked set sample of size mk from $F^*(y_1, y_2)$, as explained in Section 4.4. If the sample is ordered by the Z_{2it}, the Z_{1it} variate associated with $Z_{2(i)t}$ will be $Z_{1[i]t}$. Here $Z_{1[i]t}$ is the ith concomitant of the ith order statistic $Z_{2(i)t}$ for the tth cycle. Following Samawi and Tawalbeh (2002), we consider a simple linear regression model based on this double ranked set sample of size mk, given by

$$Z_{1[i]t} = \mu_{Y_1} + \rho\frac{\sigma_{Y_1}}{\sigma_{Y_2}}(Z_{2(i)t} - \mu_{Y_2}) + \epsilon_{[i]t}, \quad i = 1, 2, ..., m, \tag{4.4}$$

where $Z_{(2)t}$ and $\epsilon_{[i]t}$ are mutually independent. Here $\epsilon_{[i]t}$ is the particular value of ϵ_{it} associated with $Z_{2(i)t}$. Then, it follows that $E(\epsilon_{it}) = E(\epsilon_{[i]t}) = 0$, $\mathrm{Var}(\epsilon_{it}) = \mathrm{Var}(\epsilon_{[i]t}) = \sigma^2_{Y_1}(1-\rho^2)$.

Let $\upsilon^*_{(i)} = E\left(\frac{Z_{2(i)t}-\mu_{Y_2}}{\sigma_{Y_2}}\right)$, $\omega^*_{(ii)} = \mathrm{Var}\left(\frac{Z_{2(i)t}-\mu_{Y_2}}{\sigma_{Y_2}}\right)$ and $\omega_{(ij)} = \mathrm{Cov}(\frac{Z_{2(i)t}-\mu_{Y_2}}{\sigma_{Y_2}}, \frac{Z_{2(j)t}-\mu_{Y_2}}{\sigma_{Y_2}}) = 0$, for $i \neq j$, $i, j = 1, 2, ..., m$, $t = 1, 2, ..., r$.

Now from (4.4), we have

$$E(Z_{1[i]t}) = \mu_{Y_1} + \sigma_{Y_1}\upsilon^{**}_{(i)},\ \mathrm{Var}(Z_{1[i]t}) = \sigma^2_{Y_1}\omega^{**}_{(ii)},\ \mathrm{Cov}(Z_{1[i]t}, Z_{1[j]t}) = 0 \ \text{ for } i \neq j,$$

where $\upsilon^{**}_{(i)} = \rho\upsilon^*_{(i)}$ and $\omega^{**}_{(ii)} = 1 - \rho^2(1-\omega^*_{(ii)})$.

4.5.1 BLUEs using DRSS under imperfect ranking

Now the results given in Section 4.4 can be extended to the case of imperfect double ranked set sampling (IDRSS). The BLUEs of location and scale parameters under IDRSS are given below:

$$\hat{\mu}_{\mathrm{IDRSS}} = \sum_{i=1}^{m} \vartheta^*_i \bar{Z}_{1[i]} \quad \text{and} \quad \hat{\sigma}_{\mathrm{IDRSS}} = \sum_{i=1}^{m} \varphi^*_i \bar{Z}_{1[i]},$$

where $\vartheta^*_i = \frac{1}{\omega^{**}_{(ii)}\Psi^*}\{\sum_{i=1}^m(\upsilon^{**2}_{(i)}/\omega^{**}_{(ii)}) - \upsilon^{**}_{(i)}\sum_{i=1}^m(\upsilon^{**}_{(i)}/\omega^{**}_{(ii)})\}$, $\varphi^*_i = \frac{1}{\omega^{**}_{(ii)}\Psi^*}\{\upsilon^{**}_{(i)}\sum_{i=1}^m(1/\omega^{**}_{(ii)}) - \sum_{i=1}^m(\upsilon^{**}_{(i)}/\omega^{**}_{(ii)})\}$, $\bar{Z}_{[i]} = \frac{1}{k}\sum_{i=1}^k Z_{[i]}$ and $\Psi^* = \sum_{i=1}^m(\upsilon^{**}_{(i)}/\omega^{**}_{(ii)})\sum_{i=1}^m(1/\omega^{**}_{(ii)}) - \{\sum_{i=1}^m(\upsilon^{**}_{(i)}/\omega^{**}_{(ii)})\}^2$.

The variances and covariance of $\hat{\mu}_{\mathrm{IDRSS}}$ and $\hat{\sigma}_{\mathrm{IDRSS}}$, respectively, are

$$\mathrm{Var}(\hat{\mu}_{\mathrm{IDRSS}}) = \frac{\sigma^2}{k\Psi^*}\sum_{i=1}^{m}(\upsilon^{**2}_{(i)}/\omega^{**}_{(ii)}),\ \mathrm{Var}(\hat{\sigma}_{\mathrm{IDRSS}}) = \frac{\sigma^2}{k\Psi^*}\sum_{i=1}^{m}(1/\omega^{**}_{(ii)}) \quad \text{and}$$

$$\mathrm{Cov}(\hat{\mu}_{\mathrm{IDRSS}}, \hat{\sigma}_{\mathrm{IDRSS}}) = -\frac{\sigma^2}{k\Psi^*}\sum_{i=1}^{m}(\upsilon^{**}_{(i)}/\omega^{**}_{(ii)}).$$

For symmetric populations, we have $\sum_{i=1}^{m}(v_{(i)}^{**}/\omega_{(ii)}) = 0$. The simplified expressions of BLUEs and their variances under IDRSS are

$$\begin{aligned}
\hat{\mu}_{\text{IDRSS}}^{s} &= \frac{\sum_{i=1}^{m}(\bar{Z}_{1[i]}/\omega_{(ii)}^{**})}{\sum_{i=1}^{m}(1/\omega_{(ii)}^{**})}, \quad & \hat{\sigma}_{\text{IDRSS}}^{s} &= \frac{\sum_{i=1}^{m}(v_{(i)}^{**}\bar{Z}_{1[i]}/\omega_{(ii)}^{**})}{\sum_{i=1}^{m}(v_{(ii)}^{**2}/\omega_{(ii)}^{**})}, \\
\text{Var}(\hat{\mu}_{\text{IDRSS}}^{s}) &= \frac{\sigma^2}{k\sum_{i=1}^{m}(1/\omega_{(ii)}^{**})}, \quad & \text{Var}(\hat{\sigma}_{\text{IDRSS}}^{s}) &= \frac{\sigma^2}{k\sum_{i=1}^{m}(v_{(i)}^{**2}/\omega_{(ii)}^{**})}.
\end{aligned}$$

4.5.2 BLIEs using DRSS under imperfect ranking

Following the results give in Section 4.5.1, we obtain BLIEs of location and scale parameters under imperfect ranking using DRSS as follows.

Let $\tilde{\mu}_{\text{IDRSS}}$ and $\tilde{\sigma}_{\text{IDRSS}}$ be the BLIEs of μ and σ under IDRSS, respectively, given by

$$\tilde{\mu}_{\text{IDRSS}} = \hat{\mu}_{\text{IDRSS}} - \hat{\sigma}_{\text{IDRSS}}\psi_{12}^{**}(1+\psi_{22}^{**})^{-1} \quad \text{and} \quad \tilde{\sigma}_{\text{IDRSS}} = \hat{\sigma}_{\text{IDRSS}}(1+\psi_{22}^{**})^{-1},$$

where $\psi_{11}^{**} = \sigma^{-2}\text{Var}(\hat{\mu}_{\text{IDRSS}})$, $\psi_{12}^{**} = \sigma^{-2}\text{Cov}(\hat{\mu}_{\text{IDRSS}}, \hat{\sigma}_{\text{IDRSS}})$ and $\psi_{22}^{**} = \sigma^{-2}\text{Var}(\hat{\sigma}_{\text{IDRSS}})$.

The MSEs of $\tilde{\mu}_{\text{IDRSS}}$ and $\tilde{\sigma}_{\text{IDRSS}}$ are given by

$$\begin{aligned}
\text{MSE}(\tilde{\mu}_{\text{IDRSS}}) &= \sigma^2\{\psi_{11}^{**} - \psi_{12}^{**2}(1+\psi_{22}^{**})^{-1}\}, \\
\text{MSE}(\tilde{\sigma}_{\text{IDRSS}}) &= \sigma^2\psi_{22}^{**}(1+\psi_{22}^{**})^{-1}, \\
E\{(\tilde{\mu}_{\text{IDRSS}} - \mu)(\tilde{\sigma}_{\text{IDRSS}} - \sigma)\} &= \sigma^2\psi_{12}^{**}(1+\psi_{22}^{**})^{-1}.
\end{aligned}$$

The simplified expressions of $\tilde{\mu}_{\text{IDRSS}}$ and $\tilde{\sigma}_{\text{IDRSS}}$ are given by

$$\tilde{\mu}_{\text{IDRSS}} = \sum_{i=1}^{m}\varsigma_i^{**}\bar{Z}_{[i]} \quad \text{and} \quad \tilde{\sigma}_{\text{IDRSS}} = \sum_{i=1}^{m}\zeta_i^{**}\bar{Z}_{[i]},$$

where $\varsigma_i^{**} = \vartheta_i^{*} + \frac{\varphi_i^{*}}{\lambda^{**}}\sum_{i=1}^{m}(v_{(i)}^{**}/\omega_{(ii)}^{**})$, $\zeta_i^{**} = \Psi^{*}\varphi_i^{*}(\lambda^{**})^{-1}$, $\lambda^{**} = k\Psi^{*} + \sum_{i=1}^{m}(1/\omega_{(ii)}^{**})$.

Similarly, the simplified MSEs of $\tilde{\mu}_{\text{IDRSS}}$ and $\tilde{\sigma}_{\text{IDRSS}}$ are given by

$$\begin{aligned}
\text{MSE}(\tilde{\mu}_{\text{IDRSS}}) &= \frac{\sigma^2}{k\Psi^{*}}\left[\sum_{i=1}^{m}(v_{(i)}^{**2}/\omega_{(ii)}^{**}) - \frac{1}{\lambda^{**}}\left\{\sum_{i=1}^{m}(v_{(i)}^{**}/\omega_{(ii)}^{**})\right\}^2\right], \\
\text{MSE}(\tilde{\sigma}_{\text{IDRSS}}) &= \frac{\sigma^2}{\lambda^{**}}\sum_{i=1}^{m}(1/\omega_{(ii)}^{**}), \\
\text{Cov}(\tilde{\mu}_{\text{IDRSS}}, \tilde{\sigma}_{\text{IDRSS}}) &= -\frac{k\Psi^{*}\sigma^2}{\lambda^{**2}}\sum_{i=1}^{m}(v_{(i)}^{**}/\omega_{(ii)}^{**}).
\end{aligned}$$

Since in case of symmetric distribution, we have $\sum_{i=1}^{m}(\upsilon_{(i)}^{**}/\omega_{(ii)}^{**})=0$.

The BLIEs and their MSEs can be further simplified to following forms:

$$\tilde{\mu}_{\mathrm{IDRSS}}^{\mathrm{s}}=\frac{\sum_{i=1}^{m}(\bar{Z}_{1[i]}/\omega_{(ii)}^{**})}{\sum_{i=1}^{m}(1/\omega_{(ii)}^{**})},\quad \tilde{\sigma}_{\mathrm{IDRSS}}^{\mathrm{s}}=\frac{\sum_{i=1}^{m}(\upsilon_{(i)}^{**}\bar{Z}_{1[i]}/\omega_{(ii)}^{**})}{1+\sum_{i=1}^{m}(\upsilon_{(i)}^{**2}/\omega_{(ii)}^{**})},$$

$$\mathrm{Var}(\tilde{\mu}_{\mathrm{IDRSS}}^{\mathrm{s}})=\mathrm{Var}(\hat{\mu}_{\mathrm{IDRSS}}^{\mathrm{s}}),\quad \mathrm{MSE}(\tilde{\sigma}_{\mathrm{IDRSS}}^{\mathrm{s}})=\frac{\sigma^2}{1+k\sum_{i=1}^{m}(\upsilon_{(i)}^{**2}/\omega_{(ii)}^{**})}.$$

4.6 Comparisons between BLUEs and BLIEs based on RSS designs

In this section, we compare BLUEs and BLIEs based on RSS, DRSS and IDRSS for symmetric and asymmetric location-scale families. For this purpose, we present the following relative efficiencies (REs) based on different estimators.

$$\mathrm{RE}_1=\frac{\mathrm{Var}(\hat{\mu}_{\mathrm{RSS}})}{\mathrm{MSE}(\tilde{\mu}_{\mathrm{RSS}})},\quad \mathrm{RE}_2=\frac{\mathrm{Var}(\hat{\sigma}_{\mathrm{RSS}})}{\mathrm{MSE}(\tilde{\sigma}_{\mathrm{RSS}})},\quad \mathrm{RE}_3=\frac{\mathrm{Var}(\hat{\mu}_{\mathrm{DRSS}})}{\mathrm{MSE}(\tilde{\mu}_{\mathrm{DRSS}})},\quad \mathrm{RE}_4=\frac{\mathrm{Var}(\hat{\sigma}_{\mathrm{DRSS}})}{\mathrm{MSE}(\tilde{\sigma}_{\mathrm{DRSS}})},$$
$$\mathrm{RE}_5=\frac{\mathrm{Var}(\hat{\mu}_{\mathrm{RSS}})}{\mathrm{MSE}(\hat{\mu}_{\mathrm{DRSS}})},\quad \mathrm{RE}_6=\frac{\mathrm{Var}(\hat{\sigma}_{\mathrm{RSS}})}{\mathrm{MSE}(\hat{\sigma}_{\mathrm{DRSS}})},\quad \mathrm{RE}_7=\frac{\mathrm{Var}(\tilde{\mu}_{\mathrm{RSS}})}{\mathrm{MSE}(\tilde{\mu}_{\mathrm{DRSS}})},\quad \mathrm{RE}_8=\frac{\mathrm{Var}(\tilde{\sigma}_{\mathrm{RSS}})}{\mathrm{MSE}(\tilde{\sigma}_{\mathrm{DRSS}})},$$
$$\mathrm{RE}_9=\frac{\mathrm{Var}(\hat{\mu}_{\mathrm{IRSS}})}{\mathrm{MSE}(\tilde{\mu}_{\mathrm{IRSS}})},\quad \mathrm{RE}_{10}=\frac{\mathrm{Var}(\hat{\sigma}_{\mathrm{IRSS}})}{\mathrm{MSE}(\tilde{\sigma}_{\mathrm{IRSS}})},\quad \mathrm{RE}_{11}=\frac{\mathrm{Var}(\hat{\mu}_{\mathrm{IDRSS}})}{\mathrm{MSE}(\tilde{\mu}_{\mathrm{IDRSS}})},\quad \mathrm{RE}_{12}=\frac{\mathrm{Var}(\hat{\sigma}_{\mathrm{IDRSS}})}{\mathrm{MSE}(\tilde{\sigma}_{\mathrm{IDRSS}})},$$
$$\mathrm{RE}_{13}=\frac{\mathrm{Var}(\hat{\mu}_{\mathrm{IRSS}})}{\mathrm{MSE}(\hat{\mu}_{\mathrm{IDRSS}})},\quad \mathrm{RE}_{14}=\frac{\mathrm{Var}(\hat{\sigma}_{\mathrm{IRSS}})}{\mathrm{MSE}(\hat{\sigma}_{\mathrm{IDRSS}})},\quad \mathrm{RE}_{15}=\frac{\mathrm{Var}(\tilde{\mu}_{\mathrm{IRSS}})}{\mathrm{MSE}(\tilde{\mu}_{\mathrm{IDRSS}})},\quad \mathrm{RE}_{16}=\frac{\mathrm{Var}(\tilde{\sigma}_{\mathrm{IRSS}})}{\mathrm{MSE}(\tilde{\sigma}_{\mathrm{IDRSS}})}.$$

The REs of these estimators are calculated by considering several location-scale families, which are given below:

(i) Normal (μ,σ)

$$f(y;\mu,\sigma)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\},\ -\infty<y,\mu<\infty,\ \sigma>0.$$

(ii) Logistic (μ,σ)

$$f(y;\mu,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{y-\mu}{\sigma}\right)\left\{1+\exp\left(-\frac{y-\mu}{\sigma}\right)\right\}^{-2},\ -\infty<y,\mu<\infty,\ \sigma>0.$$

(iii) Laplace (μ,σ)

$$f(y;\mu,\sigma)=\frac{1}{2\sigma}\exp\left(-\frac{|y-\mu|}{\sigma}\right),\ -\infty<y,\mu<\infty,\ \sigma>0.$$

(iv) Extreme value (μ,σ)

$$f(y;\mu,\sigma)=\frac{1}{\sigma}\exp\left\{-\exp\left(-\frac{y-\mu}{\sigma}\right)-\frac{y-\mu}{\sigma}\right\},\ -\infty<y,\mu<\infty,\ \sigma>0.$$

(v) Weibull (α,μ,σ)

$$f(y;\alpha,\mu,\sigma)=\frac{\alpha}{\sigma}\left(\frac{y-\mu}{\sigma}\right)^{\alpha-1}\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)^{\alpha}\right\},\ y>\mu,\ -\infty<\mu<\infty,\ \sigma>0.$$

Table 4.1: Means of the order statistics from different sample sizes under RSS and DRSS

		$m=2$		$m=3$		$m=4$		$m=5$	
Distribution	i	RSS	DRSS	RSS	DRSS	RSS	DRSS	RSS	DRSS
Normal	1	−0.5642	−0.6632	−0.8463	−0.9646	−1.0294	−1.1525	−1.1630	−1.2870
(0, 1)	2	0.5642	0.6632	0.0000	0.0000	−0.2970	−0.3233	−0.4950	−0.5316
	3	——	——	0.8463	0.9646	0.2970	0.3233	0.0000	0.0000
	4	——	——	——	——	1.0294	1.1525	0.4950	0.5316
	5	——	——	——	——	——	——	1.1630	1.2870
Logistic	1	−1.0000	−1.1667	−1.5000	−1.7000	−1.8333	−2.0461	−2.0833	−2.3027
(0, 1)	2	1.0000	1.1667	0.0000	0.0000	−0.5000	−0.5331	−0.8333	−0.8788
	3	——	——	1.5000	1.7000	0.5000	0.5331	0.0000	0.0000
	4	——	——	——	——	1.8333	2.0461	0.8333	0.8788
	5	——	——	——	——	——	——	2.0833	2.3027
Laplace	1	−0.7500	−0.8646	−1.1250	−1.2641	−1.3854	−1.5413	−1.5885	−1.7575
(0, 1)	2	0.7500	0.8646	0.0000	0.0000	−0.3438	−0.3465	−0.5729	−0.5761
	3	——	——	1.1250	1.2641	0.3438	0.3465	0.0000	0.0000
	4	——	——	——	——	1.3854	1.5413	0.5729	0.5761
	5	——	——	——	——	——	——	1.5885	1.7575
Extreme value	1	−0.1159	−0.2337	−0.4036	−0.5267	−0.5735	−0.6918	−0.6902	−0.8029
(0, 1)	2	1.2704	1.3882	0.4594	0.4234	0.1061	0.0491	−0.1069	−0.1672
	3	——	——	1.6758	1.8349	0.8128	0.8086	0.4256	0.3945
	4	——	——	——	——	1.9635	2.1430	1.0709	1.0835
	5	——	——	——	——	——	——	2.1867	2.3783
Weibull	1	0.7088	0.6758	0.6192	0.5805	0.5625	0.5230	0.5222	0.4831
(3, 0, 5)	2	1.0772	1.1102	0.8880	0.8867	0.7890	0.7783	0.7238	0.7095
	3	——	——	1.1718	1.2118	0.9869	0.9954	0.8868	0.8858
	4	——	——	——	——	1.2335	1.2752	1.0537	1.0661
	5	——	——	——	——	——	——	1.2784	1.3204
Gamma	1	0.5000	0.4167	0.3333	0.2599	0.2500	0.1887	0.2000	0.1481
(1, 1, 0, 1)	2	1.5000	1.5833	0.8333	0.7802	0.5833	0.5217	0.4500	0.3930
	3	——	——	1.8333	1.9599	1.0833	1.0548	0.7833	0.7364
	4	——	——	——	——	2.0833	2.2348	1.2833	1.2718
	5	——	——	——	——	——	——	2.2833	2.4507
Gamma	1	1.2500	1.1212	0.9630	0.8366	0.8047	0.6899	0.7021	0.5979
(2, 1, 0, 3)	2	2.7500	2.8788	1.8241	1.7685	1.4378	1.3600	1.2151	1.1365
	3	——	——	3.2130	3.3949	2.2104	2.1939	1.7718	1.7230
	4	——	——	——	——	3.5472	3.7562	2.5027	2.5092
	5	——	——	——	——	——	——	3.8083	4.0334

(vi) Gamma $(\alpha, \beta, \mu, \sigma)$

$$f(y; \alpha, \beta, \mu, \sigma) = \frac{\beta}{\Gamma(\alpha)\sigma} \left(\frac{y-\mu}{\sigma}\right)^{\alpha\beta-1} \exp\left\{-\left(\frac{y-\mu}{\sigma}\right)^{\beta}\right\}, \; y > \mu, \; -\infty < \mu < \infty, \; \sigma > 0.$$

The values considered for the sample size m are 2, 3, 4, 5 each with $k = 1$. In Tables 4.1–4.2, means and variances of the order statistics under RSS and DRSS are given for different values of m. The values of the coefficients, i.e., ϑ_i and φ_i, needed for computing the BLUEs of μ and σ under DRSS are given in Table 4.3. The exact REs of BLUEs and BLIEs are reported in Table 4.4.

From Table 4.4, it is observed that for symmetric populations, under RSS or DRSS, the BLUEs and BLIEs of the location parameters are equally efficient whereas in estimation of scale parameters, the BLIEs are more efficient than the BLUEs. In case of asymmetric distributions, for estimation of the location and scale parameters, BLIEs perform better than the BLUEs. If the estimators are compared with respect to the

Table 4.2: Variances of the order statistics from different sample sizes under RSS and DRSS

		$m = 2$		$m = 3$		$m = 4$		$m = 5$	
Distribution	i	RSS	DRSS	RSS	DRSS	RSS	DRSS	RSS	DRSS
Normal	1	0.6817	0.5602	0.5595	0.4313	0.4917	0.3690	0.4475	0.3313
$(0, 1)$	2	0.6817	0.5602	0.4487	0.2767	0.3605	0.1982	0.3115	0.1615
	3	———	———	0.5595	0.4313	0.3605	0.1982	0.2868	0.1366
	4	———	———	———	———	0.4917	0.3690	0.3115	0.1615
	5	———	———	———	———	———	———	0.4475	0.3313
Logistic	1	2.2899	1.9288	2.0399	1.6646	1.9288	1.5574	1.8663	1.4997
$(0, 1)$	2	2.2899	1.9288	1.2899	0.7605	1.0399	0.5516	0.9288	0.4698
	3	———	———	2.0399	1.6646	1.0399	0.5516	0.7899	0.3613
	4	———	———	———	———	1.9288	1.5574	0.9288	0.4698
	5	———	———	———	———	———	———	1.8663	1.4997
Laplace	1	1.4375	1.2525	1.4149	1.2341	1.4417	1.2494	1.4703	1.2616
$(0, 1)$	2	1.4375	1.2525	0.6389	0.3362	0.5207	0.2549	0.5025	0.2486
	3	———	———	1.4149	1.2341	0.5207	0.2549	0.3512	0.1378
	4	———	———	———	———	1.4417	1.2494	0.5025	0.2486
	5	———	———	———	———	———	———	1.4703	1.2616
Extreme value	1	0.6840	0.4952	0.4485	0.3003	0.3440	0.2248	0.2849	0.1846
$(0, 1)$	2	1.6449	1.4794	0.6585	0.3910	0.4155	0.2148	0.3085	0.1497
	3	———	———	1.6449	1.4193	0.6518	0.3556	0.4060	0.1869
	4	———	———	———	———	1.6449	1.3901	0.6491	0.3382
	5	———	———	———	———	———	———	1.6449	1.3730
Weibull	1	0.7088	0.0531	0.6192	0.0371	0.5625	0.0293	0.5222	0.0246
$(3, 0, 5)$	2	1.0772	0.0632	0.8880	0.0311	0.7890	0.0215	0.7238	0.0169
	3	———	———	1.1718	0.0485	0.9869	0.0228	0.8868	0.0156
	4	———	———	———	———	1.2335	0.0411	1.0537	0.0186
	5	———	———	———	———	———	———	1.2784	0.0366
Gamma	1	0.2500	0.1458	0.1111	0.0521	0.0625	0.0262	0.0400	0.0157
$(1, 1, 0, 1)$	2	1.2500	1.1736	0.3611	0.2054	0.1736	0.0802	0.1025	0.0424
	3	———	———	1.3611	1.2250	0.4236	0.2306	0.2136	0.0941
	4	———	———	———	———	1.4236	1.2483	0.4636	0.2441
	5	———	———	———	———	———	———	1.4636	1.2615
Gamma	1	0.6875	0.4578	0.3813	0.2209	0.2548	0.1389	0.1879	0.0992
$(2, 1, 0, 3)$	2	2.1875	1.9978	0.8055	0.4788	0.4603	0.2320	0.3119	0.1448
	3	———	———	2.2355	1.9478	0.8522	0.4713	0.4970	0.2288
	4	———	———	———	———	2.2499	1.9101	0.8753	0.4632
	5	———	———	———	———	———	———	2.2526	1.8821

sampling design, then it is worth mentioning that all of the proposed estimators (BLUEs and BLIEs) under DRSS are having high precision than the estimators with RSS design.

Similarly, from Table 4.5, it is clear that for all values of ρ, under both IRSS and IDRSS, the BLUEs and BLIEs are equivalent in estimation of the location parameter. When estimating the scale parameters, the BLIEs outperform BLUEs for all values of m and ρ. In comparison of estimators with respect to the sampling schemes, i.e., IRSS versus IDRSS, the BLUEs are efficient than their competitors for all values of ρ, but for small values of ρ, the REs converge to unity. Furthermore, the BLIEs under IDRSS are uniformly better than the BLIEs based on IRSS for all cases considered here.

A simulation study is conducted in order to study the robustness of the BLUEs and BLIEs under bivariate normal distribution. Let ρ_a be the correlation coefficient between the study variable and the auxiliary variable. Here, the assumed values of ρ_a are 0.25, 0.50, 0.75 and 1. For brevity, we consider $m = 5$. The main steps involved in the simulation approach are as follows: for a given value of m and ρ_a, the coefficients of both

Table 4.3: The values of coefficients needed for computing the BLUEs of μ and σ under DRSS

		$m=2$		$m=3$		$m=4$		$m=5$	
Distribution	i	ϑ_i	φ_i	ϑ_i	φ_i	ϑ_i	φ_i	ϑ_i	φ_i
Normal	1	0.5000	−0.7539	0.2810	−0.5184	0.1747	−0.3784	0.1173	−0.2878
$(0,1)$	2	0.5000	0.7539	0.4380	0.0000	0.3253	−0.1976	0.2406	−0.2439
	3	———	———	0.2810	0.5184	0.3253	0.1976	0.2843	0.0000
	4	———	———	———	———	0.1747	0.3784	0.2406	0.2439
	5	———	———	———	———	———	———	0.1173	0.2878
Logistic	1	0.5000	−0.4286	0.2387	−0.2941	0.1308	−0.2051	0.0798	−0.1482
$(0,1)$	2	0.5000	0.4286	0.5225	0.0000	0.3692	−0.1509	0.2547	−0.1806
	3	———	———	0.2387	0.2941	0.3692	0.1509	0.3311	0.0000
	4	———	———	———	———	0.1308	0.2051	0.2547	0.1806
	5	———	———	———	———	———	———	0.0798	0.1482
Laplace	1	0.5000	−0.5783	0.1763	-0.3956	0.0847	−0.2600	0.0469	−0.1841
$(0,1)$	2	0.5000	0.5783	0.6473	0.0000	0.4153	−0.2865	0.2382	−0.3062
	3	———	———	0.1763	0.3956	0.4153	0.2865	0.4298	0.0000
	4	———	———	———	———	0.0847	0.2600	0.2382	0.3062
	5	———	———	———	———	———	———	0.0469	0.1841
Extreme value	1	0.8559	−0.6166	0.5579	−0.5595	0.3883	−0.4741	0.2860	−0.3967
$(0,1)$	2	0.1441	0.6166	0.3665	0.2277	0.3698	−0.0177	0.3262	−0.1349
	3	———	———	0.0756	0.3318	0.2007	0.2856	0.2425	0.1428
	4	———	———	———	———	0.0412	0.2062	0.1213	0.2489
	5	———	———	———	———	———	———	0.0240	0.1400
Weibull	1	2.5556	−2.3020	1.7715	−1.6529	1.3297	−1.2718	1.0466	−1.0188
$(3,0,5)$	2	−1.5556	2.3020	0.2873	0.1338	0.6910	−0.4380	0.7605	−0.6009
	3	———	———	−1.0587	1.5190	−0.2440	0.6229	0.1843	0.0903
	4	———	———	———	———	−0.7767	1.0870	−0.3955	0.7109
	5	———	———	———	———	———	———	-0.5959	0.8184
Gamma	1	1.3571	−0.8571	1.1844	−0.9888	1.0540	−1.0279	0.9550	−1.0247
$(1,1,0,1)$	2	−0.3571	0.8571	−0.0454	0.5772	0.1096	0.3173	0.1847	0.1377
	3	———	———	−0.1390	0.4116	−0.0929	0.4742	−0.0241	0.3893
	4	———	———	———	———	-0.0707	0.2363	−0.0738	0.3468
	5	———	———	———	———	———	———	−0.0418	0.1509
Gamma	1	1.6380	−0.5690	1.3622	−0.5620	1.1524	−0.5222	0.9927	−0.4761
$(2,1,0,3)$	2	−0.6380	0.5690	−0.0553	0.2692	0.2142	0.0733	0.3314	−0.0367
	3	———	———	−0.3069	0.2929	−0.1860	0.2724	−0.0307	0.1764
	4	———	———	———		−0.1806	0.1765	−0.1744	0.2193
	5	———	———	———	———	———	———	−0.1189	0.1170

BLUEs and BLIEs have been calculated. Then, imperfect ranked and imperfect double ranked set samples are drawn from standard bivariate normal distribution by assuming different values of ρ, i.e., $\rho = 0.1$, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8, 0.9 and 1. For each sample, the values of BLUEs and BLIEs have been calculated. This process is repeated 10^6 times and the estimated MSEs (EMSEs) are calculated. The EMSE of any estimator, say $\hat{E}$, of parameter E is $\text{EMSE}(\hat{E}) = \frac{1}{10^6}(\hat{E}_i - E)^2$. The EMSEs of both BLUEs and BLIEs based on different values of ρ_a are plotted again ρ in Figure 4.1.

Table 4.4: REs of BLUEs and BLIEs under RSS and DRSS

Distribution	m	RE_1 $(\tilde{\mu}_{\text{RSS}}, \hat{\mu}_{\text{RSS}})$	RE_2 $(\tilde{\sigma}_{\text{RSS}}, \hat{\sigma}_{\text{RSS}})$	RE_3 $(\tilde{\mu}_{\text{DRSS}}, \hat{\mu}_{\text{DRSS}})$	RE_4 $(\tilde{\sigma}_{\text{DRSS}}, \hat{\sigma}_{\text{DRSS}})$	RE_5 $(\hat{\mu}_{\text{DRSS}}, \hat{\mu}_{\text{RSS}})$	RE_6 $(\hat{\sigma}_{\text{DRSS}}, \hat{\sigma}_{\text{RSS}})$	RE_7 $(\tilde{\mu}_{\text{DRSS}}, \tilde{\mu}_{\text{RSS}})$	RE_8 $(\tilde{\sigma}_{\text{DRSS}}, \tilde{\sigma}_{\text{RSS}})$
Normal	2	1.0000	2.0708	1.0000	1.6368	1.2169	1.6815	1.2169	1.3291
$(0, 1)$	3	1.0000	1.3906	1.0000	1.2318	1.4218	1.6851	1.4218	1.4927
	4	1.0000	1.2084	1.0000	1.1211	1.6132	1.7200	1.6132	1.5958
	5	1.0000	1.1313	1.0000	1.0741	1.7906	1.7721	1.7906	1.6825
Logistic	2	1.0000	2.1449	1.0000	1.7085	1.1872	1.6159	1.1872	1.2872
$(0, 1)$	3	1.0000	1.4533	1.0000	1.2880	1.4333	1.5741	1.4333	1.3950
	4	1.0000	1.2522	1.0000	1.1561	1.6586	1.6154	1.6586	1.4915
	5	1.0000	1.1627	1.0000	1.0965	1.8611	1.6853	1.8611	1.5894
Laplace	2	1.0000	2.2778	1.0000	1.8378	1.1477	1.5252	1.1477	1.2306
$(0, 1)$	3	1.0000	1.5590	1.0000	1.3862	1.5427	1.4475	1.5427	1.2871
	4	1.0000	1.3209	1.0000	1.2108	1.8072	1.5226	1.8072	1.3956
	5	1.0000	1.2110	1.0000	1.1322	2.0624	1.5966	2.0624	1.4926
Extreme value	2	1.1063	2.2119	1.0251	1.7507	1.4889	1.6143	1.3796	1.2777
$(0, 1)$	3	1.0202	1.4468	1.0027	1.2706	1.5924	1.6515	1.5653	1.4503
	4	1.0063	1.2382	1.0012	1.1387	1.7346	1.7171	1.7258	1.5792
	5	1.0027	1.1495	1.0007	1.0834	1.8801	1.7916	1.8766	1.6886
Weibull	2	1.9655	2.0519	1.5608	1.6163	1.7014	1.7067	1.3510	1.3444
$(3, 0, 5)$	3	1.3415	1.3711	1.1952	1.2137	1.7380	1.7364	1.5485	1.5371
	4	1.1781	1.1938	1.0998	1.1089	1.7852	1.7789	1.6666	1.6525
	5	1.1104	1.1203	1.0603	1.0657	1.8399	1.8308	1.7571	1.7416
Gamma	2	1.8421	2.5000	1.5142	1.9694	2.0918	1.5474	1.7195	1.2189
$(1, 1, 0, 1)$	3	1.2993	1.5403	1.1688	1.3269	2.3990	1.6529	2.1581	1.4239
	4	1.1498	1.2808	1.0780	1.1574	2.5895	1.7842	2.4277	1.6123
	5	1.0881	1.1720	1.0427	1.0897	2.7205	1.9182	2.6070	1.7834
Gamma	2	1.9393	2.2778	1.5676	1.7950	1.8762	1.6073	1.5166	1.2666
$(2, 1, 0, 3)$	3	1.3378	1.4575	1.1950	1.2715	1.9947	1.6850	1.7818	1.4700
	4	1.1743	1.2382	1.0958	1.1336	2.0712	1.7830	1.9328	1.6324
	5	1.1061	1.1466	1.0558	1.0779	2.1328	1.8830	2.0358	1.7701

Table 4.5: REs of BLUEs and BLIEs under IRSS versus IDRSS

		RE_9	RE_{10}	RE_{11}	RE_{12}	RE_{13}	RE_{14}	RE_{15}	RE_{16}
ρ	m	$(\tilde{\mu}_{\mathrm{IRSS}}, \hat{\mu}_{\mathrm{IRSS}})$	$(\tilde{\sigma}_{\mathrm{IRSS}}, \hat{\sigma}_{\mathrm{IRSS}})$	$(\tilde{\mu}_{\mathrm{IDRSS}}, \hat{\mu}_{\mathrm{IDRSS}})$	$(\tilde{\sigma}_{\mathrm{IDRSS}}, \hat{\sigma}_{\mathrm{IDRSS}})$	$(\hat{\mu}_{\mathrm{IDRSS}}, \hat{\mu}_{\mathrm{IRSS}})$	$(\hat{\sigma}_{\mathrm{IDRSS}}, \hat{\sigma}_{\mathrm{IRSS}})$	$(\tilde{\mu}_{\mathrm{IDRSS}}, \tilde{\mu}_{\mathrm{IRSS}})$	$(\tilde{\sigma}_{\mathrm{IDRSS}}, \tilde{\sigma}_{\mathrm{IRSS}})$
±0.9	2	1.0000	2.4392	1.0000	1.9085	1.1529	1.5930	1.1529	1.2431
	3	1.0000	1.5543	1.0000	1.3628	1.2454	1.5491	1.2454	1.3533
	4	1.0000	1.3111	1.0000	1.2088	1.2977	1.5196	1.2977	1.4013
	5	1.0000	1.2056	1.0000	1.1408	1.3249	1.4972	1.3249	1.4124
±0.8	2	1.0000	2.9544	1.0000	2.2963	1.1082	1.5313	1.1082	1.1798
	3	1.0000	1.7833	1.0000	1.5541	1.1561	1.4666	1.1561	1.2616
	4	1.0000	1.4543	1.0000	1.3365	1.1753	1.4204	1.1753	1.2891
	5	1.0000	1.3088	1.0000	1.2370	1.1808	1.3853	1.1808	1.2944
±0.7	2	1.0000	3.7057	1.0000	2.8650	1.0759	1.4866	1.0759	1.1313
	3	1.0000	2.1172	1.0000	1.8361	1.1023	1.4121	1.1023	1.1947
	4	1.0000	1.6628	1.0000	1.5251	1.1100	1.3608	1.1100	1.2170
	5	1.0000	1.4588	1.0000	1.3791	1.1103	1.3232	1.1103	1.2216
±0.6	2	1.0000	4.8633	1.0000	3.7378	1.0520	1.4536	1.0520	1.0933
	3	1.0000	2.6317	1.0000	2.2672	1.0669	1.3744	1.0669	1.1423
	4	1.0000	1.9840	1.0000	1.8124	1.0700	1.3218	1.0700	1.1622
	5	1.0000	1.6896	1.0000	1.5952	1.0690	1.2841	1.0690	1.1681
±0.5	2	1.0000	6.7832	1.0000	5.1723	1.0341	1.4289	1.0341	1.0632
	3	1.0000	3.4850	1.0000	2.9691	1.0425	1.3476	1.0425	1.0997
	4	1.0000	2.5165	1.0000	2.2777	1.0437	1.2949	1.0437	1.1172
	5	1.0000	2.0723	1.0000	1.9440	1.0426	1.2579	1.0426	1.1245
±0.3	2	1.0000	17.9533	1.0000	13.3763	1.0114	1.3975	1.0114	1.0221
	3	1.0000	8.4495	1.0000	6.9107	1.0136	1.3148	1.0136	1.0373
	4	1.0000	5.6142	1.0000	4.8633	1.0137	1.2632	1.0137	1.0469
	5	1.0000	4.2981	1.0000	3.8683	1.0132	1.2277	1.0132	1.0530
±0.1	2	1.0000	157.5800	1.0000	114.5860	1.0012	1.3834	1.0012	1.0024
	3	1.0000	70.5056	1.0000	54.8413	1.0014	1.3007	1.0014	1.0043
	4	1.0000	44.3347	1.0000	36.0210	1.0014	1.2499	1.0014	1.0056
	5	1.0000	32.1189	1.0000	26.9125	1.0014	1.2152	1.0014	1.0067

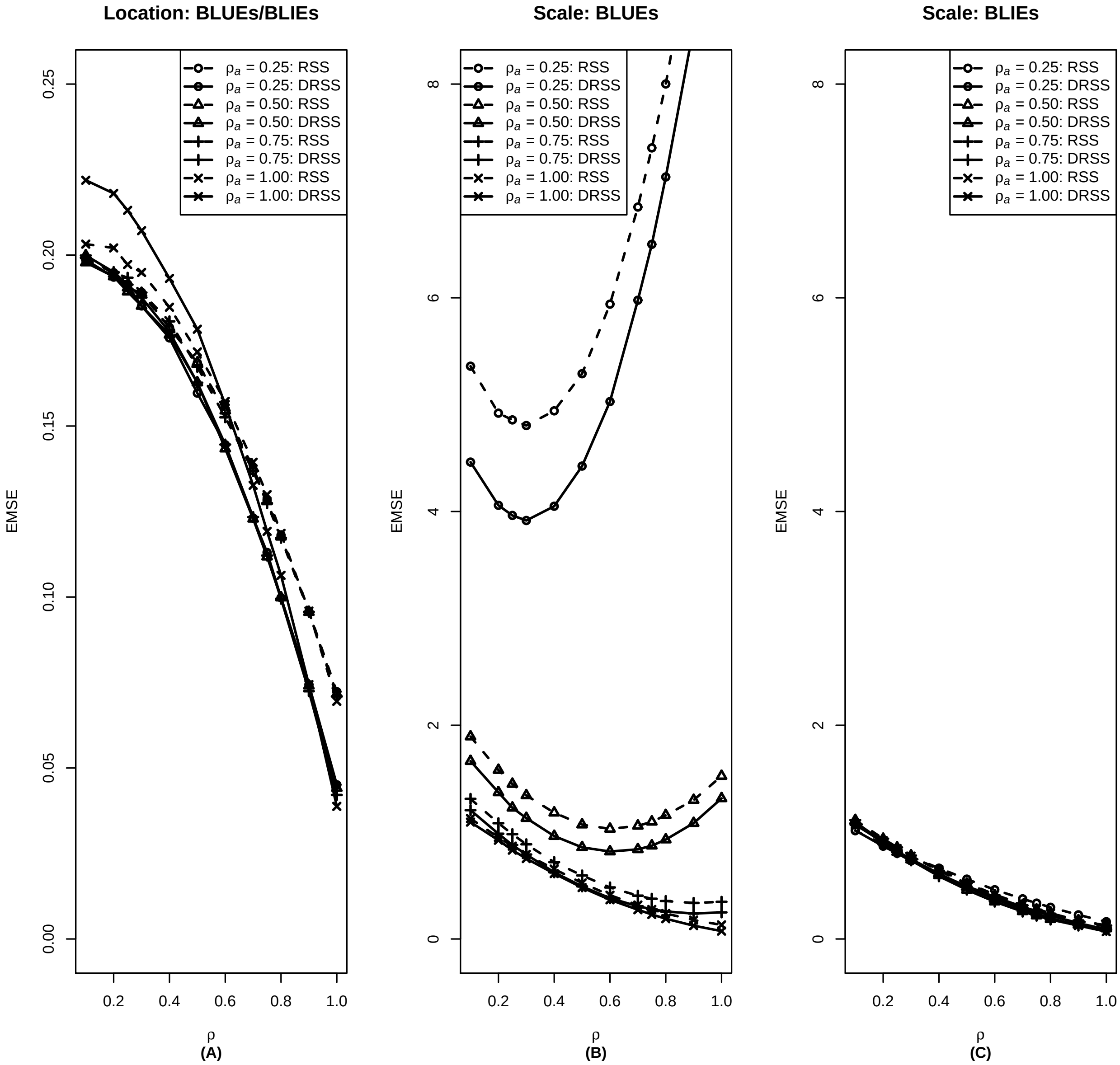

Figure 4.1: Comparison of EMSEs of BLUEs and BLIEs based on IRSS versus IDRSS when $m = 5$

Table 4.6: EREs of BLUEs and BLIEs under IRSS and IDRSS

Distribution	σ_ϵ^2	m	RE_9 $(\tilde{\mu}_{\mathrm{RSS}}, \hat{\mu}_{\mathrm{RSS}})$	RE_{10} $(\tilde{\sigma}_{\mathrm{RSS}}, \hat{\sigma}_{\mathrm{RSS}})$	RE_{11} $(\tilde{\mu}_{\mathrm{DRSS}}, \hat{\mu}_{\mathrm{DRSS}})$	RE_{12} $(\tilde{\sigma}_{\mathrm{DRSS}}, \hat{\sigma}_{\mathrm{DRSS}})$	RE_{13} $(\hat{\mu}_{\mathrm{DRSS}}, \hat{\mu}_{\mathrm{RSS}})$	RE_{14} $(\hat{\sigma}_{\mathrm{DRSS}}, \hat{\sigma}_{\mathrm{RSS}})$	RE_{15} $(\tilde{\mu}_{\mathrm{DRSS}}, \tilde{\mu}_{\mathrm{RSS}})$	RE_{16} $(\tilde{\sigma}_{\mathrm{DRSS}}, \tilde{\sigma}_{\mathrm{RSS}})$
Normal	1	2	1.0000	1.8985	1.0000	1.4846	1.0730	1.4455	1.0730	1.1303
$(0,1)$	1	3	1.0000	1.2039	1.0000	1.0792	1.0727	1.3399	1.0727	1.2011
	1	4	1.0000	1.0396	1.0000	0.9968	1.0570	1.2547	1.0570	1.2030
	1	5	1.0000	0.9825	1.0000	0.9742	1.0543	1.1727	1.0543	1.1627
	3	2	1.0000	1.8601	1.0000	1.4649	1.0335	1.3501	1.0335	1.0633
	3	3	1.0000	1.1907	1.0000	1.0820	1.0109	1.2168	1.0109	1.1057
	3	4	1.0000	1.0445	1.0000	1.0090	0.9824	1.1423	0.9824	1.1035
	3	5	1.0000	0.9973	1.0000	0.9908	0.9729	1.0835	0.9729	1.0764
Logistic	1	2	1.0000	2.0517	1.0000	1.6122	1.1197	1.5219	1.1197	1.1959
$(0,1)$	1	3	1.0000	1.3328	1.0000	1.1747	1.1896	1.4316	1.1896	1.2618
	1	4	1.0000	1.1246	1.0000	1.0542	1.2101	1.3474	1.2101	1.2630
	1	5	1.0000	1.0404	1.0000	1.0049	1.2260	1.2975	1.2260	1.2533
	3	2	1.0000	1.9654	1.0000	1.5558	1.0829	1.4314	1.0829	1.1331
	3	3	1.0000	1.2626	1.0000	1.1293	1.0575	1.3077	1.0575	1.1696
	3	4	1.0000	1.0767	1.0000	1.0278	1.0571	1.1993	1.0571	1.1448
	3	5	1.0000	1.0063	1.0000	0.9936	1.0618	1.1339	1.0618	1.1196
Laplace	1	2	1.0000	2.1145	1.0000	1.7127	1.0464	1.3821	1.0464	1.1195
$(0,1)$	1	3	1.0000	1.3885	1.0000	1.2343	1.1240	1.3106	1.1240	1.1650
	1	4	1.0000	1.1641	1.0000	1.0950	1.1552	1.1887	1.1552	1.1182
	1	5	1.0000	1.0648	1.0000	1.0334	1.1340	1.1449	1.1340	1.1112
	3	2	1.0000	2.0688	1.0000	1.6612	1.0319	1.3440	1.0319	1.0792
	3	3	1.0000	1.3255	1.0000	1.1889	0.9808	1.2353	0.9808	1.1080
	3	4	1.0000	1.1303	1.0000	1.0892	0.9994	1.0735	0.9994	1.0344
	3	5	1.0000	1.0444	1.0000	1.0349	0.9902	1.0387	0.9902	1.0293
Extreme	1	2	1.0872	2.0553	1.0152	1.6148	1.2802	1.4543	1.1954	1.1427
Value	1	3	1.0003	1.2783	0.9940	1.1310	1.2252	1.3437	1.2174	1.1889
$(0,1)$	1	4	0.9904	1.0816	0.9940	1.0182	1.1803	1.2537	1.1846	1.1802
	1	5	0.9903	1.0025	0.9948	0.9818	1.1348	1.1859	1.1400	1.1615
	3	2	1.1007	2.0047	1.0230	1.5739	1.2167	1.3735	1.1309	1.0784
	3	3	1.0048	1.2599	0.9954	1.1310	1.1307	1.2246	1.1201	1.0993
	3	4	0.9931	1.0736	0.9950	1.0296	1.1002	1.1462	1.1023	1.0993
	3	5	0.9920	1.0101	0.9956	0.9985	1.0714	1.0786	1.0752	1.0661

Table 4.6: (Continued).

Distribution	σ_ϵ^2	m	RE$_9$ ($\tilde{\mu}_{\text{RSS}}, \hat{\mu}_{\text{RSS}}$)	RE$_{10}$ ($\tilde{\sigma}_{\text{RSS}}, \hat{\sigma}_{\text{RSS}}$)	RE$_{11}$ ($\tilde{\mu}_{\text{DRSS}}, \hat{\mu}_{\text{DRSS}}$)	RE$_{12}$ ($\tilde{\sigma}_{\text{DRSS}}, \hat{\sigma}_{\text{DRSS}}$)	RE$_{13}$ ($\hat{\mu}_{\text{DRSS}}, \hat{\mu}_{\text{RSS}}$)	RE$_{14}$ ($\hat{\sigma}_{\text{DRSS}}, \hat{\sigma}_{\text{RSS}}$)	RE$_{15}$ ($\tilde{\mu}_{\text{DRSS}}, \tilde{\mu}_{\text{RSS}}$)	RE$_{16}$ ($\tilde{\sigma}_{\text{DRSS}}, \tilde{\sigma}_{\text{RSS}}$)
Weibull	1	2	1.8531	1.9305	1.4631	1.5118	1.5643	1.5733	1.2350	1.2320
$(3, 0, 5)$	1	3	1.2112	1.2338	1.0875	1.0988	1.4846	1.4890	1.3330	1.3261
	1	4	1.0521	1.0601	0.9981	1.0012	1.4166	1.4200	1.3439	1.3411
	1	5	0.9932	0.9962	0.9715	0.9716	1.3696	1.3767	1.3396	1.3426
	3	2	1.8092	1.8798	1.4266	1.4699	1.4351	1.4450	1.1316	1.1299
	3	3	1.1661	1.1837	1.0595	1.0688	1.3173	1.3248	1.1968	1.1963
	3	4	1.0231	1.0283	0.9891	0.9914	1.2410	1.2418	1.1998	1.1973
	3	5	0.9755	0.9767	0.9708	0.9709	1.1762	1.1769	1.1705	1.1700
Gamma	1	2	1.5957	2.3082	1.2661	1.7793	1.5417	1.4002	1.2233	1.0794
$(1, 1, 0, 1)$	1	3	1.1194	1.3830	1.0374	1.2211	1.3145	1.1963	1.2182	1.0562
	1	4	1.0161	1.1424	0.9991	1.0813	1.1803	1.0645	1.1606	1.0076
	1	5	0.9889	1.0557	0.9909	1.0310	1.1278	1.0105	1.1301	0.9868
	3	2	1.5513	2.2663	1.2621	1.7834	1.3855	1.3100	1.1272	1.0309
	3	3	1.1356	1.3948	1.0596	1.2539	1.2025	1.1013	1.1220	0.9900
	3	4	1.0404	1.1713	1.0175	1.1134	1.1192	0.9971	1.0945	0.9478
	3	5	1.0124	1.0874	1.0044	1.0568	1.1062	0.9732	1.0974	0.9457
Gamma	1	2	1.8857	2.2268	1.5113	1.7549	1.8107	1.5648	1.4512	1.2331
$(2, 1, 0, 3)$	1	3	1.2783	1.4041	1.1333	1.2204	1.8409	1.6054	1.6321	1.3954
	1	4	1.1099	1.1783	1.0333	1.0784	1.8264	1.6362	1.7004	1.4975
	1	5	1.0373	1.0835	0.9929	1.0181	1.7670	1.6444	1.6914	1.5452
	3	2	1.8317	2.1929	1.4751	1.7404	1.7269	1.5202	1.3907	1.2065
	3	3	1.2205	1.3562	1.0839	1.1804	1.6510	1.4859	1.4662	1.2932
	3	4	1.0536	1.1261	0.9929	1.0374	1.5520	1.4455	1.4626	1.3316
	3	5	0.9895	1.0349	0.9644	0.9858	1.4648	1.3894	1.4277	1.3236

In Figure 4.1, sub-figure A compares the BLUEs of μ based on IRSS and IDRSS schemes. Note that in symmetric populations, the BLUE and BLIE of μ are equivalent. It is observed that, for a given value of ρ_a, as the value of ρ increases, the EMSEs of both BLUEs decrease and vice-versa. For small values of ρ_a, the BLUEs under IDRSS are more efficient than the IRSS based BLUEs for different values of ρ. Sub-figure B compares the BLUEs of σ under IRSS and IDRSS. It is clear that the BLUEs under IDRSS dominate the BLUEs based on IRSS for all values of ρ. Generally, for large values of ρ_a, i.e., $\rho_a \geq 0.75$, the EMSEs of BLUEs decrease as the value of ρ increases. However, their behavior is different for small and moderate values of ρ_a, i.e., $\rho_a \leq 0.5$. In sub-figure C, IRSS based BLIEs of σ are compared with the BLIEs obtained under IDRSS. The BLIEs under IDRSS are generally more efficient than the BLIEs with IRSS. Comparing the results of sub-figures B and C, it is evident that generally the scale BLIEs are more robust than the scale BLUEs under both IRSS schemes.

We also study the robustness of both BLUEs and BLIEs when the underlying population is not bivariate normal. Following Dell and Clutter (1972) and Zheng and Al-Saleh (2003), another simulation study is conducted in order to examine the effect of judgment error on the performance of BLUEs and BLIEs. Let Y_h, $h = 1, 2, ..., m$, represent a simple random sample of size m from a known distribution. Here we consider different distributions for the study variable Y, i.e., Normal (0,1), Logistic (0,1) and so on. Let ϵ_h is the random error term and it is normally distributed with mean zero and variance σ_ϵ^2, i.e., $\epsilon \sim N(0, \sigma_\epsilon^2)$, $h = 1, 2, ..., m$, where ϵ_h is independent of X_h. Here, the assumed values of m and σ_ϵ^2 are 2, 3, 4, 5 and 1, 3, respectively. Given m and σ_ϵ^2, we compute $W_h = X_h + \epsilon_h$ for $h = 1, 2, ..., m$. Based on the values of W_h, we select both ranked and double ranked set samples of size m. Suppose RSS is performed on the values of W in order to observe the pair $(W_{(i)}, Y_{[i]})$, for $i = 1, 2, ..., m$. We name this scheme as IRSS. Similarly, if a double ranked set sample of size m is observed using the value of W, then we observe a pair $(W^*_{(i)}, Y^*_{[i]})$ for $i = 1, 2, ..., m$. We name this scheme as IDRSS. For each imperfect sampling scheme, the above procedure is repeated 10^6 times and the EMSEs of the BLUEs and BLIEs have been calculated. Note that the coefficient of both BLUEs and BLIEs depend on the underlying distribution of Y_t. Based on the EMSEs of both BLUEs and BLIEs, the estimated REs (EREs) have been calculated and reported in Table 4.6. The ERE of an estimator E_1 with respect to E_2 is: $\text{ERE}(E_1, E_2) = \frac{\text{EMSE}(E_2)}{\text{EMSE}(E_1)}$.

From Table 4.6, it is clear that when the underlying distribution is symmetric, the EREs tend to decrease as the values of σ_ϵ^2 increases and vice-versa. Under both imperfect ranking schemes when estimating location or scale parameter, generally, the BLIEs are more robust than the BLUEs for $m \leq 3$. Moreover, the BLUEs and BLIEs under IDRSS dominate their counterparts based on IRSS in most of the cases for different underlying distributions.

4.7 Best linear unbiased and invariant quantile estimators

In this section, we propose some quantile estimators based on the BLUEs and BLIEs of the location and scale parameters of normal distribution under RSS, IRSS, DRSS and IDRSS.

4.7.1 Quantile estimators based on BLUEs for RSS and IRSS

Barnett and Bown (2002) derived the BLUQ estimator of the normal quantile $Q_{1-p} = \mu + \pi_{1-p}\sigma$ for $p \in \{0.05, 0.01\}$ with different choice of sample size under RSS scheme. Here π_{1-p} is the upper pth quantile of standard normal distribution. Based on the BLUEs of μ and σ, obtained under RSS scheme, Barnett and Bown (2002) suggested a BLUQ estimator of Q_{1-p}, given by

$$\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{RSS},1-p} = \hat{\mu}_{\mathrm{RSS}} + \pi_{1-p}\hat{\sigma}_{RSS} = -\boldsymbol{\alpha}'\boldsymbol{\Gamma}\boldsymbol{Y} + \pi_{1-p}\mathbf{1}'\boldsymbol{\Gamma}\boldsymbol{Y}.$$

As $\hat{\mu}_{\mathrm{RSS}}$ and $\hat{\sigma}_{\mathrm{RSS}}$ are uncorrelated for symmetric populations, therefore, the variance of $\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{RSS},1-p}$ is

$$\mathrm{Var}(\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{RSS},1-p}) = \sigma^2\Delta^{-1}\left(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha} + \pi^2_{1-p}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right).$$

Similarly, the BLUQ estimator based on IRSS is

$$\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{IRSS},1-p} = \hat{\mu}_{\mathrm{IRSS}} + \pi_{1-p}\hat{\sigma}_{\mathrm{IRSS}} = \sum_{i=1}^{m}\gamma_i\bar{Y}_{[i]} + \pi_{1-p}\sum_{i=1}^{m}\eta_i\bar{Y}_{[i]}.$$

The variance of $\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{IRSS},1-p}$ is

$$\mathrm{Var}(\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{IRSS},1-p}) = \frac{\sigma^2}{k\Xi}\left\{\sum_{i=1}^{m}(\alpha^2_{X_{(i)}}/\sigma^*_{X_{(ii)}}) + \pi^2_{1-p}\sum_{i=1}^{m}(1/\sigma^*_{X_{(ii)}})\right\}.$$

4.7.2 Quantile estimators based on BLIEs for RSS and IRSS

In this section, we extend the work of quantile estimation and provide BLIQ estimators under RSS and IRSS schemes.

The BLIEs of the unknown parameters of location-scale distribution under SRS was considered by Mann (1969). Based on these BLIEs of location and scale parameters, a unique best linear invariant estimator of Q_{1-p} is also derived. For brevity of discussion, we name it as BLIQ estimator. On similar lines, the BLIQ estimator based on RSS is given by

$$\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{RSS},1-p} = \tilde{\mu}_{\mathrm{RSS}} + \pi_{1-p}\tilde{\sigma}_{\mathrm{RSS}} = \hat{\mu}_{\mathrm{RSS}} - \hat{\sigma}_{\mathrm{RSS}}(\xi_{12} - \pi_{1-p})(1+\xi_{22})^{-1},$$

which is a biased estimator of Q_{1-p}.
The MSE of $\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{RSS},1-p}$ is

$$\mathrm{MSE}(\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{RSS},1-p}) = \mathrm{MSE}(\tilde{\mu}_{\mathrm{RSS}}) + \pi^2_{1-p}\mathrm{MSE}(\tilde{\sigma}_{\mathrm{RSS}}) + 2\pi_{1-p}E\{(\tilde{\mu}_{\mathrm{RSS}} - \mu)(\tilde{\sigma}_{\mathrm{RSS}} - \sigma)\}.$$

As the parent distribution is normal, therefore, $E\{(\tilde{\mu}_{\mathrm{RSS}} - \mu)(\tilde{\sigma}_{\mathrm{RSS}} - \sigma)\} = 0$.

It follows that

$$\mathrm{MSE}(\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{RSS},1-p}) = \sigma^2\{\xi_{11} + \pi^2_{1-p}\xi_{22}(1+\xi_{22})^{-1}\}.$$

Similarly, the BLIQ estimator based on IRSS procedure is obtained by replacing $\tilde{\mu}_{\mathrm{RSS}}$ and $\tilde{\sigma}_{\mathrm{RSS}}$ in $\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{RSS},1-p}$ by $\tilde{\mu}_{\mathrm{IRSS}}$ and $\tilde{\sigma}_{\mathrm{IRSS}}$, given by

$$\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{IRSS},1-p} = \tilde{\mu}_{\mathrm{IRSS}} + \pi_{1-p}\tilde{\sigma}_{\mathrm{IRSS}} = \hat{\mu}_{\mathrm{IRSS}} - \hat{\sigma}_{\mathrm{IRSS}}(\xi^*_{12} - \pi_{1-p})(1+\xi^*_{22})^{-1},$$

which is a biased estimator of Q_{1-p}.

The MSE of $\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{IRSS},1-p}$, under symmetry assumption, is

$$\mathrm{MSE}(\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{RSS},1-p}) = \sigma^2\{\xi^*_{11} + \pi^2_{1-p}\xi^*_{22}(1+\xi^*_{22})^{-1}\}.$$

4.7.3 Quantile estimators based on BLUEs for DRSS and IDRSS

Following Barnett and Bown (2002), under DRSS BLUEs, the BLUQ estimator is

$$\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{DRSS},1-p} = \hat{\mu}_{\mathrm{DRSS}} + \pi_{1-p}\hat{\sigma}_{\mathrm{DRSS}} = \sum_{i=1}^{m}\vartheta_i \bar{Z}_{(i)} + \pi_{1-p}\sum_{i=1}^{m}\varphi_i \bar{Z}_{(i)}.$$

Here $\hat{\mu}_{\mathrm{DRSS}}$ and $\hat{\sigma}_{\mathrm{DRSS}}$ are uncorrelated due to symmetric underlying population.

The variance of $\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{DRSS},1-p}$ is

$$\mathrm{Var}(\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{DRSS},1-p}) = \frac{\sigma^2}{k\Psi}\left\{\sum_{i=1}^{m}(\upsilon^2_{(i)}/\omega_{(ii)}) + \pi^2_{1-p}\sum_{i=1}^{m}(1/\omega_{(ii)})\right\}.$$

Similarly, we suggest a BLUQ estimator based on IDRSS, given by

$$\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{IDRSS},1-p} = \hat{\mu}_{\mathrm{IDRSS}} + \pi_{1-p}\hat{\sigma}_{\mathrm{IDRSS}} = \sum_{i=1}^{m}\vartheta^*_i \bar{Z}_{1[i]} + \pi_{1-p}\sum_{i=1}^{m}\varphi^*_i \bar{Z}_{1[i]}.$$

The variance of $\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{IDRSS},1-p}$ is

$$\mathrm{Var}(\hat{Q}^{\mathrm{BLUQ}}_{\mathrm{IDRSS},1-p}) = \frac{\sigma^2}{k\Psi^*}\left\{\sum_{i=1}^{m}(\upsilon^{**2}_{(i)}/\omega^{**}_{(ii)}) + \pi^2_{1-p}\sum_{i=1}^{m}(1/\omega^{**}_{(ii)})\right\}.$$

4.7.4 Quantile estimators based on BLIEs for DRSS and IDRSS

Following Mann (1969), the BLIQ estimator based on DRSS procedure is

$$\tilde{Q}^{\mathrm{BLIQ}}_{\mathrm{DRSS},1-p} = \tilde{\mu}_{\mathrm{DRSS}} + \pi_{1-p}\tilde{\sigma}_{\mathrm{DRSS}} = \sum_{i=1}^{m}\varsigma_i Z_{(i)} + \pi_{1-p}\sum_{i=1}^{m}\zeta_i Z_{(i)},$$

which is a biased estimator of Q_{1-p}.

The MSE of $\tilde{Q}^{\text{BLIQ}}_{\text{DRSS},1-p}$ is

$$\text{MSE}(\tilde{Q}^{\text{BLIQ}}_{\text{DRSS},1-p}) = \sigma^2 \left[\left\{ k \sum_{i=1}^{m} (1/\omega_{(ii)}) \right\}^{-1} + \pi^2_{1-p} \left\{ 1 + k \sum_{i=1}^{m} (\upsilon^2_{(i)}/\omega_{(ii)}) \right\}^{-1} \right].$$

Similarly, the BLIQ estimator based on IDRSS procedure is

$$\tilde{Q}^{\text{BLIQ}}_{\text{IDRSS},1-p} = \tilde{\mu}_{\text{IDRSS}} + \pi_{1-p}\tilde{\sigma}_{\text{IDRSS}} = \sum_{i=1}^{m} \varsigma_i^{**} \bar{Z}_{1[i]} + \pi_{1-p} \sum_{i=1}^{m} \zeta_i^{**} \bar{Z}_{1[i]},$$

which is also a biased estimator of Q_{1-p}.

The MSE of $\tilde{Q}^{\text{BLIQ}}_{\text{IDRSS},1-p}$, under symmetric assumption, is

$$\text{MSE}(\tilde{Q}^{\text{BLIQ}}_{\text{IDRSS},1-p}) = \sigma^2 \left[\left\{ k \sum_{i=1}^{m} (1/\omega^{**}_{(ii)}) \right\}^{-1} + \pi^2_{1-p} \left\{ 1 + k \sum_{i=1}^{m} (\upsilon^{**2}_{(i)}/\omega^{**}_{(ii)}) \right\}^{-1} \right].$$

4.8 Comparisons between quantile estimators based on RSS designs

In this section, we compare BLUQ and BLIQ estimators based on RSS, IRSS, DRSS and IDRSS for normal distribution. For this purpose, we present the following REs based on different estimators.

$$\begin{aligned}
\text{RE}^*_1 &= \frac{\text{Var}(\hat{Q}^{\text{BLUQ}}_{\text{RSS},1-p})}{\text{MSE}(\tilde{Q}^{\text{BLIQ}}_{\text{RSS},1-p})}, \ \text{RE}^*_2 = \frac{\text{Var}(\hat{Q}^{\text{BLUQ}}_{\text{DRSS},1-p})}{\text{MSE}(\tilde{Q}^{\text{BLIQ}}_{\text{DRSS},1-p})}, \\
\text{RE}^*_3 &= \frac{\text{Var}(\hat{Q}^{\text{BLUQ}}_{\text{RSS},1-p})}{\text{MSE}(\hat{Q}^{\text{BLIQ}}_{\text{DRSS},1-p})}, \ \text{RE}^*_4 = \frac{\text{Var}(\tilde{Q}^{\text{BLUQ}}_{\text{RSS},1-p})}{\text{MSE}(\tilde{Q}^{\text{BLIQ}}_{\text{DRSS},1-p})}.
\end{aligned}$$

The REs based on different quantile estimators are calculated for different m and ρ for both RSS schemes. We consider $k = 1$ for all cases. Note that if $\rho = \pm 1$, then we have perfect ranking and all other cases belong to imperfect ranking. From Table 4.7, it is noteworthy that under RSS, the proposed BLIQ estimators are uniformly better than the BLUQ estimators suggested by Barnett and Bown (2002). The similar trend is observed for BLIQ estimators under DRSS scheme. In Table 4.8, we compare the quantile estimators based on RSS versus DRSS. It is worth mentioning that the proposed (BLUQ and BLIQ) estimators under DRSS are more efficient than their competitors based on RSS for all cases considered here.

Table 4.7: REs of quantile estimators based on BLUE and BLIE under perfect and imperfect rankings

	$p \rightarrow$	0.01	0.02	0.03	0.04	0.05	0.01	0.02	0.03	0.04	0.05
	RE $\rightarrow$	RE_1^*	RE_1^*	RE_1^*	RE_1^*	RE_1^*	RE_2^*	RE_2^*	RE_2^*	RE_2^*	RE_2^*
ρ	m	$\left(\tilde{Q}^{\mathrm{BLUQ}}_{\mathrm{RSS},1-p}, \hat{Q}^{\mathrm{BLIQ}}_{\mathrm{RSS},1-p}\right)$					$\left(\tilde{Q}^{\mathrm{BLUQ}}_{\mathrm{DRSS},1-p}, \hat{Q}^{\mathrm{BLIQ}}_{\mathrm{DRSS},1-p}\right)$				
±1.0	2	1.9545	1.9261	1.9026	1.8813	1.8610	1.5621	1.5440	1.5291	1.5157	1.5030
	3	1.3508	1.3410	1.3329	1.3254	1.3184	1.2071	1.2132	1.1961	1.1915	1.1872
	4	1.1875	1.1823	1.1780	1.1741	1.1704	1.1091	1.1061	1.1037	1.1014	1.0993
	5	1.1182	1.1149	1.1123	1.1098	1.1075	1.0671	1.0654	1.0639	1.0626	1.0613
±0.9	2	2.2894	2.2525	2.2220	2.1942	2.1678	1.8029	1.7783	1.7581	1.7398	1.7224
	3	1.5015	1.4883	1.4774	1.4674	1.4578	1.3211	1.3120	1.3045	1.2977	1.2912
	4	1.2820	1.2747	1.2687	1.2632	1.2579	1.1843	1.1792	1.1750	1.1712	1.1675
	5	1.1864	1.1816	1.1776	1.1740	1.1705	1.1237	1.1204	1.1176	1.1150	1.1126
±0.8	2	2.7588	2.7103	2.6702	2.6336	2.5987	2.1412	2.1080	2.0805	2.0556	2.0319
	3	1.7139	1.6965	1.6819	1.6686	1.6558	1.4829	1.4701	1.4595	1.4499	1.4406
	4	1.4155	1.4058	1.3976	1.3901	1.3829	1.2902	1.2827	1.2766	1.2709	1.2655
	5	1.2827	1.2761	1.2706	1.2656	1.2607	1.2024	1.1973	1.1931	1.1892	1.1854
±0.7	2	3.4446	3.3796	3.3257	3.2764	3.2294	2.6363	2.5908	2.5532	2.5189	2.4862
	3	2.0256	2.0023	1.9829	1.9650	1.9479	1.7213	1.7037	1.6891	1.6757	1.6628
	4	1.6119	1.5989	1.5880	1.5780	1.5683	1.4464	1.4361	1.4275	1.4195	1.4120
	5	1.4246	1.4158	1.4084	1.4016	1.3951	1.3185	1.3113	1.3053	1.2998	1.2945
±0.6	2	4.5026	4.4125	4.3375	4.2689	4.2035	3.4009	3.3369	3.2839	3.2354	3.1892
	3	2.5077	2.4760	2.4494	2.4248	2.4012	2.0911	2.0667	2.0463	2.0275	2.0094
	4	1.9163	1.8988	1.8841	1.8705	1.8573	1.6892	1.6750	1.6631	1.6521	1.6415
	5	1.6447	1.6330	1.6232	1.6141	1.6053	1.4991	1.4893	1.4810	1.4734	1.4661
±0.5	2	6.2587	6.1270	6.0175	5.9172	5.8213	4.6708	4.5767	4.4984	4.4268	4.3584
	3	3.3093	3.2640	3.2260	3.1908	3.1569	2.7071	2.6719	2.6425	2.6152	2.5890
	4	2.4231	2.3987	2.3781	2.3590	2.3405	2.0942	2.0742	2.0574	2.0418	2.0268
	5	2.0116	1.9956	1.9821	1.9696	1.9574	1.8006	1.7871	1.7756	1.7650	1.7547
±0.3	2	16.4819	16.1107	15.8012	15.5172	15.2453	12.0682	11.8003	11.5770	11.3722	11.1762
	3	7.9826	7.8609	7.7582	7.6629	7.5708	6.3029	6.2085	6.1288	6.0550	5.9837
	4	5.3807	5.3188	5.2663	5.2173	5.1696	4.4621	4.4117	4.3690	4.3291	4.2904
	5	4.1549	4.1166	4.0839	4.0534	4.0236	3.5654	3.5331	3.5056	3.4798	3.4548
±0.1	2	144.2983	140.9380	138.1346	135.5584	133.0908	104.5706	102.1392	100.1106	98.2471	96.4619
	3	66.4367	65.3699	64.4680	63.6293	62.8168	51.2998	50.4776	49.7824	49.1362	48.5101
	4	42.3898	41.8705	41.4283	41.0145	40.6114	34.1087	33.6918	33.3366	33.0046	32.6808
	5	30.9826	30.6756	30.4134	30.1670	29.9260	25.6690	25.4154	25.1984	24.9946	24.7954

Table 4.8: REs of quantile estimators based on BLUE and BLIE of RSS versus DRSS under perfect and imperfect rankings

	$p \rightarrow$	0.01	0.02	0.03	0.04	0.05	0.01	0.02	0.03	0.04	0.05
	RE $\rightarrow$	RE_3^*	RE_3^*	RE_3^*	RE_3^*	RE_3^*	RE_4^*	RE_4^*	RE_4^*	RE_4^*	RE_2^*
ρ	m	$\left(\hat{Q}^{\text{BLUQ}}_{\text{DRSS},1-p}, \hat{Q}^{\text{BLUQ}}_{\text{RSS},1-p}\right)$					$\left(\tilde{Q}^{\text{BLIQ}}_{\text{DRSS},1-p}, \tilde{Q}^{\text{BLIQ}}_{\text{RSS},1-p}\right)$				
±1.0	2	1.9545	1.9261	1.9026	1.8813	1.8610	1.5621	1.5440	1.5291	1.5157	1.5030
	3	1.3508	1.3410	1.3329	1.3254	1.3184	1.2071	1.2132	1.1961	1.1915	1.1872
	4	1.1875	1.1823	1.1780	1.1741	1.1704	1.1091	1.1061	1.1037	1.1014	1.0993
	5	1.1182	1.1149	1.1123	1.1098	1.1075	1.0671	1.0654	1.0639	1.0626	1.0613
±0.9	2	2.2894	2.2525	2.2220	2.1942	2.1678	1.8029	1.7783	1.7581	1.7398	1.7224
	3	1.5015	1.4883	1.4774	1.4674	1.4578	1.3211	1.3120	1.3045	1.2977	1.2912
	4	1.2820	1.2747	1.2687	1.2632	1.2579	1.1843	1.1792	1.1750	1.1712	1.1675
	5	1.1864	1.1816	1.1776	1.1740	1.1705	1.1237	1.1204	1.1176	1.1150	1.1126
±0.8	2	2.7588	2.7103	2.6702	2.6336	2.5987	2.1412	2.1080	2.0805	2.0556	2.0319
	3	1.7139	1.6965	1.6819	1.6686	1.6558	1.4829	1.4701	1.4595	1.4499	1.4406
	4	1.4155	1.4058	1.3976	1.3901	1.3829	1.2902	1.2827	1.2766	1.2709	1.2655
	5	1.2827	1.2761	1.2706	1.2656	1.2607	1.2024	1.1973	1.1931	1.1892	1.1854
±0.7	2	3.4446	3.3796	3.3257	3.2764	3.2294	2.6363	2.5908	2.5532	2.5189	2.4862
	3	2.0256	2.0023	1.9829	1.9650	1.9479	1.7213	1.7037	1.6891	1.6757	1.6628
	4	1.6119	1.5989	1.5880	1.5780	1.5683	1.4464	1.4361	1.4275	1.4195	1.4120
	5	1.4246	1.4158	1.4084	1.4016	1.3951	1.3185	1.3113	1.3053	1.2998	1.2945
±0.6	2	4.5026	4.4125	4.3375	4.2689	4.2035	3.4009	3.3369	3.2839	3.2354	3.1892
	3	2.5077	2.4760	2.4494	2.4248	2.4012	2.0911	2.0667	2.0463	2.0275	2.0094
	4	1.9163	1.8988	1.8841	1.8705	1.8573	1.6892	1.6750	1.6631	1.6521	1.6415
	5	1.6447	1.6330	1.6232	1.6141	1.6053	1.4991	1.4893	1.4810	1.4734	1.4661
±0.5	2	6.2587	6.1270	6.0175	5.9172	5.8213	4.6708	4.5767	4.4984	4.4268	4.3584
	3	3.3093	3.2640	3.2260	3.1908	3.1569	2.7071	2.6719	2.6425	2.6152	2.5890
	4	2.4231	2.3987	2.3781	2.3590	2.3405	2.0942	2.0742	2.0574	2.0418	2.0268
	5	2.0116	1.9956	1.9821	1.9696	1.9574	1.8006	1.7871	1.7756	1.7650	1.7547
±0.3	2	16.4819	16.1107	15.8012	15.5172	15.2453	12.0682	11.8003	11.5770	11.3722	11.1762
	3	7.9826	7.8609	7.7582	7.6629	7.5708	6.3029	6.2085	6.1288	6.0550	5.9837
	4	5.3807	5.3188	5.2663	5.2173	5.1696	4.4621	4.4117	4.3690	4.3291	4.2904
	5	4.1549	4.1166	4.0839	4.0534	4.0236	3.5654	3.5331	3.5056	3.4798	3.4548
±0.1	2	144.2983	140.9380	138.1346	135.5584	133.0908	104.5706	102.1392	100.1106	98.2471	96.4619
	3	66.4367	65.3699	64.4680	63.6293	62.8168	51.2998	50.4776	49.7824	49.1362	48.5101
	4	42.3898	41.8705	41.4283	41.0145	40.6114	34.1087	33.6918	33.3366	33.0046	32.6808
	5	30.9826	30.6756	30.4134	30.1670	29.9260	25.6690	25.4154	25.1984	24.9946	24.7954

4.9 Conclusion

In this paper, we considered the estimation of the unknown parameters of location-scale family of distributions under DRSS and IDRSS schemes. Explicit mathematical expressions for both BLUEs and BLIEs of the location and scale parameters are derived. It is worth mentioning that, under perfect DRSS scheme, the proposed estimators are uniformly better than their counterparts obtain under RSS. The DRSS based BLIEs of scale parameters are more precise than the BLUEs based on RSS and DRSS schemes. Under imperfect ranking schemes, generally, the BLIEs under IDRSS are more robust than the BLIEs computed under IRSS. The work is then extended to the estimation of normal quantiles. The suggested estimators under DRSS are better than the existing quantile estimators based on RSS for both perfect and imperfect rankings. Finally, we recommend using the BLIEs under perfect DRSS scheme whereas in case of imperfect rankings, for most cases, the BLIEs under DRSS are able to perform better than the BLIEs constructed under RSS.

Chapter 5

Improved Best Linear Unbiased Estimators for the Simple Linear Regression Model using Double Ranked Set Sampling Schemes

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In this chapter, we consider the best linear unbiased estimators (BLUEs) based on double ranked set sampling (DRSS) and ordered DRSS (ODRSS) schemes for the simple linear regression model with replicated observations. We assume three symmetric distributions for the random error term, i.e., normal, Laplace and some scale contaminated normal distributions. The proposed BLUEs under DRSS (BLUEs-DRSS) and ODRSS (BLUEs-ODRSS) are compared with the BLUEs based on ordered simple random sampling (OSRS), ranked set sampling (RSS) and ordered RSS (ORSS) schemes. These estimators are compared in terms of relative efficiency (RE), RE of determinant (RED), RE of trace (RET). It is found that the BLUEs-ODRSS are uniformly better than the BLUEs based on OSRS, RSS, ORSS and DRSS schemes. We also compare the estimators based on imperfect RSS (IRSS) schemes. It is worth mentioning here that the BLUEs under ordered imperfect DRSS (OIDRSS) are better than their counterparts based on IRSS, ordered IRSS (OIRSS) and imperfect DRSS (IDRSS) methods. Moreover, for sensitivity analysis of the BLUEs, we calculate REs and REDs of the BLUEs under the assumption of normality when in fact the parent distribution follows a non-normal symmetric distribution. It turns out that even under violation of normality assumptions, BLUEs of the intercept and the slope parameters are found to be unbiased with equal REs under each sampling scheme. It is also observed that the BLUEs under ODRSS are more efficient than the existing BLUEs.

5.1 Introduction

The main focus of many agricultural, ecological and environmental studies is to develop well-designed, cost-effective and efficient sampling designs. Ranked set sampling (RSS) is one of those sampling methods that can accomplish such objectives because it uses expert knowledge and prior information. The RSS method is an efficient alternative to the traditional simple random sampling (SRS) when the variable of interest is either difficult or expensive to measure but is easy to rank and to make an economical assessment of the rank order for the selected sampling units. For example, in ecological assessment of hazardous waste sites, expensive radio-chemical techniques may need to be used. However, the hazardous waste sites with different levels of contamination can be ranked by a visual inspection of soil discoloration.

The RSS scheme was first introduced by McIntyre (1952) for estimating mean pasture and forage yields. The mathematical setup of RSS was derived by Takahasi and Wakimoto (1968). Lloyd (1952) obtained the best linear unbiased estimators (BLUEs) of the location and scale parameters of location-scale family of distributions based on ordered SRS (OSRS) or order statistics by using generalized least-squares approach. Muttlak (1995) developed the simple linear regression model (SLRM) based on RSS. It is shown that the intercept and slope estimators under RSS are more efficient than those based on SRS. Barreto and Barnett (1999) considered the SLRM with replicated observations under perfect RSS and obtained the BLUEs of the intercept, slope and error standard deviation when the response variable is normally distributed. Al-Saleh and Al-Kadiri (2000) introduced a two-stage RSS procedure, namely, double RSS (DRSS) for estimation of the population mean. Balakrishnan and Li (2005, 2008) introduced ordered RSS (ORSS) scheme and used it to obtain the BLUEs of the location and scale parameters of generalized geometric, normal, logistic and exponential distributions. They found that the BLUEs based on ORSS (BLUEs-ORSS) are uniformly better than the BLUEs-OSRS and BLUEs-RSS. Li and Balakrishnan (2008) obtained the BLUEs-ORSS of the unknown parameters of the SLRM when the response variable is normally distributed. They showed that when estimating the parameters of the SLRM, the BLUEs-ORSS outperform their counterparts based on OSRS and ORSS schemes.

In this chapter, we consider the BLUEs of the unknown parameters of the SLRM based on DRSS and ordered DRSS (ODRSS) schemes, namely BLUEs-DRSS and BLUEs-ODRSS. We study the performance of the BLUEs when the random error term follows normal, Laplace and three scale contaminated normal distribution. The performance of the proposed and existing BLUEs is also evaluated for imperfect RSS (IRSS) schemes, i.e., IRSS, ordered IRSS (OIRSS), imperfect DRSS (IDRSS) and ordered IDRSS (OIDRSS). The comparisons of the BLUEs are based on individual and joint relative efficiencies, i.e., relative efficiency (RE), RE of determinant (RED) and RE of trace (RET). It is shown that the BLUEs-ODRSS and BLUEs based on OIDRSS (BLUEs-OIDRSS) schemes are uniformly better than their counterparts based on all perfect and IRSS schemes considered here. For sensitivity analysis of the BLUEs, we calculate REs of the BLUEs assuming normality when the parent distribution is actually a non-normal symmetric distributions.

The rest of the chapter is organized as follows: Section 5.2 explains both RSS and ORSS schemes. The DRSS and ODRSS schemes along with their mathematical setups are given in Section 5.3. In Section 5.4, we consider the SLRM with replicated observations under DRSS schemes. The performance comparisons of the BLUEs based on perfect and IRSS schemes are given in Section 5.5. Sensitivity analysis of the BLUEs based on different RSS schemes is considered in Section 5.6. Finally, Section 5.7 summarizes the main findings.

5.2 Ranked set sampling

In this section, we explain the traditional RSS and ORSS schemes.

Let Y be the study variable with probability density function (PDF) $f(y)$ and cumulative distribution function $F(y)$. Let $Y_1, Y_2, ..., Y_n$ be n independent and identically distributed (IID) random variables, i.e., $Y_j \sim f(y)$, for $j = 1, 2, ..., n$. Let $Y_{(1:n)}^{\text{OSRS}}, Y_{(2:n)}^{\text{OSRS}}, ..., Y_{(n:n)}^{\text{OSRS}}$ denote the ordered simple random sample of size n obtained by arranging Y_js in an increasing order. Then, the PDF and CDF of the jth order statistic, $Y_{(j:n)}^{\text{OSRS}}$, are respectively given by

$$\begin{aligned}
f_{(j:n)}^{\text{OSRS}}(y) &= n\binom{n-1}{j-1}\{F(y)\}^{j-1}\{1-F(y)\}^{n-j}f(y), \quad -\infty < y < \infty, \\
F_{(j:n)}^{\text{OSRS}}(y) &= \sum_{l=j}^{n}\binom{n}{l}\{F(y)\}^{l}\{1-F(y)\}^{n-l}.
\end{aligned}$$

The corresponding mean and variance of $Y_{(j:n)}^{\text{OSRS}}$ are

$$\mu_{(j:n)}^{\text{OSRS}} = \int y f_{(j:n)}^{\text{OSRS}}(y)\mathrm{d}y \quad \text{and} \quad \sigma_{(j,j:n)}^{(\text{OSRS})} = \int (y-\mu_{(j:n)}^{\text{OSRS}})^2 f_{(j:n)}^{\text{OSRS}}(y)\mathrm{d}y, \tag{5.1}$$

respectively. Similarly, the covariance between the $Y_{(j:n)}^{\text{OSRS}}$ and $Y_{(j':n)}^{\text{OSRS}}$ is $\sigma_{(j,j':n)}^{\text{OSRS}}$, for $j \neq j' = 1, 2, ..., n$. For more details, see David and Nagaraja (2003).

The traditional RSS scheme is explained as follows: identify m^2 units from the target population. Randomly allocate these units to m sets each of size m units. Rank the units within each set with respect to the study variable or by any inexpensive method. Then, select the jth smallest ranked unit from the jth set for $j = 1, 2, ..., m$. This gives a ranked set sample of size m for one cycle. The whole process can be repeated r times to get a ranked set sample of size $n = mr$. Let $Y_{(j:m)t}^{\text{RSS}}$, $j = 1, 2, ..., m$, $t = 1, 2, ..., r$, denote a ranked set sample of size $n = mr$, where $Y_{(j:m)t}^{\text{RSS}} = j\text{th}\ \min\{Y_{j1t}, Y_{j2t}, ..., Y_{jmt}\}$. Let $\mu_{(j:m)}^{\text{RSS}}$ and $\sigma_{(j,j':m)}^{\text{RSS}}$ be the mean and variance of $Y_{(j:m)t}^{\text{RSS}}$, respectively. Given t, it is interesting to note that, under perfect ranking, both $Y_{(j:m)t}^{\text{RSS}}$ and $Y_{(j:n)t}^{\text{OSRS}}$ are identically distributed when $n = m$. Therefore, it is easy to find the mean and variance of $Y_{(j:m)t}^{\text{RSS}}$ from (5.1).

The ordered ranked set sample of size m is easily obtained by arranging the ranked set sample an increasing order of magnitude, i.e., $Y^{\text{ORSS}}_{(j:m)t} = j\text{th}\ \min\{Y^{\text{RSS}}_{(1:m)t}, Y^{\text{RSS}}_{(2:m)t}, ..., Y^{\text{RSS}}_{(m:m)t}\}$ for $t = 1, 2, ..., r$. Let $\mu^{\text{ORSS}}_{(j:m)}$ and $\sigma^{\text{ORSS}}_{(j,j:m)}$ be the mean and variance of $Y^{\text{ORSS}}_{(j:m)t}$, respectively. These quantities can be obtained by deriving the density and joint density functions of $Y^{\text{ORSS}}_{(j:m)t}$. For more details, see Balakrishnan and Li (2005).

5.3 Double ranked set sampling schemes

In this section, we explain DRSS and ODRSS schemes. We derive the explicit mathematical expressions for random variables based on both DRSS and ODRSS schemes.

Let $S_{1t}, S_{2t}, ..., S_{mt}$ be m sets each of size m^2 units in the tth cycle for $t = 1, 2, ..., r$. Given t, divide the jth set S_{jt} into m subsets, say s_{hjt}, each of size m units, i.e., $S_{jt} = \{s_{hjt}\} = \{s_{1jt}, s_{2jt}, ..., s_{mjt}\}$, for $h = 1, 2, ..., m$, where $s_{hjt} = \{Y_{h1t}, Y_{h2t}, ..., Y_{hmt}\}$. Apply RSS scheme on each set S_j to get m ranked set samples each of size m. Let S^*_{jt} be the jth set that contains the jth ranked sample, i.e., $S^*_{jt} = \{Y^{\text{RSS}}_{(1:m)t}, Y^{\text{RSS}}_{(2:m)t}, ..., Y^{\text{RSS}}_{(m:m)t}\}$ for $j = 1, 2, .., m$. Again apply the RSS procedure on S^*_{jt} to get a double ranked set sample of size m. Let $Y^{\text{DRSS}}_{(j:m)t} = j\text{th}\ \min\{S^*_{jt}\}$ $(j = 1, 2, ..., m; t = 1, 2, ..., r)$ denote a double ranked set sample of size m. Note that $Y^{\text{DRSS}}_{(1:m)t}, Y^{\text{DRSS}}_{(2:m)t}, ..., Y^{\text{DRSS}}_{(m:m)t}$ all are independent and non-identically distributed (INID) random variables for the tth cycle. However, under DRSS, for fixed j, the random variables $Y^{\text{DRSS}}_{(j:m)1}, Y^{\text{DRSS}}_{(j:m)2}, ..., Y^{\text{DRSS}}_{(j:m)r}$ are IID. Therefore, for simplicity, we set $Y^{\text{DRSS}}_{(j:m)} = Y^{\text{DRSS}}_{(j:m)t}$ for $j = 1, 2, ..., m$.

Suppose $\boldsymbol{A} = ((a_{i,j}))$ is a square matrix of order m. Then, the permanent of the matrix $\boldsymbol{A}$ is defined to be $\text{Per}(\boldsymbol{A}) = \sum_P \sum_{j=1}^{m} a_{j,i_j}$, where $\sum_P(\cdot)$ denotes the sum over all $m!$ permutations $(i_1, i_2, ..., i_m)$ of $(1, 2, ..., m)$. Following Vaughan and Venables (1972), and Bapat and Beg (1989), the CDF of $Y^{\text{DRSS}}_{(j:m)}$ is

$$G^{\text{DRSS}}_{(j:m)}(y) = \sum_{i=j}^{m} \frac{1}{i!(m-i)!}\ \text{Per}(\boldsymbol{A}_1),$$

where $\text{Per}(\boldsymbol{A}_1)$ is the permanent of the matrix $\boldsymbol{A}_1$. Here $\boldsymbol{A}_1$ is defined as

$$\boldsymbol{A}_1 = \begin{array}{c} \begin{pmatrix} F^{\text{RSS}}_{(1:m)}(y) & F^{\text{RSS}}_{(2:m)}(y) & \cdots & F^{\text{RSS}}_{(m:m)}(y) \\ 1 - F^{\text{RSS}}_{(1:m)}(y) & 1 - F^{\text{RSS}}_{(2:m)}(y) & \cdots & 1 - F^{\text{RSS}}_{(m:m)}(y) \end{pmatrix} \end{array} \begin{array}{l} \}\, i \\ \}\, m-i \end{array},$$

where the first and second rows are repeated i and $m - i$ times respectively.

Similarly, the PDF of $Y^{\text{DRSS}}_{(j:m)}$ is given by

$$g^{(\text{DRSS})}_{(j:m)}(y) = \frac{1}{(j-1)!(m-j)!}\ \text{Per}(\boldsymbol{A}_2),$$

where $$\boldsymbol{A}_2 = \begin{pmatrix} F^{\text{RSS}}_{(1:m)}(y) & F^{\text{RSS}}_{(2:m)}(y) & \cdots & F^{\text{RSS}}_{(m:m)}(y) \\ f^{\text{RSS}}_{(1:m)}(y) & f^{\text{RSS}}_{(2:m)}(y) & \cdots & f^{\text{RSS}}_{(m:m)}(y) \\ 1 - F^{\text{RSS}}_{(1:m)}(y) & 1 - F^{\text{RSS}}_{(2:m)}(y) & \cdots & 1 - F^{\text{RSS}}_{(m:m)}(y) \end{pmatrix} \begin{array}{l} \}\, j-1 \\ \}\, 1 \\ \}\, m-j \end{array}.$$

Fore more details, see Balakrishnan (2007) and references cited therein.

Let $\mu_{(j:m)}^{\text{DRSS}}$ and $\sigma_{(j,j:m)}^{\text{DRSS}}$ be the mean and variance of $Y_{(j:m)}^{\text{DRSS}}$, respectively, defined by the following expressions:

$$\mu_{(j:m)}^{\text{DRSS}} = \int y g_{(j:m)}^{\text{DRSS}}(y)\mathrm{d}y \quad \text{and} \quad \sigma_{(j,j:m)}^{\text{DRSS}} = \int (y - \mu_{(j:m)}^{\text{DRSS}})^2 g_{(j:m)}^{\text{DRSS}}(y)\mathrm{d}y.$$

By using above formulae, it is easy to calculate the mean and variances of order statistics obtained under DRSS scheme. The numerical integration can be easily implemented in Mathematica.

In ODRSS scheme, we order the random variables under DRSS in an increasing order of magnitude. Let $Y_{(j:m)t}^{\text{ODRSS}} = j\text{th}\ \min\{Y_{(1:m)t}^{\text{DRSS}}, Y_{(2:m)t}^{\text{DRSS}}, ..., Y_{(m:m)t}^{\text{DRSS}}\}$ for $j = 1, 2, ..., m$, $t = 1, 2, ..., r$, is the ordered double ranked set sample of size $n = mr$. It is clear that for fixed t, $Y_{(j:m)t}^{\text{ODRSS}}$ is the jth order statistic from INID random variables $Y_{(j:m)t}^{\text{DRSS}}$ for $j = 1, 2, ..., m$. Also note that for fixed j, $Y_{(j:m)1}^{\text{ODRSS}}, Y_{(j:m)2}^{\text{ODRSS}}, ..., Y_{(j:m)r}^{\text{ODRSS}}$ are identically distributed random variables. Therefore, for simplicity, without loss of generality, we set $Y_{(j:m)t}^{\text{ODRSS}} = Y_{(j:m)}^{\text{ODRSS}}$. The PDF of $Y_{(j:m)}^{\text{ODRSS}}$ is given by

$$g_{(j:m)}^{\text{ODRSS}}(y) = \frac{1}{(j-1)!(m-j)!}\ \text{Per}(\boldsymbol{B}_2),$$

$$\text{where } \boldsymbol{B}_2 = \begin{array}{c} \begin{pmatrix} G_{(1:m)}^{\text{DRSS}}(y) & G_{(2:m)}^{\text{DRSS}}(y) & \cdots & G_{(m:m)}^{\text{DRSS}}(y) \\ g_{(1:m)}^{\text{DRSS}}(y) & g_{(2:m)}^{\text{DRSS}}(y) & \cdots & g_{(m:m)}^{\text{DRSS}}(y) \\ 1 - G_{(1:m)}^{\text{DRSS}}(y) & 1 - G_{(2:m)}^{\text{DRSS}}(y) & \cdots & 1 - G_{(m:m)}^{\text{DRSS}}(y) \end{pmatrix} \end{array} \begin{array}{l} \}\, j-1 \\ \}\, 1 \\ \}\, m-j \end{array}.$$

The joint density function of $Y_{(j:m)}^{\text{ODRSS}}$ and $Y_{(j':m)}^{\text{ODRSS}}$ $(1 \le j < j' \le m)$ is

$$g_{(j,j':m)}^{\text{ODRSS}}(y_1, y_2) = \frac{1}{(j-1)!(j'-j-1)!(m-j')!}\ \text{Per}(\boldsymbol{B}_3), \quad -\infty < y_1 < y_2 < \infty,$$

where

$$\boldsymbol{B}_3 = \begin{pmatrix} G_{(1:m)}^{\text{DRSS}}(y_1) & G_{(2:m)}^{\text{DRSS}}(y_1) & \cdots & G_{(m:m)}^{\text{DRSS}}(y_1) \\ g_{(1:m)}^{\text{DRSS}}(y_1) & g_{(2:m)}^{\text{DRSS}}(y_1) & \cdots & g_{(m:m)}^{\text{DRSS}}(y_1) \\ G_{(1:m)}^{\text{DRSS}}(y_2) - G_{(1:m)}^{\text{DRSS}}(y_1) & G_{(2:m)}^{\text{DRSS}}(y_2) - G_{(2:m)}^{\text{DRSS}}(y_1) & \cdots & G_{(m:m)}^{\text{DRSS}}(y_2) - G_{(m:m)}^{\text{DRSS}}(y_1) \\ g_{(1:m)}^{\text{DRSS}}(y_2) & g_{(2:m)}^{\text{DRSS}}(y_2) & \cdots & g_{(m:m)}^{\text{DRSS}}(y_2) \\ 1 - G_{(1:m)}^{\text{DRSS}}(y_2) & 1 - G_{(2:m)}^{\text{DRSS}}(y_2) & \cdots & 1 - G_{(m:m)}^{\text{DRSS}}(y_2) \end{pmatrix} \begin{array}{l} \}\, j-1 \\ \}\, 1 \\ \}\, j'-j-1 \\ \}\, 1 \\ \}\, m-j' \end{array}.$$

Let $\mu_{(j:m)}^{\text{ODRSS}}$ and $\sigma_{(j,j:m)}^{\text{ODRSS}}$ be the mean and variance of $Y_{(j:m)}^{\text{ODRSS}}$, respectively, given by

$$\mu_{(j:m)}^{\text{ODRSS}} = \int y g_{(j:m)}^{\text{ODRSS}}(y)\mathrm{d}y \quad \text{and} \quad \sigma_{(j,j:m)}^{\text{ODRSS}} = \int (y - \mu_{(j:m)}^{\text{ODRSS}})^2 g_{(j:m)}^{\text{ODRSS}}(y)\mathrm{d}y.$$

Let $\sigma_{(j,j':m)}^{\text{ODRSS}}$ be the covariance between $Y_{(j:m)}^{\text{ODRSS}}$ and $Y_{(j':m)}^{\text{ODRSS}}$ for $1 \le j < j' \le m$, defined as $\sigma_{(j,j':m)}^{\text{ODRSS}} = \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} y_1 y_2 g_{(j,j':m)}^{\text{ODRSS}}(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2 - \mu_{(j:m)}^{\text{ODRSS}}\mu_{(j':m)}^{\text{ODRSS}}$. Based on these formulae, it is easy to calculate the mean and variance of order statistics obtained under ODRSS scheme. The numerical integration can be implemented in Mathematica.

5.4 A simple linear regression model

In this section, we consider a SLRM based on replicated observations under DRSS and ODRSS schemes.

Sometimes, the main objective of the statistical analysis is to study the relationship between the study (dependent) variable, say Y, and the predictor (independent) variable, say X. Mostly in experimental studies, the dependent variable (Y) is observed for preset values of the independent variable (X). At each distinct value of X, say $X = x_i$ $(i = 1, 2, ..., k)$, we observe n_i replicated observations of Y, say $Y = y_{ij}(j = 1, 2, ..., n_i)$. For sake of simplification, we consider the case of equal n_is, i.e., $n_1 = n_2 = \cdots = n_k = n$, but the results presented here correspond to the general case.

Moussa-Hamouda and Leone (1974) proposed the BLUEs of the unknown parameters of the SLRM based on order statistics by ordering the replicated observations $(Y^{\text{OSRS}}_{i(j:n)})$ that were observed against each level of x, namely BLUEs-OSRS. Barreto and Barnett (1999) estimated the unknown parameters of the SLRM using replicated observation that were obtained via RSS procedure. At each level of x, a ranked set sample of size m is observed, i.e., $Y^{\text{RSS}}_{i(j:m)t}$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$ and $t = 1, 2, ..., r$. Li and Balakrishnan (2008) extended the work of Moussa-Hamouda and Leone (1974) and Barreto and Barnett (1999), and proposed BLUEs of the unknown parameters of the SLRM using replicated observations $(Y^{\text{ORSS}}_{i(j:m)t})$ based on ORSS. They showed that the BLUEs-ORSS are uniformly better than the BLUEs-RSS when errors are normally distributed.

As an alternative to OSRS, RSS and ORSS methods, some efficient estimators of the unknown parameters of the SLRM are proposed under DRSS and ODRSS schemes. For brevity of the discussion, consider the SLRM based on ODRSS. Suppose an ordered double ranked set sample of size $n = mr$, i.e., $Y^{\text{ODRSS}}_{i(j:m)t}$, $j = 1, 2, ..., m$, $t = 1, 2, ..., r$, is observed for each level of the independent variable $X = x_i$, for $i = 1, 2, ..., k$. Then, the SLRM based on ODRSS can be stated as follows:

$$Y^{\text{ODRSS}}_{i(j:m)t} = \alpha + \beta(x_i - \bar{x}) + \xi_{ijt}, \tag{5.2}$$

where $j = 1, 2, ..., m$, $t = 1, 2, ..., r$ and $i = 1, 2, ..., k$. ξ_{ijt} are IID error terms from a continuous symmetric distribution with mean zero and variance σ^2. Moreover, let $Z^{\text{ODRSS}}_{i(j:m)t} = \frac{\{Y^{\text{ODRSS}}_{i(j:m)t} - \alpha - \beta(x_i - \bar{x})\}}{\sigma}$ be the jth order statistic under ODRSS from the standardized symmetric distribution, say $g^{\text{ODRSS}}_{(j:m)}(z)$, with mean zero and variance unity.

Let $E(Z^{\text{ODRSS}}_{i(j:m)t}) = \mu^{\text{ODRSS}}_{(j:m)}$, $\text{Cov}(Z^{\text{ODRSS}}_{i(j:m)t}, Z^{\text{ODRSS}}_{i(j':m)t}) = \sigma^{\text{ODRSS}}_{(j,j':m)}$, $\boldsymbol{\mu}^{\text{ODRSS}}_t = (\boldsymbol{\mu}_{\text{ODRSS}}, \boldsymbol{\mu}_{\text{ODRSS}}, ..., \boldsymbol{\mu}_{\text{ODRSS}})'_{1\times k}$ for $t = 1, 2, ..., r$, where $\boldsymbol{\mu}_{\text{ODRSS}} = (\mu^{\text{ODRSS}}_{(1:m)}, \mu^{\text{ODRSS}}_{(2:m)}, ..., \mu^{\text{ODRSS}}_{(m:m)})'_{1\times m}$. Let $\boldsymbol{\Omega}^{*}_{\text{ODRSS}} = \text{diag}(\boldsymbol{\Omega}^{\text{ODRSS}}_1, \boldsymbol{\Omega}^{\text{ODRSS}}_2, ..., \boldsymbol{\Omega}^{\text{ODRSS}}_r)_{r\times r}$, where $\boldsymbol{\Omega}^{\text{ODRSS}}_t = \text{diag}(\boldsymbol{\Omega}_{\text{ODRSS}}, \boldsymbol{\Omega}_{\text{ODRSS}}, ..., \boldsymbol{\Omega}_{\text{ODRSS}})_{k\times k}$, for $t = 1, 2, ..., r$, and $\boldsymbol{\Omega}_{\text{ODRSS}} = (\sigma^{\text{ODRSS}}_{(j,j':m)})_{m\times m}$. Here 'diag' indicates the diagonal matrix. Then, the model given in (5.2), can be represented in a matrix notation, given by

$$\boldsymbol{Y} = \boldsymbol{W}\boldsymbol{\theta} + \boldsymbol{\xi},$$

where $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, ..., \boldsymbol{Y}_r)'_{1\times r}$, $\boldsymbol{Y}_t = (\boldsymbol{Y}_{1t}^{\text{ODRSS}}, \boldsymbol{Y}_{2t}^{\text{ODRSS}}, ..., \boldsymbol{Y}_{kt}^{\text{ODRSS}})'_{1\times k}$, $\boldsymbol{Y}_{it}^{\text{ODRSS}} = (Y_{i(1:m)t}^{\text{ODRSS}}, Y_{i(2:m)t}^{\text{ODRSS}}, ..., Y_{i(m:m)t}^{\text{ODRSS}})'_{1\times m}$, $i = 1, 2, ..., k$ and $t = 1, 2, ..., r$. Here $\boldsymbol{W} = \begin{pmatrix} \mathbf{1}_1 & \boldsymbol{x}_1^* & \boldsymbol{\mu}_1^{\text{ODRSS}} \\ \vdots & \vdots & \vdots \\ \mathbf{1}_r & \boldsymbol{x}_r^* & \boldsymbol{\mu}_r^{\text{ODRSS}} \end{pmatrix}$, $\boldsymbol{\theta} = (\alpha, \beta, \sigma)'_{1\times 3}$, $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, ..., \boldsymbol{\xi}_r)'_{1\times r}$, $\mathbf{1}_t = (\mathbf{1}, \mathbf{1}, ..., \mathbf{1})'_{1\times k}$, $\boldsymbol{x}_t^* = (\boldsymbol{x}_1 - \bar{\boldsymbol{x}}, \boldsymbol{x}_2 - \bar{\boldsymbol{x}}, ..., \boldsymbol{x}_k - \bar{\boldsymbol{x}})'_{1\times k}$, $\boldsymbol{\xi}_t = (\boldsymbol{\xi}_{1t}, \boldsymbol{\xi}_{2t}, ..., \boldsymbol{\xi}_{kt})'_{1\times k}$, for $t = 1, 2, ..., r$, $\mathbf{1} = (1, 1, ..., 1)'_{1\times m}$, $\boldsymbol{x}_i = (x_i, x_i, ..., x_i)'_{1\times m}$, $\bar{\boldsymbol{x}} = (\bar{x}, \bar{x}, ..., \bar{x})'_{1\times m}$, $\boldsymbol{\xi}_{it} = (\xi_{i1t}, \xi_{i2t}, ..., \xi_{imt})'_{1\times m}$ for $i = 1, 2, ..., k$, and $E(\boldsymbol{\xi}) = 0$, $\text{Var}(\boldsymbol{\xi}) = \text{Var}(\boldsymbol{Y}) = \sigma^2 \boldsymbol{\Omega}^*_{\text{ODRSS}}$.

Following Lloyd (1952), by using generalized least-squares approach, the BLUE-ODRSS of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}_{\text{ODRSS}} = (\hat{\alpha}_{\text{ODRSS}}, \hat{\beta}_{\text{ODRSS}}, \hat{\sigma}_{\text{ODRSS}})'_{1\times 3}$, and its variance are respectively given by

$$\hat{\boldsymbol{\theta}}_{\text{ODRSS}} = (\boldsymbol{W}'\boldsymbol{\Omega}^{*-1}_{\text{ODRSS}}\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{\Omega}^{*-1}_{\text{ODRSS}}\boldsymbol{Y}. \tag{5.3}$$

$$\text{Var}(\hat{\boldsymbol{\theta}}_{\text{ODRSS}}) = \sigma^2(\boldsymbol{W}'\boldsymbol{\Omega}^{*-1}_{\text{ODRSS}}\boldsymbol{W})^{-1}, \tag{5.4}$$

where $(\boldsymbol{W}'\boldsymbol{\Omega}^{*-1}_{\text{ODRSS}}\boldsymbol{W}) =$

$$\begin{pmatrix} rk(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\mathbf{1}) & 0 & rk(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\boldsymbol{\mu}_{\text{ODRSS}}) \\ 0 & r(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\mathbf{1})\sum_{i=1}^{k}(x_i - \bar{x})^2 & 0 \\ rk(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\boldsymbol{\mu}_{\text{ODRSS}}) & 0 & rk(\boldsymbol{\mu}'_{\text{ODRSS}}\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\boldsymbol{\mu}_{\text{ODRSS}}) \end{pmatrix}.$$

It is interesting to note that when the underlying distribution of the dependent variable Y is symmetric with mean zero, then it is easy to show that $(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\boldsymbol{\mu}_{\text{ODRSS}}) = 0$. Thus, it helps in further simplifying the mathematical expressions of the BLUEs-ODRSS and their corresponding variances. The simplified forms of (5.3) and (5.4) are respectively given by

$$\hat{\boldsymbol{\theta}}_{\text{ODRSS}} = (\hat{\alpha}_{\text{ODRSS}}, \hat{\beta}_{\text{ODRSS}}, \hat{\sigma}_{\text{ODRSS}})'$$

and

$$\text{Var}(\hat{\boldsymbol{\theta}}_{\text{ODRSS}}) = \sigma^2 \begin{pmatrix} \text{Var}(\hat{\alpha}_{\text{ODRSS}}) & 0 & 0 \\ 0 & \text{Var}(\hat{\beta}_{\text{ODRSS}}) & 0 \\ 0 & 0 & \text{Var}(\hat{\sigma}_{\text{ODRSS}}) \end{pmatrix},$$

where $\hat{\alpha}_{\text{ODRSS}} = \frac{\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\sum_{t=1}^{r}\sum_{i=1}^{k}\boldsymbol{Y}_{it}^{\text{ODRSS}}}{rk(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\mathbf{1})}$, $\hat{\beta}_{\text{ODRSS}} = \frac{\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\sum_{t=1}^{r}\sum_{i=1}^{k}(x_i-\bar{x})\boldsymbol{Y}_{it}^{\text{ODRSS}}}{r(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\mathbf{1})\sum_{i=1}^{k}(x_i-\bar{x})^2}$, $\hat{\sigma}_{\text{ODRSS}} = \frac{\boldsymbol{\mu}'_{\text{ODRSS}}\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\sum_{t=1}^{r}\sum_{i=1}^{k}\boldsymbol{Y}_{it}^{\text{ODRSS}}}{rk(\boldsymbol{\mu}'_{\text{ODRSS}}\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\boldsymbol{\mu}_{\text{ODRSS}})}$, and $\text{Var}(\hat{\alpha}_{\text{ODRSS}}) = \frac{1}{rk(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\mathbf{1})}$, $\text{Var}(\hat{\beta}_{\text{ODRSS}}) = \frac{1}{r(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\mathbf{1})\sum_{i=1}^{k}(x_i-\bar{x})^2}$, , $\text{Var}(\hat{\sigma}_{\text{ODRSS}}) = \frac{1}{rk(\boldsymbol{\mu}'_{\text{ODRSS}}\boldsymbol{\Omega}^{-1}_{\text{ODRSS}}\boldsymbol{\mu}_{\text{ODRSS}})}$.

Similarly, let $\hat{\boldsymbol{\theta}}_{\text{DRSS}} = (\hat{\alpha}_{\text{DRSS}}, \hat{\beta}_{\text{DRSS}}, \hat{\sigma}_{\text{DRSS}})'$ and $\text{Var}(\hat{\boldsymbol{\theta}}_{\text{DRSS}})$ be the BLUE-DRSS and variance-covariance matrix of $\hat{\boldsymbol{\theta}}_{\text{DRSS}}$, respectively, which can be obtained on similar steps by using (5.3) and (5.4) under DRSS scheme.

It is clear that in order to obtain the BLUEs-DRSS and BLUEs-ODRSS of $\boldsymbol{\theta}$, we need $\boldsymbol{\mu}_{\text{DRSS}}$, $\boldsymbol{\mu}_{\text{ODRSS}}$,

Table 5.1: Means of order statistics from symmetric distributions under different sampling schemes

		OSRS				
		Normal	Laplace	Scale Contaminated Normal		
m	j			$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.10$
2	2	0.564190	0.530330	0.556287	0.535203	0.522495
3	2	0.000000	0.000000	0.000000	0.000000	0.000000
3	3	0.846284	0.795495	0.834431	0.802805	0.783743
4	3	0.297011	0.243068	0.288669	0.264510	0.247037
4	4	1.029375	0.979638	1.016351	0.982237	0.962645
5	3	0.000000	0.000000	0.000000	0.000000	0.000000
5	4	0.495019	0.405113	0.481116	0.440850	0.411729
5	5	1.162964	1.123269	1.150160	1.117584	1.100374
m	j			ORSS		
2	2	0.663193	0.611353	0.652510	0.623374	0.604841
3	2	0.000000	0.000000	0.000000	0.000000	0.000000
3	3	0.964558	0.893827	0.949426	0.908373	0.882612
4	3	0.323265	0.244981	0.313662	0.285089	0.263019
4	4	1.152538	1.089893	1.136317	1.093335	1.068040
5	3	0.000000	0.000000	0.000000	0.000000	0.000000
5	4	0.531580	0.407373	0.515865	0.469179	0.433278
5	5	1.286984	1.242766	1.271219	1.230860	1.209451
m	j			ODRSS		
2	2	0.707860	0.645505	0.695855	0.662791	0.641236
3	2	0.000000	0.000000	0.000000	0.000000	0.000000
3	3	1.011511	0.931552	0.995022	0.949966	0.921192
4	3	0.328321	0.239003	0.318401	0.288723	0.265443
4	4	1.198446	1.131254	1.180981	1.134414	1.106614
5	3	0.000000	0.000000	0.000000	0.000000	0.000000
5	4	0.537555	0.399607	0.521420	0.473234	0.435618
5	5	1.331560	1.286803	1.314677	1.271211	1.247882

Missing values can be found by the symmetry relation $\mu_{(j:m)} = -\mu_{(m-j+1:m)}$.

$\boldsymbol{\Omega}_{\text{DRSS}}$ and $\boldsymbol{\Omega}_{\text{ODRSS}}$. In Tables 5.1, 5.3 and 5.4, we report the means and covariances of random variables based on DRSS and ODRSS schemes for some symmetric distributions such as normal, Laplace and scale contaminated normal distributions. Note that under perfect ranking, DRSS is a special case of ORSS. See next section for more details.

5.5 Performance comparison of estimators

In this section, we provide a comprehensive comparison of the BLUEs based on perfect and imperfect RSS schemes when estimating the unknown parameters of the SLRM.

5.5.1 Perfect ranking

In this section, we compare the performances of the BLUEs based on OSRS, RSS, ORSS, DRSS and ODRSS schemes. Let $\hat{\boldsymbol{\theta}}_{\text{H}}$ and $\text{Var}(\hat{\boldsymbol{\theta}}_{\text{H}})$ be the BLUE and variance-covariance of $\hat{\boldsymbol{\theta}}$ under H sampling scheme, respectively, where H = OSRS, RSS and ORSS.

Note that $\boldsymbol{\mu}_{\text{OSRS}}$ and $\boldsymbol{\Omega}_{\text{OSRS}}$ are the mean and variance-covariance matrix of the standardized statistics obtained from $Y_{i(j:n)}^{\text{OSRS}}$ $(j = 1, 2, ..., n)$, $\boldsymbol{\mu}_{\text{RSS}}$ and $\boldsymbol{\Omega}_{\text{RSS}}$ are the mean and variance-covariance matrix of the standardized statistics obtained from $Y_{i(j:m)t}^{\text{RSS}}$, and $\boldsymbol{\mu}_{\text{ORSS}}$ and $\boldsymbol{\Omega}_{\text{ORSS}}$ are the mean and variance-

Table 5.2: Variances and Covariances of order statistics from symmetric distributions under OSRS

			Normal	Laplace	Scale Contaminated Normal		
m	j	j'			$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.10$
2	1	1	0.681690	0.718750	0.690545	0.713557	0.726999
2	1	2	0.318310	0.281250	0.309455	0.286443	0.273001
3	1	1	0.559467	0.707465	0.591528	0.675830	0.726739
3	1	2	0.275664	0.236111	0.266989	0.244183	0.230431
3	1	3	0.164868	0.160590	0.162302	0.156131	0.153395
3	2	2	0.448671	0.319444	0.424395	0.359348	0.318016
4	1	1	0.491715	0.720866	0.542636	0.675862	0.755299
4	1	2	0.245593	0.234104	0.240588	0.228554	0.223053
4	1	3	0.158008	0.137424	0.153801	0.142409	0.135090
4	1	4	0.104684	0.115940	0.104614	0.105462	0.107688
4	2	2	0.360455	0.260362	0.341065	0.289383	0.256988
4	2	3	0.235944	0.159776	0.222919	0.187369	0.163747
5	1	1	0.447534	0.735125	0.514849	0.689434	0.791124
5	1	2	0.224331	0.241286	0.222508	0.220244	0.222814
5	1	3	0.148148	0.132014	0.145398	0.137740	0.132464
5	1	4	0.105772	0.099738	0.103775	0.098523	0.095469
5	1	5	0.074215	0.092357	0.075332	0.079598	0.084219
5	2	2	0.311519	0.251230	0.295688	0.255200	0.232610
5	2	3	0.208436	0.140087	0.196881	0.165296	0.144237
5	2	4	0.149943	0.107242	0.141893	0.120182	0.106170
5	3	3	0.286834	0.175590	0.270247	0.224047	0.191843

Missing values can be found by the symmetry relation $\sigma^{\text{OSRS}}_{(j,j':m)} = \sigma^{\text{OSRS}}_{(m-j'+1,m-j+1:m)}$.

covariance matrix of the standardized statistics obtained from $Y^{\text{ORSS}}_{i(j:m)t}$, where $i = 1, 2, ..., k$, $j = 1, 2, ..., m$ and $t = 1, 2, ..., r$. It is interesting to note that under perfect ranking, $\boldsymbol{\mu}_{\text{OSRS}} = \boldsymbol{\mu}_{\text{RSS}}$, $\boldsymbol{\mu}_{\text{ORSS}} = \boldsymbol{\mu}_{\text{DRSS}}$, $\boldsymbol{\Omega}_{\text{RSS}} = \text{diag}(\boldsymbol{\Omega}_{\text{OSRS}})$ and $\boldsymbol{\Omega}_{\text{DRSS}} = \text{diag}(\boldsymbol{\Omega}_{\text{ORSS}})$. Tables 5.1–5.4 provide the means, variances and covariances of the order statistics under these sampling schemes for different value of m considered here.

We compare the performance of the BLUEs of parameters, α, β and σ, for all sampling schemes based on relative efficiencies (REs). The RE of $\hat{\alpha}_{\text{H}}$ with respect to $\hat{\alpha}_{\text{OSRS}}$ is defined as $\text{RE}(\hat{\alpha}_{\text{H}}, \hat{\alpha}_{\text{OSRS}}) = \frac{\text{Var}(\hat{\alpha}_{\text{OSRS}})}{\text{Var}(\hat{\alpha}_{\text{H}})} = \frac{\mathbf{1}'\boldsymbol{\Omega}_{\text{H}}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}_{\text{OSRS}}^{-1}\mathbf{1}}$, where H = RSS, ORSS, DRSS and ODRSS. Similarly, we define $\text{RE}(\hat{\beta}_{\text{H}}, \hat{\beta}_{\text{OSRS}})$ and $\text{RE}(\hat{\sigma}_{\text{H}}, \hat{\sigma}_{\text{OSRS}})$. It is interesting to note that all REs are independent of k and r, and $\text{RE}(\hat{\alpha}_{\text{H}}, \hat{\alpha}_{\text{OSRS}}) = \text{RE}(\hat{\beta}_{\text{H}}, \hat{\beta}_{\text{OSRS}})$. Moreover, the overall efficiency of the BLUE of $\boldsymbol{\theta}$ is also evaluated in terms of RED and RET. These REs are defined as follows:

$$\text{RED}(\hat{\boldsymbol{\theta}}_{\text{H}}, \hat{\boldsymbol{\theta}}_{\text{OSRS}}) = \frac{\text{Det}\{\text{Var}(\hat{\boldsymbol{\theta}}_{\text{OSRS}})\}}{\text{Det}\{\text{Var}(\hat{\boldsymbol{\theta}}_{\text{H}})\}} \quad \text{and} \quad \text{RET}(\hat{\boldsymbol{\theta}}_{\text{H}}, \hat{\boldsymbol{\theta}}_{\text{OSRS}}) = \frac{\text{Trace}\{\text{Var}(\hat{\boldsymbol{\theta}}_{\text{OSRS}})\}}{\text{Trace}\{\text{Var}(\hat{\boldsymbol{\theta}}_{\text{H}})\}}.$$

In Tables 5.5 and 5.6, we report the exact REs and REDs of the BLUEs based on all RSS schemes for m =2, 3, 4, 5. We have considered three underlying distributions for the random error term, ξ_{ijt}, i.e., standard normal, standard Laplace and scale contaminated normal distributions. A random variable ξ is said to possess a scale contaminated normal distribution with scale factor δ and a proportion of contamination ϵ, if the distribution function $F(\xi)$ is a mixture of two normal distribution with same means and different

Table 5.3: Variance and Covariance of order statistics from symmetric distributions under ORSS

			Normal	Laplace	Scale Contaminated Normal		
m	j	j'			$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.10$
2	1	1	0.560175	0.626248	0.574230	0.611405	0.634167
2	1	2	0.121516	0.092502	0.116314	0.102152	0.092831
3	1	1	0.431297	0.617032	0.468273	0.566874	0.628606
3	1	2	0.096915	0.075202	0.092835	0.081784	0.074618
3	1	3	0.020345	0.015711	0.019466	0.017077	0.015515
3	2	2	0.276661	0.168083	0.260634	0.215968	0.184779
4	1	1	0.368974	0.624687	0.423554	0.567957	0.656383
4	1	2	0.082712	0.077081	0.080046	0.073211	0.069519
4	1	3	0.021048	0.015297	0.020110	0.017546	0.015835
4	1	4	0.003695	0.003054	0.003535	0.003106	0.002836
4	2	2	0.198181	0.127430	0.186845	0.155386	0.133728
4	2	3	0.073800	0.041301	0.069455	0.057282	0.048633
5	1	1	0.331298	0.630819	0.400999	0.583565	0.692175
5	1	2	0.073722	0.081116	0.072116	0.068555	0.067756
5	1	3	0.020201	0.016056	0.019434	0.017380	0.016078
5	1	4	0.004754	0.003554	0.004538	0.003954	0.003571
5	1	5	0.000692	0.000615	0.000662	0.000582	0.000534
5	2	2	0.161475	0.124319	0.152669	0.128618	0.112964
5	2	3	0.061210	0.035122	0.057622	0.047584	0.040476
5	2	4	0.020910	0.012260	0.019689	0.016275	0.013861
5	3	3	0.136644	0.068886	0.128435	0.105341	0.088718

Missing values can be found by the symmetry relation $\sigma^{\text{ORSS}}_{(j,j':m)} = \sigma^{\text{ORSS}}_{(m-j'+1,m-j+1:m)}$.

variances. For example, without loss of generality, we have

$$F(\xi) = (1-\epsilon)F_1(\xi) + \epsilon F_2(\xi), \quad 0 \le \epsilon \le 1,$$

where $F_1(\cdot)$ and $F_2(\cdot)$ are the distribution functions of $N(0,1)$ and $N(0,\delta^2)$, respectively. Following Leone and Moussa-Hamouda (1973), we set $\delta = 3$, and take ϵ =0.01, 0.05, 0.10.

From Table 5.4, it is observed that both REs and REDs are increasing with the set size m. When estimating α or β, the proposed BLUEs under both DRSS and ODRSS schemes perform uniformly better than their counterparts. In case of standard-deviation estimation of the error term, when $m \le 3$, the BLUEs under both RSS and DRSS are less efficient than the estimates under OSRS and ORSS. However, when the set size m increases, the error estimates under DRSS tend to be more precise as compared with the estimates under OSRS, RSS and ORSS schemes. Furthermore, it is clear from the REs based on determinants that both DRSS and ODRSS provide more efficient BLUEs than the existing BLUEs based on OSRS, RSS and ORSS methods. Almost similar trend of both REs and REDs is observed in Table 5.6. In estimation of individual parameter α, when $m \ge 4$, REs under ORSS and ODRSS tend to increase as the value of ϵ increases and vice-versa. However, under ORSS, DRSS and ODRSS schemes when estimating σ with $m \le 4$, REs tend to decrease as the value of ϵ increases. It is worth mentioning here that for all cases, the estimates under ODRSS are uniformly better than their competitors.

It is clear that the RETs depend on the values of x_i $(i = 1, 2, ..., k)$. Therefore, in order to study the effect

Table 5.4: Variances and Covariances of order statistics from symmetric distributions under ODRSS

			Normal	Laplace	Scale Contaminated Normal		
m	j	j'			$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.10$
2	1	1	0.498934	0.583324	0.515785	0.560708	0.588817
2	1	2	0.061241	0.042924	0.058445	0.050697	0.045350
3	1	1	0.377156	0.577034	0.416143	0.520326	0.585964
3	1	2	0.044542	0.033156	0.042569	0.037135	0.033455
3	1	3	0.003699	0.002550	0.003520	0.003023	0.002681
3	2	2	0.199377	0.110354	0.187577	0.154478	0.130882
4	1	1	0.321365	0.581497	0.377185	0.524955	0.615625
4	1	2	0.036885	0.034441	0.035602	0.032189	0.030139
4	1	3	0.003653	0.002432	0.003475	0.002980	0.002638
4	1	4	0.000180	0.000132	0.000171	0.000145	0.000128
4	2	2	0.134566	0.081646	0.126721	0.104788	0.089320
4	2	3	0.029967	0.015097	0.028171	0.023118	0.019486
5	1	1	0.288308	0.584407	0.358631	0.542775	0.652258
5	1	2	0.032404	0.036211	0.031596	0.029616	0.028806
5	1	3	0.003447	0.002636	0.003298	0.002891	0.002621
5	1	4	0.000232	0.000154	0.000220	0.000186	0.000163
5	1	5	0.000007	0.000006	0.000007	0.000006	0.000005
5	2	2	0.106894	0.080446	0.100930	0.084434	0.073203
5	2	3	0.023737	0.012651	0.022324	0.018355	0.015512
5	2	4	0.003466	0.001819	0.003259	0.002678	0.002261
5	3	3	0.085561	0.039196	0.080367	0.065727	0.055134

Missing values can be found by the symmetry relation $\sigma^{\text{ODRSS}}_{(j,j':m)} = \sigma^{\text{ODRSS}}_{(m-j'+1,m-j+1:m)}$.

of the values of x_i on the RETs, following Li and Balakrishnan (2008), we consider equally spaced values of x_i, i.e., $x_i = x + (i-1)h$, for $i = 1, 2, ..., k$, where h is the distance between the two consecutive x_is. For this case, the RET of $\hat{\boldsymbol{\theta}}_{\text{H}}$ with respect to $\hat{\boldsymbol{\theta}}_{\text{OSRS}}$ is given by

$$\text{RET}(\hat{\boldsymbol{\theta}}_{\text{H}}, \hat{\boldsymbol{\theta}}_{\text{OSRS}}) = \frac{(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{OSRS}}\mathbf{1})^{-1} + 12\{h^2(k^2-1)(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{OSRS}}\mathbf{1})\}^{-1} + (\boldsymbol{\mu}'_{\text{OSRS}}\boldsymbol{\Omega}^{-1}_{\text{OSRS}}\boldsymbol{\mu}_{\text{OSRS}})^{-1}}{(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{H}}\mathbf{1})^{-1} + 12\{h^2(k^2-1)(\mathbf{1}'\boldsymbol{\Omega}^{-1}_{\text{H}}\mathbf{1})\}^{-1} + (\boldsymbol{\mu}'_{\text{H}}\boldsymbol{\Omega}^{-1}_{\text{H}}\boldsymbol{\mu}_{\text{H}})^{-1}},$$

where H = RSS, ORSS, DRSS and ODRSS.

In Tables 5.7–5.11, for different values of m, k and h, we report the exact RETs of the BLUEs based on

Table 5.5: REs of BLUEs based on OSRS, RSS, ORSS, DRSS and ODRSS schemes

Distribution	Standard Normal				Standard Laplace			
Relative efficiency	$m=2$	$m=3$	$m=4$	$m=5$	$m=2$	$m=3$	$m=4$	$m=5$
$\text{RE}(\hat{\alpha}_{\text{RSS}}, \hat{\alpha}_{\text{OSRS}})$	1.4669	1.9345	2.4040	2.8751	1.3913	1.7560	2.1725	2.5945
$\text{RE}(\hat{\alpha}_{\text{ORSS}}, \hat{\alpha}_{\text{OSRS}})$	1.4669	1.9382	2.4126	2.8889	1.3913	2.0720	2.6102	3.1589
$\text{RE}(\hat{\alpha}_{\text{DRSS}}, \hat{\alpha}_{\text{OSRS}})$	1.7852	2.7506	3.8781	5.1482	1.5968	2.7090	3.9261	5.3508
$\text{RE}(\hat{\alpha}_{\text{ODRSS}}, \hat{\alpha}_{\text{OSRS}})$	1.7852	2.7858	3.9503	5.2549	1.5968	3.1865	4.5757	6.2148
$\text{RE}(\hat{\sigma}_{\text{RSS}}, \hat{\sigma}_{\text{OSRS}})$	0.5331	0.7053	0.8641	1.0156	0.6087	0.7730	0.9306	1.0852
$\text{RE}(\hat{\sigma}_{\text{ORSS}}, \hat{\sigma}_{\text{OSRS}})$	1.1446	1.2474	1.3618	1.4822	1.0893	1.1482	1.2807	1.4115
$\text{RE}(\hat{\sigma}_{\text{DRSS}}, \hat{\sigma}_{\text{OSRS}})$	0.8963	1.1885	1.4863	1.7997	0.9284	1.1190	1.4170	1.7326
$\text{RE}(\hat{\sigma}_{\text{ODRSS}}, \hat{\sigma}_{\text{OSRS}})$	1.3069	1.5095	1.7589	2.0439	1.1994	1.3054	1.6026	1.9123
$\text{RED}(\hat{\boldsymbol{\theta}}_{\text{RSS}}, \hat{\boldsymbol{\theta}}_{\text{OSRS}})$	1.1471	2.6396	4.9939	8.3947	1.1783	2.3835	4.3922	7.3049
$\text{RED}(\hat{\boldsymbol{\theta}}_{\text{ORSS}}, \hat{\boldsymbol{\theta}}_{\text{OSRS}})$	2.4631	4.6856	7.9269	12.3695	2.1085	4.9292	8.7254	14.0855
$\text{RED}(\hat{\boldsymbol{\theta}}_{\text{DRSS}}, \hat{\boldsymbol{\theta}}_{\text{OSRS}})$	2.8564	8.9918	22.3525	47.6992	2.3672	8.2117	21.8418	49.6072
$\text{RED}(\hat{\boldsymbol{\theta}}_{\text{ODRSS}}, \hat{\boldsymbol{\theta}}_{\text{OSRS}})$	4.1648	11.7142	27.4481	56.4418	3.0583	13.2545	33.5535	73.8583

Here $\text{RE}(\hat{\alpha}_{\text{H}}, \hat{\alpha}_{\text{OSRS}}) = \text{RE}(\hat{\beta}_{\text{H}}, \hat{\beta}_{\text{OSRS}})$ for H = RSS, ORSS, DRSS and ODRSS.

Table 5.6: REs of BLUEs based on OSRS, RSS, ORSS, DRSS, ODRSS for scale contaminated normal distributions

Distribution	$\epsilon = 0.01$				$\epsilon = 0.05$				$\epsilon = 0.10$			
Relative efficiency	$m=2$	$m=3$	$m=4$	$m=5$	$m=2$	$m=3$	$m=4$	$m=5$	$m=2$	$m=3$	$m=4$	$m=5$
$\mathrm{RE}(\hat\alpha_{\mathrm{RSS}}, \hat\alpha_{\mathrm{OSRS}})$	1.4481	1.9032	2.3678	2.8381	1.4014	1.7980	2.2459	2.7118	1.3755	1.7230	2.1582	2.6190
$\mathrm{RE}(\hat\alpha_{\mathrm{ORSS}}, \hat\alpha_{\mathrm{OSRS}})$	1.4481	1.9250	2.4093	2.8972	1.4014	1.9130	2.4207	2.9289	1.3755	1.9371	2.4584	2.9728
$\mathrm{RE}(\hat\alpha_{\mathrm{DRSS}}, \hat\alpha_{\mathrm{OSRS}})$	1.7415	2.6895	3.8249	5.1176	1.6356	2.5546	3.7299	5.0789	1.5769	2.5111	3.7250	5.1079
$\mathrm{RE}(\hat\alpha_{\mathrm{ODRSS}}, \hat\alpha_{\mathrm{OSRS}})$	1.7415	2.7588	3.9560	5.3024	1.6356	2.7419	4.0220	5.4463	1.5769	2.7992	4.1369	5.6039
$\mathrm{RE}(\hat\sigma_{\mathrm{RSS}}, \hat\sigma_{\mathrm{OSRS}})$	0.5519	0.7256	0.8826	1.0320	0.5986	0.7690	0.9057	1.0391	0.6245	0.7889	0.9168	1.0492
$\mathrm{RE}(\hat\sigma_{\mathrm{ORSS}}, \hat\sigma_{\mathrm{OSRS}})$	1.1450	1.2381	1.3498	1.4725	1.1378	1.2102	1.3325	1.4745	1.1238	1.1860	1.3312	1.4961
$\mathrm{RE}(\hat\sigma_{\mathrm{DRSS}}, \hat\sigma_{\mathrm{OSRS}})$	0.9131	1.1867	1.4690	1.7772	0.9477	1.1737	1.4258	1.7394	0.9593	1.1567	1.4119	1.7532
$\mathrm{RE}(\hat\sigma_{\mathrm{ODRSS}}, \hat\sigma_{\mathrm{OSRS}})$	1.3039	1.4792	1.7234	2.0193	1.2843	1.4067	1.6754	2.0257	1.2582	1.3580	1.6714	2.0753
$\mathrm{RED}(\hat{\boldsymbol\theta}_{\mathrm{RSS}}, \hat{\boldsymbol\theta}_{\mathrm{OSRS}})$	1.1573	2.6282	4.9483	8.3123	1.1756	2.4861	4.5683	7.6412	1.1816	2.3421	4.2704	7.1967
$\mathrm{RED}(\hat{\boldsymbol\theta}_{\mathrm{ORSS}}, \hat{\boldsymbol\theta}_{\mathrm{OSRS}})$	2.4012	4.5878	7.8351	12.3596	2.2347	4.4289	7.8084	12.6491	2.1264	4.4505	8.0451	13.2219
$\mathrm{RED}(\hat{\boldsymbol\theta}_{\mathrm{DRSS}}, \hat{\boldsymbol\theta}_{\mathrm{OSRS}})$	2.7691	8.5835	21.4904	46.5452	2.5352	7.6601	19.8354	44.8693	2.3854	7.2937	19.5914	45.7418
$\mathrm{RED}(\hat{\boldsymbol\theta}_{\mathrm{ODRSS}}, \hat{\boldsymbol\theta}_{\mathrm{OSRS}})$	3.9542	11.2575	26.9714	56.7734	3.4358	10.5754	27.1022	60.0862	3.1286	10.6403	28.6041	65.1702

Here $\mathrm{RE}(\hat\alpha_{\mathrm{H}}, \hat\alpha_{\mathrm{OSRS}}) = \mathrm{RE}(\hat\beta_{\mathrm{H}}, \hat\beta_{\mathrm{OSRS}})$ for H = RSS, ORSS, DRSS and ODRSS.

Table 5.7: Trace REs of BLUEs based on RSS, ORSS, DRSS, ODRSS relative to OSRS for normal distribution

		$m=2$				$m=3$				$m=4$				$m=5$			
h	k	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS
0.25	2	1.4239	1.4598	1.7551	1.7740	1.8931	1.9248	2.7059	2.7565	2.3579	2.3924	3.8108	3.8971	2.8226	2.8613	5.0528	5.1725
	4	1.2938	1.4360	1.6594	1.7366	1.7611	1.8793	2.5604	2.6587	2.2088	2.3237	3.5914	3.7204	2.6514	2.7678	4.7417	4.9002
	6	1.1622	1.4076	1.5544	1.6925	1.6162	1.8240	2.3948	2.5428	2.0406	2.2399	3.3407	3.5129	2.4560	2.6541	4.3874	4.5838
	8	1.0588	1.3814	1.4655	1.6521	1.4933	1.7718	2.2493	2.4365	1.8941	2.1608	3.1197	3.3246	2.2838	2.5470	4.0759	4.2998
	10	0.9845	1.3598	1.3975	1.6194	1.3993	1.7281	2.1348	2.3498	1.7798	2.0946	2.9453	3.1725	2.1480	2.4577	3.8307	4.0723
1.00	2	1.1066	1.3940	1.5073	1.6715	1.5511	1.7970	2.3184	2.4874	1.9635	2.1990	3.2247	3.4147	2.3656	2.5987	4.2237	4.4353
	4	0.8732	1.3224	1.2891	1.5631	1.2494	1.6505	1.9458	2.2003	1.5928	1.9767	2.6564	2.9131	1.9233	2.2991	3.4257	3.6888
	6	0.8127	1.2989	1.2264	1.5282	1.1626	1.6005	1.8328	2.1070	1.4820	1.9007	2.4831	2.7530	1.7886	2.1971	3.1834	3.4546
	8	0.7897	1.2892	1.2018	1.5139	1.1286	1.5797	1.7878	2.0690	1.4380	1.8690	2.4139	2.6880	1.7348	2.1548	3.0867	3.3601
	10	0.7787	1.2845	1.1899	1.5069	1.1121	1.5694	1.7658	2.0502	1.4166	1.8533	2.3801	2.6561	1.7086	2.1338	3.0395	3.3138
2.00	2	0.8963	1.3308	1.3123	1.5756	1.2815	1.6680	1.9870	2.2336	1.6334	2.0034	2.7194	2.9705	1.9723	2.3349	3.5139	3.7731
	4	0.7912	1.2899	1.2034	1.5149	1.1308	1.5811	1.7907	2.0715	1.4409	1.8711	2.4184	2.6923	1.7384	2.1576	3.0930	3.3663
	6	0.7729	1.2819	1.1836	1.5032	1.1034	1.5638	1.7541	2.0402	1.4053	1.8449	2.3621	2.6391	1.6947	2.1225	3.0145	3.2892
	8	0.7666	1.2791	1.1767	1.4991	1.0938	1.5577	1.7413	2.0292	1.3928	1.8356	2.3424	2.6203	1.6793	2.1101	2.9870	3.2621
	10	0.7637	1.2778	1.1735	1.4972	1.0894	1.5549	1.7354	2.0240	1.3871	1.8313	2.3333	2.6116	1.6722	2.1043	2.9743	3.2495

Table 5.8: Trace REs of BLUEs based on RSS, ORSS, DRSS, ODRSS relative to OSRS for Laplace distribution

		$m=2$				$m=3$				$m=4$				$m=5$			
h	k	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS
0.25	2	1.3507	1.3823	1.5704	1.5845	1.7081	2.0358	2.6267	3.0883	2.1115	2.5528	3.7813	4.3991	2.5183	3.0761	5.1183	5.9248
	4	1.2310	1.3533	1.4883	1.5450	1.5649	1.9234	2.3837	2.7991	1.9296	2.3774	3.3642	3.8942	2.2922	2.8271	4.4665	5.1222
	6	1.1142	1.3205	1.4016	1.5006	1.4225	1.8043	2.1466	2.5176	1.7491	2.1965	2.9716	3.4235	2.0699	2.5775	3.8758	4.4075
	8	1.0253	1.2919	1.3308	1.4623	1.3123	1.7068	1.9662	2.3038	1.6096	2.0520	2.6817	3.0789	1.8994	2.3828	3.4533	3.9037
	10	0.9629	1.2695	1.2783	1.4326	1.2339	1.6344	1.8393	2.1537	1.5105	1.9466	2.4825	2.8434	1.7789	2.2435	3.1692	3.5683
1.00	2	1.0661	1.3054	1.3638	1.4804	1.3630	1.7523	2.0489	2.4018	1.6738	2.1190	2.8137	3.2355	1.9777	2.4725	3.6442	4.1306
	4	0.8717	1.2328	1.1972	1.3842	1.1179	1.5222	1.6540	1.9350	1.3642	1.7867	2.1980	2.5091	1.6021	2.0363	2.7724	3.1042
	6	0.8232	1.2111	1.1517	1.3557	1.0555	1.4592	1.5554	1.8187	1.2856	1.6986	2.0497	2.3356	1.5076	1.9242	2.5694	2.8688
	8	0.8050	1.2024	1.1342	1.3445	1.0319	1.4349	1.5183	1.7750	1.2559	1.6649	1.9944	2.2711	1.4719	1.8816	2.4944	2.7822
	10	0.7963	1.1982	1.1258	1.3390	1.0206	1.4232	1.5007	1.7542	1.2417	1.6487	1.9682	2.2406	1.4549	1.8613	2.4590	2.7413
2.00	2	0.8904	1.2408	1.2143	1.3946	1.1418	1.5459	1.6920	1.9798	1.3944	1.8201	2.2558	2.5768	1.6385	2.0792	2.8522	3.1971
	4	0.8062	1.2030	1.1353	1.3452	1.0334	1.4365	1.5207	1.7778	1.2578	1.6671	1.9980	2.2753	1.4742	1.8844	2.4992	2.7877
	6	0.7918	1.1960	1.1213	1.3361	1.0147	1.4170	1.4914	1.7433	1.2342	1.6402	1.9545	2.2246	1.4460	1.8507	2.4405	2.7200
	8	0.7868	1.1935	1.1165	1.3329	1.0082	1.4102	1.4813	1.7314	1.2261	1.6309	1.9396	2.2072	1.4363	1.8390	2.4204	2.6968
	10	0.7845	1.1924	1.1142	1.3314	1.0053	1.4071	1.4767	1.7259	1.2224	1.6266	1.9327	2.1992	1.4319	1.8337	2.4111	2.6862

Table 5.9: Trace REs of BLUEs based on RSS, ORSS, DRSS, ODRSS relative to OSRS for scale contaminated normal distribution with $\epsilon = 0.01$

		$m=2$				$m=3$				$m=4$				$m=5$			
h	k	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS
0.25	2	1.4057	1.4410	1.7126	1.7307	1.8606	1.9100	2.6423	2.7255	2.3187	2.3857	3.7492	3.8925	2.7805	2.8644	5.0063	5.2024
	4	1.2781	1.4174	1.6210	1.6949	1.7264	1.8599	2.4905	2.6160	2.1618	2.3067	3.5064	3.6856	2.5956	2.7549	4.6510	4.8790
	6	1.1501	1.3894	1.5212	1.6529	1.5815	1.7998	2.3211	2.4889	1.9887	2.2125	3.2368	3.4498	2.3900	2.6251	4.2598	4.5158
	8	1.0502	1.3639	1.4373	1.6150	1.4606	1.7443	2.1752	2.3751	1.8413	2.1258	3.0057	3.2423	2.2134	2.5062	3.9273	4.2010
	10	0.9788	1.3432	1.3735	1.5844	1.3694	1.6986	2.0622	2.2841	1.7283	2.0546	2.8276	3.0789	2.0771	2.4092	3.6726	3.9560
1.00	2	1.0963	1.3761	1.4767	1.6331	1.5173	1.7710	2.2441	2.4294	1.9107	2.1674	3.1147	3.3408	2.2967	2.5632	4.0838	4.3499
	4	0.8725	1.3075	1.2725	1.5325	1.2259	1.6192	1.8792	2.1311	1.5471	1.9314	2.5405	2.8090	1.8571	2.2423	3.2652	3.5569
	6	0.8150	1.2854	1.2144	1.5005	1.1439	1.5690	1.7718	2.0378	1.4419	1.8540	2.3728	2.6474	1.7283	2.1382	3.0289	3.3212
	8	0.7932	1.2763	1.1917	1.4876	1.1121	1.5484	1.7294	2.0002	1.4006	1.8223	2.3067	2.5829	1.6775	2.0957	2.9361	3.2279
	10	0.7827	1.2719	1.1808	1.4812	1.0967	1.5383	1.7088	1.9818	1.3805	1.8067	2.2746	2.5514	1.6529	2.0747	2.8912	3.1824
2.00	2	0.8945	1.3154	1.2941	1.5440	1.2564	1.6369	1.9187	2.1647	1.5860	1.9589	2.6023	2.8678	1.9045	2.2794	3.3525	3.6432
	4	0.7946	1.2769	1.1932	1.4884	1.1141	1.5498	1.7322	2.0027	1.4033	1.8244	2.3110	2.5871	1.6808	2.0985	2.9422	3.2339
	6	0.7773	1.2695	1.1750	1.4779	1.0885	1.5328	1.6979	1.9720	1.3699	1.7983	2.2577	2.5347	1.6398	2.0636	2.8674	3.1584
	8	0.7713	1.2669	1.1686	1.4741	1.0796	1.5268	1.6859	1.9613	1.3583	1.7891	2.2390	2.5164	1.6255	2.0513	2.8413	3.1320
	10	0.7685	1.2657	1.1657	1.4724	1.0755	1.5241	1.6804	1.9563	1.3529	1.7848	2.2304	2.5079	1.6189	2.0456	2.8293	3.1197

Table 5.10: Trace REs of BLUEs based on RSS, ORSS, DRSS, ODRSS relative to OSRS for scale contaminated normal distribution with $\epsilon = 0.05$

		$m=2$				$m=3$				$m=4$				$m=5$			
h	k	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS
0.25	2	1.3605	1.3942	1.6094	1.6256	1.7525	1.8917	2.4976	2.6922	2.1876	2.3856	3.6244	3.9230	2.6391	2.8803	4.9173	5.2933
	4	1.2393	1.3705	1.5275	1.5931	1.6137	1.8227	2.3216	2.5364	2.0094	2.2731	3.3051	3.6189	2.4169	2.7252	4.4337	4.8283
	6	1.1205	1.3433	1.4402	1.5560	1.4718	1.7451	2.1380	2.3694	1.8262	2.1483	2.9816	3.3035	2.1884	2.5545	3.9517	4.3539
	8	1.0296	1.3191	1.3682	1.5234	1.3591	1.6779	1.9897	2.2309	1.6802	2.0414	2.7271	3.0500	2.0061	2.4096	3.5779	3.9783
	10	0.9656	1.3001	1.3145	1.4977	1.2773	1.6255	1.8807	2.1269	1.5738	1.9590	2.5437	2.8643	1.8734	2.2986	3.3115	3.7064
1.00	2	1.0713	1.3306	1.4018	1.5389	1.4113	1.7097	2.0587	2.2957	1.7479	2.0918	2.8447	3.1677	2.0906	2.4778	3.7500	4.1522
	4	0.8717	1.2683	1.2308	1.4553	1.1540	1.5400	1.7138	1.9644	1.4129	1.8264	2.2689	2.5814	1.6724	2.1212	2.9170	3.2970
	6	0.8216	1.2492	1.1836	1.4298	1.0864	1.4896	1.6211	1.8721	1.3243	1.7490	2.1192	2.4248	1.5619	2.0184	2.7044	3.0730
	8	0.8027	1.2415	1.1653	1.4197	1.0606	1.4696	1.5855	1.8363	1.2904	1.7185	2.0623	2.3647	1.5196	1.9781	2.6239	2.9876
	10	0.7937	1.2377	1.1565	1.4147	1.0482	1.4598	1.5683	1.8190	1.2742	1.7037	2.0350	2.3359	1.4993	1.9585	2.5854	2.9467
2.00	2	0.8910	1.2752	1.2485	1.4645	1.1797	1.5585	1.7488	1.9988	1.4465	1.8549	2.3260	2.6406	1.7144	2.1592	2.9985	3.3823
	4	0.8039	1.2420	1.1665	1.4204	1.0623	1.4709	1.5878	1.8386	1.2926	1.7205	2.0659	2.3686	1.5223	1.9807	2.6291	2.9931
	6	0.7890	1.2358	1.1518	1.4121	1.0417	1.4547	1.5593	1.8099	1.2657	1.6958	2.0207	2.3207	1.4887	1.9482	2.5653	2.9252
	8	0.7839	1.2336	1.1467	1.4092	1.0346	1.4490	1.5495	1.7999	1.2563	1.6873	2.0051	2.3042	1.4770	1.9368	2.5433	2.9017
	10	0.7815	1.2325	1.1444	1.4079	1.0314	1.4464	1.5450	1.7953	1.2520	1.6833	1.9979	2.2965	1.4716	1.9316	2.5331	2.8909

Table 5.11: Trace REs of BLUEs based on RSS, ORSS, DRSS, ODRSS relative to OSRS for scale contaminated normal distribution with $\epsilon = 0.10$

		$m=2$				$m=3$				$m=4$				$m=5$			
h	k	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS	RSS	ORSS	DRSS	ODRSS
0.25	2	1.3355	1.3679	1.5519	1.5670	1.6754	1.9082	2.4425	2.7297	2.0937	2.4119	3.5912	4.0026	2.5363	2.9102	4.9033	5.4036
	4	1.2180	1.3432	1.4748	1.5351	1.5345	1.8177	2.2391	2.5216	1.9036	2.2686	3.2061	3.6107	2.2931	2.7180	4.3222	4.8245
	6	1.1044	1.3154	1.3938	1.4994	1.3963	1.7217	2.0393	2.3139	1.7182	2.1190	2.8438	3.2345	2.0571	2.5189	3.7854	4.2763
	8	1.0187	1.2913	1.3279	1.4686	1.2906	1.6428	1.8862	2.1527	1.5772	1.9981	2.5766	2.9521	1.8783	2.3591	3.3955	3.8697
	10	0.9588	1.2725	1.2793	1.4447	1.2161	1.5840	1.7781	2.0378	1.4781	1.9091	2.3929	2.7557	1.7531	2.2422	3.1304	3.5892
1.00	2	1.0580	1.3027	1.3586	1.4831	1.3391	1.6796	1.9565	2.2270	1.6418	2.0543	2.6982	3.0811	1.9602	2.4333	3.5723	4.0550
	4	0.8719	1.2420	1.2045	1.4059	1.1069	1.4927	1.6196	1.8674	1.3336	1.7728	2.1308	2.4718	1.5709	2.0642	2.7562	3.1873
	6	0.8260	1.2239	1.1628	1.3831	1.0486	1.4413	1.5348	1.7755	1.2567	1.6971	1.9942	2.3222	1.4743	1.9658	2.5629	2.9770
	8	0.8087	1.2167	1.1467	1.3741	1.0266	1.4214	1.5029	1.7407	1.2278	1.6680	1.9433	2.2661	1.4380	1.9281	2.4913	2.8986
	10	0.8005	1.2132	1.1390	1.3697	1.0161	1.4118	1.4877	1.7241	1.2141	1.6540	1.9192	2.2395	1.4208	1.9100	2.4573	2.8613
2.00	2	0.8897	1.2486	1.2203	1.4143	1.1293	1.5120	1.6522	1.9026	1.3632	1.8014	2.1841	2.5298	1.6083	2.1015	2.8318	3.2690
	4	0.8098	1.2172	1.1477	1.3746	1.0280	1.4227	1.5049	1.7430	1.2297	1.6698	1.9466	2.2698	1.4404	1.9306	2.4959	2.9036
	6	0.7962	1.2114	1.1349	1.3674	1.0107	1.4067	1.4797	1.7154	1.2068	1.6466	1.9066	2.2256	1.4117	1.9005	2.4396	2.8419
	8	0.7915	1.2094	1.1304	1.3648	1.0047	1.4012	1.4710	1.7059	1.1990	1.6386	1.8928	2.2104	1.4019	1.8901	2.4203	2.8206
	10	0.7894	1.2084	1.1284	1.3636	1.0019	1.3987	1.4670	1.7015	1.1954	1.6349	1.8865	2.2034	1.3973	1.8853	2.4115	2.8109

Table 5.12: EMSEs of unknown parameters of SLRM under RSS schemes for normal distribution

		RSS			ORSS			DRSS			ODRSS		
m	σ_V^2	$\hat{\alpha}_{\text{RSS}}$	$\hat{\beta}_{\text{RSS}}$	$\hat{\sigma}_{\text{RSS}}$	$\hat{\alpha}_{\text{ORSS}}$	$\hat{\beta}_{\text{ORSS}}$	$\hat{\sigma}_{\text{ORSS}}$	$\hat{\alpha}_{\text{DRSS}}$	$\hat{\beta}_{\text{DRSS}}$	$\hat{\sigma}_{\text{DRSS}}$	$\hat{\alpha}_{\text{ODRSS}}$	$\hat{\beta}_{\text{ODRSS}}$	$\hat{\sigma}_{\text{ODRSS}}$
2	0.05	0.0698	0.0348	0.2196	0.0698	0.0348	0.0996	0.0581	0.0290	0.1328	0.0581	0.0291	0.0877
	0.15	0.0723	0.0362	0.2317	0.0724	0.0361	0.0992	0.0618	0.0309	0.1448	0.0618	0.0308	0.0888
	0.30	0.0755	0.0378	0.2529	0.0754	0.0377	0.0990	0.0662	0.0331	0.1655	0.0661	0.0330	0.0907
	0.50	0.0786	0.0394	0.2815	0.0788	0.0394	0.0991	0.0707	0.0353	0.1943	0.0707	0.0354	0.0928
3	0.05	0.0361	0.0180	0.0815	0.0360	0.0180	0.0447	0.0264	0.0132	0.0498	0.0261	0.0131	0.0376
	0.15	0.0387	0.0193	0.0907	0.0387	0.0194	0.0454	0.0302	0.0151	0.0588	0.0300	0.0151	0.0396
	0.30	0.0420	0.0210	0.1072	0.0421	0.0211	0.0465	0.0347	0.0174	0.0755	0.0347	0.0173	0.0423
	0.50	0.0456	0.0227	0.1323	0.0456	0.0227	0.0478	0.0395	0.0197	0.1003	0.0392	0.0196	0.0453
4	0.05	0.0223	0.0111	0.0445	0.0222	0.0111	0.0271	0.0149	0.0074	0.0271	0.0147	0.0074	0.0218
	0.15	0.0247	0.0124	0.0522	0.0248	0.0124	0.0282	0.0184	0.0092	0.0352	0.0182	0.0091	0.0241
	0.30	0.0278	0.0139	0.0676	0.0279	0.0139	0.0296	0.0225	0.0112	0.0507	0.0223	0.0112	0.0268
	0.50	0.0309	0.0154	0.0908	0.0310	0.0155	0.0310	0.0267	0.0134	0.0742	0.0265	0.0132	0.0297
5	0.05	0.0153	0.0076	0.0286	0.0152	0.0076	0.0186	0.0095	0.0048	0.0174	0.0094	0.0047	0.0144
	0.15	0.0175	0.0087	0.0356	0.0174	0.0087	0.0198	0.0126	0.0063	0.0249	0.0124	0.0062	0.0167
	0.30	0.0203	0.0101	0.0500	0.0203	0.0101	0.0213	0.0163	0.0082	0.0397	0.0161	0.0081	0.0193
	0.50	0.0231	0.0115	0.0726	0.0230	0.0116	0.0227	0.0202	0.0101	0.0624	0.0197	0.0099	0.0219

Table 5.13: EMSEs of unknown parameters of SLRM under RSS schemes for Laplace distribution

		RSS			ORSS			DRSS			ODRSS		
m	σ_V^2	$\hat{\alpha}_{\text{RSS}}$	$\hat{\beta}_{\text{RSS}}$	$\hat{\sigma}_{\text{RSS}}$	$\hat{\alpha}_{\text{ORSS}}$	$\hat{\beta}_{\text{ORSS}}$	$\hat{\sigma}_{\text{ORSS}}$	$\hat{\alpha}_{\text{DRSS}}$	$\hat{\beta}_{\text{DRSS}}$	$\hat{\sigma}_{\text{DRSS}}$	$\hat{\alpha}_{\text{ODRSS}}$	$\hat{\beta}_{\text{ODRSS}}$	$\hat{\sigma}_{\text{ODRSS}}$
2	0.05	0.1451	0.0729	0.5182	0.1455	0.0729	0.2853	0.1278	0.0639	0.3430	0.1278	0.0639	0.2606
	0.15	0.1486	0.0741	0.5318	0.1488	0.0746	0.2847	0.1320	0.0662	0.3561	0.1321	0.0661	0.2616
	0.30	0.1519	0.0760	0.5531	0.1518	0.0763	0.2839	0.1374	0.0686	0.3798	0.1373	0.0686	0.2626
	0.50	0.1565	0.0782	0.5852	0.1564	0.0781	0.2825	0.1425	0.0714	0.4129	0.1426	0.0712	0.2643
3	0.05	0.0690	0.0345	0.2285	0.0587	0.0294	0.1513	0.0463	0.0232	0.1596	0.0397	0.0199	0.1338
	0.15	0.0719	0.0361	0.2396	0.0620	0.0310	0.1524	0.0516	0.0258	0.1712	0.0445	0.0222	0.1369
	0.30	0.0764	0.0382	0.2574	0.0662	0.0330	0.1539	0.0583	0.0291	0.1897	0.0505	0.0252	0.1408
	0.50	0.0813	0.0407	0.2826	0.0708	0.0354	0.1555	0.0663	0.0331	0.2162	0.0569	0.0285	0.1444
4	0.05	0.0399	0.0200	0.1328	0.0336	0.0167	0.0945	0.0237	0.0119	0.0897	0.0206	0.0103	0.0772
	0.15	0.0430	0.0215	0.1424	0.0366	0.0182	0.0967	0.0280	0.0140	0.1022	0.0246	0.0123	0.0816
	0.30	0.0469	0.0234	0.1594	0.0402	0.0201	0.0987	0.0337	0.0168	0.1221	0.0296	0.0147	0.0869
	0.50	0.0513	0.0256	0.1840	0.0442	0.0221	0.1012	0.0400	0.0200	0.1506	0.0347	0.0173	0.0921
5	0.05	0.0260	0.0130	0.0881	0.0216	0.0108	0.0663	0.0141	0.0070	0.0580	0.0124	0.0062	0.0509
	0.15	0.0287	0.0143	0.0972	0.0243	0.0122	0.0687	0.0180	0.0090	0.0700	0.0160	0.0080	0.0556
	0.30	0.0322	0.0161	0.1138	0.0276	0.0138	0.0711	0.0230	0.0115	0.0899	0.0202	0.0101	0.0612
	0.50	0.0363	0.0181	0.1377	0.0310	0.0155	0.0740	0.0285	0.0143	0.1181	0.0244	0.0122	0.0660

Table 5.14: EMSEs of unknown parameters of SLRM under RSS schemes for scale contaminated normal distribution with $\epsilon = 0.05$

		RSS			ORSS			DRSS			ODRSS		
m	σ_V^2	$\hat{\alpha}_{\mathrm{RSS}}$	$\hat{\beta}_{\mathrm{RSS}}$	$\hat{\sigma}_{\mathrm{RSS}}$	$\hat{\alpha}_{\mathrm{ORSS}}$	$\hat{\beta}_{\mathrm{ORSS}}$	$\hat{\sigma}_{\mathrm{ORSS}}$	$\hat{\alpha}_{\mathrm{DRSS}}$	$\hat{\beta}_{\mathrm{DRSS}}$	$\hat{\sigma}_{\mathrm{DRSS}}$	$\hat{\alpha}_{\mathrm{ODRSS}}$	$\hat{\beta}_{\mathrm{ODRSS}}$	$\hat{\sigma}_{\mathrm{ODRSS}}$
2	0.05	0.1016	0.0508	0.3552	0.1015	0.0507	0.1825	0.0878	0.0440	0.2263	0.0877	0.0438	0.1631
	0.15	0.1043	0.0522	0.3703	0.1045	0.0520	0.1827	0.0919	0.0459	0.2405	0.0919	0.0460	0.1646
	0.30	0.1078	0.0539	0.3921	0.1077	0.0541	0.1828	0.0966	0.0484	0.2641	0.0966	0.0482	0.1660
	0.50	0.1114	0.0558	0.4229	0.1116	0.0557	0.1823	0.1014	0.0509	0.2964	0.1012	0.0507	0.1687
3	0.05	0.0507	0.0253	0.1514	0.0477	0.0238	0.0940	0.0370	0.0186	0.1006	0.0346	0.0173	0.0816
	0.15	0.0538	0.0269	0.1611	0.0508	0.0254	0.0949	0.0418	0.0209	0.1109	0.0392	0.0196	0.0841
	0.30	0.0577	0.0288	0.1805	0.0548	0.0273	0.0960	0.0477	0.0239	0.1300	0.0447	0.0224	0.0876
	0.50	0.0616	0.0308	0.2072	0.0586	0.0293	0.0978	0.0539	0.0269	0.1577	0.0503	0.0250	0.0911
4	0.05	0.0300	0.0150	0.0877	0.0280	0.0140	0.0581	0.0196	0.0098	0.0577	0.0182	0.0091	0.0475
	0.15	0.0331	0.0165	0.0971	0.0310	0.0155	0.0597	0.0239	0.0120	0.0683	0.0222	0.0111	0.0511
	0.30	0.0367	0.0184	0.1146	0.0344	0.0172	0.0618	0.0293	0.0147	0.0879	0.0271	0.0135	0.0553
	0.50	0.0406	0.0203	0.1411	0.0381	0.0190	0.0638	0.0348	0.0174	0.1159	0.0318	0.0159	0.0592
5	0.05	0.0200	0.0100	0.0578	0.0186	0.0093	0.0395	0.0120	0.0060	0.0370	0.0112	0.0056	0.0304
	0.15	0.0227	0.0113	0.0666	0.0212	0.0106	0.0416	0.0157	0.0079	0.0478	0.0147	0.0073	0.0346
	0.30	0.0260	0.0130	0.0837	0.0243	0.0122	0.0437	0.0204	0.0102	0.0677	0.0188	0.0094	0.0387
	0.50	0.0295	0.0147	0.1097	0.0276	0.0138	0.0459	0.0251	0.0125	0.0968	0.0228	0.0114	0.0427

different RSS schemes for normal, Laplace and scale contaminated normal distributions. From Tables 5.7–5.11, it is observed that for all cases, the RETs under ODRSS are uniformly better than its counterparts. Under each sampling scheme, given h, the RETs tend to increase as the value of k decreases. The similar trend is observed when k is fixed and h varies. For small spacing, i.e., $h = 0.25$, DRSS provides more efficient estimates than the estimates with RSS and ORSS for all values of m. Moreover, it also gives uniformly better estimates than its counterparts when $m \geq 3$ except the corresponding estimates with ODRSS.

5.5.2 Imperfect ranking

In this section, a detailed simulation study is conducted in order to study the performances of the BLUEs under imperfect RSS schemes.

The performance of an estimator obtained under DRSS scheme depends on the accuracy of ranking. Accurate rankings make it possible to get efficient estimates of the population parameters. However, in practice, errors in ranking are inevitable, particularly when dealing with large set sizes. These errors adversely effect the efficiency of estimators obtained under RSS scheme. Dell and Clutter (1972) were the first one to investigate the effect of imperfect rankings on the performance of RSS-based mean estimator. They showed that even with imperfect rankings, the mean estimator under RSS remains unbiased and it is more precise than the mean estimator with SRS.

Following Dell and Clutter (1972) approach, we study the performance of the BLUEs based on the imperfect RSS methods. For a comprehensive comparison of the BLUEs under imperfect RSS schemes, consider IRSS, OIRSS, IDRSS and OIDRSS procedures. For brevity of the discussion, ξ_{ijt} is assumed to be a standard normal random variable, i.e., $\xi_{ijt} \sim N(0,1)$, for $i = 1, 2, ..., k$, $j = 1, 2, ..., m$ and $t = 1, 2, ..., r$. However, in the rest of the chapter, other symmetric distributions are also considered for ξ_{ijt}, i.e., Laplace and scale contaminated normal distributions. Without loss of generality, consider $k = 5$, $r = 1$ and m =2, 3, 4, 5. Consider $x_i = i$, for $i = 1, 2, ..., 5$, and treat x_is as fixed constants. Then, in the SLRM $Y_{ijt} = Y_{ijt}|X_i = x_i$ is also a normal random variable with mean $\alpha + \beta(x_i - \bar{x})$ and variance unity, i.e., $Y_{ijt}|x_i \sim N(\alpha + \beta(x_i - \bar{x}), 1)$ for $j = 1, 2, ..., m$, $t = 1, 2, ..., r$. Consider $\alpha = 1$ and $\beta = 2$. Let V_{ijt} be another random error term, and it is normally distributed with mean zero and variance σ_V^2, i.e., $V_{ijt} \sim N(0, \sigma_V^2)$. Note that both Y_{ijt} and V_{ijt} are independent. Compute $Q_{ijt} = Y_{ijt} + V_{ijt}$. Given x_i, generate Y_{ijt}, for $j = 1, 2, ..., m$, $t = 1, 2, ..., r$, and find Q_{ijt}. Finally for a particular level of x_i, select a ranked set sample of size n based on the values of Q_{ijt}, i.e., $Q^{\text{RSS}}_{i(j:m)t}$, for $j = 1, 2, .., m$, $t = 1, 2, ..., r$. For each $Q^{\text{RSS}}_{i(j:m)t}$, also observe the corresponding values of Y_{ijt}, i.e., $Y^{\text{IRSS}}_{i[j:m]t}$, for $i = 1, 2, ..., k$, $j = 1, 2, ..., m$, $t = 1, 2, ..., r$. This scheme is named as IRSS. It is to be noted that if the values of $Y^{\text{IRSS}}_{i[j:m]t}$ are sorted in an increasing order, then an ordered imperfect ranked set sample of size n is obtained, denoted by $Y^{\text{OIRSS}}_{i[j:m]t}$. This scheme is named as OIRSS. Similarly, if DRSS is performed on the values of Q_{ijt}, then we get $Y^{\text{IDRSS}}_{i[j:m]t}$, for $i = 1, 2, .., k$, $j = 1, 2, ..., m$, $t = 1, 2, ..., r$, which is an imperfect double ranked set sample of size n, and the analogous sampling scheme is named as IDRSS. Moreover, after sorting the values obtained under IDRSS, i.e., $Y^{\text{IDRSS}}_{i[j:m]t}$ in an increasing order, we obtain an ordered imperfect double ranked set sample of size n, denoted by $Y_{i[j:m]t}^{\text{OIDRSS}}$. The corresponding scheme is then named as OIDRSS.

In order to study the effect of imperfect rankings on the performance of the BLUEs, we consider σ_V^2 =0.05, 0.15, 0.30, 0.50. It is difficult to find the explicit mathematical expressions for the BLUEs constructed under IRSS schemes. Therefore, the mean squared errors (MSEs) of the BLUEs are estimated using extensive Monte Carlo simulations. In Tables 5.12–5.14, under different IRSS schemes, the estimated MSEs (EMSEs) of the BLUEs for normal, Laplace and scale contaminated normal distributions are reported based on 10^6 replications.

5.5.2.1 Semi-algorithm for OIDRSS

In this section, a semi-algorithm for calculating the EMSEs of estimators under OIDRSS scheme is provided. Mathematica 8.0 is used for simulations since it is widely available and easily accessible.

Step 1: Let $m = 5$, $k = 5$, $\sigma_V^2 = 0.05$, $\alpha = 1$, $\beta = 2$, $r = 1$ and $x_i = i$ for $i = 1, 2, ..., k$.

Step 2: Start with $i = 1$.

Step 3: Generate m^2 values of $Y_{ijt}|x_i$ and V_{ijt}. Calculate $Q_{ijt} = Y_{ijt} + V_{ijt}$.

Step 4: Partition Q_{ijt} into a $m \times m$ matrix, named M_Q. Similarly, corresponding to Q_{ijt}, the values of Y_{ijt} are also partitioned into another $m \times m$ matrix, named M_Y.

Step 5: Sort each column of M_Q, and sort the columns of M_Y with respect to the ranks of the columns of M_Q. Select diagonals of both matrices (M_Q and M_Y), named dM_Q and dM_Y, each containing m values. Again sort the values of dM_Q, and dM_Y is sorted with respect to the ranks of dM_Q. Select the wth smallest value of dM_Y. Repeat the Steps 3–5 for $w = 1, 2, ..., m$.

Step 6: Sort m values obtained from Step 5 to get an ordered imperfect double ranked set sample of size m. In case when $r > 1$, repeat the Steps 2–5 r times to get a sample of size mr.

Step 7: Repeat Steps 2–6 for $i = 1, 2, ..., 5$. This gives ordered imperfect double ranked set samples of size $m = 5$ for each value of x.

Step 8: Calculate the values of $\hat{\alpha}_{\text{OIDRSS}}$, $\hat{\alpha}_{\text{OIDRSS}}$ and $\hat{\alpha}_{\text{OIDRSS}}$ using (5.5).

Step 9: Repeat above Steps 2-8 one million times, and calculate the EMSE of each estimator.

The BLUE of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}_{\text{I}}$ under IRSS scheme, named I, is given by

$$\hat{\boldsymbol{\theta}}_{\text{I}} = \begin{pmatrix} \frac{1}{rk(\mathbf{1}'\boldsymbol{\Omega}_{\text{H}}^{-1}\mathbf{1})}\, \mathbf{1}'\boldsymbol{\Omega}_{\text{H}}^{-1} \sum_{t=1}^{r}\sum_{i=1}^{k} \boldsymbol{Y}_{it}^{\text{I}} \\ \frac{1}{r(\mathbf{1}'\boldsymbol{\Omega}_{\text{h}}^{-1}\mathbf{1})\sum_{i=1}^{k}(x_i-\bar{x})^2}\, \mathbf{1}'\boldsymbol{\Omega}_{\text{H}}^{-1} \sum_{t=1}^{r}\sum_{i=1}^{k}(x_i - \bar{x})\boldsymbol{Y}_{it}^{\text{I}} \\ \frac{1}{rk(\boldsymbol{\mu}_{\text{H}}'\boldsymbol{\Omega}_{\text{H}}^{-1}\boldsymbol{\mu}_{\text{H}})}\, \boldsymbol{\mu}_{\text{H}}'\boldsymbol{\Omega}_{\text{H}}^{-1} \sum_{t=1}^{r}\sum_{i=1}^{k} \boldsymbol{Y}_{it}^{\text{I}} \end{pmatrix}, \tag{5.5}$$

where H = RSS, ORSS, DRSS, ODRSS and I = IRSS, OIRSS, IDRSS, OIDRSS.

From Tables 5.12–5.14, it is observed that when estimating intercept or slope parameters, the estimates under DRSS are more precise than their counterparts based on RSS and ORSS schemes. For all cases, the estimates under ODRSS are uniformly better than the corresponding estimates with other RSS schemes. As expected, the values of EMSEs tend to decrease as the value of m increases and vice-versa. Similarly, given m, with an increase in σ_V^2, the values of EMSEs also increase.

5.6 Sensitivity of the BLUEs

In this section, we calculate REs of the BLUEs based on different RSS schemes assuming normality when the parent distribution of the study variable Y is a non-normal symmetric distribution, say C-distribution, where C is either Laplace or scale contaminated normal.

Let $\hat{\boldsymbol{\theta}}_{\mathrm{H}}^{\mathrm{N}}$ be the BLUE of $\boldsymbol{\theta}$ that is based on the normality assumption of Y, obtained under H sampling scheme, where H = OSRS, RSS, ORSS, DRSS and ODRSS. Let $\boldsymbol{Y}_{\mathrm{C}}$ be the dependent variable that follows C-distribution. Then from (5.5), we have

$$\begin{aligned}
E(\hat{\alpha}_{\mathrm{H,N}}) &= \frac{\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\sum_{t=1}^{r}\sum_{i=1}^{k}E(\boldsymbol{Y}_{it,\mathrm{C}}^{\mathrm{H}})}{rk(\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1})}, \\
&= \frac{\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\sum_{t=1}^{r}\sum_{i=1}^{k}\{\alpha\mathbf{1}+\beta(x_i-\bar{x})\mathbf{1}+\sigma\boldsymbol{\mu}_{\mathrm{H,C}}\}}{rk(\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1})}, \\
&= \alpha+\sigma\left(\frac{\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\boldsymbol{\mu}_{\mathrm{H,C}}}{\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1}}\right),
\end{aligned}$$

where $\boldsymbol{\mu}_{\mathrm{H,N}}$ and $\boldsymbol{\mu}_{\mathrm{H,C}}$ are the mean vectors of the standardized order statistics obtained under H scheme when Y follows normal distribution and C-distribution, respectively. For any symmetric distribution with mean zero, we have $\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H}}^{-1}\boldsymbol{\mu}_{\mathrm{H,C}}=0$, see Leone and Moussa-Hamouda (1973). Therefore, $E(\hat{\alpha}_{\mathrm{H,N}})=\alpha$, which shows that $\hat{\alpha}_{\mathrm{H,N}}$ is a linear unbiased estimator of α.

On similar steps, it is easy to show

$$\begin{aligned}
E(\hat{\beta}_{\mathrm{H,N}}) &= \frac{\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\sum_{t=1}^{r}\sum_{i=1}^{k}(x_i-\bar{x})\{\alpha\mathbf{1}+\beta(x_i-\bar{x})\mathbf{1}+\sigma\boldsymbol{\mu}_{\mathrm{H,C}}\}}{r(\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1})\sum_{i=1}^{k}(x_i-\bar{x})^2}, \\
&= \beta,
\end{aligned}$$

which is also a linear unbiased estimator of β. Similarly, it can be shown that

$$\begin{aligned}
E(\hat{\sigma}_{\mathrm{H,N}}) &= \frac{\boldsymbol{\mu}_{\mathrm{H,N}}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\sum_{t=1}^{r}\sum_{i=1}^{k}\{\alpha\mathbf{1}+\beta(x_i-\bar{x})\mathbf{1}+\sigma\boldsymbol{\mu}_{\mathrm{H,C}}\}}{rk(\boldsymbol{\mu}_{\mathrm{H,N}}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\boldsymbol{\mu}_{\mathrm{H,N}})}, \\
&= \sigma\left(\frac{\mu_{\mathrm{H,N}}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\boldsymbol{\mu}_{\mathrm{H,C}}}{\mu_{\mathrm{H,N}}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\boldsymbol{\mu}_{\mathrm{H,N}}}\right),
\end{aligned}$$

which shows that $\hat{\sigma}_{\mathrm{H,N}}$ is not in the class of unbiased estimators.

The variances and MSE of $\hat{\alpha}_{\mathrm{H,N}}$, $\hat{\beta}_{\mathrm{H,N}}$ and $\hat{\sigma}_{\mathrm{H,N}}$ are respectively given by

$$\begin{aligned}
\mathrm{Var}(\hat{\alpha}_{\mathrm{H,N}}) &= \sigma^2\left(\frac{\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\boldsymbol{\Omega}_{\mathrm{H,C}}\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1}}{rk(\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1})^2}\right), \\
\mathrm{Var}(\hat{\beta}_{\mathrm{H,N}}) &= \sigma^2\left(\frac{\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\boldsymbol{\Omega}_{\mathrm{H,C}}\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1}}{r(\mathbf{1}'\boldsymbol{\Omega}_{\mathrm{H,N}}^{-1}\mathbf{1})^2\sum_{i=1}^{k}(x_i-\bar{x})^2}\right) \quad \text{and}
\end{aligned}$$

Table 5.15: REs of the BLUEs under different RSS schemes when normality assumptions do not hold

Distribution	Laplace				Scale Contaminated Normal: $\epsilon = 0.01$			
Relative efficiency	$m=2$	$m=3$	$m=4$	$m=5$	$m=2$	$m=3$	$m=4$	$m=5$
$\text{RE}(\hat{\alpha}_{\text{RSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.3913	1.8383	2.3184	2.8207	1.4481	1.9069	2.3737	2.8465
$\text{RE}(\hat{\alpha}_{\text{ORSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.3913	1.9246	2.4931	3.0752	1.4481	1.9232	2.4078	2.8976
$\text{RE}(\hat{\alpha}_{\text{DRSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.5968	2.5704	3.8400	5.3609	1.7415	2.6893	3.8227	5.1174
$\text{RE}(\hat{\alpha}_{\text{ODRSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.5968	2.7927	4.2824	6.0063	1.7415	2.7540	3.9514	5.3044
$\text{RE}(\hat{\sigma}_{\text{RSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	0.6152	0.7815	0.9412	1.0826	0.5523	0.7263	0.8870	1.0409
$\text{RE}(\hat{\sigma}_{\text{ORSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	1.1256	1.1549	1.2248	1.2984	1.1514	1.2422	1.3514	1.4707
$\text{RE}(\hat{\sigma}_{\text{DRSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	0.9667	1.1278	1.3286	1.4800	0.9186	1.1908	1.4741	1.7784
$\text{RE}(\hat{\sigma}_{\text{ODRSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	1.2515	1.3005	1.4270	1.5069	1.3136	1.4845	1.7226	2.0055
Distribution	Scale Contaminated Normal: $\epsilon = 0.05$				Scale Contaminated Normal: $\epsilon = 0.10$			
Relative efficiency	$m=2$	$m=3$	$m=4$	$m=5$	$m=2$	$m=3$	$m=4$	$m=5$
$\text{RE}(\hat{\alpha}_{\text{RSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.4014	1.8395	2.3013	2.7800	1.3755	1.8041	2.2655	2.7507
$\text{RE}(\hat{\alpha}_{\text{ORSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.4014	1.8889	2.4036	2.9344	1.3755	1.8753	2.4149	2.9786
$\text{RE}(\hat{\alpha}_{\text{DRSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.6356	2.5455	3.7006	5.0695	1.5769	2.4744	3.6565	5.0912
$\text{RE}(\hat{\alpha}_{\text{ODRSS,N}}, \hat{\alpha}_{\text{OSRS,N}})$	1.6356	2.6822	3.9766	5.4798	1.5769	2.6552	4.0315	5.6634
$\text{RE}(\hat{\sigma}_{\text{RSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	0.6035	0.7756	0.9344	1.0811	0.6339	0.8005	0.9547	1.0903
$\text{RE}(\hat{\sigma}_{\text{ORSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	1.1555	1.2105	1.2945	1.3868	1.1463	1.1790	1.2446	1.3140
$\text{RE}(\hat{\sigma}_{\text{DRSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	0.9677	1.1761	1.4009	1.6207	0.9878	1.1529	1.3371	1.4921
$\text{RE}(\hat{\sigma}_{\text{ODRSS,N}}, \hat{\sigma}_{\text{OSRS,N}})$	1.3077	1.3964	1.5697	1.7552	1.2842	1.3291	1.4546	1.5637

Here $\text{RE}(\hat{\alpha}_{\text{H,N}}, \hat{\alpha}_{\text{OSRS,N}}) = \text{RE}(\hat{\beta}_{\text{H,N}}, \hat{\beta}_{\text{OSRS,N}})$ for H = RSS, ORSS, DRSS and ODRSS.

$$\text{MSE}(\hat{\sigma}_{\text{H,N}}) \quad = \quad \sigma^2 \left(\frac{\boldsymbol{\mu}'_{\text{H,N}} \boldsymbol{\Omega}^{-1}_{\text{H,N}} \boldsymbol{\Omega}_{\text{H,C}} \boldsymbol{\Omega}^{-1}_{\text{H,N}} \boldsymbol{\mu}_{\text{H,N}}}{rk(\boldsymbol{\mu}'_{\text{H,N}} \boldsymbol{\Omega}^{-1}_{\text{H,N}} \boldsymbol{\mu}_{\text{H,N}})^2} \right) + \sigma^2 \left(\frac{\boldsymbol{\mu}'_{\text{H,N}} \boldsymbol{\Omega}^{-1}_{\text{H,N}} \boldsymbol{\mu}_{\text{H,C}}}{\boldsymbol{\mu}'_{\text{H,N}} \boldsymbol{\Omega}^{-1}_{\text{H,N}} \boldsymbol{\mu}_{\text{H,N}}} - 1 \right)^2,$$

where $\boldsymbol{\Omega}_{\text{H,N}}$ and $\boldsymbol{\Omega}_{\text{H,C}}$ are the variance-covariance matrices of standardized order statistics when Y follows normal and C-distributions, respectively.

In Table 5.15, we report the exact REs of $\hat{\alpha}_{\text{H,N}}$, $\hat{\beta}_{\text{H,N}}$ and $\hat{\sigma}_{\text{H,N}}$ when one mistakenly assumes normality. Almost similar trend of REs is observed here as aforementioned. It is clear from Table 5.15 that for a particular distribution when using any RSS scheme, the RE of an estimator is increasing with m. In both individual and joint estimation of the unknown parameters of the SLRM, the estimates under ODRSS are uniformly better than their analogies. Similarly, under DRSS, when estimating intercept or slope parameters, it is possible to get more efficient estimates than the estimates with RSS and ORSS schemes. However, when estimating σ with $m \leq 3$, the estimates under DRSS are less efficient than those with ORSS. In joint estimation of parameters, DRSS outperforms RSS and ORSS schemes for all values of m.

5.7 Conclusion

In this paper, we proposed improved BLUEs of the unknown parameters of the SLRM based on DRSS and ODRSS schemes. We considered several symmetric distribution for the random error term of the SLRM, and made a comprehensive comparison of the BLUEs based on perfect and imperfect RSS schemes. We also studied the behavior of the BLUEs when normality assumptions are violated. It is worth mentioning that the BLUEs under ODRSS and OIDRSS schemes are uniformly better than the existing BLUEs for all cases considered here.

Chapter 6

Improved Exponentially Weighted Moving Average Control Charts for Monitoring Process Mean and Dispersion

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Exponentially weighted moving average (EWMA) control charts are mostly used to monitor the manufacturing processes. In this chapter, we propose some improved EWMA control charts for detecting the random shifts in the process mean and process dispersion. These EWMA control charts are based on the best linear unbiased estimators obtained under ordered ranked set sampling (ORSS) and ordered imperfect ranked set sampling (OIRSS), named EWMA-ORSS and EWMA-OIRSS charts, respectively. Monte Carlo simulations are used to estimate the average run length, median run length and standard deviation of run length of the proposed EWMA control charts. It is observed that the EWMA-ORSS mean control chart is able to detect the random shifts in the process mean substantially quicker than the Shewhart-cumulative sum and Shewhart-EWMA control charts based on the RSS scheme. Both EWMA-ORSS and EWMA-OIRSS location charts perform better than the classical EWMA, hybrid EWMA, Shewhart-EWMA and fast initial response-EWMA charts. The EWMA-ORSS dispersion control chart performs better than the simple random

sampling based CS-EWMA and other EWMA control charts in efficient detection of the random shifts that occur in the process variability. An application to real data is also given to explain the implementation of the proposed EWMA control charts.

6.1 Introduction

Control chart is a powerful statistical process monitoring tool that is frequently used to identify the unusual variations of the production processes. In manufacturing industries, the most commonly used control charts are Shewhart, cumulative sum (CUSUM) and exponentially weighted moving average (EWMA). The Shewhart-type control charts are based on the current information due to which they remain insensitive to small shifts that occur in the process parameters. Roberts (1959) was the first one to introduce the EWMA or geometric moving average control chart for monitoring the process mean. The Shewhart control chart becomes a special case of the EWMA control chart. This makes the EWMA control chart at least as effective as the Shewhart control chart. It is well known that the EWMA control charts are superior to the Shewhart control charts for detecting random shifts of smaller magnitudes. In recent years, these control charts have gained considerable attention in various fields such as signal segmentation, navigation system monitoring, nuclear engineering, health care and education (see Montgomery, 2009; Hawkins and Olwell, 1998; Masson, 2007; Hwang et al., 2008; Woodall, 2006; Yashchin, 1989, and references therein).

In the last decades, there have been substantial advancements and improvements in the control charting methodologies. Recently, Abbas et al. (2013b) suggested a mixed EWMA-CUSUM control chart for monitoring the process mean. It is shown that for detection of small shifts in the process location, this control chart performs better than its competitors. Riaz et al. (2011) and Abbas et al. (2011) increased the efficiency of CUSUM and EWMA control charts by applying several run rules, respectively. Haq (2013) proposed a hybrid EWMA control chart for monitoring the process mean by mixing the plotting-statistics of the EWMA control charts. It is shown that both Shewhart and EWMA control charts are its special cases. Riaz et al. (2013) suggested some Shewhart-type control charts based on the auxiliary information. Some important literature in the direction of location control charts may bee seen in Riaz (2008b), Nazir et al. (2013), Ahmad et al. (2014), Abbas et al. (2013), Schoonhoven et al. (2009, 2011) and references cited therein.

The ranked set sampling (RSS) scheme has had popularity in the construction of quality control charts. The RSS method was introduced by McIntyre (1952). Takahasi and Wakimoto (1968) developed the statistical foundation of the RSS method. Salazar and Sinha (1997) developed the Shewhart-type control chart based on RSS for monitoring the process mean. Their work was then further extended by Muttlak and Al-Sabah (2003), who made some improved quality control charts based on perfect and imperfect RSS schemes. They showed that the RSS-based control charts are far better than the control charts based on simple random sampling (SRS). Abujiya and Muttlak (2004) suggested control charts for monitoring process mean based on double RSS methods. These control charts are better in detecting changes in the process mean compared with their counterparts based on SRS and RSS methods. Balakrishnan and Li (2005, 2008) introduced the ordered RSS (ORSS) and used this scheme to obtain the best linear unbiased estimators (BLUEs) of the unknown parameters of location-scale family of distributions. It is shown that the BLUEs based on ORSS (BLUEs-ORSS) are uniformly better than the BLUEs based on SRS and RSS. Recently, Al-Omari and Haq (2012) suggested some improved quality control charts for monitoring process mean based on some double RSS methods. Abujiya et al. (2013a,b) suggested Shewhart-EWMA and Shewhart-CUSUM control charts based on RSS (i.e., Shewhart-EWMA-RSS and Shewhart-CUSUM-RSS) and median RSS schemes for monitoring the process mean. They showed that these control charts are far better than the CUSUM and EWMA control charts based on SRS. A comprehensive comparison of EWMA location control charts based on different RSS schemes is given in Abujiya et al. (2014).

Dispersion control charts are also common in the statistical quality control literature. Page (1954) developed the CUSUM control chart based on samples ranges for monitoring increases in the process dispersion. Crowder and Hamilton (1992) applied logarithmic transformation to the unbiased sample variance, S^2, and proposed an EWMA control chart for monitoring changes in the process standard deviation. Acosta-Mejia et al. (1999) suggested CUSUM control charts for monitoring process dispersion by applying normalizing transformations to S^2 and made a comprehensive comparison of the dispersion control charts. Later on, Castagliola (2005) suggested S^2-EWMA control chart based on three-parameter logarithmic transformation. Following the same transformation on S^2, Castagliola et al. (2009) proposed the CUSUM-S^2 control chart for monitoring process dispersion. Recently, Abbas et al. (2013a) extended the work and proposed CS-EWMA control chart under SRS (CS-EWMA-SRS) by mixing the effects of both EWMA and CUSUM control charts. It is shown that for small shifts in the process dispersion, the CS-EWMA chart detects the random shifts in the process variation significantly quicker than the S^2-EWMA and CUSUM-S^2 charts. Note that all of these dispersion control charts are based on SRS method. Haq (2014) suggested an EWMA control chart for monitoring process variability based on mean deviation under RSS scheme. Some important literature in the direction of dispersion charts may be seen in Huwang et al. (2010), Abbasi et al. (2012), Abbasi and Miller (2012), Riaz and Saghir (2009), Riaz and Does (2008, 2009), Abbas et al. (2013b), Ahmad et al. (2013) and references cited therein.

In this chapter, we introduce two improved EWMA control charts for monitoring process mean and process dispersion based on ORSS and ordered imperfect RSS (OIRSS) schemes. The proposed EWMA control charts are based on the BLUEs-ORSS and BLUEs-OIRSS, named EWMA-ORSS and EWMA-OIRSS charts, respectively. The complete structure of the control limits of each of the EWMA-ORSS control chart is developed. The statistical properties of these control charts are evaluated in terms of average run length (ARL), median run length (MDRL) and standard deviation of run length (SDRL). ARL is the average number of observations that are required to issue a particular size shift in the process location or dispersion or both. Based on extensive Monte Carlo simulations, ARLs, MDRLs and SDRLs of the proposed EWMA-ORSS control charts have been calculated. We compare EWMA-ORSS control charts with some of the recently proposed control charts. It turns out that our proposed control charts (location and dispersion) are detecting random shift faster than their competitors while maintaining an equal in-control ARL. An application to real data is also considered to explain the implementation of the proposed EWMA control charts.

The rest of the chapter is organized as follows: In Section 6.2, we explain the method to obtain BLUEs-ORSS and BLUEs-OIRSS. The proposed EWMA-ORSS and EWMA-OIRSS control charts for process mean and process dispersion are constructed in Section 6.3. It also contains the values of ARLs, MDRLs and SDRLs of the proposed EWMA control charts. These control charts are compared with some recent control charts in Section 6.4. Real data applications are given in Section 6.5, and Section 6.6 summarizes the main findings.

6.2 BLUEs and ordered ranked set sampling

In this section, we explain the RSS, ORSS and OIRSS procedures. The BLUEs of the unknown parameters of the location-scale family of distributions under ORSS are also explained in detail.

The RSS procedure is explained as follows: identify m^2 units from the target population. Randomly allocate these units to m sets each of size m. Now, rank the units within each set visually with respect to the study variable or by any inexpensive method. Select the smallest ranked unit from the first set. Similarly, select the second smallest ranked unit from the second set. The procedure continues, and the largest ranked unit is selected from the last set. This completes one cycle of a ranked set sample of size m. The procedure can be repeated r times to obtain a ranked set sample of size mr. In order to obtain an ordered ranked set sample of size m, we sort the ranked set sample in an increasing order of magnitude.

Let $Y_1, Y_2, ..., Y_m$ be m independent and identically distributed (IID) random variables that follow an absolutely continuous distribution having cumulative distribution function (CDF) $F\left(\frac{y-\mu}{\sigma}\right)$ and probability density function (PDF) $\frac{1}{\sigma} f\left(\frac{y-\mu}{\sigma}\right)$, where μ is the location parameter and $\sigma(>0)$ is the scale parameter. For brevity of the discussion, let $F^*(y) = F\left(\frac{y-\mu}{\sigma}\right)$ and $f^*(y) = \frac{1}{\sigma} f\left(\frac{y-\mu}{\sigma}\right)$. Let $Y_{(1:m)}, Y_{(2:m)}, ..., Y_{(m:m)}$ be the m order statistics obtained from a simple random sample of size m, i.e., $Y_1, Y_2, ..., Y_m$. Then, the CDF and PDF of the ith order statistic, $Y_{(i:m)}$, are respectively given by

$$\begin{aligned} F^*_{(i:m)}(y) &= \{F^*(y)\}^i \sum_{j=0}^{m-i} \binom{i+j-1}{i-1} \{1 - F^*(y)\}^j, \quad -\infty < x < \infty, \\ f^*_{(i:m)}(y) &= m\binom{m-1}{i-1} \{F^*(y)\}^{i-1} \{1 - F^*(y)\}^{m-i} f^*(y). \end{aligned}$$

Let $Y_{11}, Y_{12}, ..., Y_{1m}, Y_{21}, Y_{22}, ..., Y_{2m}, ..., Y_{m1}, Y_{m2}, ..., Y_{mm}$ be m independent simple random samples each of size m. Apply RSS procedure to these m samples to obtain a ranked set sample of size m, denoted by $Y_{1(1:m)}, Y_{2(2:m)}, ..., Y_{m(m:m)}$. The mean of the ranked set sample is $\bar{Y}_{\text{RSS}} = \frac{1}{m} \sum_{i=1}^{m} Y_{i(i:m)}$. Takahasi and Wakimoto (1968) showed that under perfect ranking, $\bar{Y}_{\text{RSS}}$ is an unbiased estimator of μ_Y and it is more precise than the sample mean based on SRS, $\bar{Y}_{\text{SRS}} = \frac{1}{m} \sum_{i=1}^{m} Y_i$, i.e., $E(\bar{Y}_{\text{RSS}}) = \mu_Y$ and $\text{Var}(\bar{Y}_{\text{RSS}}) = \text{Var}(\bar{Y}_{\text{SRS}}) - \frac{1}{m^2} \sum_{i=1}^{m} (\mu_{Y(i:m)} - \mu_Y)^2$, where $E(Y_{i(i:m)}) = \mu_Y$, for $i = 1, 2, ..., m$.

Let $W_{(1:m)} \leq W_{(2:m)} \leq \cdots \leq W_{(m:m)}$ represent an ordered ranked set sample obtained by arranging

$Y_{1(1:m)}, Y_{2(2:m)}, ..., Y_{m(m:m)}$ in an increasing order of magnitude, i.e., $W_{1:m} = \min(Y_{1(1:m)}, Y_{2(2:m)}, ..., Y_{m(m:m)})$. Note that the random variables $Y_{i(i:m)}$, $i = 1, 2, ..., m$, are independent but not identically distributed (INID). Therefore, the CDF of $W_{(r:m)} (r = 1, 2, ..., m)$ is given by

$$F_{r:m}^{\text{ORSS}}(w) = \sum_{i=r}^{m} \sum_{S_i} \left[\prod_{l=1}^{i} F_{(j_l:m)}^{*}(w) \prod_{l=i+1}^{m} \{1 - F_{(j_l:m)}^{*}(w)\} \right], \quad -\infty < w < \infty,$$

where the summation S_i extends over all permutations $(j_1, j_2, ..., j_m)$ of $(1, 2, ..., m)$ for which $j_1 < j_2 < \cdots < j_i$ and $j_{i+1} < j_{i+2} < \cdots < j_m$.

Similarly, the PDF of $W_{(r:m)} (r = 1, 2, ..., m)$ is given by

$$f_{(r:m)}^{\text{ORSS}}(w) = \frac{1}{(r-1)!(m-r)!} \sum_{U} \left[\prod_{l=1}^{r-1} F_{(j_l:m)}^{*}(w) \prod_{l=r+1}^{m} \{1 - F_{(j_l:m)}^{*}(w)\} f_{(j_r:m)}^{*}(w) \right],$$

where $\sum_U$ denotes the sum over all $m!$ permutations $(j_1, j_2, ..., j_m)$ of $(1, 2, ..., m)$.

The joint PDF of $W_{(r:m)}$ and $W_{(s:m)}$ $(1 \le r < s \le m)$ is given by

$$\begin{aligned} f_{(r,s:m)}^{\text{ORSS}}(w_r, w_s) &= \frac{1}{(r-1)!(s-r-1)!(m-s)!} \sum_{U} \left[\prod_{l=1}^{r-1} F_{(j_l:m)}^{*}(w_r) \prod_{l=r+1}^{s-1} \{F_{(j_l:m)}^{*}(w_s) - F_{(j_l:m)}^{*}(w_r)\} \right. \\ & \left. \prod_{l=s+1}^{m} \{1 - F_{(j_l:m)}^{*}(w_s)\} f_{(j_r:m)}^{*}(w_r) f_{(j_s:m)}^{*}(w_s) \right], \quad w_r < w_s. \end{aligned}$$

Based on previously defined PDFs, it is easy to calculate the moments of order statistics under ORSS. For more details, see David and Nagaraja (2003) and Balakrishnan and Li (2005).

Let $\boldsymbol{W}_{\text{ORSS}} = (W_{(1:m)}, W_{(2:m)}, ..., W_{(m:m)})'$ be the vector obtained under ORSS from a general location-scale distribution, with location parameter μ and $\sigma(> 0)$ be its scale parameter. Define $Z_{(r:m)} = (W_{(r:m)} - \mu)/\sigma$ as the standardized variate under ORSS such that the distribution of $Z_{(r:m)}$ is independent of μ and σ. Also let $E(Z_{(r:m)}) = \mu_{(r:m)}^{*}$, $1 \le r \le m$, $\text{Cov}(Z_{(r:m)}, Z_{(s:m)}) = \sigma_{(r,s:m)}^{*}$, $1 \le r < s \le m$. Then $E(W_{(r:m)}) = \mu + \sigma\mu_{(r:m)}^{*}$ and $\text{Cov}(W_{(r:m)}, W_{(s:m)}) = \sigma^2 \sigma_{(r,s:m)}^{*}$. Following Balakrishnan and Li (2005), the BLUE-ORSS, say $\hat{\pi}_{\text{ORSS}} = (\hat{\mu}_{\text{ORSS}}, \hat{\sigma}_{\text{ORSS}})'$, of $\pi = (\mu, \sigma)'$ is $\hat{\sigma}_{\text{ORSS}} = (\boldsymbol{\theta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\theta})^{-1}\boldsymbol{\theta}'\boldsymbol{\Omega}^{-1}\boldsymbol{W}_{\text{ORSS}}$, where $\boldsymbol{\theta} = (\mathbf{1}, \boldsymbol{\mu}_{\text{ORSS}})_{m\times 2}$, $\boldsymbol{\Omega} = (\sigma_{(r,s:m)}^{*})_{m\times m}$, $\mathbf{1} = (1, 1, ..., 1)_{1\times m}$ and $\boldsymbol{\mu}_{\text{ORSS}} = (\mu_{(1:m)}^{*}, \mu_{(2:m)}^{*}, ..., \mu_{(m:m)}^{*})'$. The variance-covariance matrix of $\hat{\pi}_{\text{ORSS}}$ is $\text{Cov}(\hat{\pi}_{\text{ORSS}}) = \sigma^2(\boldsymbol{\theta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\theta})^{-1}$. The BLUEs of μ and σ can be written as linear combinations of $\boldsymbol{W}_{\text{ORSS}}$, i.e., $\hat{\mu}_{\text{ORSS}} = \sum_{r=1}^{m} \alpha_r W_{(r:m)}$ and $\hat{\sigma}_{\text{ORSS}} = \sum_{r=1}^{m} \beta_r W_{(r:m)}$. The values of the coefficients (α_r and β_r), $\mu_{(r:m)}^{*}$ and $\sigma_{(r:m)}^{*}$ for normal, exponential and logistic distributions are reported in Balakrishnan and Li (2008) for several choices of m. If the underlying distribution is symmetric, then the covariance between $\hat{\mu}_{\text{ORSS}}$ and $\hat{\sigma}_{\text{ORSS}}$ becomes zero, i.e., $\text{Cov}(\hat{\mu}_{\text{ORSS}}, \hat{\sigma}_{\text{ORSS}}) = 0$. This helps in simplifying the variances of the BLUEs-ORSS, i.e., $\text{Var}(\hat{\mu}_{\text{ORSS}}) = \sigma^2(\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1})^{-1}$ and $\text{Var}(\hat{\sigma}_{\text{ORSS}}) = \sigma^2(\boldsymbol{\mu}_{\text{ORSS}}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}_{\text{ORSS}})^{-1}$.

It is clear that the performance of RSS depends on how perfect the judgment ranking of the randomly selected units is accomplished. The correct ordering helps in achieving stratification without quantification and utilizes the prior experience and expertise of the investigator. However, in practice, the judgment error is inevitable, particularly for large m. Errors in ranking cause the units to be assigned ranks different from their true ranks according to the study variable. This, in turn, leads to imprecise estimates. Dell and Clutter (1972) were the first to study the effect of imperfect ranking on the performance of RSS-based mean estimator. They showed that even under imperfect ranking, the RSS-based mean estimator remains unbiased, but imperfect ranking should be better than the random ordering of the selected units.

Here, we examine the effect of judgment error on the performance of BLUEs of the location and scale parameters under ORSS. For brevity, we assume that the underlying process is normally distributed with mean μ_0 and variance σ_0^2 at time t, i.e., $Y_t \sim N(\mu_0, \sigma_0^2)$, for $t = 1, 2, ...$ Given the value of m, generate m sets each of size m, from the underlying process distribution, i.e., Y_{ijt}, for $i, j = 1, 2, ..., m$. We also generate the random errors, V_{ijt}, with mean zero and variance σ_V^2, i.e., $V_{ijt} \sim N(0, \sigma_V^2)$ at time t, where V_{ijt} is independent of Y_{ijt}. Then, we compute $X_{ijt} = Y_{ijt} + V_{ijt}$, for $i, j = 1, 2, ..., m$. Apply the RSS procedure on m^2 values of X and also measure the corresponding values of Y. Then, a pair $(X_{i(i:m)t}, Y_{i[i:m]t})$, for $i = 1, 2, ..., m$, is selected based on the values of X, where $Y_{i[i:m]t}$ is the ith concomitant of the ith order statistic $X_{i(i:m)t}$ at time t. Let $W_{[1:m]t} \leq W_{[2:m]t} \leq \cdots \leq W_{[m:m]t}$ represent an ordered imperfect ranked set sample obtained by rearranging $Y_{i[i:m]t}$, $i = 1, 2, ..., m$, in an increasing order, i.e., $W_{[i:m]t} = i\text{th}\min(Y_{1[1:m]t}, Y_{2[2:m]t}, ..., Y_{m[m:m]t})$, for $i = 1, 2, ..., m$. We name this scheme as OIRSS. The BLUEs of μ and σ based on OIRSS scheme, at time t, are $\hat{\mu}_{\text{ORSS},t} = \sum_{r=1}^{m} \alpha_r W_{[r:m]t}$ and $\hat{\sigma}_{\text{ORSS},t} = \sum_{r=1}^{m} \beta_r W_{[r:m]t}$, respectively, where the values of coefficients (α_r, β_r) are the same as mentioned earlier. It is noteworthy that OIRSS involves order statistics from independents concomitants. Therefore, it is difficult to derive the exact PDF of $W_{[r:m]t}$, $r = 1, 2, ..., t$. Here, we use Monte Carlo simulations to estimate the variances of the BLUEs obtained under OIRSS.

6.3 Proposed EWMA-ORSS control charts for monitoring process mean and dispersion

In this section, we develop some improved EWMA control charts for monitoring process mean and dispersion based on the BLUEs-ORSS and BLUEs-OIRSS.

6.3.1 EWMA-ORSS control chart for monitoring process mean

It is assumed that the underlying process is in-control, and let Y_t be the observation at time t, obtained from a normally distributed process with mean $\mu_0 = 0$ and standard deviation $\sigma_0 = 1$. Let $\mu_{\text{ORSS},t}$ be the estimate of the underlying process mean μ_0 under ORSS, obtained from a subgroup of size m at time t, for $t = 1, 2, ...$ Let $\hat{\mu}_{\text{ORSS},1}, \hat{\mu}_{\text{ORSS},2}, ..., \hat{\mu}_{\text{ORSS},t}, ...$ be the sequence of IID random variables and let $\xi \in [0, 1]$ be a constant. From this sequence, we define another sequence $M_1, M_2, ..., M_t, ...$, by using a recurrence formula, given by the following:

$$M_t = \xi \hat{\mu}_{\text{ORSS},t} + (1 - \xi) M_{t-1}, \quad 0 < \xi \leq 1, \tag{6.1}$$

which is called an EWMA sequence. For a positive integer $t \geq 1$, $E(M_t) = \mu_0$ for $E(\hat{\mu}_{\text{ORSS},t}) = \mu_0$ and $\text{Var}(M_t) = \left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}\sigma^2(\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1})^{-1}$ for $M_0 = \mu_0$. If both μ_0 and σ_0 are known constants, then the upper control limit (UCL_t), center limit (CL_t) and lower control limit (LCL_t) of the EWMA-ORSS control chart based on the statistic M_t, at time t, are given by

$$\begin{aligned} UCL_t &= \mu_0 + L\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1})^{-1}}, \\ CL_t &= \mu_0, \\ LCL_t &= \mu_0 - L\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1})^{-1}}, \end{aligned} \tag{6.2}$$

where L is a positive control chart multiplier and its value is determined such that the in-control ARL of the EWMA-ORSS control chart reaches to a particular level. If the underlying distribution is normally distributed, then for $\xi = 1$, the previous EWMA control chart reduces to the Shewhart control chart. The statistic M_t given in (6.1) is plotted with the control limits given in (6.2) against time t. The EWMA-ORSS control chart detects an out-of-control signal if the plotting-statistic M_t exceeds either UCL_t or LCL_t. If $M_t > UCL_t$, then there is a positive shift in the process mean at time t, or if $M_t < LCL_t$, then there is a negative shift in the process mean. Let $\delta = (\sqrt{m}/\sigma_0)|\mu_1 - \mu_0|$ represents the random shift in the process mean that is measured in $\sigma_0/\sqrt{m}$ units. Here, μ_0 is the in-control process mean and μ_1 is the out-of-control process mean. Note that as the time t increases, i.e., $t \to \infty$, then the term $\{1-(1-\xi)^{2t}\}$ approaches unity. Based on extensive Monte Carlo simulations, when the underlying process is normally distributed with $\mu_0 = 0$ and $\sigma_0 = 1$, we find values of the out-of-control ARLs, MDRLs and SDRLs with different values of δ using the control limits given in (6.2). The subgroup size is taken to be $m = 5$. With different choices of ξ, the calculated values of ARLs, MDRLs and SDRLs are given in Table 6.1. Each result is based on 10^5 replications.

Based on the results given in Table 6.1, we conclude that for fixed value of ξ, ARLs, MDRLs and SDRLs are decreasing function of δ, i.e., as the the value of δ increases, values of the out-of-control ARLs decrease and vice-versa. For example, the proposed EWMA-ORSS control chart detects on average a shift of $\delta = 0.25$ in the process mean at the 31st sample when $\xi = 0.05$. Similarly, the EWMA-ORSS chart quickly detect the random shifts in the process location for large values of δ.

In order to study the effect of imperfect ranking on the performance of EWMA-ORSS control chart, a detailed simulation study is conducted here. We name the EWMA chart based on OIRSS as EWMA-OIRSS control chart. The plotting-statistic of the EWMA-OIRSS location control chart is given by

$$G_t = \xi\hat{\mu}_{\text{OIRSS},t} + (1-\xi)G_{t-1}, \quad 0 < \xi \leq 1,$$

where ξ is a smoothing constant and $G_0 = \bar{\hat{\mu}}_{\text{OIRSS}}$.

As previously mentioned, it is difficult to find the exact mathematical expressions for the means and

Table 6.1: Run length properties of EWMA-ORSS (two-sided) process mean control chart

	$\xi\rightarrow$	0.05	0.10	0.25	0.50
δ	$L\rightarrow$	2.6400	2.8250	3.0050	3.0810
0.00	ARL	500.79	500.92	500.58	500.37
	MDRL	343.00	344.00	348.00	347.00
	SDRL	515.14	507.69	499.14	501.73
0.25	ARL	31.50	39.42	67.03	122.00
	MDRL	26.00	30.00	48.00	86.00
	SDRL	24.97	33.30	63.46	119.61
0.50	ARL	9.65	10.92	14.77	26.60
	MDRL	8.00	9.00	12.00	19.00
	SDRL	6.50	7.26	11.66	24.64
0.75	ARL	4.90	5.45	6.47	9.37
	MDRL	4.00	5.00	6.00	7.00
	SDRL	3.01	3.25	4.18	7.64
1.00	ARL	3.10	3.43	3.85	4.79
	MDRL	3.00	3.00	3.00	4.00
	SDRL	1.74	1.87	2.15	3.29
1.50	ARL	1.72	1.87	2.03	2.20
	MDRL	2.00	2.00	2.00	2.00
	SDRL	0.82	0.88	0.96	1.14
2.00	ARL	1.24	1.31	1.38	1.43
	MDRL	1.00	1.00	1.00	1.00
	SDRL	0.46	0.51	0.56	0.61
3.00	ARL	1.01	1.01	1.02	1.02
	MDRL	1.00	1.00	1.00	1.00
	SDRL	0.08	0.11	0.13	0.14

variances of BLUEs under OIRSS. Therefore, we estimate the mean and variances of the BLUEs under OIRSS from preliminary samples that were taken when the process was in-control. Let $\hat{\mu}_{1,\text{OIRSS}}, \hat{\mu}_{2,\text{OIRSS}}, ..., \hat{\mu}_{w,\text{OIRSS}}$ be the estimated values of location-BLUEs based on w subgroups, each of size m, where $\hat{\mu}_{i,\text{OIRSS}} = \sum_{r=1}^{m} \alpha_r W_{i,[r:m]}$, for $i = 1, 2, ..., w$. Then, the estimated upper control limit ($EUCL_t$), estimated center limit (ECL_t) and estimated lower control limit ($ELCL_t$) of EWMA-OIRSS location control chart, at time t, are given by

$$
\begin{aligned}
EUCL_t &= \bar{\hat{\mu}}_{\text{OIRSS}} + L\hat{\sigma}_{\hat{\mu}_{\text{OIRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},\\
ECL_t &= \bar{\hat{\mu}}_{\text{OIRSS}},\\
ELCL_t &= \bar{\hat{\mu}}_{\text{OIRSS}} - L\hat{\sigma}_{\hat{\mu}_{\text{OIRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},
\end{aligned}
$$

where $\bar{\hat{\mu}}_{\text{OIRSS}} = \frac{1}{w}\sum_{i=1}^{w}\hat{\mu}_{i,\text{OIRSS}}$, $\hat{\sigma}_{\hat{\mu}_{\text{OIRSS}}} = \sqrt{\left(\frac{1}{w-1}\right)\sum_{i=1}^{w}(\hat{\mu}_{i,\text{OIRSS}} - \bar{\hat{\mu}}_{\text{OIRSS}})^2}$ and L is the positive control chart multiplier. In order to find the values of ARLs, MDRLs and SDRLs of EWMA-OIRSS control chart, we first estimate the control limits based on one million samples, obtained under OIRSS by following Dell and Clutter (1972) approach. For brevity, we assume four values for the error variance, i.e., $\sigma_V^2 = 0.05, 0.15, 0.30, 0.50$. Then, based on 10^5 replications, the estimated values of ARLs, MDRLs and SDRLs are calculated and given in Table 6.2.

Note that in order to study the robustness of the EWMA-OIRSS chart, we keep the same values of L as

Table 6.2: Run length properties of EWMA-OIRSS (two-sided) process mean control chart

	$\xi\rightarrow$	0.05	0.10	0.25	0.50	0.05	0.10	0.25	0.50
	$L\rightarrow$	2.6400	2.8250	3.0050	3.0810	2.6400	2.8250	3.0050	3.0810
δ		$\sigma_V^2 = 0.05$				$\sigma_V^2 = 0.15$			
0.00	ARL	499.92	501.11	501.45	502.62	501.20	498.71	500.38	502.64
	MDRL	341.00	347.00	346.00	348.00	344.00	343.00	348.00	347.00
	SDRL	516.67	506.35	502.80	500.60	515.67	504.26	499.27	501.61
0.25	ARL	33.92	42.89	73.54	132.29	38.44	48.91	84.04	149.06
	MDRL	27.00	33.00	52.00	92.00	31.00	37.00	59.00	104.00
	SDRL	27.08	36.58	69.97	130.69	31.38	42.83	80.96	148.15
0.50	ARL	10.43	11.81	16.28	29.66	11.67	13.42	18.98	35.50
	MDRL	9.00	10.00	13.00	21.00	10.00	11.00	14.00	25.00
	SDRL	7.06	7.98	13.11	27.87	8.00	9.31	15.71	33.63
0.75	ARL	5.28	5.89	7.05	10.49	5.92	6.62	8.09	12.60
	MDRL	5.00	5.00	6.00	8.00	5.00	6.00	7.00	9.00
	SDRL	3.27	3.55	4.66	8.70	3.74	4.03	5.54	10.79
1.00	ARL	3.33	3.68	4.18	5.28	3.72	4.12	4.72	6.22
	MDRL	3.00	3.00	4.00	4.00	3.00	4.00	4.00	5.00
	SDRL	1.92	2.02	2.38	3.75	2.16	2.34	2.78	4.59
1.50	ARL	1.84	2.00	2.17	2.38	2.03	2.21	2.42	2.69
	MDRL	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	SDRL	0.89	0.96	1.04	1.27	1.02	1.10	1.20	1.52
2.00	ARL	1.30	1.38	1.46	1.52	1.40	1.49	1.60	1.68
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00
	SDRL	0.51	0.56	0.61	0.67	0.59	0.64	0.70	0.78
3.00	ARL	1.01	1.02	1.03	1.04	1.03	1.04	1.06	1.07
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.11	0.14	0.17	0.19	0.17	0.20	0.24	0.26
δ		$\sigma_V^2 = 0.30$				$\sigma_V^2 = 0.50$			
0.00	ARL	502.41	500.51	505.91	510.16	503.35	500.61	507.38	513.18
	MDRL	343.00	346.50	351.00	354.00	344.00	345.00	353.00	355.00
	SDRL	517.28	505.86	505.09	509.60	519.28	505.74	507.65	513.61
0.25	ARL	43.61	56.30	97.28	167.61	49.20	64.00	110.56	186.32
	MDRL	35.00	42.00	69.00	117.00	39.00	47.00	78.00	129.00
	SDRL	36.12	50.38	93.62	166.28	41.49	57.87	107.56	186.46
0.50	ARL	13.35	15.30	22.53	42.47	14.82	17.23	26.09	49.57
	MDRL	11.00	13.00	17.00	30.00	13.00	14.00	19.00	35.00
	SDRL	9.34	10.88	19.22	40.58	10.46	12.43	22.68	47.92
0.75	ARL	6.70	7.52	9.38	15.18	7.47	8.41	10.74	18.13
	MDRL	6.00	7.00	8.00	11.00	7.00	7.00	9.00	13.00
	SDRL	4.29	4.72	6.68	13.36	4.85	5.36	7.86	16.42
1.00	ARL	4.20	4.63	5.37	7.41	4.65	5.17	6.08	8.68
	MDRL	4.00	4.00	5.00	6.00	4.00	5.00	5.00	7.00
	SDRL	2.51	2.68	3.28	5.72	2.83	3.03	3.85	6.96
1.50	ARL	2.25	2.46	2.71	3.08	2.47	2.71	3.02	3.50
	MDRL	2.00	2.00	2.00	3.00	2.00	2.00	3.00	3.00
	SDRL	1.17	1.25	1.38	1.82	1.32	1.41	1.59	2.15
2.00	ARL	1.52	1.64	1.77	1.88	1.65	1.79	1.94	2.08
	MDRL	1.00	2.00	2.00	2.00	1.00	2.00	2.00	2.00
	SDRL	0.68	0.74	0.80	0.92	0.77	0.83	0.90	1.06
3.00	ARL	1.06	1.08	1.11	1.13	1.10	1.13	1.18	1.20
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.23	0.28	0.32	0.35	0.30	0.35	0.39	0.42

given in Table 6.1. It is interesting to note that when $\sigma_v^2 \leq 0.15$, for all values of ξ, the in-control ARLs of the EWMA-OIRSS control chart are almost equivalent to the in-control ARLs of the EWMA-ORSS control chart. This shows that for small errors in ranking, the EWMA-OIRSS control chart is more robust to the in-control ARL. However, with an increase in the values of σ_V^2, the out-of-control ARLs also tend to increase and vice-versa. It is also observed that the in-control ARL of the EWMA-OIRSS chart increases for large values of ξ and σ_V^2, i.e., $\sigma_V^2 \geq 0.3$ with $\xi \geq 0.25$.

6.3.2 EWMA-ORSS control chart for monitoring process dispersion

Following Section 6.3.1, it is assumed that Y_t follows a normally distributed process with mean $\mu_0 = 0$ and standard deviation $\sigma_0 = 1$. Let $\hat{\sigma}_{\text{ORSS},t}$ be the best linear unbiased estimate of the underlying process standard deviation based on ORSS, obtained from a subgroup of size m at time t. Let $\hat{\sigma}_{\text{ORSS},1}, \hat{\sigma}_{\text{ORSS},2}, ..., \hat{\sigma}_{\text{ORSS},t}, ...$ be the sequence of IID random variables. Based on this sequence, we define another sequence $D_1, D_2, ..., D_t, ...$, by using a recurrence formula, given by

$$D_t = \xi\hat{\sigma}_{\text{ORSS},t} + (1-\xi)D_{t-1}, \quad 0 < \xi \leq 1,$$

which is also an EWMA sequence. For $t \geq 1$, we have $E(D_t) = \sigma_0$ for $E(\hat{\sigma}_{\text{ORSS},t}) = \sigma_0$ and $\text{Var}(D_t) = \left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}\sigma_0^2(\boldsymbol{\mu}'_{\text{ORSS}}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}_{\text{ORSS}})^{-1}$ for $D_0 = \sigma_0$. If both μ_0 and σ_0 are known constants, then the control limits of the EWMA-ORSS control chart based on the statistic D_t, at time t, are given by

$$\begin{aligned} UCL_t &= \sigma_0 + H_2\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\boldsymbol{\mu}'_{\text{ORSS}}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}_{\text{ORSS}})^{-1}}, \\ CL_t &= \sigma_0, \\ LCL_t &= \sigma_0 - H_1\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\boldsymbol{\mu}'_{\text{ORSS}}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}_{\text{ORSS}})^{-1}}, \end{aligned} \tag{6.3}$$

where $H = H_1 = H_2$ is the positive control chart multiplier, and its values is determined such that the in-control ARL of the EWMA-ORSS control chart reaches to a specific level. Similar to the EWMA-ORSS control chart developed in Section 6.3.1; here, statistic D_t is the plotting-statistic, and the EWMA-ORSS chart detects an out-of-control signal if the plotting-statistic D_t exceeds either UCL_t or LCL_t. If $D_t > UCL_t$, then there is a positive shift in the process dispersion at time t, or if $D_t < LCL_t$, then there is a negative shift in the process dispersion. Let $\tau = \sigma_1/\sigma_0$ represents the random shift in the process dispersion. Here, σ_0 is the in-control process standard deviation, and σ_1 is the out-of-control or shifted standard deviation.

Based on extensive Monte Carlo simulations, when the underlying process follows a normal distribution with $\mu_0 = 0$ and $\sigma_0 = 1$, we find the values of out-of-control ARLs, MDRLs and SDRLs with different values of τ by using the control limits given in (6.3). For brevity of the discussion, we consider $m = 5$. For different choices of ξ, the estimated values of ARLs, MDRLs and SDRLs are given in Table 6.3. Each result is based on 10^5 replications. On similar lines, the run length properties of the one-sided EWMA-ORSS chart are given in Table 6.4.

Based on the results presented in Tables 6.3 and 6.4, it is clear that the out-of-control ARL is a decreasing function of τ when $\tau \geq 1$. For fixed value of τ, as the value of ξ increases, the values of out-of-control ARLs

Table 6.3: Run length properties of the EWMA-ORSS (two-sided) dispersion control chart

		Symmetric limits						Asymmetric limits		
	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.30	0.40	0.50
	$H_1\rightarrow$	2.2785	2.4780	2.6460	2.7210	2.7670	2.7938	2.5800	2.6660	2.6660
τ	$H_2\rightarrow$	2.2785	2.4780	2.6460	2.7210	2.7670	2.7938	2.8350	2.8220	2.8600
0.50	ARL	2.44	2.79	3.15	3.45	3.87	4.65	3.12	3.55	4.01
	MDRL	2.00	3.00	3.00	3.00	4.00	4.00	3.00	3.00	4.00
	SDRL	0.87	0.95	1.05	1.22	1.58	2.37	1.12	1.44	1.94
0.60	ARL	3.53	4.08	4.73	5.47	6.82	9.55	4.82	6.01	7.63
	MDRL	3.00	4.00	4.00	5.00	6.00	8.00	4.00	5.00	6.00
	SDRL	1.53	1.67	1.99	2.65	4.01	6.87	2.29	3.42	5.18
0.70	ARL	5.75	6.71	8.25	10.73	15.71	25.82	8.99	13.03	18.80
	MDRL	5.00	6.00	7.00	9.00	12.00	19.00	8.00	10.00	14.00
	SDRL	3.03	3.40	4.55	7.18	12.41	23.12	5.83	10.02	16.17
0.80	ARL	11.43	13.74	19.58	30.19	50.27	86.85	23.01	38.61	57.47
	MDRL	10.00	12.00	15.00	22.00	36.00	61.00	17.00	28.00	41.00
	SDRL	7.26	8.74	14.81	26.18	46.94	83.99	19.30	35.63	54.95
0.90	ARL	35.52	47.08	79.76	128.13	199.58	295.94	87.72	144.04	190.68
	MDRL	28.00	35.00	57.00	90.00	139.00	205.00	62.00	101.00	133.00
	SDRL	29.15	40.70	75.61	124.76	196.30	294.55	84.73	141.56	188.35
0.95	ARL	94.22	123.69	183.03	240.31	297.60	343.43	174.84	237.88	265.71
	MDRL	67.00	87.00	127.00	167.00	207.00	238.00	122.00	166.00	184.00
	SDRL	91.88	120.89	181.36	238.89	296.28	343.03	173.07	235.70	266.49
1.00	ARL	200.92	200.66	200.57	199.54	200.15	200.20	200.38	199.90	200.91
	MDRL	133.00	137.00	138.00	138.00	139.00	139.00	139.00	138.00	140.00
	SDRL	215.20	205.66	201.97	199.67	199.80	199.71	200.76	200.54	200.42
1.05	ARL	73.07	79.74	85.73	87.89	91.26	92.61	104.10	98.04	101.61
	MDRL	51.00	56.00	60.00	61.00	63.00	65.00	72.00	68.00	71.00
	SDRL	75.03	79.73	84.84	87.18	91.25	91.94	103.61	97.32	100.72
1.10	ARL	29.48	33.21	37.71	40.65	43.53	46.10	48.15	47.47	50.99
	MDRL	22.00	24.00	27.00	29.00	31.00	32.00	34.00	33.00	36.00
	SDRL	28.14	31.27	36.17	39.37	42.41	45.24	46.55	46.54	50.05
1.20	ARL	10.39	11.58	12.87	13.91	15.10	16.18	15.67	16.12	17.56
	MDRL	8.00	9.00	10.00	10.00	11.00	12.00	12.00	12.00	13.00
	SDRL	9.16	9.88	11.23	12.46	13.88	15.16	14.16	14.87	16.45
1.30	ARL	5.73	6.28	6.82	7.23	7.67	8.12	7.89	8.00	8.63
	MDRL	4.00	5.00	5.00	6.00	6.00	6.00	6.00	6.00	6.00
	SDRL	4.81	5.06	5.48	5.97	6.50	7.10	6.53	6.81	7.59
1.40	ARL	3.82	4.17	4.50	4.66	4.84	5.06	4.99	5.04	5.32
	MDRL	3.00	3.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
	SDRL	3.04	3.19	3.42	3.57	3.82	4.13	3.85	3.99	4.35
1.50	ARL	2.85	3.08	3.30	3.40	3.50	3.62	3.61	3.62	3.78
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
	SDRL	2.13	2.27	2.36	2.48	2.60	2.76	2.62	2.69	2.88
2.00	ARL	1.44	1.50	1.56	1.58	1.60	1.61	1.63	1.62	1.64
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.76	0.81	0.86	0.88	0.89	0.91	0.91	0.91	0.93
3.00	ARL	1.07	1.08	1.09	1.10	1.10	1.10	1.10	1.10	1.10
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.26	0.29	0.31	0.32	0.32	0.32	0.33	0.32	0.33

also increase and vice-versa. From Table 6.3, it is also observed that for small values of ξ, i.e., $0 < \xi \leq 0.20$, the out-of-control ARLs remains unbiased, i.e., they are less than the fixed in-control ARL for $\tau = 1$. However, in other cases, when $\xi \geq 0.3$, in detecting a downward shift in the process dispersion, for some values of τ, i.e., $0.9 \leq \tau < 1$, the ARLs are biased. This issue can be resolved by considering asymmetric values of the

Table 6.4: Run length properties of EWMA-ORSS (one-sided) dispersion control chart

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50
τ	$H\rightarrow$	1.9120	2.1990	2.4550	2.5840	2.6640	2.7200
1.00	ARL	200.92	200.80	200.39	200.70	199.94	199.89
	MDRL	131.00	135.00	138.00	140.00	138.00	139.00
	SDRL	221.11	209.47	202.94	200.09	199.25	199.66
1.10	ARL	20.78	24.63	29.72	33.93	37.39	40.79
	MDRL	14.00	18.00	21.00	24.00	26.00	28.00
	SDRL	21.10	23.73	28.32	32.81	36.56	39.96
1.20	ARL	7.96	9.37	10.84	12.18	13.58	14.82
	MDRL	6.00	7.00	8.00	9.00	10.00	11.00
	SDRL	7.51	8.21	9.46	10.90	12.43	13.78
1.30	ARL	4.52	5.25	6.01	6.50	7.03	7.60
	MDRL	3.00	4.00	5.00	5.00	5.00	6.00
	SDRL	3.99	4.38	4.86	5.34	5.97	6.60
1.40	ARL	3.11	3.58	4.01	4.31	4.54	4.81
	MDRL	2.00	3.00	3.00	3.00	4.00	4.00
	SDRL	2.54	2.82	3.06	3.32	3.56	3.91
1.50	ARL	2.41	2.71	3.01	3.17	3.32	3.47
	MDRL	2.00	2.00	2.00	3.00	3.00	3.00
	SDRL	1.83	2.00	2.17	2.29	2.45	2.63
2.00	ARL	1.33	1.40	1.48	1.53	1.56	1.58
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.65	0.72	0.79	0.83	0.85	0.88
3.00	ARL	1.05	1.06	1.08	1.08	1.09	1.09
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.23	0.26	0.28	0.30	0.31	0.31

control chart multiplier. In the last three columns of Table 6.3, we calculate the values of ARLs by using asymmetric values of chart multiplier. With these choices, when $\xi \geq 0.3$, there is a substantial improvement in the performance of the EWMA-ORSS dispersion control chart in detection of small downward shifts in the process variability. However, there is an increase in the values of out-of-control ARLs when $\tau \geq 1$. In case of one-sided EWMA-ORSS control chart, the ARLs remain unbiased for all values of τ.

We also study the effect of imperfect ranking on the performance of the EWMA-OIRSS dispersion control chart. A detailed simulation study is conducted in order to estimate the mean and variance of scale-BLUE under OIRSS. The plotting-statistic of the EWMA-OIRSS scale control chart is given by

$$Q_t = \xi\hat{\sigma}_{\text{OIRSS},t} + (1-\xi)Q_{t-1}, \quad 0 < \xi \leq 1,$$

where ξ is a smoothing constant and $Q_0 = \bar{\hat{\sigma}}_{\text{OIRSS}}$.

Here, we estimate the mean and variance of the scale-BLUE from preliminary samples that were taken when the process was in-control. Let $\hat{\sigma}_{1,\text{OIRSS}}, \hat{\sigma}_{2,\text{OIRSS}}, ..., \hat{\sigma}_{w,\text{OIRSS}}$ be the estimated values of scale-BLUEs based on w subgroups, each of size m, where $\hat{\sigma}_{i,\text{OIRSS}} = \sum_{r=1}^{m} \beta_r W_{i,[r:m]}$, $i = 1, 2, ..., w$. Then, the estimated control limits of EWMA-OIRSS dispersion control chart, at time t, are given by

$$EUCL_t = \bar{\hat{\sigma}}_{\text{OIRSS}} + H\hat{\sigma}_{\hat{\sigma}_{\text{OIRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},$$

$$
\begin{aligned}
ECL_t &= \bar{\hat{\sigma}}_{\text{OIRSS}}, \\
ELCL_t &= \bar{\hat{\sigma}}_{\text{OIRSS}} - H\hat{\sigma}_{\hat{\sigma}_{\text{OIRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},
\end{aligned}
$$

where $\bar{\hat{\sigma}}_{\text{OIRSS}} = \frac{1}{w}\sum_{i=1}^{w}\hat{\sigma}_{i,\text{OIRSS}}$, $\hat{\sigma}_{\hat{\sigma}_{\text{OIRSS}}} = \sqrt{\left(\frac{1}{w-1}\right)\sum_{i=1}^{w}(\hat{\sigma}_{i,\text{OIRSS}} - \bar{\hat{\sigma}}_{\text{OIRSS}})^2}$ and H is the positive control chart multiplier. We first estimate the control limits based on one million samples obtained under OIRSS by using the approach of Dell and Clutter (1972). For brevity, we assume the same values of σ_V^2 as taken previously. Based on 10^5 replications, the estimated values of ARLs, MDRLs and SDRLs are calculated and reported in Table 6.5. Similarly, we also consider one-sided EWMA-OIRSS control chart for detecting increases in the process dispersion. The ARLs, MDRLs and SDRLs of the one-sided EWMA-OIRSS dispersion control chart are given in Table 6.6 for different values of ξ and σ_V^2.

Based on the results given in Table 6.5, for $\xi \leq 0.2$ and $\sigma_V^2 \leq 0.15$, generally, the in-control ARLs are close to 200, which shows the robustness of the EWMA-OIRSS control chart. As the value of σ_V^2 increases, both in-control and out-of-control ARLs tend to increase for fixed value of τ and vice-versa. Secondly, for $\xi \leq 0.2$ with $\sigma_V^2 \leq 0.3$, the ARLs of EWMA-OIRSS control chart remain unbiased and biased otherwise. It is of interest to note that when detecting an increase in the process dispersion, from Table 6.6, it is evident that for all values values of ξ and σ_V^2, the in-control ARLs are roughly closer to 200. This shows that the scale one-sided EWMA-OIRSS dispersion chart is robust to the assumption of perfect ranking. Moreover, as σ_V^2 increases, the values of out-of-control ARLs also increase for fixed values of ξ and τ.

Table 6.5: Run length properties of EWMA-OIRSS (two-sided) dispersion chart

	$\xi \rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.05	0.10	0.20	0.30	0.40	0.50
	$H \rightarrow$	2.2785	2.4780	2.6460	2.7210	2.7670	2.7938	2.2785	2.4780	2.6460	2.7210	2.7670	2.7938
τ		$\sigma_V^2 = 0.05$						$\sigma_V^2 = 0.15$					
0.50	ARL	2.49	2.84	3.22	3.52	3.97	4.76	2.58	2.96	3.36	3.68	4.18	5.09
	MDRL	2.00	3.00	3.00	3.00	4.00	4.00	2.00	3.00	3.00	3.00	4.00	4.00
	SDRL	0.92	0.99	1.11	1.30	1.69	2.49	0.97	1.05	1.18	1.39	1.83	2.77
0.60	ARL	3.59	4.15	4.82	5.59	6.95	9.68	3.73	4.31	5.03	5.86	7.37	10.41
	MDRL	3.00	4.00	4.00	5.00	6.00	8.00	3.00	4.00	5.00	5.00	6.00	8.00
	SDRL	1.59	1.75	2.08	2.78	4.17	7.01	1.69	1.85	2.22	2.97	4.53	7.74
0.70	ARL	5.86	6.84	8.42	10.98	15.92	25.85	6.09	7.08	8.75	11.56	17.08	28.03
	MDRL	5.00	6.00	7.00	9.00	12.00	19.00	5.00	6.00	8.00	9.00	13.00	20.00
	SDRL	3.12	3.52	4.75	7.46	12.63	22.98	3.29	3.69	5.02	8.01	13.85	25.24
0.80	ARL	11.66	13.94	19.83	30.76	50.97	87.28	12.01	14.58	20.81	32.29	54.05	94.61
	MDRL	10.00	12.00	16.00	23.00	36.00	61.00	10.00	12.00	16.00	24.00	39.00	66.00
	SDRL	7.49	9.06	15.13	26.77	47.82	84.58	7.79	9.54	16.12	28.15	50.57	92.06
0.90	ARL	36.12	48.01	80.28	128.36	199.37	300.50	37.45	49.81	83.80	133.84	212.29	322.58
	MDRL	29.00	36.00	57.00	90.00	139.00	208.00	30.00	37.00	60.00	94.00	148.00	225.00
	SDRL	29.67	41.89	75.94	124.76	197.47	298.90	30.98	43.49	79.05	130.43	210.03	319.48
0.95	ARL	94.96	125.09	183.99	241.13	299.81	348.89	98.13	127.34	187.77	246.29	310.18	365.65
	MDRL	67.00	88.00	128.00	168.00	207.00	242.00	70.00	89.00	131.00	171.00	216.00	254.00
	SDRL	92.99	122.01	182.39	240.00	298.93	348.14	96.34	124.75	184.95	246.25	307.19	365.31
1.00	ARL	201.24	200.04	202.98	201.85	203.44	204.18	203.26	201.63	204.43	204.94	207.82	208.78
	MDRL	133.00	136.00	141.00	140.00	141.00	142.00	135.00	138.00	142.00	141.00	144.00	145.00
	SDRL	216.33	205.36	203.34	201.53	203.27	203.69	218.61	206.21	205.41	205.63	207.78	208.81
1.05	ARL	74.17	81.18	87.11	90.01	92.22	94.35	75.96	83.18	89.76	92.79	96.16	97.48
	MDRL	51.00	56.00	60.00	62.00	64.00	65.00	52.00	58.00	62.00	65.00	67.00	68.00
	SDRL	76.82	81.25	86.76	89.62	92.02	93.82	78.00	83.47	89.22	91.97	94.98	96.35
1.10	ARL	30.03	33.99	38.68	41.82	44.98	46.96	31.08	35.04	39.94	43.34	46.36	48.84
	MDRL	22.00	25.00	27.00	30.00	32.00	33.00	23.00	26.00	28.00	30.00	32.00	34.00
	SDRL	28.82	32.10	37.23	40.55	43.85	45.96	29.66	33.26	38.46	42.43	45.53	48.17
1.20	ARL	10.62	11.86	13.22	14.29	15.43	16.61	10.98	12.24	13.64	14.82	16.08	17.28
	MDRL	8.00	9.00	10.00	11.00	11.00	12.00	8.00	10.00	10.00	11.00	12.00	12.00
	SDRL	9.32	10.06	11.48	12.78	14.15	15.60	9.68	10.42	11.92	13.32	14.86	16.38
1.30	ARL	5.81	6.40	6.98	7.37	7.84	8.28	5.98	6.63	7.20	7.63	8.13	8.65
	MDRL	4.00	5.00	6.00	6.00	6.00	6.00	5.00	5.00	6.00	6.00	6.00	6.00
	SDRL	4.84	5.10	5.60	6.07	6.68	7.24	5.03	5.32	5.82	6.33	7.01	7.59
1.40	ARL	3.86	4.25	4.53	4.77	4.96	5.18	3.99	4.38	4.69	4.90	5.13	5.37
	MDRL	3.00	3.00	4.00	4.00	4.00	4.00	3.00	3.00	4.00	4.00	4.00	4.00
	SDRL	3.09	3.27	3.42	3.66	3.91	4.23	3.18	3.38	3.55	3.81	4.07	4.43
1.50	ARL	2.90	3.15	3.34	3.46	3.57	3.69	2.98	3.24	3.44	3.57	3.68	3.81
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00	2.00	3.00	3.00	3.00	3.00	3.00
	SDRL	2.18	2.30	2.41	2.51	2.65	2.82	2.26	2.39	2.48	2.61	2.76	2.92
2.00	ARL	1.45	1.51	1.57	1.59	1.62	1.62	1.47	1.54	1.60	1.63	1.64	1.66
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.77	0.82	0.87	0.88	0.90	0.92	0.79	0.85	0.89	0.92	0.93	0.95
3.00	ARL	1.07	1.08	1.09	1.10	1.10	1.10	1.07	1.09	1.10	1.10	1.11	1.11
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.27	0.29	0.31	0.32	0.32	0.33	0.28	0.30	0.32	0.33	0.33	0.34

Table 6.5: (Continued).

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.05	0.10	0.20	0.30	0.40	0.50
	$H\rightarrow$	2.2785	2.4780	2.6460	2.7210	2.7670	2.7938	2.2785	2.4780	2.6460	2.7210	2.7670	2.7938
τ		$\sigma_V^2 = 0.30$						$\sigma_V^2 = 0.50$					
0.50	ARL	2.71	3.11	3.54	3.92	4.51	5.65	2.84	3.26	3.73	4.16	4.87	6.30
	MDRL	3.00	3.00	3.00	4.00	4.00	5.00	3.00	3.00	4.00	4.00	4.00	5.00
	SDRL	1.02	1.10	1.25	1.50	2.04	3.20	1.06	1.14	1.31	1.60	2.25	3.70
0.60	ARL	3.92	4.53	5.31	6.26	8.07	11.90	4.10	4.75	5.61	6.72	8.90	13.75
	MDRL	4.00	4.00	5.00	6.00	7.00	9.00	4.00	4.00	5.00	6.00	7.00	11.00
	SDRL	1.77	1.94	2.37	3.24	5.10	9.10	1.85	2.02	2.52	3.52	5.78	10.81
0.70	ARL	6.37	7.45	9.32	12.46	18.96	32.55	6.65	7.82	9.90	13.56	21.43	38.77
	MDRL	6.00	7.00	8.00	10.00	14.00	24.00	6.00	7.00	9.00	11.00	16.00	28.00
	SDRL	3.46	3.91	5.43	8.77	15.67	29.60	3.60	4.10	5.86	9.72	17.92	35.74
0.80	ARL	12.59	15.28	22.15	35.22	60.61	110.10	13.12	16.01	23.79	38.83	69.22	133.40
	MDRL	11.00	13.00	17.00	26.00	43.00	78.00	11.00	14.00	18.00	28.00	49.00	93.00
	SDRL	8.25	10.11	17.26	31.24	57.02	107.24	8.61	10.63	18.85	34.46	65.86	131.00
0.90	ARL	38.87	52.47	88.73	144.40	234.66	369.18	40.51	54.48	94.79	158.76	263.85	427.79
	MDRL	31.00	39.00	63.00	101.00	164.00	256.00	32.00	40.00	67.00	111.00	184.00	297.00
	SDRL	32.54	46.19	84.18	141.48	232.24	368.18	34.07	48.23	90.31	155.92	259.56	426.01
0.95	ARL	100.85	131.38	195.28	258.93	328.02	391.45	103.16	136.56	204.70	273.76	348.40	412.26
	MDRL	72.00	92.00	136.00	180.00	228.00	272.00	73.00	96.00	142.00	190.00	242.00	283.00
	SDRL	98.66	128.02	193.05	258.17	328.12	389.57	101.14	133.78	202.69	272.40	345.85	413.76
1.00	ARL	202.34	201.73	205.58	207.08	211.53	213.58	203.37	202.69	205.74	209.26	213.01	214.08
	MDRL	134.00	138.00	143.00	144.00	147.00	148.00	134.00	139.00	142.00	145.00	148.00	147.00
	SDRL	217.80	207.75	205.85	206.65	210.91	213.49	217.76	207.28	206.93	209.09	212.08	215.32
1.05	ARL	77.76	84.54	91.36	93.57	97.19	99.40	79.48	86.38	92.43	95.10	97.91	99.43
	MDRL	53.00	59.00	64.00	65.00	67.00	69.00	54.00	60.00	64.00	66.00	68.00	69.00
	SDRL	80.65	84.80	90.58	93.02	96.64	98.99	82.68	86.52	92.09	94.20	97.63	99.52
1.10	ARL	32.07	36.12	41.21	44.46	47.59	50.25	33.10	37.35	42.12	45.73	48.37	50.55
	MDRL	23.00	26.00	29.00	31.00	33.00	35.00	24.00	27.00	30.00	32.00	34.00	35.00
	SDRL	30.85	34.25	39.89	42.92	46.48	49.38	31.83	35.39	40.74	44.71	47.41	49.66
1.20	ARL	11.44	12.69	14.15	15.39	16.69	17.93	11.83	13.09	14.55	15.84	17.11	18.28
	MDRL	9.00	10.00	11.00	11.00	12.00	13.00	9.00	10.00	11.00	12.00	12.00	13.00
	SDRL	10.09	10.82	12.46	13.90	15.43	17.02	10.50	11.27	12.80	14.47	15.93	17.27
1.30	ARL	6.20	6.84	7.43	7.94	8.41	8.96	6.42	7.08	7.70	8.14	8.72	9.26
	MDRL	5.00	5.00	6.00	6.00	6.00	7.00	5.00	6.00	6.00	6.00	7.00	7.00
	SDRL	5.20	5.51	6.00	6.62	7.20	7.92	5.42	5.75	6.25	6.83	7.55	8.27
1.40	ARL	4.11	4.51	4.83	5.07	5.32	5.58	4.25	4.69	4.98	5.24	5.44	5.73
	MDRL	3.00	4.00	4.00	4.00	4.00	4.00	3.00	4.00	4.00	4.00	4.00	4.00
	SDRL	3.30	3.47	3.67	3.94	4.27	4.61	3.43	3.63	3.84	4.12	4.40	4.77
1.50	ARL	3.05	3.33	3.55	3.69	3.82	3.93	3.14	3.43	3.65	3.80	3.91	4.05
	MDRL	2.00	3.00	3.00	3.00	3.00	3.00	2.00	3.00	3.00	3.00	3.00	3.00
	SDRL	2.31	2.47	2.58	2.71	2.88	3.03	2.40	2.57	2.66	2.81	2.96	3.17
2.00	ARL	1.49	1.56	1.62	1.66	1.68	1.70	1.52	1.59	1.66	1.68	1.71	1.72
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.82	0.88	0.92	0.93	0.96	0.98	0.85	0.91	0.95	0.97	0.99	1.01
3.00	ARL	1.08	1.09	1.10	1.11	1.11	1.12	1.08	1.10	1.11	1.12	1.12	1.12
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.28	0.31	0.33	0.34	0.34	0.35	0.29	0.32	0.33	0.35	0.35	0.36

Table 6.6: Run length properties of EWMA-OIRSS (one-sided) dispersion control chart

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.05	0.10	0.20	0.30	0.40	0.50
	$H\rightarrow$	1.9120	2.1990	2.4550	2.5840	2.6640	2.7200	1.9120	2.1990	2.4550	2.5840	2.6640	2.7200
τ				$\sigma_V^2 = 0.05$						$\sigma_V^2 = 0.15$			
1.00	ARL	203.13	199.80	201.15	202.32	202.44	202.91	202.46	201.09	201.33	202.77	204.17	205.23
	MDRL	132.00	135.00	139.00	139.00	140.00	140.00	131.00	137.00	139.00	140.00	142.00	142.00
	SDRL	225.48	207.74	203.23	204.77	203.18	202.89	224.53	208.42	202.94	203.02	203.88	205.91
1.10	ARL	21.11	25.06	30.28	34.37	38.47	41.97	21.56	25.82	31.11	35.71	39.30	43.27
	MDRL	14.00	18.00	22.00	24.00	27.00	29.00	15.00	19.00	22.00	25.00	28.00	30.00
	SDRL	21.59	24.05	29.10	33.29	37.44	41.13	21.95	24.78	30.10	34.46	38.41	42.40
1.20	ARL	8.07	9.46	11.16	12.46	13.74	15.10	8.30	9.84	11.46	12.97	14.32	15.81
	MDRL	6.00	7.00	8.00	9.00	10.00	11.00	6.00	7.00	9.00	10.00	10.00	11.00
	SDRL	7.59	8.33	9.73	11.16	12.54	14.09	7.88	8.68	10.07	11.60	13.23	14.77
1.30	ARL	4.59	5.39	6.12	6.62	7.21	7.83	4.73	5.53	6.32	6.87	7.50	8.07
	MDRL	3.00	4.00	5.00	5.00	5.00	6.00	3.00	4.00	5.00	5.00	6.00	6.00
	SDRL	4.05	4.48	4.97	5.46	6.14	6.88	4.18	4.63	5.15	5.71	6.44	7.09
1.40	ARL	3.16	3.65	4.08	4.36	4.67	4.93	3.25	3.75	4.21	4.52	4.79	5.09
	MDRL	2.00	3.00	3.00	3.00	4.00	4.00	2.00	3.00	3.00	4.00	4.00	4.00
	SDRL	2.58	2.87	3.12	3.36	3.71	4.01	2.67	2.97	3.23	3.50	3.81	4.17
1.50	ARL	2.43	2.75	3.07	3.23	3.38	3.53	2.50	2.83	3.13	3.33	3.49	3.67
	MDRL	2.00	2.00	2.00	3.00	3.00	3.00	2.00	2.00	3.00	3.00	3.00	3.00
	SDRL	1.85	2.05	2.22	2.33	2.49	2.70	1.91	2.11	2.28	2.43	2.61	2.81
2.00	ARL	1.33	1.42	1.50	1.54	1.58	1.60	1.35	1.43	1.52	1.57	1.60	1.63
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.66	0.74	0.80	0.84	0.87	0.90	0.68	0.76	0.82	0.86	0.89	0.92
3.00	ARL	1.05	1.06	1.08	1.09	1.09	1.10	1.05	1.07	1.08	1.09	1.10	1.10
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.23	0.26	0.29	0.31	0.31	0.32	0.24	0.27	0.30	0.31	0.32	0.32
				$\sigma_V^2 = 0.30$						$\sigma_V^2 = 0.50$			
1.00	ARL	201.37	199.83	200.19	202.24	202.66	204.33	200.47	197.45	197.72	199.53	199.23	201.19
	MDRL	130.00	136.00	137.00	140.00	141.00	141.00	129.00	133.00	137.00	138.00	138.00	139.00
	SDRL	223.25	207.91	203.38	202.93	202.92	205.35	222.90	206.57	200.16	200.45	198.99	201.06
1.10	ARL	22.31	26.75	32.17	36.66	40.64	44.43	23.18	27.37	32.55	37.09	41.26	45.17
	MDRL	15.00	19.00	23.00	26.00	28.00	31.00	16.00	20.00	23.00	26.00	29.00	32.00
	SDRL	22.79	25.88	31.00	35.59	39.93	43.63	23.65	26.61	31.34	35.83	40.53	44.36
1.20	ARL	8.56	10.16	11.89	13.35	14.83	16.44	8.85	10.43	12.18	13.77	15.21	16.70
	MDRL	6.00	8.00	9.00	10.00	11.00	12.00	6.00	8.00	9.00	10.00	11.00	12.00
	SDRL	8.13	9.02	10.48	11.97	13.72	15.40	8.43	9.34	10.84	12.46	14.09	15.64
1.30	ARL	4.88	5.72	6.54	7.11	7.74	8.37	5.00	5.87	6.71	7.34	7.93	8.62
	MDRL	4.00	4.00	5.00	5.00	6.00	6.00	4.00	4.00	5.00	6.00	6.00	6.00
	SDRL	4.33	4.79	5.35	5.93	6.65	7.43	4.49	4.93	5.52	6.16	6.84	7.62
1.40	ARL	3.32	3.85	4.33	4.65	4.96	5.25	3.42	3.97	4.48	4.78	5.09	5.42
	MDRL	2.00	3.00	3.00	4.00	4.00	4.00	3.00	3.00	4.00	4.00	4.00	4.00
	SDRL	2.76	3.05	3.33	3.61	3.97	4.31	2.84	3.15	3.49	3.72	4.07	4.48
1.50	ARL	2.56	2.91	3.24	3.41	3.58	3.78	2.62	2.97	3.30	3.51	3.68	3.87
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00	2.00	2.00	3.00	3.00	3.00	3.00
	SDRL	1.97	2.19	2.37	2.52	2.68	2.92	2.04	2.25	2.44	2.59	2.78	3.00
2.00	ARL	1.37	1.46	1.54	1.60	1.63	1.65	1.39	1.48	1.57	1.62	1.66	1.69
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.71	0.78	0.85	0.89	0.92	0.94	0.72	0.80	0.87	0.91	0.95	0.98
3.00	ARL	1.06	1.07	1.09	1.10	1.10	1.11	1.06	1.08	1.09	1.10	1.11	1.11
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.24	0.27	0.31	0.32	0.33	0.34	0.25	0.28	0.31	0.33	0.33	0.35

6.4 Performance comparison of control charts

In this section, we compare the proposed EWMA control charts based on ORSS and OIRSS with some CUSUM and EWMA control charts for monitoring process location and process variability based on SRS and RSS methods. The performance of each control chart is evaluated in terms of logarithm of the out-of-control ARLs.

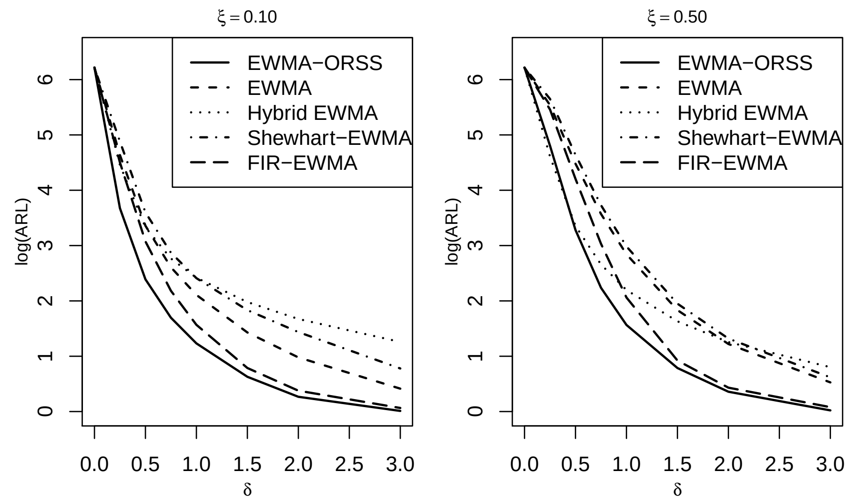

Figure 6.1: Comparison of the EWMA-ORSS location control chart with some classical EWMA charts based on SRS

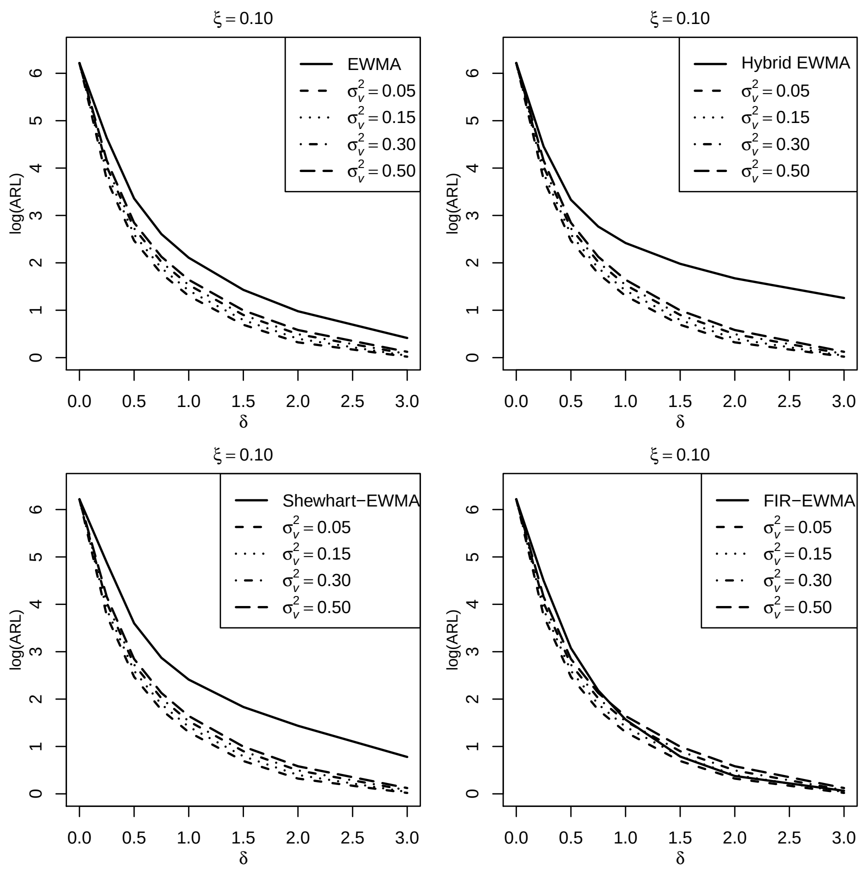

Figure 6.2: Comparison of EWMA-OIRSS location chart with some classical EWMA charts based on SRS

(i) EWMA-ORSS and EWMA-OIRSS location charts versus EWMA location charts

We compare the proposed EWMA-ORSS chart with some existing EWMA charts based on SRS, i.e., EWMA, hybrid EWMA, Shewhart-EWMA, fast initial response (FIR) based EWMA (FIR-EWMA) control charts. Note that the in-control ARL of all EWMA charts is fixed to 500. From Figure 6.1, it is clear that the EWMA-ORSS chart is more powerful than the other controlling schemes considered here. With $\xi = 0.5$, when $\delta \leq 0.5$, hybrid EWMA chart performs better than the EWMA-ORSS chart. In Figure 6.2, we compare the EWMA-OIRSS chart with the classical EWMA control charts. With $\xi = 0.1$, EWMA-OIRSS chart has smaller out-of-control ARLs than its competitors. However, for large shifts, i.e., $\delta \geq 1.5$, FIR-EWMA control chart is slightly better than the EWMA-OIRSS control chart. The performance of EWMA-OIRSS chart increases as the value of σ_V^2 decreases and vice-versa.

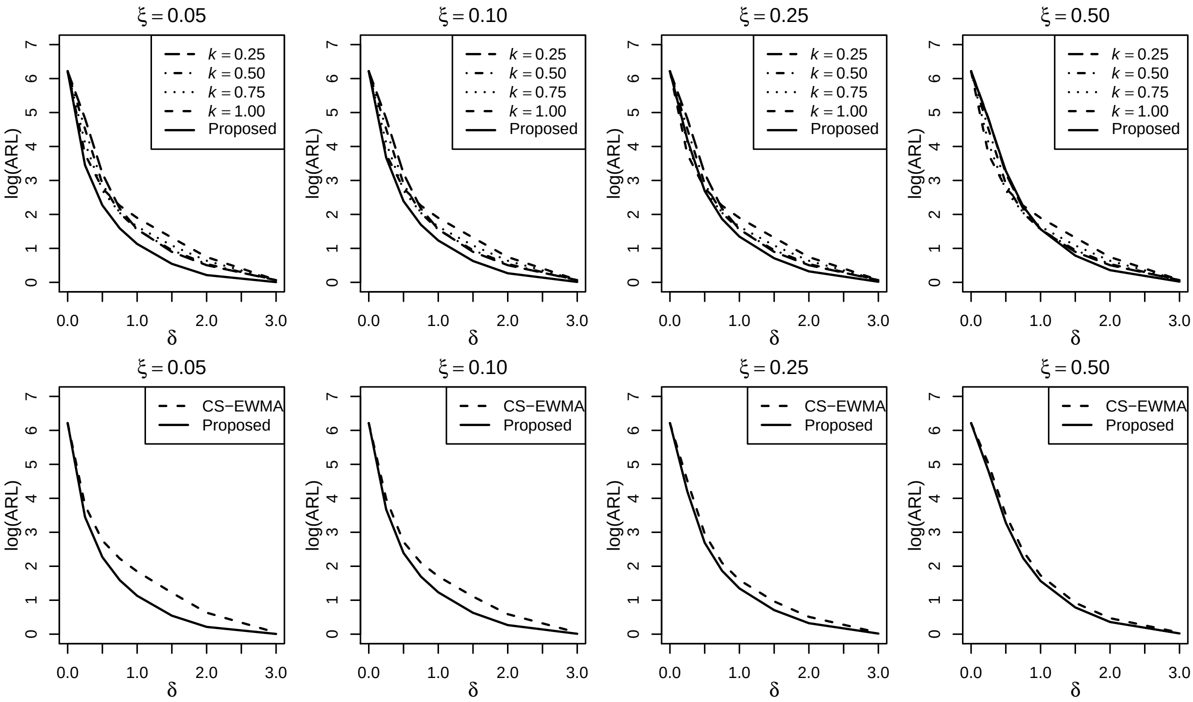

Figure 6.3: Comparison of EWMA-ORSS location control charts versus Shewhart-CUSUM-RSS and Shewhart-EWMA-RSS control charts

(ii) EWMA-ORSS location chart versus Shewhart-CUSUM-RSS location chart

In Figure 6.3, we compare the EWMA-ORSS control chart with the Shewhart-CUSUM-RSS control chart when the in-control ARL is fixed to 500. Here k is the reference values of the plotting-statistics based on the Shewhart-CUSUM-RSS control chart. It is observed that for all kinds of shifts, the values of out-of-control ARLs are uniformly less than their counterparts based on Shewhart-CUSUM-RSS control chart when $\xi \leq 0.10$. For large values of δ, say $\delta \geq 1.5$, the EWMA-ORSS control chart detects the random shifts substantially quicker than the Shewhart-CUSUM-RSS control chart. Therefore, for small shifts in the process mean, the proposed EWMA chart is better in terms of small out-of-control ARLs.

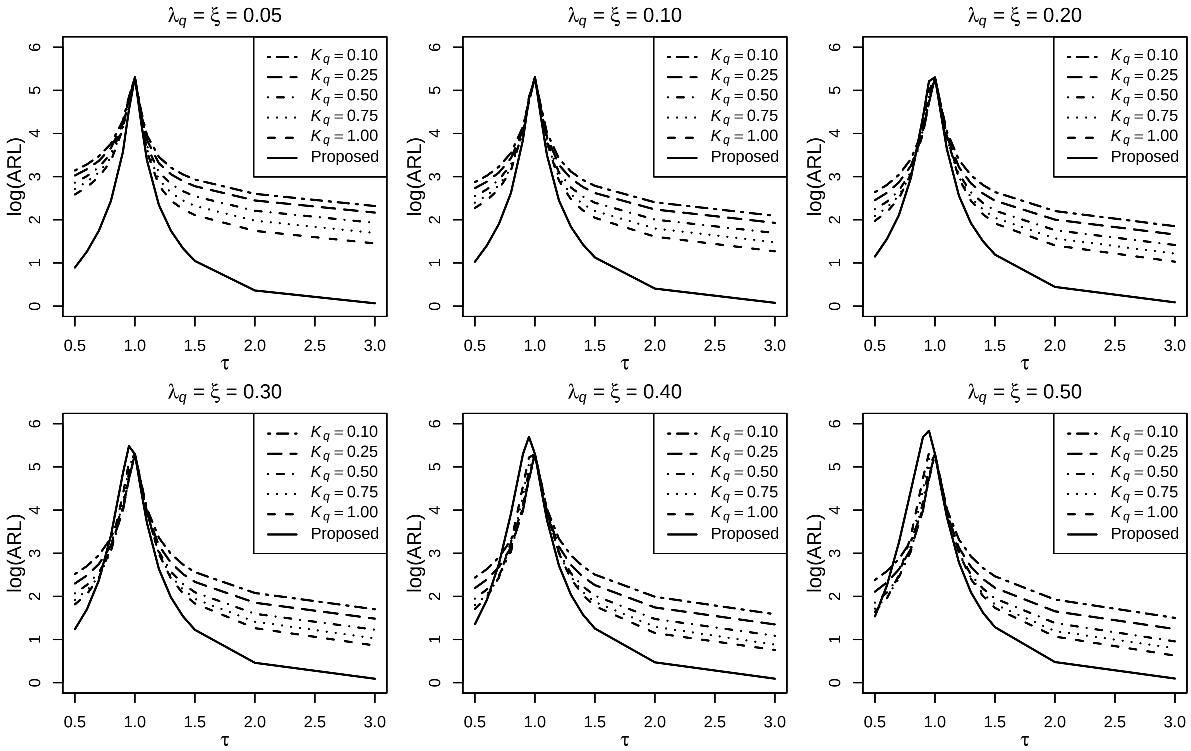

Figure 6.4: Comparison of the proposed EWMA-ORSS chart versus CS-EWMA-SRS chart for monitoring process dispersion

(iii) EWMA-ORSS location chart versus Shewhart-EWMA-RSS location chart

The EWMA-ORSS chart is also compared with the Shewhart-EWMA-RSS control chart in Figure 6.3 when the in-control ARL is fixed to 500. For a fair comparison of both control charts, we have assumed same values for their smoothing constants. It turns out that for all kinds of shifts, the performance of the proposed EWMA control chart is better than the Shewhart-EWMA-RSS control chart. The values of out-of-control ARLs under both charts come closer for large values of ξ. It is worth noting here that for large values of ξ, the proposed EWMA-ORSS control chart still dominates the Shewhart-EWMA-RSS control chart in detecting both small and large shifts in the process mean.

(iv) EWMA-ORSS chart versus CS-EWMA-SRS chart for monitoring process dispersion

Abbas et al. (2013a) suggested a CUSUM control chart based on the EWMA-statistic for monitoring process dispersion. They showed that CS-EWMA-SRS control chart is better than the S^2-EWMA (cf. Castagliola, 2005) and (cf. Castagliola et al., 2009) control charts when detecting small shifts in process dispersion. In Figure 6.4, we compare the proposed EWMA-ORSS control chart with the CS-EWMA-SRS control chart based on different values of the constants λ_q and K_q. Here, λ_q and K_q are the parameters of the CS-EWMA control chart (cf. Abbas et al., 2013a). The values of logarithms of the out-of-control ARLs computed for CS-EWMA-SRS and EWMA-ORSS charts are plotted against different values of τ when the in-control ARL is fixed to 200. From Figure 6.4, it is clear that for large shifts (upper or lower) in the process dispersion, the proposed EWMA-ORSS control chart is far better than the CS-EWMA-SRS control chart. It is also noted that for moderate values of ξ, i.e., $\xi \geq 0.3$, when detecting downward shifts in the process dispersion, CS-EWMA-SRS control chart is able to perform better than the proposed EWMA-ORSS control chart.

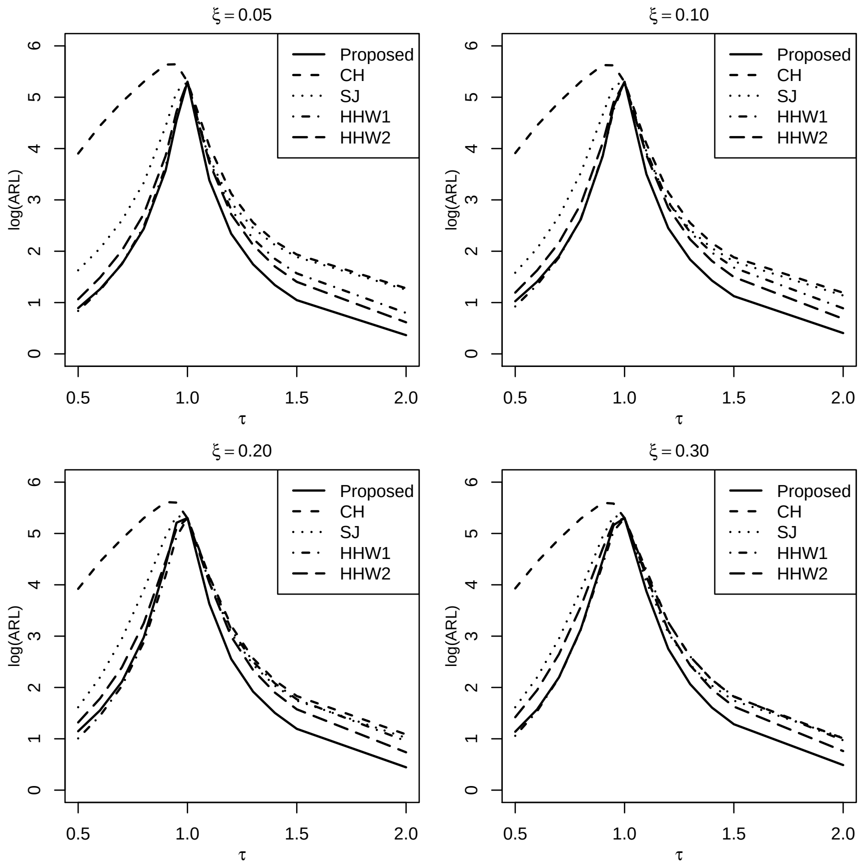

Figure 6.5: Comparison of EWMA-ORSS scale control chart versus EWMA control charts for monitoring process dispersion

(v) EWMA-ORSS dispersion chart versus EWMA dispersion charts

Crowder and Hamilton (1992) applied the logarithmic transformation to the unbiased sample variance based on SRS and proposed and EWMA control chart for monitoring increases in the process standard deviation, named CH. Shu and Jiang (2008) extended the work and suggested another EWMA chart, named SJ, and claimed that their chart is better than the EWMA chart proposed by Crowder and Hamilton (1992). Recently, Huwang et al. (2010) suggested two new EWMA-type dispersion control charts for monitoring changes in the process dispersion, named HHW1 and HHW2. For a fair comparison, we compare the proposed EWMA-ORSS control chart with these EWMA control charts in Figure 6.5 for different values of ξ. It is noteworthy that the proposed EWMA-ORSS control scheme outperforms all EWMA control charts for positive shifts in the process dispersion for all values of ξ. However, in some cases, HHW1-EWMA control chart dominates the EWMA-ORSS control chart when detecting decreases in the process variability. In this comparison, for $\xi = 0.3$, we considered the asymmetric control limits for the EWMA-ORSS chart.

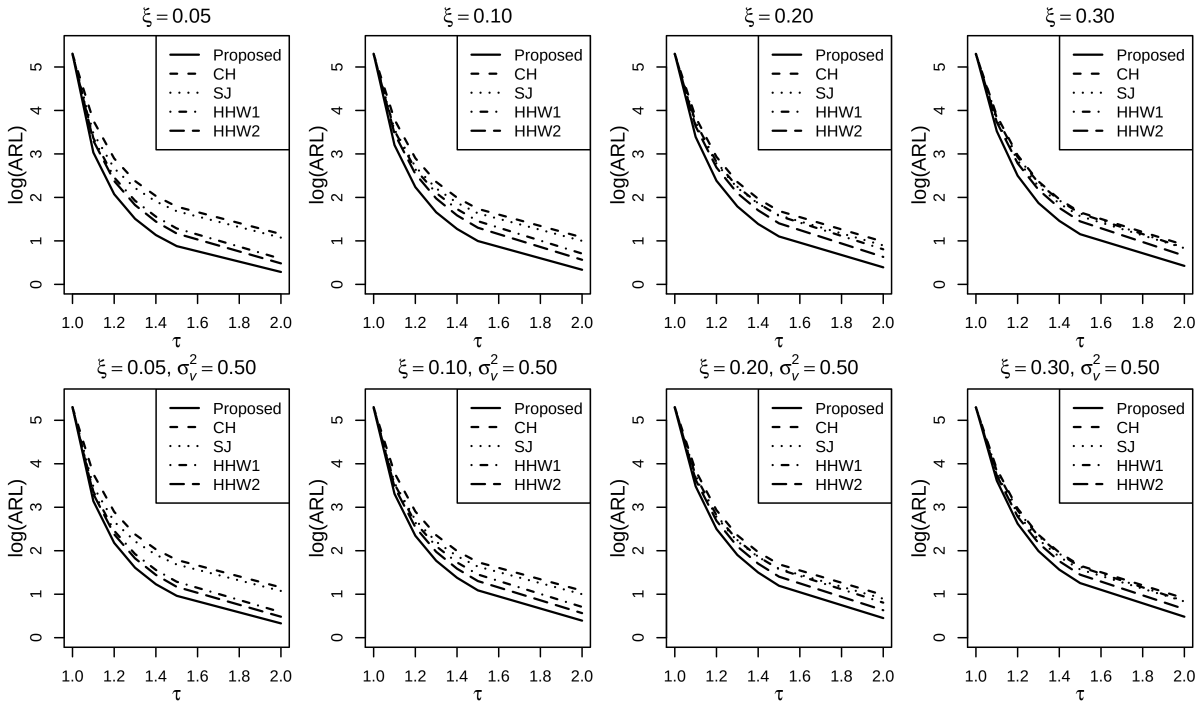

Figure 6.6: Comparison of one-sided EWMA-ORSS and EWMA-OIRSS charts versus one-sided EWMA dispersion control charts

Similarly, in Figure 6.6, the one-sided EWMA dispersion charts are compared with the one-sided EWMA-ORSS and EWMA-OIRSS charts for detecting increases in the process dispersion. It is clear that the EWMA-ORSS scheme dominates all EWMA charts when detecting increases in the process variability for all values of ξ. We also compare the worst EWMA-OIRSS chart based on $\sigma_V^2 = 0.50$ with these EWMA dispersion charts. It is worth mentioning here that the values of out-of-control ARLs under EWMA-OIRSS chart are less than the values of its counterparts. This shows that both EWMA-ORSS and EWMA-OIRSS charts are efficient alternatives to the dispersion charts considered here.

6.5 An application to real data

In this section, a real data set is used to explain the implementation of the proposed EWMA control charts based on SRS, RSS and ORSS schemes.

Suppose we wish to establish statistical control of the inside diameter of the piston rings for an automotive engine manufactured by a forging process (cf. Montgomery, 2009). Forty samples, each of size 5, have been taken from this process. The inside diameters are measured in millimeters (mm). We combine all samples such that we have 200 measurements of the inside diameters of the piston rings. Then, we apply three goodness-of-fit tests on this data set. The p-values for the Shapiro-Wilk, Kolmogorov-Smirnov (by ignoring ties) and Anderson-Darling tests are 0.2175, 0.1216 and 0.2259, respectively. It is clear that the data set follows normal distribution.

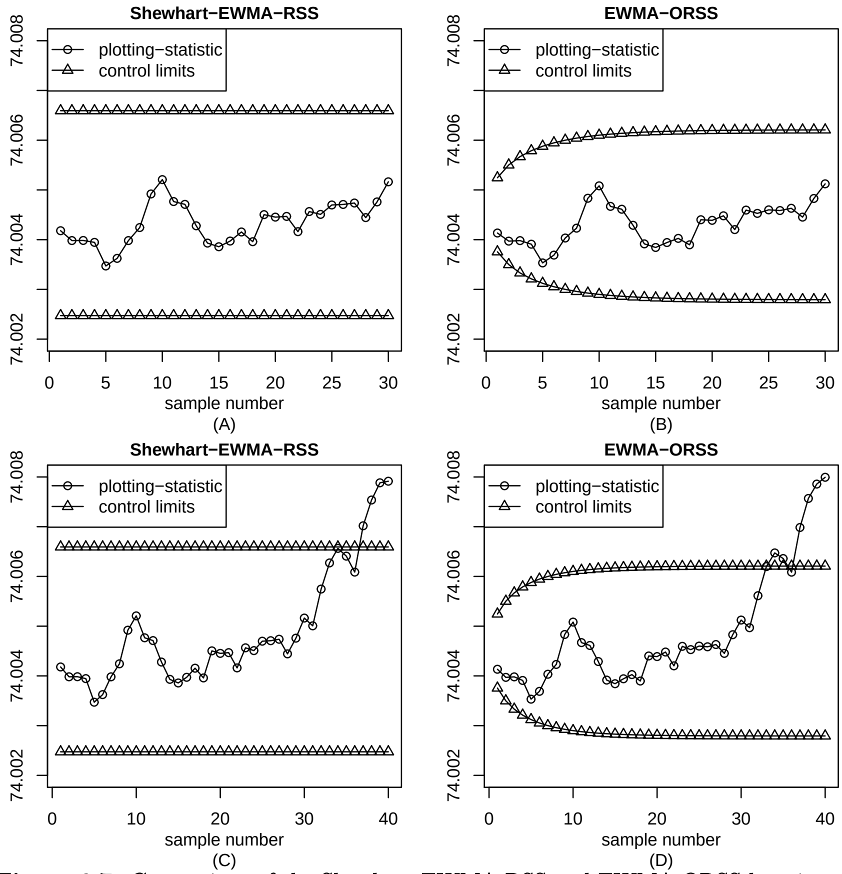

Figure 6.7: Comparison of the Shewhart-EWMA-RSS and EWMA-ORSS location control charts for real data

In order to compare the Shewhart-EWMA-RSS and EWMA-ORSS control charts, we need to collect data under both RSS and ORSS schemes. For this purpose, we assume that the process is in-control, and we draw 30 samples, each of size 5, from the 200 measurements under both RSS and ORSS schemes. Note that these samples are drawn by using with replacement sampling scheme. Based on these 30 samples, control limits of the Shewhart-EWMA-RSS and EWMA-ORSS control charts are estimated and plotted along with the values of the corresponding plotting-statistics against sample number in Figure 6.7. For both EWMA control charts, the in-control ARL is fixed to 500. From Figure 6.7, it is clear that both sub-figures (A) and (B) show that the process is in control state. Now, suppose that after 30th sample, the process gets out-of-control. For this purpose, we again draw 10 samples, each of size 5, from 200 measurements. We add 0.005 to all values within each sample, that were obtained under RSS and ORSS schemes. For both EWMA control charts, the values of their plotting-statistics have been computed for these 10 samples and are plotted in sub-figures (C) and (D) in Figure 6.7. It is evident that both EWMA control charts are showing out-of-control signals after 30th sample. It is interesting to note that the Shewhart-EWMA-RSS control chart detects the random shift at the 37th sample, whereas the proposed EWMA-ORSS control chart detects the same shift at the 34th sample. This shows that the EWMA-ORSS chart dominates the Shewhart-EWMA-RSS control chart and detects the random shift in the process mean substantially quicker than its competitor.

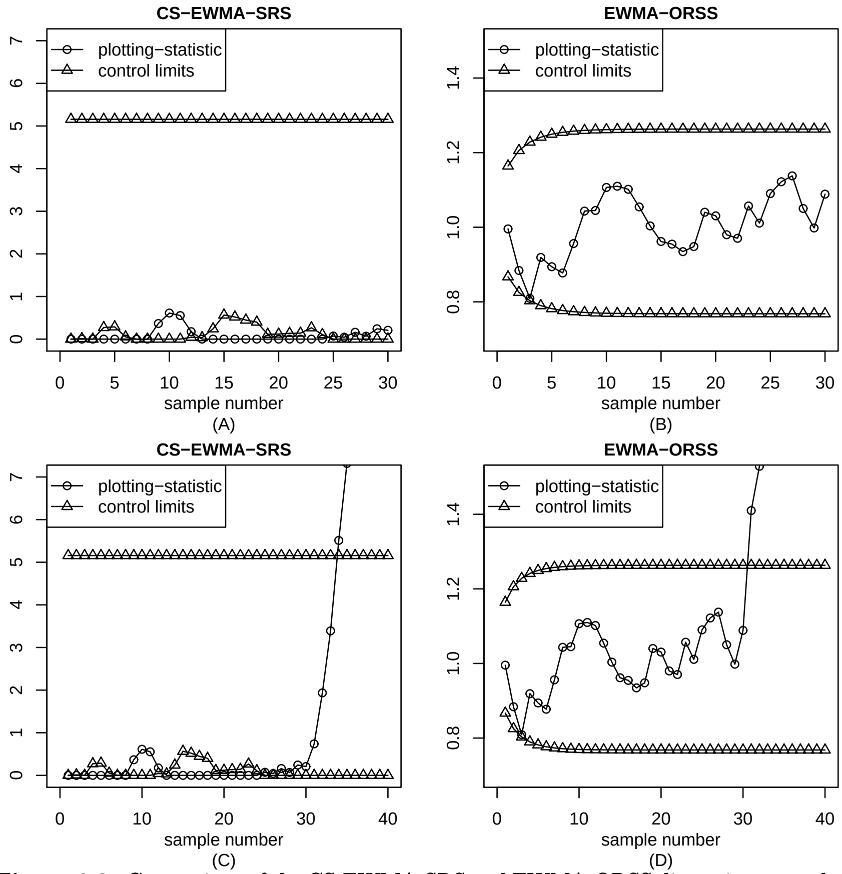

Figure 6.8: Comparison of the CS-EWMA-SRS and EWMA-ORSS dispersion control charts for real data

For a fair comparison of the dispersion charts, we consider the CS-EWMA-SRS and EWMA-ORSS control charts. We draw 30 samples with replacement, each of size 5, from the 200 measurements by using SRS and ORSS schemes. For brevity, we standardize the values obtained under each sampling scheme. Based on these samples, the control limits of each chart are computed. For both EWMA control charts, the in-control ARL is fixed to 200. We consider $\xi = \lambda_q = 0.2$, $K'_q = 0.167$ and $H'_q = 5.157$ (Abbas et al., 2013a). The control limits and plotting-statistics of both EWMA control charts are displayed in Figure 6.8. It is clear from sub-figures (A) and (B) in Figure 6.8 that the process is in-control state. Suppose that after a certain time, the process gets out-of-control. For this purpose, we again draw 10 samples each of size 5 from the 200 measurements using both sampling schemes. The sample values under each scheme are then multiplied by two. The plotting-statistics of both EWMA charts based on 40 samples each of size 5 are displayed in Figure 6.8. From Figure 6.8, the sub-figures (C) and (D) show that the CS-EWMA-SRS chart detects a random shift in the process dispersion at the 34th sample, whereas the EWMA-ORSS chart detects the same shift at the 31st sample. This earlier detection makes the EWMA-ORSS control chart as an efficient and powerful alternative to the CS-EWMA-SRS control chart for monitoring the process variability.

6.6 Conclusion

In this chapter, we propose some improved EWMA control charts based on the BLUEs-ORSS for monitoring process location and process dispersion. Monte Carlo simulations have been used to estimate the ARLs, MDRLs and SDRLs of the proposed EWMA control charts. It is observed that the EWMA-ORSS control chart is more efficient in detecting small random shifts in the process mean as compared with the Shewhart-CUSUM and the Shewhart-EWMA control charts based on RSS. It is worth mentioning that the one-sided EWMA-ORSS and EWMA-OIRSS dispersion control charts are uniformly better than their counterparts considered here. The two-sided EWMA-ORSS dispersion control chart is also sensitive to the large upward or downward shifts in the process variability. Finally, we considered a real data application of the proposed EWMA control charts. The current work can be improved by developing EWMA control charts based on double RSS schemes for monitoring process mean and dispersion.

Chapter 7

An Improved Maximum Exponentially Weighted Moving Average Control Chart for Monitoring Process Mean and Variability

This chapter appeared in:

Haq, A., Brown, J., Moltchanova, E., 2013, An improved maximum exponentially weighted moving average control chart for monitoring process mean and variability, Quality and Reliability Engineering International, Early view, DOI: 10.1002/qre.1586.

Maximum exponentially weighted moving average (MaxEWMA) control charts have gained considerable attention for detecting changes in both process mean and process variability. In this chapter, we propose improved MaxEWMA control charts based on ordered ranked set sampling (ORSS) and ordered imperfect ranked set sampling (OIRSS) schemes for simultaneous detection of both increases and decreases in the process mean and/or variability, named MaxEWMA-ORSS and MaxEWMA-OIRSS control charts. These MaxEWMA control charts are based on the best linear unbiased estimators of location and scale parameters obtained under ORSS and OIRSS methods. Extensive Monte Carlo simulations have been used to estimate the average run length and standard deviation of run length of the proposed MaxEWMA control charts. These control charts are compared with their counterparts based on simple random sampling (SRS), i.e., MaxEWMA-SRS and MaxGWMA-SRS control charts. These proposed MaxEWMA-ORSS and MaxEWMA-OIRSS control charts are able to perform better than the MaxEWMA-SRS and MaxGWMA-SRS control charts for detecting shifts in the process mean and dispersion. An application to real data is provided to illustrate the implementation of the proposed MaxEWMA control charts.

7.1 Introduction

The main objective of statistical process control (SPC) is to detect variations in parameters of production processes as early as possible. Statistical quality control charts are well-known process monitoring tools of SPC that are mainly used to track unusual variation in the manufacturing processes. The basic concept of control chart was firstly introduced by Walter A. Shewhart in the 1920s. Later on, this concept led to the introduction of modern SPC. The advanced statistical process monitoring techniques currently used are exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) control charts.

Control charts consider either location or dispersion. Location charts are used to monitor the process mean while dispersion charts monitor the process variability. Roberts (1959) introduced the EWMA control chart for monitoring the process mean while the CUSUM control chart was suggested by Page (1954). Both of these control charts are more sensitive to the small changes in the process parameters than the classical Shewhart control chart. For this reason, EWMA and CUSUM control charts are widely used in chemical and process industries, where small disturbances often have serious financial consequences (cf. Montgomery, 2009). In the last decades, the EWMA control charts are mostly used to monitor changes in the process mean and process dispersion. For discussion of some recent improvements and advancement in the EWMA and CUSUM quality control schemes for monitoring the process mean or dispersion, see Abbas et al. (2011, 2013b,a), Riaz et al. (2011), Haq (2013) and references cited therein.

In recent years, many researchers have suggested control charts for simultaneously monitoring the process mean and dispersion of normally distributed processes. Generally, two control charts are used to jointly monitor the process mean and variance. Chen and Cheng (1998) suggested a new control chart, named Max-chart, to simultaneously monitor the process mean and the process standard deviation. Lee and Lin (2012) combined Max-chart and adaptive chart to propose an improved adaptive Max-chart. They showed that the adaptive Max-charts with variable parameters are more sensitive to small shifts in the process mean and variance than that of EWMA, CUSUM and double sampling charts. Chen et al. (2001) proposed the maximum EWMA (MaxEWMA) chart based on inverse normal transformations, which takes the maximum of test statistics of two EWMA control charts. Li et al. (2010) proposed a self-starting control chart based on the likelihood ratio test and the EWMA procedure for monitoring both mean and variability when the process parameters are unknown. Recently, Sheu et al. (2012) suggested an extended maximum generally weighted moving average (MaxGWMA) control chart for monitoring process mean and variability. It is shown that the MaxGWMA chart is more sensitive than the MaxEWMA chart. Since the estimators of mean and variance in both MaxEWMA and MaxGWMA charts are based on simple random sampling (SRS), therefore, we name these charts as MaxEWMA-SRS and MaxGWMA-SRS control charts. Some important literature on the joint monitoring of the process mean and variability may be seen in Gan (1995), Reynolds Jr and

Stoumbos (2001), Costa and Rahim (2004, 2006a,b), Chen et al. (2004), Wu and Tian (2005), Zhang and Wu (2006), Wu et al. (2010), Zhang et al. (2011), Yang et al. (2012), Zhang and Zhang (2013) and references cited therein.

The traditional ranked set sampling (RSS) scheme has gained considerable attention from researchers in the last decades. The RSS scheme becomes an efficient alternative to SRS when it is easy to rank a small set of selected units without knowing the actual values. However, in some cases, ranking cost cannot be ignored. The traditional RSS scheme was first introduced by McIntyre (1952). Later on, Takahasi and Wakimoto (1968) developed the statistical background of RSS scheme. They showed that the mean estimator based on RSS is unbiased and it is more precise than the mean estimator based on SRS. Dell and Clutter (1972) were the first to study the effect of imperfect ranking on the performance of mean estimator. It is shown that even under imperfect rankings, the mean estimator under RSS remains unbiased and it is still better than the mean estimator with SRS. Salazar and Sinha (1997) were the first to propose a Shewhart-type control chart for monitoring process mean based on RSS scheme. Muttlak and Al-Sabah (2003) extended their work, and suggested some improved quality control charts for monitoring process mean based on perfect and imperfect RSS schemes. They showed that the RSS-based control charts are more powerful than the control chart based on SRS. Al-Saleh and Al-Kadiri (2000) introduced double RSS (DRSS) scheme for efficient estimation of the population mean and showed that the mean estimator based on DRSS scheme is better than the mean estimator with RSS. Using this fact, Abujiya and Muttlak (2004) suggested some Shewhart-type control charts for detecting changes in the process mean based on DRSS scheme. DRSS scheme based control charts are better than their counterparts based on SRS and RSS methods. Balakrishnan and Li (2005, 2008) introduced ordered RSS (ORSS), and used it to obtain the best linear unbiased estimators (BLUEs) of the unknown parameters of location-scale family of distributions. They showed that the BLUEs based on ORSS (BLUEs-ORSS) are uniformly better than the BLUEs constructed under SRS and RSS schemes. Recently, Haq et al. (2013a) suggested improved EWMA control charts for monitoring process mean and dispersion based on ORSS and ordered imperfect RSS (OIRSS) schemes. They showed that these control charts are better than the Shewhart-EWMA and Shewhart-CUSUM control charts based on RSS for detecting small shifts in the process mean. For more details about the control charts based on different RSS schemes, see Al-Omari and Haq (2012), Abujiya et al. (2014), Abujiya et al. (2013a,b), Mehmood et al. (2013), Haq (2014), Haq et al. (2013a) and references cited therein.

Following the motivation from Haq et al. (2013a), in this chapter we propose improved MaxEWMA control charts based on ORSS and OIRSS schemes, named MaxEWMA-ORSS and MaxEWMA-OIRSS control charts, for simultaneously monitoring the process mean and variance of a normally distributed process. The MaxEWMA-ORSS and MaxEWMA-OIRSS charts are based on the BLUEs of location and scale parameters obtained under ORSS and OIRSS schemes. Utilizing extensive Monte Carlo simulations, we estimate the average run length (ARL) and standard deviation of run length (SDRL) of both MaxEWMA control charts. ARL is the average number of samples that are required to issue a particular size shift in the process location or dispersion or both. The performance of each MaxEWMA control chart is evaluated in terms of the ARL and SDRL. We compare the performance of MaxEWMA-ORSS and MaxEWMA-OIRSS charts with the MaxEWMA-SRS and MaxGWMA-SRS charts. It is remarkable that the proposed charts perform better than their counterparts considered here.

The rest of the article is organized as follows: in Section 7.2, we briefly explain the MaxEWMA-SRS and MaxGWMA-SRS control charts. In Section 7.3, we explain the ORSS scheme and use it to find the BLUEs of the unknown parameters of location-scale family of distributions. Moreover, we also explain the OIRSS scheme. The proposed MaxEWMA control charts based on ORSS and OIRSS schemes are constructed in Section 7.4. The proposed control charts are compared with their counterparts in Section 7.5. An application to real data is provided in Section 7.6. Section 7.7 finally summarizes the main findings.

7.2 Control charts available in literature

In this section, we provide a brief explanation about some recent control charts that are mostly used to simultaneously monitor the mean and variance of normally distributed processes.

Let X be a certain quality characteristic of a process and assume this characteristic is normally distributed with mean $\mu+\delta\sigma$ and standard deviation $\rho\sigma$, i.e., $X \sim N(\mu+\delta\sigma, \rho\sigma)$, where μ and σ are the standard values for the process mean and process standard deviation, respectively. The underlying process is said to be in-control when $\delta=0$ and $\rho=1$; otherwise, the process has changed or drifted. Let X_{it}, $i=1,2,...,n$, $t=1,2,...$, be the measurements of X arranged in groups of size n_t, where t indexes the group number. Let $\bar{X}_t$ and S_t^2 be the sample mean and sample variance computed from the tth subgroup, respectively, where $\bar{X}_t=(X_{1t}+X_{2t}+\cdots+X_{n_t t})/n_t$ and $S_t^2=\sum_{i=1}^{n_t}(X_{it}-\bar{X}_t)^2/(n_t-1)$. Then, $\bar{X}_t$, $t=1,2,...$, are independent and identically distributed (IID) random variables with mean $\mu+\delta\sigma$ and standard deviation $\rho\sigma/\sqrt{n_t}$, i.e., $\bar{X}_t \sim N(\mu+\delta\sigma, \rho\sigma/\sqrt{n_t})$; $(n_t-1)S_t^2/\rho^2\sigma^2$, $t=1,2,...$, are IID chi-square random variables with n_t-1 as degrees of freedom, i.e., $(n_t-1)S_t^2/\rho^2\sigma^2 \sim \chi^2_{n_t-1}$. Note that both $\bar{X}_t$ and S_t^2 are independent of each other. Now we define two statistics based on the following transformations:

$$U_t=\frac{\bar{X}_t-\mu}{\sigma/\sqrt{n_t}} \quad \text{and} \quad V_t=\Phi^{-1}\left[F\left\{\frac{(n_t-1)S_t^2}{\sigma^2}\right\};n_t-1\right], \tag{7.1}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse distribution function of normal distribution and $F(h;v)$ is the chi-square distribution function with v as degrees of freedom. For more details on these transformations and their applications, see Quesenberry (1995). When the process is in-control, both U_t and V_t defined in (7.1) are independent and identically distributed (IID) standard normal random variables, i.e., $U_t \sim N(0,1)$ and $V_t \sim N(0,1)$.

(i) MaxEWMA-SRS control chart

Xie (1999) and Chen et al. (2001) were the first to discuss the concept of the MaxEWMA chart. The test-statistic of the MaxEWMA-SRS chart effectively combines the plotting-statistics of two EWMA control charts into a single test-statistic. This enables MaxEWMA-SRS chart to simultaneously monitor the changes in the process mean and process variability.

Based on U_t and V_t, given in (7.1), we can define two EWMA sequences, U_t^* and V_t^*, respectively, by using the following recurrence formulae:

$$U_t^* = \xi U_t + (1-\xi)U_{t-1}^*, \quad 0 < \xi \leq 1,\ U_0^* = 0,\ t = 1, 2, ..., \tag{7.2}$$

$$V_t^* = \xi V_t + (1-\xi)V_{t-1}^*, \quad 0 < \xi \leq 1,\ V_0^* = 0,\ t = 1, 2, ..., \tag{7.3}$$

where U_0^* and V_0^* are the starting values of U_t^* and V_t^*, respectively, and ξ is a smoothing constant. Note that U_t^* and V_t^* are also independent of each other because of the independence of U_t and V_t. For an in-control process, we have $U_t^* \sim N(0, \sigma_{U_t^*})$ and $V_t^* \sim N(0, \sigma_{V_t^*})$, where $\sigma^2_{U_t^*} = \sigma^2_{V_t^*} = \frac{\xi}{(2-\xi)}\{1-(1-\xi)^{2t}\}$, for $t = 1, 2, ...$

The test-statistic of the MaxEWMA-SRS chart based on U_t^* and V_t^* is defined as

$$ME_t = \max\{|U_t^*|, |V_t^*|\}, \quad t = 1, 2, ..., \tag{7.4}$$

where $\max(A, B)$ is the maximum of A and B, and $|\cdot|$ represents the absolute value. Since ME_t is non-negative, therefore, the initial state of the MaxEWMA-SRS chart is based only on an upper control limit (UCL_t) at time t, which is given by

$$UCL_t = E(ME_t) + L\sqrt{\mathrm{Var}(ME_t)}, \tag{7.5}$$

where L is the positive control chart multiplier, and its values is determined such that the in-control ARL of the MaxEWMA-SRS chart reaches to a particular level. Here $E(ME_t)$ and $\mathrm{Var}(ME_t)$ are the expected value and variance of ME_t, respectively. For more details about the computation of $E(ME_t)$ and $\mathrm{Var}(ME_t)$, see Xie (1999) and Chen et al. (2001).

(ii) MaxGWMA-SRS control chart

Sheu et al. (2012) extended the work of Xie (1999) and Chen et al. (2001), and proposed the MaxGWMA-SRS control chart to simultaneously detect both increases and decreases in the process mean and dispersion.

As GWMA is a moving average of past data where each data point is assigned a weight. Let M be the number of samples until the first occurrence of event A since the previous occurrence of event A. Thus, we can write

$$\sum_{m=1}^{\infty} P(M = m) = P(M = 1) + P(M = 2) + \cdots + P(M = t) + P(M > t) = 1.$$

Here $P(M = 1), P(M = 2), ..., P(M = t)$ are the weights of the current sample, the previous sample,..., the most out-of-date sample, respectively. Therefore, $P(M > t)$ is weighted with the target value of the underlying process mean. For further details, see Sheu and Lin (2003).

Based on U_t and V_t, given in (7.1), we can define two GWMA statistics, G_t^* and H_t^*, respectively, by using following formulae:

$$\begin{aligned} G_t^* &= P(M=1)U_t + P(M=2)U_{t-1} + \cdots + P(M=t)U_1 + P(M>t)G_0^*, \quad G_0^*=0, \\ H_t^* &= P(M=1)H_t + P(M=2)H_{t-1} + \cdots + P(M=t)H_1 + P(M>t)H_0^*, \quad H_0^*=0, \end{aligned}$$

where $t = 1, 2, ...$ Here G_0^* and H_0^* are the initial values of G_t^* and H_t^*, respectively, and are set to zero. Sheu et al. (2012) set the weights as $P(M=t) = q^{(t-1)^\alpha} - q^{(t)^\alpha}$, for $t = 1, 2, ...,$ where $q(0 \le q \le 1)$ is the design parameter and α is the adjustment parameter determined by the practitioner. Note that these weights follow the discrete Weibull distribution (cf. Nakagawa and Osaki, 1975). The GWMA statistics, G_t^* and H_t^*, can now be simplified as

$$G_t^* = \sum_{j=1}^{t} \left(q^{(j-1)^\alpha} - q^{(j)^\alpha} \right) U_{t-j+1} + q^{(t)^\alpha} G_0^*, \quad G_0^* = 0,\ t = 1, 2, ..., \tag{7.6}$$

$$H_t^* = \sum_{j=1}^{t} \left(q^{(j-1)^\alpha} - q^{(j)^\alpha} \right) V_{t-j+1} + q^{(t)^\alpha} H_0^*, \quad H_0^* = 0,\ t = 1, 2, ... \tag{7.7}$$

Here G_t^* and H_t^* are independent of each other due to the independence of U_t and V_t. When the underlying process is in-control, we have $G_t^* \sim N(0, \sigma_{G_t^*})$ and $H_t^* \sim N(0, \sigma_{H_t^*})$, where $\sigma^2_{G_t^*} = \sigma^2_{H_t^*} = \sum_{j=1}^{t} \left(q^{(j-1)^\alpha} - q^{(j)^\alpha} \right)^2$, for $t = 1, 2, ...$

The plotting-statistic of the MaxGWMA-SRS chart based on G_t^* and H_t^* is defined as

$$GE_t = \max\{|G_t^*|, |H_t^*|\}, t = 1, 2, ... \tag{7.8}$$

Similar to the ME_t defined in (7.4), here GE_t is also non-negative, therefore, the MaxGWMA-SRS chart only needs UCL_t, which is given by

$$UCL_t = E(GE_t) + L\sqrt{\text{Var}(GE_t)}, \tag{7.9}$$

where L is the positive control chart multiplier, and its value is determined such that the in-control ARL of the MaxGWMA-SRS chart reaches to a particular level. For more details about the computation of $E(GE_t)$ and $\text{Var}(GE_t)$, see Sheu et al. (2012).

7.3 Ordered ranked set sampling and BLUEs

In this section, we briefly explain RSS, ORSS and OIRSS procedures. Based on these sampling schemes, we obtain the BLUEs of the unknown parameters of the location-scale family of distributions.

In order to select a ranked set sample of size n, the traditional RSS scheme is as follows: start with n^2 units from the population. Randomly divide these units to n sets each of size n units. Rank the units within each set with respect to the study variable visually or by any inexpensive method. Select the rth smallest ranked unit from the rth set, for $r = 1, 2, ..., n$. This completes one cycle of a ranked set sample of size n. The whole procedure can be repeated k times to get a ranked set sample of size nk. In order to get an ordered ranked set sample of size n, we sort the obtained ranked set sample in an increasing order of magnitude.

Symbolically, let $X_{11}, X_{12}, ..., X_{1n}, X_{21}, X_{22}, ..., X_{2n}, ..., X_{n1}, X_{n2}, ..., X_{nn}$ be n independent simple random samples, each of size n, drawn from an absolutely continuous distribution having cumulative distribution function (CDF) $F\{(x-\mu)/\sigma\}$ and probability density function (PDF) $(1/\sigma)f\{(x-\mu)/\sigma\}$, where μ is the location parameter and $\sigma(>0)$ is the scale parameter. For simplicity, let $F^*(x) = F\{(x-\mu/\sigma\}$ and $f^*(x) = (1/\sigma)f\{(x-\mu)/\sigma\}$. Apply RSS procedure to these n independent samples to obtain a ranked set sample of size n, denoted by $X_{r(r:n)}$, for $r = 1, 2, ..., n$. Here $X_{r(r:n)}$ is the rth ordered statistic obtained from the rth simple random sample of size n, i.e., $X_{r(r:n)} = r\text{th}\ \min(X_{r1}, X_{r2}, ..., X_{rn})$.
The CDF and PDF of $X_{r(r:n)}(r = 1, 2, ..., n)$ are given by

$$\begin{aligned} F^*_{(r:n)}(x) &= \sum_{i=r}^{n} \binom{n}{i} \{F^*(x)\}^i \{1-F^*(x)\}^{n-i}, \quad -\infty < x < \infty, \\ f^*_{(r:n)}(x) &= \frac{n!}{(r-1)!(n-r)!} \{F^*(x)\}^{r-1} \{1-F^*(x)\}^{n-r} f^*(x), \quad -\infty < x < \infty, \end{aligned}$$

respectively. For more details, see David and Nagaraja (2003).
Let $X^{\text{ORSS}}_{(1:n)} \le X^{\text{ORSS}}_{(2:n)} \le \cdots \le X^{\text{ORSS}}_{(n:n)}$ represent an ordered ranked set sample of size n obtained by arranging $X_{1(1:n)}, X_{2(2:n)}, ..., X_{n(n:n)}$ is an increasing order of magnitude, i.e., $X^{\text{ORSS}}_{(r:n)} = r\text{th}\ \min(X_{1(1:n)}, X_{2(2:n)}, ..., X_{n(n:n)})$, for $r = 1, 2, ..., n$. Note that $X^{\text{ORSS}}_{(r:n)}$, $i = 1, 2, ..., n$, are independent but not identically distributed (INID) random variables. Therefore, the PDF of $X^{\text{ORSS}}_{(r:n)}(r = 1, 2, ..., n)$ is given by

$$f^{\text{ORSS}}_{(r:n)}(x) = \frac{1}{(r-1)!(n-r)!} \sum_{P[n]} \left[\prod_{k=1}^{r-1} F^*_{(i_k:n)}(x) \prod_{k=r+1}^{n} \{1 - F^*_{(i_k:n)}(x)\} f^*_{(i_r:n)}(x) \right], \quad -\infty < x < \infty, \tag{7.10}$$

where $\sum_{P[n]}$ denotes the summation over all $n!$ permutations $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$.
Similarly, the joint PDF of $X^{\text{ORSS}}_{(r:n)}$ and $X^{\text{ORSS}}_{(s:n)}$ $(1 \le r < s \le n)$ is given by

$$\begin{aligned} f^{\text{ORSS}}_{(r,s:n)}(x_r, x_s) &= \frac{1}{(r-1)!(s-r-1)!(n-s)!} \sum_{P[n]} \left[\prod_{k=1}^{r-1} F^*_{(i_k:n)}(x_r) \prod_{k=r+1}^{s-1} \{F^*_{(i_k:n)}(x_s) - F^*_{(i_k:n)}(x_r)\} \right. \\ &\quad \left. \prod_{k=s+1}^{n} \{1 - F^*_{(i_k:n)}(x_s)\} f^*_{(i_r:n)}(x_r) f^*_{(i_s:n)}(x_s) \right], \quad x_r < x_s. \end{aligned} \tag{7.11}$$

From (7.10) and (7.11), it is easy to compute the moments and cross-moments of order statistics based on ORSS. For further details, see Balakrishnan and Li (2005, 2008).

Suppose $\boldsymbol{X}_{\text{ORSS}} = (X^{\text{ORSS}}_{(1:n)}, X^{\text{ORSS}}_{(2:n)}, ..., X^{\text{ORSS}}_{(n:n)})'_{1\times n}$ is an ordered ranked set sample of size from a general location-scale distribution with location parameter μ and scale parameter $\sigma(>0)$. Let $Z^{\text{ORSS}}_{(r:n)} = (X^{\text{ORSS}}_{(r:n)} - \mu)/\sigma$ be the standardized variate under ORSS. Note that the PDF of $Z^{\text{ORSS}}_{(r:n)}$ is independent of μ and σ. We denote

$E(Z_{(r:n)}^{\mathrm{ORSS}}) = \mu_{(r:n)}^{\mathrm{ORSS}}$ and $\mathrm{Cov}(Z_{(r:n)}^{\mathrm{ORSS}}, Z_{(s:n)}^{\mathrm{ORSS}}) = \sigma_{(r,s:n)}^{\mathrm{ORSS}}$, $1 \leq r, s \leq n$. Then, $E(X_{(r:n)}^{\mathrm{ORSS}}) = \mu + \sigma\mu_{(r:n)}^{\mathrm{ORSS}}$, and $\mathrm{Cov}(X_{(r:n)}^{\mathrm{ORSS}}, X_{(s:n)}^{\mathrm{ORSS}}) = \sigma^2\sigma_{(r,s:n)}^{\mathrm{ORSS}}$. Following Balakrishnan and Li (2008), the BLUE-ORSS, say $\hat{\boldsymbol{\theta}}_{\mathrm{BLUE}}^{\mathrm{ORSS}} = (\hat{\mu}_{\mathrm{BLUE}}^{\mathrm{ORSS}}, \hat{\sigma}_{\mathrm{BLUE}}^{\mathrm{ORSS}})'$ of $\boldsymbol{\theta} = (\mu, \sigma)'_{1\times 2}$, is $\hat{\boldsymbol{\theta}}_{\mathrm{BLUE}}^{\mathrm{ORSS}} = (\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{ORSS}}$, where $\boldsymbol{B} = (\mathbf{1}, \boldsymbol{\mu}_{\mathrm{ORSS}})_{n\times 2}$ and $\boldsymbol{\Sigma} = \{\sigma_{(r,s:n)}^{\mathrm{ORSS}}\}_{n\times n}$. Here $\mathbf{1} = (1, 1, ..., 1)'_{1\times n}$ and $\boldsymbol{\mu}_{\mathrm{ORSS}} = (\mu_{(1:n)}^{\mathrm{ORSS}}, \mu_{(2:n)}^{\mathrm{ORSS}}, ..., \mu_{(n:n)}^{\mathrm{ORSS}})'_{n\times n}$. The variance-covariance matrix of $\hat{\boldsymbol{\theta}}_{\mathrm{BLUE}}^{\mathrm{ORSS}}$ is $\mathrm{Cov}(\hat{\boldsymbol{\theta}}_{\mathrm{BLUE}}^{\mathrm{ORSS}}) = \sigma^2(\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{B})^{-1}$. When the underlying distribution of X is symmetric about μ, then, the covariance between $\hat{\mu}_{\mathrm{BLUE}}^{\mathrm{ORSS}}$ and $\hat{\sigma}_{\mathrm{BLUE}}^{\mathrm{ORSS}}$ becomes zero, i.e., $\mathrm{Cov}(\hat{\mu}_{\mathrm{BLUE}}^{\mathrm{ORSS}}, \hat{\sigma}_{\mathrm{BLUE}}^{\mathrm{ORSS}}) = 0$. This helps in simplifying the expressions of the BLUEs-ORSS, i.e., $\hat{\mu}_{\mathrm{BLUE}}^{\mathrm{ORSS}} = (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^{-1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{ORSS}}$ and $\hat{\sigma}_{\mathrm{BLUE}}^{\mathrm{ORSS}} = (\boldsymbol{\mu}'_{\mathrm{ORSS}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{\mathrm{ORSS}})^{-1}\boldsymbol{\mu}'_{\mathrm{ORSS}}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{ORSS}}$. Similarly, the simplified expressions of their variances are $\mathrm{Var}(\hat{\mu}_{\mathrm{BLUE}}^{\mathrm{ORSS}}) = \sigma^2(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^{-1}$ and $\mathrm{Var}(\hat{\sigma}_{\mathrm{BLUE}}^{\mathrm{ORSS}}) = \sigma^2(\boldsymbol{\mu}'_{\mathrm{ORSS}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{\mathrm{ORSS}})^{-1}$, respectively.

It is obvious that the performance of the BLUEs obtained under ORSS depends on how accurately the judgment ranking of the randomly selected units is accomplished. Errors in ranking affect the efficiency of the estimator and lead to imprecise estimates. The problem of imperfect ranking was first put forward by Dell and Clutter (1972). In their study, they showed that, even under imperfect RSS scheme, the mean estimator remain unbiased and it is still better than the mean estimator based on SRS scheme.

In this study, we assess the efficiencies of the BLUEs under OIRSS scheme. Recall that for an in-control process, $X \sim N(\mu, \sigma)$. Following Haq et al. (2013a), the steps required to select an ordered imperfect ranked set sample of size n are as follows: given the values of n, generate n^2 values from the underlying distribution and divide them randomly into n sets each of size n, i.e., X_{ij}, $i, j = 1, 2, ..., n$. Let E be a random error term and it is normally distributed with mean zero and standard deviation σ_E, i.e., $E \sim N(0, \sigma_E)$. Also generate n^2 values of E, i.e., E_{ij}, $i, j = 1, 2, ..., n$. For imperfect ranking, we consider the model $Y_{ij} = X_{ij} + E_{ij}$, for $i, j = 1, 2, ..., n$. Apply RSS procedure to n^2 values of Y, and also observe the corresponding values of X. Then, a pair $(Y_{r(r:n)}, X_{r[r:n]})$, for $r = 1, 2, ..., n$, is selected based on the values of Y, where $X_{r[r:m]}$ is the rth judgment ordered statistic corresponding to the rth ordered statistic $Y_{i(i:n)}$. In order to select an ordered imperfect ranked set sample of size n, we sort $X_{1[1:n]}, X_{2[2:n]}, ..., X_{n[n:n]}$ in an increasing order, i.e., $X_{(1:n)}^{\mathrm{OIRSS}}, X_{(2:n)}^{\mathrm{OIRSS}}, ..., X_{(n:n)}^{\mathrm{OIRSS}}$, where $X_{(r:n)}^{\mathrm{OIRSS}} = r\text{th} \min\{X_{1[1:n]}, X_{2[2:n]}, ..., X_{n[n:n]}\}$, for $r = 1, 2, ..., n$. Let $\boldsymbol{X}_{\mathrm{OIRSS}} = (X_{(1:n)}^{\mathrm{OIRSS}}, X_{(2:n)}^{\mathrm{OIRSS}}, ..., X_{(n:n)}^{\mathrm{OIRSS}})'_{1\times n}$ be the vector of ordered imperfect ranked set sample of size n. The linear estimators of μ and σ under OIRSS (LEs-OIRSS) are $\hat{\mu}_{\mathrm{LE}}^{\mathrm{OIRSS}} = (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^{-1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{OIRSS}}$ and $\hat{\sigma}_{\mathrm{LE}}^{\mathrm{OIRSS}} = (\boldsymbol{\mu}'_{\mathrm{ORSS}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{\mathrm{OIRSS}})^{-1}\boldsymbol{\mu}'_{\mathrm{ORSS}}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{OIRSS}}$. Note that the estimators $(\hat{\mu}_{\mathrm{LE}}^{\mathrm{OIRSS}}, \hat{\sigma}_{\mathrm{LE}}^{\mathrm{OIRSS}})$ will no longer be the BLUE because their known coefficients are based on ORSS. However, these estimators approach to the BLUEs when errors in ranking reduce and vice-versa. Since OIRSS scheme is based on order statistics, $X_{(r:n)}^{\mathrm{OIRSS}}$, for $r = 1, 2, ..., n$, from independent judgment ordered statistics $X_{1[1:n]}, X_{2[2:n]}, ..., X_{n[n:n]}$. Therefore, it is difficult to obtain the explicit mathematical expressions for the CDF and PDF of $X_{(r:n)}^{\mathrm{OIRSS}}$, for $r = 1, 2, ..., n$. Following Haq et al. (2013a), we use extensive Monte Carlo simulations to estimate the mean and variances of the LEs-OIRSS.

7.4 Proposed control chart

In this section, we propose improved MaxEWMA control charts for monitoring process mean and dispersion based on ORSS and OIRSS schemes.

7.4.1 MaxEWMA-ORSS control chart

Recall that for an in-control process $X_t \sim N(\mu, \sigma)$, $t = 1, 2, ...$ Let $\{\hat{\mu}^{\text{ORSS}}_{\text{BLUE},t}\}$ and $\{\hat{\sigma}^{\text{ORSS}}_{\text{BLUE},t}\}$ be the sequences of IID random variables for $t = 1, 2, ...$ Based on these sequences, we define EWMA sequences based on following recurrence formulae:

$$A_t = \xi \hat{\mu}^{\text{ORSS}}_{\text{BLUE},t} + (1-\xi)A_{t-1}, \quad 0 < \xi \leq 1, \ A_0 = \mu, \tag{7.12}$$

$$B_t = \xi \hat{\sigma}^{\text{ORSS}}_{\text{BLUE},t} + (1-\xi)B_{t-1}, \quad 0 < \xi \leq 1, \ B_0 = \sigma, \tag{7.13}$$

where ξ is a smoothing parameter. Here A_0 and B_0 are the initial starting values of the EWMA sequences A_t and B_t, respectively. These values are usually set by quality practitioners. Note that A_t and B_t are independent of each other due to the independence of $\hat{\mu}^{\text{ORSS}}_{\text{BLUE},t}$ and $\hat{\sigma}^{\text{ORSS}}_{\text{BLUE},t}$.
Based on A_t and B_t, we define the following two standardized statistics, i.e.,

$$A^*_t = \frac{A_t - \mu}{\sqrt{\frac{\xi}{(2-\xi)}\{1-(1-\xi)^{2t}\}\text{Var}(\hat{\mu}^{\text{ORSS}}_{\text{BLUE},t})}} \quad \text{and} \quad B^*_t = \frac{B_t - \sigma}{\sqrt{\frac{\xi}{(2-\xi)}\{1-(1-\xi)^{2t}\}\text{Var}(\hat{\sigma}^{\text{ORSS}}_{\text{BLUE},t})}}, \tag{7.14}$$

where $\text{Var}(\hat{\mu}^{\text{ORSS}}_{\text{BLUE},t}) = \sigma^2(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})^{-1}$ and $\text{Var}(\hat{\sigma}^{\text{ORSS}}_{\text{BLUE},t}) = \sigma^2(\boldsymbol{\mu}'_{\text{ORSS}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}_{\text{ORSS}})^{-1}$. It is clear that the resulting distribution of A^*_t or B^*_t is independent of μ and σ. Here A^*_t and B^*_t are also mutually independent because A_t and B_t are independent.
Similar to the MaxEWMA-SRS chart, the plotting-statistic of the MaxEWMA-ORSS chart is defined as

$$ME^{\text{ORSS}}_t = \max\{|A^*_t|, |B^*_t|\}, \quad t = 1, 2, ... \tag{7.15}$$

Since ME^{ORSS}_t is non-negative, the initial state of the MaxEWMA-ORSS chart only needs an UCL, which is given by

$$UCL = E(ME^{\text{ORSS}}_t) + L\sqrt{\text{Var}(ME^{\text{ORSS}}_t)}, \tag{7.16}$$

where L is a positive control chart multiplier, and its value is selected such that the in-control ARL of the MaxEWMA-ORSS chart reaches to a specific level. Here $E(ME^{\text{ORSS}}_t)$ and $\text{Var}(ME^{\text{ORSS}}_t)$ are the mean and variance of ME^{ORSS}_t, respectively. Due to the complexity involved in deriving the probability distributions of the BLUEs-ORSS, we estimate the values of $E(ME^{\text{ORSS}}_t)$ and $\text{Var}(ME^{\text{ORSS}}_t)$ by using Monte Carlo simulations.

The main steps involved in the implementation of the MaxEWMA-ORSS control chart are as follows:

1. Estimate the unknown parameter(s). It is customary to estimate the unknown parameters(s) using large historical data that were obtained when the process was in control state. Suppose a preliminary data set is available that comprises of w subgroups, each of size n, obtained under ORSS scheme. Then, μ and σ can be estimated by their unbiased estimators, say $\bar{\hat{\mu}}_{\text{BLUE}}^{\text{ORSS}} = (1/w)\sum_{i=1}^{w}\hat{\mu}_{\text{BLUE},w}^{\text{ORSS}}$ and $\bar{\hat{\sigma}}_{\text{BLUE}}^{\text{ORSS}} = (1/w)\sum_{i=1}^{w}\hat{\sigma}_{\text{BLUE},w}^{\text{ORSS}}$, respectively.

2. Select the desired (ξ, L) combination from Tables 7.1–7.3 depending on the in-control ARL.

3. Set the UCL according to (7.16). The initial values of A_t and B_t can be estimated by $\hat{A}_0 = \bar{\hat{\mu}}_{\text{BLUE}}^{\text{ORSS}}$ and $\hat{B}_0 = \bar{\hat{\sigma}}_{\text{BLUE}}^{\text{ORSS}}$, respectively. Similarly, estimate the standard deviations of $\hat{\mu}_{\text{BLUE}}^{\text{ORSS}}$ and $\hat{\sigma}_{\text{BLUE}}^{\text{ORSS}}$ by their unbiased estimators, $\bar{\hat{\sigma}}_{\text{BLUE}}^{\text{ORSS}}(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})^{-1/2}$ and $\bar{\hat{\sigma}}_{\text{BLUE}}^{\text{ORSS}}(\boldsymbol{\mu}'_{\text{ORSS}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}_{\text{ORSS}})^{-1/2}$, respectively. Note that both $\boldsymbol{\mu}_{\text{ORSS}}$ and $\mathbf{\Sigma}^{-1}$ are known quantities because they are computed when the underlying distribution is standard normal. Then, compute values of the statistics A_t, B_t, A_t^*, B_t^* and ME_t^{ORSS} for each sample.

4. Plot ME_t^{ORSS} versus t on the control chart with UCL. Plot a dot against t when $ME_t^{\text{ORSS}} \leq UCL$. When $ME_t^{\text{ORSS}} > UCL$, check both $|A_t^*|$ and $|B_t^*|$. If $|A_t^*|$ alone is greater than UCL, then plot "$m+$" against t when $A_t^* > 0$ to show there is a positive shift in the process mean, and plot "$m-$" against t when $A_t^* < 0$ to show there is a negative shift in the process mean. Similarly, if $|B_t^*|$ is greater than UCL, then plot "$v+$" against t when $B_t^* > 0$ to show there is a positive shift in the process variance, and plot "$v-$" against t when $B_t^* < 0$ to show there is a negative shift in the process variance. If both $|A_t^*|$ and $|B_t^*|$ are greater than UCL, plot "$++$" if $A_t^* > 0$ and $B_t^* > 0$, then both process mean process variance have increased simultaneously. Similarly, plot "$+-$" if $A_t^* > 0$ and $B_t^* < 0$, plot "$-+$" if $A_t^* < 0$ and $B_t^* > 0$, plot "$--$" if $A_t^* < 0$ and $B_t^* < 0$, with similar interpretations.

5. Finally, examine the cause(s) for each out-of-control point.

Based on extensive Monte Carlo simulations, when the underlying process is normally distributed with mean zero and standard deviation unity, i.e., $X_t \sim N(0, 1)$, we compute the values of out-of-control ARLs and SDRLs of the MaxEWMA-ORSS chart for different values of δ and ρ. The changes in the process mean are from μ to $\mu + \delta\sigma$, where $\delta = 0.00$, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 2.50 and 3.00. Similarly, the changes in the process standard deviation are from σ to $\rho\sigma$, where $\rho = 0.25$, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00, 2.50 and 3.00. The subgroup size is taken to be $n = 5$. The values of smoothing constant ξ are taken in the interval $0.05 \leq \xi \leq 0.3$, which is mostly considered for quick detection of small to moderate changes in the process mean and/or dispersion. The in-control ARL of the MaxEWMA-ORSS chart is matched to 185, 250 and 370. Each result is based on 10^5 replications. The computed values of ARLs and SDRLs of the MaxEWMA-ORSS chart are given in Tables 7.1–7.3. From Tables 7.1–7.3, it is observed that the out-of-control ARL of the MaxEWMA-ORSS chart is a decreasing function of δ for fixed values of ξ and ρ. Having fixed δ and ρ, the performance of the MaxEWMA-ORSS chart increases as the value of ξ decreases and vice-versa.

7.4.2 MaxEWMA-OIRSS control chart

As mentioned in Section 7.3, it is difficult to obtain the explicit mathematical expressions for the mean and variances of the LE-OIRSS. Therefore, we estimate the means and variances of LEs-OIRSS based on large historical data.

Let $\hat{\mu}_{\mathrm{LE},i}^{\mathrm{OIRSS}}$ and $\hat{\sigma}_{\mathrm{LE},i}^{\mathrm{OIRSS}}$, for $i = 1,2,...,w$, be the estimated values based on w subgroups, each of size n, where $\hat{\mu}_{\mathrm{LE},i}^{\mathrm{OIRSS}} = (\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{OIRSS},i}$ and $\hat{\sigma}_{\mathrm{LE},i}^{\mathrm{OIRSS}} = (\boldsymbol{\mu}_{\mathrm{ORSS}}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}_{\mathrm{ORSS}})^{-1}\boldsymbol{\mu}_{\mathrm{ORSS}}'\mathbf{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{OIRSS},i}$. Let $\bar{\hat{\mu}}_{\mathrm{LE}}^{\mathrm{OIRSS}} = (1/w)\sum_{i=1}^{w}\hat{\mu}_{\mathrm{LE},i}^{\mathrm{OIRSS}}$ and $\bar{\hat{\sigma}}_{\mathrm{LE}}^{\mathrm{OIRSS}} = (1/w)\sum_{i=1}^{w}\hat{\sigma}_{\mathrm{LE},i}^{\mathrm{OIRSS}}$, which can be used to estimate μ and σ, respectively. Similarly, the estimated standard deviations of $\hat{\mu}_{\mathrm{LE}}^{\mathrm{OIRSS}}$ and $\hat{\sigma}_{\mathrm{LE}}^{\mathrm{OIRSS}}$ are $\hat{\sigma}_{\hat{\mu}_{\mathrm{LE}}^{\mathrm{OIRSS}}} = \sqrt{\frac{1}{(w-1)}\sum_{i=1}^{w}(\hat{\mu}_{\mathrm{LE},i}^{\mathrm{OIRSS}} - \bar{\hat{\mu}}_{\mathrm{LE}}^{\mathrm{OIRSS}})^2}$ and $\hat{\sigma}_{\hat{\sigma}_{\mathrm{LE}}^{\mathrm{OIRSS}}} = \sqrt{\frac{1}{(w-1)}\sum_{i=1}^{w}(\hat{\sigma}_{\mathrm{LE},i}^{\mathrm{OIRSS}} - \bar{\hat{\sigma}}_{\mathrm{LE}}^{\mathrm{OIRSS}})^2}$, respectively. Based on estimated means and standard deviations of $\hat{\mu}_{\mathrm{LE}}^{\mathrm{OIRSS}}$ and $\hat{\sigma}_{\mathrm{LE}}^{\mathrm{OIRSS}}$, it is easy to construct the MaxEWMA-OIRSS control chart.

Let $\{\hat{\mu}_{\mathrm{LE},t}^{\mathrm{OIRSS}}\}$ and $\{\hat{\sigma}_{\mathrm{LE},t}^{\mathrm{OIRSS}}\}$ be the sequences of IID random variable for $t = 1,2,....$ Based on these sequences, we define the EWMA sequences by using following recurrence formulae:

$$C_t = \xi\hat{\mu}_{\mathrm{LE},t}^{\mathrm{OIRSS}} + (1-\xi)C_{t-1}, \quad 0 < \xi \leq 1,\ C_0 = \bar{\hat{\mu}}_{\mathrm{LE}}^{\mathrm{OIRSS}}, \tag{7.17}$$

$$D_t = \xi\hat{\sigma}_{\mathrm{LE},t}^{\mathrm{OIRSS}} + (1-\xi)D_{t-1}, \quad 0 < \xi \leq 1,\ D_0 = \bar{\hat{\sigma}}_{\mathrm{LE}}^{\mathrm{OIRSS}}. \tag{7.18}$$

Based on C_t and D_t, we can define two standardized statistics, given below

$$C_t^* = \frac{C_t - \bar{\hat{\mu}}_{\mathrm{LE}}^{\mathrm{OIRSS}}}{\sqrt{\frac{\xi}{(2-\xi)}\{1-(1-\xi)^{2t}\}(\hat{\sigma}_{\hat{\mu}_{\mathrm{LE},t}^{\mathrm{OIRSS}}})^2}} \quad \text{and} \quad D_t^* = \frac{D_t - \bar{\hat{\sigma}}_{\mathrm{LE}}^{\mathrm{OIRSS}}}{\sqrt{\frac{\xi}{(2-\xi)}\{1-(1-\xi)^{2t}\}(\hat{\sigma}_{\hat{\sigma}_{\mathrm{LE},t}^{\mathrm{OIRSS}}})^2}}. \tag{7.19}$$

The plotting-statistic of the MaxEWMA-OIRSS chart is defined as

$$ME_t^{\mathrm{OIRSS}} = \max\{|C_t^*|, |D_t^*|\},\ t = 1,2,.... \tag{7.20}$$

Since ME_t^{OIRSS} is non-negative, therefore, the initial state of the MaxEWMA-OIRSS chart only needs an UCL, which is given by

$$UCL = E(ME_t^{\mathrm{OIRSS}}) + L\sqrt{\mathrm{Var}(ME_t^{\mathrm{OIRSS}})}, \tag{7.21}$$

where L is a positive control chart multiplier, and its value is selected such that the in-control ARL of the MaxEWMA-OIRSS chart reaches to a particular level. Due to the complexity involved in deriving probability distributions of the LEs-OIRSS, we estimate the values of $E(ME_t^{\mathrm{OIRSS}})$ and $\mathrm{Var}(ME_t^{\mathrm{OIRSS}})$ using Monte Carlo simulations.

In order to find the values of out-of-control ARLs and SDRLs of the MaxEWMA-OIRSS chart, we first estimate means and variances of the LEs-OIRSS using one million replications. The subgroup size is taken to be $n = 5$. Then, we estimate UCL of the MaxEWMA-OIRSS chart based on one million replications.

For brevity of discussion, we consider several values of error variance, i.e., $\sigma_E^2 = 0.05$, 0.15, 0.30 and 0.50. Based on extensive Monte Carlo simulations (10^5), we estimate out-of-control ARLs and SDRLs of the MaxEWMA-OIRSS control chart for different values of δ and ρ. Note that for the MaxEWMA-OIRSS chart, we keep the same values of the control charting multiplier L, which were used to match the in-control ARL of the MaxEWMA-ORSS chart. The reason for using same values of L is to study the effect the imperfect ranking on the performance of the MaxEWMA-ORSS chart. The calculated values of ARLs and SDRLs of the MaxEWMA-OIRSS chart are given in Tables 7.4–7.6.

From Tables 7.4–7.6, a similar trend is present in the values of the out-of-control ARLs of the MaxEWMA-OIRSS chart as observed for the MaxEWMA-ORSS chart. The detection ability of the MaxEWMA-OIRSS chart increases as the value of error variance (σ_E^2) decreases and vice-versa. Note that instead of fixing the in-control ARL of the MaxEWMA-OIRSS control chart to a particular level, we have used the same values of control chart multiplier L of the MaxEWMA-ORSS chart for the MaxEWMA-OIRSS chart. It is interesting to note that the in-control ARL of the MaxEWMA-OIRSS chart remains closer to the in-control ARL of MaxEWMA-ORSS chart. Moreover, the false alarm or incorrect out-of-control signal generally remains low for the MaxEWMA-OIRSS chart as compared with the MaxEWMA-OIRSS chart. However, the decrease in the false alarm rate for the MaxEWMA-OIRSS chart leads to an increase in its out-of-control ARLs.

7.5 Performance comparison of control charts

In this section, we evaluate the detection abilities of the MaxEWMA control charts for detecting changes in the process mean and process dispersion.

(i) MaxEWMA-ORSS chart versus optimal MaxEWMA-SRS and optimal MaxGWMA-SRS charts

In Tables 7.7–7.9, we compare the proposed MaxEWMA-ORSS chart based on $\xi = 0.05$ with optimal MaxEWMA-SRS and optimal MaxGWMA-SRS quality control schemes when the in-control ARL is fixed at 185, 250 and 370, respectively. The optimal values of the ARLs of both MaxEWMA-SRS and MaxGWMA-SRS charts are taken from Sheu et al. (2012). It is interesting to note that the MaxEWMA-ORSS chart performs uniformly better than the optimal MaxEWMA-SRS and MaxGWMA-SRS chart for all values of δ when $\rho \geq 0.5$. However, when $\rho = 0.25$, MaxEWMA-ORSS chart is less sensitive as compared with its counterparts for small values of δ in the interval $0 \leq \delta \leq 0.5$. Moreover, the performance of the MaxEWMA-ORSS chart increases as δ increases, i.e., $\delta > 0.5$, and it detects random shifts in the process mean substantially quicker than its competitors.

(ii) MaxEWMA-OIRSS chart versus optimal MaxEWMA-SRS and optimal MaxGWMA-SRS charts

In Table 7.10, we compare the MaxEWMA-OIRSS chart using $\xi = 0.05$ with the optimal MaxEWMA-SRS and optimal MaxGWMA-SRS schemes. Note that for the optimal control charts the in-control ARL is fixed to 370. Even under imperfect rankings, the proposed MaxEWMA-OIRSS chart is still able to perform uniformly better than the other optimal MaxEWMA schemes when $\rho \geq 0.5$ for all values of δ considered here. However, when $\rho = 0.25$ and δ in the interval $0 \leq \delta \leq 0.5$, the MaxEWMA-OIRSS chart remains less effective as compared with other control charts considered here.

(iii) Diagnostic abilities: MaxEWMA-ORSS chart versus MaxGWMA-SRS chart

Sheu et al. (2012) showed that both MaxGWMA-SRS and MaxEWMA-SRS charts have same diagnostic abilities and the former has better ARL and SDRL performances than the latter. Therefore, here we compare the MaxGWMA-SRS chart with the proposed MaxEWMA-ORSS chart. For the MaxGWMA-SRS chart, the assumed parameter values are $q = 0.25$, $\alpha = 0.80$ and $L = 2.8430$. Similarly, the parameters set for the MaxEWMA-ORSS chart are $\xi = 0.05$ and $L = 2.7650$. For both control charts, the in-control ARL is fixed to 370. In Table 7.11, we compare the diagnostic abilities of the MaxEWMA-ORSS chart with that of MaxGWMA-SRS control chart. The proposed MaxEWMA-ORSS chart is more efficient than the MaxGWMA-SRS chart in terms of having better diagnostic abilities when there are increases in the process mean and dispersion. For example, when detecting a positive change of 1.5 in both mean and variance, the MaxGWMA shows 127 samples out of 1000 showing an out-of-control signal. However, for the same shift, the MaxEWMA-ORSS chart shows 209 samples out of 1000 signaling an out-of-control signal. This shows that the MaxEWMA-ORSS chart is better at signaling random shifts of different magnitudes in the process mean and variation as compared with its competitors.

Table 7.1: ARLs and SDRLs of the MaxEWMA-ORSS control chart when in-control ARL is fixed to 185

		$\delta \rightarrow$	0.00		0.25		0.50		0.75		1.00		1.50		2.00		2.50		3.00	
ρ	ξ	L	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.25	0.05	2.3363	1.54	0.50	1.54	0.50	1.53	0.50	1.06	0.23	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.05	2.3363	2.89	0.95	2.87	0.94	2.13	0.64	1.26	0.44	1.01	0.07	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.05	2.3363	9.57	5.14	5.99	2.95	2.37	1.05	1.36	0.52	1.05	0.21	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.05	2.3363	184.91	198.75	7.17	4.92	2.46	1.35	1.44	0.64	1.10	0.31	1.00	0.03	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.05	2.3363	7.86	6.84	4.59	3.58	2.27	1.42	1.47	0.70	1.15	0.38	1.00	0.07	1.00	0.00	1.00	0.00	1.00	0.00
1.50	0.05	2.3363	2.88	2.21	2.49	1.81	1.82	1.11	1.39	0.66	1.16	0.40	1.01	0.11	1.00	0.02	1.00	0.00	1.00	0.00
2.00	0.05	2.3363	1.40	0.74	1.37	0.69	1.28	0.58	1.18	0.45	1.10	0.32	1.02	0.14	1.00	0.04	1.00	0.01	1.00	0.00
2.50	0.05	2.3363	1.13	0.38	1.12	0.37	1.10	0.33	1.08	0.29	1.05	0.23	1.02	0.12	1.00	0.05	1.00	0.01	1.00	0.00
3.00	0.05	2.3363	1.05	0.23	1.05	0.23	1.04	0.21	1.03	0.19	1.02	0.15	1.01	0.10	1.00	0.05	1.00	0.02	1.00	0.01
0.25	0.10	2.6412	1.81	0.40	1.81	0.40	1.81	0.40	1.24	0.43	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.10	2.6412	3.25	1.03	3.23	1.01	2.39	0.68	1.40	0.50	1.02	0.12	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.10	2.6412	11.22	6.06	6.99	3.32	2.64	1.13	1.47	0.57	1.07	0.26	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.10	2.6412	185.61	190.51	8.15	5.40	2.70	1.45	1.54	0.69	1.14	0.36	1.00	0.04	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.10	2.6412	8.77	7.29	5.15	3.85	2.50	1.53	1.57	0.77	1.19	0.43	1.01	0.09	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.10	2.6412	3.16	2.34	2.73	1.94	1.98	1.23	1.47	0.73	1.20	0.44	1.02	0.13	1.00	0.02	1.00	0.00	1.00	0.00
2.00	0.10	2.6412	1.47	0.79	1.43	0.75	1.34	0.64	1.22	0.49	1.12	0.36	1.02	0.16	1.00	0.05	1.00	0.01	1.00	0.00
2.50	0.10	2.6412	1.16	0.42	1.15	0.41	1.13	0.37	1.09	0.31	1.06	0.25	1.02	0.14	1.00	0.06	1.00	0.02	1.00	0.01
3.00	0.10	2.6412	1.06	0.26	1.06	0.25	1.05	0.23	1.04	0.21	1.03	0.18	1.01	0.11	1.00	0.06	1.00	0.03	1.00	0.01
0.25	0.20	2.8874	1.95	0.26	1.94	0.26	1.94	0.25	1.50	0.50	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.20	2.8874	3.64	1.17	3.62	1.14	2.64	0.74	1.52	0.52	1.03	0.17	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.20	2.8874	15.49	10.45	8.65	4.64	2.90	1.24	1.57	0.61	1.11	0.31	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.20	2.8874	184.76	185.84	9.50	6.77	2.94	1.57	1.64	0.73	1.18	0.40	1.00	0.05	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.20	2.8874	9.60	8.06	5.63	4.18	2.70	1.63	1.65	0.82	1.23	0.46	1.01	0.10	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.20	2.8874	3.37	2.46	2.91	2.02	2.11	1.29	1.54	0.78	1.23	0.48	1.02	0.15	1.00	0.03	1.00	0.00	1.00	0.00
2.00	0.20	2.8874	1.53	0.84	1.49	0.80	1.38	0.67	1.25	0.52	1.14	0.39	1.03	0.17	1.00	0.05	1.00	0.01	1.00	0.00
2.50	0.20	2.8874	1.18	0.44	1.17	0.43	1.14	0.39	1.11	0.34	1.07	0.27	1.02	0.15	1.00	0.07	1.00	0.02	1.00	0.01
3.00	0.20	2.8874	1.07	0.27	1.07	0.27	1.06	0.25	1.05	0.22	1.04	0.19	1.02	0.13	1.00	0.07	1.00	0.03	1.00	0.01
0.25	0.30	3.0000	1.99	0.22	1.99	0.23	1.99	0.22	1.61	0.49	1.00	0.02	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.30	3.0000	4.03	1.42	4.01	1.40	2.84	0.83	1.58	0.53	1.04	0.20	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.30	3.0000	24.94	20.51	11.45	7.68	3.10	1.38	1.63	0.64	1.12	0.33	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.30	3.0000	185.98	185.94	11.29	8.82	3.10	1.70	1.69	0.76	1.20	0.42	1.00	0.06	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.30	3.0000	10.33	8.95	5.99	4.60	2.79	1.71	1.70	0.85	1.25	0.48	1.01	0.11	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.30	3.0000	3.47	2.53	3.01	2.09	2.17	1.33	1.58	0.80	1.25	0.50	1.03	0.16	1.00	0.03	1.00	0.00	1.00	0.00
2.00	0.30	3.0000	1.55	0.85	1.51	0.82	1.40	0.69	1.27	0.54	1.16	0.40	1.03	0.18	1.00	0.06	1.00	0.01	1.00	0.00
2.50	0.30	3.0000	1.19	0.46	1.18	0.44	1.15	0.40	1.12	0.35	1.08	0.29	1.03	0.16	1.01	0.07	1.00	0.03	1.00	0.00
3.00	0.30	3.0000	1.07	0.28	1.07	0.27	1.06	0.26	1.05	0.23	1.04	0.20	1.02	0.13	1.01	0.07	1.00	0.03	1.00	0.01

Table 7.2: ARLs and SDRLs of the MaxEWMA-ORSS control chart when in-control ARL is fixed to 250

		$\delta\rightarrow$	0.00		0.25		0.50		0.75		1.00		1.50		2.00		2.50		3.00	
ρ	ξ	L	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.25	0.05	2.5247	1.71	0.45	1.71	0.45	1.71	0.45	1.15	0.36	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.05	2.5247	3.11	0.99	3.09	0.98	2.28	0.66	1.34	0.48	1.01	0.10	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.05	2.5247	10.42	5.42	6.58	3.13	2.53	1.10	1.42	0.56	1.06	0.24	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.05	2.5247	249.55	263.99	7.81	5.22	2.61	1.41	1.50	0.67	1.13	0.34	1.00	0.03	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.05	2.5247	8.61	7.31	5.01	3.86	2.42	1.51	1.52	0.74	1.17	0.41	1.01	0.08	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.05	2.5247	3.08	2.34	2.65	1.91	1.92	1.19	1.44	0.71	1.18	0.43	1.01	0.12	1.00	0.02	1.00	0.00	1.00	0.00
2.00	0.05	2.5247	1.45	0.78	1.41	0.74	1.31	0.62	1.20	0.48	1.12	0.35	1.02	0.15	1.00	0.05	1.00	0.01	1.00	0.00
2.50	0.05	2.5247	1.15	0.41	1.14	0.39	1.12	0.36	1.09	0.30	1.06	0.24	1.02	0.13	1.00	0.06	1.00	0.02	1.00	0.00
3.00	0.05	2.5247	1.06	0.25	1.06	0.24	1.05	0.23	1.04	0.20	1.03	0.17	1.01	0.11	1.00	0.06	1.00	0.03	1.00	0.01
0.25	0.10	2.8249	1.92	0.29	1.92	0.29	1.92	0.29	1.43	0.49	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.10	2.8249	3.47	1.07	3.45	1.06	2.56	0.71	1.49	0.51	1.03	0.16	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.10	2.8249	12.31	6.60	7.66	3.56	2.81	1.18	1.54	0.60	1.10	0.30	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.10	2.8249	250.94	254.67	8.80	5.74	2.87	1.51	1.61	0.72	1.17	0.39	1.00	0.04	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.10	2.8249	9.57	7.87	5.59	4.09	2.66	1.62	1.63	0.81	1.22	0.45	1.01	0.10	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.10	2.8249	3.37	2.49	2.90	2.04	2.10	1.29	1.53	0.78	1.22	0.47	1.02	0.14	1.00	0.02	1.00	0.00	1.00	0.00
2.00	0.10	2.8249	1.52	0.84	1.48	0.79	1.37	0.67	1.24	0.52	1.14	0.38	1.03	0.17	1.00	0.05	1.00	0.01	1.00	0.00
2.50	0.10	2.8249	1.17	0.44	1.17	0.43	1.14	0.39	1.10	0.33	1.07	0.27	1.02	0.15	1.00	0.07	1.00	0.02	1.00	0.01
3.00	0.10	2.8249	1.07	0.27	1.07	0.26	1.06	0.25	1.05	0.22	1.04	0.19	1.01	0.12	1.00	0.07	1.00	0.03	1.00	0.01
0.25	0.20	3.0561	2.00	0.20	2.00	0.20	2.00	0.19	1.67	0.47	1.00	0.02	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.20	3.0561	3.89	1.23	3.86	1.21	2.82	0.77	1.61	0.52	1.05	0.21	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.20	3.0561	17.55	11.93	9.54	5.14	3.07	1.29	1.64	0.64	1.14	0.35	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.20	3.0561	249.21	247.83	10.44	7.42	3.09	1.63	1.71	0.77	1.21	0.42	1.00	0.06	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.20	3.0561	10.54	8.85	6.09	4.50	2.84	1.70	1.71	0.85	1.26	0.49	1.01	0.12	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.20	3.0561	3.56	2.59	3.09	2.15	2.20	1.34	1.60	0.82	1.27	0.51	1.03	0.16	1.00	0.03	1.00	0.00	1.00	0.00
2.00	0.20	3.0561	1.58	0.88	1.53	0.83	1.41	0.71	1.28	0.56	1.17	0.42	1.04	0.19	1.00	0.06	1.00	0.01	1.00	0.00
2.50	0.20	3.0561	1.20	0.47	1.19	0.45	1.16	0.41	1.12	0.35	1.08	0.29	1.03	0.17	1.01	0.07	1.00	0.03	1.00	0.01
3.00	0.20	3.0561	1.08	0.28	1.08	0.28	1.07	0.26	1.05	0.24	1.04	0.20	1.02	0.13	1.01	0.07	1.00	0.03	1.00	0.01
0.25	0.30	3.1681	2.03	0.22	2.03	0.22	2.03	0.22	1.77	0.42	1.00	0.04	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.30	3.1681	4.34	1.54	4.32	1.53	3.03	0.87	1.68	0.53	1.06	0.24	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.30	3.1681	30.42	25.72	13.38	9.29	3.31	1.47	1.71	0.66	1.15	0.36	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.30	3.1681	249.54	248.38	12.62	9.97	3.27	1.79	1.76	0.80	1.23	0.44	1.00	0.06	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.30	3.1681	11.43	10.00	6.55	5.00	2.96	1.82	1.76	0.89	1.28	0.51	1.02	0.12	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.30	3.1681	3.69	2.71	3.18	2.22	2.27	1.40	1.63	0.84	1.28	0.52	1.03	0.17	1.00	0.03	1.00	0.00	1.00	0.00
2.00	0.30	3.1681	1.60	0.90	1.56	0.85	1.44	0.73	1.30	0.57	1.18	0.42	1.04	0.20	1.00	0.06	1.00	0.01	1.00	0.00
2.50	0.30	3.1681	1.20	0.48	1.19	0.46	1.16	0.42	1.13	0.37	1.09	0.30	1.03	0.17	1.01	0.08	1.00	0.03	1.00	0.01
3.00	0.30	3.1681	1.08	0.30	1.08	0.29	1.07	0.27	1.06	0.24	1.04	0.21	1.02	0.14	1.01	0.08	1.00	0.04	1.00	0.01

Table 7.3: ARLs and SDRLs of the MaxEWMA-ORSS control chart when in-control ARL is fixed to 370

		$\delta \rightarrow$	0.00		0.25		0.50		0.75		1.00		1.50		2.00		2.50		3.00	
ρ	ξ	L	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.25	0.05	2.7650	1.89	0.32	1.89	0.33	1.88	0.33	1.37	0.48	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.05	2.7650	3.38	1.04	3.37	1.03	2.50	0.70	1.46	0.51	1.02	0.14	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.05	2.7650	11.56	5.79	7.38	3.37	2.75	1.16	1.52	0.59	1.09	0.29	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.05	2.7650	370.47	383.99	8.67	5.57	2.82	1.50	1.59	0.71	1.16	0.38	1.00	0.05	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.05	2.7650	9.69	7.92	5.61	4.19	2.61	1.61	1.61	0.80	1.21	0.44	1.01	0.09	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.05	2.7650	3.35	2.51	2.87	2.07	2.06	1.29	1.51	0.77	1.22	0.47	1.02	0.14	1.00	0.02	1.00	0.00	1.00	0.00
2.00	0.05	2.7650	1.51	0.83	1.47	0.79	1.36	0.67	1.24	0.52	1.14	0.38	1.03	0.17	1.00	0.05	1.00	0.01	1.00	0.00
2.50	0.05	2.7650	1.17	0.43	1.16	0.42	1.13	0.38	1.10	0.33	1.07	0.27	1.02	0.15	1.00	0.06	1.00	0.02	1.00	0.01
3.00	0.05	2.7650	1.07	0.27	1.06	0.26	1.06	0.24	1.05	0.22	1.03	0.19	1.01	0.12	1.00	0.06	1.00	0.03	1.00	0.01
0.25	0.10	3.0529	2.00	0.20	1.99	0.19	1.99	0.19	1.67	0.47	1.00	0.02	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.10	3.0529	3.77	1.12	3.75	1.11	2.78	0.74	1.61	0.52	1.05	0.21	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.10	3.0529	13.67	7.25	8.54	3.87	3.04	1.24	1.64	0.63	1.13	0.34	1.00	0.01	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.10	3.0529	370.28	375.66	9.80	6.26	3.09	1.60	1.70	0.76	1.21	0.43	1.00	0.06	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.10	3.0529	10.71	8.59	6.23	4.47	2.86	1.72	1.72	0.86	1.26	0.49	1.01	0.12	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.10	3.0529	3.62	2.64	3.14	2.19	2.22	1.38	1.60	0.83	1.26	0.51	1.03	0.16	1.00	0.03	1.00	0.00	1.00	0.00
2.00	0.10	3.0529	1.59	0.90	1.54	0.84	1.42	0.72	1.28	0.56	1.16	0.41	1.04	0.19	1.00	0.06	1.00	0.01	1.00	0.00
2.50	0.10	3.0529	1.20	0.47	1.19	0.46	1.16	0.41	1.12	0.36	1.08	0.29	1.03	0.17	1.01	0.07	1.00	0.03	1.00	0.01
3.00	0.10	3.0529	1.08	0.29	1.07	0.28	1.07	0.26	1.05	0.24	1.04	0.20	1.02	0.13	1.01	0.07	1.00	0.04	1.00	0.01
0.25	0.20	3.2752	2.05	0.23	2.05	0.23	2.05	0.23	1.84	0.37	1.00	0.05	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.20	3.2752	4.21	1.31	4.20	1.29	3.06	0.82	1.73	0.51	1.08	0.27	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.20	3.2752	20.87	14.72	10.99	6.04	3.32	1.36	1.75	0.66	1.17	0.38	1.00	0.02	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.20	3.2752	369.55	368.70	11.71	8.32	3.33	1.73	1.80	0.80	1.25	0.46	1.00	0.07	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.20	3.2752	11.95	10.00	6.72	4.91	3.05	1.82	1.81	0.91	1.30	0.52	1.02	0.13	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.20	3.2752	3.85	2.79	3.33	2.29	2.35	1.44	1.68	0.87	1.30	0.54	1.03	0.18	1.00	0.04	1.00	0.00	1.00	0.00
2.00	0.20	3.2752	1.64	0.93	1.59	0.88	1.47	0.75	1.32	0.59	1.19	0.44	1.04	0.21	1.01	0.07	1.00	0.02	1.00	0.00
2.50	0.20	3.2752	1.22	0.49	1.21	0.48	1.18	0.44	1.14	0.38	1.09	0.31	1.03	0.18	1.01	0.08	1.00	0.03	1.00	0.01
3.00	0.20	3.2752	1.09	0.30	1.08	0.30	1.07	0.28	1.06	0.25	1.05	0.22	1.02	0.14	1.01	0.08	1.00	0.04	1.00	0.02
0.25	0.30	3.3799	2.10	0.31	2.10	0.31	2.10	0.31	1.90	0.31	1.01	0.07	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.50	0.30	3.3799	4.77	1.71	4.75	1.69	3.30	0.94	1.79	0.52	1.10	0.30	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
0.75	0.30	3.3799	40.18	34.89	16.52	12.00	3.57	1.57	1.81	0.69	1.20	0.40	1.00	0.02	1.00	0.00	1.00	0.00	1.00	0.00
1.00	0.30	3.3799	369.69	368.55	14.57	11.67	3.54	1.92	1.86	0.84	1.28	0.48	1.01	0.08	1.00	0.00	1.00	0.00	1.00	0.00
1.25	0.30	3.3799	13.10	11.48	7.34	5.66	3.19	1.96	1.86	0.94	1.33	0.54	1.02	0.14	1.00	0.01	1.00	0.00	1.00	0.00
1.50	0.30	3.3799	3.98	2.92	3.43	2.39	2.42	1.49	1.72	0.90	1.32	0.56	1.04	0.19	1.00	0.04	1.00	0.00	1.00	0.00
2.00	0.30	3.3799	1.67	0.95	1.61	0.89	1.49	0.77	1.34	0.61	1.20	0.46	1.05	0.22	1.01	0.07	1.00	0.02	1.00	0.00
2.50	0.30	3.3799	1.23	0.50	1.21	0.48	1.18	0.44	1.15	0.39	1.10	0.32	1.03	0.19	1.01	0.09	1.00	0.03	1.00	0.01
3.00	0.30	3.3799	1.09	0.31	1.09	0.30	1.08	0.29	1.07	0.26	1.05	0.23	1.02	0.15	1.01	0.08	1.00	0.04	1.00	0.02

Table 7.4: ARLs and SDRLs of the MaxEWMA-OIRSS control chart when in-control ARL of MaxEWMA-ORSS chart is fixed to 185

		$\delta \rightarrow$	0.00		0.25		0.50		1.00		2.00		0.00		0.25		0.50		1.00		2.00	
ρ	ξ	L	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
							$\sigma_E^2 = 0.05$										$\sigma_E^2 = 0.15$					
0.25	0.05	2.3363	1.57	0.50	1.56	0.50	1.55	0.50	1.00	0.02	1.00	0.00	1.67	0.47	1.67	0.47	1.67	0.47	1.00	0.07	1.00	0.00
0.50	0.05	2.3363	2.94	1.01	2.91	0.98	2.21	0.70	1.02	0.15	1.00	0.00	3.07	1.07	3.03	1.04	2.38	0.78	1.07	0.26	1.00	0.00
1.00	0.05	2.3363	185.10	198.63	7.74	5.35	2.62	1.47	1.14	0.36	1.00	0.00	184.75	197.97	8.65	6.12	2.92	1.68	1.21	0.43	1.00	0.00
1.50	0.05	2.3363	2.96	2.25	2.58	1.87	1.90	1.18	1.18	0.43	1.00	0.02	3.06	2.35	2.69	1.97	2.01	1.28	1.23	0.49	1.00	0.03
2.00	0.05	2.3363	1.43	0.77	1.38	0.71	1.29	0.60	1.11	0.34	1.00	0.05	1.46	0.80	1.42	0.75	1.33	0.64	1.13	0.37	1.00	0.06
0.25	0.10	2.6412	1.81	0.40	1.81	0.40	1.81	0.40	1.00	0.05	1.00	0.00	1.90	0.34	1.90	0.33	1.90	0.33	1.02	0.15	1.00	0.00
0.50	0.10	2.6412	3.31	1.09	3.29	1.06	2.50	0.75	1.05	0.22	1.00	0.00	3.45	1.15	3.42	1.12	2.69	0.84	1.13	0.34	1.00	0.00
1.00	0.10	2.6412	185.51	190.09	8.79	5.92	2.90	1.58	1.19	0.41	1.00	0.00	186.38	191.46	9.91	6.81	3.22	1.79	1.27	0.49	1.00	0.01
1.50	0.10	2.6412	3.23	2.40	2.82	2.01	2.07	1.29	1.23	0.48	1.00	0.02	3.36	2.50	2.96	2.11	2.19	1.38	1.29	0.54	1.00	0.04
2.00	0.10	2.6412	1.50	0.82	1.45	0.77	1.36	0.66	1.14	0.38	1.00	0.06	1.53	0.85	1.49	0.81	1.39	0.69	1.16	0.42	1.00	0.07
0.25	0.20	2.8874	1.95	0.28	1.95	0.28	1.95	0.28	1.01	0.10	1.00	0.00	2.01	0.25	2.01	0.25	2.01	0.25	1.07	0.25	1.00	0.00
0.50	0.20	2.8874	3.71	1.24	3.69	1.21	2.77	0.82	1.08	0.28	1.00	0.00	3.87	1.31	3.85	1.29	3.01	0.93	1.19	0.39	1.00	0.00
1.00	0.20	2.8874	186.90	188.60	10.41	7.53	3.16	1.71	1.23	0.45	1.00	0.00	186.24	186.78	11.94	8.95	3.52	1.97	1.33	0.53	1.00	0.01
1.50	0.20	2.8874	3.44	2.51	3.02	2.10	2.18	1.35	1.27	0.52	1.00	0.03	3.58	2.61	3.16	2.20	2.34	1.47	1.34	0.58	1.00	0.05
2.00	0.20	2.8874	1.56	0.86	1.51	0.82	1.41	0.70	1.16	0.41	1.00	0.06	1.59	0.89	1.55	0.85	1.44	0.74	1.19	0.45	1.01	0.08
0.25	0.30	3.0000	2.00	0.26	2.00	0.26	2.00	0.26	1.02	0.13	1.00	0.00	2.07	0.28	2.07	0.28	2.06	0.27	1.10	0.29	1.00	0.00
0.50	0.30	3.0000	4.11	1.51	4.09	1.49	2.98	0.93	1.11	0.31	1.00	0.00	4.33	1.64	4.30	1.61	3.25	1.07	1.22	0.42	1.00	0.00
1.00	0.30	3.0000	186.61	186.68	12.37	9.91	3.35	1.88	1.26	0.47	1.00	0.00	187.57	187.24	14.48	11.94	3.78	2.20	1.36	0.55	1.00	0.01
1.50	0.30	3.0000	3.58	2.62	3.10	2.17	2.26	1.40	1.29	0.53	1.00	0.03	3.72	2.75	3.27	2.29	2.41	1.51	1.36	0.60	1.00	0.05
2.00	0.30	3.0000	1.58	0.88	1.54	0.83	1.42	0.71	1.18	0.43	1.00	0.07	1.62	0.91	1.58	0.87	1.47	0.75	1.21	0.47	1.01	0.09
							$\sigma_E^2 = 0.30$										$\sigma_E^2 = 0.50$					
0.25	0.05	2.3363	1.80	0.42	1.80	0.41	1.79	0.42	1.03	0.16	1.00	0.00	1.88	0.35	1.89	0.35	1.88	0.35	1.09	0.28	1.00	0.00
0.50	0.05	2.3363	3.23	1.12	3.20	1.10	2.58	0.85	1.15	0.36	1.00	0.00	3.37	1.16	3.35	1.14	2.79	0.91	1.24	0.43	1.00	0.00
1.00	0.05	2.3363	185.06	199.17	9.80	7.03	3.26	1.93	1.30	0.51	1.00	0.01	184.78	198.11	10.86	7.90	3.59	2.18	1.39	0.59	1.00	0.02
1.50	0.05	2.3363	3.18	2.44	2.83	2.10	2.13	1.39	1.29	0.55	1.00	0.05	3.28	2.54	2.93	2.19	2.24	1.49	1.35	0.61	1.00	0.07
2.00	0.05	2.3363	1.49	0.82	1.46	0.79	1.37	0.68	1.16	0.41	1.01	0.08	1.52	0.85	1.48	0.81	1.39	0.71	1.19	0.45	1.01	0.10
0.25	0.10	2.6412	1.98	0.27	1.98	0.26	1.98	0.26	1.10	0.30	1.00	0.00	2.03	0.25	2.03	0.25	2.03	0.25	1.24	0.43	1.00	0.00
0.50	0.10	2.6412	3.63	1.20	3.61	1.19	2.94	0.92	1.24	0.43	1.00	0.00	3.81	1.26	3.79	1.24	3.17	0.98	1.36	0.49	1.00	0.00
1.00	0.10	2.6412	187.83	192.96	11.22	7.93	3.62	2.08	1.37	0.57	1.00	0.01	185.79	190.24	12.50	9.06	4.01	2.35	1.48	0.64	1.00	0.02
1.50	0.10	2.6412	3.50	2.59	3.11	2.25	2.34	1.52	1.36	0.61	1.00	0.06	3.57	2.69	3.22	2.33	2.47	1.63	1.43	0.68	1.01	0.09
2.00	0.10	2.6412	1.57	0.89	1.53	0.85	1.43	0.74	1.20	0.46	1.01	0.09	1.60	0.92	1.57	0.88	1.47	0.78	1.23	0.50	1.01	0.12
0.25	0.20	2.8874	2.08	0.28	2.08	0.28	2.07	0.27	1.21	0.40	1.00	0.00	2.13	0.34	2.13	0.34	2.13	0.34	1.40	0.49	1.00	0.00
0.50	0.20	2.8874	4.11	1.40	4.08	1.38	3.29	1.03	1.33	0.48	1.00	0.00	4.33	1.47	4.32	1.46	3.58	1.12	1.46	0.53	1.00	0.00
1.00	0.20	2.8874	189.07	190.17	13.70	10.56	3.98	2.27	1.45	0.61	1.00	0.02	188.32	188.77	15.50	12.26	4.42	2.61	1.57	0.69	1.00	0.04
1.50	0.20	2.8874	3.72	2.71	3.33	2.34	2.50	1.59	1.42	0.66	1.01	0.07	3.82	2.81	3.45	2.47	2.63	1.69	1.51	0.73	1.01	0.10
2.00	0.20	2.8874	1.64	0.93	1.60	0.90	1.49	0.78	1.23	0.49	1.01	0.10	1.67	0.96	1.63	0.92	1.53	0.82	1.27	0.53	1.02	0.13
0.25	0.30	3.0000	2.14	0.35	2.14	0.35	2.14	0.35	1.27	0.44	1.00	0.00	2.22	0.42	2.22	0.42	2.22	0.42	1.49	0.50	1.00	0.00
0.50	0.30	3.0000	4.63	1.78	4.60	1.76	3.61	1.21	1.37	0.50	1.00	0.00	4.94	1.93	4.92	1.92	3.95	1.34	1.51	0.54	1.00	0.00
1.00	0.30	3.0000	190.16	189.72	16.88	14.38	4.27	2.59	1.49	0.64	1.00	0.02	189.83	189.42	19.27	16.63	4.80	3.01	1.62	0.72	1.00	0.04
1.50	0.30	3.0000	3.86	2.87	3.45	2.45	2.58	1.64	1.46	0.68	1.01	0.08	3.96	2.95	3.57	2.57	2.73	1.77	1.54	0.75	1.01	0.11
2.00	0.30	3.0000	1.66	0.95	1.63	0.91	1.52	0.80	1.25	0.51	1.01	0.11	1.71	0.99	1.66	0.94	1.56	0.84	1.29	0.55	1.02	0.14

Table 7.5: ARLs and SDRLs of the MaxEWMA-OIRSS control chart when in-control ARL of MaxEWMA-ORSS chart is fixed to 250

		$\delta\rightarrow$	0.00		0.25		0.50		1.00		2.00		0.00		0.25		0.50		1.00		2.00	
ρ	ξ	L	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
							$\sigma_E^2 = 0.05$										$\sigma_E^2 = 0.15$					
0.25	0.05	2.5247	1.72	0.45	1.73	0.45	1.72	0.45	1.00	0.03	1.00	0.00	1.82	0.40	1.82	0.40	1.82	0.39	1.01	0.12	1.00	0.00
0.50	0.05	2.5247	3.16	1.05	3.14	1.03	2.38	0.73	1.04	0.19	1.00	0.00	3.29	1.11	3.26	1.08	2.56	0.82	1.11	0.31	1.00	0.00
1.00	0.05	2.5247	252.10	266.41	8.42	5.65	2.80	1.55	1.17	0.39	1.00	0.00	250.75	265.71	9.49	6.49	3.11	1.76	1.25	0.47	1.00	0.00
1.50	0.05	2.5247	3.17	2.42	2.73	1.97	2.01	1.26	1.21	0.46	1.00	0.02	3.28	2.47	2.88	2.10	2.12	1.36	1.26	0.52	1.00	0.04
2.00	0.05	2.5247	1.47	0.80	1.43	0.75	1.34	0.64	1.13	0.37	1.00	0.05	1.51	0.83	1.47	0.79	1.37	0.68	1.15	0.40	1.00	0.07
0.25	0.10	2.8249	1.92	0.31	1.92	0.31	1.92	0.30	1.01	0.08	1.00	0.00	1.99	0.25	1.99	0.25	1.98	0.25	1.05	0.22	1.00	0.00
0.50	0.10	2.8249	3.54	1.13	3.52	1.10	2.68	0.79	1.07	0.26	1.00	0.00	3.69	1.20	3.66	1.17	2.89	0.88	1.17	0.38	1.00	0.00
1.00	0.10	2.8249	252.58	258.46	9.54	6.30	3.07	1.65	1.22	0.44	1.00	0.00	253.14	258.06	10.81	7.32	3.43	1.88	1.31	0.51	1.00	0.01
1.50	0.10	2.8249	3.44	2.53	3.00	2.12	2.17	1.35	1.26	0.51	1.00	0.03	3.58	2.64	3.17	2.25	2.32	1.48	1.33	0.57	1.00	0.04
2.00	0.10	2.8249	1.55	0.86	1.50	0.82	1.40	0.70	1.16	0.41	1.00	0.07	1.59	0.90	1.54	0.85	1.43	0.74	1.19	0.44	1.01	0.08
0.25	0.20	3.0561	2.02	0.24	2.02	0.24	2.01	0.24	1.02	0.14	1.00	0.00	2.07	0.27	2.07	0.27	2.07	0.27	1.12	0.32	1.00	0.00
0.50	0.20	3.0561	3.95	1.29	3.93	1.27	2.96	0.86	1.12	0.32	1.00	0.00	4.14	1.38	4.10	1.36	3.21	0.98	1.24	0.43	1.00	0.00
1.00	0.20	3.0561	250.38	251.51	11.45	8.37	3.34	1.79	1.27	0.47	1.00	0.00	250.88	250.32	13.09	9.84	3.74	2.05	1.37	0.55	1.00	0.01
1.50	0.20	3.0561	3.65	2.64	3.19	2.22	2.31	1.43	1.30	0.55	1.00	0.04	3.80	2.76	3.36	2.34	2.46	1.54	1.38	0.61	1.00	0.05
2.00	0.20	3.0561	1.60	0.89	1.56	0.86	1.44	0.73	1.18	0.43	1.01	0.07	1.65	0.94	1.60	0.89	1.49	0.78	1.22	0.48	1.01	0.09
0.25	0.30	3.1681	2.06	0.27	2.06	0.28	2.06	0.27	1.03	0.18	1.00	0.00	2.13	0.34	2.13	0.34	2.13	0.34	1.16	0.36	1.00	0.00
0.50	0.30	3.1681	4.43	1.64	4.41	1.60	3.20	0.99	1.14	0.35	1.00	0.00	4.67	1.77	4.64	1.74	3.50	1.14	1.28	0.45	1.00	0.00
1.00	0.30	3.1681	253.02	252.21	14.01	11.30	3.56	1.98	1.29	0.50	1.00	0.00	256.46	257.08	16.36	13.62	4.02	2.32	1.41	0.57	1.00	0.01
1.50	0.30	3.1681	3.80	2.79	3.30	2.31	2.38	1.47	1.33	0.57	1.00	0.04	3.96	2.92	3.49	2.46	2.54	1.60	1.41	0.63	1.00	0.06
2.00	0.30	3.1681	1.63	0.92	1.58	0.87	1.47	0.75	1.20	0.45	1.01	0.08	1.68	0.96	1.63	0.91	1.51	0.79	1.23	0.49	1.01	0.10
							$\sigma_E^2 = 0.30$										$\sigma_E^2 = 0.50$					
0.25	0.05	2.5247	1.92	0.31	1.92	0.31	1.92	0.31	1.07	0.25	1.00	0.00	1.99	0.26	1.99	0.26	1.98	0.26	1.17	0.38	1.00	0.00
0.50	0.05	2.5247	3.47	1.17	3.45	1.14	2.79	0.89	1.21	0.41	1.00	0.00	3.63	1.21	3.61	1.20	3.01	0.95	1.31	0.47	1.00	0.00
1.00	0.05	2.5247	252.94	267.14	10.68	7.42	3.49	2.03	1.34	0.55	1.00	0.01	250.76	265.10	11.94	8.46	3.87	2.29	1.45	0.62	1.00	0.03
1.50	0.05	2.5247	3.41	2.60	3.03	2.22	2.27	1.48	1.34	0.59	1.00	0.05	3.52	2.70	3.13	2.33	2.39	1.61	1.40	0.65	1.01	0.08
2.00	0.05	2.5247	1.54	0.87	1.50	0.83	1.41	0.72	1.18	0.45	1.01	0.08	1.57	0.90	1.54	0.86	1.44	0.76	1.21	0.48	1.01	0.11
0.25	0.10	2.8249	2.05	0.25	2.05	0.25	2.05	0.25	1.18	0.38	1.00	0.00	2.10	0.30	2.10	0.30	2.10	0.30	1.36	0.48	1.00	0.00
0.50	0.10	2.8249	3.89	1.26	3.87	1.25	3.16	0.97	1.30	0.47	1.00	0.00	4.08	1.32	4.07	1.30	3.42	1.03	1.43	0.52	1.00	0.00
1.00	0.10	2.8249	255.59	260.46	12.37	8.59	3.86	2.17	1.43	0.60	1.00	0.02	253.07	257.85	13.73	9.72	4.28	2.45	1.55	0.68	1.00	0.03
1.50	0.10	2.8249	3.71	2.72	3.32	2.37	2.48	1.59	1.41	0.65	1.00	0.07	3.83	2.84	3.46	2.49	2.62	1.72	1.49	0.72	1.01	0.10
2.00	0.10	2.8249	1.63	0.93	1.59	0.90	1.48	0.78	1.22	0.48	1.01	0.10	1.66	0.96	1.63	0.93	1.52	0.82	1.26	0.53	1.02	0.13
0.25	0.20	3.0561	2.14	0.35	2.14	0.35	2.14	0.35	1.30	0.46	1.00	0.00	2.23	0.42	2.23	0.42	2.23	0.42	1.53	0.50	1.00	0.00
0.50	0.20	3.0561	4.39	1.49	4.37	1.46	3.53	1.08	1.40	0.50	1.00	0.00	4.63	1.56	4.61	1.53	3.84	1.18	1.53	0.54	1.00	0.00
1.00	0.20	3.0561	254.83	255.66	15.25	11.83	4.23	2.40	1.50	0.64	1.00	0.02	253.33	254.49	17.32	13.74	4.71	2.74	1.63	0.72	1.00	0.04
1.50	0.20	3.0561	3.95	2.87	3.54	2.50	2.64	1.68	1.47	0.69	1.01	0.08	4.08	2.98	3.67	2.59	2.79	1.80	1.56	0.77	1.01	0.11
2.00	0.20	3.0561	1.69	0.98	1.65	0.93	1.53	0.82	1.26	0.52	1.01	0.12	1.73	1.01	1.68	0.96	1.58	0.86	1.30	0.57	1.02	0.15
0.25	0.30	3.1681	2.23	0.43	2.24	0.43	2.24	0.43	1.38	0.49	1.00	0.00	2.35	0.49	2.35	0.49	2.35	0.49	1.61	0.49	1.00	0.00
0.50	0.30	3.1681	5.02	1.95	5.00	1.94	3.89	1.30	1.44	0.52	1.00	0.00	5.35	2.13	5.36	2.12	4.29	1.47	1.60	0.55	1.00	0.00
1.00	0.30	3.1681	259.99	260.90	19.32	16.43	4.59	2.76	1.54	0.66	1.00	0.02	257.33	257.70	22.32	19.45	5.15	3.21	1.68	0.75	1.00	0.05
1.50	0.30	3.1681	4.10	3.05	3.67	2.61	2.74	1.75	1.50	0.72	1.01	0.08	4.23	3.16	3.82	2.76	2.88	1.88	1.60	0.79	1.01	0.12
2.00	0.30	3.1681	1.72	1.00	1.68	0.95	1.56	0.84	1.27	0.54	1.01	0.12	1.76	1.03	1.72	0.98	1.61	0.88	1.32	0.58	1.02	0.15

Table 7.6: ARLs and SDRLs of the MaxEWMA-OIRSS control chart when in-control ARL of MaxEWMA-ORSS chart is fixed to 370

		δ	0.00		0.25		0.50		1.00		2.00		0.00		0.25		0.50		1.00		2.00	
ρ	ξ	L	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
							$\sigma_E^2 = 0.05$										$\sigma_E^2 = 0.15$					
0.25	0.05	2.7650	1.89	0.33	1.89	0.34	1.89	0.34	1.01	0.07	1.00	0.00	1.96	0.27	1.96	0.28	1.96	0.27	1.04	0.20	1.00	0.00
0.50	0.05	2.7650	3.44	1.09	3.43	1.08	2.61	0.77	1.06	0.25	1.00	0.00	3.60	1.17	3.57	1.14	2.82	0.87	1.16	0.37	1.00	0.00
1.00	0.05	2.7650	370.38	386.23	9.38	6.09	3.03	1.65	1.21	0.43	1.00	0.00	370.95	386.85	10.57	6.94	3.38	1.87	1.30	0.51	1.00	0.01
1.50	0.05	2.7650	3.42	2.56	2.98	2.16	2.16	1.36	1.25	0.50	1.00	0.03	3.57	2.67	3.13	2.26	2.30	1.47	1.32	0.56	1.00	0.04
2.00	0.05	2.7650	1.53	0.86	1.49	0.81	1.39	0.69	1.15	0.40	1.00	0.06	1.57	0.89	1.53	0.85	1.43	0.73	1.18	0.43	1.01	0.07
0.25	0.10	3.0529	2.01	0.23	2.01	0.23	2.01	0.23	1.02	0.14	1.00	0.00	2.06	0.26	2.06	0.26	2.06	0.25	1.11	0.32	1.00	0.00
0.50	0.10	3.0529	3.84	1.18	3.82	1.16	2.91	0.83	1.12	0.32	1.00	0.00	4.00	1.25	3.98	1.24	3.16	0.93	1.24	0.43	1.00	0.00
1.00	0.10	3.0529	373.49	377.12	10.63	6.85	3.31	1.73	1.27	0.47	1.00	0.00	375.25	379.85	12.06	8.02	3.71	1.99	1.37	0.55	1.00	0.01
1.50	0.10	3.0529	3.72	2.70	3.25	2.29	2.33	1.46	1.31	0.55	1.00	0.03	3.85	2.79	3.41	2.39	2.49	1.58	1.38	0.62	1.00	0.05
2.00	0.10	3.0529	1.61	0.91	1.57	0.86	1.44	0.74	1.18	0.44	1.00	0.07	1.65	0.95	1.60	0.90	1.49	0.78	1.22	0.48	1.01	0.09
0.25	0.20	3.2752	2.09	0.29	2.09	0.29	2.08	0.28	1.05	0.23	1.00	0.00	2.16	0.37	2.16	0.37	2.16	0.36	1.21	0.41	1.00	0.00
0.50	0.20	3.2752	4.30	1.38	4.28	1.36	3.21	0.92	1.17	0.38	1.00	0.00	4.49	1.48	4.48	1.46	3.49	1.04	1.32	0.47	1.00	0.00
1.00	0.20	3.2752	371.57	371.99	12.90	9.42	3.58	1.89	1.32	0.51	1.00	0.00	373.46	372.39	14.95	11.31	4.05	2.20	1.44	0.59	1.00	0.01
1.50	0.20	3.2752	3.94	2.85	3.44	2.37	2.46	1.51	1.35	0.58	1.00	0.04	4.10	2.95	3.62	2.53	2.63	1.64	1.43	0.65	1.00	0.06
2.00	0.20	3.2752	1.67	0.96	1.62	0.91	1.50	0.78	1.21	0.47	1.01	0.08	1.72	0.99	1.66	0.94	1.55	0.82	1.25	0.51	1.01	0.10
0.25	0.30	3.3799	2.15	0.36	2.15	0.36	2.14	0.35	1.08	0.27	1.00	0.00	2.25	0.44	2.25	0.44	2.25	0.44	1.26	0.44	1.00	0.00
0.50	0.30	3.3799	4.87	1.82	4.86	1.79	3.49	1.07	1.20	0.40	1.00	0.00	5.15	1.99	5.12	1.96	3.83	1.24	1.36	0.49	1.00	0.00
1.00	0.30	3.3799	375.82	379.43	16.37	13.42	3.84	2.14	1.34	0.53	1.00	0.00	375.09	373.74	19.31	16.28	4.35	2.52	1.47	0.61	1.00	0.01
1.50	0.30	3.3799	4.09	3.00	3.57	2.49	2.54	1.58	1.37	0.60	1.00	0.04	4.29	3.16	3.77	2.66	2.74	1.73	1.46	0.67	1.00	0.07
2.00	0.30	3.3799	1.70	0.97	1.65	0.91	1.52	0.79	1.23	0.48	1.01	0.09	1.75	1.01	1.70	0.96	1.57	0.84	1.27	0.53	1.01	0.11
							$\sigma_E^2 = 0.30$										$\sigma_E^2 = 0.50$					
0.25	0.05	2.7650	2.03	0.24	2.03	0.24	2.02	0.24	1.15	0.36	1.00	0.00	2.07	0.27	2.07	0.27	2.07	0.27	1.32	0.47	1.00	0.00
0.50	0.05	2.7650	3.78	1.22	3.77	1.21	3.07	0.94	1.28	0.46	1.00	0.00	3.97	1.27	3.95	1.26	3.32	1.01	1.41	0.51	1.00	0.00
1.00	0.05	2.7650	375.87	391.03	12.02	7.99	3.80	2.14	1.41	0.59	1.00	0.02	369.22	382.04	13.39	9.11	4.21	2.43	1.53	0.67	1.00	0.03
1.50	0.05	2.7650	3.70	2.77	3.29	2.41	2.45	1.60	1.39	0.63	1.00	0.06	3.82	2.87	3.43	2.52	2.60	1.73	1.47	0.71	1.01	0.09
2.00	0.05	2.7650	1.61	0.92	1.57	0.89	1.47	0.78	1.22	0.48	1.01	0.10	1.65	0.96	1.61	0.93	1.50	0.81	1.25	0.52	1.01	0.12
0.25	0.10	3.0529	2.13	0.34	2.13	0.34	2.13	0.34	1.30	0.46	1.00	0.00	2.21	0.41	2.21	0.41	2.21	0.41	1.53	0.50	1.00	0.00
0.50	0.10	3.0529	4.23	1.33	4.22	1.31	3.45	1.02	1.39	0.50	1.00	0.00	4.44	1.38	4.43	1.38	3.73	1.09	1.54	0.54	1.00	0.00
1.00	0.10	3.0529	380.95	384.16	13.78	9.40	4.17	2.29	1.50	0.64	1.00	0.02	375.68	380.89	15.50	10.82	4.64	2.59	1.63	0.72	1.00	0.04
1.50	0.10	3.0529	4.02	2.92	3.60	2.55	2.68	1.73	1.47	0.70	1.01	0.08	4.13	3.01	3.74	2.67	2.82	1.84	1.56	0.77	1.01	0.11
2.00	0.10	3.0529	1.70	0.99	1.66	0.95	1.54	0.83	1.26	0.52	1.01	0.11	1.73	1.02	1.69	0.98	1.59	0.88	1.30	0.57	1.02	0.14
0.25	0.20	3.2752	2.27	0.45	2.28	0.45	2.27	0.45	1.46	0.50	1.00	0.00	2.40	0.50	2.40	0.50	2.40	0.50	1.69	0.46	1.00	0.00
0.50	0.20	3.2752	4.78	1.58	4.77	1.57	3.85	1.16	1.49	0.52	1.00	0.00	5.05	1.67	5.04	1.66	4.20	1.27	1.64	0.55	1.00	0.00
1.00	0.20	3.2752	377.80	380.38	17.61	13.71	4.59	2.57	1.58	0.68	1.00	0.03	374.68	375.45	20.17	16.23	5.11	2.93	1.72	0.76	1.00	0.05
1.50	0.20	3.2752	4.28	3.07	3.83	2.68	2.83	1.79	1.54	0.73	1.01	0.09	4.40	3.19	3.97	2.79	2.99	1.92	1.64	0.81	1.02	0.13
2.00	0.20	3.2752	1.76	1.02	1.72	0.98	1.60	0.87	1.30	0.56	1.02	0.13	1.80	1.06	1.76	1.02	1.64	0.91	1.34	0.60	1.03	0.16
0.25	0.30	3.3799	2.40	0.51	2.40	0.51	2.40	0.50	1.53	0.50	1.00	0.00	2.55	0.53	2.55	0.53	2.55	0.53	1.76	0.43	1.00	0.00
0.50	0.30	3.3799	5.57	2.20	5.54	2.18	4.30	1.45	1.54	0.54	1.00	0.00	5.97	2.42	5.98	2.41	4.76	1.64	1.70	0.57	1.00	0.00
1.00	0.30	3.3799	382.46	382.26	23.16	20.06	5.00	3.00	1.62	0.70	1.00	0.03	378.85	380.09	26.94	23.83	5.68	3.56	1.77	0.79	1.00	0.06
1.50	0.30	3.3799	4.44	3.30	3.98	2.83	2.93	1.88	1.57	0.76	1.01	0.10	4.57	3.39	4.14	2.98	3.12	2.03	1.68	0.84	1.02	0.14
2.00	0.30	3.3799	1.79	1.04	1.75	1.01	1.63	0.89	1.31	0.57	1.02	0.13	1.84	1.09	1.79	1.04	1.68	0.94	1.36	0.62	1.03	0.17

Table 7.7: A comparison of ARLs and SDRLs of the MaxEWMA-ORSS ($\xi = 0.05$) with optimal MaxEWMA-SRS and optimal MaxGWMA-SRS charts when in-control ARL is fixed to 185

ρ	Chart	$\delta \rightarrow$	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00
0.25	MaxEWMA-SRS	ARL	1.49	1.49	1.49	1.49	1.43	1.00	1.00	1.00	1.00
		SDRL	0.52	0.52	0.52	0.51	0.49	0.02	0.00	0.00	0.00
	MaxGWMA-SRS	ARL	1.49	1.49	1.49	1.48	1.42	1.00	1.00	1.00	1.00
		SDRL	0.52	0.52	0.52	0.51	0.49	0.02	0.00	0.00	0.00
	MaxEWMA-ORSS	ARL	1.54	1.54	1.53	1.06	1.00	1.00	1.00	1.00	1.00
		SDRL	0.50	0.50	0.50	0.23	0.00	0.00	0.00	0.00	0.00
0.50	MaxEWMA-SRS	ARL	3.43	3.43	3.22	2.44	1.75	1.05	1.00	1.00	1.00
		SDRL	1.50	1.50	1.29	0.85	0.59	0.21	0.01	0.00	0.00
	MaxGWMA-SRS	ARL	3.43	3.43	3.22	2.44	1.75	1.05	1.00	1.00	1.00
		SDRL	1.50	1.50	1.29	0.85	0.59	0.21	0.01	0.00	0.00
	MaxEWMA-ORSS	ARL	2.89	2.87	2.13	1.26	1.01	1.00	1.00	1.00	1.00
		SDRL	0.95	0.94	0.64	0.44	0.07	0.00	0.00	0.00	0.00
0.75	MaxEWMA-SRS	ARL	12.74	10.35	5.14	2.83	1.86	1.14	1.01	1.00	1.00
		SDRL	7.91	5.99	2.64	1.32	0.80	0.35	0.07	0.01	0.00
	MaxGWMA-SRS	ARL	12.64	10.30	5.14	2.83	1.86	1.14	1.01	1.00	1.00
		SDRL	7.78	5.91	2.64	1.32	0.80	0.35	0.07	0.01	0.00
	MaxEWMA-ORSS	ARL	9.57	5.99	2.37	1.36	1.05	1.00	1.00	1.00	1.00
		SDRL	5.14	2.95	1.05	0.52	0.21	0.00	0.00	0.00	0.00
1.00	MaxEWMA-SRS	ARL	185.00	16.87	5.54	2.95	1.95	1.22	1.03	1.00	1.00
		SDRL	186.23	13.13	3.65	1.71	1.00	0.44	0.16	0.03	0.01
	MaxGWMA-SRS	ARL	185.00	16.31	5.51	2.95	1.95	1.22	1.03	1.00	1.00
		SDRL	186.23	12.40	3.60	1.71	1.00	0.44	0.16	0.03	0.01
	MaxEWMA-ORSS	ARL	184.91	7.17	2.46	1.44	1.10	1.00	1.00	1.00	1.00
		SDRL	198.75	4.92	1.35	0.64	0.31	0.03	0.00	0.00	0.00
1.25	MaxEWMA-SRS	ARL	11.22	8.05	4.50	2.78	1.95	1.27	1.06	1.01	1.00
		SDRL	10.04	6.89	3.44	1.86	1.13	0.52	0.24	0.09	0.02
	MaxGWMA-SRS	ARL	10.60	7.71	4.42	2.77	1.95	1.27	1.06	1.01	1.00
		SDRL	9.33	6.41	3.34	1.82	1.13	0.52	0.24	0.09	0.02
	MaxEWMA-ORSS	ARL	7.86	4.59	2.27	1.47	1.15	1.00	1.00	1.00	1.00
		SDRL	6.84	3.58	1.42	0.70	0.38	0.07	0.00	0.00	0.00
1.50	MaxEWMA-SRS	ARL	3.99	3.66	2.94	2.27	1.80	1.29	1.09	1.02	1.00
		SDRL	3.28	2.96	2.22	1.55	1.08	0.56	0.29	0.13	0.05
	MaxGWMA-SRS	ARL	3.90	3.59	2.90	2.26	1.80	1.29	1.09	1.02	1.00
		SDRL	3.09	2.78	2.15	1.51	1.07	0.56	0.29	0.13	0.05
	MaxEWMA-ORSS	ARL	2.88	2.49	1.82	1.39	1.16	1.01	1.00	1.00	1.00
		SDRL	2.21	1.81	1.11	0.66	0.40	0.11	0.02	0.00	0.00
2.00	MaxEWMA-SRS	ARL	1.74	1.72	1.64	1.52	1.41	1.22	1.10	1.04	1.01
		SDRL	1.11	1.09	1.00	0.87	0.74	0.50	0.32	0.19	0.10
	MaxGWMA-SRS	ARL	1.74	1.71	1.63	1.52	1.41	1.22	1.10	1.04	1.01
		SDRL	1.09	1.06	0.98	0.86	0.74	0.50	0.32	0.19	0.10
	MaxEWMA-ORSS	ARL	1.40	1.37	1.28	1.18	1.10	1.02	1.00	1.00	1.00
		SDRL	0.74	0.69	0.58	0.45	0.32	0.14	0.04	0.01	0.00
2.50	MaxEWMA-SRS	ARL	1.31	1.30	1.28	1.25	1.21	1.14	1.08	1.04	1.02
		SDRL	0.64	0.63	0.60	0.56	0.51	0.39	0.29	0.20	0.13
	MaxGWMA-SRS	ARL	1.31	1.30	1.28	1.25	1.21	1.14	1.08	1.04	1.02
		SDRL	0.64	0.63	0.60	0.56	0.51	0.39	0.29	0.20	0.13
	MaxEWMA-ORSS	ARL	1.13	1.12	1.10	1.08	1.05	1.02	1.00	1.00	1.00
		SDRL	0.38	0.37	0.33	0.29	0.23	0.12	0.05	0.01	0.00
3.00	MaxEWMA-SRS	ARL	1.20	1.19	1.18	1.17	1.15	1.11	1.07	1.04	1.02
		SDRL	0.52	0.51	0.50	0.47	0.44	0.36	0.28	0.21	0.15
	MaxGWMA-SRS	ARL	1.20	1.19	1.18	1.17	1.15	1.11	1.07	1.04	1.02
		SDRL	0.51	0.50	0.49	0.46	0.43	0.36	0.28	0.21	0.15
	MaxEWMA-ORSS	ARL	1.05	1.05	1.04	1.03	1.02	1.01	1.00	1.00	1.00
		SDRL	0.23	0.23	0.21	0.19	0.15	0.10	0.05	0.02	0.01

Table 7.8: A comparison of ARLs and SDRLs of the MaxEWMA-ORSS ($\xi = 0.05$) with optimal MaxEWMA-SRS and optimal MaxGWMA-SRS charts when in-control ARL is fixed to 250

ρ	Chart	δ	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00
0.25	MaxEWMA-SRS	ARL	1.58	1.58	1.58	1.58	1.54	1.00	1.00	1.00	1.00
		SDRL	0.53	0.53	0.53	0.52	0.50	0.04	0.00	0.00	0.00
	MaxGWMA-SRS	ARL	1.58	1.58	1.58	1.58	1.54	1.00	1.00	1.00	1.00
		SDRL	0.53	0.53	0.53	0.52	0.50	0.04	0.00	0.00	0.00
	MaxEWMA-ORSS	ARL	1.71	1.71	1.71	1.15	1.00	1.00	1.00	1.00	1.00
		SDRL	0.45	0.45	0.45	0.36	0.00	0.00	0.00	0.00	0.00
0.50	MaxEWMA-SRS	ARL	3.68	3.68	3.47	2.64	1.87	1.08	1.00	1.00	1.00
		SDRL	1.57	1.57	1.36	0.89	0.60	0.26	0.01	0.00	0.00
	MaxGWMA-SRS	ARL	3.68	3.68	3.47	2.64	1.87	1.08	1.00	1.00	1.00
		SDRL	1.57	1.57	1.36	0.89	0.60	0.26	0.01	0.00	0.00
	MaxEWMA-ORSS	ARL	3.11	3.09	2.28	1.34	1.01	1.00	1.00	1.00	1.00
		SDRL	0.99	0.98	0.66	0.48	0.10	0.00	0.00	0.00	0.00
0.75	MaxEWMA-SRS	ARL	13.90	11.44	5.61	3.04	1.98	1.17	1.01	1.00	1.00
		SDRL	8.39	6.41	2.79	1.38	0.84	0.39	0.07	0.01	0.00
	MaxGWMA-SRS	ARL	13.77	11.37	5.61	3.04	1.98	1.17	1.01	1.00	1.00
		SDRL	8.24	6.33	2.79	1.38	0.84	0.38	0.09	0.01	0.00
	MaxEWMA-ORSS	ARL	10.42	6.58	2.53	1.42	1.06	1.00	1.00	1.00	1.00
		SDRL	5.42	3.13	1.10	0.56	0.24	0.00	0.00	0.00	0.00
1.00	MaxEWMA-SRS	ARL	250.00	18.77	6.00	3.15	2.06	1.25	1.04	1.00	1.00
		SDRL	249.93	14.15	3.84	1.79	1.05	0.47	0.18	0.04	0.01
	MaxGWMA-SRS	ARL	250.00	18.08	5.96	3.15	2.06	1.25	1.04	1.00	1.00
		SDRL	249.93	13.30	3.79	1.79	1.05	0.47	0.18	0.04	0.01
	MaxEWMA-ORSS	ARL	249.55	7.81	2.61	1.50	1.13	1.00	1.00	1.00	1.00
		SDRL	263.99	5.22	1.41	0.67	0.34	0.03	0.00	0.00	0.00
1.25	MaxEWMA-SRS	ARL	12.53	8.94	4.89	2.97	2.06	1.31	1.07	1.01	1.00
		SDRL	10.84	7.43	3.66	1.97	1.19	0.55	0.26	0.10	0.02
	MaxGWMA-SRS	ARL	11.84	8.54	4.81	2.96	2.06	1.31	1.07	1.01	1.00
		SDRL	10.11	6.94	3.56	1.93	1.19	0.55	0.26	0.10	0.02
	MaxEWMA-ORSS	ARL	8.61	5.01	2.42	1.52	1.17	1.01	1.00	1.00	1.00
		SDRL	7.31	3.86	1.51	0.74	0.41	0.08	0.01	0.00	0.00
1.50	MaxEWMA-SRS	ARL	4.33	3.97	3.16	2.42	1.90	1.33	1.10	1.02	1.00
		SDRL	3.50	3.16	2.37	1.65	1.15	0.59	0.32	0.15	0.06
	MaxGWMA-SRS	ARL	4.22	3.88	3.12	2.40	1.89	1.33	1.10	1.02	1.00
		SDRL	3.29	2.98	2.30	1.61	1.13	0.59	0.32	0.15	0.06
	MaxEWMA-ORSS	ARL	3.08	2.65	1.92	1.44	1.18	1.01	1.00	1.00	1.00
		SDRL	2.34	1.91	1.19	0.71	0.43	0.12	0.02	0.00	0.00
2.00	MaxEWMA-SRS	ARL	1.83	1.80	1.71	1.59	1.46	1.24	1.11	1.04	1.01
		SDRL	1.19	1.15	1.06	0.93	0.79	0.53	0.34	0.21	0.11
	MaxGWMA-SRS	ARL	1.82	1.79	1.70	1.58	1.46	1.24	1.11	1.04	1.01
		SDRL	1.16	1.13	1.04	0.91	0.78	0.53	0.34	0.21	0.11
	MaxEWMA-ORSS	ARL	1.45	1.41	1.31	1.20	1.12	1.02	1.00	1.00	1.00
		SDRL	0.78	0.74	0.62	0.48	0.35	0.15	0.05	0.01	0.00
2.50	MaxEWMA-SRS	ARL	1.34	1.33	1.31	1.28	1.24	1.15	1.09	1.04	1.02
		SDRL	0.68	0.67	0.64	0.59	0.54	0.42	0.31	0.21	0.14
	MaxGWMA-SRS	ARL	1.34	1.33	1.31	1.28	1.24	1.15	1.09	1.04	1.02
		SDRL	0.67	0.66	0.63	0.58	0.54	0.42	0.31	0.21	0.14
	MaxEWMA-ORSS	ARL	1.15	1.14	1.12	1.09	1.06	1.02	1.00	1.00	1.00
		SDRL	0.41	0.39	0.36	0.30	0.24	0.13	0.06	0.02	0.00
3.00	MaxEWMA-SRS	ARL	1.22	1.21	1.20	1.19	1.16	1.12	1.08	1.05	1.02
		SDRL	0.55	0.54	0.52	0.50	0.46	0.38	0.29	0.22	0.16
	MaxGWMA-SRS	ARL	1.22	1.21	1.20	1.18	1.16	1.12	1.08	1.05	1.02
		SDRL	0.54	0.53	0.51	0.49	0.45	0.38	0.29	0.22	0.16
	MaxEWMA-ORSS	ARL	1.06	1.06	1.05	1.04	1.03	1.01	1.00	1.00	1.00
		SDRL	0.25	0.24	0.23	0.20	0.17	0.11	0.06	0.03	0.01

Table 7.9: A comparison of ARLs and SDRLs of the MaxEWMA-ORSS ($\xi = 0.05$) with optimal MaxEWMA-SRS and optimal MaxGWMA-SRS charts when in-control ARL is fixed to 370

ρ	Chart	δ	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00
0.25	MaxEWMA-SRS	ARL	1.71	1.71	1.71	1.71	1.66	1.01	1.00	1.00	1.00
		SDRL	0.53	0.53	0.53	0.53	0.48	0.09	0.00	0.00	0.00
	MaxGWMA-SRS	ARL	1.71	1.71	1.71	1.70	1.66	1.01	1.00	1.00	1.00
		SDRL	0.53	0.53	0.53	0.53	0.48	0.09	0.00	0.00	0.00
	MaxEWMA-ORSS	ARL	1.89	1.89	1.88	1.37	1.00	1.00	1.00	1.00	1.00
		SDRL	0.32	0.33	0.33	0.48	0.01	0.00	0.00	0.00	0.00
0.50	MaxEWMA-SRS	ARL	4.02	4.02	3.82	2.90	2.04	1.13	1.00	1.00	1.00
		SDRL	1.66	1.66	1.46	0.94	0.62	0.33	0.02	0.00	0.00
	MaxGWMA-SRS	ARL	4.02	4.02	3.82	2.90	2.03	1.13	1.00	1.00	1.00
		SDRL	1.66	1.66	1.46	0.94	0.62	0.33	0.02	0.00	0.00
	MaxEWMA-ORSS	ARL	3.38	3.37	2.50	1.46	1.02	1.00	1.00	1.00	1.00
		SDRL	1.04	1.03	0.70	0.51	0.14	0.00	0.00	0.00	0.00
0.75	MaxEWMA-SRS	ARL	15.52	12.96	6.25	3.33	2.14	1.23	1.01	1.00	1.00
		SDRL	9.06	6.98	2.98	1.46	0.88	0.43	0.11	0.01	0.00
	MaxGWMA-SRS	ARL	15.36	12.88	6.25	3.33	2.14	1.23	1.01	1.00	1.00
		SDRL	8.85	6.87	2.98	1.46	0.88	0.43	0.11	0.01	0.00
	MaxEWMA-ORSS	ARL	11.56	7.38	2.75	1.52	1.09	1.00	1.00	1.00	1.00
		SDRL	5.79	3.37	1.16	0.59	0.29	0.01	0.00	0.00	0.00
1.00	MaxEWMA-SRS	ARL	370.00	21.42	6.63	3.42	2.21	1.31	1.05	1.00	1.00
		SDRL	368.91	85.57	4.10	1.90	1.11	0.51	0.21	0.05	0.01
	MaxGWMA-SRS	ARL	370.00	20.57	6.58	3.42	2.21	1.31	1.05	1.00	1.00
		SDRL	368.92	14.53	4.04	1.90	1.11	0.51	0.21	0.05	0.01
	MaxEWMA-ORSS	ARL	370.47	8.67	2.82	1.59	1.16	1.00	1.00	1.00	1.00
		SDRL	383.99	5.57	1.50	0.71	0.38	0.05	0.00	0.00	0.00
1.25	MaxEWMA-SRS	ARL	14.37	10.21	5.44	3.24	2.21	1.37	1.09	1.01	1.00
		SDRL	11.92	8.15	3.96	2.11	1.28	0.60	0.29	0.11	0.03
	MaxGWMA-SRS	ARL	13.56	9.72	5.34	3.22	2.21	1.37	1.09	1.01	1.00
		SDRL	11.12	7.63	3.78	2.07	1.28	0.60	0.29	0.11	0.03
	MaxEWMA-ORSS	ARL	9.69	5.61	2.61	1.61	1.21	1.01	1.00	1.00	1.00
		SDRL	7.92	4.19	1.61	0.80	0.44	0.09	0.01	0.00	0.00
1.50	MaxEWMA-SRS	ARL	4.82	4.39	3.48	2.63	2.03	1.38	1.12	1.03	1.00
		SDRL	3.80	3.42	2.57	1.79	1.24	0.64	0.35	0.17	0.07
	MaxGWMA-SRS	ARL	4.69	4.29	3.43	2.61	2.02	1.38	1.12	1.03	1.00
		SDRL	3.59	3.24	2.45	1.75	1.22	0.64	0.35	0.17	0.07
	MaxEWMA-ORSS	ARL	3.35	2.87	2.06	1.51	1.22	1.02	1.00	1.00	1.00
		SDRL	2.51	2.07	1.29	0.77	0.47	0.14	0.02	0.00	0.00
2.00	MaxEWMA-SRS	ARL	1.95	1.91	1.81	1.67	1.53	1.28	1.13	1.05	1.02
		SDRL	1.28	1.24	1.14	1.00	0.85	0.57	0.37	0.23	0.13
	MaxGWMA-SRS	ARL	1.94	1.90	1.80	1.67	1.53	1.28	1.13	1.05	1.02
		SDRL	1.25	1.22	1.12	0.98	0.84	0.57	0.37	0.23	0.12
	MaxEWMA-ORSS	ARL	1.51	1.47	1.36	1.24	1.14	1.03	1.00	1.00	1.00
		SDRL	0.83	0.79	0.67	0.52	0.38	0.17	0.05	0.01	0.00
2.50	MaxEWMA-SRS	ARL	1.39	1.38	1.36	1.32	1.27	1.18	1.10	1.05	1.02
		SDRL	0.73	0.72	0.69	0.64	0.58	0.45	0.34	0.23	0.15
	MaxGWMA-SRS	ARL	1.39	1.38	1.36	1.32	1.27	1.18	1.10	1.05	1.02
		SDRL	0.72	0.71	0.68	0.63	0.58	0.45	0.34	0.23	0.15
	MaxEWMA-ORSS	ARL	1.17	1.16	1.13	1.10	1.07	1.02	1.00	1.00	1.00
		SDRL	0.43	0.42	0.38	0.33	0.27	0.15	0.06	0.02	0.01
3.00	MaxEWMA-SRS	ARL	1.25	1.24	1.23	1.21	1.19	1.14	1.09	1.05	1.03
		SDRL	0.60	0.58	0.57	0.54	0.50	0.41	0.32	0.24	0.17
	MaxGWMA-SRS	ARL	1.25	1.24	1.23	1.21	1.19	1.14	1.09	1.05	1.03
		SDRL	0.58	0.57	0.55	0.53	0.49	0.40	0.32	0.24	0.17
	MaxEWMA-ORSS	ARL	1.07	1.06	1.06	1.05	1.03	1.01	1.00	1.00	1.00
		SDRL	0.27	0.26	0.24	0.22	0.19	0.12	0.06	0.03	0.01

Table 7.10: ARLs and SDRLs of the MaxEWMA-OIRSS chart versus optimal MaxEWMA-SRS and optimal MaxEWMA-GWMA-SRS control charts when in-control ARL is fixed to 370

ρ	Chart/σ_E^2	$\delta\rightarrow$	0.00	0.25	0.50	1.00	2.00
0.25	MaxEWMA-SRS	ARL	1.71	1.71	1.71	1.66	1.00
		SDRL	0.53	0.53	0.53	0.48	0.00
	MaxGWMA-SRS	ARL	1.71	1.71	1.71	1.66	1.00
		SDRL	0.53	0.53	0.53	0.48	0.00
	0.05	ARL	1.89	1.89	1.89	1.01	1.00
		SDRL	0.33	0.34	0.34	0.07	0.00
	0.15	ARL	1.96	1.96	1.96	1.04	1.00
		SDRL	0.27	0.28	0.27	0.20	0.00
	0.30	ARL	2.03	2.03	2.02	1.15	1.00
		SDRL	0.24	0.24	0.24	0.36	0.00
	0.50	ARL	2.07	2.07	2.07	1.32	1.00
		SDRL	0.27	0.27	0.27	0.47	0.00
0.50	MaxEWMA-SRS	ARL	4.02	4.02	3.82	2.04	1.00
		SDRL	1.66	1.66	1.46	0.62	0.02
	MaxGWMA-SRS	ARL	4.02	4.02	3.82	2.03	1.00
		SDRL	1.66	1.66	1.46	0.62	0.02
	0.05	ARL	3.44	3.43	2.61	1.06	1.00
		SDRL	1.09	1.08	0.77	0.25	0.00
	0.15	ARL	3.60	3.57	2.82	1.16	1.00
		SDRL	1.17	1.14	0.87	0.37	0.00
	0.30	ARL	3.78	3.77	3.07	1.28	1.00
		SDRL	1.22	1.21	0.94	0.46	0.00
	0.50	ARL	3.97	3.95	3.32	1.41	1.00
		SDRL	1.27	1.26	1.01	0.51	0.00
1.00	MaxEWMA-SRS	ARL	370.00	21.42	6.63	2.21	1.05
		SDRL	368.91	85.57	4.10	1.11	0.21
	MaxGWMA-SRS	ARL	370.00	20.57	6.58	2.21	1.05
		SDRL	368.92	14.53	4.04	1.11	0.21
	0.05	ARL	370.38	9.38	3.03	1.21	1.00
		SDRL	386.23	6.09	1.65	0.43	0.00
	0.15	ARL	370.95	10.57	3.38	1.30	1.00
		SDRL	386.85	6.94	1.87	0.51	0.01
	0.30	ARL	375.87	12.02	3.80	1.41	1.00
		SDRL	391.03	7.99	2.14	0.59	0.02
	0.50	ARL	369.22	13.39	4.21	1.53	1.00
		SDRL	382.04	9.11	2.43	0.67	0.03
1.50	MaxEWMA-SRS	ARL	4.82	4.39	3.48	2.03	1.12
		SDRL	3.80	3.42	2.57	1.24	0.35
	MaxGWMA-SRS	ARL	4.69	4.29	3.43	2.02	1.12
		SDRL	3.59	3.24	2.45	1.22	0.35
	0.05	ARL	3.42	2.98	2.16	1.25	1.00
		SDRL	2.56	2.16	1.36	0.50	0.03
	0.15	ARL	3.57	3.13	2.30	1.32	1.00
		SDRL	2.67	2.26	1.47	0.56	0.04
	0.30	ARL	3.70	3.29	2.45	1.39	1.00
		SDRL	2.77	2.41	1.60	0.63	0.06
	0.50	ARL	3.82	3.43	2.60	1.47	1.01
		SDRL	2.87	2.52	1.73	0.71	0.09
2.00	MaxEWMA-SRS	ARL	1.95	1.91	1.81	1.53	1.13
		SDRL	1.28	1.24	1.14	0.85	0.37
	MaxGWMA-SRS	ARL	1.94	1.90	1.80	1.53	1.13
		SDRL	1.25	1.22	1.12	0.84	0.37
	0.05	ARL	1.53	1.49	1.39	1.15	1.00
		SDRL	0.86	0.81	0.69	0.40	0.06
	0.15	ARL	1.57	1.53	1.43	1.18	1.01
		SDRL	0.89	0.85	0.73	0.43	0.07
	0.30	ARL	1.61	1.57	1.47	1.22	1.01
		SDRL	0.92	0.89	0.78	0.48	0.10
	0.50	ARL	1.65	1.61	1.50	1.25	1.01
		SDRL	0.96	0.93	0.81	0.52	0.12

Table 7.11: A comparison of diagnostic abilities of the MaxGWMA-SRS and MaxEWMA-ORSS control charts when in-control ARL is fixed to 370

		MaxGWMA-SRS									MaxEWMA-ORSS								
ρ	$\delta\rightarrow$	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00
0.25	$m+$	0	0	0	0	50	677	688	688	688	0	0	0	526	877	889	889	872	870
	$m-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$v+$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$v-$	1000	1000	1000	958	389	7	0	0	0	1000	1000	699	43	0	0	0	0	0
	$++$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$+-$	0	0	0	42	561	316	312	312	312	0	0	301	431	123	111	111	128	130
	$-+$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.50	$m+$	0	1	90	503	799	957	972	972	972	0	6	547	916	994	995	995	992	993
	$m-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$v+$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$v-$	1000	998	823	299	61	6	0	0	0	1000	975	165	2	0	0	0	0	0
	$++$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$+-$	0	1	87	198	140	37	28	28	28	0	19	288	82	6	5	5	8	7
	$-+$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.75	$m+$	1	256	810	938	977	994	997	997	997	0	584	965	992	998	999	1000	998	999
	$m-$	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
	$v+$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$v-$	998	721	155	35	10	0	0	0	0	999	351	11	0	0	0	0	0	0
	$++$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$+-$	0	23	35	27	13	6	3	3	3	0	65	24	8	2	1	0	2	1
	$-+$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.00	$m+$	271	950	981	989	991	996	997	997	997	247	973	989	992	994	994	994	993	996
	$m-$	239	1	0	0	0	0	0	0	0	251	0	0	0	0	0	0	0	0
	$v+$	251	27	9	4	4	1	0	0	0	291	21	5	6	2	0	0	0	0
	$v-$	237	21	6	3	2	0	0	0	0	211	5	1	0	0	0	0	0	0
	$++$	1	1	4	2	1	2	3	3	3	0	1	4	2	2	5	6	6	3
	$+-$	1	0	0	2	2	1	0	0	0	0	0	1	0	2	1	0	1	1
	$-+$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.25	$m+$	66	427	741	861	907	949	960	966	966	32	555	827	898	920	925	935	940	935
	$m-$	72	4	2	1	0	0	0	0	0	54	1	0	0	0	0	0	0	0
	$v+$	852	548	220	89	53	14	1	0	0	899	388	112	47	18	1	0	0	0
	$v-$	1	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0
	$++$	5	20	36	48	39	37	39	34	34	9	55	61	55	62	74	65	60	65
	$+-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$-+$	4	0	0	0	0	0	0	0	0	5	1	0	0	0	0	0	0	0

Table 7.11: (Continued).

		MaxGWMA-SRS									MaxEWMA-ORSS								
ρ	$\delta\rightarrow$	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.50	$m+$	72	205	413	587	703	818	862	874	879	59	245	499	625	720	779	782	774	779
	$m-$	70	21	4	1	1	0	0	0	0	43	8	2	0	0	0	0	0	0
	$v+$	829	733	479	285	180	55	19	5	2	854	661	335	152	80	12	0	0	0
	$v-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$++$	13	38	102	126	116	127	119	121	119	19	85	164	223	200	209	218	226	221
	$+-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$-+$	16	3	2	1	0	0	0	0	0	25	1	0	0	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.00	$m+$	67	122	192	262	331	461	539	585	600	39	93	165	276	337	394	408	425	417
	$m-$	64	32	18	11	4	1	1	0	0	35	23	2	0	0	0	0	0	0
	$v+$	753	717	639	520	408	218	101	39	11	784	693	547	328	191	48	2	0	0
	$v-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$++$	56	97	135	202	254	319	359	376	389	70	168	277	394	472	558	590	575	583
	$+-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$-+$	60	32	16	5	3	1	0	0	0	72	23	9	2	0	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.50	$m+$	58	86	119	147	188	273	331	369	396	22	52	75	105	153	176	196	188	169
	$m-$	59	38	26	15	9	1	0	0	0	21	10	3	0	0	0	0	0	0
	$v+$	664	658	616	558	476	314	193	94	37	711	664	566	453	271	103	25	2	0
	$v-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$++$	106	149	200	249	311	406	474	536	567	108	217	318	431	573	721	779	810	831
	$+-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$-+$	113	69	39	31	16	6	2	1	0	138	57	38	11	3	0	0	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3.00	$m+$	57	79	99	117	140	187	218	253	272	12	36	38	53	55	90	82	103	81
	$m-$	57	42	32	22	13	7	3	0	0	16	4	7	1	0	1	0	0	0
	$v+$	583	583	567	541	467	352	234	145	70	648	617	538	475	369	162	53	16	1
	$v-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$++$	147	182	229	270	344	439	538	599	656	167	267	362	444	560	746	863	881	918
	$+-$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$-+$	156	114	73	50	36	15	7	3	2	157	76	55	27	16	1	2	0	0
	$--$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

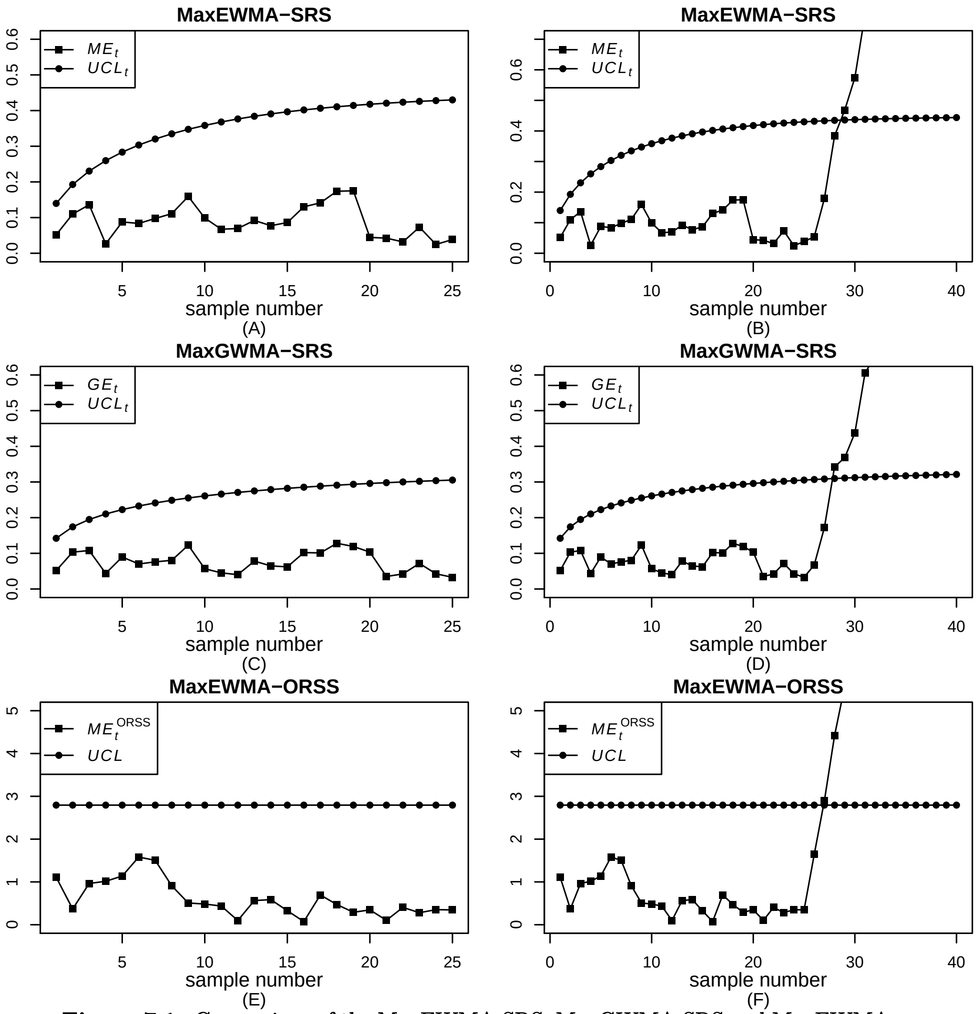

Figure 7.1: Comparison of the MaxEWMA-SRS, MaxGWMA-SRS and MaxEWMA-ORSS control charts for piston rings data

7.6 An application to real data

In this section, we consider a real data to explain the implementation of the MaxEWMA-SRS, MaxGWMA-SRS and MaxEWMA-ORSS control charts.

Suppose a quality practitioner is interested in establishing a statistical control of the inside diameters of the piston rings from an automotive engine manufactured by a forging process. For this purpose, 40 subgroups each of size five from an in-control process are observed. The complete data set is given in Montgomery (2009). The inside diameters of the piston rings are measured in millimeters (mm). We combine all samples such that we have 200 measurements of the inside diameters of piston rings. The data set reasonably satisfies the normality assumptions (cf. Haq et al., 2013a).

In order to apply the MaxEWMA-SRS, MaxGWMA-SRS and MaxEWMA-ORSS control charts, we need to generate data under both SRS and ORSS schemes. We draw 25 samples each of size five from the 200 measurements using SRS and ORSS methods. Both simple random and ordered ranked set samples are obtained by using with replacement sampling scheme. Based on these 25 samples under both sampling schemes, we estimate the control limits of each of the control chart considered here. In Figure 7.1, sub-figures

A, C and E, and plot the UCLs of these control charts along with their plotting-statistics. The in-control ARL for all control charts is fixed to 370 with smoothing constant $\xi = 0.05$. For the MaxEWMA-SRS chart, we consider $L = 2.764$; for the MaxGWMA-SRS chart, we consider $q = 0.95$, $\alpha = 0.80$ and $L = 2.8430$. It is clear from sub-figures A, C and E that the underlying process is in-control for all control charts. Now suppose that due to some interval cause, the process gets out-of-control. In order to capture that situation, 15 samples, each of size five, from 200 measurements under both sampling schemes are drawn. We multiply the values within each subgroup, obtained under SRS or ORSS scheme, by 1.00005 and add 0.01. The plotting-statistics of all control charts are then calculated for these 15 samples and are plotted in sub-figures B, D and F. From these sub-figures, it is evident that all control charts are signaling out-of-control signals after 25th sample. It is interesting to note that the MaxEWMA-SRS, MaxGWMA-SRS and MaxEWMA-ORSS charts detect a random shift in the process parameters at 29th, 28th and 27th samples, respectively. This shows that the proposed MaxEWMA-ORSS chart dominates the MaxEWMA-SRS and MaxGWMA-SRS control charts, and is able to detect random shift in the process parameters substantially quicker than its counterparts.

7.7 Conclusion

In this chapter, we proposed improved MaxEWMA control charts based on ORSS and OIRSS methods for simultaneously monitoring the process mean and dispersion, named MaxEWMA-ORSS and MaxEWMA-OIRSS charts. Extensive Monte Carlo simulations have been used to estimate the ARLs and SDRLs of the proposed MaxEWMA control charts. In order to fairly access the detection abilities of the proposed control charts, these control charts are compared with their counterparts based on SRS, i.e., MaxEWMA-SRS and MaxGWMA-SRS charts. It is worth mentioning here that both MaxEWMA-ORSS and MaxEWMA-OIRSS charts perform uniformly better than the MaxEWMA-SRS and MaxGWMA-SRS charts for all values of δ when $\rho \geq 0.50$. Finally, we considered a real data set to explain the implementation of the proposed MaxEWMA-ORSS control chart. Therefore, we recommend the use of the MaxEWMA-ORSS control chart for an improved monitoring of process mean and dispersion. The current work can be further improved by constructing the MaxEWMA control chart based on DRSS scheme.

Chapter 8

New Exponentially Weighted Moving Average Control Charts for Monitoring Process Mean and Process Dispersion

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Exponentially weighted moving average (EWMA) control charts have been widely accepted because of their excellent performance in detecting small to moderate shifts in the process parameters. In this chapter, we propose new EWMA control charts for monitoring the process mean and the process dispersion. These EWMA control charts are based on the best linear unbiased estimators obtained under ordered double ranked set sampling (ODRSS) and ordered imperfect double ranked set sampling (OIDRSS) schemes, named EWMA-ODRSS and EWMA-OIDRSS charts, respectively. We use Monte Carlo simulations to estimate the average run length, median run length, and standard deviation of run length of the proposed EWMA charts. We compare the performances of the proposed EWMA charts with the existing EWMA charts when detecting shifts in the process mean and in the process variability. It turns out that the EWMA-ODRSS mean chart performs uniformly better than the classical EWMA, fast initial response-based EWMA, Shewhart-EWMA, and hybrid EWMA mean charts. The EWMA-ODRSS mean chart also outperforms the Shewhart-EWMA

mean charts based on ranked set sampling (RSS) and median RSS schemes and the EWMA mean chart based on ordered RSS scheme. Moreover, the graphical comparisons of the EWMA dispersion charts reveal that the proposed EWMA-ODRSS and EWMA-OIDRSS charts are more sensitive than their counterparts. We also provide illuminating examples to illustrate the implementation of the proposed EWMA mean and dispersion charts.

8.1 Introduction

Statistical quality control charts are well-known process monitoring tool of statistical process control (SPC). The main objective of these control charts is to detect infrequent variations in the industrial processes as early as possible. The monitoring and identification of special cause of variations in the production processes are key features of the SPC and can be used to ensure that necessary corrective actions are taken before defective items are produced. The basic concept of the control chart was first introduced by Walter A. Shewhart in 1920s. Later on, this concept led to the introduction of modern SPC. Presently, the advanced statistical process monitoring techniques include exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) control charts. In recent years, quality control charts have found extensive applications in various fields, like signal segmentation, nuclear engineering, epidemiology, navigation system monitoring, fisheries, health care and public health surveillance, and eduction, refer to Montgomery (2009), Hawkins and Olwell (1998), Masson (2007), Hwang et al. (2008), Woodall (2006), Yashchin (1989), Pazhayamadom et al. (2013) and references cited therein.

The EWMA chart was first introduced by Roberts (1959) for monitoring the process mean. The traditional Shewhart chart is a special case of the EWMA chart. The CUSUM chart was suggested by Page (1954). In recent years, there have been substantial improvements and advancements in both EWMA and CUSUM charts. Abbas et al. (2013) proposed an improved EWMA-CUSUM chart for monitoring the process mean. They showed that the CUSUM chart based on the EWMA-statistic is more powerful than the existing charts when detecting small shifts in the process mean. Riaz et al. (2011) and Abbas et al. (2011) increased the detection abilities of the CUSUM and EWMA charts using several run rules. Haq (2013) proposed a hybrid EWMA chart by mixing the plotting-statistics of two EWMA charts for monitoring the process mean. It is shown that the hybrid EWMA chart is more sensitive than the EWMA-CUSUM chart for detecting small shifts in the process mean. Recently, Haq et al. (2014) suggested improved fast initial response features for both EWMA and CUSUM charts. Note that all of these control charts are based on simple random sampling (SRS) method. For more literature on the process mean control charts, refer to Riaz (2008b), Nazir et al. (2013), Ahmad et al. (2014), Abbas et al. (2013), Schoonhoven et al. (2009, 2011) and references therein.

The traditional ranked set sampling (RSS) was introduced by McIntyre (1952). The RSS scheme has now been applied to different fields, including biological and environmental studies, reliability theory, education, and statistical quality control. The RSS scheme becomes an efficient alternative to SRS when taking actual measurement of the quality characteristic is very costly or involves breaking the product that is expensive, hard to construct, and so on, while ranking a small set of selected units is cheap based on their quality level or by using non-destructive tests, for example, testing the weights or shape of products using expert knowledge (cf. Jafari Jozani and Mirkamali, 2011). As previously mentioned, control charts have many applications in health care and public health monitoring and surveillance. Control charts can be used for hospital monitoring with respect to patient infection rates, patient falls or accidents, emergency waiting room times, and so on. For these purposes, data on several different variables are collected on either weekly or monthly basis from patients in different hospitals (cf. Montgomery, 2009). The data from patients can be obtained via RSS schemes using expert's knowledge or using auxiliary variables. It is customary to use the auxiliary variables in order to judge the ranks of the study variable. Salazar and Sinha (1997) suggested an improved Shewhart-type control chart for monitoring the process mean based on RSS scheme. Later on, their work was extended by Muttlak and Al-Sabah (2003) who constructed the Shewhart-type quality control charts for detecting changes in the process mean based on perfect and imperfect RSS schemes. They showed that the RSS-based mean charts are more efficient than the classical Shewhart mean chart based on SRS. Al-Saleh and Al-Kadiri (2000) introduced double RSS (DRSS) scheme for efficient estimation of the population mean. They showed that the mean estimator based on DRSS is more efficient than the mean estimators with RSS and SRS schemes. Using this fact, Abujiya and Muttlak (2004) proposed the Shewhart-type quality control charts for monitoring process mean based on DRSS schemes. It is shown that the DRSS-based mean charts outperform their counterparts based on RSS and SRS methods. The concept of ordered RSS (ORSS) scheme was introduced by Balakrishnan and Li (2005, 2008). They obtained the best linear unbiased estimators (BLUEs) under ORSS (BLUEs-ORSS) of the unknown parameters (location and scale) of several location-scale families of distributions. They showed that the BLUEs-ORSS are more efficient than their counterparts based on RSS and SRS schemes. Abujiya et al. (2013b,a) proposed the Shewhart-CUSUM and the Shewhart-EWMA control charts for monitoring the process mean based on RSS and median RSS (MRSS) schemes. They showed that these control charts dominate their counterparts based on SRS scheme under both perfect and imperfect RSS scenarios. Recently, Haq et al. (2013a) extended the work of Balakrishnan and Li (2005) in statistical quality control and constructed some improved EWMA quality control charts for monitoring the process mean and the process dispersion based on ORSS and ordered imperfect RSS (OIRSS) schemes and named them EWMA-ORSS and EWMA-OIRSS charts. For more details on RSS-based control charts, refer to Al-Omari and Haq (2012), Abujiya et al. (2014) and references cite therein.

Dispersion charts are also frequently used to detect the changes in the process dispersion. Page (1954) suggested a CUSUM chart using a sample range for monitoring the process variability. Later on, Crowder and Hamilton (1992) applied the logarithmic transformation to the unbiased sample variance (S^2) and proposed an EWMA dispersion chart for detecting increases in the process standard deviation. A comprehensive comparison of the one-sided dispersion charts was considered by Acosta-Mejia et al. (1999). Castagliola (2005) applied three-parametric logarithmic transformation to S^2 and proposed a S^2-EWMA chart for monitoring the process dispersion. Later on, based on the same transformation, Castagliola et al. (2009) proposed a CUSUM-S^2 chart for monitoring the process variability. Following Castagliola (2005), Abbas et al. (2013a) proposed a CUSUM chart based on the EWMA-statistic and named it CS-EWMA chart, for monitoring the process variance. They showed that the CS-EWMA chart performs better than the S^2-EWMA and the CUSUM-S^2 charts in detecting small shifts in the process dispersion. Abbasi and Miller (2013) proposed a mean deviation based EWMA chart for monitoring the process dispersion. Note that all of these dispersion charts are based on the traditional SRS scheme. Haq (2014) extended the work of Abbasi and Miller (2013) and proposed an improved mean deviation based EWMA chart for monitoring the process dispersion using RSS scheme. For more literature on the dispersion charts, refer to Haq et al. (2013a), Abbasi et al. (2012), Abbasi and Miller (2012), Riaz and Does (2009), Riaz (2008a), Abbas et al. (2013b) and references therein.

Haq et al. (2014) introduced ordered double RSS (ODRSS) scheme for estimating the unknown parameters of a location-scale family of distributions. They showed that the BLUEs based on ODRSS (BLUEs-ODRSS) scheme are more efficient than the BLUEs-ORSS. In this chapter, we extend their work in statistical quality control and propose new EWMA charts for monitoring the process mean and the process dispersion based on ODRSS and ordered imperfect DRSS (OIDRSS) schemes, named EWMA-ODRSS and EWMA-OIDRSS charts, respectively. Based on extensive Monte Carlo simulations, we estimate the run length characteristics, i.e., average run length (ARL), median run length (MDRL), and standard deviation of run length (SDRL), of the proposed EWMA charts. ARL is defined as the expected numbers of observations that are required to signal a particular size shift in the process location or dispersion or both. The proposed EWMA charts are compared with their counterparts based on SRS, RSS, MRSS and ORSS schemes. It is observed that the proposed EWMA charts are better at signaling different shifts in the process mean and in the process dispersion that their existing counterparts.

The rest of the paper is as follows: in Section 8.2, we explain the ODRSS scheme and use it to obtain the BLUEs of the unknown parameters of a location-scale family of distributions. Section 8.3 contains the proposed EWMA control charts based on ODRSS and OIDRSS schemes. In this section, we also estimate the run length characteristics of the proposed EWMA charts. The performance comparisons of the EWMA charts are considered in Section 8.4. Illustrative examples are presented in Section 8.5, and Section 8.6 concludes the paper.

8.2 Ordered double ranked set sampling and mathematical setup

In this section, the traditional RSS, DRSS, and ODRSS schemes are explained. We obtain the BLUEs-ODRSS of the unknown parameters of a location-scale family of distributions.

The traditional RSS scheme is as follows: start with m^2 units from the target population and partition them into m sets, each having m units. Rank the units within each set with respect to the study variable or by any inexpensive method. Select the rth smallest ranked unit from the rth set, for $r = 1, 2, ..., m$. This completes one cycle of a ranked set sample of size m. If we arrange this ranked set sample in an increasing order of magnitude, we get an ordered ranked set sample of size m (cf. Balakrishnan and Li, 2008).

The DRSS scheme was suggested by Al-Saleh and Al-Kadiri (2000). DRSS scheme is as follows: identify m^3 units from the target population and allocate them into m sets, each having m^2 units. Apply the RSS procedure on each set to get m ranked set samples, each having m units. Again, apply the RSS scheme on m ranked set samples to get a double ranked set sample of size m. This completes one cycle of a double ranked set sample of size m. If we arrange this double ranked sample in an increasing order of magnitude, we obtain an ordered double ranked set sample of size m, and the corresponding sampling scheme is named ODRSS (cf. Haq et al., 2014).

Let Y be the study variable with probability density function (PDF) $f(y)$ and cumulative distribution function $F(y)$. Let $Y_1, Y_2, ..., Y_m$ be m independent and identically distributed (IID) random variables with PDF $f(y)$, i.e., $Y_r \sim f(y)$ for $r = 1, 2, ..., m$. Let $Y_{(1:m)}^{\text{OSRS}}, Y_{(2:m)}^{\text{OSRS}}, ..., Y_{(m:m)}^{\text{OSRS}}$ be an ordered simple random sample (OSRS) of size m obtained by arranging $Y_1, Y_2, ..., Y_m$ in an increasing order. The PDF and CDF of the rth order statistic $Y_{(r:m)}^{\text{OSRS}}$ $(1 \leq r \leq m)$ are, respectively, given by

$$\begin{aligned} f_{(r:m)}^{\text{OSRS}}(y) &= \frac{m!}{(r-1)!(m-r)!}\{F(y)\}^{r-1}\{1-F(y)\}^{m-r}f(y), \quad -\infty < y < \infty, \\ F_{(r:m)}^{\text{OSRS}}(y) &= \sum_{i=r}^{m}\binom{m}{i}\{F(y)\}^{i}\{1-F(y)\}^{m-i}, \quad -\infty < y < \infty, \end{aligned}$$

see Arnold et al. (1992) and David and Nagaraja (2003).

Let $(Y_{11}, Y_{12}, ..., Y_{1m}), (Y_{21}, Y_{22}, ..., Y_{2m}), ..., (Y_{m1}, Y_{m2}, ..., Y_{mm})$ be m independent simple random samples, each of size m. Apply the RSS procedure to obtain a ranked set sample of size m, denoted by $Y_{(r:m)}^{\text{RSS}} = r\text{th} \min\{Y_{r1}, Y_{r2}, ..., Y_{rm}\}$ for $r = 1, 2, ..., m$. Then, it is easy to show that $F_{(r:m)}^{\text{RSS}}(t) = F_{(r:m)}^{\text{OSRS}}(y)$, where $F_{(r:m)}^{\text{RSS}}(y)$ is the CDF of $Y_{(r:m)}^{\text{RSS}}$. Note that $Y_{(r:m)}^{\text{RSS}}$ $(1 \leq r \leq m)$ are independent and not identically distributed (INID) random variables. Let $\{Y_{(1:m),j}^{\text{RSS}}, Y_{(2:m),j}^{\text{RSS}}, ..., Y_{(m:m),j}^{\text{RSS}}\}$ denote a ranked set sample of size m obtained in the jth cycle, $j = 1, 2, ..., m$. Then, let $Y_{(r:m)}^{\text{DRSS}} = r\text{th} \min\{Y_{(1:m),r'}^{\text{RSS}}, Y_{(2:m),r'}^{\text{RSS}}, ..., Y_{(m:m),r'}^{\text{RSS}}\}$ for $r = 1, 2, ..., m$ denote a double ranked set sample of size m.

Suppose $\boldsymbol{A} = ((a_{i,j}))$ is a square matrix of order m. Then, the permanent of the matrix $\boldsymbol{A}$ is defined as $\text{Per}(\boldsymbol{A}) = \sum_P \prod_{j=1}^{m} a_{j,i_j}$, where $\sum_P(\cdot)$ denotes the sum over all $m!$ permutations $(i_1, i_2, ..., i_m)$ of $(1, 2, ..., m)$.

Following the work of Bapat and Beg (1989) and Vaughan and Venables (1972), the CDF of $Y_{(r:m)}^{\text{DRSS}}$ $(1 \leq r \leq m)$ is given by

$$F_{(r:m)}^{\text{DRSS}}(y) = \sum_{i=r}^{m} \frac{1}{i!(m-i)!}\text{Per}(\boldsymbol{A}_1), \quad -\infty < y < \infty, \tag{8.1}$$

where $\boldsymbol{A}_1 = \begin{pmatrix} F_{(1:m)}^{\text{RSS}}(y) & F_{(2:m)}^{\text{RSS}}(y) & \cdots & F_{(m:m)}^{\text{RSS}}(y) \\ 1-F_{(1:m)}^{\text{RSS}}(y) & 1-F_{(2:m)}^{\text{RSS}}(y) & \cdots & 1-F_{(m:m)}^{\text{RSS}}(y) \end{pmatrix} \begin{matrix} \}\,i \\ \}\,m-i \end{matrix}$. Here $\}\,i$ and $\}\,m-i$ show that the first row is repeated i times and the second row is repeated $m-i$ times, respectively.

Similarly, the PDF of $Y_{(r:m)}^{\text{DRSS}}$ $(1 \leq r \leq m)$ is given by

$$f_{(r:m)}^{\text{DRSS}}(y) = \frac{1}{(r-1)!(m-r)!}\text{Per}(\boldsymbol{A}_2), \quad -\infty < y < \infty,$$

where $\boldsymbol{A}_2 = \begin{array}{c} \begin{pmatrix} F_{(1:m)}^{\text{RSS}}(y) & F_{(2:m)}^{\text{RSS}}(y) & \cdots & F_{(m:m)}^{\text{RSS}}(y) \\ f_{(1:m)}^{\text{RSS}}(y) & f_{(2:m)}^{\text{RSS}}(y) & \cdots & f_{(m:m)}^{\text{RSS}}(y) \\ 1-F_{(1:m)}^{\text{RSS}}(y) & 1-F_{(2:m)}^{\text{RSS}}(y) & \cdots & 1-F_{(m:m)}^{\text{RSS}}(y) \end{pmatrix} \end{array} \begin{array}{l} \}\, r-1 \\ \}\, 1 \\ \}\, m-r \end{array}.$

In ODRSS scheme, we order the random variables under DRSS in an increasing order of magnitude. Let $Y_{(r:m)}^{\text{ODRSS}} = r\text{th}\min\{Y_{(1:m)}^{\text{DRSS}}, Y_{(2:m)}^{\text{DRSS}}, ..., Y_{(m:m)}^{\text{DRSS}}\}$ for $r = 1, 2, ..., m$ denote an ordered double ranked set sample of size m. Note that here, $Y_{(r:m)}^{\text{DRSS}}(1 \leq r \leq m)$ are INID random variable. The CDF of $Y_{(r:m)}^{\text{DRSS}}$ $(1 \leq r \leq m)$ is given by

$$F_{(r:m)}^{\text{ODRSS}}(y) = \sum_{i=r}^{m} \frac{1}{i!(m-i)!}\text{Per}(\boldsymbol{A}_3), \quad -\infty < y < \infty,$$

where $\boldsymbol{A}_3 = \begin{pmatrix} F_{(1:m)}^{\text{DRSS}}(y) & F_{(2:m)}^{\text{DRSS}}(y) & \cdots & F_{(m:m)}^{\text{DRSS}}(y) \\ 1-F_{(1:m)}^{\text{DRSS}}(y) & 1-F_{(2:m)}^{\text{DRSS}}(y) & \cdots & 1-F_{(m:m)}^{\text{DRSS}}(y) \end{pmatrix} \begin{array}{l} \}\, i \\ \}\, m-i \end{array}.$

Similarly, the PDF of $Y_{(r:m)}^{\text{ODRSS}}$ $(1 \leq r \leq m)$ is given by

$$f_{(r:m)}^{\text{ODRSS}}(y) = \frac{1}{(r-1)!(m-r)!}\text{Per}(\boldsymbol{A}_4), \quad -\infty < y < \infty,$$

where $\boldsymbol{A}_4 = \begin{pmatrix} F_{(1:m)}^{\text{DRSS}}(y) & F_{(2:m)}^{\text{DRSS}}(y) & \cdots & F_{(m:m)}^{\text{DRSS}}(y) \\ f_{(1:m)}^{\text{DRSS}}(y) & f_{(2:m)}^{\text{DRSS}}(y) & \cdots & f_{(m:m)}^{\text{DRSS}}(y) \\ 1-F_{(1:m)}^{\text{DRSS}}(y) & 1-F_{(2:m)}^{\text{DRSS}}(y) & \cdots & 1-F_{(m:m)}^{\text{DRSS}}(y) \end{pmatrix} \begin{array}{l} \}\, r-1 \\ \}\, 1 \\ \}\, m-r \end{array}.$

The joint density function of $Y_{(r:m)}^{\text{ODRSS}}$ and $Y_{(s:m)}^{\text{ODRSS}}$ $(1 \leq r < s \leq m)$ is given by

$$f_{(r,s:m)}^{\text{ODRSS}}(y,z) = \frac{1}{(r-1)!(s-r-1)!(m-s)!}\text{Per}(\boldsymbol{A}_5), \quad -\infty < y < z < \infty,$$

where $\boldsymbol{A}_5 = \begin{pmatrix} F_{(1:m)}^{\text{DRSS}}(y) & F_{(2:m)}^{\text{DRSS}}(y) & \cdots & F_{(m:m)}^{\text{DRSS}}(y) \\ f_{(1:m)}^{\text{DRSS}}(y) & f_{(2:m)}^{\text{DRSS}}(y) & \cdots & f_{(m:m)}^{\text{DRSS}}(y) \\ F_{(1:m)}^{\text{DRSS}}(z) - F_{(1:m)}^{\text{DRSS}}(y) & F_{(2:m)}^{\text{DRSS}}(z) - F_{(2:m)}^{\text{DRSS}}(y) & \cdots & F_{(m:m)}^{\text{DRSS}}(z) - F_{(m:m)}^{\text{DRSS}}(y) \\ f_{(1:m)}^{\text{DRSS}}(z) & f_{(2:m)}^{\text{DRSS}}(z) & \cdots & f_{(m:m)}^{\text{DRSS}}(z) \\ 1-F_{(1:m)}^{\text{DRSS}}(z) & 1-F_{(2:m)}^{\text{DRSS}}(z) & \cdots & 1-F_{(m:m)}^{\text{DRSS}}(z) \end{pmatrix} \begin{array}{l} \}\, r-1 \\ \}\, 1 \\ \}\, s-r-1 \\ \}\, 1 \\ \}\, m-s \end{array}.$

Let $\mu_{(r:m)}^{\text{ODRSS}}$ and $\sigma_{(r:m)}^{\text{ODRSS}}$ be the mean and variance of $Y_{(r:m)}^{\text{ODRSS}}(1 \leq r \leq m)$, respectively, defined as

$$\mu_{(r:m)}^{\text{ODRSS}} = \int y f_{(r:m)}^{\text{ODRSS}}(y)\text{d}y \quad \text{and} \quad \sigma_{(r,r,:m)}^{\text{ODRSS}} = \int (y - \mu_{(r:m)}^{\text{ODRSS}})^2 f_{(r:m)}^{\text{ODRSS}}(y)\text{d}y.$$

Similarly, the covariance between $Y_{(r:m)}^{\text{ODRSS}}$ and $Y_{(s:m)}^{\text{ODRSS}}$, say $\sigma_{(r,s:m)}^{\text{ODRSS}}$ $(1 \le r < s \le m)$, is given by

$$\sigma_{(r,s:m)}^{\text{ODRSS}} = \int_{-\infty}^{\infty} \int_{-\infty}^{z} yz f_{(r,s:m)}^{\text{ODRSS}}(y,z)\mathrm{d}y\mathrm{d}z - \mu_{(r:m)}^{\text{ODRSS}} \mu_{(s:m)}^{\text{ODRSS}}.$$

Let $\boldsymbol{Y}_{\text{ODRSS}} = (Y_{(1:m)}^{\text{ODRSS}}, Y_{(2:m)}^{\text{ODRSS}}, ..., Y_{(m:m)}^{\text{ODRSS}})'_{1\times m}$ be an ordered double ranked set sample of size m from a general location-scale family of distributions, with μ and $\sigma(>0)$ be the location and scale parameters, respectively. Let $Q_{(r:m)}^{\text{ODRSS}} = (Y_{(r:m)}^{\text{ODRSS}} - \mu)/\sigma$ be the standardized variate under ODRSS scheme. Therefore, the PDF of $Q_{(r:m)}^{\text{ODRSS}}$ is independent of μ and σ. Denote $\upsilon_{(r:m)}^{\text{ODRSS}} = E(Q_{(r:m)}^{\text{ODRSS}})$ for $1 \le r \le m$, $\vartheta_{(r,s:m)}^{\text{ODRSS}} = \text{Cov}(Q_{(r:m)}^{\text{ODRSS}}, Q_{(s:m)}^{\text{ODRSS}})$ for $1 \le r < s \le m$. Then, we can write $E(Y_{(r:m)}^{\text{ODRSS}}) = \mu + \sigma \upsilon_{(r:m)}^{\text{ODRSS}}$ and $\text{Cov}(Y_{(r:m)}^{\text{ODRSS}}, Y_{(s:m)}^{\text{ODRSS}}) = \sigma^2 \vartheta_{(r,s:m)}^{\text{ODRSS}}$. Following Lloyd (1952) and David and Nagaraja (2003), the BLUE-ODRSS, say $\hat{\boldsymbol{\theta}}_{\text{ODRSS}}^{\text{BLUE}} = (\hat{\mu}_{\text{ODRSS}}^{\text{BLUE}}, \hat{\sigma}_{\text{ODRSS}}^{\text{BLUE}})'_{1\times 2}$ of $\boldsymbol{\theta} = (\mu, \sigma)'_{1\times 2}$, is $\hat{\boldsymbol{\theta}}_{\text{ODRSS}}^{\text{BLUE}} = (\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}_{\text{ODRSS}}$, where $\boldsymbol{B} = (\boldsymbol{1}, \boldsymbol{\upsilon})_{m\times 2}$ and $\boldsymbol{\Sigma} = (\vartheta_{(r,s:m)}^{\text{ODRSS}})_{m\times m}$. Here, $\boldsymbol{1} = (1,1,...,1)'_{1\times m}$ and $\boldsymbol{\upsilon} = (\upsilon_{(1:m)}^{\text{ODRSS}}, \upsilon_{(2:m)}^{\text{ODRSS}}, ..., \upsilon_{(m:m)}^{\text{ODRSS}})'_{1\times m}$. The variance-covariance matrix of $\hat{\boldsymbol{\theta}}_{\text{ODRSS}}^{\text{BLUE}}$ is $\text{Cov}(\hat{\boldsymbol{\theta}}_{\text{ODRSS}}^{\text{BLUE}}) = \sigma^2(\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{B})^{-1}$.

The BLUEs-ODRSS of μ and σ can also be written as a linear combinations of $\boldsymbol{Y}_{\text{ODRSS}}$, i.e.,

$$\hat{\mu}_{\text{ODRSS}}^{\text{BLUE}} = -\boldsymbol{\upsilon}'\boldsymbol{\Gamma}\boldsymbol{Y}_{\text{ODRSS}} \quad \text{and} \quad \hat{\sigma}_{\text{ODRSS}}^{\text{BLUE}} = \boldsymbol{1}'\boldsymbol{\Gamma}\boldsymbol{Y}_{\text{ODRSS}},$$

where $\boldsymbol{\Gamma} = \frac{\boldsymbol{B}^{-1}(\boldsymbol{1}\boldsymbol{\upsilon}' - \boldsymbol{\upsilon}\boldsymbol{1}')\boldsymbol{B}^{-1}}{\Delta}$ is the skew-symmetric matrix with $\Delta = |\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{B}|$.

Similarly, we can write

$$\text{Var}(\hat{\mu}_{\text{ODRSS}}^{\text{BLUE}}) = \frac{\sigma^2(\boldsymbol{\upsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon})}{\Delta}, \ \text{Var}(\hat{\sigma}_{\text{ODRSS}}^{\text{BLUE}}) = \frac{\sigma^2(\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1})}{\Delta} \ \text{and}$$
$$\text{Cov}(\hat{\mu}_{\text{ODRSS}}^{\text{BLUE}}, \hat{\sigma}_{\text{ODRSS}}^{\text{BLUE}}) = -\frac{\sigma^2(\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon})}{\Delta}.$$

It is interesting to note that when the underlying location-scale distribution is symmetric, then it is easy to show that $\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon} = -\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon} = 0$. Thus, the covariance between $\hat{\mu}_{\text{ODRSS}}^{\text{BLUE}}$ and $\hat{\sigma}_{\text{ODRSS}}^{\text{BLUE}}$ becomes zero. It helps in further simplifying the mathematical expressions of the BLUEs-ODRSS and their corresponding variances, given by

$$\hat{\mu}_{\text{ODRSS}}^{\text{BLUE}} = (\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1})^{-1}\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}_{\text{ODRSS}}, \ \hat{\sigma}_{\text{ODRSS}}^{\text{BLUE}} = (\boldsymbol{\upsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon})^{-1}\boldsymbol{\upsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}_{\text{ODRSS}} \ \text{and}$$
$$\text{Var}(\hat{\mu}_{\text{ODRSS}}^{\text{BLUE}}) = \sigma^2(\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1})^{-1}, \ \text{Var}(\hat{\sigma}_{\text{ODRSS}}^{\text{BLUE}}) = \sigma^2(\boldsymbol{\upsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon})^{-1}.$$

The performance of RSS-based estimators depends on how perfectly the judgmental ordering of the randomly selected units is accomplished. The accurate ordering, in turn, helps in attaining stratification without quantification and utilizes the prior knowledge, experience, and the expertise of the investigators. Nonetheless, in practice, the judgmental ordering may not continuously match with the actual ordering. Thus, the judgment error is inevitable, particularly when dealing with large m. Moreover, the errors in ranking procedures cause the units to be assigned ranks different from their true ranks. Consequently, ranking errors adversely affect the performances of the estimates obtained via RSS scheme.

Dell and Clutter (1972) were the first to study the effect of imperfect ranking on the performance of the RSS-based mean estimator. They showed that, under imperfect ranking, the RSS-based mean estimator is unbiased and better than the SRS-based mean estimator, but imperfect orderings should be better than the random ordering of the selected units. Here, we examine the effect of the judgment error on the performances of the BLUEs-ODRSS. For brevity of discussion, it is assumed that the underlying quality characteristic Y is normally distributed with mean μ and variance σ^2, i.e., $Y \sim N(\mu, \sigma^2)$. For imperfect rankings, we follow the method suggested by Dell and Clutter (1972). Let Y_{ij}, $i = 1, 2, ..., m$ and $j = 1, 2, ..., m^2$, denote m^3 values drawn from a normal distributions. Partition these values into m sets, each of size m^2. Also, generate random errors V_{ij} from a normal distribution with mean zero and variance σ_V^2. Then, compute $X_{ij} = Y_{ij} + V_{ij}$. Note that V_{ij} is independent of Y_{ij}. Based on the values of X_{ij}, we select a double ranked set sample of size m, denoted by $X_{(r:m)}^{\text{DRSS}}$, $r = 1, 2, ..., m$. In fact, we select a pair $(X_{(r:m)}^{\text{DRSS}}, Y_{[r:m]}^{\text{IDRSS}})$ based on the ranks of X. Here $Y_{[r:m]}^{\text{IDRSS}}$ is the rth concomitant corresponding to the rth order statistic $X_{(r:m)}^{\text{DRSS}}$ obtained under DRSS scheme. Note that here, the ranking is performed with respect to the values of X; therefore, the corresponding values of Y are measured with error. Let $\boldsymbol{Y}_{\text{OIDRSS}} = (Y_{(1:m)}^{\text{OIDRSS}}, Y_{(2:m)}^{\text{OIDRSS}}, ..., Y_{(m:m)}^{\text{OIDRSS}})'_{1\times m}$ be the vector of an imperfect double ranked set sample of size m, where $Y_{(r:m)}^{\text{OIDRSS}} = r\text{th} \min\{Y_{[1:m]}^{\text{IDRSS}}, Y_{[2:m]}^{\text{IDRSS}}, ..., Y_{[m:m]}^{\text{IDRSS}}\}$ for $r = 1, 2, ..., m$. The BLUEs of μ and σ under OIDRSS scheme are $\hat{\mu}_{\text{OIDRSS}}^{\text{BLUE}} = -\boldsymbol{v}'\boldsymbol{\Gamma}\boldsymbol{Y}_{\text{OIDRSS}}$ and $\hat{\sigma}_{\text{OIDRSS}}^{\text{BLUE}} = \boldsymbol{1}'\boldsymbol{\Gamma}\boldsymbol{Y}_{\text{OIDRSS}}$, respectively. Note that since we are using the same coefficients of the BLUEs-ODRSS under OIDRSS scheme, therefore, the BLUEs based on OIDRSS (BLUEs-OIDRSS) are only linear estimators and do not hold the minimum variance property. However, when the ranking error reduces, the BLUEs-OIDRSS approach to the BLUEs-ODRSS. As OIDRSS scheme involves order statistics from independent concomitants obtained under IDRSS scheme. Therefore, it is difficult to obtain the mathematical expressions for the PDF and CDF of $Y_{(r:m)}^{\text{OIDRSS}}$. Hence, extensive Monte Carlo simulations are used to estimate the means and variances of the BLUEs-OIDRSS.

8.3 Proposed EWMA-ODRSS control charts

In this section, we propose new EWMA charts for monitoring the process mean and the process dispersion based on ODRSS scheme.

8.3.1 EWMA-ODRSS control chart for monitoring the process mean

Assume that the underlying quality characteristics Y_t is normally distributed with mean μ_0 and variance σ_0^2 at time t, i.e., $Y_t \sim N(\mu_0, \sigma_0^2)$. Without loss of generality, we set $\mu_0 = 0$ and $\sigma_0^2 = 1$. Let $\hat{\mu}_{\text{ODRSS},t}^{\text{BLUE}}$ be the BLUE of μ_0 under ODRSS scheme, obtained from a subgroup of size m, at time t for $t = 1, 2, ...$ Let $\{\hat{\mu}_{\text{ODRSS},t}^{\text{BLUE}}\}$ be the sequence of IID random variables, and let $\xi \in [0, 1]$ be a smoothing constant. The EWMA sequence $\{W_t\}$ based on $\{\hat{\mu}^{\text{BLUE}}_{\text{ODRSS},t}\}$ can be defined by using a following recurrence formula:

$$W_t = \xi\hat{\mu}^{\text{BLUE}}_{\text{ODRSS},t} + (1-\xi)W_{t-1}, \quad W_0 = \mu_0,\ 0 < \xi \leq 1. \tag{8.2}$$

For a positive integer t, it is easy to show that $E(W_t) = \mu_0$ and $\text{Var}(W_t) = \left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}\sigma_0^2(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})^{-1}$. Here, W_t is the test-statistic of the EWMA mean chart based on ODRSS scheme. If the underlying process parameters (μ_0, σ_0) are known, then the upper control limit (UCL_t), center limit (CL_t) and lower control limit (LCL_t) of the EWMA-ODRSS mean chart, at time t, are given by

$$\begin{aligned} UCL_t &= \mu_0 + L\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})^{-1}}, \\ CL_t &= \mu_0, \\ LCL_t &= \mu_0 - L\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})^{-1}}, \end{aligned} \tag{8.3}$$

respectively, where L is a positive control chart multiplier, and its value is selected such that the in-control ARL of the EWMA-ODRSS mean chart reaches to a particular level. Note that as the time t increases, i.e., $t \to \infty$, then the term $\{1-(1-\xi)^{2t}\}$ approaches unity. It is interesting to note that the EWMA-ODRSS mean chart becomes equivalent to the Shewhart-ODRSS mean chart when $\xi = 1$. For detecting unusual variations in the process mean under ODRSS scheme, the test-statistic W_t, given in (8.2), is plotted with the control limits, given in (8.3), against time t. The EWMA-ODRSS mean chart detects an out-of-control signal if W_t exceeds either UCL_t or LCL_t. In case when $W_t > UCL_t$ $(W_t < LCL_t)$, then there is a positive (negative) shift in the process mean at time t. Let $\delta = (\sqrt{m}/\sigma_0)\,|\mu_1 - \mu_0|$ represents the amount of shift in the process mean measured in standard deviation units. Here, μ_1 is the out-of-control process mean. The underlying process is said to be in statistical control when $\delta = 0$ and out-of-control when $\delta > 0$. In order to study the run length properties of the EWMA-ODRSS mean chart, we use extensive Monte Carlo simulations to estimate the ARL, MDRL, and SDRL of the EWMA-ODRSS mean chart by using the control limits given in (8.3). We consider different values of mean shift δ, i.e., $\delta = 0.00$, 0.25, 0.50, 0.75, 1.00, 1.50, 2, and 3. The subgroup size is taken to be $m = 5$, and the in-control ARL is fixed to 500. Based on different values of ξ and δ, the estimated values of ARLs, MDRLs, and SDRLs are given in Table 8.1. Each result is based on 10^5 replications.

From Table 8.1, we observe that, having fixed ξ, the ARLs, MDRLs, and SDRLs tend to decrease as the value of δ increases and vice-versa. Similarly, for a fixed value of δ, the detection ability of the EWMA-ODRSS mean chart increases as the value of ξ decreases. For example, with $\xi = 0.25$, the EWMA-ODRSS mean chart detects a shift $\delta = 0.25$ in the process mean on average at the 19th sample, whereas the same shift is detected on average at the 70th sample with $\xi = 0.50$.

A detailed simulation study is conducted to study the effect of imperfect ranking on the performance of the EWMA-ODRSS mean chart. We name the EWMA chart based on the BLUEs-OIDRSS as the

Table 8.1: Run length properties of the EWMA-ODRSS process mean control chart

	$\xi\rightarrow$	0.05	0.10	0.25	0.50
δ	$L\rightarrow$	2.6402	2.8274	3.0085	3.0910
0.00	ARL	499.64	500.50	500.36	499.63
	MDRL	340.00	344.00	346.00	344.00
	SDRL	510.21	505.42	501.12	501.87
0.25	ARL	18.78	22.44	35.61	69.21
	MDRL	16.00	18.00	26.00	49.00
	SDRL	13.59	17.17	32.23	67.32
0.50	ARL	5.84	6.54	7.94	12.53
	MDRL	5.00	6.00	7.00	9.00
	SDRL	3.67	3.98	5.38	10.73
0.75	ARL	3.04	3.37	3.79	4.72
	MDRL	3.00	3.00	3.00	4.00
	SDRL	1.71	1.84	2.11	3.23
1.00	ARL	1.99	2.18	2.38	2.65
	MDRL	2.00	2.00	2.00	2.00
	SDRL	1.00	1.08	1.18	1.47
1.50	ARL	1.22	1.29	1.37	1.41
	MDRL	1.00	1.00	1.00	1.00
	SDRL	0.45	0.49	0.55	0.60
2.00	ARL	1.03	1.04	1.06	1.07
	MDRL	1.00	1.00	1.00	1.00
	SDRL	0.16	0.20	0.23	0.25
3.00	ARL	1.00	1.00	1.00	1.00
	MDRL	1.00	1.00	1.00	1.00
	SDRL	0.01	0.00	0.01	0.01

EWMA-OIDRSS chart. Let W_t^* be the plotting-statistic of the EWMA-OIDRSS mean chart, which is defined as follows:

$$W_t^* = \xi\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS},t} + (1-\xi)W^*_{t-1}, \quad W_0^* = \bar{\hat{\mu}}^{\text{BLUE}}_{\text{OIDRSS}}, \; 0 < \xi \leq 1,$$

where ξ is a smoothing constant and $\bar{\hat{\mu}}^{\text{BLUE}}_{\text{OIDRSS}} = \frac{1}{k}\sum_{i=1}^{k}\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS},i}$.

As already mentioned in 8.2, it is difficult to derive the mathematical expressions for the mean and variances of the BLUEs-OIDRSS. Hence, we estimate the mean and the variance of the BLUE-OIDRSS based on a large historical data set, obtained from an in-control process. Let $\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS},i}$, $i = 1, 2, ..., k$, be the estimated values of the BLUEs-OIDRSS obtained from k subgroups, each of size m, where $\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS},i} = (\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^{-1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}_{\text{OIDRSS},i}$. Then, the estimated UCL_t ($EUCL_t$), estimated CL_t (ECL_t), estimated LCL_t ($ELCL_t$) of the EWMA-OIDRSS mean chart are, respectively, given by

$$\begin{aligned}
EUCL_t &= \bar{\hat{\mu}}^{\text{BLUE}}_{\text{OIDRSS}} + L\hat{\sigma}_{\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},\\
ECL_t &= \bar{\hat{\mu}}^{\text{BLUE}}_{\text{OIDRSS}},\\
ELCL_t &= \bar{\hat{\mu}}^{\text{BLUE}}_{\text{OIDRSS}} - L\hat{\sigma}_{\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},
\end{aligned}$$

where $\hat{\sigma}_{\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS}}} = \sqrt{\frac{1}{(k-1)}\sum_{i=1}^{k}(\hat{\mu}^{\text{BLUE}}_{\text{OIDRSS},i} - \bar{\hat{\mu}}^{\text{BLUE}}_{\text{OIDRSS}})^2}$ and L is a positive control chart multiplier.

In order to study the run length properties of the EWMA-OIDRSS mean chart, we first estimate the control limits based on 10^6 samples obtained under OIDRSS scheme (cf. Haq et al., 2013a). For brevity of

Table 8.2: Run length properties of the EWMA-OIDRSS process mean control chart

		$\sigma_V^2 = 0.05$				$\sigma_V^2 = 0.15$			
	$\xi\rightarrow$	0.05	0.10	0.25	0.50	0.05	0.10	0.25	0.50
δ	$L\rightarrow$	2.6402	2.8274	3.0085	3.0910	2.6402	2.8274	3.0085	3.0910
0.00	ARL	499.28	499.91	506.27	509.81	502.09	503.07	510.02	522.06
	MDRL	343.00	343.00	350.00	353.00	344.00	347.00	354.00	360.00
	SDRL	515.71	505.93	509.20	511.73	516.86	506.30	509.68	524.33
0.25	ARL	22.59	27.42	45.13	86.35	28.78	35.56	60.71	114.37
	MDRL	19.00	22.00	32.00	61.00	24.00	28.00	43.00	80.00
	SDRL	16.88	21.81	41.74	84.49	22.32	29.52	57.09	112.04
0.50	ARL	6.96	7.80	9.84	16.37	8.80	9.94	13.27	23.60
	MDRL	6.00	7.00	8.00	12.00	8.00	9.00	11.00	17.00
	SDRL	4.48	4.90	7.08	14.52	5.84	6.51	10.24	21.55
0.75	ARL	3.59	3.97	4.54	5.97	4.50	5.01	5.89	8.36
	MDRL	3.00	4.00	4.00	5.00	4.00	4.00	5.00	6.00
	SDRL	2.08	2.22	2.64	4.34	2.71	2.93	3.70	6.67
1.00	ARL	2.33	2.54	2.82	3.23	2.86	3.16	3.53	4.32
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00	3.00	4.00
	SDRL	1.22	1.30	1.44	1.94	1.59	1.70	1.93	2.86
1.50	ARL	1.37	1.45	1.56	1.63	1.62	1.74	1.89	2.02
	MDRL	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
	SDRL	0.57	0.62	0.67	0.75	0.74	0.80	0.87	1.02
2.00	ARL	1.07	1.10	1.14	1.16	1.18	1.23	1.30	1.34
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.26	0.30	0.35	0.37	0.41	0.45	0.50	0.54
3.00	ARL	1.00	1.00	1.00	1.00	1.00	1.01	1.01	1.01
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.02	0.02	0.03	0.03	0.06	0.07	0.09	0.11
δ		$\sigma_V^2 = 0.30$				$\sigma_V^2 = 0.50$			
	ARL	497.62	504.47	511.40	526.52	502.91	503.32	507.73	525.08
0.00	MDRL	340.00	348.00	354.00	364.00	345.00	347.00	351.00	364.00
	SDRL	513.64	512.45	512.32	526.57	518.77	508.38	508.11	523.53
	ARL	35.84	45.56	78.73	143.52	42.55	54.82	95.15	168.52
0.25	MDRL	29.00	35.00	56.00	100.00	34.00	41.00	67.00	118.00
	SDRL	28.75	39.13	75.81	142.19	35.29	48.69	92.12	167.23
	ARL	10.90	12.55	17.47	32.65	12.97	15.02	21.78	42.00
0.50	MDRL	9.00	11.00	13.00	23.00	11.00	13.00	16.00	30.00
	SDRL	7.41	8.58	14.25	30.71	9.01	10.64	18.38	39.97
	ARL	5.55	6.21	7.51	11.51	6.55	7.38	9.11	14.94
0.75	MDRL	5.00	6.00	6.00	9.00	6.00	6.00	8.00	11.00
	SDRL	3.47	3.75	5.05	9.72	4.20	4.60	6.43	13.11
	ARL	3.49	3.87	4.41	5.72	4.09	4.55	5.26	7.22
1.00	MDRL	3.00	3.00	4.00	5.00	4.00	4.00	5.00	6.00
	SDRL	2.00	2.16	2.56	4.15	2.44	2.62	3.19	5.54
	ARL	1.91	2.08	2.27	2.52	2.20	2.41	2.66	3.02
1.50	MDRL	2.00	2.00	2.00	2.00	2.00	2.00	2.00	3.00
	SDRL	0.94	1.02	1.10	1.37	1.14	1.21	1.35	1.77
	ARL	1.34	1.43	1.52	1.59	1.50	1.61	1.73	1.86
2.00	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00
	SDRL	0.54	0.60	0.65	0.71	0.66	0.72	0.78	0.90
	ARL	1.02	1.03	1.04	1.05	1.05	1.08	1.10	1.12
3.00	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.14	0.17	0.20	0.22	0.22	0.27	0.31	0.33

discussion, we consider different values for error variance (σ_V^2), i.e., $\sigma_V^2 = 0.05$, 0.15, 0.30 and 0.50. Based on 10^5 replications, we estimate the ARLs, MDRLs, and SDRLs of the EWMA-OIDRSS mean chart and are presented in Table 8.2. Here, we consider the same values of L as considered in Table 8.1. However, it is also

Table 8.3: Run length properties of the EWMA-ODRSS dispersion chart

		Symmetric limits						Asymmetric limits		
	$\xi \rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.30	0.40	0.50
	$L_1 \rightarrow$	2.2777	2.4800	2.6501	2.7358	2.7856	2.8220	2.5800	2.6512	2.7600
τ	$L_2 \rightarrow$	2.2777	2.4800	2.6501	2.7358	2.7856	2.8220	2.8500	2.8575	2.8500
0.50	ARL	1.90	2.17	2.41	2.59	2.78	3.08	2.35	2.53	2.92
	MDRL	2.00	2.00	2.00	2.00	3.00	3.00	2.00	2.00	3.00
	SDRL	0.69	0.72	0.77	0.84	0.98	1.25	0.79	0.90	1.18
0.60	ARL	2.72	3.11	3.54	3.92	4.48	5.55	3.49	3.98	5.12
	MDRL	3.00	3.00	3.00	4.00	4.00	5.00	3.00	4.00	4.00
	SDRL	1.15	1.24	1.40	1.66	2.19	3.28	1.49	1.91	2.96
0.70	ARL	4.36	5.05	5.96	7.20	9.41	13.82	6.15	7.83	12.28
	MDRL	4.00	5.00	5.00	6.00	8.00	10.00	5.00	6.00	9.00
	SDRL	2.22	2.44	3.01	4.19	6.51	11.14	3.51	5.22	9.79
0.80	ARL	8.65	10.20	13.44	19.30	30.42	51.75	15.16	22.58	43.14
	MDRL	8.00	9.00	11.00	15.00	22.00	37.00	12.00	17.00	31.00
	SDRL	5.34	6.16	9.22	15.55	27.17	49.20	11.81	19.57	40.58
0.90	ARL	27.13	35.07	57.44	94.80	152.23	240.27	64.02	101.11	193.79
	MDRL	22.00	27.00	42.00	67.00	106.00	167.00	46.00	70.00	136.00
	SDRL	21.27	28.94	52.62	91.04	149.96	238.76	60.84	99.03	190.37
0.95	ARL	75.39	101.86	157.71	221.02	283.57	341.03	151.27	205.08	303.72
	MDRL	55.00	72.00	110.00	154.00	197.00	238.00	106.00	143.00	208.00
	SDRL	71.68	98.17	154.39	219.96	282.31	339.17	149.31	203.90	305.98
1.00	ARL	199.72	201.03	199.71	201.05	198.71	199.36	200.23	199.56	200.62
	MDRL	130.00	138.00	138.00	140.00	137.00	138.00	138.00	138.00	139.00
	SDRL	215.71	206.06	200.23	200.41	198.59	199.99	199.85	199.60	200.57
1.05	ARL	61.25	68.31	74.95	78.88	81.93	84.66	94.03	91.17	88.50
	MDRL	43.00	48.00	52.00	55.00	57.00	59.00	65.00	64.00	62.00
	SDRL	61.87	67.63	74.34	77.95	81.51	84.28	93.33	90.20	87.68
1.10	ARL	23.61	26.92	30.77	34.30	37.07	39.65	40.20	41.01	41.64
	MDRL	18.00	20.00	22.00	24.00	26.00	28.00	28.00	29.00	29.00
	SDRL	21.85	24.71	28.97	32.94	36.01	38.61	38.86	39.87	40.63
1.20	ARL	8.28	9.19	10.23	11.21	12.17	13.29	12.48	13.14	13.71
	MDRL	6.00	7.00	8.00	8.00	9.00	10.00	9.00	10.00	10.00
	SDRL	7.03	7.52	8.52	9.75	10.90	12.12	10.88	11.84	12.64
1.30	ARL	4.55	5.02	5.47	5.79	6.16	6.62	6.28	6.49	6.71
	MDRL	4.00	4.00	4.00	5.00	5.00	5.00	5.00	5.00	5.00
	SDRL	3.61	3.83	4.15	4.53	5.04	5.56	4.93	5.31	5.67
1.40	ARL	3.06	3.36	3.60	3.77	3.95	4.12	4.02	4.10	4.20
	MDRL	2.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
	SDRL	2.27	2.42	2.55	2.73	2.92	3.18	2.89	3.05	3.25
1.50	ARL	2.32	2.52	2.68	2.79	2.87	2.97	2.94	2.97	3.00
	MDRL	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	SDRL	1.58	1.71	1.78	1.87	1.99	2.11	1.98	2.05	2.13
2.00	ARL	1.26	1.30	1.34	1.36	1.38	1.39	1.40	1.40	1.40
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.54	0.58	0.62	0.64	0.65	0.67	0.67	0.67	0.68
3.00	ARL	1.02	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.04
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.15	0.16	0.18	0.19	0.19	0.19	0.20	0.20	0.19

possible to select the value of L such that the in-control ARL of the EWMA-OIDRSS mean chart reaches to a particular level.

From the results presented in Table 8.2, it is observed that, when $\xi \leq 0.10$, the in-control ARLs of the EWMA-OIDRSS mean chart remain roughly closer to 500 for all values of σ_V^2. This shows that, even under imperfect rankings, the false-alarm of the EWMA-OIDRSS mean chart is stable. Moreover, for other cases, when $\xi > 0.10$, the false-alarm of the EWMA-OIDRSS mean chart decreases. As expected, having fixed ξ and δ, the performance of the EWMA-OIDRSS mean chart increases as the value of σ_V^2 decreases and vice-versa.

8.3.2 EWMA-ODRSS control chart for monitoring the process dispersion

Recall that the underlying quality characteristic $Y_t \sim N(\mu, \sigma_0^2)$ for $t \geq 1$. Let $\hat{\sigma}^{\mathrm{BLUE}}_{\mathrm{ODRSS},t}$ be the BLUE of σ under ODRSS scheme, obtained from a subgroup of size m, at time t for $t = 1, 2, \ldots$. Let $\{\hat{\sigma}^{\mathrm{BLUE}}_{\mathrm{ODRSS},t}\}$ be the sequence of IID random variables. An EWMA sequence $\{H_t\}$ based on $\{\hat{\sigma}^{\mathrm{BLUE}}_{\mathrm{ODRSS},t}\}$ can be defined by using a following recurrence formula:

$$H_t = \xi \hat{\sigma}^{\mathrm{BLUE}}_{\mathrm{ODRSS},t} + (1-\xi)H_{t-1}, \quad H_0 = \sigma_0, \ 0 < \xi \leq 1,$$

where ξ is a smoothing constant. For $t \geq 1$, it is easy to show that $E(H_t) = \sigma_0$ and $\mathrm{Var}(H_t) = \{\frac{\xi}{2-\xi}\}\{1-(1-\xi)^{2t}\}\sigma_0^2(\boldsymbol{v}'\boldsymbol{\Sigma}^{-1}\boldsymbol{v})^{-1}$. We name the EWMA chart based on H_t as the EWMA-ODRSS dispersion chart. If the underlying process parameters (μ_0, σ_0) are known, then the control limits of the EWMA-ODRSS dispersion chart, at time t, are given by

$$\begin{aligned}
UCL_t &= \sigma_0 + L_2\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\boldsymbol{v}'\boldsymbol{\Sigma}^{-1}\boldsymbol{v})^{-1}},\\
CL_t &= \sigma_0,\\
LCL_t &= \sigma_0 - L_1\sigma_0\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}(\boldsymbol{v}'\boldsymbol{\Sigma}^{-1}\boldsymbol{v})^{-1}},
\end{aligned}$$

where L_1 and L_2 are the control charting multipliers, and their values are selected such that the in-control ARL of the EWMA-ODRSS chart reaches to a specific level. Here, the EWMA-ODRSS dispersion chart detects an out-of-control signal as soon as the plotting-statistic H_t exceeds UCL_t or LCL_t. If at time t, $H_t > UCL_t$, then there is a positive shift in the process dispersion, or if $H_t < LCL_t$, then there is a negative shift in the process dispersion. Let $\tau = \sigma_1/\sigma_0$ represents the amount of shift in the underlying process standard deviation, where σ_1 denotes an out-of-control standard deviation or shifted process standard deviation. The underlying process is said to be in control state when $\tau = 1$, i.e., $\sigma_1 = \sigma_0$. Note that if the quality practitioner is interested in detecting a shift in the process dispersion, then, both upper and lower control limits are used for this purpose. The EWMA chart based on both UCL_t and LCL_t is called a two-sided EWMA chart. However, if the interest lies in only detecting an increase (decrease) in the process variation, then an upper (lower) control limit is used. The EWMA charts based on a single upper or lower control limit is called a one-sided EWMA chart. In this study, we consider both two-sided and one-sided EWMA control charts for detecting overall changes in the process dispersion. Based on extensive Monte Carlo simulations from the standard normal distribution, for different values of ξ and τ, we compute the run length characteristics, i.e., ARL, MDRL, and SDRL, of the two-sided and one-sided EWMA-ODRSS dispersion charts. For brevity of discussion, we consider subgroup size $m = 5$. For each case, the in-control ARL of the EWMA-ODRSS dispersion chart is fixed to 200. Each result is based on 10^5 replications. In

Tables 8.3 and 8.4, we report the run length properties of the two-sided and one-sided EWMA-OIDRSS dispersion charts, respectively.

From Tables 8.3 and 8.4, it is observed that, having fixed τ, the performance of the EWMA-ODRSS dispersion chart increases as the value of ξ decreases and vice-versa. From Table 8.3, note that the out-of-control ARLs are unbiased for small values of ξ in the range $0 < \xi \leq 0.20$. It means that each out-of-control ARL is less than the in-control ARL. However, for $0.3 \leq \xi \leq 0.50$, the out-of-control ARLs of the EWMA-ODRSS dispersion chart are biased when $0.9 \leq \tau < 1$. Most quality practitioners are interested in detecting an increase in the process dispersion because small decreases lead to an improvement in the quality of the product. On the other hand, it is also possible to design an EWMA dispersion chart such that its out-of-control ARL becomes unbiased. This task is accomplishable when using asymmetric control limits for the EWMA-ODRSS dispersion chart. For this purpose, in the last three columns of Table 8.3, we calculate the out-of-control ARLs of the EWMA-ODRSS dispersion chart using asymmetric control limits. It is interesting to note that there are some improvements when detecting small changes in the process dispersion. However, a reduction in the out-of-control ARL when detecting a decrease in the process dispersion with asymmetric control limits leads to large values of the out-of-control ARLs when detecting an increase in the process variability. In Table 8.4, we report the run length characteristics of the one-sided EWMA-ODRSS dispersion charts. It is observed that all out-of-control ARLs are unbiased for all values of τ, and the performance of the EWMA-ODRSS dispersion chart increases as the value of ξ decreases and vice-versa.

Following Section 8.3.1, we also study the effect of imperfect ranking on the performance of the EWMA-ODRSS dispersion chart. Let $\{\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},t}\}$ be the sequence of IID random variables. Define an EWMA sequence $\{H^*_t\}$ based on $\{\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},t}\}$, by using a following recurrence formula:

$$H^*_t = \xi\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},t} + (1-\xi)H^*_{t-1}, \quad H^*_0 = \bar{\hat{\sigma}}^{\text{BLUE}}_{\text{OIDRSS}},\ 0 < \xi \leq 1,$$

where ξ is smoothing constant.

Based on a large historical data set, we estimate the mean and variance of $\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS}}$. Let $\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},1}, \hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},2}, ..., \hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},k}$ be the estimated values of $\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS}}$ obtained from k subgroups, each of size m, where $\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},i} = (\boldsymbol{v}'\boldsymbol{\Sigma}^{-1}\boldsymbol{v})^{-1}\boldsymbol{v}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}_{\text{OIDRSS},i}$, $i = 1, 2, ..., k$. Then, at time t, the estimated control limits of the EWMA-OIDRSS dispersion chart are given by

$$\begin{aligned}
EUCL_t &= \bar{\hat{\sigma}}^{\text{BLUE}}_{\text{OIDRSS}} + L\hat{\sigma}_{\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},\\
ECL_t &= \bar{\hat{\sigma}}^{\text{BLUE}}_{\text{OIDRSS}},\\
ELCL_t &= \bar{\hat{\sigma}}^{\text{BLUE}}_{\text{OIDRSS}} - L\hat{\sigma}_{\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS}}}\sqrt{\left(\frac{\xi}{2-\xi}\right)\{1-(1-\xi)^{2t}\}},
\end{aligned}$$

where $\bar{\hat{\sigma}}^{\text{BLUE}}_{\text{OIDRSS}} = \frac{1}{k}\sum_{i=1}^{k}\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},i}$, $\hat{\sigma}_{\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS}}} = \sqrt{\frac{1}{(k-1)}\sum_{i=1}^{k}(\hat{\sigma}^{\text{BLUE}}_{\text{OIDRSS},i} - \bar{\hat{\sigma}}^{\text{BLUE}}_{\text{OIDRSS}})^2}$ and L is a positive control chart multiplier. Following Haq et al. (2013a), we first estimate the control limits based on 10^6

Table 8.4: Run length properties of the EWMA-ODRSS (one-sided) dispersion charts

	$\xi \rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	$\xi \rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50
$\tau \downarrow$	$L \rightarrow$	1.9123	2.2055	2.4710	2.6107	2.6985	2.7593	$\tau \downarrow$ $L \rightarrow$	1.8380	2.0737	2.2395	2.2923	2.3100	2.3113
1.00	ARL	200.75	200.08	199.69	200.08	200.98	200.43	1.00	200.78	200.56	200.90	200.66	200.90	200.09
	MDRL	130.00	136.00	137.00	139.00	140.00	139.00		131.00	137.00	138.00	140.00	139.00	139.00
	SDRL	222.67	208.90	202.21	202.45	200.80	199.35		218.43	206.15	202.62	198.85	200.83	201.38
1.10	ARL	16.87	20.34	24.87	28.91	32.74	36.43	0.95	42.71	52.34	66.63	77.27	87.64	97.39
	MDRL	12.00	15.00	18.00	20.00	23.00	25.00		30.00	37.00	47.00	54.00	61.00	68.00
	SDRL	16.73	19.01	23.34	27.69	31.69	35.59		42.53	50.23	64.50	75.27	86.20	95.80
1.20	ARL	6.32	7.55	8.85	10.04	11.22	12.38	0.90	17.06	21.16	27.63	34.11	41.08	48.80
	MDRL	5.00	6.00	7.00	8.00	8.00	9.00		13.00	16.00	20.00	25.00	29.00	34.00
	SDRL	5.72	6.41	7.44	8.67	10.06	11.32		14.92	17.80	24.58	31.48	39.07	47.17
1.30	ARL	3.66	4.26	4.87	5.33	5.78	6.28	0.85	9.26	11.36	14.19	17.28	21.11	25.85
	MDRL	3.00	3.00	4.00	4.00	4.00	5.00		7.00	9.00	11.00	13.00	15.00	19.00
	SDRL	3.00	3.36	3.74	4.18	4.68	5.27		7.26	8.41	11.12	14.58	18.81	23.99
1.40	ARL	2.55	2.92	3.28	3.52	3.74	3.97	0.80	5.90	7.17	8.65	10.12	12.05	14.61
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00		5.00	6.00	7.00	8.00	9.00	11.00
	SDRL	1.91	2.15	2.36	2.55	2.77	3.04		4.15	4.72	5.90	7.54	9.80	12.73
1.50	ARL	1.99	2.23	2.48	2.62	2.75	2.88	0.75	4.14	5.00	5.84	6.60	7.58	8.94
	MDRL	2.00	2.00	2.00	2.00	2.00	2.00		4.00	4.00	5.00	6.00	6.00	7.00
	SDRL	1.35	1.51	1.66	1.76	1.89	2.03		2.63	2.96	3.50	4.31	5.46	7.04
1.60	ARL	1.67	1.84	2.02	2.11	2.19	2.27	0.70	3.10	3.73	4.28	4.70	5.19	5.88
	MDRL	1.00	1.00	2.00	2.00	2.00	2.00		3.00	3.00	4.00	4.00	4.00	5.00
	SDRL	1.01	1.14	1.25	1.32	1.40	1.48		1.78	2.01	2.28	2.67	3.27	4.12
1.70	ARL	1.47	1.60	1.73	1.79	1.85	1.90	0.65	2.43	2.91	3.32	3.55	3.82	4.18
	MDRL	1.00	1.00	1.00	1.00	2.00	2.00		2.00	3.00	3.00	3.00	3.00	4.00
	SDRL	0.80	0.91	1.00	1.04	1.09	1.15		1.26	1.42	1.59	1.77	2.08	2.54
1.80	ARL	1.34	1.43	1.53	1.58	1.63	1.66	0.60	1.98	2.35	2.66	2.81	2.96	3.15
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00		2.00	2.00	2.00	3.00	3.00	3.00
	SDRL	0.65	0.73	0.82	0.85	0.89	0.93		0.94	1.05	1.15	1.25	1.40	1.65
1.90	ARL	1.25	1.32	1.39	1.44	1.47	1.49	0.55	1.66	1.95	2.20	2.31	2.39	2.49
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00		2.00	2.00	2.00	2.00	2.00	2.00
	SDRL	0.54	0.62	0.68	0.72	0.74	0.77		0.72	0.80	0.87	0.93	1.01	1.14
2.00	ARL	1.18	1.24	1.29	1.33	1.35	1.37	0.50	1.42	1.66	1.86	1.94	2.00	2.04
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00		1.00	2.00	2.00	2.00	2.00	2.00
	SDRL	0.46	0.52	0.58	0.61	0.62	0.65		0.56	0.64	0.69	0.72	0.76	0.82
2.50	ARL	1.05	1.06	1.08	1.09	1.10	1.11	0.40	1.11	1.24	1.38	1.43	1.45	1.46
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00		1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.22	0.26	0.29	0.31	0.32	0.33		0.31	0.43	0.50	0.52	0.53	0.54
3.00	ARL	1.01	1.02	1.03	1.03	1.03	1.03	0.30	1.00	1.02	1.06	1.07	1.08	1.08
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00		1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.12	0.14	0.16	0.17	0.18	0.19		0.06	0.14	0.23	0.26	0.28	0.28

samples obtained under OIDRSS scheme. For brevity of discussion, we consider the same values of m and σ_V^2 as considered in Section 8.3.1. For the EWMA-OIDRSS dispersion chart, we consider the same values of control charting multiplier L as considered for the EWMA-ODRSS dispersion chart, i.e., $L = L_1 = L_2$. Based on extensive Monte Carlo simulations, we estimate the run length characteristics of the two-sided and one-sided EWMA-OIDRSS dispersion charts for different values of τ and ξ and are presented in Tables 8.5–8.7.

From Table 8.5, it is observed that, when $0 < \xi \leq 0.2$, the in-control ARLs roughly remain closer to 200, and the out-of-control ARLs are unbiased. For fixed values of ξ and τ, the performance of the two-sided EWMA-OIDRSS dispersion chart decreases as the value of σ_V^2 increases and vice-versa. Note that in Tables 8.6 and 8.7, we have considered the same values of L_1 and L_2 of the one-sided EWMA-ODRSS dispersion charts for the one-sided EWMA-OIDRSS dispersion charts. However, it is possible to select the values of L_1 and L_2 for the one-sided EWMA-OIDRSS dispersion charts such that the in-control ARL reaches to 200. From Table 8.6, it is clear that, given the values of τ and σ_V^2, both in-control and out-of-control ARLs tend to increase with an increase in the value of ξ. From Table 8.7, we observe that when $\xi \geq 0.10$ and $\sigma_V^2 < 0.50$, the in-control ARLs are less than 200. This shows that with very small errors in rankings, the false-alarm of the EWMA-OIDRSS dispersion chart increases. Therefore, when using the EWMA-OIDRSS chart for detecting decreases in the process dispersion, the value of L_2 should be selected such that the in-control ARL of the EWMA-OIDRSS chart reaches to 200.

Table 8.5: Run length properties of the EWMA-OIDRSS (two-sided) dispersion charts

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.05	0.10	0.20	0.30	0.40	0.50
	$L\rightarrow$	2.2777	2.4800	2.6501	2.7358	2.7856	2.8220	2.2777	2.4800	2.6501	2.7358	2.7856	2.8220
τ				$\sigma_V^2=0.05$						$\sigma_V^2=0.15$			
0.50	ARL	2.02	2.30	2.57	2.77	2.99	3.35	2.13	2.43	2.72	2.95	3.21	3.64
	MDRL	2.00	2.00	2.00	3.00	3.00	3.00	2.00	2.00	3.00	3.00	3.00	3.00
	SDRL	0.77	0.82	0.89	0.99	1.16	1.50	0.83	0.89	0.96	1.08	1.30	1.73
0.60	ARL	2.88	3.31	3.77	4.22	4.86	6.08	3.05	3.50	4.00	4.50	5.29	6.68
	MDRL	3.00	3.00	4.00	4.00	4.00	5.00	3.00	3.00	4.00	4.00	5.00	5.00
	SDRL	1.29	1.40	1.58	1.93	2.56	3.86	1.38	1.51	1.72	2.12	2.90	4.40
0.70	ARL	4.65	5.40	6.39	7.76	10.17	14.90	4.92	5.70	6.80	8.37	11.08	16.47
	MDRL	4.00	5.00	6.00	7.00	8.00	11.00	4.00	5.00	6.00	7.00	9.00	12.00
	SDRL	2.46	2.72	3.39	4.74	7.33	12.29	2.66	2.94	3.71	5.28	8.27	13.84
0.80	ARL	9.22	10.89	14.48	20.67	31.93	52.94	9.71	11.51	15.38	22.23	34.44	56.78
	MDRL	8.00	9.00	12.00	16.00	23.00	37.00	8.00	10.00	12.00	17.00	25.00	40.00
	SDRL	5.82	6.78	10.30	16.94	28.74	50.51	6.24	7.31	11.13	18.54	31.41	54.12
0.90	ARL	28.64	37.16	60.24	96.26	150.09	229.28	30.17	39.20	63.20	100.42	155.93	236.43
	MDRL	23.00	29.00	43.00	68.00	105.00	159.00	24.00	30.00	45.00	71.00	109.00	164.00
	SDRL	23.01	31.23	55.57	92.68	147.35	228.44	24.48	33.56	58.86	97.22	152.75	234.87
0.95	ARL	78.62	104.41	157.68	215.68	275.60	331.50	82.24	108.59	160.82	218.95	279.03	338.00
	MDRL	56.00	73.00	110.00	151.00	191.00	230.00	59.00	77.00	112.00	151.00	194.00	235.00
	SDRL	75.91	100.91	155.46	213.65	275.34	331.36	79.07	105.46	158.34	218.71	278.17	337.60
1.00	ARL	202.11	201.26	200.54	202.59	204.75	207.13	201.52	201.21	202.80	206.36	209.58	214.90
	MDRL	134.00	138.00	139.00	139.00	141.00	143.00	133.00	138.00	140.00	143.00	145.00	149.00
	SDRL	219.11	206.23	201.87	204.24	206.52	207.82	217.70	206.75	204.58	206.71	209.54	214.14
1.05	ARL	64.72	72.45	79.67	84.85	87.85	91.39	67.58	74.92	83.11	88.04	91.76	96.50
	MDRL	45.00	51.00	56.00	59.00	61.00	63.00	47.00	52.00	58.00	61.00	64.00	67.00
	SDRL	65.47	72.06	78.87	83.81	87.36	91.04	69.01	74.57	82.72	87.00	91.26	96.45
1.10	ARL	25.08	28.50	32.83	37.02	40.15	43.38	26.32	30.06	35.01	38.78	42.56	46.01
	MDRL	19.00	21.00	24.00	26.00	28.00	30.00	20.00	22.00	25.00	27.00	30.00	32.00
	SDRL	23.29	26.30	30.90	35.70	39.08	42.60	24.53	27.78	33.34	37.45	41.54	44.92
1.20	ARL	8.80	9.79	10.96	12.07	13.21	14.53	9.24	10.27	11.54	12.69	14.02	15.50
	MDRL	7.00	8.00	9.00	9.00	10.00	10.00	7.00	8.00	9.00	10.00	10.00	11.00
	SDRL	7.46	8.01	9.18	10.52	11.94	13.44	7.82	8.45	9.76	11.10	12.66	14.35
1.30	ARL	4.80	5.31	5.83	6.21	6.61	7.15	5.05	5.58	6.07	6.52	6.98	7.57
	MDRL	4.00	4.00	5.00	5.00	5.00	5.00	4.00	4.00	5.00	5.00	5.00	6.00
	SDRL	3.83	4.06	4.46	4.86	5.41	6.05	4.04	4.29	4.67	5.19	5.78	6.49
1.40	ARL	3.22	3.55	3.81	4.00	4.19	4.40	3.36	3.69	3.98	4.20	4.39	4.64
	MDRL	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	4.00	4.00
	SDRL	2.40	2.57	2.72	2.92	3.15	3.43	2.52	2.69	2.86	3.07	3.33	3.65
1.50	ARL	2.44	2.65	2.83	2.93	3.03	3.16	2.52	2.75	2.94	3.06	3.17	3.30
	MDRL	2.00	2.00	2.00	2.00	2.00	3.00	2.00	2.00	2.00	3.00	3.00	3.00
	SDRL	1.69	1.81	1.90	2.00	2.10	2.29	1.76	1.89	1.99	2.10	2.23	2.40
2.00	ARL	1.28	1.33	1.38	1.40	1.42	1.43	1.30	1.35	1.41	1.43	1.45	1.47
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.57	0.62	0.66	0.68	0.69	0.71	0.60	0.64	0.69	0.70	0.73	0.74
3.00	ARL	1.03	1.03	1.04	1.04	1.04	1.04	1.03	1.03	1.04	1.04	1.05	1.05
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.16	0.18	0.19	0.20	0.20	0.21	0.17	0.19	0.20	0.21	0.21	0.22

Table 8.5: (Continued).

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.05	0.10	0.20	0.30	0.40	0.50
	$L\rightarrow$	2.2777	2.4800	2.6501	2.7358	2.7856	2.8220	2.2777	2.4800	2.6501	2.7358	2.7856	2.8220
τ		$\sigma_V^2 = 0.30$						$\sigma_V^2 = 0.50$					
0.50	ARL	2.46	2.82	3.18	3.49	3.93	4.72	2.67	3.07	3.49	3.88	4.46	5.64
	MDRL	2.00	3.00	3.00	3.00	4.00	4.00	3.00	3.00	3.00	4.00	4.00	5.00
	SDRL	0.95	1.03	1.13	1.32	1.71	2.50	1.02	1.10	1.24	1.49	2.01	3.18
0.60	ARL	3.53	4.07	4.71	5.44	6.70	9.23	3.84	4.44	5.20	6.15	7.90	11.68
	MDRL	3.00	4.00	4.00	5.00	6.00	7.00	4.00	4.00	5.00	5.00	7.00	9.00
	SDRL	1.64	1.79	2.09	2.73	3.99	6.66	1.77	1.94	2.32	3.17	4.96	8.87
0.70	ARL	5.69	6.63	8.11	10.42	14.79	24.02	6.21	7.25	8.97	12.00	18.18	32.07
	MDRL	5.00	6.00	7.00	9.00	11.00	18.00	6.00	7.00	8.00	10.00	14.00	23.00
	SDRL	3.14	3.50	4.62	7.01	11.69	21.11	3.41	3.82	5.21	8.41	14.86	28.98
0.80	ARL	11.18	13.46	18.65	28.30	46.23	81.19	12.21	14.76	21.13	33.46	58.19	111.24
	MDRL	10.00	12.00	15.00	21.00	33.00	57.00	11.00	13.00	17.00	25.00	42.00	78.00
	SDRL	7.35	8.80	14.19	24.39	42.87	78.42	8.00	9.76	16.28	29.36	54.40	108.08
0.90	ARL	34.65	45.62	75.34	121.03	192.91	307.32	37.49	50.04	84.50	142.08	235.16	392.65
	MDRL	27.00	34.00	54.00	85.00	135.00	214.00	30.00	37.00	60.00	100.00	163.00	274.00
	SDRL	28.75	39.72	70.72	117.71	189.89	305.32	31.30	44.13	79.68	137.95	232.17	391.03
0.95	ARL	91.53	119.35	177.38	242.27	310.85	381.77	97.23	127.26	191.27	263.53	342.04	420.28
	MDRL	65.00	84.00	124.00	169.00	216.00	264.00	69.00	89.00	133.00	183.00	238.00	292.00
	SDRL	89.47	116.53	175.71	241.21	311.26	380.87	95.32	124.40	190.06	261.87	340.40	421.23
1.00	ARL	201.86	201.95	206.03	213.93	220.06	228.42	201.69	202.16	206.47	215.30	221.85	227.96
	MDRL	134.00	138.00	142.00	149.00	153.00	159.00	134.00	139.00	143.00	149.00	154.00	158.00
	SDRL	216.16	207.84	208.15	213.33	220.04	227.77	217.66	206.72	206.96	215.65	221.46	226.96
1.05	ARL	73.63	81.42	89.54	95.35	100.35	105.16	77.11	84.50	92.13	97.34	101.57	105.44
	MDRL	51.00	57.00	62.00	66.00	70.00	73.00	53.00	58.00	64.00	68.00	71.00	74.00
	SDRL	75.55	80.73	88.83	94.53	99.65	104.47	79.66	85.24	91.85	96.52	100.69	104.29
1.10	ARL	29.53	33.60	39.16	43.32	47.46	51.49	31.26	35.49	41.15	45.61	49.27	52.81
	MDRL	22.00	25.00	28.00	30.00	33.00	36.00	23.00	26.00	29.00	32.00	35.00	37.00
	SDRL	27.87	31.41	37.42	42.07	46.27	50.63	29.80	33.34	39.43	44.11	48.16	51.94
1.20	ARL	10.38	11.60	13.08	14.51	16.13	17.67	11.14	12.39	14.04	15.47	16.89	18.60
	MDRL	8.00	9.00	10.00	11.00	12.00	13.00	8.00	10.00	11.00	11.00	12.00	13.00
	SDRL	8.94	9.69	11.21	12.91	14.79	16.61	9.72	10.45	12.19	13.91	15.64	17.48
1.30	ARL	5.65	6.25	6.87	7.45	7.98	8.64	6.03	6.65	7.31	7.90	8.48	9.20
	MDRL	4.00	5.00	6.00	6.00	6.00	6.00	5.00	5.00	6.00	6.00	6.00	7.00
	SDRL	4.59	4.88	5.39	6.06	6.73	7.58	4.99	5.26	5.78	6.51	7.27	8.08
1.40	ARL	3.73	4.10	4.45	4.68	4.94	5.24	3.97	4.37	4.72	4.99	5.24	5.59
	MDRL	3.00	3.00	4.00	4.00	4.00	4.00	3.00	4.00	4.00	4.00	4.00	4.00
	SDRL	2.87	3.04	3.26	3.54	3.85	4.23	3.11	3.29	3.50	3.81	4.14	4.57
1.50	ARL	2.79	3.04	3.25	3.39	3.51	3.69	2.94	3.22	3.45	3.60	3.73	3.90
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00	2.00	3.00	3.00	3.00	3.00	3.00
	SDRL	2.01	2.13	2.27	2.38	2.54	2.77	2.16	2.32	2.43	2.58	2.74	2.98
2.00	ARL	1.37	1.44	1.49	1.52	1.55	1.57	1.42	1.49	1.56	1.58	1.61	1.63
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.67	0.73	0.77	0.79	0.81	0.84	0.72	0.79	0.83	0.85	0.87	0.90
3.00	ARL	1.04	1.04	1.05	1.06	1.06	1.06	1.05	1.05	1.06	1.07	1.07	1.07
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.20	0.21	0.23	0.24	0.25	0.25	0.22	0.23	0.25	0.26	0.27	0.27

Table 8.6: Run length properties of the EWMA-OIDRSS (one-sided) dispersion chart for monitoring increases in the process dispersion

	$\xi\to$	0.05	0.10	0.20	0.30	0.40	0.50	0.05	0.10	0.20	0.30	0.40	0.50
	$L\to$	1.9123	2.2055	2.4710	2.6107	2.6985	2.7593	1.9123	2.2055	2.4710	2.6107	2.6985	2.7593
τ				$\sigma_V^2=0.05$						$\sigma_V^2=0.15$			
1.00	ARL	199.86	200.69	203.93	210.71	211.59	213.87	204.37	204.33	210.54	217.37	222.67	226.33
	MDRL	130.00	136.00	140.00	147.00	147.00	148.00	132.00	139.00	146.00	149.00	154.00	158.00
	SDRL	221.09	209.45	206.88	210.53	212.12	213.22	226.70	211.82	212.43	219.56	222.50	225.12
1.10	ARL	17.79	21.42	26.49	31.18	35.35	39.24	19.32	23.16	28.94	34.15	38.84	43.59
	MDRL	12.00	16.00	19.00	22.00	25.00	27.00	13.00	17.00	21.00	24.00	27.00	31.00
	SDRL	17.72	20.16	25.14	29.84	34.26	38.34	19.35	21.89	27.42	32.82	37.78	42.64
1.20	ARL	6.74	7.97	9.45	10.71	12.04	13.47	7.25	8.67	10.29	11.78	13.30	15.00
	MDRL	5.00	6.00	7.00	8.00	9.00	10.00	5.00	7.00	8.00	9.00	10.00	11.00
	SDRL	6.09	6.78	7.99	9.39	10.79	12.35	6.63	7.40	8.75	10.36	12.06	13.93
1.30	ARL	3.82	4.51	5.15	5.68	6.20	6.76	4.15	4.86	5.61	6.21	6.82	7.43
	MDRL	3.00	4.00	4.00	5.00	5.00	5.00	3.00	4.00	4.00	5.00	5.00	6.00
	SDRL	3.18	3.57	3.99	4.49	5.07	5.72	3.49	3.87	4.40	4.96	5.62	6.36
1.40	ARL	2.67	3.06	3.45	3.71	3.98	4.24	2.85	3.29	3.74	4.04	4.31	4.62
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00	2.00	3.00	3.00	3.00	3.00	4.00
	SDRL	2.02	2.26	2.48	2.69	2.99	3.31	2.20	2.46	2.71	3.00	3.28	3.63
1.50	ARL	2.07	2.34	2.61	2.77	2.90	3.04	2.19	2.49	2.78	2.97	3.13	3.31
	MDRL	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	3.00	3.00
	SDRL	1.42	1.60	1.76	1.87	2.01	2.18	1.54	1.74	1.91	2.05	2.21	2.42
2.00	ARL	1.20	1.26	1.33	1.36	1.39	1.41	1.23	1.30	1.38	1.42	1.45	1.48
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.48	0.55	0.61	0.64	0.67	0.69	0.52	0.59	0.66	0.70	0.73	0.75
3.00	ARL	1.02	1.02	1.03	1.03	1.04	1.04	1.02	1.03	1.04	1.04	1.05	1.05
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.13	0.15	0.18	0.18	0.20	0.20	0.14	0.17	0.19	0.21	0.21	0.22
				$\sigma_V^2=0.30$						$\sigma_V^2=0.50$			
1.00	ARL	202.58	203.64	210.16	216.81	222.11	228.42	202.24	201.40	206.99	213.36	218.43	222.94
	MDRL	131.00	139.00	146.00	150.00	155.00	159.00	132.00	137.00	143.00	148.00	151.50	154.00
	SDRL	225.11	211.92	211.07	217.71	221.07	227.62	224.06	209.92	209.71	214.02	218.95	223.03
1.10	ARL	20.78	24.99	30.91	36.28	41.48	46.40	22.08	26.47	32.36	38.10	42.80	47.93
	MDRL	14.00	18.00	22.00	26.00	29.00	32.00	15.00	19.00	23.00	27.00	30.00	33.00
	SDRL	20.95	23.85	29.56	35.09	40.40	45.59	22.32	25.35	30.86	36.83	41.74	47.35
1.20	ARL	7.86	9.39	11.16	12.82	14.46	16.25	8.37	9.97	11.86	13.59	15.35	17.29
	MDRL	6.00	7.00	9.00	9.00	11.00	12.00	6.00	8.00	9.00	10.00	11.00	12.00
	SDRL	7.28	8.13	9.63	11.41	13.14	15.16	7.87	8.67	10.27	12.14	14.08	16.11
1.30	ARL	4.41	5.24	6.05	6.69	7.36	8.14	4.72	5.57	6.41	7.12	7.83	8.64
	MDRL	3.00	4.00	5.00	5.00	6.00	6.00	3.00	4.00	5.00	6.00	6.00	6.00
	SDRL	3.77	4.25	4.79	5.44	6.14	7.05	4.08	4.59	5.15	5.82	6.68	7.55
1.40	ARL	3.05	3.53	3.99	4.36	4.67	5.03	3.22	3.74	4.24	4.62	4.95	5.32
	MDRL	2.00	3.00	3.00	4.00	4.00	4.00	2.00	3.00	3.00	4.00	4.00	4.00
	SDRL	2.41	2.69	2.97	3.27	3.62	4.05	2.60	2.90	3.17	3.52	3.89	4.33
1.50	ARL	2.33	2.65	2.97	3.19	3.37	3.55	2.44	2.80	3.14	3.37	3.57	3.76
	MDRL	2.00	2.00	2.00	3.00	3.00	3.00	2.00	2.00	3.00	3.00	3.00	3.00
	SDRL	1.70	1.89	2.07	2.25	2.42	2.65	1.81	2.04	2.23	2.42	2.62	2.86
2.00	ARL	1.27	1.35	1.43	1.48	1.51	1.54	1.30	1.39	1.48	1.53	1.57	1.60
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.57	0.65	0.72	0.75	0.78	0.81	0.61	0.69	0.77	0.81	0.84	0.87
3.00	ARL	1.02	1.03	1.05	1.05	1.06	1.06	1.03	1.04	1.05	1.06	1.07	1.07
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	SDRL	0.16	0.18	0.22	0.23	0.24	0.24	0.17	0.20	0.23	0.25	0.26	0.26

Table 8.7: Run length properties of the EWMA-OIDRSS (one-sided) dispersion chart for monitoring decreases in the process dispersion

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.40	0.50	0.05	0.10	0.20	0.30	0.40	0.50
	$L\rightarrow$	1.8380	2.0737	2.2395	2.2923	2.3100	2.3113	1.8380	2.0737	2.2395	2.2923	2.3100	2.3113
τ		$\sigma_V^2 = 0.05$						$\sigma_V^2 = 0.15$					
1.00	ARL	201.85	198.07	196.99	193.30	191.39	190.80	200.71	195.59	192.75	190.83	188.44	186.71
	MDRL	133.00	136.00	135.00	134.00	133.00	132.00	130.00	133.00	134.00	132.00	131.00	129.00
	SDRL	219.93	202.73	198.24	192.81	190.56	190.91	220.28	201.87	193.31	190.44	187.03	186.75
1.10	ARL	44.77	54.31	67.92	77.60	86.23	93.65	47.38	57.85	70.35	79.78	88.47	96.70
	MDRL	31.00	39.00	48.00	54.00	60.00	65.00	33.00	41.00	49.00	56.00	61.00	68.00
	SDRL	44.75	52.23	65.65	76.14	84.66	92.78	48.03	56.13	68.68	78.00	87.21	95.24
1.20	ARL	18.09	22.37	29.06	35.13	41.59	48.43	19.49	24.27	31.32	37.71	44.15	51.34
	MDRL	14.00	17.00	21.00	25.00	29.00	34.00	15.00	19.00	23.00	27.00	31.00	36.00
	SDRL	16.12	19.11	26.10	32.73	39.75	46.57	17.48	21.07	28.53	35.47	42.30	49.67
1.30	ARL	6.27	7.65	9.18	10.76	12.69	15.25	6.85	8.38	10.17	11.97	14.24	16.99
	MDRL	5.00	7.00	8.00	9.00	10.00	11.00	6.00	7.00	8.00	9.00	11.00	12.00
	SDRL	4.52	5.15	6.44	8.28	10.55	13.41	5.08	5.79	7.38	9.49	12.04	15.19
1.40	ARL	3.30	3.96	4.58	5.03	5.56	6.34	3.63	4.35	5.06	5.59	6.27	7.22
	MDRL	3.00	4.00	4.00	4.00	5.00	5.00	3.00	4.00	4.00	5.00	5.00	6.00
	SDRL	1.97	2.21	2.55	2.99	3.67	4.58	2.24	2.51	2.93	3.45	4.28	5.45
1.50	ARL	2.11	2.51	2.83	3.01	3.19	3.40	2.31	2.75	3.13	3.33	3.55	3.86
	MDRL	2.00	2.00	3.00	3.00	3.00	3.00	2.00	3.00	3.00	3.00	3.00	3.00
	SDRL	1.05	1.18	1.30	1.42	1.61	1.90	1.19	1.34	1.48	1.64	1.90	2.29
2.00	ARL	1.51	1.77	1.98	2.07	2.12	2.19	1.66	1.94	2.18	2.28	2.37	2.46
	MDRL	1.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	SDRL	0.63	0.71	0.77	0.81	0.87	0.95	0.72	0.81	0.88	0.93	1.01	1.13
3.00	ARL	1.17	1.33	1.48	1.54	1.56	1.58	1.27	1.47	1.64	1.70	1.74	1.76
	MDRL	1.00	1.00	1.00	2.00	2.00	2.00	1.00	1.00	2.00	2.00	2.00	2.00
	SDRL	0.38	0.49	0.54	0.56	0.58	0.60	0.46	0.55	0.59	0.61	0.63	0.67
		$\sigma_V^2 = 0.30$						$\sigma_V^2 = 0.50$					
1.00	ARL	196.90	197.22	195.31	194.03	195.29	199.25	198.41	199.46	201.05	204.12	210.22	218.87
	MDRL	128.00	135.00	135.00	135.00	136.00	138.00	130.00	135.00	139.00	142.00	146.00	152.00
	SDRL	216.47	202.80	195.05	192.80	193.76	199.64	215.96	205.86	201.14	202.38	209.38	217.84
1.10	ARL	50.33	61.03	75.28	85.63	95.10	104.86	53.02	64.45	80.04	92.59	104.56	117.84
	MDRL	35.00	43.00	53.00	60.00	67.00	73.00	37.00	46.00	56.00	65.00	73.00	82.00
	SDRL	51.23	59.27	73.00	83.76	93.61	103.40	54.28	62.70	78.11	90.39	103.28	116.63
1.20	ARL	21.17	26.43	34.25	41.70	49.32	57.34	22.66	28.55	37.36	46.00	55.23	65.69
	MDRL	16.00	20.00	25.00	30.00	35.00	40.00	17.00	21.00	27.00	33.00	39.00	46.00
	SDRL	19.26	23.30	31.33	39.52	47.51	55.88	20.68	25.27	34.37	43.59	53.31	64.10
1.30	ARL	7.52	9.19	11.30	13.50	16.23	19.69	8.10	9.96	12.45	15.10	18.54	23.02
	MDRL	6.00	8.00	9.00	10.00	12.00	14.00	7.00	8.00	10.00	12.00	14.00	17.00
	SDRL	5.64	6.48	8.36	10.87	14.00	17.81	6.08	7.04	9.35	12.42	16.17	20.99
1.40	ARL	3.96	4.80	5.61	6.30	7.19	8.37	4.30	5.21	6.16	6.98	8.14	9.77
	MDRL	3.00	4.00	5.00	5.00	6.00	6.00	4.00	5.00	5.00	6.00	7.00	7.00
	SDRL	2.48	2.81	3.29	4.01	5.08	6.48	2.70	3.06	3.67	4.56	5.90	7.79
1.50	ARL	2.52	3.02	3.46	3.72	4.01	4.44	2.72	3.27	3.78	4.09	4.50	5.09
	MDRL	2.00	3.00	3.00	3.00	4.00	4.00	2.00	3.00	3.00	4.00	4.00	4.00
	SDRL	1.32	1.49	1.67	1.87	2.20	2.76	1.43	1.59	1.81	2.07	2.53	3.23
2.00	ARL	1.79	2.13	2.42	2.54	2.65	2.78	1.93	2.31	2.62	2.78	2.93	3.13
	MDRL	2.00	2.00	2.00	2.00	2.00	3.00	2.00	2.00	2.00	3.00	3.00	3.00
	SDRL	0.79	0.89	0.96	1.03	1.15	1.31	0.86	0.95	1.03	1.13	1.27	1.51
3.00	ARL	1.37	1.61	1.81	1.89	1.94	1.98	1.46	1.75	1.98	2.06	2.13	2.19
	MDRL	1.00	2.00	2.00	2.00	2.00	2.00	1.00	2.00	2.00	2.00	2.00	2.00
	SDRL	0.52	0.60	0.63	0.65	0.68	0.74	0.56	0.62	0.65	0.68	0.72	0.80

8.4 Performance comparisons of control charts

In this section, we compare the performances of the proposed EWMA-ODRSS and EWMA-OIDRSS charts with some of the recent EWMA charts when detecting changes in the process mean and in the process dispersion. The performance of each control chart is evaluated in terms of logarithm of out-of-control ARLs.

(i) EWMA-ODRSS and EWMA-OIDRSS mean charts versus EWMA mean charts

In Figure 8.1, we compare the run length performance of the proposed EWMA chart with some existing powerful EWMA charts based on SRS and ORSS schemes, respectively, i.e., classical EWMA, fast initial response based EWMA (FIR-EWMA), Shewhart-EWMA, hybrid EWMA, and EWMA-ORSS charts. Note that the in-control ARL for each EWMA chart is fixed to 500 with $\xi = 0.10, 0.50$. It is worth mentioning that the proposed EWMA-ODRSS chart performs uniformly better than the EWMA charts considered here.

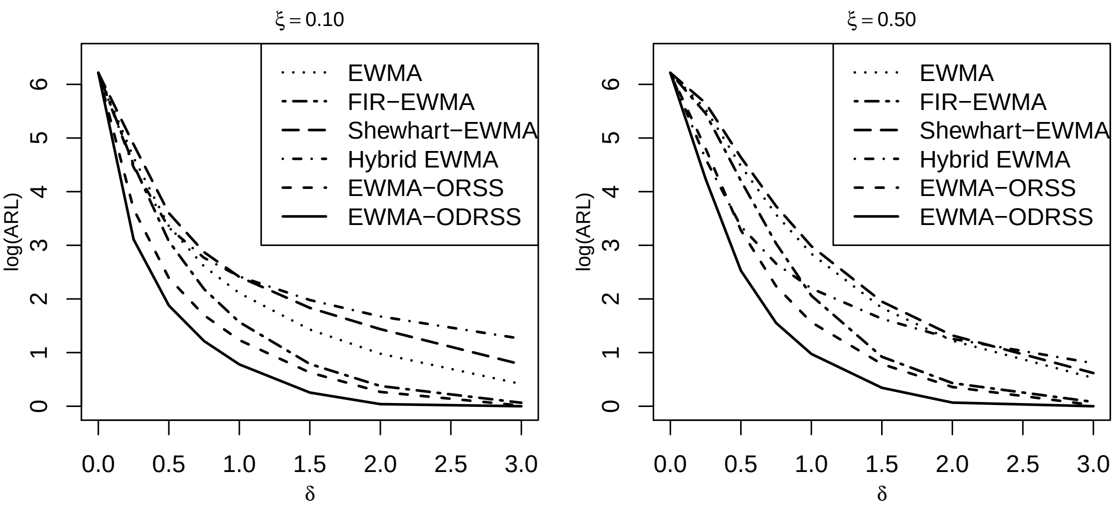

Figure 8.1: Comparison of the EWMA-ODRSS mean chart with some classical and recent EWMA mean charts

Similarly, in Figure 8.2, we compare the proposed EWMA-OIDRSS chart with EWMA charts considered in Figure 8.1. It is clear that the proposed EWMA-OIDRSS chart has better run length performance than the existing EWMA charts. However, for large shifts, the proposed EWMA chart is less efficient than the FIR-EWMA chart when $\sigma_V^2 = 0.50$. Moreover, for all values of δ, the EWMA-OIDRSS chart is able to perform substantially better than the EWMA-OIRSS chart.

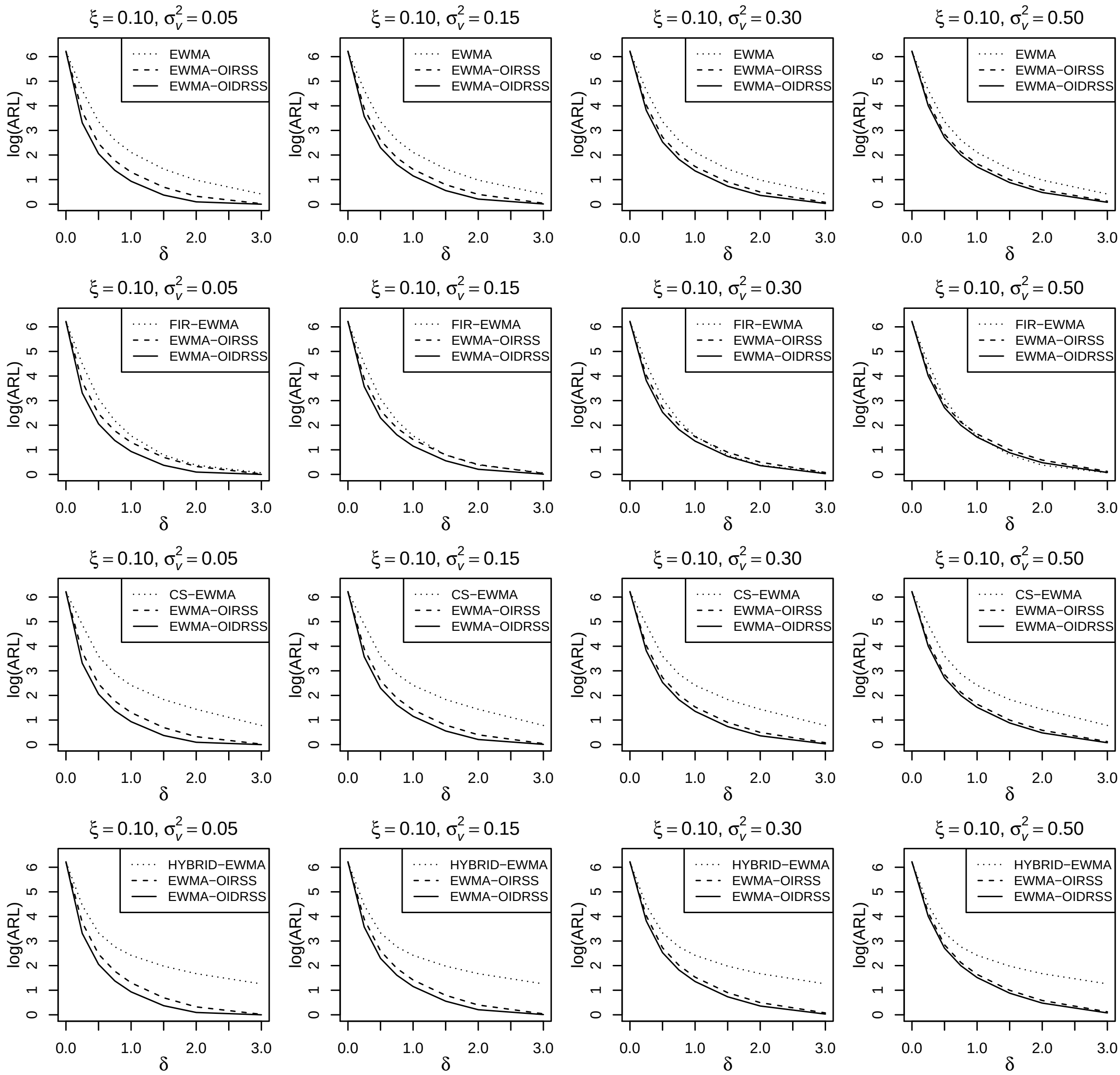

Figure 8.2: Comparison of the EWMA-OIDRSS mean chart with some classical and recent EWMA mean charts

(ii) EWMA-ODRSS mean chart versus CS-EWMA-RSS, CS-EWMA-MRSS and EWMA-ORSS mean charts

Recently, Abujiya et al. (2013a) proposed combined Shewhart-EWMA charts based on RSS and MRSS schemes for monitoring the process mean and named them CS-EWMA-RSS and CS-EWMA-MRSS charts. They showed that these charts perform better than many other mean charts based on SRS. In Figure 8.3, we compare the proposed EWMA-ODRSS chart with the CS-EWMA-RSS and CS-EWMA-MRSS mean charts for different values of ξ. Figure 8.3 demonstrates that the EWMA-ODRSS is more sensitive than the EWMA charts based on RSS, MRSS and ORSS methods.

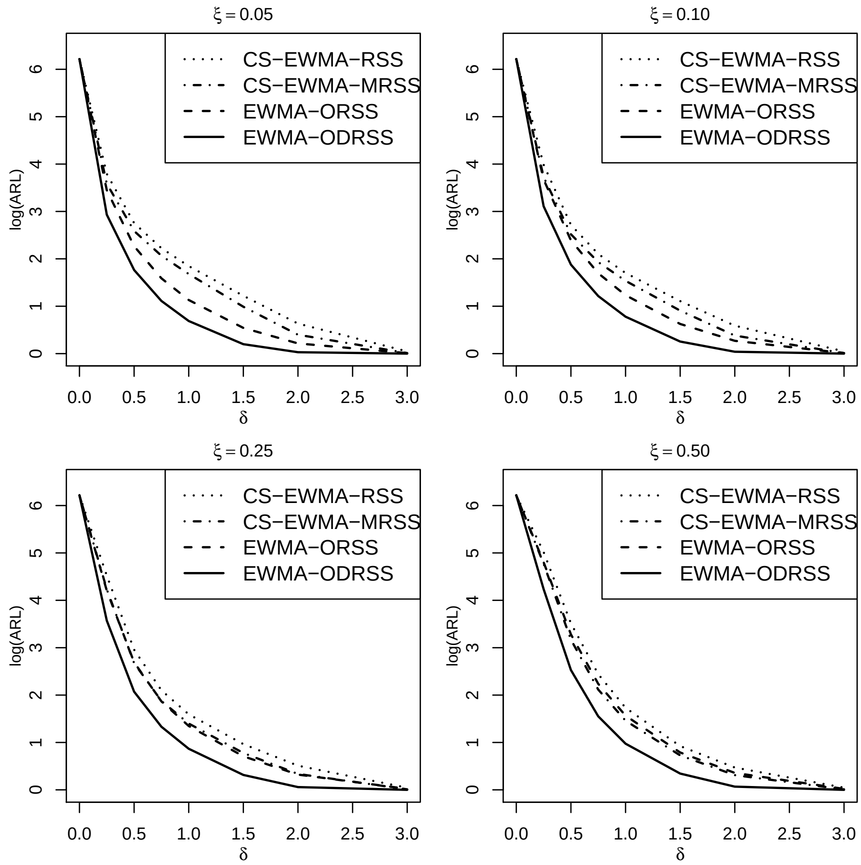

Figure 8.3: Comparison of the EWMA-OIDRSS mean chart versus combined Shewhart-EWMA-RSS, combined Shewhart-EWMA-MRSS and EWMA-ORSS mean charts

(iii) EWMA-ODRSS dispersion chart versus EWMA dispersion charts

Crowder and Hamilton (1992) applied the logarithmic transformation to S^2 based on SRS and suggested an EWMA control chart for monitoring increases in the process standard deviation. Later on, their work was extended by Shu and Jiang (2008), and they proposed another EWMA chart for monitoring the process dispersion. For simplicity, we denote the EWMA charts suggested by Crowder and Hamilton (1992) and Shu and Jiang (2008) by CH-EWMA and SJ-EWMA charts, respectively. Huwang et al. (2010) suggested new EWMA charts for monitoring the process dispersion and named them HHW1-EWMA and HHW2-EWMA. Recently, Haq et al. (2013a) proposed an improved EWMA chart based on ORSS (EWMA-ORSS) for monitoring the process dispersion. For a fair comparison of these dispersion charts, in Figure 8.4, we compare the proposed EWMA-ODRSS dispersion chart with these EWMA dispersion charts for different values of ξ and τ. For each chart, the in-control ARL is fixed to 200. It is notable from Figure 8.4 that the proposed EWMA-ODRSS dispersion chart outperforms all EWMA dispersion charts considered here.

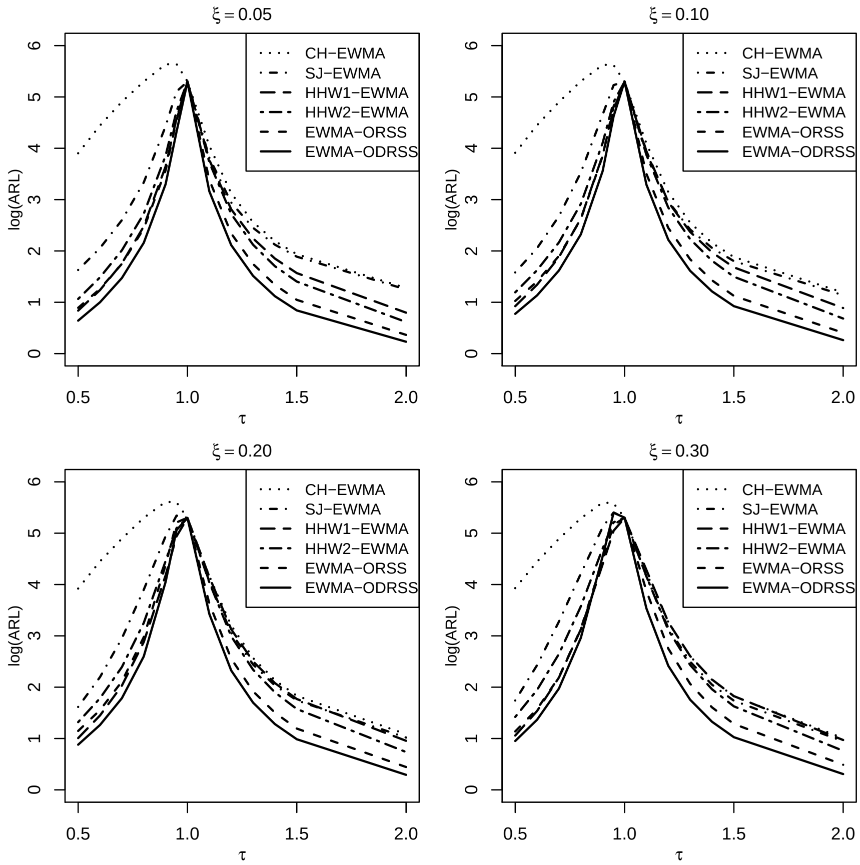

Figure 8.4: Comparison of the two-sided EWMA-ODRSS dispersion chart versus two-sided EWMA dispersion charts

Similarly, in Figure 8.5, we compare the performances of the one-sided EWMA charts for detecting increases in the process dispersion. It is worth mentioning that the the EWMA-ODRSS chart performs uniformly better than its existing counterparts. Moreover, in Figure 8.6, we also compare the performances of the one-sided EWMA-OIDRSS and EWMA-OIRSS charts with the existing one-sided EWMA dispersion charts. Figure 8.6 shows that, even under imperfect rankings, the proposed one-sided EWMA-OIDRSS dispersion chart has better run length performance than that of its counterparts.

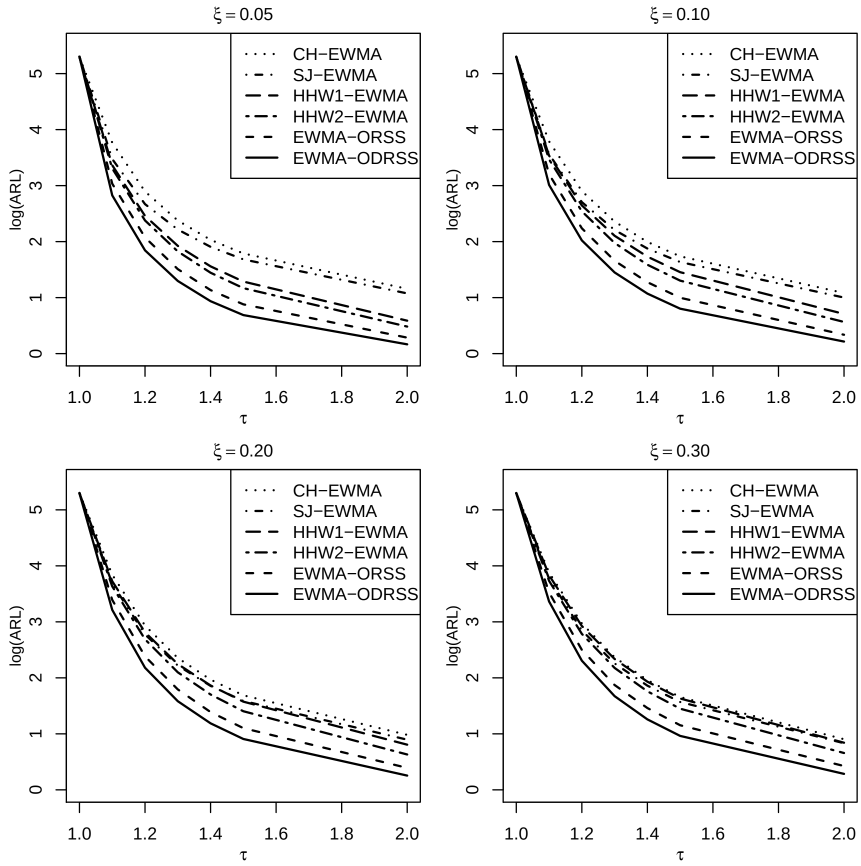

Figure 8.5: Comparison of the one-sided EWMA-ODRSS dispersion chart versus one-sided EWMA dispersion charts

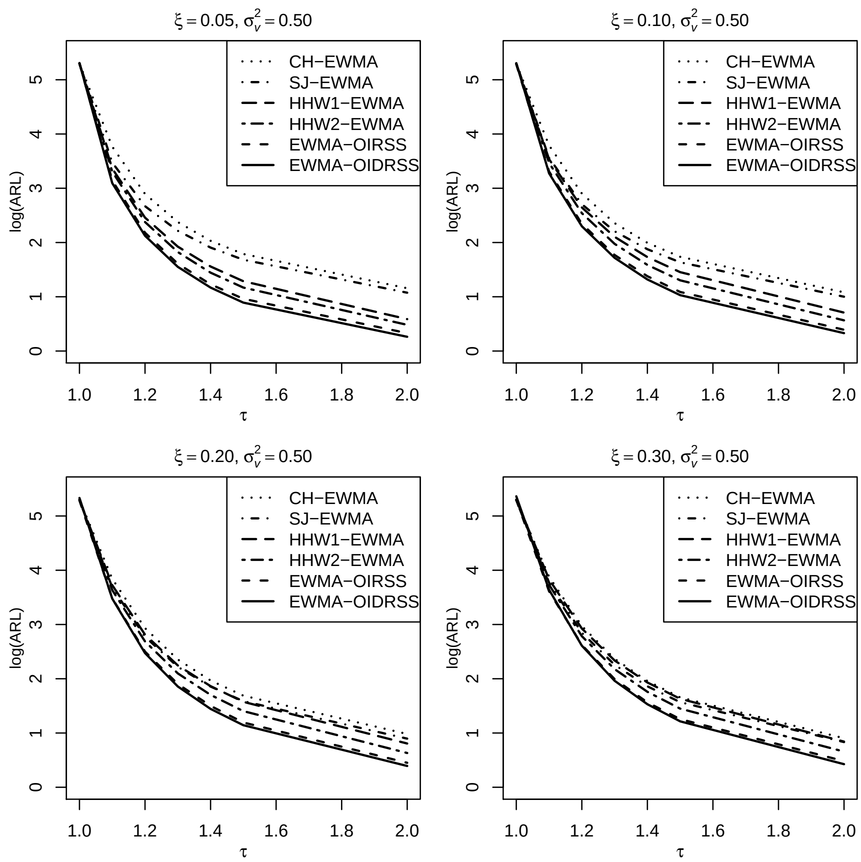

Figure 8.6: Comparison of the one-sided EWMA-OIDRSS dispersion chart versus one-sided EWMA dispersion charts

8.5 Illustrative examples

Many authors provide illuminating examples for a better understanding of the quality control schemes. These examples include real or simulated data sets. Following the works of Abbas et al. (2013, 2011), Riaz et al. (2011) and Haq (2013), in this section, we provide examples to show how the proposed EWMA charts can be easily implemented in real-life practical situations.

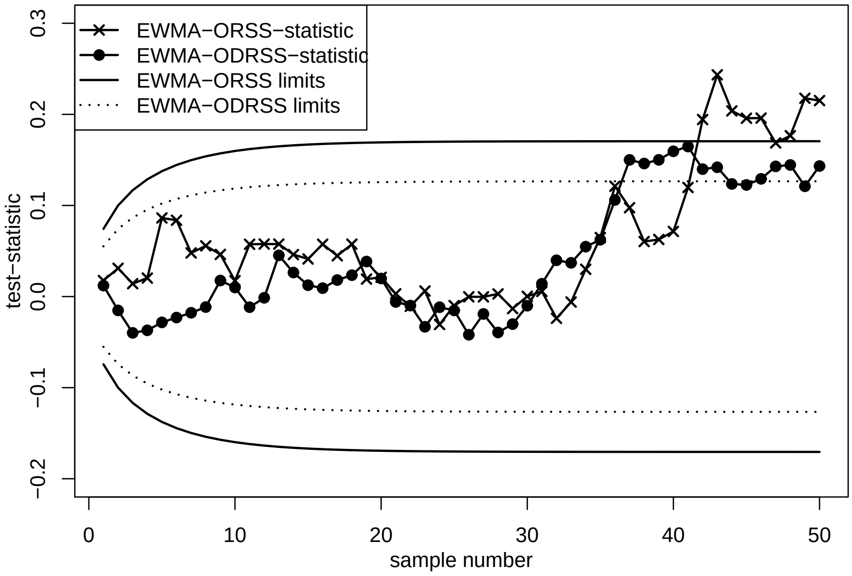

Figure 8.7: EWMA-ODRSS and EWMA-ORSS mean charts for simulated data

Recently, Haq et al. (2013a) showed that the EWMA-ORSS charts are more powerful than the existing EWMA charts in detecting overall changes in the process mean and in the process dispersion. Therefore, for brevity of discussion, we compare the proposed EWMA-ODRSS charts with the EWMA-ORSS charts. Assume that, at time t, the underlying quality characteristic is normally distributed with mean zero and variance unity, i.e., $Y_t \sim N(0, 1)$ for $t \geq 1$. For a fair comparison of both EWMA-ODRSS and EWMA-ORSS mean charts, the in-control ARLs of these charts are fixed to 500 with $\xi = 0.10$. We generate 30 samples, each of size 5, from a standard normal distribution under both ORSS and ODRSS schemes, i.e., $Y_t \sim N(0, 1)$ for $t \leq 30$. Now suppose that, when $t > 30$, the underlying process gets out-of-control due to an unknown shift in the underlying process mean. In order to capture this situation, we again generate 20 samples, each of size 5, under both sampling schemes, from a normal distribution with mean 0.2 and variance unity, i.e., $Y_t \sim N(0.2, 1)$ for $t > 30$. Then, we apply the proposed EWMA-ODRSS and EWMA-ORSS mean charts on the generated samples. The values of the plotting-statistics and control limits of both EWMA mean charts are displayed in Figure 8.7. Figure 8.7 shows that the underlying process remains in-control when $t \leq 30$. However, when $t > 30$, both EWMA charts are showing out-of-control signals. The proposed EWMA-ORSS mean chart detects an out-of-control signal at the 37th sample, whereas the EWMA-ORSS mean chart detects an out-of-control signal at the 46th sample for this data set.

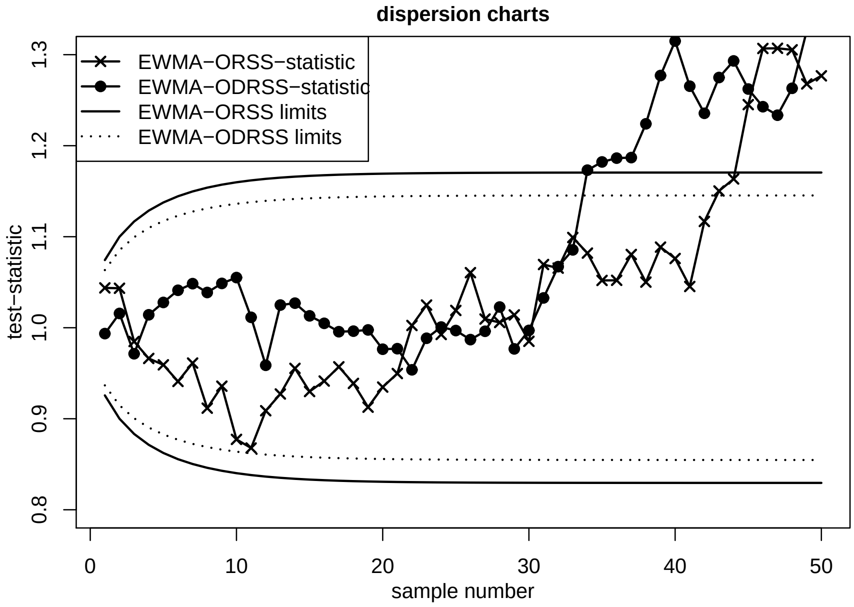

Figure 8.8: EWMA-ODRSS and EWMA-ORSS dispersion charts for simulated data

Similarly, we also compare the performances of the EWMA-ODRSS and EWMA-ORSS dispersion charts. Assume that the underlying process remains in control state when $t \leq 30$ and gets out-of-control when $t > 30$. For this purpose, we first generate 30 samples, each of size 5, from a standard normal distribution under both sampling schemes. Next we generate 20 samples, each of size 5, from a normal distribution with mean zero and variance 1.69, i.e., $Y_t \sim N(0, 1.69)$ for $t > 30$. The in-control ARLs of both EWMA dispersion charts are fixed to 200 with $\xi = 0.10$. Based on these 50 samples, under both sampling schemes, the test-statistics and control limits of both EWMA dispersion charts calculated and displayed in Figure 8.8. It is clear from Figure 8.8 that both charts are showing out-of-control signals when $t > 30$. It is interesting to note that the proposed EWMA-ODRSS chart detects an upward shift in the process dispersion at the 34th sample, while the EWMA-ORSS chart detects the same shift at the 46th sample.

8.6 Conclusion

In this chapter, we proposed some improved EWMA control charts for monitoring changes in the process mean and in the process dispersion. These EWMA control charts are based on the BLUEs-ODRSS and BLUEs-OIDRSS obtained under ODRSS and OIDRSS schemes, respectively. Extensive Monte Carlo simulations have been used to estimate the run length characteristics of the proposed EWMA charts. In order to evaluate the detection abilities of the proposed EWMA charts, we compared their run length performances with some of the recently proposed EWMA charts. It is worth mentioning that the EWMA-ODRSS and EWMA-OIDRSS charts perform uniformly better than the EWMA-ORSS and EWMA-OIRSS charts when detecting overall changes in the process mean and in the process dispersion. Moreover, these charts are also able to perform substantially better than their counterparts based on SRS, RSS, and MRSS schemes. Finally, we considered some illuminating examples to explain the implementation of the proposed EWMA-ODRSS charts.

Chapter 9

New Exponentially Weighted Moving Average Control Charts for Monitoring Process Dispersion

This chapter appeared in:

Haq, A., Brown, J., Moltchanova, E., 2013, New exponentially weighted moving average control charts for monitoring process dispersion, Quality and Reliability Engineering International, Early view, DOI: 10.1002/qre.1553.

Exponentially weighted moving average (EWMA) control charts have received considerable attention for detecting small changes in the process mean or the process variability. Several EWMA control charts are constructed using logarithmic and normalizing transformations on unbiased sample variance for monitoring changes in the process dispersion. In this chapter, we propose new EWMA control charts for monitoring process dispersion based on the best linear unbiased absolute estimators obtained under simple random sampling (SRS) and ranked set sampling (RSS) schemes, named EWMA-SRS and EWMA-RSS control charts. The performance of the proposed EWMA control charts is evaluated in terms of average run length and standard deviation of run length, estimated by using Monte Carlo simulations. The proposed EWMA control charts are then compared with their existing counterparts for detecting increases and decreases in the process dispersion. It turns out that the EWMA-RSS control chart performs uniformly better than its analogues for detecting overall changes in process dispersion. Moreover, the EWMA-SRS chart significantly outperforms the existing EWMA charts for detecting increases in process variability. A real data set is also used to explain the construction and operations of the proposed EWMA control charts.

9.1 Introduction

Statistical quality control charts are well-known process monitoring tools, primarily used to track the unusual variations in industrial processes. These charts include location and dispersion control charts. The location charts are used to monitor changes in the process mean level whereas dispersion charts identify changes in process dispersion. In practice, it is vital to monitor changes in the process dispersion rather than the mean, because an increase in process dispersion leads to an increase in the number of defective items while a decrease in the process variance implies an improvement in the production process. The identification and monitoring of special cause of variations in the manufacturing processes are fundamental features of statistical process control (SPC) that help in improving the process productivity and the quality of products.

In order to detect the infrequent changes in the process dispersion, rational subgrouping is often used. The efficient measures of dispersion, such as the unbiased sample variance S^2, sample range R and many others are then computed from each subgroup. Then, it is customary to apply the classical Shewhart, cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts on these subgroup statistics for monitoring the process variance. In the last decades, dispersion control charts have gained a great deal of attention. Therefore, the literature on these control charts is enormous and growing at a fast pace. Roberts (1959) was the first one to introduce the EWMA control chart for monitoring process mean. The CUSUM control chart based on the sample range for monitoring the standard deviation of a normally distributed process was suggested by Page (1954). It is clear that when the underlying process is normally distributed, then S^2 is a chi-square random variable. Therefore, when a control chart is constructed based on S^2, it is difficult to obtain an unbiased average run length (ARL) for that chart. Here, ARL is the average number of observations or subgroups that are required to issue a particular size shift in the process location or dispersion or both. An ARL is said to be unbiased if there does not exist any out-of-control ARL greater than the in-control ARL. Crowder and Hamilton (1992) applied logarithmic transformation to S^2, i.e., $\ln(S^2/\sigma_0^2)$, and suggested one-sided EWMA control charts based on $\log(S^2)$ for monitoring increases in the process standard deviation. Here, σ_0^2 is the in-control process variance. The performance comparison of the CUSUM control charts based on S^2 and $\ln(S^2)$ was done by Chang and Gan (1995). They showed that S^2-CUSUM chart is partially better than the $\ln(S^2)$-CUSUM chart when detecting an increase in the process dispersion. However, for monitoring overall changes in the process dispersion, the out-of-control ARLs of the $\ln(S^2)$-CUSUM control chart are more likely to be unbiased than the corresponding ARLs of the S^2-CUSUM chart. Acosta-Mejia et al. (1999) made a comprehensive comparison of the dispersion control charts. They suggested new CUSUM control charts based on normalizing transformation and a likelihood ratio test for monitoring increases and decreases in process dispersion. Castagliola (2005) applied a three-parameter logarithmic transformation to S^2 and suggested and improved S^2-EWMA control chart for monitoring changes in process variance. Shu and Jiang (2008) suggested a new EWMA control chart for monitoring process dispersion by truncating the distribution of $\ln(S^2/\sigma_0^2)$ to its in-control approximated mean whenever it is less than its approximated mean. Recently, Huwang et al. (2010) suggested some new EWMA control charts for monitoring process dispersion by applying some normalizing transformations to S^2/σ_0^2 and $(n-1)S^2/\sigma_0^2$, where n is the sample size. They showed that their proposed EWMA control charts are better than the EWMA charts suggested by Crowder and Hamilton (1992) and Shu and Jiang (2008). For some recent literature review and advancements related to dispersion control charts, see Reynolds Jr and Stoumbos (2006), Maravelakis and Castagliola (2009), Riaz (2008a), Abbasi and Miller (2012), Abbas et al. (2013a) and references therein.

The ranked set sampling (RSS) was first suggested by McIntyre (1952) for estimating mean pasture and forage yields. This scheme now has many applications in ecological and environmental studies (cf. Dell and Clutter, 1972; Al-Saleh and Zheng, 2002), reliability theory (cf. Kvam and Samaniego, 1994), medical studies (cf. Samawi and Al-Sagheer, 2001) and quality control (cf. Abujiya et al., 2013a; Al-Omari and Haq, 2012; Haq, 2014; Jafari Jozani and Mirkamali, 2011, and references therein). RSS scheme is useful when measurements of interest are expensive or time-consuming, but it is easy to rank a small set of selected units visually with respect to the study variable or by any correlated variable (cf. Stokes, 1977). Takahasi and Wakimoto (1968) were the first to develop the statistical theory of RSS. They showed that, under perfect ranking, the sample mean based on RSS is an unbiased estimator of the population mean, and at the same time, it is more efficient than the sample mean based on simple random sampling (SRS). Dell and Clutter (1972) examined the effect of imperfect ranking on the efficiency of RSS-based mean estimator. They showed that even under imperfect RSS (IRSS), the RSS mean estimator remain unbiased, and it is better than the SRS mean estimator, but ranking should be better than random ordering. The RSS-based control chart for monitoring process mean was first suggested by Salazar and Sinha (1997). Muttlak and Al-Sabah (2003) extended their work and suggested some improved Shewhart-type control charts for monitoring process mean based on RSS, median RSS and extreme RSS methods. They showed that the RSS-based control charts detect random shift in the process location substantially quicker than the Shewhart control chart based on SRS. The performance of RSS schemes can be increased by using double RSS (DRSS) schemes. Using this fact, Abujiya and Muttlak (2004) suggested Shewhart-type control charts for monitoring process mean based on DRSS methods. They proved that the DRSS-based control charts are better than the control charts with RSS. Al-Omari and Haq (2012) suggested Shewhart-type control charts for monitoring process mean based on some efficient DRSS schemes. Abujiya et al. (2013a) suggested Shewhart-EWMA control charts for detecting changes in process mean based on RSS and MRSS schemes. Recently, Haq (2014) proposed an improved mean deviation-based EWMA control chart for monitoring process dispersion under RSS. For more details about RSS-based control charts, see Abujiya et al. (2014), Mehmood et al. (2013) and references cited therein.

Rao et al. (1991) and Rosaiah et al. (1991) derived the best linear unbiased estimator (BLUE) for the scale parameter using the absolute value of the order statistics obtained under SRS. They showed that, when estimating the scale parameter, the variance of the BLUE obtained from absolute values of order statistics is more precise than the BLUE obtained without taking the absolute value. Later on, Zheng and Al-Saleh (2003) extended the same work by using RSS and derived the BLUE of the scale parameter by using the absolute value of the order statistics coming from RSS method. They named this estimator as best linear unbiased absolute estimator (BLUAE) and showed that the BLUAE based on RSS is more efficient than the BLUAE with SRS.

In this chapter, we propose new EWMA control charts for monitoring overall changes in the variance of a normally distributed process. The proposed EWMA control charts are based on BLUAEs obtained under SRS, RSS and IRSS schemes, named EWMA-SRS, EWMA-RSS and EWMA-IRSS charts, respectively. We use Monte Carlo simulations to estimate the run length characteristics of these control charts. The proposed EWMA control charts are then compared with their counterparts. It is noteworthy that the proposed EWMA control charts outperform their analogues for detecting increases and decreases in process dispersion.

The rest of the paper is as follows: In Section 9.2, we provide a brief introduction related to the existing EWMA control charts. Section 9.3 contains the details about the proposed EWMA control charts. Section 9.4 provides a comprehensive comparison of the EWMA control charts. A real data example is given in Section 9.5, and Section 9.6 provides the concluding remarks.

9.2 Dispersion control charts available in literature

In this section, we provide a brief introduction about some EWMA control charts that were designed to monitor the process dispersion.

Let $Y_{1,t}, Y_{2,t}, ..., Y_{n,t}$ be a random sample of size n, at time t, from a normally distributed process with mean μ_t and variance σ_t^2, i.e., $Y_{i,t} \sim N(\mu_t, \sigma_t^2)$, for $i = 1, 2, ..., n$, $t = 1, 2, ...$ Here, our objective is to monitor the changes in process dispersion. It is assumed that the process remains in-control when $t < \tau$ with variance $\sigma_t^2 = \sigma_0^2$, and the process gets out-of-control when $t \geq \tau$ with variance $\sigma_t^2 \neq \sigma_0^2$. Let $\delta_t = \sigma_t/\sigma_0$, where δ_t represents the amount of shift in the nominal process standard deviation σ_0. Let $S_t^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_{i,t} - \bar{Y})^2$ be the unbiased sample variance based on $Y_{1,t}, Y_{2,t}, ..., Y_{n,t}$, where $\bar{Y}_t$ is the sample mean of the tth subgroup of size n. For brevity, without loss of generality, we set $\mu_t = 0$.

(i) Crowder and Hamilton (1992) EWMA control chart

In order to monitor the changes in process dispersion, Crowder and Hamilton (1992) applied natural logarithmic (ln) transformation (suggested by Box, 1954) to S_t^2, i.e., $\ln(S_t^2/\sigma_0^2)$. Let $A_t = \ln(S_t^2/\sigma_0^2)$, where S_t^2/σ_0^2 is a gamma random variable with shape parameter $(n-1)/2$ and scale parameter $2\delta_t^2/(n-1)$, i.e., $S_t^2/\sigma_0^2 \sim \Gamma\left((n-1)/2, 2\delta_t^2/(n-1)\right)$. The resulting distribution of A_t is log-gamma distribution, which can be approximated by a normal distribution (cf. Lawless, 2003), i.e., $A_t \approx N(\mu_A, \sigma_A^2)$, where

$$\mu_A = \ln(\delta_t^2) - \frac{1}{n-1} - \frac{1}{3(n-1)^2} + \frac{2}{15(n-1)^4} \quad \text{and} \quad \sigma_A^2 = \frac{2}{n-1} + \frac{2}{(n-1)^2} + \frac{4}{3(n-1)^3} - \frac{16}{15(n-1)^5}.$$

Following this transformation, Crowder and Hamilton (1992) proposed an EWMA chart for monitoring the process dispersion, named CH-EWMA chart. The upper and lower plotting-statistics of the CH-EWMA chart are

$$\begin{aligned} C_{U,t} &= \max[0, \lambda A_t + (1-\lambda)C_{U,t-1}], \quad C_{U,0} = 0, \\ C_{L,t} &= \min[0, \lambda A_t + (1-\lambda)C_{L,t-1}], \quad C_{L,0} = 0, \end{aligned}$$

respectively, where $\lambda(0 < \lambda \leq 1)$ is a smoothing constant. In order to detect an increase in the process dispersion, the CH-EWMA chart triggers an out-of-control signal if $C_{U,t} > b_U\sqrt{\lambda(2-\lambda)^{-1}}$. The value of b_U is selected such that the in-control ARL of the CH-EWMA chart reaches to a specific level. Similarly, when detecting a decrease in the process dispersion, the CH-EWMA chart generates an out-of-control signal if $C_{L,t} < -b_L\sqrt{\lambda(2-\lambda)^{-1}}$. Here, b_L is chosen such that the in-control ARL of the CH-EWMA chart reaches to a particular level. For more details, see Crowder and Hamilton (1992).

(ii) Shu and Jiang (2008) EWMA control chart

Shu and Jiang (2008) proposed another EWMA chart for monitoring process dispersion by truncating the distribution of the transformation $\ln(S_t^2/\sigma_0^2)$ to its in-control approximated mean, i.e., $\mu_{A|\sigma_t=\sigma_0}$, whenever it becomes less that $\mu_{A|\sigma_t=\sigma_0}$. Recall that A_t is approximately a normal random variable with its approximated in-control mean $\mu_{A|\sigma_t=\sigma_0}$. Then, define the standardized quantity $Z_t = \frac{A_t - \mu_{A|\sigma_t=\sigma_0}}{\sigma_A}$. It is clear that σ_A is a function of n only; thus, the changes in σ_t^2 will only affect the approximated in-control mean of A_t. Let $Z_t^+ = \max(0, Z_t)$, Barr and Sherrill (1999) showed that when $Z_t \sim N(0,1)$, then $\mathrm{E}(Z_t^+) = 1/\sqrt{2\pi}$ and $\mathrm{Var}(Z_t^+) = \sigma^2_{Z_t^+} = 1/2 - 1/(2\pi)$. Using this fact, Shu and Jiang (2008) suggested an EWMA control chart for monitoring process variability. We name this chart as SJ-EWMA control chart. The plotting-statistic of the SJ-EWMA chart when detecting an increase in the process dispersion, at time t, is given by

$$D_t^+ = \lambda\left(Z_t^+ - 1/\sqrt{2\pi}\right) + (1-\lambda)D_{t-1}^+, \quad D_0^+ = 0,\ 0 < \lambda \leq 1.$$

This EWMA chart gives an out-of-control signal if $D_t^+ > d_U^+\sigma_{Z_t^+}\sqrt{\lambda(2-\lambda)^{-1}}$, where d_U^+ is selected such that the in-control ARL of SJ-EWMA chart reaches to a particular level. Similarly, in order to detect a decrease in the process dispersion, the plotting-statistic of the SJ-EWMA chart is defined by

$$D_t^- = \lambda\left(Z_t^- + 1/\sqrt{2\pi}\right) + (1-\lambda)D_{t-1}^-, \quad D_0^- = 0,$$

where $Z_t^- = \min(0, Z_t)$. The SJ-EWMA chart triggers an out-of-control signal as soon as $D_t^- < -d_L^-\sigma_{Z_t^-}\sqrt{\lambda(2-\lambda)^{-1}}$. Here, d_L^- is chosen to achieve the desired in-control ARL for the SJ-EWMA chart. Note that due to the symmetry of standard normal distribution, we have $\sigma_{Z_t^+} = \sigma_{Z_t^-}$. For more details, see Shu and Jiang (2008).

(iii) Huwang et al. (2010) EWMA control charts

Recently, Huwang et al. (2010) suggested some improved EWMA control charts for detecting increases and decreases in the process dispersion. As aforementioned, $S_t^2/\sigma_0^2 \sim \Gamma\left((n-1)/2, 2\delta_t^2/(n-1)\right)$. Then, the EWMA-statistic based on S_t^2/σ_0^2, at time t, is given by

$$E_t = \lambda(S_t^2/\sigma_0^2) + (1-\lambda)E_{t-1}, \quad E_0 = 1.$$

They showed that $E_t^* = E_t - (1-\lambda)^t E_0$ has an approximated gamma distribution, i.e., $E_t^* \approx \Gamma(\phi_1, \phi_2)$, where $\phi_1 = \frac{(n-1)(2-\lambda)\{1-(1-\lambda)^t\}^2}{2\lambda\{1-(1-\lambda)^{2t}\}}$ and $\phi_2 = \frac{2\lambda\{1-(1-\lambda)^{2t}\}}{(n-1)(2-\lambda)\{1-(1-\lambda)^t\}}$. Similarly, $\ln(E_t^*)$ is a log-gamma random variable, and it can be approximated by a normal random variable, i.e., $\ln(E_t^*) \approx (\mu_E^*, \sigma_E^{*2})$, where $\mu_E^* = \ln(\phi_1\phi_2) - \frac{1}{2\phi_1} - \frac{1}{12\phi_1^2} + \frac{1}{120\phi_1^4}$ and $\sigma_E^{*2} = \frac{1}{\phi_1} + \frac{1}{2\phi_1^2} + \frac{1}{6\phi_1^3} - \frac{1}{30\phi_1^5}$. Define the standardized quantity $E_t^{**} = \frac{E_t^* - \mu_E^*}{\sigma_E^*} \approx N(0,1)$. We name the EWMA chart based on E_t^{**} as HHW1-EWMA chart.

The control limits of the HHW1-EWMA chart are

$$UCL_t = g, \quad CL_t = 0, \quad LCL_t = -g,$$

where g is the upper control limit. Here, UCL_t, CL_t and LCL_t are the upper, center and lower control limits, at time t, respectively. Similarly, it is easy to derive the one-sided (upper and lower) versions of the HHW1-EWMA control chart when detecting increases or decreases in the process dispersion.

Huwang et al. (2010) also suggested another EWMA control chart by transforming S_t^2/σ_0^2 to an exact normal random variable. It is easy to show that $(n-1)S_t^2/\sigma_0^2$ is a chi-square random variable with $n-1$ as degrees of freedom. Let $G_{n-1}(\cdot)$ be the cumulative distribution function (CDF) of the chis-square random variable, i.e., $(n-1)S_t^2/\sigma_0^2$. Then, apply the CDF transformation on $(n-1)S_t^2/\sigma_0^2$, i.e., $\vartheta_t = G_{n-1}\left((n-1)S_t^2/\sigma_0^2\right)$. The resulting distribution of ϑ_t is uniform, i.e., $\vartheta_t \sim U(0,1)$. Let $\zeta_t = \phi^{-1}(\vartheta_t)$, which is a standard normal random variable, i.e., $\zeta_t \sim N(0,1)$, where $\phi(\cdot)$ is the CDF of the standard normal distribution. The plotting-statistic of the EWMA chart based on ζ_t is given by

$$H_t = \lambda\zeta_t + (1-\lambda)H_{t-1}, \quad H_0 = 0.$$

Here, H_t is also a normal random variable with mean zero and variance σ_H^2, i.e., $H_t \sim N(0, \sigma_{H_t}^2)$, where $\sigma_{H_t}^2 = \frac{\lambda\{1-(1-\lambda)^{2t}\}}{(2-\lambda)}$. Define the standardized quantity $H_t^* = \frac{H_t}{\sigma_{H_t}}$, i.e., $H_t^* \sim N(0,1)$. The EWMA chart based on H_t^* is named as HHW2-EWMA chart. Let h and $-h$ be the upper and lower control limits of the HHW2-EWMA chart, respectively. The HHW2-EWMA charts gives an out-of-control signal when either $H_t^* > h$ or $H_t^* < -h$. For more details, see Huwang et al. (2010).

Note that all above EWMA control charts are based on SRS method.

9.3 Proposed EWMA control charts

In this section, we propose new EWMA control charts for monitoring the process dispersion based on SRS, RSS and IRSS schemes, named EWMA-SRS, EWMA-RSS and EWMA-IRSS charts.

9.3.1 New EWMA-SRS chart

As aforementioned, recall that $Y_{i,t} \sim N(0, \sigma_t^2)$, for $i = 1, 2, ..., n$, at time t. Let $Y_{i,t}^* = Y_{i,t} - \mu_t$, then $Y_{i,t}^* \sim N(0, \sigma_t^2)$, which belongs to a scale family, symmetric about zero, with scale parameter σ_t. Define $Z_{i,t} = \frac{Y_{i,t}^*}{\sigma_t}$ be the standardized variate with probability density function (PDF) independent of σ_t, i.e., $Z_{i,t} \sim N(0, 1)$, at time t. Let $\boldsymbol{Y}_{\mathrm{SRS},t} = (Y_{(1:n),t}^*, Y_{(2:n),t}^*, ..., Y_{(n:n),t}^*)'$ be $n \times 1$ vector of observed order statistics obtained from a random sample of size n, i.e., $Y_{1,t}^*, Y_{2,t}^*, ..., Y_{n,t}^*$, and let $\boldsymbol{Z}_t = (Z_{(1:n),t}, Z_{(2:n),t}, ..., Z_{(n:n),t})'$ be the corresponding $n \times 1$ vector of standardized order statistics. Let $\boldsymbol{\mu}_t = (\mu_{(1:n),t}, \mu_{(2:n),t}, ..., \mu_{(n:n),t})'$ be the mean vector of $\boldsymbol{Z}_t$, where $\mu_{(i:n),t} = E(Z_{(i:n),t})$, for $i = 1, 2, ..., n$, and $\boldsymbol{\Sigma}_t = \{\sigma_{(i,j:n),t}\}$ be the covariance matrix of $\boldsymbol{Z}_t$, where $\sigma_{(i,j:n),t} = \mathrm{Cov}(Z_{(i:n),t}, Z_{(j:n),t})$, for $i, j = 1, 2, ..., n$. Then, following Lloyd (1952), at time t, the BLUE of σ_t, say $\hat{\sigma}_{\mathrm{BLUE},t}^{\mathrm{SRS}}$, and its variance are, respectively, given by

$$\hat{\sigma}_{\mathrm{BLUE},t}^{\mathrm{SRS}} = \left(\boldsymbol{\mu}_t' \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}_t\right)^{-1} \boldsymbol{\mu}_t' \boldsymbol{\Sigma}_t^{-1} \boldsymbol{Y}_{\mathrm{SRS},t} \quad \text{and} \quad \mathrm{Var}(\hat{\sigma}_{\mathrm{BLUE},t}^{\mathrm{SRS}}) = \sigma_t^2 \left(\boldsymbol{\mu}_t' \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}_t\right)^{-1}.$$

Rao et al. (1991) and Rosaiah et al. (1991) derived the BLUE for the scale parameter of a symmetric distribution using the absolute value of the order statistics obtained from a simple random sample of size n, named BLUAE. They showed that, for a symmetric distribution, when estimating the scale parameter, the variance of the BLUAE is less than the variance of the BLUE. In order to obtain the BLUAE of σ_t, say $\hat{\sigma}_{\mathrm{BLUAE},t}^{\mathrm{SRS}}$, we take absolute of $\boldsymbol{Y}_{\mathrm{SRS},t}$, i.e., $|\boldsymbol{Y}_{\mathrm{SRS},t}|$. Then, following Lloyd (1952), at time t, the BLUAE of σ_t based on $|\boldsymbol{Y}_{\mathrm{SRS},t}|$ and its variance are, respectively, given by

$$\hat{\sigma}_{\mathrm{BLUAE},t}^{\mathrm{SRS}} = \left(\boldsymbol{\alpha}_t' \boldsymbol{\Pi}_t^{-1} \boldsymbol{\alpha}_t\right)^{-1} \boldsymbol{\alpha}_t' \boldsymbol{\Pi}_t^{-1} |\boldsymbol{Y}_{\mathrm{SRS},t}| \quad \text{and} \quad \mathrm{Var}(\hat{\sigma}_{\mathrm{BLUAE},t}^{\mathrm{SRS}}) = \sigma_t^2 \left(\boldsymbol{\alpha}_t' \boldsymbol{\Pi}_t^{-1} \boldsymbol{\alpha}_t\right)^{-1},$$

where $\boldsymbol{\alpha}_t = (\alpha_{(1:n),t}, \alpha_{(2:n),t}, ..., \alpha_{(n:n),t})'$ is the mean vector of $|\boldsymbol{Z}_t|$, $\alpha_{(i:n),t} = E(|Z_{(i:n),t}|)$, for $i = 1, 2, ..., n$, and $\boldsymbol{\Pi}_t = \{\pi_{(i,j:n),t}\}$ be the covariance matrix of $|\boldsymbol{Z}_t|$, where $\pi_{(i,j:n),t} = \mathrm{Cov}(|Z_{(i:n),t}|, |Z_{(j:n),t}|)$, for $i, j = 1, 2, ..., n$.

Rosaiah et al. (1991) showed that when the underlying distribution is symmetric about zero, then it is possible to express the moments and cross-moments of the standardized absolute order statistics ($|Z_{(i:n),t}|$) in terms of moments and cross-moments of the standardized order statistics in the corresponding folded distribution. Recall that $Y_{i,t}^* \sim N(0, \sigma_t^2)$, for $i = 1, 2, ..., n$, at time t. Here, $Z_{i,t}^* = \frac{1}{\sigma_t}|Y_{i,t}^*|$, having PDF $g(z_t^*) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{z_t^{*2}}{2}\right)$, $z_t^* > 0$. Here, $g(z_t^*)$ is the folded normal or half-normal distribution.

(i) For all positive integer value of ϖ, we have

$$E(|Z_{(i:n),t}|^{\varpi}) = \frac{1}{2^n}\left\{\sum_{w=0}^{i-1} C_{n,w} E\left(Z^{*\varpi}_{(i-w:n-w),t}\right) + \sum_{w=i}^{n} C_{n,w} E(Z^{*\varpi}_{(w-i+1:w),t})\right\}, \tag{9.1}$$

where $C_{n,w} = n!\{w!(m-w)!\}^{-1}$.

(ii) For positive i, j and $1 \leq i < j \leq n$, we have

$$\begin{aligned} E(|Z_{(i:n),t}||Z_{(j:n),t}|) &= \frac{1}{2^n}\left\{\sum_{w=0}^{i-1} C_{n,w} E(Z^{*}_{(i-w,j-w:n-w),t}) + \sum_{w=i}^{j-1} C_{n,w} E(Z^{*}_{(w-i+1:w),t})\right. \\ &\quad \left. E(Z^{*}_{(j-w:n-w),t}) + \sum_{w=j}^{n} C_{n,w} E(Z^{*}_{(w-j+1,w-i+1:w),t})\right\}. \end{aligned} \tag{9.2}$$

Here, $E(Z^{*\varpi}_{(i:n),t})$ is the ϖth moments of the ith order statistic, and $E(Z^{*}_{(i,j:n),t})$ is the product of the ith and jth order statistics $(i < j)$ in a sample of size n drawn from a standardized density $g(z^*_t)$ at time t. Solving (9.1) and (9.2), it is easy to find the values of $\boldsymbol{\alpha}_t$ and $\boldsymbol{\Pi}_t$, which are needed in computing $\hat{\sigma}^{\text{SRS}}_{\text{BLUAE},t}$ and its corresponding variance.

Assume that the underlying process is in-control and let $Y^*_{i,t}$, for $i = 1, 2, ..., n$, be a simple random sample of size n, drawn from a normally distributed process with mean zero and variance σ_0^2 at time t, i.e., $Y^*_{i,t} \sim N(0, \sigma_0^2)$, for $t = 1, 2, ..., \tau$. Let $\hat{\sigma}^{\text{SRS}}_{\text{BLUAE},1}, \hat{\sigma}^{\text{SRS}}_{\text{BLUAE},2}, ..., \hat{\sigma}^{\text{SRS}}_{\text{BLUAE},t}, ...$ be a sequence of independent and identically distributed (IID) random variables and let $\xi (0 < \xi \leq 1)$ be a smoothing constant. Based on this sequence, we define another sequence, say $\{M_t\}$, by using a recurrence formula, given by

$$M_t = \xi \hat{\sigma}^{\text{SRS}}_{\text{BLUAE},t} + (1-\xi) M_{t-1}, \quad 0 < \xi \leq 1,$$

which is an EWMA sequence. We name the control chart based on this plotting-statistic as EWMA-SRS chart. It is easy to show that $E(M_t) = E(\hat{\sigma}^{\text{SRS}}_{\text{BLUAE},t}) = \sigma_0$ for $M_0 = \sigma_0$, and $\text{Var}(M_t) = R(t;\xi)\sigma_0^2(\boldsymbol{\alpha}'_t \boldsymbol{\Pi}_t^{-1} \boldsymbol{\alpha}_t)^{-1}$, where $R(t;\xi) = (\frac{\xi}{2-\xi})\{1-(1-\xi)^{2t}\}$. The control limits of the EWMA-SRS chart, at time t, are given by the following:

$$\begin{aligned} UCL_t &= \sigma_0 + I_2 \sigma_0 \sqrt{R(t;\xi)(\boldsymbol{\alpha}'_t \boldsymbol{\Pi}_t^{-1} \boldsymbol{\alpha}_t)^{-1}}, \\ CL_t &= \sigma_0, \\ LCL_t &= \sigma_0 - I_1 \sigma_0 \sqrt{R(t;\xi)(\boldsymbol{\alpha}'_t \boldsymbol{\Pi}_t^{-1} \boldsymbol{\alpha}_t)^{-1}}, \end{aligned}$$

where (I_1, I_2) is determined such that the desired in-control ARL of the EWMA-SRS chart is achieved. Similar to the aforementioned EWMA control charts in Section 9.2, here EWMA-SRS chart generates an out-of-control signal as soon as $M_t > UCL_t$ or $M_t < LCL_t$. At time t, when $M_t > UCL_t$, then there is a positive shift in the process dispersion or when $M_t < LCL_t$, this shows a negative shift in the process

Table 9.1: Run length characteristics of the two-sided EWMA-SRS control chart

		Symmetric limits				Asymmetric limits			
	$\xi\rightarrow$	0.05	0.10	0.02	0.30	0.05	0.10	0.20	0.30
	$I_1\rightarrow$	2.2729	2.4777	2.6440	2.7259	2.1291	2.3200	2.4100	2.4300
δ_t	$I_2\rightarrow$	2.2729	2.4777	2.6440	2.7259	2.4400	2.6488	2.8760	2.9847
0.50	ARL	2.87	3.31	3.78	4.26	2.58	2.96	3.20	3.37
	SDRL	1.04	1.14	1.30	1.63	0.98	1.06	1.16	1.29
0.55	ARL	3.41	3.96	4.58	5.33	3.07	3.54	3.85	4.12
	SDRL	1.37	1.50	1.78	2.37	1.29	1.40	1.55	1.81
0.60	ARL	4.16	4.85	5.74	6.99	3.73	4.33	4.77	5.20
	SDRL	1.84	2.04	2.54	3.68	1.73	1.90	2.17	2.63
0.65	ARL	5.23	6.14	7.49	9.75	4.67	5.44	6.09	6.89
	SDRL	2.55	2.85	3.79	6.05	2.38	2.64	3.11	4.05
0.70	ARL	6.81	8.04	10.30	14.66	6.04	7.08	8.16	9.61
	SDRL	3.65	4.17	6.09	10.61	3.36	3.80	4.75	6.46
0.75	ARL	9.27	11.15	15.36	24.38	8.23	9.69	11.62	14.41
	SDRL	5.43	6.47	10.55	19.96	5.04	5.78	7.81	11.06
0.80	ARL	13.50	16.65	25.46	44.56	11.85	14.31	18.06	23.30
	SDRL	8.71	11.02	20.19	40.07	7.97	9.60	13.86	19.85
0.85	ARL	21.87	28.20	48.48	90.17	18.97	23.45	31.32	41.68
	SDRL	15.89	21.58	42.73	85.78	14.19	17.90	27.02	38.10
0.90	ARL	42.02	58.05	105.23	190.18	35.28	45.49	62.06	80.69
	SDRL	35.22	51.13	100.14	186.79	29.86	40.09	58.14	77.82
0.95	ARL	106.25	143.95	220.25	301.42	86.61	108.61	133.04	153.58
	SDRL	104.36	141.44	218.27	299.57	85.46	106.17	130.44	151.31
1.00	ARL	200.26	200.87	200.10	200.29	199.89	200.94	200.41	200.28
	SDRL	214.20	205.79	201.33	199.44	213.61	205.66	200.31	200.55
1.10	ARL	33.73	37.72	41.67	44.36	39.63	45.53	56.36	63.96
	SDRL	32.91	36.29	40.52	43.34	37.78	43.20	55.13	62.52
1.20	ARL	12.20	13.49	14.86	15.97	13.80	15.45	18.46	21.02
	SDRL	10.97	11.69	13.30	14.61	12.08	13.34	16.50	19.34
1.30	ARL	6.66	7.31	7.93	8.36	7.39	8.16	9.31	10.35
	SDRL	5.71	6.04	6.56	7.12	6.18	6.63	7.67	8.86
1.40	ARL	4.43	4.85	5.17	5.42	4.88	5.35	5.96	6.37
	SDRL	3.65	3.82	4.07	4.36	3.92	4.19	4.61	5.09
1.50	ARL	3.30	3.58	3.83	3.94	3.59	3.90	4.29	4.52
	SDRL	2.58	2.72	2.86	2.97	2.77	2.92	3.18	3.42
1.60	ARL	2.63	2.85	3.02	3.11	2.84	3.06	3.34	3.49
	SDRL	1.95	2.07	2.15	2.24	2.10	2.21	2.38	2.52
1.70	ARL	2.21	2.37	2.50	2.58	2.36	2.55	2.74	2.86
	SDRL	1.55	1.64	1.72	1.77	1.65	1.77	1.88	1.97
1.80	ARL	1.93	2.06	2.16	2.21	2.05	2.19	2.35	2.44
	SDRL	1.27	1.36	1.41	1.44	1.36	1.46	1.55	1.62
1.90	ARL	1.73	1.84	1.92	1.96	1.82	1.93	2.06	2.14
	SDRL	1.07	1.15	1.20	1.22	1.14	1.22	1.30	1.36
2.00	ARL	1.59	1.67	1.74	1.77	1.66	1.75	1.85	1.91
	SDRL	0.93	0.99	1.03	1.05	0.99	1.05	1.12	1.16
2.50	ARL	1.23	1.27	1.30	1.31	1.26	1.30	1.35	1.37
	SDRL	0.52	0.56	0.59	0.60	0.55	0.59	0.64	0.65
3.00	ARL	1.11	1.12	1.14	1.15	1.12	1.14	1.17	1.18
	SDRL	0.34	0.36	0.39	0.40	0.36	0.39	0.42	0.43

variability.

The performance of a control chart is generally evaluated in terms of the run length properties, i.e., ARL and standard deviation of run length (SDRL). Here, we use Monte Carlo simulations to estimate the run length characteristics of the proposed EWMA-SRS control chart. Using extensive Monte Carlo simulations

Table 9.2: Run length characteristics of the one-sided EWMA-SRS control charts

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	$\xi\rightarrow$	0.05	0.10	0.20	0.30
δ_t	$I_2\rightarrow$	1.9225	2.2200	2.4950	2.6358	$\delta_t\downarrow$ $I_1\rightarrow$	1.8250	2.0488	2.2030	2.2360
1.00	ARL	200.92	199.79	200.20	199.69	1.00	199.93	199.82	200.99	200.91
	SDRL	222.77	207.04	204.44	200.32		218.73	203.58	201.52	200.27
1.10	ARL	23.88	28.32	34.34	38.91	0.95	56.95	68.69	86.09	95.97
	SDRL	24.60	27.57	33.30	37.88		57.98	66.80	84.10	94.28
1.20	ARL	9.28	10.96	12.99	14.58	0.90	24.63	30.94	41.05	49.73
	SDRL	8.90	9.90	11.61	13.41		22.42	27.30	38.00	47.15
1.30	ARL	5.26	6.17	7.11	7.82	0.85	13.77	17.10	22.27	27.31
	SDRL	4.75	5.26	5.90	6.64		11.24	13.53	18.86	24.49
1.40	ARL	3.63	4.19	4.77	5.13	0.80	8.85	10.81	13.64	16.50
	SDRL	3.11	3.40	3.77	4.13		6.57	7.61	10.23	13.59
1.50	ARL	2.77	3.15	3.53	3.76	0.75	6.22	7.55	9.17	10.74
	SDRL	2.20	2.43	2.65	2.84		4.23	4.78	6.12	7.96
1.60	ARL	2.27	2.54	2.82	2.98	0.70	4.63	5.61	6.64	7.54
	SDRL	1.69	1.86	2.02	2.15		2.87	3.23	3.92	4.97
1.70	ARL	1.95	2.16	2.36	2.49	0.65	3.62	4.35	5.06	5.60
	SDRL	1.34	1.48	1.61	1.70		2.04	2.28	2.64	3.22
1.80	ARL	1.73	1.89	2.05	2.15	0.60	2.91	3.50	4.02	4.33
	SDRL	1.11	1.23	1.33	1.40		1.49	1.67	1.89	2.19
1.90	ARL	1.56	1.70	1.83	1.91	0.55	2.41	2.88	3.29	3.48
	SDRL	0.93	1.03	1.13	1.19		1.12	1.25	1.38	1.56
2.00	ARL	1.45	1.56	1.68	1.73	0.50	2.05	2.43	2.75	2.89
	SDRL	0.80	0.89	0.98	1.02		0.87	0.96	1.04	1.14
2.50	ARL	1.17	1.22	1.27	1.29	0.40	1.53	1.82	2.05	2.12
	SDRL	0.45	0.51	0.56	0.58		0.58	0.62	0.64	0.67
3.00	ARL	1.08	1.10	1.13	1.14	0.30	1.17	1.40	1.61	1.66
	SDRL	0.29	0.33	0.37	0.38		0.38	0.50	0.51	0.51

(10^5) from standard normal distribution, we have estimated ARLs and SDRLs of the EWMA-SRS chart for different values of δ_t and are reported in Tables 9.1 and 9.2. In Tables 9.1 and 9.2, we report the run length properties of both two-sided and one-sided EWMA-SRS charts, respectively. For two-sided EWMA-SRS chart, we consider both symmetric and asymmetric control limits and study their effect on the performance of the EWMA-SRS chart. For the EWMA-SRS chart, the assumed values of the smoothing constant ξ are 0.05, 0.10, 0.20 and 0.30. The subgroup size is taken to be five, i.e., $n = 5$.

From Table 9.1, it is clear that when $\delta_t \geq 1$, the ARLs tend to decrease as δ_t increases and vice-versa. However, the ARLs decrease as the value of δ_t decreases when $\delta_t \leq 1$. For a fixed value of δ_t, as the value of ξ increases, the ARLs and SDRLs both tend to increase and vice-versa. Note that when detecting a small decrease in the process dispersion, i.e., $0.9 < \delta_t \leq 1$, ARLs of the EWMA-SRS chart are biased for $\xi = 0.2, 0.3$, under symmetric control limits. This shows that the probability distribution of the BLUAE under SRS is asymmetric. In order to obtain unbiased ARLs for all values of δ_t and ξ, we consider asymmetric control limits for the EWMA-SRS chart. For all values of δ_t and ξ, under asymmetric control limits, ARLs of the EWMA-SRS chart are unbiased, and there is a substantial improvement when detecting decreases in the process dispersion. However, with these control limits, there is an increase in the values of ARLs of the EWMA-SRS chart when $\delta_t \geq 1$ for each value of ξ. Table 9.2 presents the run length characteristics of the one-sided EWMA-SRS chart when detecting either increases or decreases in process dispersion. The proposed EWMA-SRS chart is more sensitive in detecting positive shifts in the process dispersion as compared with the negative shifts. The values of δ_t and ξ have similar impact on the performance of the one-sided EWMA-SRS chart as observed for the two-sided EWMA-SRS chart. For example, given the value of ξ, under one-sided EWMA-SRS charts, the estimated ARL is also a decreasing function of δ_t and vice-versa. Similarly, when δ_t is fixed, ARLs tend to increase as the value of δ_t increases.

9.3.2 New EWMA-RSS chart

The RSS scheme becomes an efficient alternative to the SRS scheme when the sampling units are difficult or expensive to measure, but it is relatively easy to rank a small set of selected units visually or by judgment without knowing the actual measurements. For example, it is easy to rank the products with respect to their sizes, volume or by using any correlated variable (cf. Jafari Jozani and Mirkamali, 2011).

The RSS procedure is as follows: identify m^2 units from the target population. Randomly allocate these units to m sets, each of size m units. Without knowing the actual values, rank the units within each set with respect to the study variable visually or by any low cost method. Then, the ith smallest ranked unit is quantified from the ith set of m units, for $i = 1, 2, ..., m$. This completes one cycle of ranked set sample of size m. The whole procedure can be repeated r times in order to obtain a ranked set sample of size $n = mr$. In usual practice, it is customary to keep small set sizes (e.g., $2 \leq m \leq 5$) and increase the number of cycles (r) in order to avoid ranking errors. Note that in some practical applications, ranking costs cannot be ignored. For brevity, we here assume that the units are ranked with negligible ranking cost.

Let $Y^*_{11k,t}, Y^*_{12k,t}, ..., Y^*_{1mk,t}, Y^*_{21k,t}, Y^*_{22k,t}, ..., Y^*_{2mk,t}, ..., Y^*_{m1k,t}, Y^*_{m2k,t}, ..., Y^*_{mmk,t}$, be m independent simple random samples, each of size m, in the kth cycle at time t, such that $Y^*_{ijk,t} \sim N(0, \sigma_t^2)$, for $i, j = 1, 2, ..., m$, $k = 1, 2, .., r$. Apply the RSS procedure on these samples to obtain a ranked set sample of size n, denoted by $Y^*_{i(i:m)k,t}$, $i = 1, 2, ..., m$ and $k = 1, 2, ..., r$. Here, $Y^*_{i(i:m)k,t} = i\text{th} \min\{Y^*_{i1k,t}, Y^*_{i2k,t}, ..., Y^*_{imk,t}\}$. Let $U_{(i:m)k,t} = \frac{1}{\sigma_t} Y^*_{i(i:m)k,t}$ be the standardized variate with PDF independent of σ_t. Let $\boldsymbol{Y}'_{\text{RSS},t} = (\boldsymbol{Y}'_{1,t}, \boldsymbol{Y}'_{2,t}, ..., \boldsymbol{Y}'_{r,t})$ be $n \times 1$ vector of observed order statistics obtained from a ranked set sample of size n, and let $\boldsymbol{U}'_t = (\boldsymbol{U}'_{1,t}, \boldsymbol{U}'_{2,t}, ..., \boldsymbol{U}'_{r,t})$ be $n \times 1$ vector of the standardized order statistics corresponding to $\boldsymbol{Y}'_{\text{RSS},t}$, where $\boldsymbol{Y}'_{k,t} = (Y^*_{1(1:m)k,t}, Y^*_{2(2:m)k,t}, ..., Y^*_{m(m:m)k,t})$ and $\boldsymbol{U}'_{k,t} = (U_{(1:m)k,t}, U_{(2:m)k,t}, ..., U_{(m:m)k,t})$, for $k = 1, 2, ..., r$. Let $\boldsymbol{v}'_t = (\boldsymbol{v}'_{1,t}, \boldsymbol{v}'_{2,t}, ..., \boldsymbol{v}'_{r,t})$ be $n \times 1$ mean vector of $\boldsymbol{U}'_t$, and $\boldsymbol{\Omega}_t = \text{diag}(\boldsymbol{\Omega}_{1,t}, \boldsymbol{\Omega}_{2,t}, ..., \boldsymbol{\Omega}_{r,t})$ is a $n \times n$ diagonal matrix. Here, $\boldsymbol{v}'_{k,t} = (v_{(1:m),t}, v_{(2:m),t}, ..., v_{(m:m),t})$ and $\boldsymbol{\Omega}_{k,t} = \text{diag}(\omega_{(1:m),t}, \omega_{(2:m),t}, ..., \omega_{(m:m),t})$, where $v_{(i:m),t} = E(U_{(i:m)k,t})$, $\omega_{(i:m),t} = \text{Var}(U_{(i:m)k,t})$, for $i = 1, 2, ..., m$ and $k = 1, 2, ..., r$. Here, 'diag' indicates the diagonal matrix. Then, following Stokes (1995), the BLUE of σ_t and its variance are as follows:

$$\hat{\sigma}^{\text{RSS}}_{\text{BLUE},t} = (\boldsymbol{v}'_t \boldsymbol{\Omega}_t^{-1} \boldsymbol{v}_t)^{-1} \boldsymbol{v}'_t \boldsymbol{\Omega}_t^{-1} \boldsymbol{Y}_{\text{RSS},t} \quad \text{and} \quad \text{Var}(\hat{\sigma}^{\text{RSS}}_{\text{BLUE},t}) = \sigma_t^2 (\boldsymbol{v}'_t \boldsymbol{\Omega}_t^{-1} \boldsymbol{v}_t)^{-1}.$$

Following the work of Rao et al. (1991) and Rosaiah et al. (1991), Zheng and Al-Saleh (2003) showed that under RSS when estimating the scale parameter of a symmetric distribution, it is possible to construct a more efficient estimator of σ_t than $\sigma^{\text{RSS}}_{\text{BLUE},t}$ by using the absolute values of ranked set sample, i.e., $|\boldsymbol{Y}_{\text{RSS},t}|$, say $\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t}$. The estimator $\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t}$ and its variance are, respectively, given by

$$\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t} = (\boldsymbol{\beta}_t'\boldsymbol{\Psi}_t^{-1}\boldsymbol{\beta}_t)^{-1}\boldsymbol{\beta}_t'\boldsymbol{\Psi}_t^{-1}|\boldsymbol{Y}_{\text{RSS},t}| \quad \text{and} \quad \text{Var}(\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t}) = \sigma_t^2(\boldsymbol{\beta}_t'\boldsymbol{\Psi}_t^{-1}\boldsymbol{\beta}_t)^{-1},$$

where $\boldsymbol{\beta}_t' = (\boldsymbol{\beta}_{1,t}', \boldsymbol{\beta}_{2,t}', ..., \boldsymbol{\beta}_{r,t}')$ is a $n \times 1$ mean vector of $|\boldsymbol{U}_t'|$, where $\boldsymbol{\beta}_{k,t}' = (\boldsymbol{\beta}_{(1:m),t}, \boldsymbol{\beta}_{(2:m),t}, ..., \boldsymbol{\beta}_{(m:m),t})$, for $k = 1, 2, ..., r$, $\beta_{(i:m),t} = E(|U_{(i:m)k,t}|)$, for $i = 1, 2, ..., m$, and $\boldsymbol{\Psi}_t = \text{diag}\{\boldsymbol{\Psi}_{1,t}, \boldsymbol{\Psi}_{2,t}, ..., \boldsymbol{\Psi}_{r,t}\}$ is a $n \times n$ diagonal matrix, where $\boldsymbol{\Psi} = \text{diag}(\psi_{(1:m),t}, \psi_{(2:m),t}, ..., \psi_{(m:m),t})$, for $k = 1, 2, ..., r$, $\psi_{(i:m),t} = \text{Var}(|U_{(i:m)k,t}|)$, for $i = 1, 2, ..., m$. Under RSS, the simplified expressions of $\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t}$ and its variance are, respectively, given by the following:

$$\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t} = \frac{\sum_{i=1}^{m} \beta_{(i:m),t}\psi_{(i:m),t}^{-1}|\bar{Y}^*_{i(i:m),t}|}{\sum_{i=1}^{m} \beta_{(i:m),t}^2\psi_{(i:m),t}^{-1}} \quad \text{and} \quad \text{Var}(\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t}) = \frac{\sigma_t^2}{r}\left(\sum_{i=1}^{m} \beta_{(i:m),t}^2\psi_{(i:m),t}^{-1}\right)^{-1},$$

where $|\bar{Y}^*_{i(i:m),t}| = \frac{1}{r}\sum_{k=1}^{r} |Y^*_{i(i:m)k,t}|$.

Following Rosaiah et al. (1991), it is easy to express the moments and cross-moments of $|U_{(i:m)k,t}|$ in terms of the moments and cross-moments of the standardized order statistics, i.e., $U_{(i:m),t}$, in the corresponding folded density. As mentioned above, $Y^*_{ijk,t} \sim N(0, \sigma_t^2)$, for $i, j = 1, 2, ..., m$, $k = 1, 2, ..., r$. Let $U^*_{(i:m)k,t} = \frac{1}{\sigma_t}|Y_{i(i:m)k,t}|$, having PDF given by

$$g_{U^*_{(i:m)k,t}}(u_t^*) = i\binom{m}{i}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{u_t^{*2}}{2}\right)\left\{\text{Erf}\left[\frac{u_t^*}{\sqrt{2}}\right]\right\}^{i-1}\left\{1 - \text{Erf}\left[\frac{u_t^*}{\sqrt{2}}\right]\right\}^{m-i}, \quad u_t^* > 0,$$

where $\text{Erf}[v] = \frac{2}{\sqrt{\pi}}\int_0^v \exp(-q^2)\text{d}q$ is the error function.

For all positive values of ϖ, we have

$$E(|U^{\varpi}_{(i:m)k,t}|) = \frac{1}{2^m}\left\{\sum_{w=0}^{i-1} C_{m,w}E(U^{*\varpi}_{(i-w:m-w)k,t}) + \sum_{w=i}^{m} E(U^{*\varpi}_{(w-i+1:w)k,t})\right\}, \tag{9.3}$$

where $C_{m,w} = m!\{w!(m-w)!\}^{-1}$. Solving (9.3), it is easy to find the values of $\boldsymbol{\beta}_t'$ and $\boldsymbol{\Psi}_t$ in order to obtain the BLUAE of σ_t and its corresponding variance.

Now, suppose that the underlying process is in-control and $Y^*_{i,t} \sim N(0, \sigma_0^2)$ for $t = 1, 2, ..., \tau$. For each t, we obtain a ranked set sample of size n from this process and generate a sequence of the BLUAEs. Let $\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},1}, \hat{\sigma}^{\text{RSS}}_{\text{BLUAE},2}, ..., \hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t}, ...$ be a sequence of IID random variables and let ξ be a smoothing constant. We define another EWMA sequence, say $\{J_t\}$, based on this sequence by using a recurrence formula, given by

$$J_t = \xi\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t} + (1-\xi)J_{t-1}, \quad 0 < \xi \le 1.$$

It is easy to show that $E(J_t) = E(\hat{\sigma}^{\text{RSS}}_{\text{BLUAE},t}) = \sigma_0$ for $j_0 = \sigma_0$, and $\text{Var}(J_t) = R(\xi; t)\sigma_0^2(\boldsymbol{\beta}_t'\boldsymbol{\Psi}_t^{-1}\boldsymbol{\beta}_t)^{-1}$. We name the EWMA chart based on J_t as EWMA-RSS chart. The control limits of the EWMA-RSS chart are given by the following:

$$\begin{aligned} UCL_t &= \sigma_0 + h_2\sigma_0\sqrt{R(\xi;t)(\boldsymbol{\beta}_t'\boldsymbol{\Psi}_t^{-1}\boldsymbol{\beta}_t)^{-1}}, \\ CL_t &= \sigma_0, \\ LCL_t &= \sigma_0 - h_1\sigma_0\sqrt{R(\xi;t)(\boldsymbol{\beta}_t'\boldsymbol{\Psi}_t^{-1}\boldsymbol{\beta}_t)^{-1}}, \end{aligned}$$

where (h_1, h_2) is selected such that the in-control ARL reaches to a fixed pre-specified level. Based on extensive Monte Carlo simulations (10^5) from standard normal distribution, we have estimated the run length characteristics (ARL and SDRL) of the EWMA-RSS chart. The estimated ARLs and SDRLs of the EWMA-SRS chart for different values of δ_t and ξ are given in Tables 3 and 4. For a fair comparison of both EWMA-SRS and EWMA-RSS charts, we consider $n = 5$ based on $m = 5$ and $r = 1$.

In Table 9.3, we report the run length properties of the EWMA-RSS chart by using symmetric and asymmetric control limits. Given the value of ξ, the estimated ARL is a decreasing function of δ_t when $\delta_t \geq 1$ and increasing function of δ_t when $\delta_t < 1$. Under symmetric control limits, the ARLs of the EWMA-RSS chart are unbiased when $\xi \leq 0.2$. However, they become biased for small values of δ_t, i.e., $0.9 < \delta_t \leq 1$ for $\xi = 0.3$. In order to obtain unbiased ARLs for all possible values of δ_t and ξ considered here, we use asymmetric control limits and estimate the ARLs and SDRLs of the EWMA-RSS chart. With asymmetric control limits, the performance of the EWMA-RSS chart is greatly improved for small values of δ_t, i.e., $0.9 < \delta_t \leq 1$. Nevertheless, the estimated ARLs for $\delta_t > 1$ are now increased. In Table 9.4, we report ARLs and SDRLs of the one-sided EWMA-RSS charts designed to monitor increases or decreases in the process dispersion. The performance of the EWMA-RSS chart is better in detecting positive ($\delta_t > 1$) shifts as compared with the negative shifts ($\delta_t < 1$) shifts in the process variability.

The performance of RSS depends on the accuracy of ranking of the selected units. The correct ordering lead to accurate and precise estimates of the population parameters. But, in practice, the judgment ordering may not match with the actual ordering. Thus, error in ranking are inevitable, particularly when dealing with large set sizes, and adversely affect the efficiency of estimator under RSS. Dell and Clutter (1972) investigated the effect of imperfect ranking on the performance of RSS mean estimator. They showed that even under imperfect ranking, the RSS-based mean estimator remains unbiased, and it is still better than the SRS-based mean estimator given that the ranking should be at least better than the random ordering of the selected units.

In this study, following Dell and Clutter (1972), we investigate the effect of imperfect ranking on the performance of BLUAE. Recall that $Y^*_{ijk,t} \sim N(0, \sigma_t^2)$, for $i, j = 1, 2, ..., m$, $k = 1, 2, ..., r$, at time t. Let $V_{ijk,t}$ be the random error term, and it is normally distributed with mean zero and variance σ_V^2, i.e., $V_{ijk,t} \sim N(0, \sigma_V^2)$, for $i, j = 1, 2, ..., m$, $k = 1, 2, ..., r$, at time t. Note that both random variables, i.e., $Y^*_{ijk,t}$ and $V_{ijk,t}$, are independent. Then, compute $X_{ijk,t} = Y^*_{ijk,t} + V_{ijk,t}$, for $i, j = 1, 2, ..., m$, $k = 1, 2, ..., r$, at time t. Based on the values of $X_{ijk,t}$, we select a ranked set sample of size $n = mr$, i.e., $X_{i(i:m)k,t}$,

Table 9.3: Run length characteristics of the two-sided EWMA-RSS control chart

		Symmetric limits				Asymmetric limits			
	$\xi \rightarrow$	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30
	$h_1 \rightarrow$	2.2764	2.4735	2.6399	2.7236	2.1000	2.3000	2.3936	2.4620
δ_t	$h_2 \rightarrow$	2.2764	2.4735	2.6399	2.7236	2.4910	2.6970	2.9500	3.0000
0.50	ARL	2.26	2.57	2.90	3.16	2.01	2.30	2.47	2.63
	SDRL	0.81	0.86	0.96	1.09	0.77	0.82	0.86	0.94
0.55	ARL	2.69	3.07	3.48	3.86	2.37	2.73	2.94	3.18
	SDRL	1.05	1.14	1.28	1.53	0.98	1.06	1.14	1.29
0.60	ARL	3.26	3.74	4.30	4.92	2.86	3.31	3.59	3.95
	SDRL	1.40	1.53	1.78	2.27	1.31	1.42	1.55	1.82
0.65	ARL	4.07	4.70	5.51	6.59	3.57	4.14	4.55	5.14
	SDRL	1.93	2.12	2.59	3.58	1.79	1.95	2.20	2.74
0.70	ARL	5.28	6.13	7.41	9.46	4.59	5.37	5.98	7.00
	SDRL	2.74	3.04	3.99	6.06	2.52	2.79	3.27	4.31
0.75	ARL	7.19	8.44	10.73	14.86	6.23	7.29	8.38	10.30
	SDRL	4.11	4.69	6.72	11.13	3.78	4.17	5.26	7.33
0.80	ARL	10.47	12.47	17.29	26.36	9.01	10.65	12.73	16.69
	SDRL	6.60	7.79	12.72	22.46	5.98	6.87	9.17	13.43
0.85	ARL	17.03	20.94	32.34	52.36	14.35	17.39	21.84	29.95
	SDRL	11.99	15.27	27.35	48.48	10.55	12.75	17.87	26.78
0.90	ARL	32.68	43.09	71.33	117.87	26.94	33.96	43.94	60.21
	SDRL	26.44	36.85	66.79	114.55	22.54	28.93	40.37	57.16
0.95	ARL	88.38	116.38	173.89	238.49	69.41	86.68	102.35	128.57
	SDRL	85.39	113.62	171.74	236.52	67.65	84.01	99.99	126.29
1.00	ARL	200.86	199.53	199.59	200.59	199.88	200.79	199.58	200.86
	SDRL	214.93	204.81	199.11	201.08	214.70	206.28	201.00	200.96
1.10	ARL	27.46	31.00	35.14	38.79	33.64	39.35	53.36	58.25
	SDRL	26.04	29.12	33.58	37.56	31.01	36.50	51.51	56.81
1.20	ARL	9.74	10.75	11.92	13.06	11.34	12.69	15.68	17.18
	SDRL	8.49	9.05	10.28	11.60	9.53	10.44	13.48	15.46
1.30	ARL	5.33	5.85	6.33	6.71	6.12	6.74	7.80	8.25
	SDRL	4.42	4.64	5.03	5.44	4.90	5.20	6.07	6.73
1.40	ARL	3.58	3.89	4.17	4.36	4.00	4.37	4.94	5.14
	SDRL	2.80	2.95	3.10	3.28	3.06	3.23	3.60	3.88
1.50	ARL	2.69	2.90	3.08	3.20	2.96	3.23	3.58	3.68
	SDRL	1.97	2.09	2.18	2.28	2.15	2.29	2.50	2.61
1.60	ARL	2.18	2.33	2.46	2.53	2.36	2.56	2.80	2.84
	SDRL	1.49	1.57	1.64	1.70	1.61	1.73	1.87	1.91
1.70	ARL	1.85	1.96	2.07	2.13	1.99	2.13	2.31	2.34
	SDRL	1.18	1.25	1.31	1.35	1.28	1.37	1.48	1.50
1.80	ARL	1.64	1.72	1.80	1.84	1.74	1.84	1.99	2.02
	SDRL	0.97	1.03	1.07	1.10	1.05	1.12	1.21	1.24
1.90	ARL	1.49	1.55	1.61	1.65	1.57	1.65	1.76	1.78
	SDRL	0.81	0.86	0.90	0.93	0.88	0.94	1.01	1.03
2.00	ARL	1.37	1.43	1.49	1.51	1.44	1.50	1.60	1.61
	SDRL	0.69	0.74	0.78	0.79	0.75	0.81	0.87	0.88
2.50	ARL	1.13	1.15	1.16	1.18	1.15	1.17	1.21	1.22
	SDRL	0.37	0.40	0.42	0.44	0.40	0.43	0.48	0.48
3.00	ARL	1.05	1.06	1.07	1.07	1.06	1.07	1.09	1.09
	SDRL	0.23	0.25	0.26	0.28	0.25	0.27	0.30	0.30

for $i = 1, 2, ..., m$, $k = 1, 2, ..., r$. We also observe the corresponding values of $X_{i(i:m)k,t}$, i.e., $Y_{i[i:m]k,t}$, for $i = 1, 2, ..., m$, $k = 1, 2, ..., r$, where $X_{i(i:m)k,t}$ is the ith order statistic and $Y^*_{i[i:m]k,t}$ is the corresponding ith concomitant, both obtained from the ith sample in the kth cycle at time t. This scheme is named IRSS. Let

Table 9.4: Run length characteristics of the one-sided EWMA-RSS control charts

	$\xi\rightarrow$	0.05	0.10	0.20	0.30	$\xi\rightarrow$	0.05	0.10	0.20	0.30
δ_t	$h_2\rightarrow$	1.9108	2.2074	2.4613	2.5900	$\delta_t\downarrow$ $h_1\rightarrow$	1.8390	2.0756	2.2460	2.2929
1.00	ARL	200.76	200.30	200.73	200.28	1.00	199.99	199.50	200.39	199.84
	SDRL	222.76	209.30	201.81	202.63		219.49	205.53	202.64	199.36
1.10	ARL	19.55	23.52	28.36	32.20	0.95	48.84	59.95	75.77	86.69
	SDRL	19.73	22.44	27.03	30.98		49.29	57.94	73.51	84.40
1.20	ARL	7.40	8.78	10.20	11.46	0.90	20.22	25.40	33.95	41.33
	SDRL	6.89	7.67	8.79	10.13		17.97	21.87	30.82	38.82
1.30	ARL	4.23	4.96	5.67	6.14	0.85	11.06	13.67	17.79	21.82
	SDRL	3.66	4.06	4.54	4.99		8.82	10.41	14.56	19.15
1.40	ARL	2.94	3.37	3.78	4.05	0.80	7.06	8.66	10.77	12.84
	SDRL	2.34	2.60	2.84	3.05		5.06	5.86	7.74	10.15
1.50	ARL	2.27	2.57	2.85	2.99	0.75	4.97	6.04	7.26	8.35
	SDRL	1.66	1.86	2.03	2.13		3.26	3.70	4.57	5.80
1.60	ARL	1.88	2.09	2.29	2.39	0.70	3.70	4.48	5.27	5.87
	SDRL	1.26	1.41	1.53	1.60		2.19	2.47	2.93	3.56
1.70	ARL	1.64	1.79	1.95	2.03	0.65	2.90	3.49	4.03	4.40
	SDRL	1.01	1.12	1.22	1.28		1.56	1.76	2.01	2.33
1.80	ARL	1.47	1.60	1.71	1.78	0.60	2.34	2.81	3.22	3.45
	SDRL	0.82	0.93	1.01	1.05		1.14	1.29	1.44	1.62
1.90	ARL	1.36	1.46	1.55	1.60	0.55	1.96	2.33	2.65	2.79
	SDRL	0.70	0.77	0.85	0.89		0.87	0.97	1.07	1.16
2.00	ARL	1.28	1.35	1.42	1.47	0.50	1.67	1.97	2.23	2.33
	SDRL	0.59	0.66	0.72	0.76		0.68	0.76	0.82	0.87
2.50	ARL	1.09	1.12	1.15	1.16	0.40	1.26	1.48	1.67	1.73
	SDRL	0.31	0.36	0.40	0.41		0.45	0.53	0.56	0.57
3.00	ARL	1.04	1.05	1.06	1.06	0.30	1.03	1.11	1.24	1.29
	SDRL	0.19	0.22	0.25	0.26		0.17	0.32	0.43	0.45

$\sigma_{\text{BLUAE},t}^{\text{IRSS}}$ be the BLUAE of σ_t based on IRSS, given by

$$\hat{\sigma}_{\text{BLUAE},t}^{\text{IRSS}} = \frac{\sum_{i=1}^{m} \beta_{(i:m),t}\psi_{(i:m),t}^{-1}|\bar{Y}_{i[i:m],t}^{*}|}{\sum_{i=1}^{m} \beta_{(i:m),t}^{2}\psi_{(i:m),t}^{-1}}$$

where $|\bar{Y}_{i[i:m],t}^{*}| = \frac{1}{r}\sum_{k=1}^{r} |Y_{i[i:m]k,t}^{*}|$.

In order to examine the effect of judgment error on the performance of $\hat{\sigma}_{\text{BLUAE},t}^{\text{IRSS}}$, we choose $\sigma_V^2 = 0.05$, 0.15, 0.30 and 0.50. It is difficult to derive the exact mathematical expression for the variance of $\hat{\sigma}_{\text{BLUAE},t}^{\text{IRSS}}$. Therefore, we estimate the variance of $\hat{\sigma}_{\text{BLUAE},t}^{\text{IRSS}}$ by using Monte Carlo simulations.

It is possible to construct an EWMA control chart based on $\hat{\sigma}_{\text{BLUAE},t}^{\text{IRSS}}$ for monitoring the process dispersion. The plotting-statistic of the EWMA chart based on $\hat{\sigma}_{\text{BLUAE},t}^{\text{IRSS}}$, named EWMA-IRSS chart, is given by

$$W_t = \xi\hat{\sigma}_{\text{BLUAE},t}^{\text{IRSS}} + (1-\xi)W_{t-1}, \quad 0 < \xi \leq 1,$$

where ξ is a smoothing constant and $W_0 = \bar{\hat{\sigma}}_{\text{BLUAE}}^{\text{IRSS}}$.

Let $\hat{\sigma}_{1,\text{BLUAE}}^{\text{IRSS}}, \hat{\sigma}_{2,\text{BLUAE}}^{\text{IRSS}}, ..., \hat{\sigma}_{\eta,\text{BLUAE}}^{\text{IRSS}}$ be the estimated values of the BLUAEs obtained from η subgroups, each of size n. Let $\bar{\hat{\sigma}}_{\text{BLUAE}}^{\text{IRSS}} = \frac{1}{\eta}\sum_{i=1}^{\eta} \hat{\sigma}_{i,\text{BLUAE}}^{\text{IRSS}}$. The estimated control limits of the EWMA-IRSS control chart, at time t, are given by

$$\begin{aligned} EUCL_t &= \bar{\hat{\sigma}}^{\text{IRSS}}_{\text{BLUAE}} + q_2 \hat{\sigma}_{\hat{\sigma}^{\text{IRSS}}_{\text{BLUAE}}} \sqrt{R(\xi;t)}, \\ ECL_t &= \bar{\hat{\sigma}}^{\text{IRSS}}_{\text{BLUAE}}, \\ ELCL_t &= \bar{\hat{\sigma}}^{\text{IRSS}}_{\text{BLUAE}} - q_1 \hat{\sigma}_{\hat{\sigma}^{\text{IRSS}}_{\text{BLUAE}}} \sqrt{R(\xi;t)}, \end{aligned}$$

where $EUCL_t$, ECL_t and $ELCL_t$ stand for estimated upper, center and lower control limits at time t, respectively, and $\hat{\sigma}_{\hat{\sigma}^{\text{IRSS}}_{\text{BLUAE}}} = \sqrt{\frac{1}{\eta-1}\sum_{i=1}^{\eta}(\hat{\sigma}^{\text{IRSS}}_{i,\text{BLUAE}} - \bar{\hat{\sigma}}^{\text{IRSS}}_{\text{BLUAE}})^2}$. Here (q_1, q_2) is selected such that the in-control ARL of the EWMA-IRSS chart reaches to a desired level. Similarly, it is easy to obtain the one-sided EWMA-IRSS chart for monitoring increases or decreases in the process dispersion.

In order to estimate the run length characteristics of the (one-sided or two-sided) EWMA-IRSS chart, we first estimate the control limits based on one million samples obtained under IRSS given that the underlying process is in-control. Then, based on 10^5 replications from standard normal distribution, we estimate ARLs and SDRLs of the EWMA-IRSS chart. The run length properties of the one-sided and two-sided EWMA-IRSS charts are given in Tables 9.5–9.7. Note that for the two-sided EWMA-IRSS chart, we have used asymmetric control limits. For all EWMA-IRSS charts, we consider same values of ξ as already taken for EWMA-RSS charts.

From Table 9.5, it is observed that for small values of ξ, i.e., 0.05 and 0.10, generally, the in-control ARL of the two-sided EWMA-IRSS chart based on different values of σ_V^2 are closer to the fixed in-control ARL, i.e., 200. However, given the values of ξ and δ_t, the out-of-control ARLs of the two-sided EWMA-IRSS chart tend to increase as the value of σ_V^2 increases and vice-versa. It is observed that when $\sigma_V^2 \geq 0.3$, the in-control ARLs become more sensitive when $\xi > 0.10$. An interesting feature of the EWMA-IRSS chart is that with an increase in the error variance σ_V^2, the performance of the IRSS charts goes down but at the same time the false alarm rate also decreases.

Tables 9.6 and 9.7 provide the run length properties of the one-sided EWMA-IRSS chart when detecting increases and decreases in the process dispersion, respectively. As expected, given ξ and δ_t, the out-of-control ARLs are increasing function of σ_V^2. From Table 9.6, it is clear that when the ranking error is small, i.e., $\sigma_V^2 \leq 0.15$, normally the in-control ARLs remain close to 200. However, when $\sigma_V^2 \geq 0.3$, the in-control ARLs tend to decrease as the value of ξ increases from 0.10, i.e., $\xi > 0.1$. Therefore, we recommend using one-sided EWMA-IRSS chart with small values of ξ, i.e., $\xi \leq 0.1$, when the objective is to monitor an increase in the process variation. On the other hand, in Table 9.7, the in-control ARLs of the one-sided EWMA-IRSS chart tend to increase as σ_V^2 increases, which shows a reduction in the false alarm rate associated with this chart. However, this decrease in the false alarm rate also affects the performance of the EWMA-IRSS chart. Therefore, it is advantageous to use this chart with small values of ξ.

Table 9.5: Run length characteristics of the two-sided EWMA-IRSS control chart

		$\sigma_V^2 = 0.05$				$\sigma_V^2 = 0.15$				$\sigma_V^2 = 0.30$				$\sigma_V^2 = 0.50$			
	$\xi \rightarrow$	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30
	$q_1 \rightarrow$	2.1000	2.3000	2.3936	2.4620	2.1000	2.3000	2.3936	2.4620	2.1000	2.3000	2.3936	2.4620	2.1000	2.3000	2.3936	2.4620
δ_t	$q_2 \rightarrow$	2.4910	2.6970	2.9500	3.0000	2.4910	2.6970	2.9500	3.0000	2.4910	2.6970	2.9500	3.0000	2.4910	2.6970	2.9500	3.0000
0.50	ARL	2.09	2.41	2.59	2.77	2.21	2.56	2.75	2.95	2.33	2.69	2.90	3.13	2.42	2.79	3.02	3.27
	SDRL	0.83	0.89	0.95	1.04	0.91	0.98	1.04	1.15	0.96	1.03	1.10	1.23	0.98	1.07	1.14	1.30
0.55	ARL	2.48	2.86	3.08	3.35	2.62	3.03	3.28	3.57	2.77	3.20	3.47	3.80	2.87	3.33	3.62	3.99
	SDRL	1.07	1.15	1.25	1.42	1.16	1.26	1.36	1.56	1.23	1.34	1.46	1.69	1.28	1.39	1.52	1.79
0.60	ARL	3.00	3.46	3.78	4.17	3.19	3.68	4.02	4.48	3.35	3.89	4.26	4.78	3.49	4.06	4.45	5.04
	SDRL	1.42	1.53	1.69	2.00	1.54	1.68	1.86	2.22	1.63	1.77	1.99	2.42	1.70	1.86	2.07	2.58
0.65	ARL	3.73	4.34	4.78	5.41	3.98	4.61	5.08	5.81	4.18	4.88	5.39	6.25	4.36	5.11	5.68	6.61
	SDRL	1.92	2.11	2.38	2.96	2.09	2.30	2.60	3.30	2.22	2.45	2.80	3.63	2.33	2.57	2.98	3.88
0.70	ARL	4.79	5.61	6.29	7.40	5.12	5.98	6.72	8.00	5.41	6.33	7.18	8.57	5.64	6.64	7.53	9.21
	SDRL	2.69	3.00	3.53	4.67	2.93	3.24	3.88	5.19	3.15	3.48	4.21	5.66	3.27	3.65	4.45	6.18
0.75	ARL	6.51	7.63	8.82	10.90	6.94	8.14	9.41	11.76	7.33	8.62	10.09	12.79	7.66	9.05	10.63	13.72
	SDRL	4.02	4.48	5.62	7.88	4.34	4.90	6.14	8.70	4.62	5.22	6.66	9.63	4.85	5.52	7.14	10.48
0.80	ARL	9.39	11.15	13.34	17.58	9.98	11.86	14.30	18.98	10.55	12.64	15.43	20.61	11.09	13.31	16.31	22.21
	SDRL	6.32	7.35	9.72	14.44	6.82	7.95	10.63	15.74	7.28	8.54	11.65	17.27	7.68	9.05	12.45	18.78
0.85	ARL	15.05	18.21	22.98	31.54	15.87	19.47	24.61	33.84	16.83	20.54	26.37	36.69	17.66	21.63	27.94	39.87
	SDRL	11.25	13.62	19.16	28.38	11.99	14.77	20.84	30.75	12.83	15.68	22.56	33.46	13.47	16.65	23.99	36.50
0.90	ARL	28.14	35.20	45.88	62.67	29.64	37.46	48.22	66.68	31.24	39.71	51.72	71.72	32.72	41.76	54.78	77.02
	SDRL	23.74	30.13	42.32	59.75	25.09	32.65	44.53	63.99	26.86	34.61	48.12	68.63	28.21	36.66	50.91	74.15
0.95	ARL	71.34	88.87	104.24	132.41	74.40	92.80	109.33	137.36	77.51	96.11	114.16	144.51	79.94	100.08	118.94	152.60
	SDRL	69.90	85.65	101.80	130.73	73.09	90.61	107.02	135.84	76.69	93.40	111.95	142.94	78.83	97.53	116.78	151.16
1.00	ARL	200.47	202.46	199.38	204.79	199.09	202.52	202.65	206.59	199.33	201.85	204.06	209.92	199.90	204.77	205.90	212.83
	SDRL	215.28	205.95	199.98	204.56	214.10	207.42	203.22	205.63	214.24	206.50	203.62	209.84	213.72	209.73	205.81	212.09
1.10	ARL	34.97	41.13	55.49	60.56	37.32	43.84	59.53	64.79	39.38	46.27	61.95	66.97	40.85	47.88	64.22	68.58
	SDRL	32.32	38.37	53.41	59.08	34.74	40.97	57.42	63.34	36.77	43.65	60.08	65.49	38.54	45.24	62.17	67.15
1.20	ARL	11.86	13.27	16.46	18.20	12.61	14.11	17.72	19.62	13.34	14.98	18.94	20.92	13.98	15.75	19.80	21.75
	SDRL	9.95	10.86	14.17	16.43	10.59	11.69	15.41	17.86	11.28	12.51	16.59	19.20	11.87	13.16	17.40	19.97
1.30	ARL	6.35	6.98	8.18	8.66	6.71	7.44	8.70	9.36	7.10	7.88	9.24	9.93	7.41	8.24	9.67	10.42
	SDRL	5.07	5.42	6.44	7.09	5.41	5.74	6.88	7.77	5.73	6.13	7.37	8.35	6.03	6.46	7.78	8.75

Table 9.5: (Continued).

		$\sigma_V^2 = 0.05$				$\sigma_V^2 = 0.15$				$\sigma_V^2 = 0.30$				$\sigma_V^2 = 0.50$			
	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30
	$q_1\rightarrow$	2.1000	2.3000	2.3936	2.4620	2.1000	2.3000	2.3936	2.4620	2.1000	2.3000	2.3936	2.4620	2.1000	2.3000	2.3936	2.4620
δ_t	$q_2\rightarrow$	2.4910	2.6970	2.9500	3.0000	2.4910	2.6970	2.9500	3.0000	2.4910	2.6970	2.9500	3.0000	2.4910	2.6970	2.9500	3.0000
1.40	ARL	4.14	4.56	5.14	5.38	4.41	4.84	5.46	5.70	4.66	5.08	5.77	6.08	4.83	5.34	6.06	6.35
	SDRL	3.18	3.37	3.78	4.08	3.41	3.59	4.03	4.39	3.62	3.82	4.32	4.72	3.77	4.03	4.55	4.94
1.50	ARL	3.06	3.35	3.72	3.80	3.22	3.51	3.91	4.02	3.39	3.71	4.14	4.26	3.53	3.84	4.31	4.45
	SDRL	2.24	2.38	2.61	2.71	2.36	2.51	2.75	2.89	2.51	2.68	2.94	3.11	2.63	2.79	3.09	3.28
1.60	ARL	2.43	2.63	2.89	2.95	2.55	2.76	3.05	3.11	2.67	2.90	3.19	3.29	2.78	3.02	3.35	3.40
	SDRL	1.68	1.79	1.94	1.99	1.78	1.90	2.06	2.14	1.88	2.00	2.17	2.27	1.97	2.11	2.30	2.37
1.70	ARL	2.04	2.19	2.38	2.42	2.13	2.29	2.49	2.54	2.22	2.38	2.61	2.67	2.30	2.49	2.71	2.77
	SDRL	1.33	1.41	1.52	1.56	1.40	1.50	1.62	1.65	1.47	1.57	1.70	1.77	1.55	1.66	1.79	1.85
1.80	ARL	1.79	1.89	2.05	2.07	1.84	1.97	2.13	2.16	1.91	2.05	2.22	2.25	1.98	2.12	2.31	2.34
	SDRL	1.09	1.16	1.25	1.26	1.14	1.22	1.32	1.35	1.20	1.28	1.38	1.41	1.27	1.35	1.46	1.49
1.90	ARL	1.60	1.69	1.81	1.83	1.65	1.74	1.87	1.90	1.71	1.82	1.94	1.98	1.76	1.87	2.01	2.05
	SDRL	0.91	0.97	1.05	1.08	0.96	1.02	1.10	1.12	1.02	1.09	1.16	1.19	1.06	1.13	1.22	1.25
2.00	ARL	1.46	1.54	1.63	1.65	1.51	1.59	1.69	1.71	1.56	1.64	1.75	1.76	1.60	1.69	1.80	1.82
	SDRL	0.77	0.83	0.90	0.91	0.82	0.88	0.94	0.97	0.86	0.92	1.00	1.01	0.91	0.97	1.04	1.06
2.50	ARL	1.16	1.19	1.22	1.23	1.18	1.20	1.24	1.25	1.19	1.23	1.27	1.28	1.21	1.25	1.29	1.30
	SDRL	0.42	0.45	0.49	0.50	0.44	0.47	0.52	0.52	0.46	0.50	0.54	0.55	0.48	0.52	0.57	0.58
3.00	ARL	1.07	1.08	1.09	1.10	1.07	1.08	1.10	1.11	1.08	1.10	1.12	1.12	1.09	1.10	1.13	1.13
	SDRL	0.26	0.28	0.31	0.31	0.27	0.30	0.32	0.33	0.29	0.32	0.34	0.35	0.30	0.33	0.36	0.37

Table 9.6: Run length characteristics of the one-sided EWMA-IRSS chart when detecting increases in the process dispersion

		$\sigma_V^2 = 0.05$				$\sigma_V^2 = 0.15$				$\sigma_V^2 = 0.30$				$\sigma_V^2 = 0.50$			
	$\xi \rightarrow$	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30
δ_t	$q_2 \rightarrow$	1.9108	2.2074	2.4613	2.5900	1.9108	2.2074	2.4613	2.5900	1.9108	2.2074	2.4613	2.5900	1.9108	2.2074	2.4613	2.5900
1.00	ARL	201.29	202.09	200.19	199.91	200.53	201.44	199.11	198.55	199.52	199.43	196.44	193.47	200.15	197.46	193.47	189.40
	SDRL	222.56	210.92	200.67	200.99	222.78	209.51	201.69	199.22	222.93	208.04	199.32	193.79	221.96	207.23	195.35	190.87
1.10	ARL	20.21	24.41	29.30	33.35	21.28	25.69	30.99	35.15	22.30	26.82	32.32	36.49	23.07	27.66	33.18	37.56
	SDRL	20.40	23.37	28.04	32.14	21.51	24.76	29.69	34.04	22.68	25.94	30.91	35.44	23.58	26.80	31.99	36.57
1.20	ARL	7.70	9.12	10.70	11.92	8.12	9.70	11.39	12.75	8.55	10.16	11.96	13.44	8.90	10.61	12.49	14.04
	SDRL	7.20	7.98	9.20	10.58	7.63	8.48	9.96	11.39	8.07	9.01	10.53	12.11	8.44	9.43	11.03	12.68
1.30	ARL	4.38	5.14	5.85	6.37	4.61	5.43	6.23	6.77	4.87	5.75	6.56	7.19	5.05	5.97	6.84	7.45
	SDRL	3.81	4.21	4.67	5.23	4.03	4.46	5.02	5.56	4.32	4.76	5.34	6.04	4.47	4.98	5.61	6.22
1.40	ARL	3.02	3.48	3.91	4.19	3.16	3.66	4.12	4.43	3.32	3.86	4.35	4.67	3.45	4.01	4.51	4.88
	SDRL	2.43	2.70	2.94	3.20	2.57	2.84	3.13	3.39	2.71	3.02	3.33	3.62	2.86	3.16	3.49	3.81
1.50	ARL	2.33	2.66	2.94	3.11	2.43	2.78	3.07	3.27	2.52	2.90	3.23	3.43	2.62	3.01	3.34	3.55
	SDRL	1.72	1.93	2.09	2.23	1.82	2.04	2.21	2.36	1.91	2.15	2.34	2.51	2.02	2.25	2.44	2.62
1.60	ARL	1.92	2.15	2.36	2.46	2.01	2.25	2.46	2.60	2.08	2.34	2.57	2.71	2.15	2.42	2.67	2.82
	SDRL	1.30	1.46	1.58	1.66	1.39	1.55	1.67	1.77	1.46	1.63	1.76	1.87	1.52	1.71	1.86	1.96
1.70	ARL	1.67	1.84	1.99	2.08	1.73	1.90	2.08	2.17	1.79	1.98	2.17	2.25	1.84	2.04	2.23	2.34
	SDRL	1.04	1.17	1.25	1.32	1.09	1.22	1.33	1.39	1.15	1.29	1.41	1.47	1.21	1.35	1.47	1.55
1.80	ARL	1.50	1.62	1.76	1.82	1.54	1.68	1.82	1.89	1.58	1.74	1.88	1.95	1.63	1.79	1.94	2.02
	SDRL	0.85	0.95	1.04	1.09	0.90	1.00	1.10	1.14	0.94	1.07	1.15	1.21	0.99	1.11	1.21	1.27
1.90	ARL	1.39	1.49	1.58	1.63	1.41	1.52	1.63	1.68	1.45	1.57	1.68	1.74	1.49	1.61	1.73	1.79
	SDRL	0.72	0.81	0.88	0.91	0.75	0.84	0.92	0.96	0.79	0.89	0.97	1.01	0.83	0.93	1.01	1.06
2.00	ARL	1.29	1.37	1.45	1.49	1.32	1.41	1.49	1.54	1.35	1.45	1.54	1.58	1.38	1.48	1.58	1.62
	SDRL	0.61	0.68	0.75	0.78	0.64	0.72	0.79	0.82	0.68	0.76	0.83	0.87	0.70	0.80	0.87	0.90
2.50	ARL	1.10	1.13	1.16	1.17	1.11	1.14	1.17	1.18	1.12	1.15	1.19	1.21	1.13	1.17	1.21	1.23
	SDRL	0.32	0.37	0.41	0.43	0.34	0.39	0.43	0.45	0.36	0.41	0.45	0.47	0.38	0.43	0.48	0.49
3.00	ARL	1.04	1.05	1.06	1.07	1.04	1.05	1.07	1.08	1.05	1.06	1.08	1.08	1.05	1.07	1.08	1.10
	SDRL	0.19	0.23	0.26	0.27	0.21	0.24	0.27	0.28	0.22	0.26	0.28	0.30	0.23	0.26	0.29	0.31

Table 9.7: Run length characteristics of the one-sided EWMA-IRSS chart when detecting decreases in the process dispersion

		$\sigma_V^2 = 0.05$				$\sigma_V^2 = 0.15$				$\sigma_V^2 = 0.30$				$\sigma_V^2 = 0.50$			
	$\xi\rightarrow$	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30	0.05	0.10	0.20	0.30
δ_t	$q_1\rightarrow$	1.8390	2.0756	2.2460	2.2929	1.8390	2.0756	2.2460	2.2929	1.8390	2.0756	2.2460	2.2929	1.8390	2.0756	2.2460	2.2929
1.00	ARL	202.57	202.20	203.98	201.48	199.69	202.21	208.72	207.08	200.95	204.51	210.40	213.34	203.96	206.96	216.81	223.92
	SDRL	220.65	207.73	205.47	200.44	218.62	207.42	207.52	206.47	219.62	209.74	210.89	212.14	223.14	212.51	217.57	221.85
0.95	ARL	50.28	61.73	78.00	89.42	52.04	63.91	81.18	92.44	54.16	66.38	84.49	97.08	56.30	68.94	88.39	102.04
	SDRL	50.81	59.71	76.13	87.98	53.11	62.14	79.01	90.30	54.88	65.05	82.16	95.15	57.26	67.17	86.37	100.12
0.90	ARL	20.89	26.28	34.83	42.73	22.11	27.77	37.07	45.49	23.23	29.16	39.14	48.11	24.25	30.45	41.16	51.11
	SDRL	18.70	22.84	31.69	40.40	19.88	24.42	33.95	42.92	21.24	25.85	36.07	45.54	22.14	26.99	37.96	48.61
0.85	ARL	11.54	14.29	18.49	22.93	12.19	15.13	19.69	24.36	12.88	15.98	20.97	26.13	13.44	16.81	22.08	27.87
	SDRL	9.32	10.94	15.19	20.06	9.97	11.81	16.39	21.73	10.63	12.59	17.63	23.17	11.03	13.30	18.64	25.09
0.80	ARL	7.38	9.02	11.23	13.45	7.80	9.60	11.99	14.46	8.22	10.13	12.80	15.54	8.61	10.63	13.45	16.54
	SDRL	5.42	6.20	8.14	10.81	5.79	6.72	8.84	11.74	6.15	7.13	9.57	12.76	6.53	7.55	10.18	13.71
0.75	ARL	5.19	6.30	7.61	8.78	5.48	6.69	8.13	9.43	5.77	7.09	8.64	10.08	6.04	7.42	9.08	10.72
	SDRL	3.45	3.93	4.90	6.25	3.73	4.24	5.34	6.83	3.98	4.54	5.77	7.37	4.17	4.77	6.10	7.94
0.70	ARL	3.87	4.69	5.53	6.17	4.12	4.97	5.88	6.59	4.32	5.26	6.22	7.08	4.51	5.49	6.55	7.50
	SDRL	2.35	2.65	3.15	3.84	2.57	2.86	3.44	4.18	2.71	3.07	3.68	4.59	2.85	3.23	3.89	4.94
0.65	ARL	3.02	3.64	4.22	4.61	3.20	3.87	4.51	4.95	3.38	4.08	4.77	5.27	3.52	4.26	4.99	5.56
	SDRL	1.66	1.88	2.16	2.51	1.82	2.05	2.37	2.80	1.94	2.19	2.55	3.04	2.02	2.28	2.67	3.23
0.60	ARL	2.45	2.93	3.37	3.61	2.60	3.12	3.59	3.86	2.73	3.29	3.79	4.10	2.83	3.42	3.96	4.31
	SDRL	1.23	1.39	1.56	1.75	1.34	1.51	1.71	1.94	1.44	1.61	1.82	2.09	1.50	1.68	1.90	2.23
0.55	ARL	2.04	2.43	2.77	2.94	2.16	2.58	2.95	3.12	2.27	2.71	3.11	3.32	2.34	2.82	3.23	3.47
	SDRL	0.94	1.06	1.16	1.28	1.03	1.15	1.27	1.40	1.10	1.22	1.35	1.51	1.13	1.27	1.41	1.58
0.50	ARL	1.74	2.06	2.33	2.45	1.83	2.18	2.48	2.59	1.92	2.29	2.61	2.75	1.99	2.38	2.72	2.86
	SDRL	0.74	0.83	0.89	0.96	0.81	0.90	0.98	1.05	0.85	0.95	1.03	1.12	0.88	0.98	1.07	1.16
0.40	ARL	1.32	1.54	1.75	1.82	1.39	1.63	1.85	1.92	1.44	1.71	1.94	2.02	1.49	1.78	2.01	2.09
	SDRL	0.49	0.57	0.60	0.62	0.53	0.60	0.63	0.65	0.56	0.63	0.65	0.68	0.58	0.64	0.66	0.69
0.30	ARL	1.06	1.18	1.33	1.38	1.09	1.24	1.41	1.47	1.12	1.31	1.49	1.55	1.15	1.35	1.56	1.62
	SDRL	0.24	0.39	0.47	0.49	0.29	0.43	0.50	0.51	0.33	0.46	0.51	0.52	0.36	0.48	0.52	0.52

9.4 Performance comparison of control charts

In this section, we compare the proposed EWMA charts with some of the recently proposed EWMA charts. The performance of each chart is evaluated in terms of logarithm of ARL, i.e., log(ARL). For a fair comparison of the EWMA charts, we fix the in-control ARL of each chart to 200. In each figure, we plot log(ARL) of different EWMA charts versus different values of δ_t.

(i) Proposed two-sided EWMA charts versus two-sided CH-EWMA, SJ-EWMA, HHW1-EWMA and HHW2-EWMA charts

In Figure 9.1, we compare the proposed two-sided EWMA charts, i.e., EWMA-SRS and EWMA-RSS charts, with some existing EWMA charts. The ARLs of the two-sided CH-EWMA, SJ-EWMA, HHW1-EWMA and HHW2-EWMA charts are taken from Huwang et al. (2010). Note that the proposed EWMA charts are based on the symmetric control limits. From Figure 9.1, it is clear that for all values of ξ, the proposed EWMA charts are more powerful in detecting positive shifts in the process dispersion. They also perform better than the CH-EWMA and SJ-EWMA charts in detecting decreases in the process variation. However, they are less sensitive to small downward shifts in the process variations as compared with HHW1-EWMA and HHW2-EWMA control charts when $\xi \geq 0.10$. The EWMA-RSS chart performs uniformly better than the EWMA-SRS chart for detecting all types of random shifts in the process variability.

In Figure 9.2, we make a similar comparison of the EWMA charts, but now the proposed EWMA charts are based on the asymmetric control limits. It is worth mentioning that the EWMA-RSS chart dominates all EWMA charts for detecting overall changes in the process dispersion. Similarly, the EWMA-SRS chart outperforms the CH-EWMA, SJ-EWMA, HHW1-EWMA and HHW2-EWMA charts when detecting an increase in the process variation. It also performs equally well for monitoring decreases in dispersion. However, it remains slightly less sensitive to small decreases in the process variation as compared with the HHW1-EWMA chart.

(ii) Proposed one-sided EWMA charts versus one-sided CH-EWMA, SJ-EWMA, HHW1-EWMA and HHW2-EWMA charts

We compare the proposed one-sided EWMA charts with their one-sided counterparts in Figures 9.3 and 9.4. In Figure 9.3, we compare all EWMA charts when detecting a positive change in the process variation. It is interesting to note that both EWMA-SRS and EWMA-RSS charts perform uniformly better than their counterparts for all types of positive shifts in the process dispersion. Similarly, in Figure 9.4, all EWMA charts are compared for monitoring decreases in the process variability. The EWMA-RSS chart dominates all EWMA charts when detecting decreases in dispersion. Moreover, EWMA-SRS chart also outperforms the CH-EWMA, SJ-EWMA and HHW2-EWMA charts for detecting changes in the process dispersion. However, HHW1-EWMA chart is able to perform slightly better than the EWMA-SRS chart.

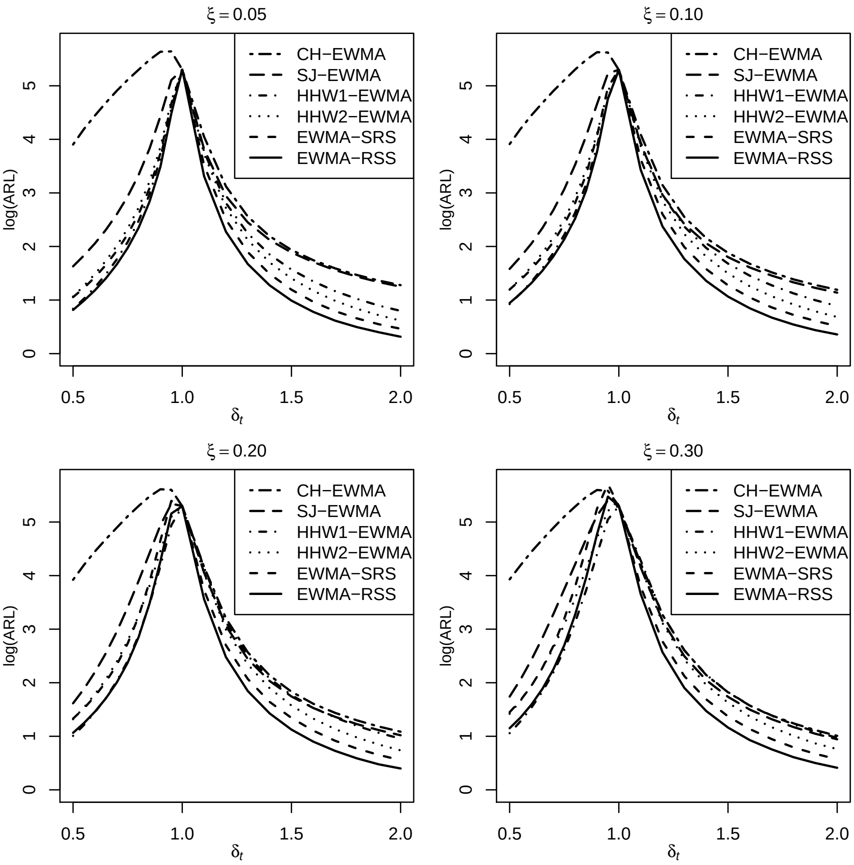

Figure 9.1: Comparison of the two-sided EWMA control charts when EWMA-SRS and EWMA-RSS charts are based on the symmetric control limits

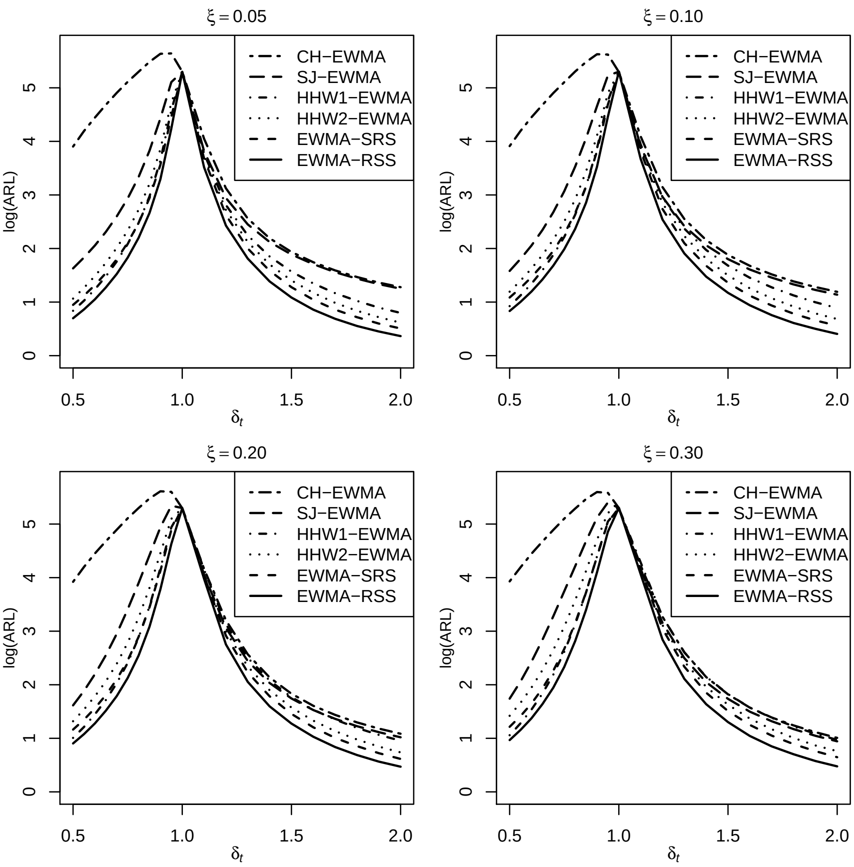

Figure 9.2: Comparison of the two-sided EWMA control charts when EWMA-SRS and EWMA-RSS charts are based on the asymmetric control limits

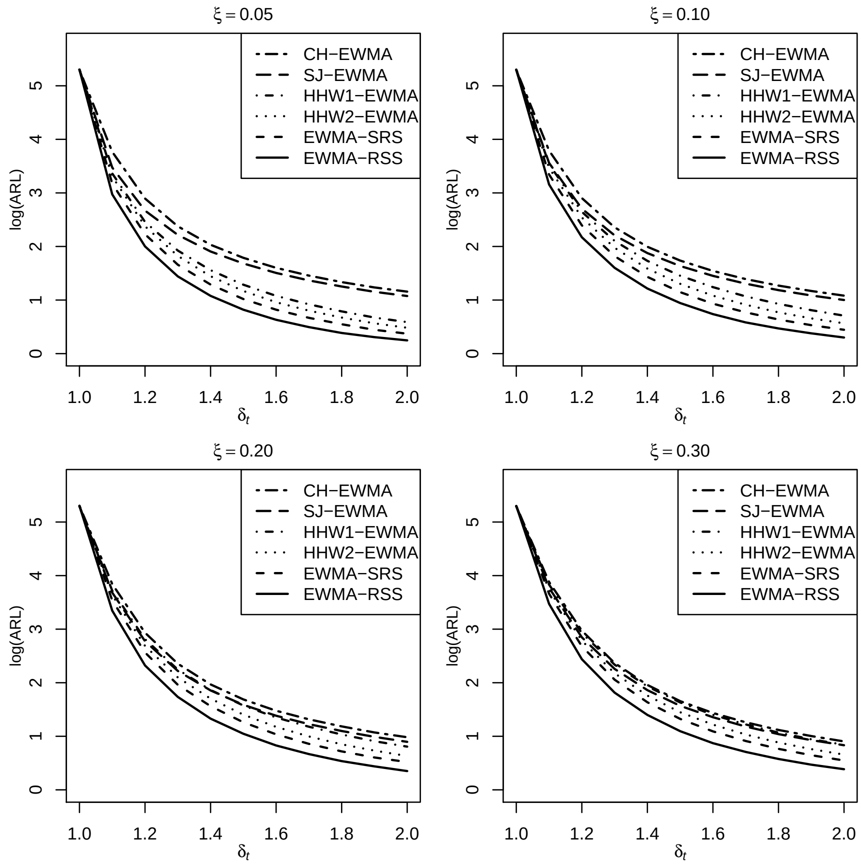

Figure 9.3: Comparisons of the one-sided EWMA control charts for monitoring increases in the process dispersion

(iii) The two-sided EWMA-IRSS charts versus two-sided CH-EWMA, SJ-EWMA, HHW-EWMA and HHW2-EWMA charts

In Figure 9.5, we compare the two-sided EWMA-IRSS charts with its analogues for detecting overall changes in the process variation. Note here that the two-sided EWMA-IRSS chart is based on the asymmetric control limits. Recall that we observed in Table 9.5 that even with errors in ranking, for small values of ξ, i.e., $\xi \leq 0.10$, the in-control ARL of EWMA-IRSS chart remains close to 200 for all values of σ_V^2. Therefore, in Figure 9.5, we compare the performance of EWMA-IRSS chart with other EWMA charts for $\xi = 0.05$, 0.10. It is noteworthy that the EWMA-IRSS chart outperforms CH-EWMA, SJ-EWMA, HHW1-EWMA and HHW2-EWMA charts for detecting overall changes in the process dispersion.

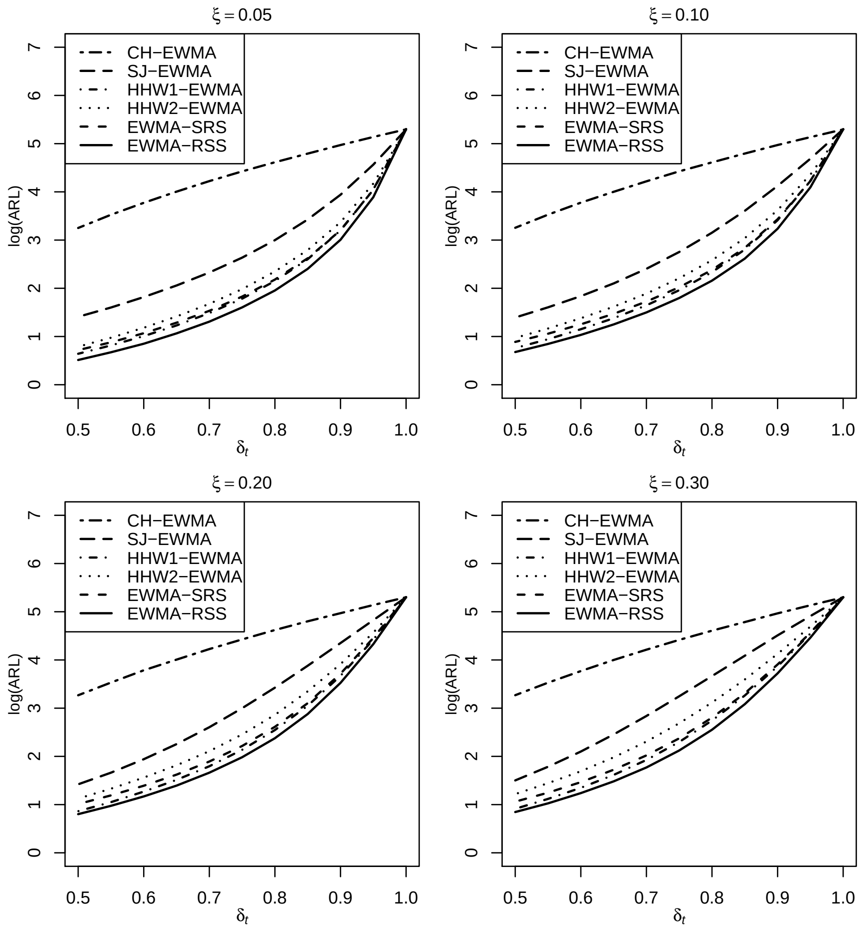

Figure 9.4: Comparisons of the one-sided EWMA control charts for monitoring decreases in the process dispersion

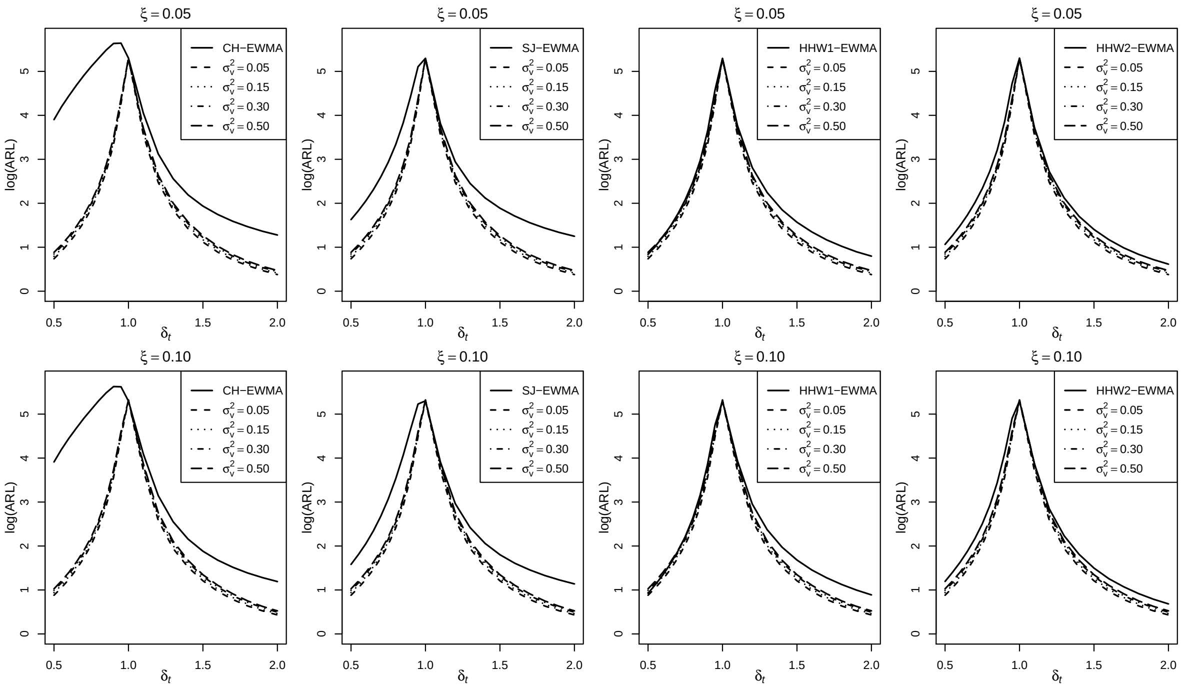

Figure 9.5: Comparisons of the two-sided EWMA control charts when EWMA-IRSS chart is based on asymmetric control limits

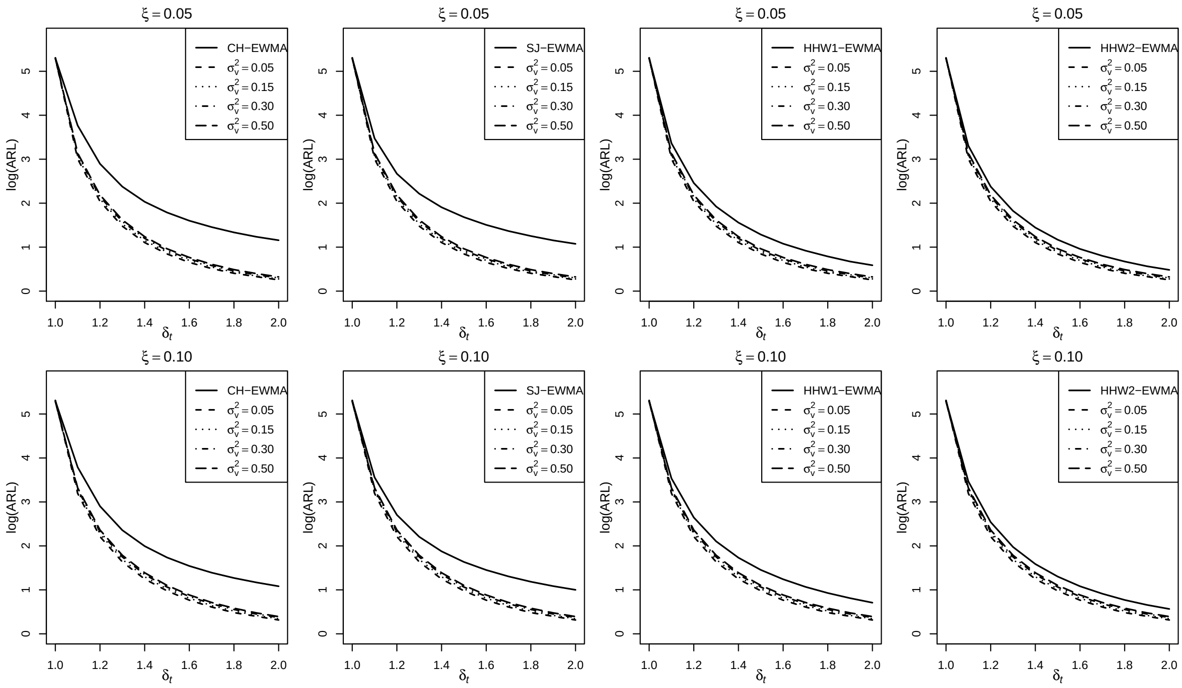

Figure 9.6: Comparisons of the one-sided EWMA control charts with EWMA-IRSS control chart for monitoring increases in the process dispersion

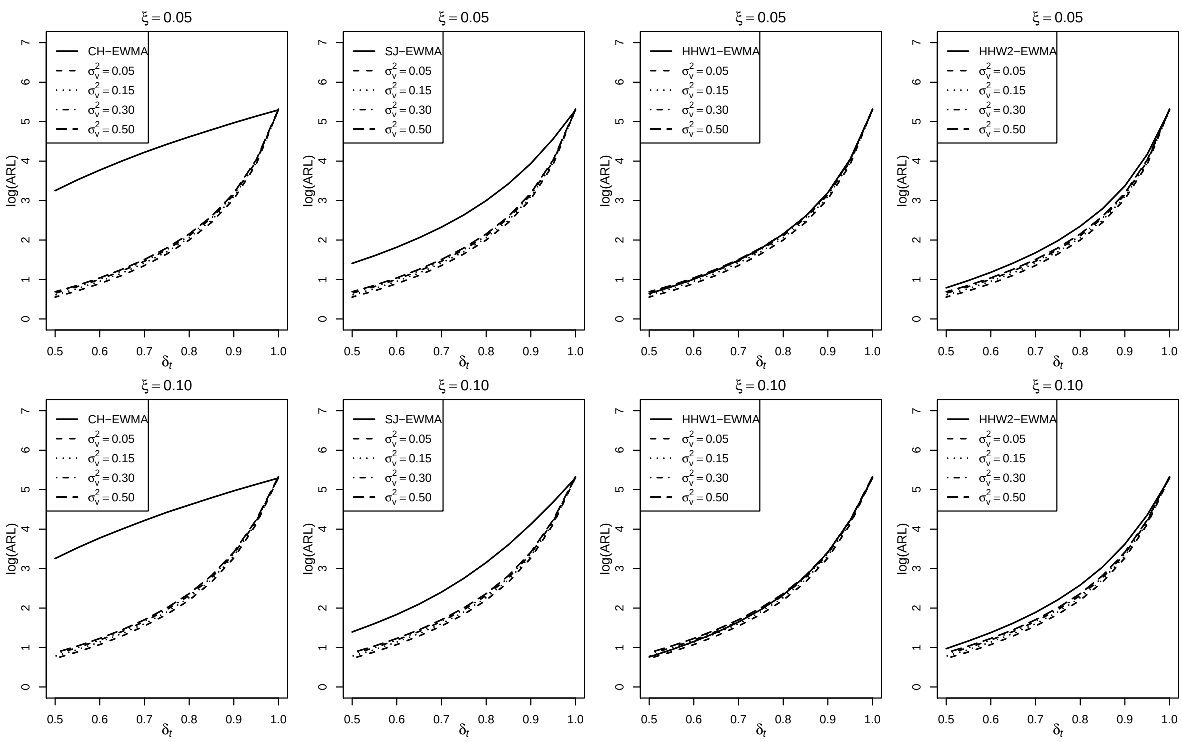

Figure 9.7: Comparisons of the one-sided EWMA control charts with EWMA-IRSS control chart for monitoring decreases in the process dispersion

(iv) The one-sided EWMA-IRSS chart versus one-sided CH-EWMA, SJ-EWMA, HHW1-EWMA, HHW2-EWMA charts

The one-sided EWMA-IRSS chart is also compared with its competitors for detecting increases or decreases in the process dispersion. Figures 9.6 and 9.7 provide a comprehensive comparison of EWMA charts when $\xi = 0.05$ and 0.10. For all kinds of positive shifts in the process variance, the EWMA-IRSS chart detects random shifts substantially quicker than its counterparts. Similarly, when detecting decreases in the process variability, EWMA-IRSS chart dominates CH-EWMA, SJ-EWMA, HHW1-EWMA and HHW2-EWMA charts for all cases. However, when $\xi = 0.10$ with $\sigma_V^2 \geq 0.15$, it remains less sensitive to the small shifts as compared with HHW1-EWMA chart.

9.5 An application to real data

In this section, we consider a real data set to illustrate the construction and applications of the proposed EWMA quality control charts based on SRS and RSS schemes.

Consider a forging process that produces piston rings for an automotive engine. We want to establish statistical control of the inside diameter of the piston ring manufactured by this process (cf. Montgomery, 2009). Forty samples, each of size five, have been taken from this process. The inside diameters of the piston rings are measured in millimeters (mm). We combine the whole data such we have 200 inside diameter measurements for automobile engine piston rings. The data reasonably satisfy the normality assumption. We then standardize the whole data.

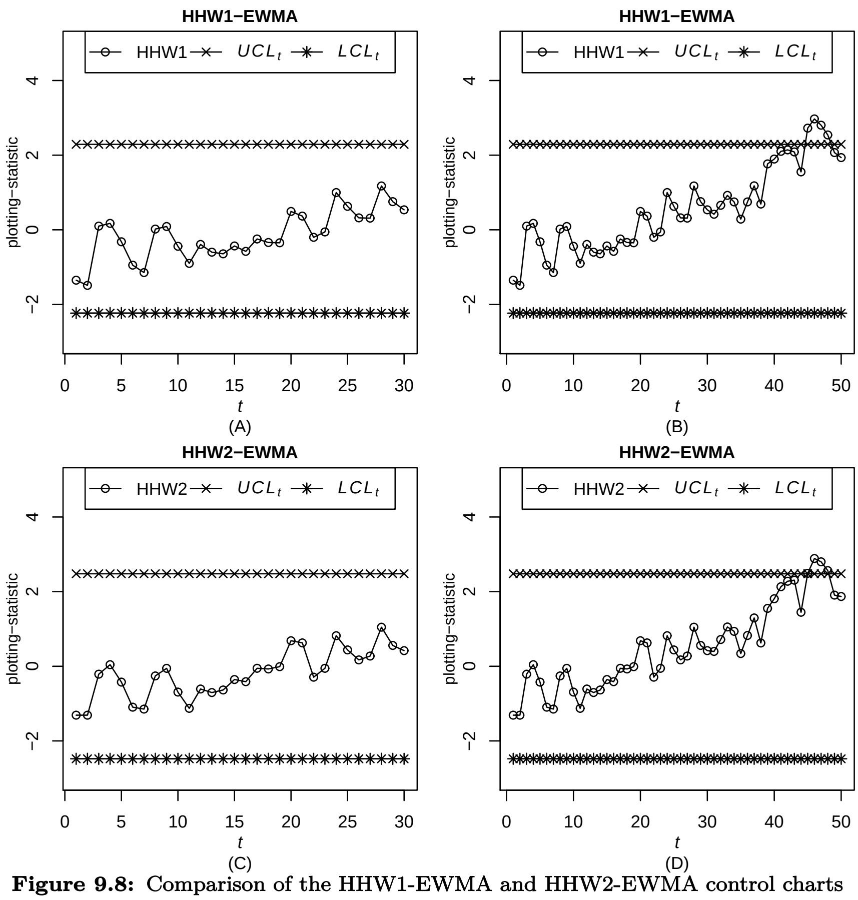

Figure 9.8: Comparison of the HHW1-EWMA and HHW2-EWMA control charts for real data

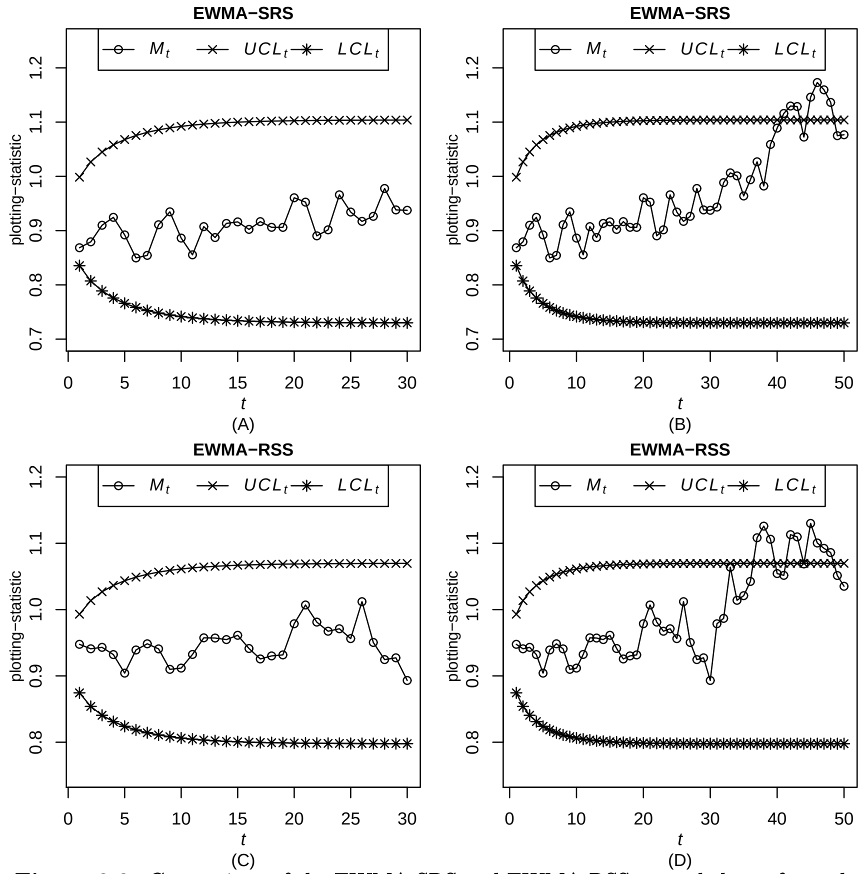

Figure 9.9: Comparison of the EWMA-SRS and EWMA-RSS control charts for real data

For a fair comparison of the EWMA charts, we consider HHW1-EWMA, HHW2-EWMA, EWMA-SRS and EWMA-RSS charts. Suppose that the underlying process is in-control, we draw 30 samples, each of size five, from the standardized measurements under SRS and RSS schemes. Note that the samples are drawn by using with replacement section. For each EWMA chart, the in-control ARL and smoothing parameter are fixed to 200 and 0.10, respectively. Based on these 30 samples, we estimate the control limits and plotting-statistics of all of the EWMA control charts considered here. Now, suppose that after 30th sample, the process gets out-of-control due to a positive shift in the process dispersion. In order to capture this situation, we again draw 20 samples, each of size five, from the standardized measurements under both sampling schemes (SRS and RSS). This time we multiply each observation within each sample by 1.2. Then, we calculate the plotting-statistics of each EWMA control chart. The plotting-statistics and control limits of HHW1-EWMA and HHW2-EWMA control charts are displayed in Figure 9.8. Similarly, in Figure 9.9, we display the plotting-statistics and control limits of the EWMA-SRS and EWMA-RSS control charts. Note that HHW1-EWMA and HHW2-EWMA and EWMA-SRS control charts use the same data obtained under SRS scheme.

In Figure 9.8, HHW1-EWMA and HHW2-EWMA control charts show that the process is in-control state when $t \leq 30$ in sub-figures A and C, respectively. However, in sub-figures B and D, the process gets out-of-control. Both HHW1-EWMA and HHW2-EWMA charts detect the random shift in the process dispersion at the 45th sample. Similarly, in Figure 9.9, both EWMA-SRS and EWMA-RSS control charts show that the process is in-control in sub-figures A and C, respectively, when $t \leq 30$. In Figure 9.9, the proposed EWMA control charts also declare out-of-control signals in sub-figures B and D. It is of interest to note that the EWMA-SRS chart triggers an out-of-control signal at 41st sample whereas the EWMA-RSS chart detects the same shift at 37th sample. Therefore, in practice, the proposed EWMA control charts can be used as an efficient alternative to the existing EWMA control charts.

9.6 Concluding remarks

In this article, we proposed new improved EWMA control charts based on SRS, RSS and IRSS schemes for detecting random shifts in the process dispersion. Extensive Monte Carlo simulations have been used to estimate the run length characteristics of these EWMA control charts. It is worth mentioning that the proposed EWMA control charts perform uniformly better than their counterparts in detecting positive shifts in the process dispersion. With asymmetric control limits, the proposed EWMA-RSS chart significantly outperforms all other EWMA control charts considered here. Similarly, EWMA-SRS control chart also performs uniformly better than the CH-EWMA and SJ-EWMA control charts for detecting overall changes in the process variability. Under IRSS with small values of σ_V^2 and ξ, the EWMA-IRSS control chart is superior to all other existing EWMA control charts in terms of its ability to quickly detect changes in the process variation. Therefore, in practice, we recommend the use of the proposed EWMA charts for SPC practitioners.

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