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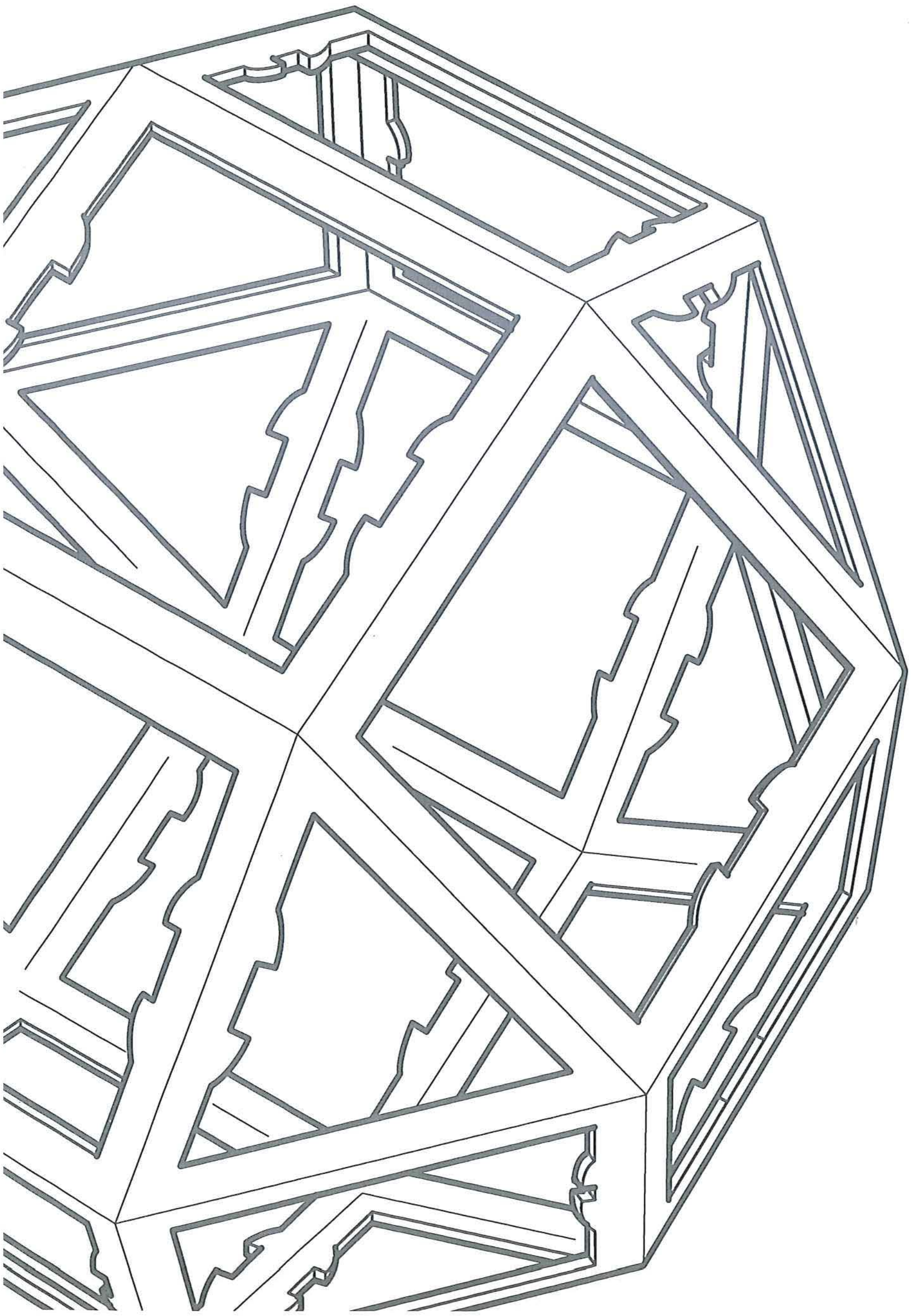
Summer Research Project

Picturing Pythagoras:

An Ancient Chinese Approach to Mathematical Reasoning

by Clare Bycroft



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Picturing Pythagoras: An Ancient Chinese approach to mathematical reasoning

Clare Bycroft

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Abstract

I consider the Theorem of Pythagoras as understood by ancient Chinese mathematicians based on texts dated to the 3rd-century AD. I have attempted to reproduce the diagrams that are indicated by the text to have existed, but are no longer extant. Through an analysis of the Chinese '*Gou-gu* Rule' and its use in the context of a mathematical problem, it can be seen that the diagram was an essential component of the communication and development of ancient Chinese mathematics. By comparison with the Greek proof as given by Euclid in his *Elements*, this study explores the different modes of mathematical reasoning employed by these two independent civilisations.

1 Introduction

Diagrams are useful, especially in mathematics. But are they essential? Euclid's *Elements* are littered with diagrams, illustrating parts of his verbal geometry. But would his proofs make any sense without them? A recent study of the *Elements* has shown that the diagrams are not merely illustrative but essential 'elements' of the text. Indeed, the working 'tool' of the Ancient Greek mathematician.[1]

I will examine whether this is also the case with Ancient Chinese mathematical diagrams of a similar antiquity (between 300BC and 300AD). Luckily for mathematicians and historians alike, many of Euclid's diagrams are still extant, but in the Chinese case many of the original diagrams have been lost - thus I have attempted to reconstruct two of the diagrams that seem to be missing from an original text. These illustrate the Chinese equivalent of the Theorem of Pythagoras as it was understood by mathematicians in the 3rd Century AD.

[1] *The Shaping of Deduction in Greek Mathematics* by Reviel Netz. (Netz, 2003, See Chapters 1 & 3)

In the process of reconstruction, along with a comparison to the Theorem as proved in Euclid's *Elements*, I investigate the role that diagrams play in the representation and development of mathematical principles. This can reveal more about the different approaches to mathematical reasoning as employed by two independent civilisations.

2 The Chinese approach: A diagram, a Rule and a Problem

2.1 The Texts

I will draw evidence from two influential mathematical texts produced by Ancient Chinese scholars in the same century:

Zhou bi suan jing (The Mathematical Classic of the Zhou Gnomon)[2] and,

Jiuzhang Suanshu (The Nine Chapters of the Mathematical Art).[3]

Both of these texts contain work by an anonymous author of an unknown antiquity, but were first compiled and annotated by Chinese scholars in the third Century AD (Cullen, 1996). Over the centuries Chinese scholars have transcribed and contributed to the text, and their annotations have been preserved and kept distinct in each successive edition. It is the earliest known commentary of these works that I will consider in this study. The *Zhou bi suan jing* was originally compiled by Zhao Shuang who lived between 200 and 300AD (Cullen, 1996, pg. 69). It deals primarily with astronomical ideas, but one chapter is dedicated purely to mathematical reasoning in the form of geometry. The other text (hereafter referred to as the *Nine Chapters*) was first compiled in 263AD by Liu Hui, who is considered to be the 'earliest notable Chinese mathematician' (Shen Kangshen et al., 1999). The text is split into nine chapters, each pertaining to a certain theme, or area of mathematics with titles such as, 'Distribution by Proportion', 'Excess and Deficit', 'Rectangular Arrays', and 'Right-Angle Triangles'. Each chapter of the work contains a set of problems which are solved using a common theorem or 'Rule' (sometimes more than one), in conjunction with a method. Both the 'Rules' and the 'Problems' were written by an anonymous author and are possibly as old as 200BC (Cullen, 1996).

These texts contain the earliest known Chinese documentation of the geometrical relationship which we

[2]I use Christopher Cullen's 1996 translation of this text (Cullen, 1996).
[3]I use the 1999 translation by Shen Kangshen et al. for this text (Shen Kangshen et al., 1999).

now learn off by heart as school pupils :

$$h^2 = a^2 + b^2 \tag{1}$$

Where h is the hypotenuse, a and b are the two other sides of a right angle triangle.

2.2 The significance of diagrams

The importance of diagrams in mathematics was well recognised and frequently acknowledged by ancient Chinese scholars. We see this in the prefaces to these texts, which encompass areas of mathematical knowledge other than pure geometry. Zhao Shuang, writing in the 3rd-century AD (Cullen, 1996), states in his preface to the *Zhou bi suan jing*:

> **...But during a few leisure days of convalescence I happened to look at the *Zhou bi*. Its prescriptions are brief but far-reaching; its words are authoritative and accurate. I feared lest it should be cast aside, [or be thought] hard to penetrate, so that those who discuss the heavens should get nothing from it. I set out at once, therefore, to construct diagrams in accordance with the text. My sincere hope was to demolish the high walls and reveal the mysteries of the halls and chambers within. Perhaps in time gentlemen with a taste for wide learning may turn their attention to this work.** (Cullen, 1996)

Zhao understands the importance of the text he has found, and hopes to enlighten others, and he states that the construction of diagrams will be the method by which he will complete this task.

Liu Hui makes a similar point (among others) in his preface to the *Nine Chapters*:

> **Things are known to belong to various classifications. Just as the branches of a tree are to its trunk, so are a multitude of things to an archetype. Therefore I have tried to explain the whole theory as concisely as possible, with spatial forms shown in diagrams, so that the reader should have a reasonably good all-round understanding of it.** (Shen Kangshen et al., 1999)

Liu Hui goes a step beyond Zhao's noble intention to educate others by saying how he thinks this goal is best achieved. He intends to demonstrate mathematical generalities via a multitude of examples, which is a common approach taken by the Chinese mathematicians of his era. Joseph W. Dauben sums it up: 'when it came to establishing the validity of their mathematics, they did so by examples, by analogy, by looking for categories of problems and showing how established methods suited a particular problem type' (Katz, 2007, Dauben's Chap 3). Furthermore, Liu Hui emphasises the need to be concise, as well as easily understood. The link between these two requirements (and the middle clause of the last sentence) is the 'spacial forms shown in diagrams'. This 3rd-century mathematician saw diagrams as essential elements of his published work.

2.3 So where are the diagrams?

Clearly, the use of diagrammatical aids was deemed centrally important for understanding mathematical concepts, but what is striking about these ancient Chinese mathematical texts is their apparent lack of diagrams. In fact, the oldest extant edition of the *Nine Chapters* does not contain a single diagram.[4] There is, however, plenty of evidence that can be found in the text for the use of diagrams or figures, whether originally included in the texts, or used by a reader in conjunction with the commentary. All of Liu Hui's original diagrams have been lost, but attempts have been made by more recent scholars to reconstruct them 'in accordance with the text.' This is not an easy task since the diagrams are more than simply a visual transcription of what the text says; they seem to serve their own independent purpose. Many of the original diagrams of the *Zhou bi* are still extant, but the chapter pertaining to mathematical theory contains only three diagrams. They are set out right at the beginning of the chapter dedicated to geometry of right-angle triangles, but two of them are considered by Christopher Cullen to be superfluous to the Zhao's original text (Cullen, 1996), so I do not include them here. Chinese mathematicians of Zhao and Liu Hui's time were clearly well aware of the importance and usefulness of the Pythagorean relationship (equation (1)). Figure 1 shows how this equation works for a 3, 4, 5 triangle. The diagram comes from a 1214 AD reprint of the *Zhao bi* and is believed to be a correct rendition of Zhao's original diagram (Cullen, 1996). It shows what Western mathematics calls the Theorem of Pythagoras (Shen Kangshen et al., 1999), although Cullen believes 'that there is nothing in the main text that could be considered an attempt at proof.'[5] The diagram itself is the closest he gets.

However, textual evidence of something nearer a mathematical proof of this identity applicable to any right-angle triangle can be found in Liu Hui's commentary of the *Nine Chapters.* It is in chapter nine, titled 'Right-angled Triangles' that we find the '*Gou-gu* Rule' and Liu Hui's 'proof'. This particular chapter contains 24 problems, all involving right-angle triangles. It is unique in the sense that the geometric object itself is the basis of the chapter's significance, rather than the aspect of Chinese technology or commerce which the mathematics can be applied to, as is the case with the other eight Chapters.[6] An analysis of the problems and solutions in chapter nine reveal a very geometrical type of reasoning using a 'non-linguistic

[4] Jock Hoe, of Christchurch, NZ has a copy.

[5] Zhao simply refers to the diagram to explain how it can be used for calculations. See Cullen (1996, Appendix 1) for full his translation and thorough analysis of the chapter of the *Zhou bi.*

[6] The other eight chapters are called: Field Measurement, Millet and Rice, Distribution by Proportion, Short Width, Construction Consultations, Fair levies, Excess and Deficit, and Rectangular Arrays (Shen Kangshen et al., 1999).

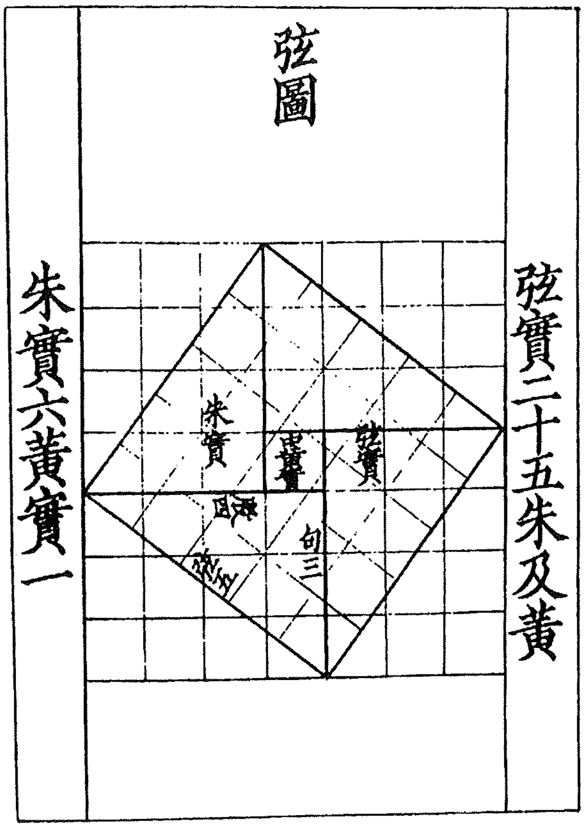


Figure 1: Zhao Shuang's 'Hypotenuse diagram' from the *Zhao bi* (Swetz and Kao, 1977).

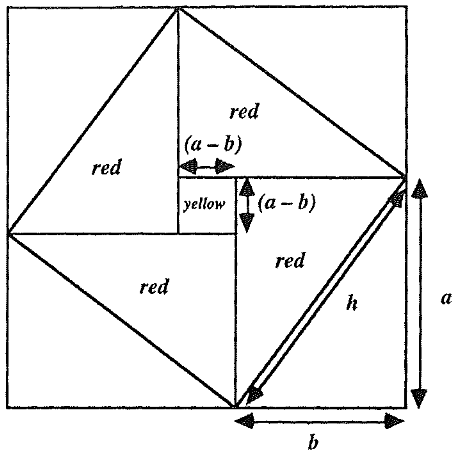


Figure 2: A modern interpretation of the Hypotenuse diagram (Figure 1) found in the *Zhou bi* (Cullen, 1996).

means of communication'[7] that is prevalent in ancient Chinese mathematical texts.

One specific method of reasoning often used is known as the "Out-in complimentary principle" (Shen Kangshen et al., 1999). It stems from the following basic principle: cutting up a 2 dimensional area and rearranging it will not change the total area. This is a sort of visual, diagrammatic algebra, where the conservation of area is equivalent to an equal sign. In these Chinese mathematical texts the Out-in method is commonly used along with a kind of colour code. The colours are essentially used to keep track of certain shapes within the diagram as they 'move' around, much like the alphabet labelling used today. Some of the well-known problems that the Chinese solved using this principle are:

- Approximating the area of a circle
- Finding the area of any triangle
- Finding the volume of a Trapezoid
- Right-angle triangle problems

It is the last of these which I will focus on. I have attempted to reconstruct the lost diagram that constitutes part of Liu Hui's explanation of the *Gou-gu* Rule which is fundamentally based on the Out-in principle.

2.4 A note on language

Before I present the *Gou-gu* Rule and Liu Hui's proof in English, it must be noted that certain aspects of the Chinese written language are inevitably lost in translation. The difficulty of properly portraying the 'semi-symbolic' quality of the Chinese text in European languages has been recently addressed by Jock Hoe, a fluent speaker of both English and Chinese. His translation of a 14th-century Chinese mathematical text[8] attempts to overcome this problem, and in his introduction Hoe notes that 'the semi-symbolic nature of the Chinese written language enabled Chinese mathematicians to express themselves with a conciseness that is almost impossible to attain in highly-inflected languages, using an alphabet, such as prevailed in Europe' (Hoe, 2007, pg 5).

Each word in Chinese is represented by one character, thus there was no need to develop a new set of symbols for mathematics as used in algebraic notation. The language itself was sufficient for both mathematical conciseness and linguistic clarity. However, an English translation fails to preserve these qualities and

[7]Part of Jean-Claude Martzloff's list of the modes of mathematical argument most commonly used by the ancient Chinese (Martzloff, 1997, pg 71).
[8]*The Jade Mirror of the Four Unknowns* (Hoe, 2007).

becomes very wordy and difficult to follow, giving an impression that the mathematics is also unnecessarily long-winded. Hoe warns that when it comes to Chinese mathematical texts, 'a verbal translation obscures the true nature of their achievements, while a translation into modern symbolism over-estimates what they achieved.' Thus, I have gathered evidence from both types of interpretation (verbal and algebraic) in the hope that I neither 'obscure', nor 'over-estimate' the nature of the mathematical reasoning found in the text.

2.5 The *Gou-gu* Rule

The *Gou-gu* Rule defines the relationship between the three sides of any right-angle triangle as described by equation (1). The *gou* and the *gu* are Chinese words for each of the orthogonal sides, hence the name, *Gou-gu.* Chapter nine of the *Nine Chapters* begins with three short 'Problems', each finding the length of one side of a 3, 4, 5 triangle, given the other two sides, then follows the statement of the Rule. This is the anonymous original text to which Liu Hui later added an explanation.

> **Add the squares of the *gou* and the *gu*, take the square root [of the sum] giving the hypotenuse.**$^{(\alpha)}$
> **Further, the square of the *gu* is subtracted from the square on the hypotenuse. The square root of the remainder is the *gou*.**$^{(\beta)}$
> **Further, the square of the *gou* is subtracted from the square on the hypotenuse. The square root of the remainder is the *gu*.**$^{(\gamma)}$

The three parts of the Rule can be expressed in modern-day algebra and we can see that they are essentially rearrangements of the Pythagorean equation (1):

$$hyp = \sqrt{gou^2 + gu^2} \qquad (\alpha)$$

$$gou = \sqrt{hyp^2 - gu^2} \qquad (\beta)$$

$$gu = \sqrt{hyp^2 - gou^2} \qquad (\gamma)$$

Expressed like this they are trivially identical, so why include all three rather than just state the first? This is because, without thinking algebraically, it is not immediately clear that the three are actually expressing the same identity. This suggests that the author understood the concept visually, using solely geometrical objects. It makes more sense to the author that (α), (β) and (γ) should be made distinct, in the way that two sides of an actual triangle are distinct. It also suggests that the Rule is meant as a computational formula, as it is useful to spell out how to find all three possible unknowns that might constitute a numerical problem.

Following the Rule Liu Hui's commentary explains:

The shorter side [of the orthogonal sides] is called the *gou*, and the longer side the *gu*. The side opposite to the right-angle is called the hypotenuse. The *gou* is shorter than the *gu*. The *gou* is shorter than the hypotenuse. They apply in various problems in terms of rates of proportion. Hence [I] mention them here so as to show the reader their origin.

Let the square on the *gou* be red in colour, the square on the *gu* be blue. Let the deficit and excess parts be mutually substituted into corresponding positions, the other parts remain unchanged. They are combined to form the square on the hypotenuse. Extract the square root to obtain the hypotenuse.

In the first paragraph Liu defines the geometrical object. It is implied from the name of the chapter and the previous three problems that he refers to a right-angle triangle. His definition clarifies that the orthogonal sides are called the *gou* and *gu*, carefully distinguishing between the shorter and the longer. He then states where the Rule is used. That is, in the various problems that are contained in the chapter, one of which I will examine later.

The second paragraph of Liu Hui's commentary is evidence of a desire to not just explain the Rule (which is used in the problems following it) but to prove that it is true in general, something which his contemporary Zhao Shuang failed to do. Here Liu describes his 'proof' of the Rule, but it is remarkably short and somewhat vague in comparison to Euclid's proof of the same identity. Sometime, somewhere there existed a diagram to accompany Liu Hui's sentence beginning "Let the square on the *gou* be red...", which would make up for the textual inadequacy in the next sentence: "Let the deficit and excess parts be mutually substituted..." etc. The original diagram has been lost, but there have been several attempts by scholars to reconstruct it.[9] Li Huang, in his *A Detailed Commentary on the Nine Chapters with Diagrams* (around 1800AD) conjectured the following diagram (Figure 2), which seems to correlate with Liu's commentary, as well as visually 'prove' the *Gou-gu* Rule. The bold lines mark out the primary triangle with sides *gou*, *gu* and the hypotenuse.

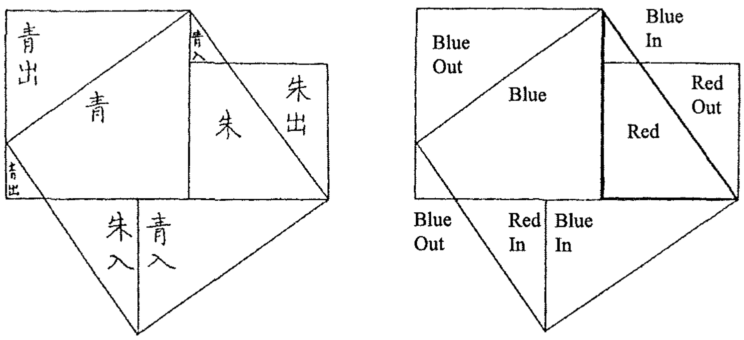


Figure 3: Li Huang's diagram in Chinese and the English translation (Martzloff, 1997, pg 297).

[9]Including two modern authors, Jacob Bronowski in 1973 (Zitarelli, 2007), and Donald B. Wagner (Wagner, 1985).

With the colours included in place of Chinese characters the proof becomes much clearer. Figures 3 and 4 constitute my reconstruction, based on Liu Hui's text and Li Huang's diagram.

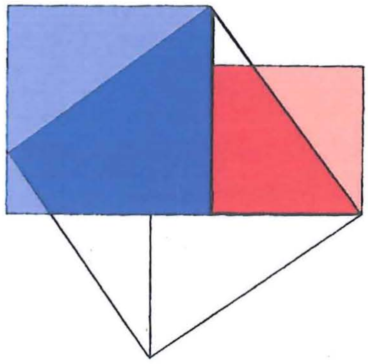


Figure 4: $gu^2 + gou^2$

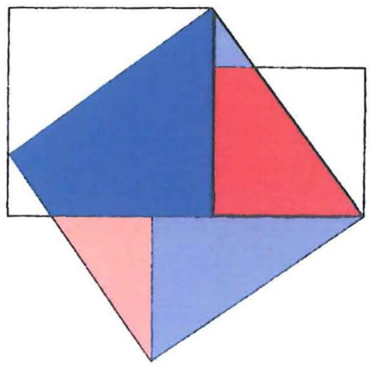


Figure 5: hyp^2

In the first diagram (Figure 3) the square shaded red corresponds to the *gou* square, and the square shaded blue corresponds to the *gu* square. It can be seen by following the 'movement' of the colours from Figure 3 to Figure 4 ('mutually substituting into corresponding positions'), that the two smaller squares 'combine to form the square on the hypotenuse.'

It must be noted that Li Huang's diagram contains no actual coloured squares. Instead, the Chinese characters label the spaces of area as certain colours. This may be due to a lack of coloured printing material at the time, but perhaps Liu Hui did not limit himself in the same way, 1600 years earlier. My diagrams illustrate more clearly how Liu Hui may have been thinking through the *Gou-gu* Rule, or how he intended his readers to follow his explanation. Indeed, the relation between the squares of the three sides is remarkably easy to see when the shapes are really coloured, thus explaining why Liu Hui did not expand his proof beyond three written sentences. Instead, he let his diagram (or diagrams) do the talking.

2.6 An application of the Rule

As seen above, ancient Chinese mathematicians understood the relationship described by the *Gou-gu* Rule, but how did they use it in the context of a geometrical problem? The structure of the problem I present here is common to all the problems in the *Nine Chapters.* It begins with a description of a geometrical object (often a real object like a door or a field) with some dimensions associated with it. Then there is the problem, or question, followed immediately by the numerical solution. Lastly, there is a method for obtaining the solution. As with the Rule there may be more than one method described, by more than one commentator.

Problem 11, of Chapter Nine ('Right-angled Triangles') uses the above *Gou-gu* Rule, and how Liu Hui explains the origins of his method is particularly insightful. This first section is the anonymous original.

Figure 6: An illustration of Problem 11, dated to the Ching dynasty (1662 - 1722AD) (Swetz and Kao, 1977).

Problem 11

Now given a door, which is 68 units longer than its width. The diagonal is 100 units. Tell: what are the height and width?
Answer: The width is 28 units and the height 96 units.
Method: Let the square of 100 units be the *shi* (given number),$^{(\alpha)}$ subtract from it twice the square of half the given difference.$^{(\beta)}$ Halve the remainder and extract its square root.$^{(\gamma)}$ from which subtract half the given difference to obtain the width of the door.$^{(\delta)}$ Adding half the given difference to the square root gives the height of the door.$^{(\epsilon)}$

The method given here describes the order of computation, so that someone could follow the procedure for calculating the height and width of the door using basic arithmetical operations. It is possible to express the method as a series of calculation steps using modern arithmetic notation.

$$
\begin{aligned}
hyp^2 &= 100^2 = 10000 && (\alpha)\\
10000 - 2(\frac{68}{2})^2 &= 7688 && (\beta)\\
\sqrt{7688/2} &= 62 && (\gamma)\\
62 - \frac{68}{2} &= 28 && (\delta)\\
62 + 34 &= 92 && (\epsilon)
\end{aligned}
$$

Thus the method gives the correct solution for the width (28 units) and height (96 units) of the door as required. What has been given is a procedure for this particular problem rather than a general formula for finding the width and height of the door. However, we can write out the general equation for this procedure using algebraic symbolism: h for the diagonal, a for the longer side, and b the shorter side. In Problem 11 $(a-b)$ and h are given as 68 and 100 respectively.

$$\text{The width: } b = \sqrt{\frac{h^2 - 2(\frac{1}{2}(a-b))^2}{2}} - \frac{(a-b)}{2}$$

$$\text{The height: } a = \sqrt{\frac{h^2 - 2(\frac{1}{2}(a-b))^2}{2}} + \frac{(a-b)}{2}$$

Not only does this algebraic representation make the given method look messy and complicated, but as with the Rule, algebraic manipulation can show that these two equations are equivalent.
Using C for the square-root term, the width equation simplifies to:

$$
\begin{aligned}
b &= C - \frac{(a-b)}{2}\\
2b &= 2C - (a-b)\\
(a-b) + 2b &= 2C\\
a + b &= 2C && (*)
\end{aligned}
$$

And the height equation simplifies to the same:

$$
\begin{aligned}
a &= C + \frac{(a-b)}{2} \\
2a &= 2C + (a-b) \\
2a - (a-b) &= 2C \\
a + b &= 2C \qquad (*)
\end{aligned}
$$

This shows that the method has been processed geometrically, and the mode of presentation is designed for computational ease rather than notational efficiency. What can this say about the purpose of the problem? It could be a handy rule for door-makers, but this is doubtful, as the format of the original problem is somewhat impractical. In practice, it is highly unlikely that the *difference* between the width and height of the door $(a - b)$ would be known, but that the actual dimensions of the door are not. The problem is only practical insofar as it uses measurable quantities and real objects to illustrate what is really a mathematical construct. Is it then, an instructive problem for someone interested in learning mathematics? The answer is given directly after the problem is stated, so perhaps not. It would make more sense (to us at least) for a pupil to follow the method first, and then be given the answer after they have worked through the problem themselves.

Notably, we are not told how the original (anonymous) author came up with the method given here. At least, not until Liu Hui's commentary in which he gives a justification of the above procedure, as well as outlining his own, new method. He also writes this in the order that it should be calculated, but he gives the problem some further thought, consequently expanding upon the mathematics. Being neither wholly practical, or entirely pedagogical, it seems the text is designed to document the *mathematical knowledge* contained within the problem/solution structure.

Liu Hui's subsequent exploration of the problem supports this hypothesis.

> **Let the width of the door be the *gou*, the height be the *gu* and the diagonal be the hypotenuse. The difference between the width and the height, 68 units, is the difference between the *gou* and the *gu*. Calculate according to the diagram.**
>
> **The square of the hypotenuse is exactly 10,000 [square] units. Double it, and subtract from it the square of the difference between the gu and the *gou*.$^{(\alpha)}$ Extract the square root from the remainder to obtain the sum of the height and width.$^{(\beta)}$ Subtract the difference from the sum and halve it to obtain the width of the door,$^{(\gamma)}$ to which add their difference to obtain the height of the door.$^{(\delta)}$**

Liu Hui first describes how the problem should be constructed geometrically, then shows how to solve it numerically. As with the anonymous text, we can follow it step-by-step and confirm the accuracy of his results.

$$2 \times 10,000 - 68^2 = 20,000 - 4624 = 15376 \qquad (\alpha)$$
$$\sqrt{15376} = 124 \qquad (\beta)$$
$$\frac{124 - 68}{2} = 28 \qquad (\gamma)$$
$$28 + 68 = 96 \qquad (\delta)$$

In algebraic notation his method can be described as:

$$2h^2 - (a - b)^2 = (a + b)^2 \qquad (\alpha^*)$$
$$\sqrt{(a + b)^2} = a + b \qquad (\beta)$$
$$\frac{(a + b) - (a - b)}{2} = b \qquad (\gamma)$$
$$b + (a - b) = a \qquad (\epsilon)$$

Written out like this the first equation (α^*) is the only algebraically unique identity, obtainable from the *Gou-gu* Rule. This can be seen by simplifying the equation, and yielding the *Gou-gu* relationship:

$$2h^2 - (a - b)^2 = (a + b)^2 \qquad (\alpha^*)$$
$$2h^2 - a^2 - b^2 + 2ab = a^2 + b^2 + 2ab$$
$$2h^2 = 2a^2 + 2b^2$$
$$h^2 = a^2 + b^2$$

Echoing the original Rule, the last three steps (β), (γ), (δ) are basic simplifications, obvious to the modern mathematician who would make use of his algebraic knowledge; tools which have been developed since Liu Hui's time. Thus, when the 3rd-century mathematician goes on to explain the reasoning behind both his method and the original method, it is purely geometric. Again he refers to a diagram that has since been lost, and the text is difficult to follow without the aid of this diagram. He begins by supplying an explanation of how the *original method* works.

The method here finds half the dimensions of the door first. For there are 4 red square areas and 1 yellow square area in a 100 unit square. Twice the square of half the difference between the width and height is 2/4 yellow square area. Half the difference between the *shi* and half the yellow square area is 2 red square areas and 1/4 yellow square, which is just 1/4 of the big square. So its square root is half the sum of the width and height of the door. From that we subtract half the difference between the height and width to obtain the width, and adding that to half the difference we obtain the height of the door.

It is not immediately clear whether Liu Hui is referring to the hypotenuse diagram (Figure 1) of the *Zhou-bi* origin, or the diagram which he points to in the aforementioned proof (see my reconstruction, Figures 3 4), or a completely different diagram altogether. One eminent Chinese scholar, Dai Zhen, conjectured the following diagram (Figure 6) in his edition of the *Nine Chapters*[10](1774). It is simply a mirror image of Zhao Shuang's Hypotenuse diagram (see Figure 1) and works in a similar way.

There is a clue in the colours mentioned by Liu Hui. For the *Gou-gu* Rule he uses Blue and Red, whereas

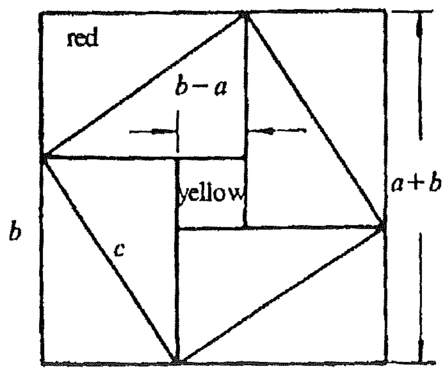


Figure 7: A reconstruction of Liu Hui's diagram by Dai Zhen in his 1774 edition of the *Nine Chapters*

in Problem 11 he uses Yellow and Red, which suggests he refers to the Hypotenuse diagram to illustrate his reasoning. Dai Zhen's diagram then, seems accurate.

However, looking more closely at Liu's commentary suggests there is more to it.

Examining Lui's account algebraically yields the following:

$$\text{“One quarter of the big square”} = \frac{(a+b)^2}{4} = \frac{h^2 - \frac{(a-b)^2}{2}}{2}$$

There appears to be a superfluous factor of a half here. A modern mathematician might wonder, why not

[10]Just as Zhao Shuang had done with the *Zhou-bi*, Dai Zhen reprinted the text and added diagrams.

simplify to:

$$\frac{(a+b)^2}{2} = h^2 - \frac{(a-b)^2}{2}$$

or further to:

$$(a+b)^2 = 2h^2 - (a-b)^2 \text{ ?}$$

When we consider that Liu Hui is not reasoning according to algebraic procedure, the explanation for the extra factors lies in the missing diagram. They are all necessary if you find the solution using the Out-in principle and "calculate according to the diagram." I supply one, with colour included.

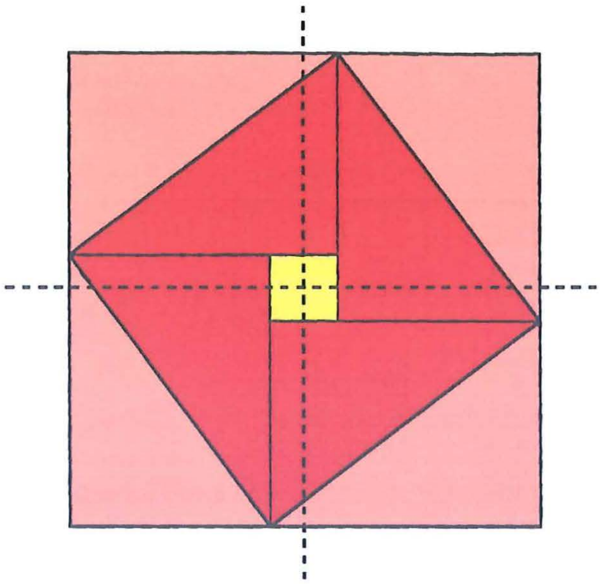


Figure 8: My reconstruction of Liu Hui's diagram. Based on Dai Zhen's, with colours added and the 'big square' cut into quarters.

One must count the shapes, all of which are exactly the same area, and also cut the whole square into quarters. The big square has in total 8 identical triangles, and 4 identical small yellow squares that make up the large yellow square in the middle. Thus, one quarter of the big square must be two red triangles and one small yellow square.

The value for half the large yellow square can be obtained by taking twice one of the small yellow squares which are simply the square of half $(a-b)$. Liu expresses this as, "Twice the square of half the difference between the width and height is 2/4 yellow square area."

Then the value for one quarter of the big square can be obtained by subtracting half the large yellow square from the hypotenuse square (*shi*) and taking half of that. For this Liu says "Half the difference between the *shi* and half the yellow square area is 2 red square areas and 1/4 yellow square."

Once the area of one quarter of the big square is obtained, taking the square root of this gives exactly half of the side of the large square. This is precisely $\frac{(a+b)}{2}$, hence explaining Liu's somewhat imprecise comment, "The method here finds half the dimensions of the door first."

From there it is simple to find a and b using 68 for the difference.

Following Lui Hui's reasoning we can see his very geometrical approach to obtaining the solution, and the addition of colour in the diagram gives the reader (ancient or modern) a better grasp of how the method works.

What comes immediately after in Liu's commentary gives further insight into the value of this diagram.

> **In the same figure, the square of the sum of the *gu* and the *gou* plus the square of their difference is twice the square on the hypotenuse. Extract the square root of half of it to obtain the hypotenuse.**
>
> **Furthermore, twice the square on the hypotenuse minus the square of the difference between the *gou* and the *gu* is the [square of the][11] sum of the *gu* and the *gou*. So it is easy to obtain the *gou* and the *gu* when the hypotenuse is known.**

This section essentially proves the accuracy of Liu's own, new method which uses the identity (α^*) found earlier, namely:

$$\text{The area of the big square: } (a+b)^2 = 2h^2 - (a-b)^2$$

This is how he reasons: Twice the hypotenuse comes to 8 red triangles and 2 lots of the big yellow square. The big square is 8 red triangles and one big yellow square. Thus, the difference is one big yellow square, i.e. "the square of the difference between the *gu* and the *gou*."

Liu Hui has come up with a simpler method for solving this problem, via close analysis of his diagram which I have attempted to reproduce. Effectively, he has removed the factor of a half from the original method by multiplying the hypotenuse square by two first. In the process he has also able to develop a general method for finding one unknown provided any two (of h, a, b) are known.

As promised in his preface, Liu Hui has sought out a mathematical generality (the trunk of a tree) based

[11] This was not in the translation I have used, but seems like a mistake of either the translators or scribes. I have found no alternative explanation.

on a seemingly practical problem (a branch of a tree). The diagram (regardless of whether my reconstruction is strictly accurate) is useful not only for the purpose of demonstrating the validity of his argument, but has been used by the mathematician as a means by which he can improve upon the mathematics involved.

3 The Greek approach: An insightful comparison

Around 500 years before Liu Hui was compiling his edition of the *Nine Chapters*, the Greek mathematician, Euclid, was busy putting together the 13 books that made up his *Elements.* In the first of these we also find a proof of the Theorem of Pythagoras, but his approach to this particular geometrical phenomena is distinctly different from the Chinese. Instead of the Out-in principle, Euclid uses the principle of proportions to prove the relationship between the squares of the sides of a right-angle triangle. It appears in Book 1 as Proposition 47. (See Figure 9)

Before discussing the proof itself it is important to note that the Chinese version (*Gou-gu* Rule) appears as part of a computational exposition, whereas Euclid's version is not concerned with how or where his 'proposition' can be used. Euclid begins with a general statement of the Theorem, but the Chinese 'Rule' begins with how to calculate any side of a right-angle triangle according to the identity (1) which is not explicitly stated until the end of Liu Hui's commentary. However if we consider Liu Hui's commentary independently, we can make genuine comparisons between the two proofs.

Like the Chinese mathematician, Euclid begins his proof with a clear description of the geometrical construct, namely the right-angle triangle, ABC. He then restates the proposition with regards to this triangle.

I say that the square on BC is equal to the squares on BA, AC.

His proof begins proper, as does Liu Hui's, with the hypothetical 'let there be...the square'. This is where the likenesses end, up until Euclid's final statement, 'Therefore the square on the side BC...etc.' which is essentially equivalent to Liu's statement, 'They are combined to form the square on the hypotenuse.'

The way Euclid shows that the areas of the squares must equal is not by physical demonstration, but through the relationships between parallels, straight lines and angles, all which have been previously proven within the earlier parts of Book 1.

[I. 46] refers to the construction of a square on a straight line.

[I. 14] refers to the proof that two adjacent right-angles will form a straight line.

PROPOSITION 47.

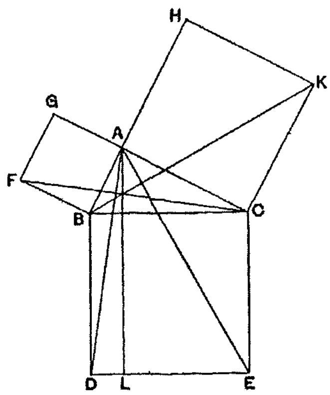
In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right;

I say that the square on BC is equal to the squares on BA, AC.

For let there be described on BC the square $BDEC$, and on BA, AC the squares GB, HC; [I. 46]

through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.



Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles;

therefore CA is in a straight line with AG. [I. 14]

For the same reason

BA is also in a straight line with AH.

And, since the angle DBC is equal to the angle FBA: for each is right:

let the angle ABC be added to each;

therefore the whole angle DBA is equal to the whole angle FBC. [C. N. 2]

And, since DB is equal to BC, and FB to BA, the two sides AB, BD are equal to the two sides FB, BC respectively,

and the angle ABD is equal to the angle FBC;

therefore the base AD is equal to the base FC,

and the triangle ABD is equal to the triangle FBC. [I. 4]

Now the parallelogram BL is double of the triangle ABD, for they have the same base BD and are in the same parallels BD, AL. [I. 41]

And the square GB is double of the triangle FBC, for they again have the same base FB and are in the same parallels FB, GC. [I. 41]

[But the doubles of equals are equal to one another.]

Therefore the parallelogram BL is also equal to the square GB.

Similarly, if AE, BK be joined, the parallelogram CL can also be proved equal to the square HC;

therefore the whole square $BDEC$ is equal to the two squares GB, HC. [C. N. 2]

And the square $BDEC$ is described on BC,

and the squares GB, HC on BA, AC.

Therefore the square on the side BC is equal to the squares on the sides BA, AC.

Therefore etc. Q. E. D.

Figure 9: Sir Thomas Heath's English translation of Euclid's proof of the Theorem of Pythagoras (Heath, 1956).

[I. 4] refers to the proof that two triangles are exactly equal if they have one angle the same and its two sides also the same.

[I. 41] refers to the proof that a parallelogram is exactly double the size of any triangle with the same base, and formed between the same parallels.

[*C.N.2*] refers to Euclid's second 'Common Notion': "If equals be added to equals the wholes are equal."

A rigorous, axiomatic, no-step-missing approach is evident here and is characteristic of Greek mathematics in general.[12] Before Euclid can show that two squares are equal he must show that the lines he has drawn in the diagram must actually be straight. This gives the impression that the text is trying to do away with any diagrams; that somehow the visual elements corrupt the mathematical generality being proposed. Thus we should ask, exactly how necessary is this diagram? Glancing over the proposition and at the pains taken to 'show' that a line must be straight through a *verbal* argument, it seems that the diagram was not put there to independently demonstrate 'straightness' but instead to be a visual helping-hand to the reader. If this were absolutely the case it would imply that the diagram is, in fact, unnecessary for understanding the proof. However, taking a pen and paper and following the proof by constructing the diagram as it is 'drawn out' by Euclid's text is more difficult than should be expected if the diagram is essentially redundant. It is not impossible, although difficult to follow without the aid of Euclid's diagram already there.[13] In contrast to the Chinese proof, there is no 'movement' described, nor is there any mention of colour in Euclid's text, or in his diagram, which consists solely of lines and letters of the Greek alphabet that index the main points of intersection. The diagram *is* necessary, but the essence of Euclid's proof is contained in his prose. As a result, Euclid's proof is far less intuitively obvious than Lui Hui's, which uses the Out-in principle in conjunction with a diagram to visually demonstrate the relationship between the squares of a right-angle triangle. In his attempt to ensure that his proof can be can be accurately presented in words and symbols, without the need for a diagram, Euclid has constructed a much longer, more complicated illustration of this important mathematical identity.

[12] G. Lloyd discusses the 'Greek pre-occupation with axiomatic-demonstration' in his book, *Understanding the World in Ancient Greece and China* (Lloyd, 2002).

[13] As an example, the letter F is undetermined. That is, it appears in the text without having been previously allocated a place in the geometrical construct [cite netz]), so I did not know where to put it in my diagram until I came across a contradiction further on.

4 An important distinction

It is generally accepted that ancient Chinese mathematicians were not concerned with defining every mathematical object or concept as Euclid does in his *Elements*. Rather, they used 'practical' problems that use measurable quantities and numerical algorithms to illustrate their understanding of mathematical generalities. Euclid's 'Definitions' include a Point, a Straight Line and a Circle, among other geometrical elements; his 'Common Notion 2' states that 'If equals be added to equals, the wholes are equal.' These are all ideas that the ancient Chinese took as plainly obvious and did not bother to state explicitly. However, they still saw it as important to clearly define the relationship between length and area, which at first seems like an uncharacteristic pre-occupation with what could be considered a 'common notion.' Evidence of this desire to make a particular distinction between length and area can be found in the first chapter of the *Nine Chapters*, entitled 'Field Measurement'. It covers methods of calculating the areas of geometrical figures and contains the simple 'Rule for Rectangular Fields.'

> **Multiply the number of *bu* in breadth by that in length to obtain the *bu* product.**

Then Liu Hui adds:

> **The product is the area of the field. Multiplying the breadth by the length is generally called *mi*.**

In the 7th century AD an eminent Chinese astronomer, Li Chungfeng, comments on Liu Hui's commentary, pointing out a mistake that seems trivial at first. *Bu* is a unit of length, whereas *Mi* is a unit of area. Liu Hui equates the product of *bu* with *mi*, which is true enough in numerical calculation, but by strict definition of *mi* and *bu* it is not, according to the astronomer.

> **The Rule says: "Multiplying the number of *bu* in breadth by that in length to obtain the *bu* product" whereas Liu says: "Multiplying the breadth by the length is generally called *mi*." Normally, it is incorrect to equate the two different concepts - product and *mi*. Why? *Mi* means a figure covered by a side shifting in one direction, whereas a product is an accumulation of numbers. In fact, as in the nomenclature, the two are, after all, quite different and [I] cannot approve of mixing them together. *Mi* denotes a rectangle with its sides, while the product is the number itself which follows the Rule: "Multiplying gives the *bu* product", which certainly shows that the product is a number. Liu considered it as *mi*, losing sight of the original idea. Liu is right when he says: "The product is the area of a field", but wrong to assert, "Multiplying the breadth by the length is generally called *mi*". In providing commentaries we should retain nothing but the truth. Here I contribute my modest opinion just for reference.**

Li has fairly pointed out an important distinction between a 'product' which is a number, and 'area' which is a geometrical entity. It is a distinction that Liu Hui has probably understood, but failed to make clearer in his commentary. However, it is the relationship between lengths and areas (width $\times$ breadth = area) that allows Liu to calculate the *gou* and *gu* of a right-angle triangle using the *Gou-gu* Rule, and forms the basis his geometric proofs. Eventually the problems involving the *Gou-gu* Rule must solve the algebraic equation $x^2 = A$ where x is *bu* and A is *mi*.

The distinction between area and length is algebraically trivial (and needs to be ignored in the process of calculation). But since the two are geometrically distinct entities, and a diagram is sensitive to this difference, Li Chungfeng's 'correction' betrays the visual basis of their reasoning.

Surprisingly, this distinction is omitted from Euclid's *Elements.* When Euclid says: 'the square on BC is equal to the squares on BA, AC' he takes it as given that he really means: the *area* of the square BC is equal to *the sum of the area* of the squares BA and BC. In general, Euclid avoids working with units of things as they imply a measurable quantity, giving rise to numbers, which are never used to prove his geometry. Perhaps this explains why he also avoids using 'area.'[14] However he is still interested in the application of his ideas in the form of construction - construction of the diagrams he presents. The *Elements* are made up of 'Propositions' which pertain either to the construction of a geometrical object, or to the proof of a theorem which may, within it, require one or more constructions.[15] The construction of geometrical objects forms the basis of his problems, these then lead to arguments that lead to mathematical theorems.

It would be easy to generalise the mathematics of the Chinese and Greeks as follows: the first emphasised numerical calculation and ease of understanding, while the other emphasised strictness of definition and incontrovertible proof. But in light of the above analysis, the delineation between these two civilisations is more complex.

5 Conclusion

The study of the mathematical identity, namely the Theorem of Pythagoras, as proved independently by both the Greeks and the Chinese reveals two different modes of mathematical enquiry. The Chinese approach appeals to the geometric and visual intuition, whereas the Greeks felt that they needed to justify their intuition first, through rigorous proof.

For ancient Chinese scholars the use of diagrammatical aids was centrally important for understanding

[14]See Robin Hartshorne's *Euclid and Beyond* for further exploration of this point (Hartshorne, 2000)

[15]For example, Proposition 46 of Book 1 states: "On a given straight line to describe a square." This is then used in the first part of Proposition 47 (Figure 9), which is the theorem that relates the squares on a right-angle triangle.

the mathematical concepts they present in the texts I have studied here. Thus I (and others) have tried to reconstruct the missing diagrams in the canonical Chinese text the *Nine Chapters of the Mathematical Art.* The analysis of the Chinese equivalent of Euclid's Proposition 47, and an application as developed by Liu Hui in the 3rd-century AD shows that the the diagram was a very useful tool, but not in the same sense as a geometrical construction is a tool. The question is not how to draw it, but how its properties can be used in multiple ways to deal with a multitude of problems.

The consequences of this approach to mathematical reasoning is currently addressed by modern scholars. According to Martzloff, 'visual elements remained an essential component of proofs in China for a long time, while in Greece they were abandoned at an early stage although figurative references were retained' (Martzloff, 1997, pg 73). Perhaps this tendency to distrust visual proof was what lead the ancient Greek, Heron of Alexandria, to write alternative proofs of Euclid's Book II, Propositions 2-10 'without figures.' Sir Thomas Heath suggests that this indicates the origin of an 'easy but uninstructive semi-algebraical method' (Heath, 1963, *Greek Mathematics*, pg 417). Lam Lay Yong's book, *Fleeting Footsteps*, traces the development of algebra in Ancient China and discusses the geometrical basis of their method of extracting the square-root, which enabled them to find a method for extracting the cubed-root (long before the Greeks) by constructing a 3-dimensional, cubic figure (Yong and Se, 2004, pg 110).

If it was the rejection of the visual mode of representation that lead the Greeks towards a symbolic algebra, then perhaps it was the acceptance, or wide use of visual reasoning that meant the Chinese mathematicians were able 'to deal with problems which in the West could not be tackled until a suitable mathematical symbolism had been developed' (Hoe, 2007, pg 5).

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