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DIVERGING TYPE - D METRICS

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Diverging Type - D Metrics

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This paper contains an investigation of algebraically special spaces with two commuting Killing vectors. It is shown that the field equations for these spaces can be reduced to two ordinary differential equations, one of which is quasi-linear in one of the variables. The metric is type D iff it possesses a two dimensional, abelian, orthogonally transitive symmetry group. Finally, the type D metrics of Kinnersley are expressed in various coordinates, including those of Plebanski and Demianski.

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1. Introduction

In this paper we shall investigate the diverging type D metrics of Kinnersley (1969). These are empty Einstein spaces, possessing two distinct non-shearing ray congruences, which, from the Goldberg-Sachs Theorem, must each be Debever-Penrose vectors for the Conformal tensor.

Kinnersley has shown that there are two distinct metrics of this type, the first being a generalisation of that Schwarzschild (Debney 1967, Carter 1968, Debney Kerr and Schild 1969) and the second a completely new solution of the Einstein equations, having the C-metric of Levi-Civita as limit. All these solutions contain two commuting Killing vectors, together with a discrete symmetry, or inversion (Carter 1968), which leads to a particularly simple canonical form for the metrics.

After this paper had been prepared, it was noticed that its main result, the canonical form for the Kinnersley metric, had been discovered earlier by Plebanski and Demiansky (1976). However, we feel that the techniques and results in this paper are sufficiently interesting to warrant publication. We inspected those empty algebraically special spaces with two commuting Killing vectors, and searched for canonical forms for the metric, whereas Plebanski and Demiansky started with a particular canonical form and looked at the resulting geometries - in both cases the Kinnersley metric appeared naturally.

The notation used in this paper is set out in the next section, and then in section III we set up the standard coordinates used in algebraically special spaces (Robinson and Trautman 1962, Kerr 1963, Debney et al. 1969) and give the field equations for type D spaces. In section IV we show why the solutions divide into two classes each of which admit two Killing vectors. We treat these separately in sections V and VI but show that Kinnersley's metric admits coordinates similar to those given by Carter (1968) for the Kerr-Nut metric.

2. Notation

We shall suppose that $\mathcal{S}$ is an algebraically special, empty Einstein space, and that $\{\mathbf{e}_{\alpha\dot{\alpha}}\}$ is a spin (hermitian) basis for the vector fields on $\mathcal{S}$,

$$\mathbf{A} = A^i \partial_i = A^{\alpha\dot{\alpha}} \mathbf{e}_{\alpha\dot{\alpha}} . \qquad (*)$$

A bar will be used to denote complex conjugation, and so $\bar{\mathbf{e}}_{\alpha\dot{\beta}} = \mathbf{e}_{\beta\dot{\alpha}}$. The vector $\mathbf{A}$ is real iff $\bar{A}^{\alpha\dot{\beta}} = A^{\beta\dot{\alpha}}$. The components of the metric are

$$g_{\alpha\dot{\alpha}\beta\dot{\beta}} = g(\mathbf{e}_{\alpha\dot{\alpha}}, \mathbf{e}_{\beta\dot{\beta}}) = \varepsilon_{\alpha\beta}\, \varepsilon_{\dot{\alpha}\dot{\beta}} , \qquad (2.1)$$

where $\varepsilon_{\alpha\beta}$ is the usual antisymmetric spin metric (Pirani 1964). $\{\omega^{\alpha\dot{\alpha}}\}$ is the dual basis for covariant vector fields on $\mathcal{S}$,

$$\omega^{\alpha\dot{\alpha}} = \omega^{\alpha\dot{\alpha}}{}_i\, dx^i , \quad \omega^{\alpha\dot{\alpha}}(\mathbf{e}_{\beta\dot{\beta}}) = \delta^{\alpha\dot{\alpha}}_{\beta\dot{\beta}} . \qquad (2.2)$$

Spin indices are raised and lowered with $\varepsilon_{\alpha\beta}$

$$\xi^{\alpha} = \varepsilon^{\alpha\beta}\, \xi_{\beta} , \; \xi_{\alpha} = -\varepsilon_{\alpha\beta}\, \xi^{\beta} , \qquad (2.3)$$

which agrees with the usual convention for ordinary tensors. The spin transformations

$$\xi^{\alpha'} = S^{\alpha'}{}_{\alpha}\, \xi^{\alpha} , \; \text{det.}\ (S^{\alpha'}{}_{\alpha}) = +1 ,$$

which generate all orthochronous Lorentz transformations

$$\omega^{\alpha'\dot{\alpha}'} = S^{\alpha'}{}_{\alpha}\, \bar{S}^{\dot{\alpha}'}{}_{\dot{\alpha}}\, \omega^{\alpha\dot{\alpha}} , \qquad (2.4)$$

are precisely those for which eq.(2.3) is invariant.

We shall write the connexion forms as $\Gamma^i_j = \Gamma^i{}_{jk} dx^k$. Since the components of the metric are constants for a spin basis,

(*) $i,j,k.... = 1,2,3,4; \quad \alpha,\beta.... = 0,1$

$dg_{\alpha\dot{\alpha}\beta\dot{\beta}} = 0$ and $\Gamma_{\alpha\dot{\alpha}\beta\dot{\beta}} = -\Gamma_{\beta\dot{\beta}\alpha\dot{\alpha}}$. For any real bivector $A_{ij}(= -A_{ji})$ there exists a unique symmetric spinor, $A_{\alpha\beta}$, such that $-A_{\alpha\dot{\alpha}\beta\dot{\beta}} = A_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + A_{\dot{\alpha}\dot{\beta}}\varepsilon_{\alpha\beta}$. Hence

$$\Gamma^{\alpha\dot{\alpha}}_{\beta\dot{\beta}} = \delta^{\alpha}_{\beta}\,\bar{\Gamma}^{\dot{\alpha}}_{\dot{\beta}} + \delta^{\dot{\alpha}}_{\dot{\beta}}\,\Gamma^{\alpha}_{\beta}\ ,\ \Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}\ ,$$

and the covariant differential of any spin tensor is

$$\nabla T^{\dot{\alpha}\dots}_{\beta\dots} = dT^{\dot{\alpha}\dots}_{\beta\dots} + \Gamma^{\dot{\alpha}}_{\dot{\rho}}\,T^{\dot{\rho}\dots}_{\beta\dots} - \Gamma^{\rho\dots}_{\beta\dots}\,T^{\dot{\alpha}\dots}_{\rho\dots} + \dots .$$

Since $\mathfrak{K}$ is empty, the Reimann and conformal tensors are identical, and

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \Psi_{\alpha\beta\gamma\delta}\ \varepsilon_{\dot{\alpha}\dot{\beta}}\ \varepsilon_{\dot{\gamma}\dot{\delta}} + \bar{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\ \varepsilon_{\alpha\beta}\ \varepsilon_{\gamma\delta}\ , \tag{2.6}$$

where $\Psi_{\alpha\beta\gamma\delta}$ is completely symmetric. We will use the standard notation for the components of this tensor

$$\Psi_0 = \psi_{0000},\ \psi_1 = \Psi_{0001},\ \text{etc.} \tag{2.7}$$

The Cartan equations can also be expressed in spinor form

$$d\omega^{\alpha\dot{\alpha}} - (\delta^{\dot{\alpha}}_{\dot{\beta}}\,\Gamma^{\alpha}_{\beta\gamma\dot{\gamma}} + \delta^{\alpha}_{\beta}\,\bar{\Gamma}^{\dot{\alpha}}_{\dot{\beta}\gamma\dot{\gamma}})\ \omega^{\beta\dot{\beta}}\wedge\omega^{\gamma\dot{\gamma}} = 0, \tag{2.8a}$$

$$d\Gamma_{\alpha\beta} + \Gamma_{\alpha\rho}\wedge\Gamma^{\rho}_{\beta} = \tfrac{1}{2}\ \Psi_{\alpha\beta\gamma\delta}\ \omega^{\gamma}{}_{\dot{\rho}}\wedge\omega^{\delta\dot{\rho}}\ . \tag{2.8b}$$

If $\mathbf{e}_{i'} = A^{i}_{i'}\,\mathbf{e}_{i}$ is any change of basis, the $\Gamma^{i'}_{j'}$ are related to the Γ^{i}_{j} by

$$\Gamma^{i'}_{j'} = A^{i'}_{i}\ A^{j}_{j'}\ \Gamma^{i}_{j} - dA^{i'}_{i}\,A^{i}_{j'}\ .$$

Since $\Gamma^{\alpha}_{\beta} = \tfrac{1}{2}\,\Gamma^{\alpha\dot{\alpha}}_{\beta\dot{\alpha}}$,

$$\Gamma^{\alpha'}_{\beta'} = S^{\alpha'}_{\alpha}\ S^{\beta}_{\beta'}\ \Gamma^{\alpha}_{\beta} - d\Gamma^{\alpha'}_{\sigma}\,\Gamma^{\sigma}_{\beta}\ , \tag{2.9}$$

where $S^{\alpha'}_{\alpha}$ is a general spin transformation.

Throughout this paper $\omega^{1\dot{1}}$ will be a fixed principal null vector (hereafter referred to as p.n.v.) of $\mathcal{S}$. The remaining spin transformations can be parametrised by two complex functions, $\mathfrak{a}$, and s,

$$S^{\alpha'}_{\alpha} = \begin{bmatrix} s^{-1} & \mathfrak{a} s^{-1} \\ 0 & s \end{bmatrix}. \tag{2.10}$$

From eq. (2.3) and (2.9)

$$\begin{aligned}
\omega^{1'\dot{1}'} &= s\bar{s}\omega^{1\dot{1}} , \\
\omega^{0'\dot{1}'} &= (\bar{s}/s)(\omega^{0\dot{1}} + \mathfrak{a}\omega^{1\dot{1}}), \\
\omega^{0'\dot{0}'} &= (s\bar{s})^{-1}(\omega^{0\dot{0}} + \mathfrak{a}\,\omega^{1\dot{0}} + \bar{\mathfrak{a}}\omega^{0\dot{1}} + \mathfrak{a}\bar{\mathfrak{a}}\omega^{1\dot{1}}), \\
\Gamma_{0'0'} &= s^2\,\Gamma_{00} , \\
\Gamma_{0'1'} &= \Gamma_{01} + \mathfrak{a}\Gamma_{00} + d(\log s), \\
\Gamma_{1'1'} &= s^{-2}(\Gamma_{11} - 2\,\mathfrak{a}\,\Gamma_{01} + \mathfrak{a}^2\,\Gamma_{00} - d\,\mathfrak{a}).
\end{aligned} \tag{2.11}$$

The parameter $\mathfrak{a}$ gives a null rotation about $\omega^{1\dot{1}}$. Ψ_0 and Ψ_1 are both zero, since $\mathcal{S}$ is algebraically special, and the remaining conformal tensor components transform as

$$\begin{aligned}
\Psi_{2'} &= \Psi_2 , \\
\Psi_{3'} &= (\Psi_3 - 3\mathfrak{a}\Psi_2)/s^2 , \\
\Psi_{4'} &= (\Psi_4 - 4\mathfrak{a}\Psi_3 + 6\,\mathfrak{a}^2\Psi_2)/s^4 .
\end{aligned} \tag{2.12}$$

The space is type D iff $\exists$ an $\mathfrak{a}$ for which $\omega^{0'\dot{0}'}$ is also a p.n.v., i.e. for which $\Psi_3' = \Psi_4' = 0$, and therefore

$$\mathcal{S} \text{ is type D} \Leftrightarrow \Psi_0 = \Psi_1 = 3\Psi_4\Psi_2 - 2\Psi_3{}^2 = 0,\ \Psi_2 \neq 0 . \tag{2.13}$$

The space will then be flat if ψ_2 is also zero.

3. Field Equations

The components of Γ_{00} are the optical scalars of Sachs for the null vector $\omega^{1\dot{1}}$ (Jordan Ehlers and Sachs 1961),

$$\kappa = \Gamma_{000\dot{0}} = \text{geodesy} \quad ,$$

$$\rho = \Gamma_{001\dot{0}} = \text{complex divergence},$$

$$\sigma = \Gamma_{000\dot{1}} = \text{shear} \quad .$$

The fourth component, $\Gamma_{001\dot{1}}$, is not invariant under a null rotation about $\omega^{1\dot{1}}$. The Goldberg-Sachs theorem states that κ and σ are zero for any p.n.v. in an empty Einstein space, and so

$$\Gamma_{00} = \rho\omega^{1\dot{0}} + \Gamma_{001\dot{1}}\,\omega^{1\dot{1}} \;. \tag{3.1}$$

Since the type D nondiverging spaces are rather trivial, we shall assume that $\rho \neq 0$ throughout the next four sections.

$$\text{Assumption:} \qquad \rho \neq 0 \;. \tag{3.2}$$

The (00) component of eq. (2.8) reduces to

$$d\Gamma_{00} + 2\Gamma_{00}\wedge\Gamma_{01} = \Psi_2\omega^{1\dot{0}}\wedge\,\omega^{1\dot{1}} \;, \tag{3.3}$$

and so, from eq. (3.1)

$$d\Gamma_{00}\wedge\Gamma_{00} = 0.$$

The complete solution of this is $\Gamma_{00} = \phi d\zeta$, where ϕ and ζ are both complex functions. A spin transformation (with $s^2 = \phi^{-1}$ and $\mathfrak{L} = \Gamma_{001\dot{1}} / \rho$) transforms Γ_{00} to $d\zeta$ and $\Gamma_{001\dot{1}}$ to zero, so that

$$\Gamma_{00} = \rho\omega^{1\dot{0}} = d\zeta \;. \tag{3.4}$$

Any transformation preserving eq.(3.4) must satisfy $d\zeta' = s^2 d\zeta$, and so

$$\zeta' = \Phi(\zeta)\,,\ s^2 = \Phi_\zeta\,,\ \mathfrak{L} = 0\ , \tag{3.5}$$

where Φ is an analytic function. Following Kerr and Debney (1970) (hereafter referred to as K.D.), ζ, $\bar{\zeta}$ and $v = R\ell(\rho^{-1})$ are used as coordinates, the last being an affine parameter along $\omega^{1\dot{1}}$. The fourth coordinate is any solution of the equations $\mathbf{e}_{0\dot{0}}(u) = 0$, $\mathbf{e}_{1\dot{1}}(u) = 1$.

It was shown in K.D. that

$$\begin{aligned}
\omega^{1\dot{1}} &= du + \Omega d\zeta + \bar{\Omega} d\bar{\zeta} \quad , \\
\omega^{0\dot{0}} &= dv + \beta d\zeta + \bar{\beta} d\bar{\zeta} + U\omega^{1\dot{1}} \\
\omega^{1\dot{0}} &= d\zeta/\rho \quad ,
\end{aligned} \tag{3.6}$$

where β, U and ρ satisfy

$$\begin{aligned}
\beta &= -v\Omega_u + \tfrac{1}{2}(\bar{\mathbf{D}}\mathbf{D}\Omega - \mathbf{D}\mathbf{D}\bar{\Omega})\,, \\
U &= -R\ell[\bar{\mathbf{D}}\Omega_u + 2M\rho] \quad , \\
\rho &= -(v + \Delta)^{-1} \quad ; \\
\Delta &= i\ \mathrm{Im}\ (\mathbf{D}\bar{\Omega})
\end{aligned} \tag{3.7}$$

The operator $\mathbf{D}$ is essentially the derivative in the direction of $\mathbf{e}_{1\dot{0}}$,

$$\mathbf{D} = \partial_\zeta - \Omega\partial_u\ , \tag{3.8}$$

and Ω and the complex mass M are independent of v.

The full group of permissible coordinate transformations, $\mathcal{C}$, is

$$\mathcal{C}: \quad \zeta' = \phi(\zeta) \quad , \tag{3.9a}$$

$$u' = (u + S(\zeta,\bar{\zeta}))\,|\Phi_\zeta| \quad , \tag{3.9b}$$

$$v' = v/|\Phi_\zeta| \quad , \tag{3.9c}$$

$$M' = M/|\Phi_\zeta|^3 \quad , \tag{3.10a}$$

$$\Omega' = (\Omega - S_\zeta - \tfrac{1}{2}(\Phi_{\zeta\zeta}/\Phi_\zeta)(u+S))\,|\Phi_\zeta|/\Phi_\zeta \quad , \tag{3.10b}$$

and the corresponding spin transformation is

$$(s,\mathfrak{m}) = (|\Phi_\zeta|,\ 0). \tag{3.11}$$

Before we write down the components of the connexion, and the Ψ_n, we shall define

$$H = \bar{\mathbf{D}}\partial_u \mathbf{D}\Omega \quad ,$$

$$I = \mathbf{D}M - 3\dot{\Omega}M \quad ,$$

$$J = \mathbf{D}\bar{\mathbf{D}}\bar{\Omega} - \bar{\mathbf{D}}\mathbf{D}\Omega \quad . \tag{3.12}$$

The only nonzero components of Γ and Ψ are then

$$\Gamma_{001\dot{0}} = \rho \quad ,$$

$$\Gamma_{011\dot{0}} = \rho\Omega_u \quad ,$$

$$\Gamma_{011\dot{1}} = -M\rho^2 ,$$

$$\Gamma_{110\dot{1}} = -\bar{\rho}(\bar{\mathbf{D}}\partial_u\Omega) - M(\rho^2 + \rho\bar{\rho}) \quad ,$$

$$\Gamma_{111\dot{1}} = \bar{\Omega}_{uu} + \rho H + \rho^2 I + \rho^3 MJ \quad ,$$

$$\Psi_2 = -2M\rho^3 \quad ,$$

$$\Psi_3 = \rho^2 H + 2\rho^3 I + 3\rho^4 MJ \quad ,$$

$$\Psi_4 = -\rho(\partial_u\partial_u \mathbf{D}\Omega) - \rho^2(\mathbf{D} - 4\Omega_u)H - \rho^3[(\mathbf{D} - 5\Omega_u)I - HJ] - \rho^4[(\mathbf{D} - 6\Omega_u)(MJ) + 2IJ] - 3MJ^2\rho^5. \tag{3.13}$$

The remaining field equations are

$$\text{f.e.I} \quad : \quad 2M_u - (\bar{\mathbf{D}} - 2\bar{\Omega}_u)H = 0 \ ,$$

$$\text{f.e. II} \quad : \quad \mathrm{Im}(2M - \bar{\mathbf{D}}\bar{\mathbf{D}}\mathbf{D}\Omega) = 0 \ ,$$

$$\text{f.e. III} \quad : \quad \bar{\mathbf{D}}M - 3\bar{\Omega}_u M = 0 \ . \tag{3.14}$$

These differ slightly from those in (K.D.), since we have replaced μ (there) by $2M$, and have rearranged eq.(f.e.I). They are necessary and sufficient conditions for the metric

$$ds^2 = 2\omega^{0\dot{0}}\omega^{1\dot{1}} - 2|\omega^{0\dot{1}}|^2 \tag{3.15}$$

with $\omega^{\alpha\dot{\alpha}}$ given by eq.(3.6) and eq.(3.7), to be algebraically special, empty Einstein space. It will then be type D iff eq.(2.13) is satisfied. Equating the various powers of ρ to zero in this gives four equations,

$$\text{D. I} \quad : \quad 3M\partial_u\partial_u\mathbf{D}\Omega - H^2 = 0 \ ,$$

$$\text{D. II} \quad : \quad \mathbf{D}(H^3M^{-4}) = 0 \ ,$$

$$\text{D. III} \quad : \quad 4(\mathbf{D}M)^2 + 3MHJ - 3M(\mathbf{D}\mathbf{D}M) + 9M^2(\partial_u\mathbf{D}\Omega) = 0 \ ,$$

$$\text{D. IV} \quad : \quad \mathbf{D}(J/M) = 0 \ . \tag{3.16}$$

The metric is flat iff $M = 0$, so we must assume

$$M \neq 0. \tag{3.17}$$

4. ISOMETRIES OF $\mathcal{S}$

In this section we will investigate the symmetries of the type D spaces. In (K.D.) it was shown that any Killing vector, K, of the metric of section 3 can be written as

$$\mathsf{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + R\ell(\alpha_\zeta)(u\partial_u + v\partial_v) + P\partial_u \,, \tag{4.1}$$

where P is a real function of ζ and $\bar{\zeta}$, and α is an analytic function of ζ. This is equivalent to the fact that K must be an infinitesimal generator of the coordinate group $\mathcal{C}$. A simple calculation gives the following transformation equation for α and P,

$$\begin{aligned} \alpha' &= \Phi_\zeta \alpha \,, \\ P' &= |\Phi_\zeta| \, [P - \mathrm{Re}(\alpha_\zeta)S - \mathsf{K}S] \,. \end{aligned} \tag{4.2}$$

If α is non-zero, then the coordinates can be chosen so that $\mathsf{K} = \partial_\zeta + \partial_{\bar{\zeta}}$, whereas if it is zero then $\mathsf{K} = P\partial_u$, which is already a simple expression.

We shall introduce the following notation:

<u>Definition</u>: $\mathcal{D}$ is the set of all type D metrics.

<u>Definition</u>: $\mathcal{A}(n)$ is the set of algebraically special spaces for which ρ and M are non-zero, and which admit at least n <u>commuting</u> Killing vectors.

We shall not need to consider $\mathcal{A}(n)$ for $n > 2$ since these were shown to be empty in K.D.. Also Kinnersley has shown that $\mathcal{D} \subset \mathcal{A}(2)$. We will give an independent proof of this later.

<u>Definition</u>: $\mathcal{A}(1,N)$ is the set of metrics which admit a Killing vector of the type $P\partial_u$; $\mathcal{A}(n,R) = \mathcal{A}(n) \setminus \mathcal{A}(1,N)$; and $\mathcal{A}(n,N)$ are those algebraically special spaces containing at least n commuting Killing vectors, one of which is $P\partial_u$.

In the above, R and N stand for "radiating" and "non-radiating" (Trim and Wainwright, 1974) respectively. In $\mathcal{a}(2,R)$ the $(\zeta,\bar{\zeta},u,v)$ coordinates can be chosen so that the metric is independent of ζ, and $\bar{\zeta}$, and so ∂_ζ is a complex Killing vector.

If $\mathcal{g} \in \mathcal{a}(1,N)$, it is convenient to introduce a slightly different coordinate system, $(\zeta,\bar{\zeta},r,s)$ defined by

$$s = u/P, \qquad r = vP \tag{4.3}$$

for which the metric is independent of s, and can be written as

$$(d\tau)^2 = -2\Sigma P^{-2}\left|d\zeta - iP^2\delta_{\bar{\zeta}}\omega/\Sigma\right|^2 + (2dr + \pi\omega/\Sigma)\omega\ , \tag{4.4}$$

where

$$\begin{aligned}
\omega &= P^{-1}\dot{\omega}^{11} = ds + \Lambda d\zeta + \bar{\Lambda}d\bar{\zeta}\\
\Lambda &= (\Omega + sP_\zeta)P^{-1}\quad , \qquad m = -2MP^3\\
\delta &= -\, i\Delta P = P^2\mathrm{Im}(\bar{\Lambda}_\zeta)\ , \quad Q = \delta P,\\
\Sigma &= P^2/\rho\bar{\rho} = r^2 + \delta^2,\\
\pi &= -2K_2 - 2\mathrm{Rl}(m)r + 2K_1 r^2,
\end{aligned} \tag{4.5}$$

and K_1 and K_2 are the two-curvatures for the metrics $d\zeta d\bar{\zeta}/P^2$ and $d\zeta d\bar{\zeta}/Q^2$ respectively,

$$K_1 = PP_{\zeta\bar{\zeta}} - P_\zeta P_{\bar{\zeta}}\ , \quad K_2 = QQ_{\zeta\bar{\zeta}} - Q_\zeta Q_{\bar{\zeta}}\ . \tag{4.6}$$

All these functions are independent of r and s, except for π and Σ which are quadratic in r.

The field equations for this metric are

$$\text{f.e.I} \qquad K_{1\zeta\bar{\zeta}} = 0 \,, \tag{4.7}$$

$$\text{f.e.II} \qquad -\mathrm{Im}(m) + PQ_{\zeta\bar{\zeta}} - P_{\zeta}Q_{\bar{\zeta}} - P_{\bar{\zeta}}Q_{\zeta} + P_{\zeta\bar{\zeta}}Q = 0, \tag{4.8}$$

$$\text{f.e.III} \qquad m_{\bar{\zeta}} = 0. \tag{4.9}$$

Any other symmetry must have the form

$$\mathbf{K} = \alpha\partial_{\zeta} + \bar{\alpha}\partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + T(\zeta,\bar{\zeta})\partial_s \,, \tag{4.10}$$

where $\alpha = \alpha(\zeta)$, T is real, and a_0 is a real constant. It was proved in (K.D.) that $\mathbf{K}$ is then a symmetry iff

$$\mathbf{K}P + (a_0 - \mathrm{Rl}(\alpha_{\zeta}))P = 0 \,, \tag{4.11a}$$

$$\mathbf{K}m + 3a_0 m = 0 \,, \tag{4.11b}$$

$$\mathbf{K}\delta + a_0\delta = 0 \,. \tag{4.11c}$$

The function T is defined up to an additive constant (corresponding to the known symmetry, ∂_{ζ}) by

$$T_{\zeta} + \mathbf{K}\Lambda + (\alpha_{\zeta} - a_0)\Lambda = 0 \,. \tag{4.12}$$

The integrability condition for this is just

$$\mathbf{K}(\delta/P^2) + (2\mathrm{Rl}(\alpha_{\zeta}) - a_0)\,(\delta/P^2) = 0 \,,$$

and this follows trivially from equation (4.11).

Kinnersley (1969) has proved the following theorem

<u>Theorem 1</u>: Every type D metric admits two commuting Killing vectors, and so $\mathcal{D}$ is the disjoint union of $\mathcal{D}(2,R)$ and $\mathcal{D}(2,N)$ where $\mathcal{D}(2,Q) = \mathcal{D} \cap \mathcal{A}(2,Q)$, $Q = R,N$.

Proof: A proof of this will be given, since rather different coordinates are being used to those of Kinnersley, and certain equations needed later will be derived.

If eq. (D.I) is differentiated with respect to $\bar{\mathbf{D}}$ and the first two field equations used, then the following equation is obtained

$$\partial_u [H^3 M^{-4}] = 0 \ ,$$

and so, from eq. (D.II)

$$H = \bar{f}(\bar{\zeta}) M^{4/3} \tag{4.13}$$

where f is an analytic function of ζ. If the coordinate transformation, eq. (3.9), is applied ,

$$(\partial_u \mathbf{D}\Omega)' = \Phi_\zeta^{-2}[\partial_u \mathbf{D}\Omega - \tfrac{1}{2}(\ln\Phi_\zeta)_{\zeta\zeta} + \tfrac{1}{4}(\ln\Phi_\zeta)^2{}_\zeta] \ , \tag{4.14a}$$

$$H' = H/|\Phi_\zeta|^2 \ \Phi_\zeta \ , \tag{4.14b}$$

and so $f' = \Phi_\zeta f$. The function f can therefore be transformed to a constant, $-3\lambda_0$ (say), by an appropriate choice of Φ. We will complete the proof of theorem 1 by proving the following two lemmas.

Lemma 1(R): The type D metrics for which H = 0 are precisely those of Type $_0\mathcal{D}(2,R)$.

Proof: From eq. (4.13) and eq. (D.I),

$$\bar{\mathbf{D}}\partial_u \mathbf{D}\Omega = -3\lambda_0^{-3}\bar{N}^{-4} \ , \tag{4.15a}$$

$$\partial_u \partial_u \mathbf{D}\Omega = 3\lambda_0^{-3} N^{-5} \ , \tag{4.15b}$$

where M has been replaced by $\bar{N}$,

$$\bar{N} = M^{-1/3}\lambda_0^{-1} \neq 0 \ . \tag{4.16}$$

If eq.(4.15a) is substituted into eq.(f.e.I) and eq.(f.e.III) used, then $\partial_u(N - \Omega) = 0$, and so N can be written as

$$N = \Omega + \varepsilon(\zeta,\bar{\zeta}) \quad . \tag{4.17}$$

From eq.(3.10) the transformation $u \rightarrow u + S(\zeta,\bar{\zeta})$ maps $\varepsilon \rightarrow \varepsilon - S_\zeta$, and so ε can be eliminated if it can be shown that $\varepsilon_{\bar{\zeta}} = \bar{\varepsilon}_\zeta$.

The following two (equivalent) equations are obtained by substituting eq.(4.17) into eq.(f.e.III),

$$(\partial_\zeta + \varepsilon\partial_u)N = 0 \ , \tag{4.18}$$

$$\mathbf{D}\Omega + \varepsilon_\zeta + N\partial_u N = 0 \ . \tag{4.19}$$

By differentiating eq.(4.19) with respect to u, we obtain

$$\partial_u \mathbf{D}\Omega = -(\tfrac{1}{2}N^2)_{uu} \ . \tag{4.20}$$

Now from eq.(4.15), $(\bar{\mathbf{D}} + \bar{N}\partial_u)\partial_u\mathbf{D}\Omega = 0$, and so

$$(\partial_{\bar{\zeta}} + \bar{\varepsilon}\partial_u)(N^2)_{uu} = 0 \ . \tag{4.21}$$

The commutators of the differential operators in eq.(4.18,21) are

$$[\partial_\zeta + \varepsilon\partial_u, \partial_{\bar{\zeta}} + \bar{\varepsilon}\partial_u] = (\bar{\varepsilon}_\zeta - \varepsilon_{\bar{\zeta}})\partial_u \ , \ [\partial_\zeta + \varepsilon\partial_u, \partial_u] = 0,$$

and therefore if eq.(4.20) is differentiated by $\partial_\zeta + \varepsilon\partial_u$,

$$(\varepsilon_{\bar{\zeta}} - \bar{\varepsilon}_\zeta)(N^2)_{uuu} = 0 \ .$$

From eq.(4.15b), $(N^2)_{uuu} \neq 0$, so that $\mathrm{Im}(\varepsilon_{\bar{\zeta}}) = 0$, and therefore ε can be transformed to zero. From eq.(4.18,21), $(N^2)_{uu}$ is a function of u alone, and so from eq.(4.20) $\partial_u\mathbf{D}\Omega$ is a function of u. Finally, from eq.(4.15b), N itself is independent of ζ and $\bar{\zeta}$, and

$$M = \tfrac{1}{2}\mu_0\bar{\Omega}^{-3} \ , \ \Omega = \Omega(u) \ , \quad \mu_0 = 2\bar{\lambda}_0^{3} \quad . \tag{4.22}$$

Since the metric is a function of M,Ω and v, ∂_ζ must be a complex Killing vector, and so $\mathcal{B} \in \mathcal{D}(2,R)$.

Lemma 1(N): The type D metrics for which $H = 0$ are precisely those of Type $\mathcal{D}(2,N)$.

Proof: When $H = 0$, eq.(f.e.I) and eq.(f.e.III) reduce to $M_u = \Omega_{uu} = 0$, and eq.(D.I) and (D.II) will be satisfied iff $\partial_u(\partial_u \mathbf{D}\Omega) = \bar{\mathbf{D}}(\partial_u \mathbf{D}\Omega) = 0$, i.e. $\partial_u \mathbf{D}\Omega = g(\zeta)$, an analytic function of ζ. This can be transformed to zero by solving the third order differential equation for $\Phi(\zeta)$ obtained by replacing $\partial_u \mathbf{D}\Omega$ by $g(\zeta)$ on the right hand side of eq.(4.14a), and equating it to zero.

$$\partial_u \mathbf{D}\Omega = \partial_{u\zeta}\Omega - \Omega^2_u = 0 \ . \tag{4.23}$$

From eq.(f.e.III),

$$\Omega_u = -(\ln \bar{M}^{-1/3})_\zeta \ , \tag{4.24}$$

and so if this is substituted into eq.(4.23)

$$(M^{-1/3})_{\bar{\zeta}\bar{\zeta}} = 0 \ , \tag{4.25}$$

so that $M^{-1/3}$ is linear in $\bar{\zeta}$. Now eq.(D.III) can be simplified to

$$(M^{-1/3})_{\zeta\zeta} = 0 \ , \tag{4.26}$$

and so $M^{-1/3}$ is bilinear in ζ and $\bar{\zeta}$. We will eventually prove that it is in fact a real bilinear function multiplied by a complex constant.

From eq.(f.e.II) and eq.(D.IV),

$$\mathbf{D}\bar{\mathbf{D}}\bar{\mathbf{D}}\mathbf{D}\Omega - \mathbf{D}\mathbf{D}\mathbf{D}\bar{\mathbf{D}}\bar{\Omega} = 2\mathbf{D}(M - \bar{M}) \ ,$$

$$M\bar{\mathbf{D}}\mathbf{D}\,[(\bar{\mathbf{D}}\mathbf{D}\Omega - \mathbf{D}\bar{\mathbf{D}}\bar{\Omega})/M] = 0 \ .$$

Subtracting one from the other, and using the commutation relation for $\mathbf{D}$ and $\bar{\mathbf{D}}$,

$$[\mathrm{D},\bar{\mathrm{D}}] = (\bar{\mathrm{D}}\Omega - \mathrm{D}\bar{\Omega})\partial_u = -2\Delta\partial_u ,$$

the following equation is obtained after a rather lengthy calculation,

$$6\bar{M}\Omega_u = 2\ln M_\zeta [\bar{M} + 2\Delta(D\partial_u\bar{\Omega})] - 6\Delta DD\partial_u\bar{\Omega} . \tag{4.27}$$

The coefficient of 2Δ in this is just

$$2(\ln M)_\zeta\bar{\Omega}_{u\zeta} - 3\bar{\Omega}_{u\zeta\zeta}$$

which is zero, as can be proved by substituting eq.(4.24) into the $\bar{\zeta}$ derivative of eq.(4.26). Eq.(4.27) now reduces to

$$\Omega_u = -(\ln M^{-1/3})_\zeta . \tag{4.28}$$

From eq.(4.24) and eq.(4.28), $(M/\bar{M})$ is independent of ζ and $\bar{\zeta}$, and so M can be written as

$$M = -\tfrac{1}{2}\mu_0 P^{-3} , \tag{4.29}$$

where μ_0 is a constant, and P is a real bilinear function of ζ and $\bar{\zeta}$,

$$\begin{aligned} &P_{\bar{\zeta}\bar{\zeta}} = P_{\zeta\zeta} = P_u = 0 , \\ &\Omega_u = -(\ln P)_\zeta , \quad \Delta_u = 0 . \end{aligned} \tag{4.30}$$

If we write $\Lambda = \Omega P^{-1} + uP_\zeta/P^2$, as in eq.(4.5), then Λ is independent of u, and the transformations $(v,u) \to (r,s)$, given by eq.(4.3), transform the metric to that of eq.(4.4). This is independent of s, and so $\partial_s(= P\partial_u)$ is a Killing vector. The mass function m is constant,

$$m = -2MP^3 = \mu_0 ,$$

as is the 2 curvature, K_1, from eq.(4.30).

The last two type D conditions, eq.(D.III,IV) are satisfied iff Q, defined in eq.(4.5) is also a real bilinear function of $\zeta,\bar{\zeta}$.

Furthermore, eq.(f.e.II) then gives $\mathrm{Im}(\mu_0)$ as a function of the coefficients of the two bilinear forms, and so an $\mathcal{A}(1,N)$ metric is type D iff $\exists$ a ζ coordinate for which P and Q are bilinear.

The coordinate transformations which preserve the form of the $\mathcal{A}(1,N)$ metric of eq.(4.4) are

$$\zeta' = \Phi(\zeta)\ ,\ r' = C_0 r\ ,\ s' = C_0^{-1}(s + A(\zeta,\bar{\zeta}))\ ,$$
$$P' = |\Phi_\zeta| C_0 P\ ,\ Q' = |\Phi_\zeta| C_0^2 Q\ ,$$
$$\Lambda' = (\Lambda - A_\zeta)/C_0\Phi_\zeta\ ,\ m' = C_0^3 m\ , \tag{4.31}$$

where C_0 is an arbitrary constant. Since eq.(4.23) must be preserved, it follows from eq.(4.14a) that

$$(\ln\Phi_\zeta)_{\zeta\zeta} = \tfrac{1}{2}((\ln\Phi_\zeta)_\zeta)^2 .$$

The complete solution of this is the bilinear transformation

$$\zeta' = \frac{a\zeta + b}{c\zeta + d}\ ,\qquad \det.\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1\ . \tag{4.32}$$

It will be convenient to use a homogeneous spinor notation, where we write

$$\zeta = \zeta^1/\zeta^0,\quad \zeta^{\dot{\alpha}} = \bar{\zeta}^{\alpha}\ , \tag{4.33}$$

so that the forms P and Q can be written as

$$P = \mathcal{P}/|\zeta^0|^2\ ,\quad Q = \mathcal{Q}/|\zeta^0|^2\ ,$$

where

$$\mathcal{P} = P_{\alpha\dot{\alpha}}\zeta^{\alpha}\zeta^{\dot{\alpha}}\ ,\ \mathcal{Q} = Q_{\alpha\dot{\alpha}}\zeta^{\alpha}\zeta^{\dot{\alpha}}\ , \tag{4.34}$$

and, since P and Q are real, the spin tensors $P_{\alpha\dot{\alpha}}$ and $Q_{\alpha\dot{\alpha}}$ are both Hermitean.

The bilinear transformations on ζ are equivalent to the group of spin transformations

$$\zeta^{\alpha'} = S^{\alpha'}_{\alpha}\zeta^{\alpha}, \quad S^{\alpha'}_{\alpha} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \det. \ S^{\alpha'}_{\alpha} = 1 ,$$

where $\zeta' = \zeta^{1'}/\zeta^{0'}$. Since

$$\Phi_{\zeta} = (ad - bc)/(c\zeta + d)^2 = (\zeta^0/c\zeta^1 + d\zeta^0)^2 = (\zeta^0/\zeta^{0'})^2 ,$$

it is easily seen from eq.(4.31) that

$$\mathcal{P}' = c_0\mathcal{P}, \quad \mathcal{Q}' = c_0^2\mathcal{Q} . \tag{4.35}$$

It was shown in (K.D.) that if $\mathbf{K}$ is any other Killing vector for this metric, then it must be an infinitesimal transformation of the coordinate group in eq.(4.31) and so it must be

$$\mathbf{K} = \alpha\partial_{\zeta} + \bar{\alpha}\partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + T(\zeta,\bar{\zeta})\partial_s , \tag{4.36}$$

where α is an analytic function of ζ, T is real, and, as in eq.(4.11), a_0 is a real constant. The metric is invariant under this iff

$$\begin{aligned} &\mathbf{K}P - (a_0 + \mathrm{Re}(\alpha_{\zeta}))P = 0 , \\ &\mathbf{K}Q - (2a_0 + \mathrm{Re}(\alpha_{\zeta}))Q = 0 , \\ &\mathbf{K}m + 3a_0 m = 0 , \end{aligned} \tag{4.37}$$

and T is then any solution of

$$T_{\zeta} + \mathbf{K}\Lambda + (\alpha_{\zeta} - a_0)\Lambda = 0 . \tag{4.38}$$

Since $m = \mu_0$, a non-zero constant, a_0 must be zero, and so

$$\mathbf{K}\delta = (\alpha\partial_{\zeta} + \bar{\alpha}\partial_{\bar{\zeta}})\delta = 0 , \tag{4.39}$$

where $\delta = Q/P$. Now $(P^2\delta_{\zeta})_{\zeta} = (PQ_{\zeta} - QP_{\zeta})_{\zeta} = 0$, and so $P^2\delta_{\zeta} = \bar{X}(\bar{\zeta})$, where X is an analytic function. From eq.(4.39), $\alpha\bar{X} + \bar{\alpha}X = 0$, and so $X = i\kappa_0\alpha(\zeta)$ where κ_0 is a real constant, i.e.

$$P^2\delta_{\zeta} = PQ_{\zeta} - QP_{\zeta} = - i\kappa_0\bar{\alpha}(\bar{\zeta}) . \tag{4.40}$$

This shows that if δ is not a constant, then α is determined up to a multiplicative constant (any multiple of a Killing vector is also a Killing vector). This α can be shown, by direct substitution, to satisfy eq.(4.37), so that there are precisely two independent Killing vectors for these metrics. If δ is a constant then α is any solution of $\mathbb{K}P - \mathrm{Re}(\alpha_\zeta)P = 0$. This has three independent solutions and so there are four independent Killing vectors. However, only two can commute, and so $\mathcal{B}$ is in $\mathcal{U}(2,N)$. This completes the proof of theorem I.

We still have to solve for Λ from eq.(4.4). If $\delta_\zeta \neq 0$ then $\kappa_0 \neq 0$ and so $(\delta^2/\alpha)_{\bar{\zeta}} = 2i\kappa_0\delta P^{-2}$, yielding

$$\delta_\zeta \neq 0 \qquad \Lambda = \delta^2/2\kappa_0\alpha = \frac{i}{2}\,\delta^2(P^2\delta_\zeta)^{-1} \quad . \tag{4.41}$$

Of course, we can always set $\kappa_0 = 1$ by an appropriate choice of α_0.

When δ is a constant, α is far from unique. The coordinates can be chosen so that $\mathcal{P}$ is diagonalised and

$$P = \zeta\bar{\zeta} + \kappa_1 . \tag{4.42}$$

Any <u>finite</u> symmetry must be a bilinear transformation in ζ, and so if $t \to \psi(\zeta,t)$ is the one parameter subgroup corresponding to α,

$$\alpha(\zeta) = \frac{\partial\psi(\zeta,t)}{\partial t}\bigg|_{t=0} ,$$

where $\psi(\zeta,t) = (a\zeta + b)/(c\zeta + d)$, and a,b,c, and d are functions of t. From this,

$$\alpha(\zeta) = -\,c_t\zeta^2 + (a_t + d_t)\zeta + b_t ,$$

so that α corresponds to a symmetric spinor of rank two,

$$\alpha(\zeta) = A_{\alpha\beta}\zeta^\alpha\zeta^\beta/(\zeta^0)^2 . \tag{4.43}$$

Substituting this into eq.(4.46) and using eq.(4.51),

$$\alpha(\zeta) = \alpha_0\zeta^2 + K_1\bar{\alpha}_0 + ie_0\zeta ,$$

where α_0 and e_0 are arbitrary constants, e_0 being real. In particular, $\alpha = i\zeta$ can always be chosen as a preferred solution of the Killing equation, and then

$$\delta = \text{constant} \Rightarrow P = \zeta\bar{\zeta} + K_1 , \quad \alpha = i\zeta , \quad \Lambda = \frac{i\delta}{\zeta(\zeta\bar{\zeta} + K_1)} . \tag{4.44}$$

5. Metrics of type $\mathcal{D}(2,R)$ and $\mathcal{Q}(2,R)$

The metrics of type $\mathcal{D}(2,R)$ are all independent of ζ and $\bar{\zeta}$, and so we shall have a close look at the larger class, $\mathcal{Q}(2,R)$, of algebraically special spaces with the same symmetries. For these, the field equations, (3.14), reduce to ordinary differential equations, since $\mathbf{D} = -\Omega\partial_u$, and can be partially integrated. From eq.(f.e.III),

$$M = -\tfrac{1}{2}\mu_0\bar{\Omega}^{-3} , \tag{5.1}$$

where μ_0 is a constant.Eq.(f.e.I) can now be written as

$$\frac{d}{du}(\bar{\Omega}^2 H) = -3\mu_0\bar{\Omega}^{-3}\bar{\Omega}_u ,$$

and so

$$2H = 2\bar{\mathbf{D}}\partial_u\mathbf{D}\Omega = 3\mu_0\bar{\Omega}^{-4} - \nu_0\bar{\Omega}^{-2} \tag{5.2}$$

where ν_0 is another constant. Since

$$\bar{\mathbf{D}}\bar{\mathbf{D}}\mathbf{D}\Omega = -\bar{\mathbf{D}}(\bar{\Omega}\partial_u\mathbf{D}\Omega) = -\bar{\Omega}H - (\bar{\mathbf{D}}\bar{\Omega})\,\partial_u\mathbf{D}\Omega ,$$

the last field equation for $\mathcal{Q}(2,R)$, eq.(f.e.II), can be written as

$$\text{Im}[\,2(\mathbf{D}\Omega)\,\partial_u\bar{\mathbf{D}}\bar{\Omega} - \mu_0\bar{\Omega}^{-3} + \nu_0\bar{\Omega}^{-1}] = 0 . \tag{5.3}$$

Eq.(5.2) and Eq.(5.3) can be simplified by the substitution

$$\Omega = g^{-\frac{1}{2}} e^{i\theta} \quad , \tag{5.4}$$

together with a change of variable from u to a variable x defined by

$$dx = g^{3/2}\, du\,, \quad \partial_u = g^{-3/2}\, \partial_x \quad . \tag{5.5}$$

The field equations then become

$$\dot{\theta}\ddot{g} - \ddot{\theta}\dot{g} + (2\dot{\theta}^3 - \dddot{\theta}).g = -\mathrm{Im}\left\{\mu_0 e^{3i\theta}\right\} \quad , \tag{5.6}$$

$$\dddot{g} + 6\dot{\theta}^2\dot{g} + 12\dot{\theta}\ddot{\theta}g = \mathrm{Re}\left\{-3\mu_0 e^{3i\theta} + \nu_0 g^{-1} e^{i\theta}\right\} \quad , \tag{5.6a}$$

$$\left\{2\dot{\theta}^3 + \dddot{\theta}\right\} g = \mathrm{Im}\left\{\nu_0 g^{-1} e^{i\theta}\right\} \quad , \tag{5.7}$$

where a dot denotes differentiation with respect to x. These equations are not independent, since

$$\frac{d}{dx}(5.6) + g^{-1}\frac{d}{dx}(5.7) = \dot{\theta}(5.6a) \quad ,$$

and so eq.(5.6a) can be ignored if $\dot{\theta}$ is non zero.

When $\nu_0 \neq 0$, we have been unable to integrate these equations any further.

When the metric is type D, eq.(5.2) can be simplified with the aid of eq.(4.22) to $\nu_0 = 0$. Conversely, if ν_0 is zero then $HM^{-4/3}$ is a constant, and so eq.(D.I.) and eq.(D.II) are satisfied. The last two type D equations (D.III,IV) follow from eq.(5.6) and eq.(5.7).

<u>Theorem IIR</u>: A space of type $\mathcal{A}(2,R)$ is type D iff ν_0 is zero, i.e. iff $H\Omega^{-4}$ is constant.

For type $\mathcal{D}(2,R)$ spaces, eq.(5.7) can be integrated, giving

$$e^{-2i\theta}(\dot{\theta}^2 - i\ddot{\theta}) = \tfrac{1}{2}\lambda_0 \quad ,$$

where λ_0 is a constant, and so

$$\dot{\theta}^2 = \tfrac{1}{2}\,\mathrm{Re}\,(\lambda_0 e^{2i\theta}) \ , \ \ddot{\theta} = -\tfrac{1}{2}\,\mathrm{Im}\,(\lambda_0 e^{2i\theta}) \ . \tag{5.8}$$

The remaining coordinate freedom is

$$\begin{aligned} \zeta' &= q_0\zeta + q_1 \ , \\ u' &= |q_0|(u + h_0) \ , \\ \Omega' &= \Omega|q_0|/q_0 \ , \end{aligned} \tag{5.9a}$$

where q_0, q_1 and h_0 are all constants. From eq.(5.4,5)

$$g' = g, \ \theta' = \theta - \arg(q_0), \ x' = |q_0|(x - x_0) \ , \tag{5.9b}$$

x_0 being another constant, and so $\lambda_0' = \lambda_0 \bar{q}_0^{\,-2}$. Providing λ_0 is non-zero, it can be transformed to any required value. Following Kinnersley (1969) we choose $\lambda_0 = 1$, and so

$$\dot{\theta}^2 = \tfrac{1}{2}\cos 2\theta \ , \quad \ddot{\theta} = -\tfrac{1}{2}\sin 2\theta \ , \tag{5.10}$$

which have as solutions

$$e^{i\theta} = \mathrm{dn}(x) + \frac{1}{\sqrt{2}}\,\mathrm{sn}(x) \ , \quad \dot{\theta} = \frac{1}{\sqrt{2}}\,\mathrm{cn}(x) \ , \tag{5.11}$$

where the constant of integration has been eliminated by the co-ordinate transformation $x \to x - x_0$. The functions in eq.(5.11) are the Jacobian elliptic functions of modulus $1/\sqrt{2}$. The only remaining co-ordinate freedom is $\zeta' = \zeta + q_1$, which corresponds to the complex Killing vector, ∂_ζ .

When $\dot{\theta} \neq 0$, eq.(5.6a) can be ignored and eq.(5.6) solved for g. The complete solution is

$$g(x) = 2\,\mathrm{Re}\,(a_0 e^{2i\theta} + b_0\dot{\theta}e^{i\theta}) \ , \tag{5.12}$$

where a_0 and b_0 are arbitrary complex constants, and the mass parameter, μ_0, is given by

$$\mu_0 = \tfrac{1}{2} i \bar{b}_0 \quad . \tag{5.13}$$

When $\dot{\theta} = 0$, i.e. $\lambda_0 = 0$, then $\mathrm{Im}(\bar{\mathbf{D}}\,\Omega) = 0$ and so $\Delta = 0$ and the complex divergence ρ is real. This is the necessary and sufficient condition for $\dot{\omega}^{11}$ to be hyper-surface orthogonal. The field equations (5.6,7) become

$$\dddot{g} + R\ell\,(3\mu_0 e^{3i\theta_0}) = 0 \ , \quad \mathrm{Im}(\mu_0 e^{3i\theta_0}) = 0 \, . \tag{5.14}$$

The transformation $\theta' = \theta - \arg(q_0)$ can be used to make $\theta_0 = 0$, so that $\Omega = g^{-\frac{1}{2}}$ and μ_0 is real.
From eq. (5.14),

$$g = -\tfrac{1}{2}\mu_0 x^3 + g_2 x^2 + g_1 x + g_0 \quad ,$$

where $\mu_0 \neq 0$. The transformation $x' = |q_0|(x - x_0)$ can be used to set $\mu_0 = -2$, and to eliminate the quadratic term. Finally, the canonical form for g is

$$g = x^3 + a_0 x + b_0 \quad . \tag{5.15}$$

We shall see that this gives the C - metrics of Ehlers and Kundt (1962).

The metric for these $\mathcal{a}(2,R)$ metrics can be transformed to

$$\begin{aligned}(ds)^2 &= -2\Sigma g\,[\,d\zeta^*\, d\bar{\zeta}^* + dx^2/4g^2\,] \\ &\quad + 2\,[\,e^{i\theta} d\zeta^* + e^{-i\theta} d\bar{\zeta}^*\,]\ [\,dR + p\,dx + \sigma d\zeta^* + \bar{\sigma} d\bar{\zeta}^*\,] \ ,\end{aligned} \tag{5.16}$$

where

$$\Sigma = (R^2 + \dot{\theta}^2) \ , \quad p = \tfrac{1}{2}(R^2 + 3\dot{\theta}^2) \quad ,$$

$$\sigma e^{-i\theta} = -\,\mathrm{Re}\,(\mu_0 e^{3i\theta}/R + i\dot{\theta}) + \tfrac{1}{2}\,(R\dot{g} - \ddot{g}) - 2g\dot{\theta}^2 + gi\ddot{\theta} - Rgi\dot{\theta} \quad , \tag{5.17}$$

and the coordinates are defined by

$$d\zeta^* = d\zeta + dx/2ge^{i\theta} \;, \; R = v/\sqrt{-g}\,. \tag{5.18}$$

We shall follow the notation of Papapetrou (1966) and write this metric as

$$ds^2 = g_{\mathfrak{a}\mathfrak{b}}\,dx^{\mathfrak{a}}dx^{\mathfrak{b}} + 2g_{\mathfrak{a}A}\,dx^{\mathfrak{a}}dx^{A} + g_{AB}dx^{A}dx^{B} \quad , \tag{5.19}$$

where $\mathfrak{a}, \mathfrak{b} = 1,2$; $A,B = 3,4$ and g_{ij} is a function of the $(x^{\mathfrak{a}})$, i.e. the $\{\partial_A\}$ are Killing vectors. In eq.(5.16), $X^i = (R,X,\zeta^*,\bar{\zeta}^*)$. We will say that the metric in eq.(5.19) is quasi-diagonalizable (q.d. for short) if the cross terms can be eliminated. This can only be done by a transformation of the type

$$x^A = x^{A'} + f^A(x^{\mathfrak{a}})\,, \qquad x^{\mathfrak{a}'} = x^{\mathfrak{a}}\,. \tag{5.20}$$

If this is substituted into eq.(5.19) the equation $g_{\mathfrak{a}'A'} = 0$ can be solved for the df^A

$$df^A = -h^{AB}g_{\mathfrak{a}B}dx^{\mathfrak{a}} \quad , \tag{5.21}$$

where $g_{AC}h^{CB} = \delta^B_A$ and so the metric is q.d. iff the right hand side of eq.(5.21) is a perfect differential.

For the $\mathcal{A}(2,R)$ metrics, eq.(5.20) gives

$$d\eta - d\zeta^* = -e^{-i\theta}(g\Sigma - \sigma^{-i\theta} + \bar{\sigma}e^{i\theta})(dR + pdx)/G \tag{5.22}$$

where $\eta = \zeta^{*\prime}$, and G is the determinant of the two metric, g_{AB},

$$G = -g^2\Sigma^2 + 4g\Sigma\,\mathrm{Re}\,(\sigma e^{-i\theta}) + 4(\mathrm{Im}(\sigma e^{-i\theta}))^2\,. \tag{5.23}$$

Such a metric is q.d. iff $d(\eta - \zeta^*)$ is a perfect differential. It is shown in the appendix that this is so iff ν_0 is zero, implying

<u>Theorem IIIR:</u> A metric of type $\mathcal{a}(2,R)$ is q.d. iff it is of type D.

When η exists, the metric can be written as

$$(ds)^2 = -2g\Sigma[d\eta d\bar{\eta} + (dx/2g)^2 + (dR + pdx)^2/G] + 2(e^{i\theta}d\eta + e^{-i\theta}d\bar{\eta})(\sigma d\eta + \bar{\sigma}d\bar{\eta}) \quad . \tag{5.24}$$

We need a convenient variable, y, whose differential is proportional to $dR + pdx$. From eq.(5.22),

$$d(\zeta^* - \eta) = (ge^{-i\theta}/G)(\Sigma + iA)(dy/y_{,R}) \quad , \tag{5.25}$$

where

$$A = 2\mathrm{Im}(\bar{\sigma}e^{i\theta}/g) = 2R\dot{\theta} + \sin 2\theta \, . \tag{5.26}$$

From eq.(5.25)

$$d(\zeta^* - \eta) = (C_1 + iC_2)dy \quad , \tag{5.27}$$

where C_1 and C_2 are real functions of y, and so the ratio C_2/C_1 must itself be a function of y. This variable will be defined by setting

$$C_2/C_1 = -\tan\phi\,(y) \tag{5.28}$$

$$\left(\frac{d\phi}{dy}\right)^2 = \tfrac{1}{2}\cos 2\phi \, . \tag{5.29}$$

This will only work if C_2/C_1 is not constant, that is, if $\dot{\theta} \neq 0$, and so the C - metric will have to be treated separately. For the Kinnersley metric, eq.(5.25) and eq.(5.28) give

$$\Sigma + iA = (\Sigma^2 + A^2)^{\frac{1}{2}} e^{i(\theta - \phi)} \, . \tag{5.30}$$

From eq. (5.10),

$$(\Sigma^2 - A^2)\cos 2\theta + 2A\Sigma \sin 2\theta = 2(R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3)^2 \quad ,$$

and so, from eq. (5.29) and eq. (5.30),

$$R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3 = (\Sigma^2 + A^2)^{\frac{1}{2}}\dot{\phi} \quad , \tag{5.31}$$

where $\dot{\phi} = \phi_y$. Eq. (5.28) is equivalent to

$$e^{2i(\theta-\phi)} = (\Sigma + iA)/(\Sigma - iA) \quad , \tag{5.32}$$

and therefore

$$(\Sigma^2 + A^2)\,d\phi = (\Sigma^2 + A^2)\,d\theta + (A d\Sigma - \Sigma dA). \tag{5.33}$$

Substituting in the known values for Σ and A,

$$(\Sigma^2 + A^2)\,d\phi = 2(R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3)(dR + p\,dx) \quad , \tag{5.34}$$

$$d(\eta - \zeta^*) = -[g(\Sigma^2 + A^2)/G]\,e^{-i\phi}\,dy/2 \ . \tag{5.35}$$

The expression inside the square brackets must be a function of y. From known results for the C metric this was guessed to be proportional to $[g(y)]^{-1}$, so that

$$G(x,R)/g(x) = -g(y)(\Sigma^2 + A^2) \ . \tag{5.36}$$

The easiest way to prove this is to substitute for $g(y)$ using eq. (5.29-31)

$$\begin{aligned}(\Sigma^2 + A^2)g(y) &= 2(\Sigma^2 + A^2)\,\mathcal{R}\ell\,(a_0 e^{2i\phi} + b_0 e^{i\phi}\dot{\phi}) \\ &= 2\mathcal{R}\ell\,[a_0 e^{2i\theta}(\Sigma - iA)^2 + b_0 e^{i\theta}(\Sigma - iA)(R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3)].\end{aligned}$$

Eq.(5.36) can then be verified by comparing this with eq.(5.23).

The function σ still has to be expressed as a function of x and y. From eq.(5.23) and eq.(5.26)

$$4\Sigma R\ell(\sigma e^{-i\theta}) = g(x)(\Sigma^2 - A^2) - g(y)(\Sigma^2 + A^2) \ .$$

From this, eq.(5.26) and eq.(5.30),

$$\sigma = \tfrac{1}{4}\Sigma e^{-i\theta} \sec^2(\theta - \phi)\,[\,g(x)e^{2i\phi} - g(y)e^{2i\theta}\,]$$

The equation for η can now be written as

$$d\eta = d\zeta + dx/2e^{i\theta}g(x) + dy/2e^{i\phi}g(y) \quad , \qquad (5.37)$$

and the metric takes the simple form

$$(ds)^2 = \tfrac{1}{2}\Sigma[\sec^2(\theta - \phi)g(x)(e^{i\phi}d\eta - e^{-i\phi}d\bar{\eta})^2 - dx^2/g(x)$$
$$- \sec^2(\theta - \phi)g(y)(e^{i\theta}d\eta + e^{-i\theta}d\bar{\eta})^2 + dy^2/g(y)] \quad , \qquad (5.38)$$

$$\Sigma = \cos(\theta - \phi)/(\cos(\theta + \phi) - (\cos 2\theta \cos 2\phi)^{\frac{1}{2}}) \ . \qquad (5.39)$$

The first p.n.v. is given by

$$2R\ell(e^{i\theta}d\zeta^*) = 2R\ell(e^{i\theta}d\eta) + \cos(\theta - \phi)dy/g(y) \ . \qquad (5.40)$$

The metric is invariant to $d\eta \rightarrow -d\eta$ and so the second p.n.v. is got from eq.(5.40) by changing the sign of $d\eta$. We shall denote the two p.n.v.'s by $K\pm$,

$$K\pm = \cos(\theta - \phi)dy/g(y) \pm 2R\ell(e^{i\theta}d\eta) \ . \qquad (5.41)$$

When $\omega^{1\dot{i}}$ is hypersurface orthogonal, $\dot{\theta} = 0$, and we can choose $e^{i\theta} = 1$, so that Ω is real. Eq.(5.22) becomes

$$d(\eta - \zeta^*) = -[g(x)R^4/2G]d(x - 2/R) \quad ,$$

and so we will replace the coordinate R by y = x - 2/R. Now

$\Sigma = R^2$, and from eq.(5.23)

$$G/R^4 g(x) = -g(x) + 4R^{-2}\, R\ell(\sigma e^{-i\theta}) \ ,$$
$$= -g(x) + O(R^{-1}) \ .$$

Since the right hand side of this must be a function of y, and since g is a polynomial, it follows that

$$G = -R^4\, g(x)g(y) \ .$$

Of course, this can also be proved by direct calculation. It is now fairly simple to show that

$$d\eta = d\zeta + dx/2g(x) + dy/2g(y) \ , \qquad (5.42)$$

and that if we write $2\eta = \eta_1 + i\eta_2$ where the η_A are real, then

$$(ds)^2 = \frac{-2}{(x-y)^2}\left[g(x)\,d\eta_1^2 + \frac{dx^2}{g(x)} + g(y)\,d\eta_2^2 - \frac{dy^2}{g(y)}\right], \qquad (5.43)$$

and the p.n.v's are

$$K\pm = d\eta_2 \pm dy/g(y) \ . \qquad (5.44)$$

The curvature invariant Ψ_2 is given by

$$\Psi_2 = \tfrac{1}{8}\,\mu_0 (x-y)^3 \ , \qquad (5.45)$$

and so the curvature tensor is regular for all finite values of y and x. This is the C metric of Ehlers and Kundt.

The similarities between eq.(5.38) and eq.(5.43) are obvious. The function g appears to be more complicated in Kinnersley's metric, since it involves elliptic functions, but it can be reduced to polynominal form by defining

$$X^2 = \tan(\theta + \tfrac{1}{4}\pi) \ , \ Y^2 = \tan(\phi + \tfrac{1}{4}\pi) \ , \ \eta = \tfrac{1}{2}(\eta_2 + i\eta_1)e^{i\pi/4} \ .$$

The metric then becomes

$$ds^2 = -2\,(X - Y)^{-2}\,[(\tilde{g}(X)(d\eta_1 + Y^2 d\eta_2)^2 + \tilde{g}(Y)(d\eta_2 - X^2 d\eta_1)^2)/(1 + X^2Y^2)$$
$$+ (dX^2/\tilde{g}(X) - dY^2/\tilde{g}(Y))(1 + X^2Y^2)] \quad , \qquad (5.46)$$

where $\tilde{g}$ is a quartic polynomial,

$$\tilde{g}(X) = c_0(1 - X^4) + c_1X + C_2X^2 + c_3X^3 \ ,$$
$$a_0 = \tfrac{1}{4}c_1 + \tfrac{1}{2}ic_0, \ b_0 = \tfrac{1}{2}c_1(1 + i) + \tfrac{1}{2}c_3(1 - i) \ ,$$
$$K\pm = d\eta_2 - X^2 d\eta_1 \pm (1 + X^2Y^2)dY/\tilde{g}(Y) \ .$$

To see that eq.(5.43) is the limit of eq.(5.45), it is necessary to let $(X,Y,\eta_1,\eta_2) \rightarrow (qX,qY,\eta_1/q,\eta_2/q)$ in eq.(5.45), so that

$$\tilde{g}(qX) = d_0(1 - q^4X^4) + d_1X + d_2X^2 + d_3X^3 \quad ,$$

where the $\{d_i\}$ are arbitrary. If we set $q = 0$ in the resultant metric, we get eq.(5.43).

Another simple limit follows from (5.46) by defining $Z = Y^{-1}$, $\tilde{Z}(Z) = -Z^4\tilde{g}(Z^{-1})$ or $\tilde{Z}(Z) = c_0(1 - Z^4) - c_3Z - c_2Z^2 - c_1Z^3$.

Eq.(5.46) becomes

$$ds^2 = 2(1 - XZ)^{-2}\left[\frac{\Sigma dX^2}{\tilde{g}(X)} + \frac{\Sigma dZ^2}{\tilde{Z}(Z)} + \frac{\tilde{g}(X)}{\Sigma}(d\eta_2 + Z^2 d\eta_1)^2\right.$$
$$\left. - \frac{\tilde{Z}(Z)}{\Sigma}(d\eta_2 - d\eta_1)^2\right] \ , \qquad (5.47)$$

$$\Sigma = X^2 + Z^2 \quad ,$$

and the Kerr Nut metric of the next section is obtained by making the substitution

$(X,Z,\eta_1,\eta_2,c_0,c_1,c_2,c_3) \to (qX,qZ,\eta_1/q,\eta_2/q^3,q^4c_0,q^3c_1,q^2c_2,q^3c_3)$,

and then putting $q = 0$.

The curvature invariant, Ψ_2, of eq.(5.47) is

$$\Psi_2 = \frac{1}{2\sqrt{2}}(c_1 - ic_3)\left(\frac{X - Z}{1 - iXZ}\right)^3 , \tag{5.48}$$

and so the curvature tensor is again regular for all finite X and Z.

6. METRICS OF TYPE $\mathcal{A}(1,N)$ and $\mathcal{D}(2,N)$

If $\mathcal{S} \in \mathcal{A}(1,N)$ then it has a Killing vector $P\partial_u$ and the metric can be transformed to that of eq.(4.4). The mass function, m, can be eliminated from eq.(4.8,9) giving

$$[PQ_{\zeta\bar\zeta} - P_\zeta Q_{\bar\zeta} - P_{\bar\zeta}Q_\zeta + P_{\zeta\bar\zeta}Q]_{\zeta\bar\zeta} = 0 , \tag{6.1}$$

which can be written as a function of K_1,K_2 and $P^2\delta_\zeta(= PQ_\zeta - QP_\zeta)$, so that the field equations for $\mathcal{A}(1,N)$ are

$$K_{1\zeta\bar\zeta} = 0 , \tag{6.2}$$

$$P^2K_{2\zeta\bar\zeta} - |(P^2\delta_\zeta)_\zeta|^2 + Q^2K_{2\zeta\bar\zeta} = 0 . \tag{6.3}$$

If these are satisfied then m is defined up to an additive real constant by eq.(4.8,9).

The following two equations will prove useful

$$P^{-2}|P^2\delta_\zeta|^2 + \delta^2K_1 - \mathrm{Im}(m)\,\delta + K_2 = 0 , \tag{6.4}$$

$$((P^2\delta_\zeta)_\zeta/PQ)_{\bar\zeta} = Q^{-2}K_{2\zeta} - P^{-2}K_{1\bar\zeta} , \tag{6.5}$$

the first of these being equivalent to eq.(4.8), and the second being an identity. Also, the derivative of m can be found by differentiating eq.(4.8) with respect to ζ, and using eq.(4.9),

$$m_\zeta = 2i\,[\,Q^2K_{1\zeta} - (PQ)_{\bar\zeta}\,(P^2\delta_\zeta)_\zeta + P^2K_{2\zeta}\,]\,/PQ\ . \tag{6.6}$$

Since Λ can be taken as any solution of $\mathrm{Im}(\bar\Lambda_\zeta) = QP^{-3}$, by an appropriate choice of U, we will assume

$$\Lambda_{\bar\zeta} = -iQP^{-3}\quad , \tag{6.7}$$

and then eq. (6.6) becomes

$$m_\zeta + 2iP^2(\,P^{-2}(P^2\delta_\zeta)_\zeta + 2i\Lambda K_{1\zeta})_{\bar\zeta} = 0\ . \tag{6.8}$$

From the analysis of section 4, an $\mathcal{S} \in \mathcal{Q}(1,N)$ is type D iff $\exists$ a ζ coordinate for which P and Q are bilinear in ζ and $\bar\zeta$. For a <u>general</u> ζ coordinate, $\Omega = P\Lambda - (P_\zeta/P)u$, so that $\partial_u \mathbf{D}\,\Omega = -P_{\zeta\zeta}/P$ and

$$\partial_u\partial_u\ \mathbf{D}\,\Omega = 0,\ H = K_{1\zeta}/P^2,\ J = 2i\delta_\zeta/P\ .$$

The type D conditions, eq. (3.16), reduce to $H = 0$ and $(J\,P^3)_\zeta = 0$, that is to $K_{1\zeta} = (P^2\delta_\zeta)_\zeta = 0$. From eq. (6.5) and eq. (6.6), K_2 and m are also constant and so the field equations, (6.2) and (6.3), are satisfied.

<u>Lemma 2N</u>: The non-radiating metric of eq. (4.4) is type D iff K_1 and K_2 are constant, $(P^2\delta_\zeta)_\zeta = 0$, and m is a constant whose imaginary part is defined in eq. (4.8) (or equivalently, eq. (6.4)).

An $\mathcal{S} \in \mathcal{Q}(2,N)$ iff its metric is given by eq. (4.4), and it admits a second Killing vector $\mathbf{K}$, given by eq. (4.10), commuting with ∂_s. Since $[\,\partial_s, \mathbf{K}\,] = a_0\partial_s$, the constant a_0 must be zero. The two curvatures, K_1 and K_2, and δ have simple transformation properties for the transformation of eq. (4.31).

$$K_1' = C_0{}^2K_1\ ,\ K_2' = C_0{}^2K_2,\quad \delta' = C_0\delta\quad , \tag{6.9}$$

and so they satisfy simple Killing equations

$$\mathbf{K}(K_1) = \mathbf{K}(K_2) = \mathbf{K}(\delta) = 0 \,. \tag{6.10}$$

If any of these three real functions are non-constant then $\mathbf{K}$ is essentially unique, since if $\mathbf{K}^* = \alpha^*\partial_\zeta + \ldots$ is another Killing vector and if $\delta_\zeta \neq 0$, then $\mathbf{K}^*\delta = \mathbf{K}\delta = 0$ gives $(\alpha^*/\alpha) = (\bar{\alpha}^*/\bar{\alpha})$ and so $\alpha^* = (\text{real constant})\ \alpha$.

Suppose we use a ζ coordinate for which $\alpha(\zeta) = \frac{1}{2}$, and write

$$d\zeta = \tfrac{1}{2}(d\phi + i d\theta), \qquad \partial_\zeta = \partial_\phi - i\partial_\theta \,. \tag{6.11}$$

From eq.(4.11) and eq.(6.7), P, Q, and Λ are all functions of θ; and, from eq.(6.7), Λ can be chosen to be real, so that $\mathbf{K} = \partial_\phi$. The metric is given by

$$\begin{aligned}(d\tau)^2 = -\tfrac{1}{2}\,\Sigma P^{-2}\,[(d\theta)^2 + [\,d\phi + (2\dot{\delta}P^2/\Sigma)(ds + \Lambda d\phi)\,]^2\,] \\ + (2dr + (\pi/\Sigma)(ds + \Lambda d\phi))(ds + \Lambda d\phi) \quad ,\end{aligned} \tag{6.12}$$

where a dot denotes differentiation with respect to θ, and g_{ij} is independent of ϕ and s. If this is q.d., then the appropriate coordinate transformation must eliminate all cross terms between $(d\theta, dr)$ and $(d\phi, ds)$. There is only one such, and therefore if $(s,\phi) \to (\eta_1,\eta_2)$,

$$\begin{aligned}d\phi + (2\dot{\delta}P^2/\Sigma)(ds + \Lambda d\phi) &= d\eta_2 + (2\dot{\delta}P^2/\Sigma)(d\eta_1 + \Lambda d\eta_2) \;, \\ ds + \Lambda d\phi &= d\eta_1 + \Lambda d\eta_2 - (\Sigma/\pi)\,dr \;,\end{aligned}$$

so that

$$d\eta_1 = ds + (\Sigma + 2\Lambda\dot{\delta}P^2)\,dr/\pi \quad , \tag{6.13a}$$

$$d\eta_2 = d\phi - (2\dot{\delta}P^2/\pi)\,dr \quad , \tag{6.13b}$$

$$\begin{aligned}-ds^2 = \frac{\Sigma}{2P^2}\,[\,d\theta^2 + (d\eta_2 + (2\dot{\delta}P^2/\Sigma)(d\eta_2 + \Lambda d\eta_2))^2\,] \\ + \frac{\Sigma}{\pi}\,dr^2 - \frac{\pi}{\Sigma}\,(d\eta_1 + \Lambda d\eta_2)^2 \,.\end{aligned} \tag{6.13c}$$

These equations are integrable iff the coefficients of dr are functions of r; and so the ratio $(r^2 + \delta^2 + 2\Lambda\dot{\delta}P^2)/\dot{\delta}P^2$ must be independent of θ; i.e. $\dot{\delta}P^2$ and $(\delta^2 + 2\Lambda\dot{\delta}P^2)$ must both be constants. From eq. (6.7) $P^2\dot{\Lambda} + \delta = 0$, $(\delta^2 + 2\Lambda\dot{\delta}P^2)_\theta = \Lambda(2\dot{\delta}P^2)_\theta$ and so $\mathscr{g}$ will be q.d. iff $\dot{\delta}P^2$ is constant and π is a function of r, that is iff K_1, K_2, $\dot{\delta}P^2$ and m are all constants. Finally, from lemma (2N),

<u>Lemma 3N</u> : A type $\mathcal{A}(2,N)$ is quasi-diagonal iff it is type D.

For an arbitrary $\mathcal{A}(2,N)$ space, eq. (6.2) and eq. (6.8) give

$$\dot{K}_1 = n_1 \quad , \tag{6.14a}$$

$$P^{-2}(P^2\dot{\delta})_\theta - 2\Lambda n_1 = n_2 \quad , \tag{6.14b}$$

where n_1 and n_2 are constants, and so the metric is q.d. iff $n_1 = n_2 = 0$. The general solution of eq. (6.14a) is not known, and so the field equation for the $\mathcal{A}(2,N)$ metrics have not been solved completely, although some special solutions are given in (K.D.). In particular, the complete solution is given when n_1 is zero.

If $\mathscr{g} \in \mathscr{D}(2,N)$ and δ is not a constant then $\Lambda = -\delta^2/2P\dot{\delta}$. For the C_0 transformation of eq. (4,31) and eq. (6.9), $(2P^2\dot{\delta})' = C_0{}^3(2P^2\dot{\delta})$, and so we will assume

$$2P^2\dot{\delta} = 1, \ \Lambda = -\delta^2 \quad . \tag{6.15}$$

If $(x,Y) = (\delta(\theta), r)$ are used as non-ignorable coordinates, then

$$-(ds)^2 = \frac{\Sigma}{\pi_x} dx^2 + \frac{\pi_x}{\Sigma}(d\eta_1 + y^2 d\eta_2)^2$$

$$+ \frac{\Sigma}{\pi_y} dy^2 - \frac{\pi_y}{\Sigma}(d\eta_1 - x^2 d\eta_2)^2 \quad , \tag{6.16}$$

where

$$\pi_x = -2(K_1x^2 - \mathrm{Im}(m)x + K_2) \quad , \tag{6.17a}$$

$$\pi_y = -2(-K_1y^2 + R\ell(m)y + K_2) \quad , \tag{6.17b}$$

$$\Sigma = x^2 + y^2 \quad . \tag{6.17c}$$

The curvature invariant Ψ_2 is given by

$$\Psi_2 = -m(r + ix)^{-3} \quad , \tag{6.17d}$$

and so the curvature tensor is regular for all points other than $r = x = 0$.

The transformation

$$\begin{aligned}(x', y', \eta_1', \eta_2') &= (qx, qy, q^{-1}\eta_1, q^{-3}\eta_2) \quad , \\ (K_1', K_2', m') &= (q^2K_1, q^4K_2, q^3m) \quad ,\end{aligned} \tag{6.18}$$

preserves the form of eq.(6.16), but modifies the parameters, showing that they are not invariants. If $2K_1 \neq 0$, then it can be chosen to be ± 1. Since both π_x and π_y have to be positive, $-K_2$ must be also whenever $K_1 > 0$ and $\mathrm{Im}(m) = 0$. This leads to the Kerr metric,

$$\pi_x = a^2 - x^2, \ \pi_y = y^2 - 2my + a^2 \quad . \tag{6.19}$$

When $\dot{\delta} = 0$, then $Q = \delta P$, $K_2 = \delta^2 K_1$ and $\mathrm{Im}(m) = 2\delta K_1$. It is convenient to introduce a new coordinate, x, defined by $\frac{dx}{d\theta} = \pi_x$ where $\pi_x = 1/2P^2$ and to choose u such that $\Lambda = 2\delta x$. It follows from the equation for K_1, eq.(4.6) and eq.(6.11), that

$$\frac{d^2\pi_x}{dx^2} + 4K_1 = 0 \ . \tag{6.20}$$

The metric can be written as

$$-(ds)^2 = \Sigma\left[\frac{dx^2}{\pi_x} + \frac{dy^2}{\pi_y} + \pi_x d\eta_2^2\right] - \frac{\pi_y}{\Sigma}(d\eta_1 + 2\delta x d\eta_2)^2 , \tag{6.21}$$

where

$$\begin{aligned} \pi_x &= -2K_1x^2 + 2qx + p > 0 , \\ \pi_y &= 2K_1(y^2 - \delta^2) - 2m_0y > 0 , \\ \Sigma &= y^2 + \delta^2 , \\ m &= m_0 + 2iK_1\delta , \\ (y, \eta_1, \eta_2) &= (r, s, \phi) . \end{aligned} \tag{6.22}$$

and p and q are arbitrary constants. This is the B[+] metric of Carter (1968), or the generalised NUT metric. If $K_1 < 0$ then p and q can both be transformed to zero. This corresponds to taking $P = (\sqrt{-K_1})i(\zeta - \bar{\zeta})$ in section 4, but does not work for $K_1 > 0$. In the latter case, the best that can be done is to eliminate one or other of p and q. Also, the transformation of eq.(6.18) can be used to transform any nonzero $|K_1|$ to ½, and so the canonical forms for π_x and π_y are

$$\begin{aligned} K_1 > 0 &: \pi_x = 1 - x^2, \quad \pi_y = y^2 - 2my - \delta^2 \\ K_1 < 0 &: \pi_x = x^2 \quad , \quad \pi_y = -y^2 - 2my + \delta^2 \\ K_1 = 0 &: \pi_x = 1 , \quad \pi_y = -2my . \end{aligned}$$

The corresponding ranges of coordinates follow from the requirement that both π_x and π_y be positive.

The curvature invariants, Ψ_2, is given by

$$\Psi_2 = -m(y + i\delta)^{-3}$$

and so the curvature tensor is nowhere singular for $\delta \neq 0$.

When $\delta = 0$ we get the usual variants of Schwarzschild which are singular when $y = 0$.

7. CONCLUSION

In this paper we have discussed the diverging empty algebraically special metrics and have shown that all (diverging) type D spaces with two commuting Killing vectors are characterised by a complex constant which is zero iff the space is quasi-diagonalizable or iff the space is type D. The non-diverging case has not been treated here, but the type D metrics with zero divergence are also quasi-diagonalizable (Kinnersley 1969) and in fact may be derived from a limiting process on Kinnersley's metric (Carter 1968, Kinnersley 1968).

Canonical forms have been found for all (diverging) type D metrics. The most general metric is typified by a quartic function of a single variable and all metrics can be found from it by simple limiting processes. This behaviour also holds for the generalised solutions containing a maxwell field and cosmological constant where again a quartic function of a single variable appears, (Plebanski et al. 1976, Debever 1971, Weir 1976).

Coordinates which have been used in the past to describe empty algebraically special spaces were introduced in section 3 but it was found in section 5 that these are unsatisfactory for the radiating type D metrics. In particular, new coordinates were introduced which linearised the remaining field equations for the radiating metrics. Whether there are coordinates which drastically simplify the field equations for the general algebraically special spaces, as there are for type D metrics, is an open question.

APPENDIX

We wish to show that $d(\eta - \zeta^*)$ is a perfect differential. Papapetrou (1966) has proved that this occurs iff $K^{[\mu;\nu}K^{\alpha}\bar{K}^{\beta]} = 0$ where $\mathbf{K} = K^{\alpha}\partial_{\alpha}$ is the complex Killing vector. Since, in our co-ordinates, $K^{\alpha} = \delta^{\alpha}_{3}$, $\bar{K}^{\alpha} = \delta^{\alpha}_{4}$, we must prove $K^{[1;2]} = 0$. Expressing $K^{[1;2]}$ in our spin frame, we find, from section III, that

$$K^{[1;2]} = 2\, R\ell[(\beta\rho + U\rho\Omega)\; T_{00} - T_{01} + \rho\Omega\, T_{\dot{1}\dot{1}}]$$

$$+ \rho\bar{\rho}(\Omega\bar{\beta} - \beta\bar{\Omega})\;.\;[T_{01} - T_{\dot{0}\dot{1}}]$$

where $T_{AB} = \Gamma_{ABC\dot{D}}\, K^{C\dot{D}}$ and

$$\mathbf{K} = K^{\mu}\partial_{\mu} = -\rho^{-1}\partial_{1\dot{0}} + \Omega\partial_{1\dot{1}} + (\beta + \Omega U)\,\partial_{0\dot{0}} \;.$$

Expanding out $K^{[1;2]}$, and using

$$M - \bar{M} = \bar{\nu}_0/\Omega - \nu_0/\bar{\Omega} + \dot{\Omega}\dot{\bar{\Omega}}(\bar{\Omega}\dot{\Omega} - \Omega\dot{\bar{\Omega}}) + \Omega\bar{\Omega}(\dot{\bar{\Omega}}\ddot{\Omega} - \dot{\Omega}\ddot{\bar{\Omega}}) \quad ,$$

we find

$$K^{[1;2]} = 2\rho\bar{\rho}\bar{\nu}_0 \quad .$$

This calculation is simplified by noting from the work of Papapetrou that $K^{[1;2]} = \rho\bar{\rho}(\text{constant})$.

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