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A NEW CLASS OF ROTATIONAL INVARIANTS USING DISCRETE ORTHOGONAL MOMENTS

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ABSTRACT
This paper presents a new class of Tchebichef moments in polar coordinate form, using which rotational invariants can be easily constructed. The structure of the invariants is very similar to that of Zernike and Pseudo-Zernike moments, and their computation does not involve discrete approximation of continuous integral terms. The invariants are thus very robust in the presence of image noise, and have far better recognition capabilities when compared with Zernike/Legendre moments. The new class of moment invariants presented in this paper can be used in pattern and character recognition tasks.

KEY WORDS
Rotational invariants, orthogonal moments, zernike moment invariants, Tchebichef moments, pattern recognition.

1. Introduction

Zernike and Pseudo-Zernike moments are popular types of orthogonal moments that are used in several pattern recognition applications. Owing to the polar coordinate representation of the kernel functions, the rotational invariants of these moments can be easily obtained. However, the computation of invariants require a coordinate space transformation to a subspace of the unit circle, and also a discrete approximation of the continuous moment integrals. The above two processes are often cited as the primary limitations of all forms of continuous orthogonal moments.

Tchebichef (sometimes also written as Chebyshev) moments were recently introduced as image feature descriptors, in [1]. The Tchebichef moments of order $p+q$ of an image $f(i, j)$ are defined using the scaled orthogonal Tchebichef polynomials, which are the simplest among discrete orthogonal functions of unit weight [2]. Tchebichef moments provide several computational advantages over moments based on continuous orthogonal functions [3].

Instead of combining two independent Tchebichef polynomials to form a kernel function in Cartesian coordinates, we can use a one-dimensional polynomial along radial directions and a circular-harmonic function for the orthogonal direction, to construct moments in radial-polar form. The advantage with this type of representation is that rotational invariants can be easily derived. Since the kernal functions are orthogonal in the image coordinate space, the invariants are expected to have a far better feature recongnition capability than its continuous counterparts. This paper presents the complete mathematical framework of radial-Tchebichef moments, and proposes rotation-invariats based on them, for pattern recognition applications. Experimental results showing the image reconstruction capability of radial Tchebichef moments and their invariant characteristics, are also included.

2. Zernike Moments

The Zernike moments have complex kernel functions based on Zernike polynomials [4,5], and are often defined with respect to a polar coordinate representation of the image intensity function $f(r, \theta)$ as

$$Z_{nl} = \frac{(n+1)}{\pi}\int_0^{2\pi}\int_0^1 V^*_{nl}(r,\theta)\, f(r,\theta)\, r\, dr\, d\theta\,, \tag{1}$$

where $r \le 1$, the function $V_{nl}(r, \theta)$ denotes a Zernike polynomial of order n and repetition l, and $*$ denotes complex conjugate. In the above equation n is a non-negative integer, and l is an integer such that $n-|l|$ is even, and $|l| \le n$. The Zernike polynomials are defined as

$$V_{nl}(r, \theta) = R_{nl}(r)\ e^{jl\theta} \tag{2}$$

where $j = (-1)^{1/2}$, and $R_{nl}()$ is the real-valued Zernike radial polynomial given by [6]

$R_{nl}(r) =$

$$\sum_{s=0}^{(n-|l|)/2} (-1)^s \frac{(n-s)!\ r^{n-2s}}{s!\left(\frac{n-2s+|l|}{2}\right)!\left(\frac{n-2s-|l|}{2}\right)!} \tag{3}$$

When an image undergoes a rotation by an angle α, the moment functions Z_{nl} in Eq.(1) get transformed into Z'_{nl} according to the equation

$$Z'_{nl} = Z_{nl}\, e^{-iq\alpha}. \tag{4}$$

From the above equation, the following primary rotation invariants can be derived:

$$\varphi_1 = Z_{p0}\,; \qquad \varphi_2 = |Z_{pq}|^2\,, \tag{5}$$

where $p > 0$, and $p-|q|$ is even.

If we define real-valued radial polynomials using the equation

$$\widetilde{R}_{nl}(r) = \sum_{s=0}^{n-|l|} (-1)^s \frac{(2n+1-s)!}{s!(n-|l|-s)!(n+|l|+1-s)!} r^{n-s} \tag{6}$$

and use them to replace $R_{nl}(r)$ in Eq. (2), then we get pseudo-Zernike moments $\widetilde{Z}_{nl}$ of order n.

3. Tchebichef Moments

The Tchebichef moments of order $p+q$ of an image $f(i, j)$ of size N are defined using the scaled orthogonal Tchebichef polynomials $t_n(i)$, as

$$T_{mn} = \frac{1}{\rho(m,N)\rho(n,N)} \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} t_m(i)t_n(j)f(i,j)$$

$$m, n = 0, 1, \ldots N-1, \tag{7}$$

and has an exact image reconstruction formula (inverse moment transform),

$$f(i, j) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} T_{mn} t_m(i) t_n(j). \tag{8}$$

The inverse moment transform allows us to reconstruct the image intensity distribution from a set of computed moments. In Eq.(7), the polynomials $t_n(i)$, satisfy the recurrence formula:

$$(n+1)t_{n+1}(x) - (2n+1)(2x-N+1)t_n(x) + n(1-n^2/N^2)t_{n-1}(x) = 0$$

$$n = 1,2 \ldots N-2; \qquad x = 0,1,\ldots N-1. \tag{9}$$

with the initial conditions

$$t_0(x) = 1, \quad t_1(x) = (2x-N+1)/N\,, \tag{10}$$

and the squared-norm $\rho(n, N)$ is given by

$$\rho(n,N) = \frac{N\left(1-\frac{1}{N^2}\right)\left(1-\frac{2^2}{N^2}\right)\cdots\left(1-\frac{n^2}{N^2}\right)}{2n+1}, \quad n = 0, 1, \ldots, N-1. \tag{11}$$

The most important property of the set $\{t_n(x)\}$ that is utilized in a moment definition, is its orthogonality in the discrete domain:

$$\sum_{x=0}^{N-1} t_m(x)t_n(x) = \rho(n,N)\ \delta_{mn}\,. \tag{12}$$

For a detailed description of the Tchebichef moment equations, refer [1].

4. Radial Tchebichef Moments

Even though Tchebichef moments have several advantages over continuous orthogonal moments such as the Legendre, Zernike and Pseudo-Zernike moments, the form given in Eq. (7) is not very convenient for generating invariants. We define the radial Tchebichef moments of order p and repetition q as

$$S_{pq} = \frac{1}{2\pi\ \rho(p,m)} \sum_{r=0}^{m-1}\sum_{\theta=0}^{2\pi} t_p(r)e^{-jq\theta} f(r,\theta) \tag{13}$$

where the image size is NxN pixels, and m denotes $(N/2)+1$. Since θ is a real quantity measured in radians, we further generalize Eq. (13) and define,

$$S_{pq} = \frac{1}{n\ \rho(p,m)} \sum_{r=0}^{m-1}\sum_{\theta=0}^{n-1} t_p(r)e^{-jq\theta} f(r,\theta) \tag{14}$$

In the above equation, both r and θ take integer values. The mapping between (r, θ) and image coordinates x, y is given by

$$x = \frac{rN}{2(m-1)}\cos\left(\frac{2\pi\theta}{n}\right) + \frac{N}{2}$$
$$y = \frac{rN}{2(m-1)}\sin\left(\frac{2\pi\theta}{n}\right) + \frac{N}{2} \tag{15}$$

It can be easily shown that the definition in Eq. (14) yields a moment set that is orthogonal in the discrete polar coordinate space of the image. The inverse moment transform is given by the following equation:

$$f(r,\theta) = \sum_{p=0}^{P}\sum_{q=0}^{Q} S_{pq} t_p(r) e^{jq\theta} \tag{16}$$

where P, Q respectively denote the maximum order for p, q ($P<m$, $Q<n$) used for image reconstruction. It may be noted that the term on the right-hand side in Eq.(14) is complex-valued. For the convenience of computing with real-valued quantities, we can rewrite the expression for S_{pq} in terms of its real-valued components as discussed below. The inverse transform can then be easily computed, and used to verify the correctness of the moments. If we write

$$S_{pq} = S_{pq}^{(c)} - jS_{pq}^{(s)} \tag{17}$$

then the real-valued radial Tchebichef moments can be defined as

$$S_{pq}^{(c)} = \frac{1}{n\ \rho(p,m)} \sum_{r=0}^{m-1}\sum_{\theta=0}^{n} t_p(r)\ \cos(q\theta)\, f(r,\theta)$$

$$S_{pq}^{(s)} = \frac{1}{n\ \rho(p,m)} \sum_{r=0}^{m-1}\sum_{\theta=0}^{n} t_p(r)\ \sin(q\theta)\, f(r,\theta) \tag{18}$$

Since

$$S_{p,-q}^{(c)} = S_{p,q}^{(c)}\ ; \qquad S_{p,-q}^{(s)} = -S_{p,q}^{(s)} \tag{19}$$

the inverse moment transform in Eq. (16) now becomes

$f(r, \theta) =$

$$\sum_{p=0}^{P} t_p(r)\left\{ S_{p0}^{(c)} + 2\sum_{q=1}^{Q} S_{pq}^{(c)}\cos(q\theta) + S_{pq}^{(s)}\sin(q\theta)\right\} \tag{20}$$

Similar to Eq. (5), we can write invariants of radial Tchebichef moments in the form

$$\varphi_1 = S_{p0} = S_{p0}^{(c)},$$

$$\varphi_2 = |S_{pq}|^2 = \left(S_{pq}^{(c)}\right)^2 + \left(S_{pq}^{(s)}\right)^2 \tag{21}$$

5. Comparison

Zernike polynomials are defined only inside a unit circle, and therefore the computation of Zernike moments given in Eq.(1) requires a linear coordinate transformation from the image space to the interior of the unit circle, followed by a mapping from the rectangular coordinate system to the polar coordinate system. In the case of radial Tchebichef moments, the generalized mapping equations are given in Eq. (15). The values of m, n can be selected to suit the desired sampling frequency. Typically m has a value which is at least $N/2$, and n is at 360 when the image is sampled at one degree intervals (Fig 1):

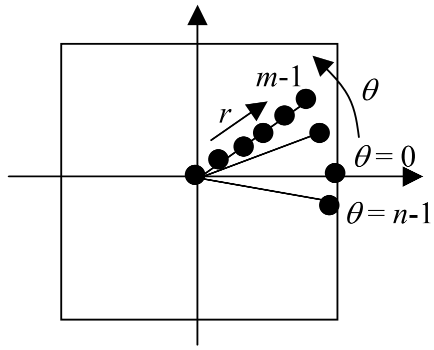


Fig. 1: Discrete Pixel Sampling in Radial-Polar Form

The main difference to be noted here is that while Zernike moment definition uses real values for r and θ, with $0 \le r \le 1$, $0 \le \theta \le 2\pi$; the Tchebichef moment definition uses integer values for both r and θ, where $0 \le r < m$, $0 \le \theta \le n$.

The discrete approximation involved in the computation of Zernike moment integrals, affect the orthogonality property of the feature descriptors. On the other hand, the radial Tchebichef moments given in Eqs.(14),(18) are perfectly orthogonal in the discrete coordinate space $0 \le r < m$, $0 \le \theta \le n$. Consequently, the radial Tchebichef moments exhibit a superior feature representation capability compared to Zernike moments.

The invariant functions of Tchebichef moments are also expected to be more robust than Zernike invariants.

The basis functions of both Zernike and radial Tchebichef moments are separable. Therefore the first equation in (18) can be equivalently expressed as follows:

$$g(q,r) = \sum_{\theta=0}^{n} \cos(q\theta)\, f(r,\theta) \ ,$$
$$S_{pq}^{(c)} = \frac{1}{n\ \rho(p,m)} \sum_{r=0}^{m-1} t_p(r)\, g(q,r). \tag{22}$$

The above decomposition allows $S_{pq}^{(c)}$ (and similarly $S_{pq}^{(s)}$) to be evaluated in $O(2N^3)$ time, instead of $O(N^4)$ time. The inverse transform in (20) can also be evaluated in two stages as

$$h(p,\theta) = S_{p0}^{(c)} + 2\sum_{q=1}^{Q} S_{pq}^{(c)} \cos(q\theta) + S_{pq}^{(s)} \sin(q\theta),$$
$$f(r,\theta) = \sum_{p=0}^{P} t_p(r)\ h(p,\theta) \tag{23}$$

6. Experimental Results

Fig. 2 shows the reconstruction of a binary image using radial Tchebichef moments (Eq. (20)). In this study, a threshold of 0.5 was applied to the reconstructed intensity values, to obtain a binary image. The reconstruction error ε in Fig.2 was computed as the total number of pixels that do not have the same value in both the original and the reconstructed binary images.

Fig. 3 shows the original and the reconstructed versions of a gray-level image of size 256x256 pixels. The values of the various parameters used for reconstruction of the ‘coin’ image are:

$$m = 128,\ n = 700,\ P = 80,\ Q = 100 \tag{24}$$

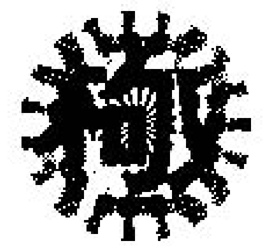
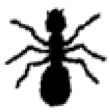
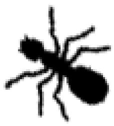
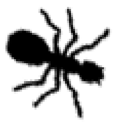
	Original Image Image size $N = 100$
	Reconstructed Image $m = 50$ $n = 360$ $P = 30$ $Q = 30$ $\varepsilon = 1711$
	Reconstructed Image $m = 50$ $n = 360$ $P = 40$ $Q = 150$ $\varepsilon = 425$

Fig. 2: Binary Image Reconstruction Using Radial Tchebichef Moments

Fig. 3: Gray-Level Image Reconstruction Using Radial Tchebichef Moments

Some preliminary results obtained for the rotational invariants, computed using a set of binary images, are given in Fig.4. The images are transformed versions of the original image got by performing a rotation about the centre by a certain angle. The Zernike and Radial-Techebichef invariants are then computed using Eqs. (5), (21) respectively. A detailed statistical analysis of the magnitude of the invariants with respect to image noise is currently being carried out, and is not included in this paper. In addition to the primary invariants given in Eq. (21), more complex forms of invariants can also be constructed as follows:

$$\varphi_3 = (S_{pq})^*(S_{rs})^k + [(S_{pq})^*(S_{rs})^k]^* \qquad (25)$$

Images	Zernike Invariants	Radial Tcheb Invariants
	0.016341 0.054553 0.102924 0.089054	0.085064 0.084679 0.092717 0.142306
	0.016164 0.054073 0.102870 0.089052	0.084764 0.086765 0.093248 0.141434
	0.014162 0.047828 0.095870 0.088109	0.084004 0.098046 0.071271 0.139676
	0.013321 0.045242 0.091937 0.084608	0.080987 0.094842 0.062781 0.132017

7. Conclusion

This paper has presented a new class of discrete orthogonal moments based on Tchebichef polynomials with a radial-polar representation of the image coordinates. The definition of radial-Tchebichef moments is very similar to that of Zernike moments, using a separable basis function which is a product of an orthogonal radial function and a circular harmonic term. However, radial-Tchebichef moments do not require the discrete approximation of continuous functions as in the case of Zernike moments, and are exactly orthogonal in the discrete space. The polar coordinate representation of the kernel functions has also been made use of in deriving the rotational invariants. Experimental results showing the recognition capability and the invariant characteristic of the proposed moment functions were also given.

Future research in this area is directed towards a more rigorous analysis of radial-Tchebichef moments, their invariant characteristics, and computational complexities with respect to different sampling frequencies along both radial and polar directions. An exhaustive comparative analysis of discrete and continuous rotational invariants using Zernike, Pseudo-Zernike and radial-Tchebichef moments also needs to be done to clearly ascertain the suitability of the proposed moments for pattern recognition tasks.

References:

[1] R. Mukundan, S.H.Ong, P.A.Lee, Image Analysis by Tchebichef Moments, *IEEE Trans. on Image Processing, 10* (9), 2001, 1357-1364.

[2] Nikiforov A.V, Suslov S.K, Uvarov V.B, *Classical orthogonal polynomials of a discrete variable* (Springer-Verlag : 1991).

[3] R. Mukundan, S.H.Ong, P.A. Lee, Discrete vs Continuous Orthognal Moments for Image Analysis, *Proc. Intnl. Conf. On Imaging Systems, Science and Technology ,* Las Vegas, NV, 2001, pp 23-29.

[4] Khotanzad A, Invariant image recognition by Zernike moments, *IEEE Trans. on Pattern Analysis and Machine Inteligence, 12*(5), 1990, 489-497.

[5] S.O. Belkasim, Pattern recognition with moment invariants -A comparative study and new results, *Pattern Recognition, 24*(12), 1991 , 1117-1138.

[6] C.W. Chong, P.Raveendran, R. Mukundan, A Comparative Analysis of Algorithms for Fast Computation of Zernike Moments, *Pattern Recognition, 36*(3), 2003, 731-742.