TAKENINGS, RAMES AND MILLS:
STRUCTURES FOR THE MODELLING OF BRANCHING

by

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ABSTRACT

A takening is a combinatorial structure composed of points and rays. A ray is a finite sequence of at least two points, and two rays have at most one point in common, which must be the foot (initial point) of at least one of them.

Walks, paths, proper walks and cycles are defined. A root is a point which is the foot of every ray on which it lies. Every connected takening is either a rame, which has one root and no cycle, or a mill, which has one cycle and no root.

The concept of a subtakening is defined.

1. INTRODUCTION

There have been a number of studies over the years of the branching of plants which have employed graph theory, or more specifically, combinatorial trees, [1],[2],[4],[5],[8] for example. Most of these made use of Horton's method of assigning an order or ranking to branches, [3], or Strahler's modification of it, [9]. All of these take as the unit of structure the section of branch between successive points at which the plant divides.

Such a model ignores an essential feature of plant growth. Branches grow by extension of the apex, a terminal bud which creates at intervals a node bearing a leaf or group of leaves and, where the leaves join the branch, axillary buds which, in the case of many temperate-climate trees, become apices and grow new branches only in the following year. This means that one branch continues through several nodes, instead of beginning afresh each time. Further, not every axillary bud develops into a new branch. A few modelling studies, like [6],[7] do consider branching in this way, and of course a great many non-mathematical studies also recognise this integrity. It so happens that for trees growing strictly to the pattern outlined above, Strahler's method, which was designed for rivers, does correctly aggregate the branches grown in a particular year.

It seemed that there was a need for a combinatorial structure devised for the modelling of botanical growth. A collection of axioms was set down which included a wider collection of structures than was originally desired. However, the mathematical theory also turned out to be richer than expected, and the purpose of this paper is to introduce the abstract theory of takenings. As 'graph' is of Greek derivation, and merely means something drawn or written, and being in the Netherlands at the time of first developing this theory, I took the Dutch word for a drawing, 'tekening', and altered the spelling to make the English natural pronunciation approximate the Dutch. By a happy chance, 'tak' in Dutch means a branch.

The use of 'ray' rather than 'branch' avoids possible confusions in botanical applications where the entity may be used to model structures other than branches. The use of 'node' is less likely to cause such problems, and though 'root' introduced later could cause ambiguities, I am there following established mathematical usage.

2. TAKENINGS

DEFINITION 1. A *takening* T = (V,R) is a non-empty finite set V of objects called *points* or *vertices*, together with a set R of *rays*, where a ray is a finite sequence of distinct points. The leading member of each ray is called its *foot*, and the remaining points are called the *nodes* of the ray. The following conditions also apply:

(A) Each ray consists of at least two points;

(B) Two distinct rays have at most one point in common, which must be the foot of at least one of them.

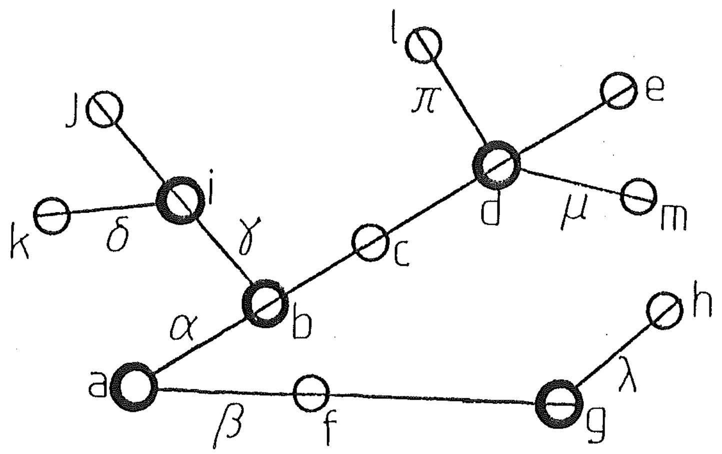

Figure 1. A takening. Each ray is shown as a line passing through its nodes, but not through its foot.

If a point b occurs in the sequence γ, we say that b is *on* γ, and that γ and b are *incident* with each other. Figure 1 shows a takening. The points are shown by circles, the rays as straight lines, passing through the nodes, but not through the foot of the ray. In set form this takening may be written:

$$V = \{a,b,c,\ldots,m\}$$

$$R = \{\alpha,\beta,\gamma,\ldots,\pi\}$$

$$\alpha = (a,b,c,d,e) \qquad \lambda = (g,h)$$

$$\beta = (a,f,g) \qquad \mu = (d,m)$$

$$\gamma = (b,i,j) \qquad \pi = (d,l)$$

$$\delta = (i,k)$$

Then a is the foot of α and β, and d is on α, μ and π and is the foot of two of them. An immediate consequence of Definition 1B is:

THEOREM 1. *A point may be on arbitrarily many rays, but is a node of at most one of them.*

DEFINITION 2. The *order* of a takening is the cardinality of its point set.

DEFINITION 3. The *length* of a ray is the number of its nodes.

DEFINITION 4. The *degree* of any point is the number of rays of which it is the foot.

THEOREM 2. *The sum of the degrees of the points of a takening is the number of its rays.*

3. WALKS

As in graph theory, the concept of a walk plays a central role; the definitions are also closely analogous.

DEFINITION 5. A *walk* W in a takening $T = (V,R)$ is a non-empty finite sequence of points, $\langle u_1, u_2, u_3, \ldots., u_k \rangle$ such that any two points u_i, u_{i+1} consecutive in W are consecutive (in either order) in some ray in R.

Among the walks in Figure 1 are:

$W_1 = \langle a,b,c,d,m \rangle$

$W_2 = \langle k,i,b,c,d,e \rangle$

$W_3 = \langle m,d,e \rangle$

$W_4 = \langle b,c,d,c,b \rangle$.

We can concatenate walks if the final point of the first walk coincides with the first point of the second walk:

$W_1 W_3 = \langle a,b,c,d,m,d,e \rangle$.

We may also reverse any walk, which we show by a superscript ^{R} :

$W_1^{\;R} = \langle m,d,c,b,a \rangle$.

We observe that for any walk W, $(W^R)^R = W$.

There is also at each point b the *trivial walk* $\langle b \rangle$.

DEFINITION 6. Two walks $\langle u_1, u_2, \ldots. u_k \rangle$, $\langle v_1, v_2, \ldots, v_m \rangle$ are equal if and only if $k = m$ and $u_i = v_i$ for all i, $1 \leqslant i \leqslant k$.

DEFINITION 7. The point c is *reachable* from the point b if there is a walk from b to c.

THEOREM 4. *Reachability is an equivalence relation.*

Reachability partitions the points into equivalence classes, which we will call *components*, as in graph theory.

DEFINITION 8. A takening is *connected* if there is a walk from each point to each other point.

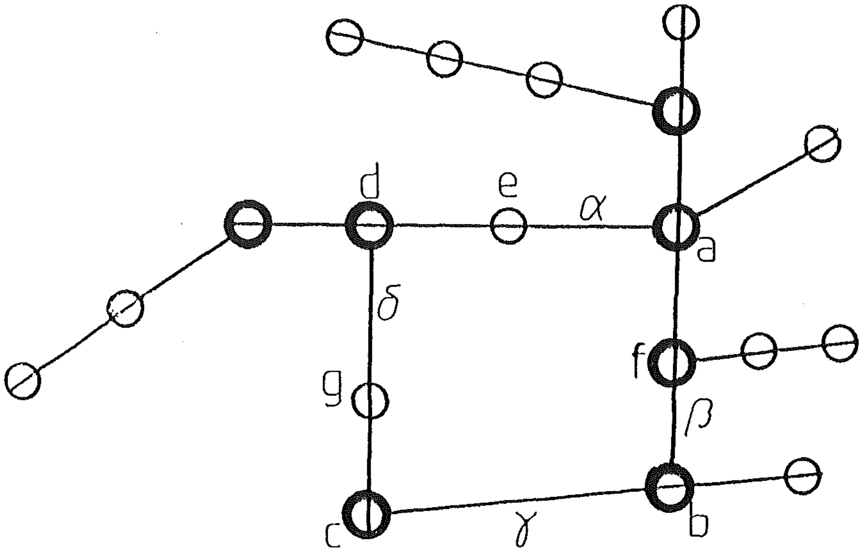

Figure 2. A takening with a cycle.

The takening in Figure 1 is connected, and so is the one in Figure 2. In this latter takening we observe a cycle, consisting of the four rays $\alpha,\beta,\gamma,\delta$, the feet of which, a,b,c,d, each lie on the next ray of the cycle. We have indeed a walk

$C = \langle a,f,b,c,g,d,e,a\rangle$

whose initial and final points are the same.

DEFINITION 9. A walk is *closed* if its initial and final points coincide.

Takenings in which there are such cycles clearly play no part in the botanical application from which we started: a main purpose of this paper is to characterise those takenings in which there are no cycles. We cannot simply forbid closed walks, for in Figure 1 W_4 is a closed walk, and every non-trivial takening will have such walks. We notice that W_4 has the pattern ..,c,d,c,..., and it turns out that this pattern is crucial.

We defer any formal definition of cycles until we have developed further machinery.

DEFINITION 10. A walk is *proper* if it does not contain any sequence of points of the form ...,u,v,u,... .

DEFINITION 11. A walk in which no point occurs more than once is called a *path*.

It is immediate that every path is proper: a proper walk may however have repetitions at greater separation.

THEOREM 5. *If there is a walk from b to c, there is a path from b to c.*

PROOF. Choose from all the walks from b to c a walk W with a minimum number of points, counted in their multiplicity.

If W has no points equal, W is a path.

Otherwise, some point d in W is repeated. Suppose

$$W = \langle b,\dots,d,\dots,d,\dots,c\rangle .$$

Then we may write $W = W_1W_2W_3$, where $W_1 = \langle b,\dots,d\rangle$, $W_2 = \langle d,\dots,d\rangle$, $W_3 = \langle d,\dots,c\rangle$. Note that W_1 and W_3 may be trivial, but W_2 contains at least one point other than the two occurrences of d.

Then W_1W_3 is also a walk from b to c, but with fewer points than W. This contradicts the choice of W.

Two points belong to the same component if and only if there is a path from one to the other. The reverse of a path is clearly a path.

4. ASCENDING AND DESCENDING WALKS

DEFINITION 12. Let u, v be consecutive points in that order in some walk W. Then they are also consecutive in some ray γ. If they occur in the order u, v in γ we say that the pair (u,v) is *ascending* in W. If they occur in the order v, u in γ we say that (u,v) is *descending* in W.

Note that if W is not a path, it is possible for (u,v) to occur in both orders in W: (c,d) in W_4 of Figure 1 is an example.

THEOREM 6. *In a proper walk, an ascending pair cannot be followed by a descending pair.*

PROOF. Let W = <b,...,u,v,w,...,c> be a walk with (u,v) ascending and (v,w) descending. Then u,v occur consecutively in that order in some ray γ, and w,v occur consecutively in that order in some ray δ. But v is not the foot of either ray, so $\gamma = \delta$ and u = w. Thus W is not proper.

However, a descending pair may be followed by an ascending pair, and a pair of either kind may be followed by another of the same kind.

DEFINITION 13. A walk is an *ascending walk* if all its pairs are ascending, and a *descending walk* if all its pairs are descending. A trivial walk is both ascending and descending.

THEOREM 7. *Any ascending walk is proper.*

PROOF. Suppose W = <...,u,v,u,...> is an ascending walk. Then both (u,v) and (v,u) must be ascending pairs. But v cannot both follow and precede u in the same ray. Hence u and v must be common to two rays, contrary to the takening axioms. The contradiction shows that an ascending walk has no sequence ...,u,v,u,.... and is proper.

THEOREM 8. *Any descending walk is proper*, is proved in exactly the same way, or by considering the reverse walk.

Definition 13 allows us to recast Theorem 6 in the form:

THEOREM 9. *Every proper walk can be written as the concatenation of two proper walks, W_1,W_2 in which W_1 is descending and W_2 is ascending.*

Naturally, either W_1 or W_2 or both may be trivial. We observe that the reverse of an ascending walk is descending, and of a descending walk is ascending.

THEOREM 10. *Let u be any point in a takening. Then there is at most one point v such that (u,v) is a descending pair.*

PROOF. If (u,v) is a descending pair, then v occurs before u in some ray γ which contains them both. Then u is a node of γ, and for given u there is at most one such ray. Then there is a unique point v which precedes u directly in γ.

THEOREM 11. *There is at most one descending path between any two points.*

PROOF. Let $W_1 = \langle u_1, u_2, \ldots, u_k \rangle$, $W_2 = \langle v_1, v_2, \ldots, v_m \rangle$ be two descending paths from $b = u_1 = v_1$ to $c = u_k = v_m$.
If $k = m$ and $u_i = v_i$ for $1 \leqslant i \leqslant m$, then $W_1 = W_2$.
If $k < m$ and $u_i = v_i$ for $1 \leqslant i \leqslant k$, then $v_k = c$, which makes W_2 not a path. Similarly if $m < k$ and $u_i = v_i$ for $1 \leqslant i \leqslant m$, W_1 is not a path. The remaining case to consider is when there is some j, $j < k$, $j < m$ such that $u_i = v_i$, $1 \leqslant i \leqslant j$, but $u_{j+1} \neq v_{j+1}$. But Theorem 10 shows that there is only one possible point following u_j in a descending walk. Hence only the first case fits the conditions of the theorem.

THEOREM 12. *There is at most one ascending path between any two points.*

PROOF. Let W_1, W_2 be ascending paths from b to c. Then W_1^R, W_2^R are descending paths from c to b. By Theorem 11 they are equal. Hence $W_1 = W_2$ also.

THEOREM 13. *If b, c, e are points such that there are descending paths from b to each of c, e, then either there is a descending path from c to e or there is a descending path from e to c.*

PROOF. Let $W_1 = \langle u_1, u_2, \ldots, u_k \rangle$, $W_2 = \langle v_1, v_2, \ldots, v_m \rangle$, with $u_1 = v_1 = b$, $u_k = c$, $v_m = e$. Theorem 10 shows that we cannot have a value j with $j < k$, $j < m$ such that $u_i = v_i$ for $1 \leqslant i \leqslant j$, but $u_{j+1} \neq v_{j+1}$. Hence we are in one of the cases:

(a) $k = m$, $u_i = v_i$ for $1 \leqslant i \leqslant k$. Then $c = e$ and the required descending path is the trivial path $\langle c \rangle$.

(b) $k < m$, $u_i = v_i$ for $1 \leqslant i \leqslant k$. Then $W_2 = W_1X_1$, where X_1 is then a descending path from c to e.

(c) $m < k$, $u_i = v_i$ for $1 \leqslant i \leqslant m$. Then $W_1 = W_2X_2$, where X_2 is a descending path from e to c.

5. CLOSED WALKS

We are now in a position to consider the structure of closed proper walks.

Our definition of proper walks, in terms of consecutive points does not prevent repetitions at a greater separation. This will influence our definition of a cycle. We also need to exclude trivial walks.

DEFINITION 14. A *cycle* is a descending, non-trivial proper walk in which no point is repeated except that the initial and final points are the same. A *reversed cycle* is the reverse of a cycle.

THEOREM 14. *Every proper closed walk X can be written in the form* $WCCCC..CW^R$, *where W is a descending proper walk, possibly trivial, and C is some cycle or reversed cycle. C may be traversed any positive number of times.*

PROOF. Let b be the initial and final point of X. If X is a descending walk, set $Y = X$, $W = \langle b \rangle$, and go to Case (ii). If X is an ascending walk, set $Y = X$, $W = \langle b \rangle$, and go to Case (iii).

Case (i). Otherwise X consists of a non-trivial descending walk followed by a non-trivial ascending walk. Let

$$X = \langle b, b_1, \ldots, b_m, b \rangle .$$

Then (b, b_1) is a descending pair and (b_m, b) is an ascending pair, so both b_1 and b_m precede b in the ray of which b is a node, implying that $b_1 = b_m$. We may write $X = W_1 X_1 W_1^R$, where $W_1 = \langle b, b_1 \rangle$ and X_1 is a proper closed walk from b_1 to b_1.

If X_1 has both ascending and descending pairs, we continue the process, separating initial and final pairs from X_1, X_2 etc. Finally one class or the other is exhausted. If both ascending and descending pairs are exhausted together, the final X_k may be written $W_k W_k^R$, which is $\langle b_{k-1}, b_k, b_{k-1} \rangle$, which is not proper, implying that X was not proper. Hence the two classes are not exhausted together, and we obtain at the end of this process

$$X = W_1 W_2 \cdots\cdots W_k X_k W_k^R W_{k-1}^R \cdots W_2^R W_1^R ,$$

where X_k is a nontrivial proper closed walk, either ascending or descending. Set $Y = X_k$ and $W = W_1 W_2 \ldots W_k$. Go to Case (ii) if Y is descending and Case (iii) if Y is ascending.

Case (ii). Y is a descending proper closed walk. If Y has no repetitions of points beyond its initial and final points, Y is a cycle and X is in the required form. Otherwise, let $Y = \langle c_0, c_1, c_2, \ldots, c_m, c_0 \rangle$, and let c_j be the first point in Y which is repeated and c_{j+p} its first repetition.

Then by Theorem 10, $c_{j+1} = c_{j+p+1}$, and applying an induction argument $c_r = c_{p+r}$ for each $r \geqslant j$. For $r = m + 1 - p$, $c_r = c_{m+1} = c_0$. Hence c_0 is a repeated point, making $j = 0$. Further, p divides m, and if $pq = m$, and we take

$$C = \langle c_0, c_1, c_2, \ldots, c_{p-1}, c_0 \rangle ,$$

$Y = CC\ldots C$ (q factors) and $X = WYW^R$ as required.

Case (iii). Y is a proper non-trivial ascending closed walk, so Y^R is a proper non-trivial descending closed walk, and by Case (ii) Y^R is a concatenation of cycles. Thus Y is a concatenation of reversed cycles, as required. □

6. ROOTS AND CYCLES

In Figure 1, the point a is a foot of two rays and is not a node of any ray. In Figure 2, every point is a node of some ray.

DEFINITION 15. (a) A point which is a node of no ray is called a *root*.

(b) A point which lies on no ray is called an *isolated point*.

Thus an isolated point is always a root.

THEOREM 15. *From every point there is a descending walk, either to some root, or to some point on a cycle, but not both.*

PROOF. Let b_0 be the point. If it is itself a root or known to be on a cycle, the trivial walk fulfills the conditions. Otherwise consider the sequence of descending walks:

$\langle b_0, b_1\rangle$

$\langle b_0, b_1, b_2\rangle$

$\langle b_0, b_1, b_2, b_3\rangle$

etc.

By Theorem 10 each b_i is completely determined and in this way we generate all the descending walks from b_0. As the set of points is finite, the process cannot continue for ever with new points. Eventually we either reach a point b_k from which no further progress is possible, or we reach a point b_m which has been visited before. In the former case, b_k is a root, and in the latter, b_m is on a cycle. Moreover, if we proceed beyond

b_m, we shall continue to circle the cycle, and can never reach a root.

It is clear from this that every component of a takening must contain at least one root or one cycle. We recall that two points in the same component are joined by a path, and as a path is certainly proper, this consists of a descending part followed by an ascending part.

THEOREM 16. *No component contains two roots.*

PROOF. For if b,c are the roots there is a path from b to c, of which at least one of the descending part from b and the ascending part to c is non-trivial. But both these things are impossible, so there is no such path.

THEOREM 17. *No component contains both a root and a cycle.*

PROOF. Let b be the root and c a point on the cycle. Any descending walk from c remains trapped within the cycle, so the ascending part of the path to b must begin on the cycle. But b cannot be reached by any non-trivial ascending walk. Hence there is no path from c to b.

To complete this group of theorems we wish to show that a component of a takening does not contain two cycles. Before we can do this, we must face what we mean by two cycles being the same. The starting point of any cycle is built into the notation, yet it is in a sense arbitrary. If a cycle C be written as a concatenation of paths XY, then YX is also a cycle, with the same points in the same cyclic order, but a different starting point. For the present purposes we wish to consider XY and YX as the same.

THEOREM 18. *Two cycles in the same component are the same cycle, possibly written with different starting points.*

PROOF. Let $B = \langle b_1, b_2, \ldots, b_k, b_1 \rangle$ and $C = \langle c_1, c_2, \ldots, c_m, c_1 \rangle$ be cycles in the same component of some takening. Then there is a path P from b_1 to c_1.

The descending part of P continues always in B, so the ascending part must begin at some point b_r of B. The ascending part of P is the reverse of the descending part of P^R, which by the above argument ends at some point c_r in C. Thus $b_r = c_s$. But then $b_{r+i} = c_{s+i}$ for all i. This makes $b_r = b_{r+k} = c_{s+k}$, so $c_s = c_{s+k}$. This is possible only if m divides k. Similarly k divides m, so m = k. Then the two cycles contain exactly the same points in the same order.

These three theorems enable us to divide the components of a takening into two types:

(a) containing a single root and no cycles;

(b) containing a single cycle and no roots.

DEFINITION 16. A component with a root is called a *rame*. A component with a cycle is called a *mill*.

The term 'rame' comes from the Latin 'ramus', a branch, and there are also connections with Dutch, German and obsolete English words related to 'framework'. The term 'mill' comes from the grooves in a millstone for grinding grain. The rames are the structures desired for applications to branching in plants. To date, no applications have been proposed for mills.

The rames in which every ray has exactly two points are the rooted trees.

In rames, the discussion of types of walk is simplified, because 'proper walk' and 'path' are synonymous. In all takenings a path is a proper walk, and:

THEOREM 19. *In a rame, every proper walk is a path.*

PROOF. Let $W = \langle b_1, b_2, \ldots, b_m \rangle$ be a proper walk in a rame. If no point is repeated, W is a path. Otherwise, let b_k be a point which is repeated, b_r one of its repetitions. Then the section of W between b_k and b_r is a closed, non-trivial proper walk, and by Theorem 12 contains a cycle or a reversed cycle. But in a rame there are no cycles, so repetition of points is impossible.

THEOREM 20. *Between any two points in a rame, there is exactly one path.*

PROOF. That there is at least one path is guaranteed by the connectedness of a rame.

Let b and c be the given points and suppose there are two paths b to c. Each is composed of a descending part followed by an ascending. These define points p and q and descending paths W_1 from b to p, W_2 from c to p, W_3 from b to q, and W_4 from c to q, in such a way that $W_1 W_2^R$ and $W_3 W_4^R$ are the paths from b to c. As there are descending paths from b to p and q, Theorem 13 assures us that there is a descending path from p to q or a descending path from q to p. Without loss of generality, assume the former. Let X be this path. Then $W_1 X$ and W_3 are descending paths from b to q, so by Theorem 11 they are equal. In the same way, $W_4 = W_2 X$. Thus

$$W_3 W_4^R = W_1 X X^R W_2 .$$

This cannot be proper unless X is trivial, but then $W_1 = W_3$ and $W_2 = W_4$, which yields the desired result.

It follows immediately from Theorems 7 and 8 that:

THEOREM 21. *In a rame there are no non-trivial closed ascending or descending walks.*

We conclude the section with a theorem on rames:

THEOREM 22. *In any rame there is at least one ray none of whose nodes is the foot of any ray.*

PROOF. Suppose that every ray had at least one node that was the foot of some ray. Then there would be at least as many feet as rays. But this leaves out the root of the rame, which is not a node of any ray, so there would be more feet than rays. But every ray has exactly one foot, so we have a contradiction. □

7. DISTAL AND AXILLARY PARTS

It is a universal procedure of mathematics when developing a structure from an axiomatic foundation to define the appropriate substructures, homomorphism, isomorphisms and quotient structures.

The concept of isomorphism is clear enough: two takenings $T_1 = (V_1, R_1)$ and $T_2 = (V_2, R_2)$ will be *isomorphic* if there is a one-one correspondence $\phi : V_1 \to V_2$ and a one-one correspondence $\psi : R_1 \to R_2$ such that if u is the ith point in ray ρ of T_1 then $\phi(u)$ is the ith point of ray $\psi(\rho)$ of T_2.

There are several possible extensions of the isomorphism property to yield definitions of homomorphism. Before deciding on the most appropriate it would be necessary to consider the use that might be made of it in botanical applications, and at present we shall make no moves in this direction.

The same sort of consideration applies to the definition of a subtakening, but we shall here discuss two kinds of substructure that arise naturally from the application and which must be included by any useful definition of a subtakening.

The most obvious way in which we might take part of a plant is to cut one of its stems at an arbitrary point. This separates the plant into two pieces, one of which contains the root, and the other everything which lies 'above' (on the distal side of) the cut.

Another way to separate part of a plant is to pluck out a shoot from a leaf axil. This is not the same as cutting the plant at the node or the shoot between its foot and first node. Figure 3 shows the effect on a plant.

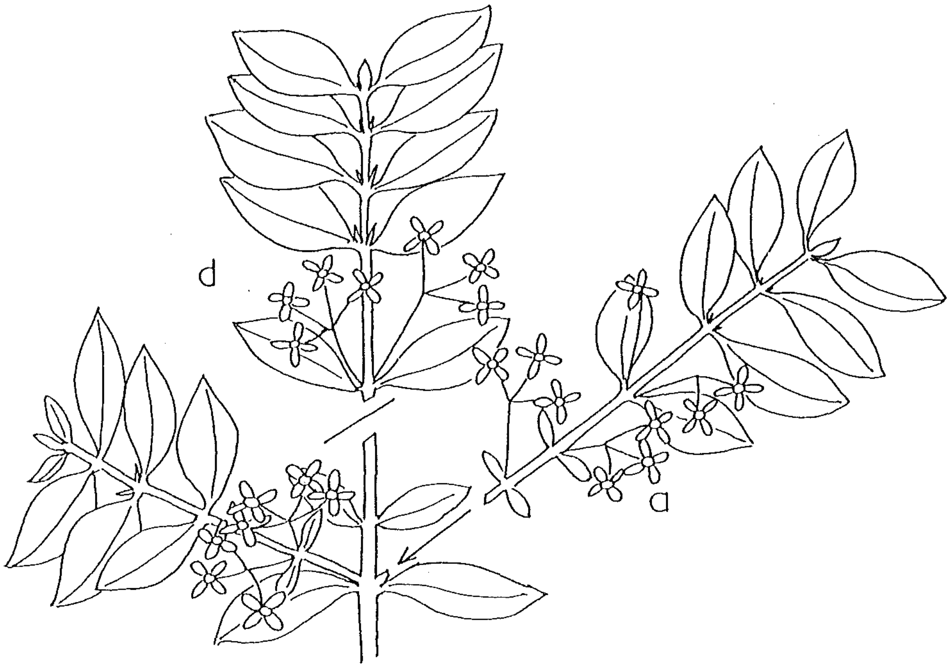

Figure 3. The portion of a plant on the distal side of a cut (d) and an axillary shoot (a).

The first method of cutting will, in general, pass through or just below a node of the plant, so that part of the shoot cut will be above the cut and part below. Our definitions must then allow us to take part of a ray. In Definition 17 we take a more general stance than is required by the immediate context.

DEFINITION 17. If $\rho = (u_0, u_1, \ldots u_i, \ldots, u_j, \ldots, u_m)$ is any ray, with $0 \leqslant i < j \leqslant m$, then $\rho_{ij} = (u_i, u_{i+1}, \ldots, u_{j-1}, u_j)$ is the *segment* of ρ *between* u_i and u_j. If $i = 0$, ρ_{ij} is the *proximal segment* of ρ defined by u_j, and may also be written ρ_j^p. If $j = m$, ρ_{ij} is the *distal segment* defined by u_i and may be written ρ_i^d.

Note that as $i < j$, every segment must have at least two points: the foot does not define a proximal segment and the terminal node does not define a distal segment. The distal segment defined by the foot and the proximal segment defined by the terminal node are each the ray ρ. A segment may be considered as a ray in its own right, for example in Definition 19: u_i is then the foot of ρ_{ij}.

We now confine our attention to rames.

DEFINITION 18. If v is a point of a rame $T = (V,R)$, we say that u is a *distal point relative to v* if the path from v to u is ascending.

THEOREM 23. *A point u is a distal point relative to v if and only if the path from the root r of T to u passes through v.*

PROOF. Suppose u is a distal point relative to v. Then there is an ascending path W from v to u, so a descenting path W^R from u to v. By Theorem 15 there is a descending walk from u to r. By Theorem 8 this walk is proper, and by Theorem 19 it is a path. Consequently Theorem 13 shows that there is either a descending path from v to r or from r to v. The latter is clearly impossible as there are no non-trivial descending walks from the root, so there must be a descending path Y from v to r. Then X and W^RY are descending paths from u to r and by Theorem 11 must be equal. Thus v lies on X and thus on X^R, the path from the root to u.

Conversely, there is a unique (ascending) path X^R from the root to u. If v lies on this path, we may write $X^R = ZW$, where Z is an ascending path from r to v and W is an ascending path from v to u. Consequently u is a distal point relative to v.

We next need the mathematical expression of the observation that cutting a plant in general severs some shoot.

THEOREM 24. *If v is a node of some ray ρ, then another point w of ρ is a distal point relative to v if and only if w is a point of the distal segment of ρ relative to v.*

PROOF. Let $\rho = (u_0, u_1, \ldots, u_m)$, $v = u_i$, $1 \leqslant i \leqslant m$, and let $w = u_j$, $0 \leqslant j \leqslant m$.

If w is a point of the distal segment of ρ relative to v, then $j \geqslant i$ and $\langle u_i, u_{i+1}, \ldots, u_j \rangle$ is an ascending walk, so w is a distal point relative to v.

Otherwise, $j < i$ and then $\langle u_j, u_{j+1}, \ldots, u_i \rangle$ is an ascending walk. But if w is also a distal point relative to v, there is an ascending walk from v to w, and concatenating the two gives a closed ascending walk in a rame, whereas by Theorem 21 all such walks are trivial. This contradiction establishes the result.

THEOREM 25. *If v is not a node of the ray ρ then either every point of ρ is a distal point relative to v or no point is.*

PROOF. Suppose first that the foot f of ρ is a distal point relative to v. Then there is an ascending walk W from v to f. If u is any point of ρ, there is an ascending walk X from f to u. Then WX is an ascending walk from v to u and u is a distal point relative to v.

On the other hand if u is any point of ρ which is a distal point relative to v, there is a descending walk Y from u to v. But every descending walk from u begins by descending ρ to its foot f, unless it ends first. As v is not a point of ρ, the latter does not happen. Thus f is on Y, so that the first part of Y^R is an ascending walk from v to f and f is a distal point relative to v.

We have shown that if any point of ρ is a distal point relative to v, then f is, and that if f is, then so is every point. Hence the theorem is proved.

DEFINITION 19. The *distal part* T_v^d of a rame T = (V,R) defined by a point v of T is (V_v^d, R_v^d) where V_v^d is the set of points of T distal relative to v and R_v^d consists of all rays whose feet are in V_v^d together with the distal segment of the ray ρ of which v is a node, if there is one and provided that v is not a terminal node.

THEOREM 26. T_v^d *is a rame*.

PROOF. We must first prove that T_v^d is a takening.

V_v^d is a finite set, being a subset of V. Every member of R_v^d is a finite sequence of points of V, and each point is shown to be in V_v^d by Theorems 24 and 25. Each ray has at least two points, and if two rays had more than one point in common, so would the corresponding rays in T. If the common point were not the foot of either of them in T_v^d, nor would it be T. Hence T_v^d is a takening.

If x and y are two points of T_v^d, there is an ascending walk X from v to x and an ascending walk Y from v to y. Thus X^RY is a walk from x to y, and T_v^d is connected.

Finally, v is not a node of any ray in R_v^d, for the ray of which it is a node in T, if there is one, has been replaced by the distal segment, of which it is the leading point and therefore the root, or, if v is the terminal node, simply omitted. Thus v is a root, and by the characterisation of connected takenings, T_v^d is a rame.

In defining the takening-theoretic analogue of an axillary shoot we must specify a ray ρ. We cannot take simply the distal part relative to its foot r, for this may include not only the distal segment of the ray of which r is a node but also other rays of which r is the foot and further rays springing from them, and so on. We could specify that the points of the axillary part defined by ρ are those points reachable from r by an ascending walk whose second point is s, the first node of ρ. We must then not forget to restore r itself. A formal version of this is:

DEFINITION 19. If $T = (V,R)$ is any rame and $\rho \in R$, the *axillary part* $T_\rho = (V_\rho, R_\rho)$, where r is the root and s the first node of ρ and $V_\rho = V_s^d \cup \{r\}$ and R_ρ consists if all rays with their feet in V_s^d together with ρ.

THEOREM 27. T_ρ *is a rame.*

The proof follows the same sequence as the proof of Theorem 26.

The distal parts and axillary parts give us some guidance as to what must be included in any general definition of a subtakening. The following generalisation seems appropriate:

DEFINITION 20. Let $T_1 = (V_1,R_1)$ and $T_2 = (V_2,R_2)$ be takenings. Then T_1 is a *subtakening* of T_2 if

(a) $V_1 \subseteq V_2$, and

(b) Every ray in R_1 is a segment of a ray in R_2.

We recall that every ray is a segment of itself, so (b) includes the possibility that rays of T_1 are also rays of T_2. Several rays of T_1 can be segments of the same ray of T_2.

It is easy to establish the following properties essential for a substructure:

THEOREM 28. *Every takening is a subtakening of itself.*

THEOREM 29. *If* T_1 *is a subtakening of* T_2 *and* T_2 *is a subtakening of* T_3 *then* T_1 *is a subtakening of* T_3.

THEOREM 30. *If* $T_1 = (V_1, R_1)$ *and* $T_2 = (V_2, R_2)$ *are takenings then* $T_3 = (V_1 \cap V_2, R_1 \cap R_2)$ *is a subtakening of each of them if* $V_1 \cap V_2 \neq \emptyset$.

Theorem 30 gives an unsubtle way of finding the common features of two takenings: it would be better to include common segments of rays. However, this opens up a whole area of algebra of subtakenings, taking us too far from the purpose of the present paper.

8. CONCLUSION

This paper has defined takenings, the motivation being derived from a botanical application. A combinatorial structure with features resembling both graphs and digraphs has been described, with definitions of walk, path, connectedness and so on. Connected takenings have been divided into two types, rames and mills, of which the former are appropriate to the botanical application.

The paper concludes with an investigation leading to the concept of subtakening.

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