Two Generalisations of the Wheels-and-Whirls Theorem

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Abstract

One of the most famous results in matroid theory is Tutte's Wheels-and-Whirls Theorem. It states that every 3-connected matroid has an element which can either be deleted or contracted while retaining 3-connectivity, except for two families of matroids: the eponymous wheels and whirls. The Wheels-and-Whirls Theorem is a powerful tool for inductive arguments on 3-connected matroids. We consider two generalisations of the Wheels-and-Whirls Theorem.

First, what are the k-connected matroids such that the deletion and contraction of every element is not k-connected? Motivated by this problem, we consider matroids in which every element is contained in a small circuit and a small cocircuit, and, in particular, when these circuits and cocircuits have a cyclic structure. The first part of this thesis is concerned with matroids in which have a cyclic ordering σ of their ground set such that every set of $s-1$ consecutive elements of σ is contained in an s-element circuit and every set of $t-1$ consecutive elements of σ is contained in a t-element circuit. We show that these matroids are highly structured by proving that they are "(s,t)-cyclic", that is, their s-element circuits and t-element cocircuits are consecutive in σ in a prescribed way. Next, we provide a characterisation of these matroids by showing that every (s,t)-cyclic matroid is a weak-map image of a particular (s,t)-cyclic matroid.

Secondly, what are the 3-connected matroids such that such that the deletion and contraction of every 2-element subset is not 3-connected? In the second part of this thesis, we find all such matroids. Roughly speaking, these matroids can be constructed in one of four ways: by attaching fans to a spike, by attaching fans to a line, by attaching particular matroids to $M(K_{3,m})$, or by attaching particular matroids to each end of a fan.

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One of my highlights over the last few years has been the Annual Nelson/Westport Matroid Workshop (unofficial title). In the midst of endless online conferences, it has been an incredible and inspiring opportunity to meet and collaborate with others, and I would like to acknowledge the organisers and attendees.

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chapter 1

Introduction

1.1 Tutte's Wheels-and-Whirls Theorem

One of the most famous results in matroid theory is Tutte's Wheels-and-Whirls Theorem [32]. It states that every 3-connected matroid has an element which can either be deleted or contracted while retaining 3-connectivity, except for two families of matroids: the eponymous wheels and whirls. The Wheels-and-Whirls Theorem is a powerful tool for inductive arguments on 3-connected matroids [20], which has motivated numerous extensions of this theorem.

We consider two natural generalisations of the Wheels-and-Whirls Theorem. First, what are the k-connected matroids such that the deletion and contraction of every element is not k-connected? And second, what are the 3-connected matroids such that the deletion of every pair of elements and the contraction of every pair of elements is not 3-connected?

We recall the definitions of wheels, whirls, and connectivity. The *rank-r wheel* is the matroid corresponding to the graph shown in Figure 1.1. The *rank-r whirl* has the same independent sets as the rank-r wheel except the set $\{b_1, b_2, \ldots, b_r\}$ is also independent.

The connectivity of a matroid was introduced by Tutte [32] as a notion analagous to that of vertex-connectivity for graphs. For a matroid M on ground set E and a subset $X \subseteq E$, the *connectivity* of X is defined

$$\lambda(X) = r(X) + r(E - X) - r(M)$$

or equivalently

$$\lambda(X) = r(X) + r^*(X) - |X|.$$

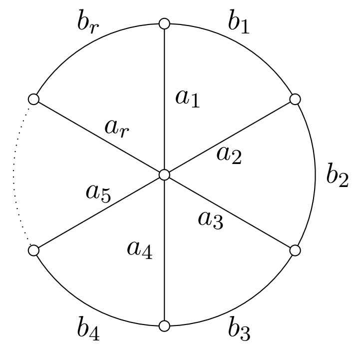


Figure 1.1: The rank-r wheel.

We say X is a *k-separation* if $\lambda(X) = k - 1$ and $|X| \geq k$ and $|E - X| \geq k$. A matroid is *k-connected* if, for all $k' \in \{0, 1, \ldots, k-1\}$, it has no k'-separations.

1.2 A Wheels-and-Whirls Theorem for Higher Connectivity

Finding a Wheels-and-Whirls-type theorem for connectivity higher than 3 is a difficult problem. To date, there is no such theorem for 4-connected matroids, let alone higher connectivity. There have been some results concerning "almost" 4-connected matroids, in which certain types of 3-separations are allowed to appear. For many applications, allowing these 3-separations makes for a more natural and useful notion of connectivity than general 4-connectivity.

The first such result was by Geelen and Whittle [12], more than 30 years after Tutte's seminal paper. A k-separation X is *sequential* if its elements can be ordered $(e_1, e_2, e_3, \ldots, e_m)$ such that, for all $i \in \{1, 2, \ldots, m\}$, we have that $\lambda(\{e_1, e_2, \ldots, e_i\}) \leq k$. A matroid is *sequentially* 4*-connected* if it is 3-connected and every 3-separation is sequential. Geelen and Whittle proved the following.

Theorem 1.2.1. *Let M be a sequentially* 4*-connected matroid. If M is not a wheel or a whirl, then M has an element x such that either $M\backslash x$ or M/x is sequentially* 4*-connected.*

Sequential 4-connectivity is a natural notion which allows only 3-separations with simple structure, but these 3-separations can have arbitrary size. Another possible weakening of 4-connectivity is to allow only small 3-separations. A matroid M is 4*-connected up to separators of size* k if it is 3-connected and, for every 3-separation X of M, either $|X| \leq k$ or $|E(M) - X| \leq k$. The following result is due to Hall [15].

Theorem 1.2.2. *Let M be a matroid which is 4-connected up to separators of size 5. Then M has a proper minor N which is 4-connected up to separators of size 5 such that $|N| \geq |E(M)| - 2$.*

This result was later extended by Chun, Mayhew, and Oxley [9][10].

Theorem 1.2.3. *Let M be a matroid which is 4-connected up to separators of size 3. Then M has a proper minor N which is 4-connected up to separators of size 3 such that $|N| \geq |E(M)| - 6$.*

A result due to Oxley, Semple, and Whittle [24] concerned 3-connected matroids in which every 3-separation is both small and sequential.

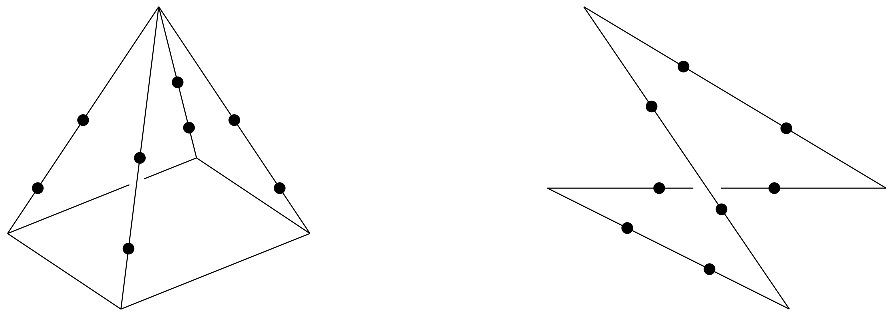
Theorem 1.2.4. *Let M be a matroid which is both sequentially 4-connected and 4-connected up to separators of size 5. Then M has a proper minor N which is both sequentially 4-connected and 4-connected up to separators of size 5 such that $|N| \geq |E(M)| - 2$.*

These results will not extend easily to 4-connected matroids. Rajan [28] showed that, for all $m \geq 1$, there is a 4-connected matroid M such that every minor of M with size at least $|E(M)| - m$ is not 4-connected.

1.3 Small Circuits and Cocircuits

So a Wheels-and-Whirls theorem for k-connectivity, with $k > 3$, is currently infeasible. However, a starting point to work towards such a theorem is to identify matroids which have no single-element deletions or contractions that are k-connected. The reason wheels and whirls are the exceptional matroids for the Wheels-and-Whirls Theorem is that they are precisely the 3-connected matroids in which every element is contained in both a 3-element circuit and a 3-element cocircuit. Contracting an element, therefore, produces a 2-element circuit, a 2-separation, so the resulting matroid is not 3-connected. Similarly, deleting an element produces a 2-element cocircuit, and so the resulting matroid is not 3-connected. More generally, a matroid in which every element is contained in both a k-element circuit and a k-element cocircuit will have no single-element deletions or single-element contractions which are k-connected. This has motivated the study of matroids in which every element is contained in a small circuit and a small cocircuit.

Miller [19] investigated matroids in which every pair of elements is contained in both a 4-element circuit and a 4-element cocircuit, and showed that such a matroid is a tipless spike. A *tipless spike* is a matroid which can be partitioned into pairs such that the union



(a) (b)

Figure 1.2: (a) A rank-4 spike, and (b) a rank-4 swirl.

of any two pairs is both a circuit and a cocircuit. An example of a rank-4 spike is shown in Figure 1.2a.

Theorem 1.3.1. *Let M be a matroid such that every pair of elements is contained in both a 4-element circuit and a 4-element cocircuit. If $|E(M)| \geq 13$, then M is a tipless spike.*

Oxley, Pfeil, Semple and Whittle [22] considered matroids in which every pair of elements is contained in a 4-element circuit, and either every element is contained in a 3-element cocircuit, or every element is contained in a 4-element cocircuit. They proved the following two theorems.

Theorem 1.3.2. *Let M be a 3-connected matroid such that every pair of elements is contained in a 4-element circuit, and every element is contained in a 3-element cocircuit. If $|E(M)| \geq 9$, then M is isomorphic to $M(K_{3,m})$, for some $m \geq 3$.*

Theorem 1.3.3. *Let M be a 4-connected matroid such that every pair of elements is contained in a 4-element circuit, and every element is contained in a 4-element cocircuit. If $|E(M)| \geq 16$, then M is isomorphic to $M(K_{4,m})$, for some $m \geq 4$.*

Brettell, Campbell, Chun, Grace, and Whittle [3] considered matroids in which every t-element subset is contained in both an ℓ-element circuit and an ℓ-element cocircuit. They showed that when $\ell < 2t$, there are finitely many matroids satisfying this property, and when $\ell = 2t$, the following holds.

Theorem 1.3.4. *Let M be a matroid such that every t-element subset is contained in a $2t$-element circuit and a $2t$-element cocircuit. If M has sufficiently many elements, then $E(M)$ has a partition into pairs such that the union of any t pairs is both a circuit and a cocircuit.*

Brettell and Grace [5] recently extended this result to the case in which the circuits and cocircuits need not have the same size.

Theorem 1.3.5. *Let M be a matroid such that every s-element subset is contained in a a $2s$-element circuit and every t-element subset is contained in a $2t$-element cocircuit. If M has sufficiently many elements, then $E(M)$ has a partition into pairs such that the union of any s pairs is a circuit, and the union of any t pais is a cocircuit.*

1.4 Cyclic Matroids

Of particular relevance to this thesis, is work done by Brettell, Chun, Fife, and Semple in [4]. We have already seen that every element of a wheel or a whirl is contained in both a 3-element circuit and a 3-element cocircuit, but they also satisfy a stronger property. If M is a wheel or a whirl, then there is a cyclic ordering σ of its ground set such that every set of two consecutive elements of σ is contained in a 3-element circuit and a 3-element cocircuit. For example, $\sigma_1 = (a_1, b_1, a_2, b_2, \dots, a_r, b_r)$ is such an ordering for the wheel in Figure 1.1. Brettell *et al.* [4] generalised this structure. They considered matroids which have a cyclic ordering σ such that every set of $s-1$ consecutive elements is contained in both an s-element circuit and an s-element cocircuit. In Chapter 2, we consider a further generalisation of these matroids in which the circuits and cocircuits do not necessarily have the same size.

A matroid M is *nearly (s,t)-cyclic* if there exists a cyclic ordering σ of $E(M)$ such that every set of $s-1$ consecutive elements of σ is contained in an s-element circuit and every set of $t-1$ consecutive elements of σ is contained in a t-element cocircuit. We say that σ is a *nearly (s,t)-cyclic ordering* of M. Wheels and whirls are examples of nearly $(3,3)$-cyclic matroids. As we have seen, a matroid M is a spike if $E(M)$ can be partitioned into pairs $L_1, L_2, \dots, L_r$ such that, for all distinct $i, j \in \{1, 2, \dots, r\}$, the set $L_i \cup L_j$ is both a 4-element circuit and a 4-element cocircuit. A matroid M is a *swirl* (Figure 1.2b) if $E(M)$ can be partitioned into pairs $L_1, L_2, \dots, L_r$ such that, for all distinct $i, j \in \{1, 2, \dots, r\}$, the set $L_i \cup L_j$ is both a 4-element circuit and a 4-element cocircuit if $j \in \{i-1, i+1\}$, where subscripts are interpreted modulo r, and independent and coindependent otherwise. In either case, if we write $L_i = \{a_i, b_i\}$ for all $i \in \{1, 2, \dots, r\}$, then $(a_1, b_1, a_2, b_2, \dots, a_r, b_r)$ is a nearly $(4,4)$-cyclic ordering of M. Thus, spikes and swirls are nearly $(4,4)$-cyclic matroids.

We can construct nearly (s,t)-cyclic matroids for large values of s and t using elementary quotients and elementary lifts. An *elementary quotient* of a matroid M is a matroid M' obtained by extending M by an element e, then contracting e. In particular, when M is freely extended by e, we say M' is the *truncation* of M. The truncation of a nearly (s,t)-cyclic matroid is a nearly $(s,t+2)$-cyclic matroid. Dually, an *elementary lift* of a matroid M is found by coextending M by an element e, then deleting e. When this coextension is free, it is called the *Higg's lift* of M. The Higg's lift of a nearly (s,t)-cyclic matroid is a nearly $(s+2,t)$-cyclic matroid. By repeatedly applying truncations and Higg's lifts, therefore, we can construct nearly (s,t)-cyclic matroids for arbitrarily large s and t.

Consider again the nearly (s,t)-cyclic ordering $\sigma_1 = (a_1, b_1, a_2, b_2, \dots, a_r, b_r)$ of the wheel in Figure 1.1. The 3-element circuits and 3-element cocircuits of this matroid consist of sets of consecutive elements of σ_1 — for all $i \in \{1, 2, \dots, r\}$, the set $\{a_i, b_i, a_{i+1}\}$ is a circuit and the set $\{b_i, a_{i+1}, b_{i+1}\}$ is a cocircuit. In [4], Brettell *et al.* proved that every (sufficiently large) nearly (s,s)-cyclic ordering has this structure. We generalise this result to nearly (s,t)-cyclic orderings with $s \neq t$.

More precisely, a matroid M is *(s,t)-cyclic* if there exists a cyclic ordering $\sigma = (e_1, e_2, \dots, e_n)$ of $E(M)$ such that each of the following holds, where subscripts are interpreted modulo n:

(i) either $\{e_1, e_2, \dots, e_s\}$ or $\{e_2, e_3, \dots, e_{s+1}\}$ is an s-element circuit of M,

(ii) either $\{e_1, e_2, \dots, e_t\}$ or $\{e_2, e_3, \dots, e_{t+1}\}$ is a t-element cocircuit of M,

(iii) if $\{e_i, e_{i+1}, \dots, e_{i+s-1}\}$ is an s-element circuit of M for some $i \in \{1, 2, \dots, n\}$, then $\{e_{i+2}, e_{i+3}, \dots, e_{i+s+1}\}$ is also an s-element circuit of M, and

(iv) if $\{e_i, e_{i+1}, \dots, e_{i+t-1}\}$ is a t-element cocircuit of M for some $i \in \{1, 2, \dots, n\}$, then $\{e_{i+2}, e_{i+3}, \dots, e_{i+t+1}\}$ is also a t-element cocircuit of M.

Such an ordering is called an *(s,t)-cyclic ordering* of $E(M)$. Note the terminology here differs to that in [4]: what we call a nearly (t,t)-cyclic ordering was called a cyclic $(t-1,t)$-ordering there, and what we call a (t,t)-cyclic ordering was called a t-cyclic ordering.

The first main result of Chapter 2 is the following.

Theorem 1.4.1. *Let M be a matroid on n elements, and suppose that σ is a nearly (s,t)-cyclic ordering of M, where $s,t \geq 3$. Let $t_1 = \min\{s,t\}$ and $t_2 = \max\{s,t\}$. If $n \geq 3t_1 + t_2 - 5$ and $n \geq t_1 + 2t_2 - 1$, then σ is an (s,t)-cyclic ordering of M.*

The remainder of Chapter 2 is concerned with characterising the class of (s,t)-cyclic matroids. We have seen that (s,t)-cyclic matroids can be constructed by a sequence of elementary quotients and elementary lifts. It was our hope, and a conjecture in [4], that every (s,t)-cyclic matroid could be constructed in this way. However, we define an (s,s)-cyclic matroid Ψ_s^n which is not a quotient of any $(s,s-2)$-cyclic matroid. In fact, it is not a quotient of any matroid with the s-element circuits of an $(s,s-2)$-cyclic matroid.

Theorem 1.4.2. *Let $s \geq 3$, and let $n \geq 4s-8$ be even. Let M be a matroid on n elements with cyclic ordering $\sigma = (e_1, e_2, \dots, e_n)$ such that, for all odd $i \in \{1, 2, \dots, n\}$, the set $\{e_i, e_{i+1}, \dots, e_{i+s-1}\}$ is an s-element circuit. Then Ψ_s^n is not a quotient of M.*

The next result of Chapter 2 shows that the matroid Ψ_s^n can be thought of as "the most free" (s, s)-cyclic matroid. Let M_1 and M_2 be matroids, and let $\varphi : E(M_1) \to E(M_2)$ be a bijection. Then φ is a *weak map* from M_1 to M_2 if for all circuits C of M_1, the set $\varphi(C)$ contains a circuit of M_2. We say M_2 is a *weak-map image* of M_1.

Theorem 1.4.3. *Let M be an (s,t)-cyclic matroid such that $|E(M)| \geq s+t-1$ and $t \geq s$. Then M is a weak-map image of the $(\frac{t-s}{2})$-th truncation of Ψ_s^n, which is an (s,t)-cyclic matroid.*

1.5 The Splitter Theorem and Excluded Minors

Perhaps the most important and widely used extension of the Wheels-and-Whirls Theorem is the Splitter Theorem, due to Seymour [30], and independently Tan [31].

Theorem 1.5.1. *Let M and N be 3-connected matroids such that N is a proper minor of M, and $|E(N)| \geq 4$, and if N is a wheel then M has no larger wheels as a minor, and if N is a whirl then M has no larger whirls as a minor. Then there exists an element e of M such that either $M\backslash e$ or M/e is 3-connected and has a minor isomorphic to N.*

The Splitter Theorem enables an element to be removed from a 3-connected matroid M while retaining not just 3-connectivity, but also a certain minor of M. One of the primary ways of understanding the structure of a class of matroids is through its minors, and the Splitter Theorem has proven itself to be invaluable for such analyses [20][29]. In particular, an *excluded minor* of a minor-closed class of matroids $\mathcal{M}$ is a matroid M such that M is not contained in $\mathcal{M}$ but every minor of M is. A matroid is a member of $\mathcal{M}$ if and only if it does not have a minor isomorphic to an excluded minor of $\mathcal{M}$. Hence, the class $\mathcal{M}$ can be characterised by its set of excluded minors.

For example, Geelen, Gerard, and Kapoor [13] used the Splitter Theorem to find the excluded minors for matroids representable over $GF(4)$. The strategy of their proof is as follows. For an excluded minor M and distinct elements $e, f \in E(M)$, the matroids $M\backslash e$, $M\backslash f$, and $M\backslash e\backslash f$ are representable over $GF(4)$. Use representations of these matroids to construct a matroid N which is representable over $GF(4)$ and $N\backslash e = M\backslash e$ and $N\backslash f = M\backslash f$. But M is not representable over $GF(4)$ and N is representable over $GF(4)$, so compare M and N to reach a bound on the size of M. This strategy generalises to different fields (and partial fields). For example, it is used by Hall, Mayhew, and van Zwam [16] to find the excluded minors for matroids representable over all fields with at least three elements.

One major difficulty in the above is that a matroid may have multiple inequivalent representations over a field. The theory of stabilizers, introduced by Whittle in [33], is used to limit these inequivalent representations. A *stabilizer over a partial field* $\mathbb{P}$ is a matroid N such that, for every 3-connected matroid M which is representable over $\mathbb{P}$ and has a minor isomorphic to N, two representations of M in which the representation of a particular N-minor are equivalent are themselves equivalent. The representation of an N-minor of M uniquely determines the representation of M. So, in the proof strategy above, we want to choose e and f such that $M\backslash e$, $M\backslash f$, and $M\backslash e\backslash f$ are each 3-connected and have a minor isomorphic to a stabilizer. The Splitter Theorem enables us to find one such element. How can we find both e and f? The proofs in [13] and [16] use the following result, proved in [33].

Theorem 1.5.2. *Let M be a 3-connected matroid with a 3-connected minor N such that $|E(M)| \geq |E(N)| + 4$. Then there is a pair of distinct elements $e, f \in E(M)$ such that either all of $M\backslash e$, $M\backslash f$, and $M\backslash e\backslash f$ have a minor isomorphic to N and are 3-connected up to 2-element cocircuits, or all of M/e, M/f, and $M/e/f$ have a minor isomorphic to N and are 3-connected up to 2-element circuits.*

Removing the possibility of 2-element circuits and cocircuits would simplify the proofs of these excluded-minor characterisations and strengthen the toolbox available for future attacks on excluded-minor problems. This motivates the following question: for a 3-connected matroid M with 3-connected minor N, when does there exist $\{e, f\} \subseteq E(M)$ such that either $M\backslash e\backslash f$ or $M/e/f$ is 3-connected and has a minor isomorphic to N?

1.6 Detachable Pairs

Let M be a 3-connected matroid. A pair $\{e, f\} \subseteq E(M)$ is called a *detachable pair* if either $M\backslash e\backslash f$ or $M/e/f$ is 3-connected. If N is a 3-connected minor of M, then a pair $\{e, f\} \subseteq E(M)$ is called an *N-detachable pair* if either $M\backslash e\backslash f$ or $M/e/f$ is 3-connected and has a minor isomorphic to N. Hence, the question at the end of the previous section can be rephrased as "when does M have an N-detachable pair?". An answer to this question was provided by Brettell, Whittle, and Williams [6][7][8]. They showed that, if M is sufficiently large relative to N, either M has an N-detachable pair, or a matroid with an N-detachable pair can be constructed from M using a certain operation called a Δ-Y exchange (or its inverse, a Y-Δ exchange), or M is, roughly speaking, N with a spike attached.

One of the major obstructions to a matroid having a detachable pair is the presence of 3-element circuits and 3-element cocircuits. The result of Brettell *et al.* uses the Δ-Y exchange to avoid this obstruction. This is because their result is specifically aimed at finding excluded minors for matroids representable over a field (or partial field), and if a matroid is representable over a partial field, then a Δ-Y exchange of that matroid is also representable over that partial field. However, it remains an open problem to determine the 3-connected matroids which have either a 3-element circuit or a 3-element cocircuit and an N-detachable pair. Such an analysis would combine with [6][7][8] to form a complete Splitter-type theorem in which two-element sets are deleted or contracted rather than single elements. This would be a powerful asset for finding excluded minors of classes which are not closed under Δ-Y exchange.

In Chapter 3, we consider the Wheels-and-Whirls analogue of this result. In particular, we find precisely the 3-connected matroids which have a 3-element circuit or a 3-element cocircuit and do not have a detachable pair. A Wheels-and-Whirls-type theorem is typically the first step towards a Splitter-type theorem, and it is our hope that the result in Chapter 3 can be extended to the Splitter-type theorem described above. Indeed, [6][7][8] developed from the corresponding Wheels-and-Whirls-type theorem in Williams' PhD thesis [34].

Theorem 1.6.1. *Let M be a 3-connected matroid with $|E(M)| \geq 13$. Then either*

(i) *M has a detachable pair,*

(ii) *there exists a matroid M' such that M' can be constructed by performing a single Δ-Y or Y-Δ exchange on M and M' has a detachable pair, or*

(iii) *M is a spike.*

By combining the results of Chapter 3 with Theorem 1.6.1, we obtain the following.

Theorem 1.6.2. *Let M be a 3-connected matroid with $|E(M)| \geq 13$. Then one of the following holds:*

(i) *M has a detachable pair,*

(ii) *M is a wheel or a whirl,*

(iii) *M is an accordion,*

(iv) *M is an even-fan-spike or a degenerate even-fan-spike,*

(v) *M is an even-fan-spike with tip and cotip or a degenerate even-fan-spike with tip and cotip,*

(vi) *M or M^* is a degenerate even-fan-paddle, or*

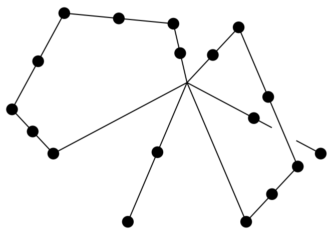
(vii) *M' has a paddle $(P_1, P_2, \ldots, P_m)$ for some $M' \in \{M, M^*\}$ and $m \geq 3$, and either*

(a) *M' is an even-fan-paddle,*
(b) $M' \cong M(K_{3,m})$,
(c) *there exists $x \in E(M)$ and $0 \leq t \leq m$ such that $M' \backslash (\{x\} \cup \bigcup_{i=1}^{t} P_i) \cong M(K_{3,m-t})$ and, for all $i \in \{1, 2, \ldots, m\}$, the set $P_i - \{x\}$ is a triad and $x \in \mathrm{cl}(P_i - \{x\})$, and for all $j \in \{1, 2, \ldots, t\}$, distinct from i, the set $P_j \cup \{x\}$ is a 4-element-fan-petal relative to $P_i - \{x\}$, or*
(d) *$M' \backslash P_1 \cong M(K_{3,m-1})$, and, for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and either*
  (I) *$M'|P_1 \cong M(K_{3,t})$ for some $t \geq 2$,*
  (II) *P_1 is an augmented-fan-petal relative to P_i,*
  (III) *P_1 is a co-augmented-fan-petal relative to P_i, or*
  (IV) *P_1 is a quad-petal relative to P_i.*

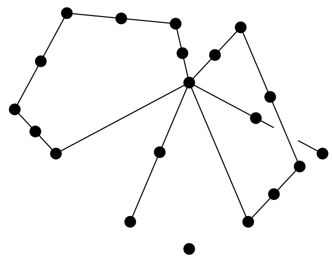
Formal definitions of the matroids in this theorem are deferred until Chapter 3, but we give a brief description here. It is not surprising that "partial wheels", called fans, are a crucial structure for these exceptional matroids. A *fan* of a matroid M is a set F with ordering $(e_1, e_2, \ldots, e_{|F|})$ such that $\{e_1, e_2, e_3\}$ is either a triangle or a triad, and, for all $i \in \{1, 2, \ldots, |F|-3\}$, if $\{e_i, e_{i+1}, e_{i+2}\}$ is a triangle, then $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a triad, and if $\{e_i, e_{i+1}, e_{i+2}\}$ is a triad, then $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a triangle. The exceptional matroids in Theorem 1.6.2 fall roughly into four categories. Firstly, matroids formed by attaching fans to a spike (Figure 1.3). Secondly, matroids formed by attaching fans to a line (Figure 1.4). Thirdly, matroids formed by attaching particular matroids to $M(K_{3,m})$, for some $m \geq 2$ (Figure 1.5). Finally, those matroids which we call accordions. The nine types of accordions are shown in Figure 1.6.

1.7 Other Generalisations

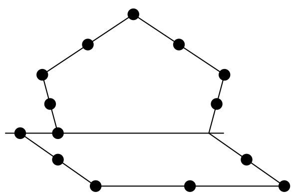
We discuss other generalisations of Tutte's Wheels-and-Whirls Theorem. An element e in a 3-connected matroid M is said to be *non-essential* if either $M \backslash e$ or M/e is 3-connected. The Wheels-and-Whirls Theorem states that wheels and whirls are precisely the 3-connected matroids with no non-essential elements. What are the 3-connected matroids with exactly k non-essential elements? Oxley and Wu [26][27] showed that there are no matroids with exactly one non-essential element, and found the two families of matroids with exactly two non-essential elements, calling them twisted wheels and multi-dimensional wheels. These families appear in Chapter 3 — we refer to a twisted wheel as a degenerate even-fan-spike with tip and cotip (Figure 1.3d), while a single-element deletion of a multi-dimensional wheel produces an even-fan-paddle (Figure 1.4a).



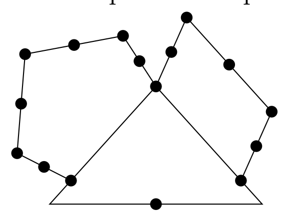
(a) An even-fan-spike.



(b) An even-fan-spike with tip and cotip.

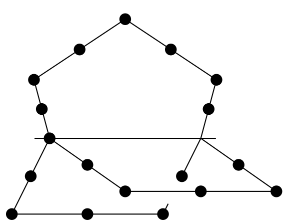


(c) A degenerate even-fan-spike.

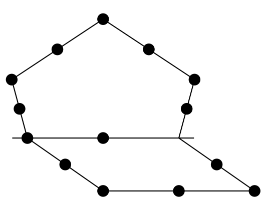


(d) A degenerate even-fan-spike with tip and cotip.

Figure 1.3: Matroids with no detachable pairs formed by attaching fans to spikes.

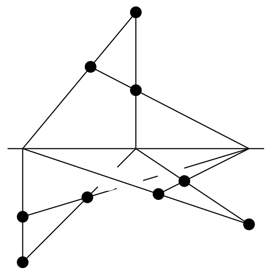


(a) An even-fan-paddle.

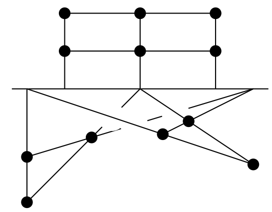


(b) A degenerate even-fan-paddle.

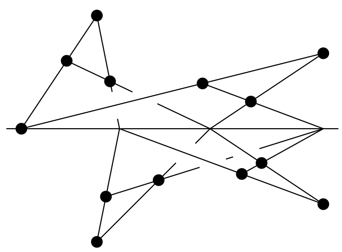
Figure 1.4: Matroids with no detachable pairs formed by attaching fans to lines.



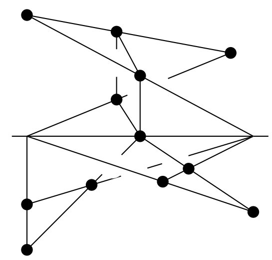
(a) $M(K_{3,3})$.



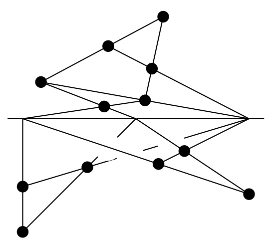
(b) $M(K_{3,2})$ with a $M^*(K_{3,2})$ petal.



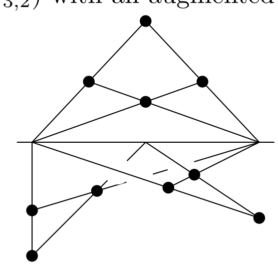
(c) $M(K_{3,2})$ with two 4-element-fan-petals.



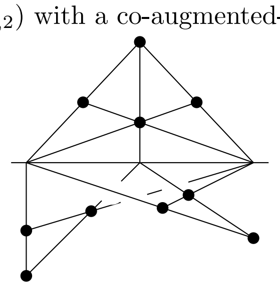
(d) $M(K_{3,2})$ with an augmented-fan-petal.



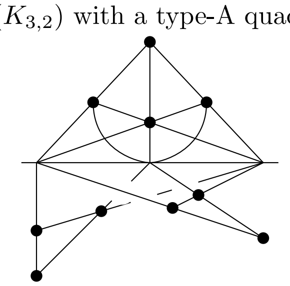
(e) $M(K_{3,2})$ with a co-augmented-fan-petal.



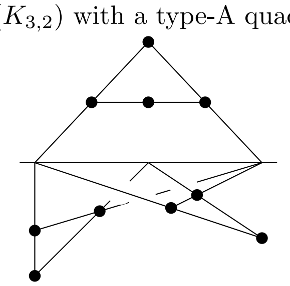
(f) $M(K_{3,2})$ with a type-A quad-petal.



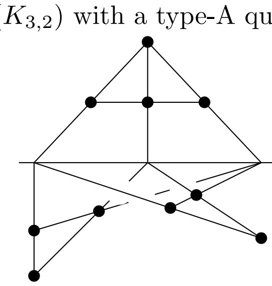
(g) $M(K_{3,2})$ with a type-A quad-petal.



(h) $M(K_{3,2})$ with a type-A quad-petal.

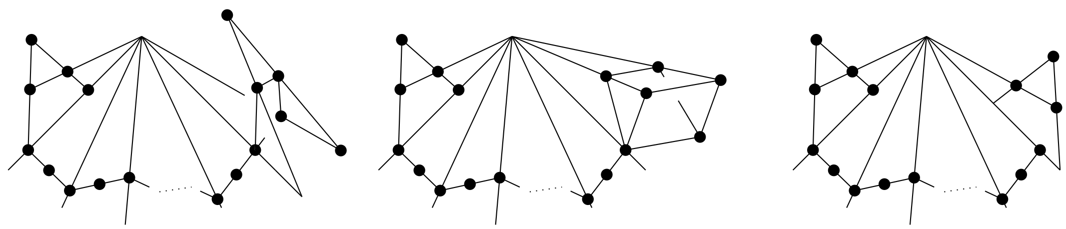


(i) $M(K_{3,2})$ with a type-B quad-petal.



(j) $M(K_{3,2})$ with a type-B quad-petal.

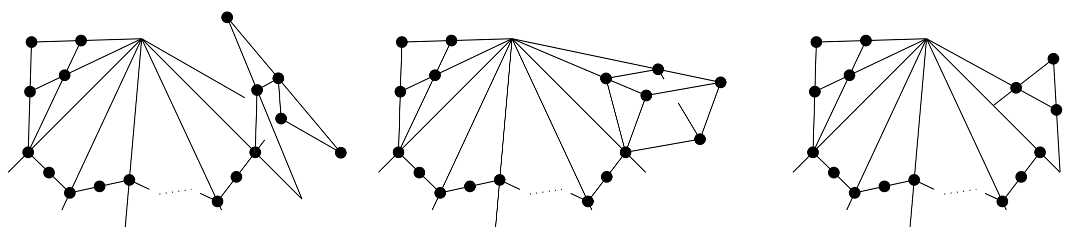
Figure 1.5: Matroids with no detachable pairs formed by attaching a matroid to $M(K_{3,m})$.



(a) An accordion with a fan-type and a co-fan-type accordion end.

(b) An accordion with a fan-type and a co-K_4-type accordion end.

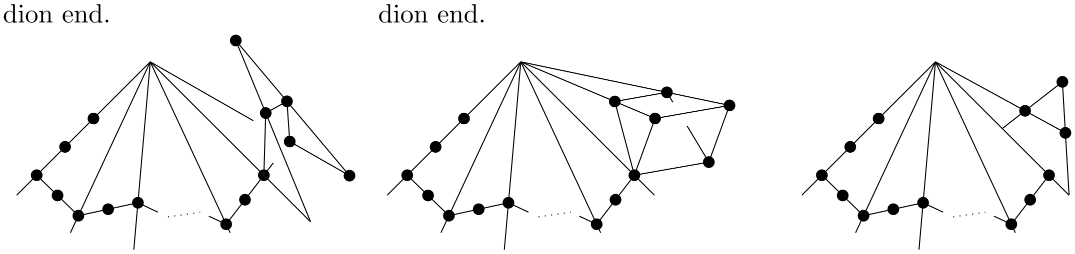
(c) An accordion with a fan-type and a triad-type accordion end.



(d) An accordion with a K_4-type and a co-fan-type accordion end.

(e) An accordion with a K_4-type and a co-K_4-type accordion end.

(f) An accordion with a K_4-type and a triad-type accordion end.



(g) An accordion with a triangle-type and a co-fan-type accordion end.

(h) An accordion with a triangle-type and a co-K_4-type accordion end.

(i) An accordion with a triangle-type and a triad-type accordion end.

Figure 1.6: The nine types of accordion.

A Wheels-and-Whirls-type theorem has also been considered for the class of matroids which are 3-connected and have no 3-element circuits. Lemos [17][18] described seven reduction operations (regular deletion and contraction are two of these seven operations), and found the 3-connected matroids with no 3-element circuits which cannot be reduced to a smaller 3-connected matroid with no 3-element circuits by applying one of these seven operations. Dos Santos [11] considered those matroids M which are 3-connected and have no 3-element circuits such that every minor N of M with $|E(N)| \geq |E(M)| - 2$ is either not 3-connected or has a 3-element circuit.

1.8 Outline

Recall the two motivating questions from the start of this chapter. The first of these questions, to find a generalisation of Tutte's Wheels-and-Whirls Theorem for connectivity higher than three, motivates the simpler problem of analysing matroids in which every element is contained in a small circuit and a small cocircuit. Chapter 2 contains results concerning this problem in the case where the circuits and cocircuits are cyclically arranged. The second question is to find a generalisation of Tutte's Wheels-and-Whirls Theorem in which two-element sets are removed, rather than single elements. In Chapter 3, we find such a generalisation.

The research throughout this thesis is original, and the content of Chapter 2 has been accepted for publication by SIAM Journal on Discrete Mathematics. Relevant preliminaries will be introduced at the start of each chapter, however there is some notation in common which we mention now. We say two sets X and Y *intersect* if $X \cap Y$ is non-empty; otherwise, X and Y *do not intersect*. For a positive integer m, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. The following well-known lemma will be used frequently throughout. When applying this lemma, we use the phrase *by orthogonality*.

Lemma 1.8.1. *Let M be a matroid. If C is a circuit of M and C^* is a cocircuit of M, then $|C \cap C^*| \neq 1$.*

Unless stated otherwise, notation and terminology follows [21].

chapter 2

Cyclic Matroids

2.1 Introduction

In this chapter, we build on work in [4] by considering matroids with a cyclic arrangement of circuits and cocircuits. In particular, we prove Theorems 1.4.1, 1.4.3, and 1.4.2.

We recall the definitions of nearly (s,t)-cyclic and (s,t)-cyclic matroids from the introduction. Let s and t be positive integers exceeding one. A matroid M is *nearly* (s,t)*-cyclic* if there exists a cyclic ordering σ of $E(M)$ such that every set of $s-1$ consecutive elements of σ is contained in an s-element circuit and every set of $t-1$ consecutive elements of σ is contained in a t-element cocircuit, in which case we say that σ is a *nearly* (s,t)*-cyclic ordering* of $E(M)$. Although not explicitly stated, there is an implicit assumption that if M is nearly (s,t)-cyclic, then M has at least $\max\{s,t\}-1$ elements, so it has at least one s-element circuit and at least one t-element cocircuit.

A matroid M is (s,t)*-cyclic* if there exists a cyclic ordering $\sigma=(e_1,e_2,\dots,e_n)$ of $E(M)$ such that each of the following holds, where subscripts are interpreted modulo n:

(i) either $\{e_1,e_2,\dots,e_s\}$ or $\{e_2,e_3,\dots,e_{s+1}\}$ is an s-element circuit of M,

(ii) either $\{e_1,e_2,\dots,e_t\}$ or $\{e_2,e_3,\dots,e_{t+1}\}$ is a t-element cocircuit of M,

(iii) if $\{e_i,e_{i+1},\dots,e_{i+s-1}\}$ is an s-element circuit of M for some $i\in\{1,2,\dots,n\}$, then $\{e_{i+2},e_{i+3},\dots,e_{i+s+1}\}$ is also an s-element circuit of M, and

(iv) if $\{e_i,e_{i+1},\dots,e_{i+t-1}\}$ is a t-element cocircuit of M for some $i\in\{1,2,\dots,n\}$, then $\{e_{i+2},e_{i+3},\dots,e_{i+t+1}\}$ is also a t-element cocircuit of M.

A cyclic ordering satisfying (i)–(iv) is called an (s,t)-*cyclic ordering* of $E(M)$.

If M is nearly $(2,2)$-cyclic, then, as noted in [4], M is obtained by taking direct sums of copies of $U_{1,2}$, and so M is $(2,2)$-cyclic. Brettell *et al.* [4, Theorem 1.1] showed that, for all $s \geq 3$, if σ is a nearly (s,s)-cyclic ordering of a matroid M on n elements and $n \geq 6s-10$, then σ is an (s,s)-cyclic ordering of M. The first main result, Theorem 1.4.1, of this chapter generalises that theorem.

Theorem 1.4.1. *Let M be a matroid on n elements, and suppose that σ is a nearly (s,t)-cyclic ordering of M, where $s,t \geq 3$. Let $t_1 = \min\{s,t\}$ and $t_2 = \max\{s,t\}$. If $n \geq 3t_1 + t_2 - 5$ and $n \geq t_1 + 2t_2 - 1$, then σ is an (s,t)-cyclic ordering of M.*

The proof of Theorem 1.4.1 takes a different approach to that used in [4]. Equating s and t in Theorem 1.4.1, we have the following corollary, improving the lower bound in [4, Theorem 1.1].

Corollary 2.1.1. *Let M be a matroid on n elements, and suppose that σ is a nearly (s,s)-cyclic ordering of M for $s \geq 3$. If $n \geq \max\{8, 4s-5\}$, then σ is an (s,s)-cyclic ordering of M.*

For all positive integers s and t exceeding one, we will show that if a matroid on n elements is nearly (s,t)-cyclic, then $n \geq s+t-2$. Observe that, for all such s and t, the uniform matroid $U_{s-1,s+t-2}$ is nearly (s,t)-cyclic with $s+t-2$ elements. Thus this lower bound is sharp. Furthermore, if a matroid on n elements is (s,t)-cyclic and $n > s+t-2$, then we will also show that n is even and $s \equiv t \bmod 2$. Hence, if a matroid M is (s,t)-cyclic and $s \not\equiv t \bmod 2$, then M has exactly $s+t-2$ elements. Lastly, we suspect the inequalities $n \geq 3t_1 + t_2 - 5$ and $n \geq t_1 + 2t_2 - 1$ in Theorem 1.4.1 are not tight, and leave it as an open problem to determine, for all positive integers $s,t \geq 2$, tight lower bounds on the size of the ground set of a matroid M having the property that if σ is a nearly (s,t)-cyclic ordering of $E(M)$, then σ is an (s,t)-cyclic ordering of $E(M)$.

The second main result of this chapter, Theorem 1.4.3, shows that, given positive integers s and t exceeding one, such that $t \geq s$, an (s,t)-cyclic matroid M on n elements, where $n > s+t-2$, is a weak-map image of the $\left(\frac{t-s}{2}\right)$-th truncation of a certain (s,s)-cyclic matroid, which we define now.

For vertices u and v of a graph, u is a *neighbour* of v if u is adjacent to v, and we let $N(v)$ denote the set of neighbours of v. Note that here, as well as elsewhere in the chapter, we adopt the convention of writing singletons without set braces provided there is no ambiguity.

Now let s be an integer exceeding one and let n be a positive even integer. Let G_s^n be the bipartite graph with vertex parts $E = \{e_1, e_2, \ldots, e_n\}$ and $\{1, 2, \ldots, \frac{n}{2}\}$ such that, for all

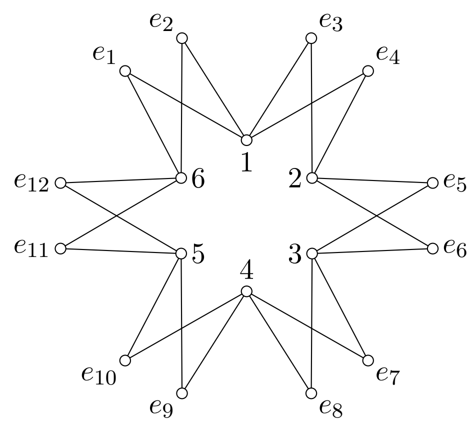


Figure 2.1: The bipartite graph G_4^{12}.

$i \in \{1, 2, \ldots, \frac{n}{2}\}$, the set of neighbours of i is

$$N(i) = \{e_{2i-1}, e_{2i}, \ldots, e_{2i+s-2}\},$$

where subscripts are interpreted modulo n. For example, if $n = 12$ and $s = 4$, then G_4^{12} is the bipartite graph shown in Figure 2.1. The transversal matroid on E in which

$$(N(1), N(2), \ldots, N(\tfrac{n}{2}))$$

is a presentation is an example of a multi-path matroid [2]. Denote the dual of this transversal matroid by Ψ_s^n. Multi-path matroids have the property that their duals are transversal [2, Theorem 3.8], so Ψ_s^n is a transversal matroid. In fact, we shall show that Ψ_s^n is a self-dual matroid. If $s = 2$, then Ψ_s^n is isomorphic to the rank-$\frac{n}{2}$ matroid obtained by taking direct sums of copies of $U_{1,2}$; while if $s = 3$ or $s = 4$, then Ψ_s^n is isomorphic to the rank-$\frac{n}{2}$ whirl or rank-$\frac{n}{2}$ free swirl, respectively. For example, the dual of the transversal matroid realised by G_4^{12} is the rank-6 free swirl. More generally, it turns out that, for all $s \geq 2$, the matroid Ψ_s^n is (s, s)-cyclic.

Let M be a matroid. If $r(M) > 0$, then the matroid obtained from M by freely adding an element f and then contracting f is called the *truncation* of M and is denoted by $T(M)$. If $r(M) = 0$, we set $T(M) = M$. For all positive integers i, the *i-th truncation* of M, denoted $T^i(M)$, is defined iteratively as $T^i(M) = T(T^{i-1}(M))$, where $T^0(M) = M$. The second main result of this paper is the following theorem.

Theorem 1.4.3. *Let M be an (s, t)-cyclic matroid such that $|E(M)| \geq s + t - 1$ and $t \geq s$. Then M is a weak-map image of the $(\frac{t-s}{2})$-th truncation of Ψ_s^n, which is an (s, t)-cyclic matroid.*

The chapter is organised as follows. The next section contains some preliminaries, while Section 2.3 establishes some basic properties of cyclic matroids. These properties are used in the proofs of Theorems 1.4.1 and 1.4.3 which are given in Sections 2.4 and 2.5, respectively. The proof of Theorem 1.4.3 follows from a more general result concerning the duals of multi-path matroids. Lastly, in Section 2.6, we prove Theorem 1.4.2, a counterexample to a conjecture concerning (s, s)-cyclic matroids, given in [4]. This conjecture says that if s is an integer exceeding two and M is an (s, s)-cyclic matroid, then M can be obtained from either a wheel or a whirl (if s is odd), or either a spike or a swirl (if s is even) by a sequence of elementary quotients and elementary lifts.

2.2 Preliminaries

For a positive integer m, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. Furthermore, for $i, j \in [m]$, we let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$ if $i \leq j$ and the set $\{i, i+1, \ldots, m, 1, 2, \ldots, j\}$ if $i > j$. Now let $\sigma = (e_1, e_2, \ldots, e_n)$ be a cyclic ordering of $\{e_1, e_2, \ldots, e_n\}$. For all $i, j \in [n]$, the notation $\sigma(i, j)$ denotes the set of elements $\{e_i, e_{i+1}, \ldots, e_j\}$, where subscripts are interpreted modulo n.

The next lemma concerns the independent sets of the i-th truncation of a matroid (see, for example, [21, Proposition 7.3.10]).

Lemma 2.2.1. *Let M be a matroid with $r(M) \geq 1$, and let i be a non-negative integer such that $i \leq r(M)$. Then*

$$\mathcal{I}(T^i(M)) = \{X \in \mathcal{I}(M) : |X| \leq r(M) - i\}.$$

2.3 Properties of Cyclic Matroids

In this section, we establish various properties of nearly (s, t)-cyclic and (s, t)-cyclic matroids on n elements. The first lemma is used frequently in this section.

Lemma 2.3.1. *Let M be an (s, t)-cyclic matroid on n elements, where $n > s + t - 2$, and let $\sigma = (e_1, e_2, \ldots, e_n)$ be an (s, t)-cyclic ordering of M. Then,*

(i) *if $\sigma(i, i+s-1)$ is a circuit, then $\sigma(i-t, i-1)$ and $\sigma(i+s, i+s+t-1)$ are cocircuits, and*

(ii) *if* $\sigma(i, i+t-1)$ *is a cocircuit, then* $\sigma(i-s, i-1)$ *and* $\sigma(i+t, i+s+t-1)$ *are circuits.*

Proof. We will prove (i). The proof of (ii) follows by duality as M^* is a (t,s)-cyclic matroid. Since σ is an (s,t)-cyclic ordering of M, it follows that one of $\sigma(i-t, i-1)$ and $\sigma(i-t+1, i)$ is a t-element cocircuit of M. But, as $n > s+t-2$, the set $\sigma(i-t+1, i)$ intersects the circuit $\sigma(i, i+s-1)$ in one element, and so $\sigma(i-t+1, i)$ is not a cocircuit. Therefore $\sigma(i-t, i-1)$ is a cocircuit of M. Similarly, $\sigma(i+s-1, i+s+t-2)$ is not a cocircuit as it intersects $\sigma(i, i+s-1)$ in one element, and so $\sigma(i+s, i+s+t-1)$ is a cocircuit. □

The next two lemmas consider the relationships amongst s, t, and n.

Lemma 2.3.2. *Let* M *be a nearly* (s,t)*-cyclic matroid on* n *elements. Then* $n \geq s+t-2$.

Proof. Since M contains an s-element circuit, we have that $r(M) \geq s-1$. Similarly, as M contains a t-element cocircuit, $r^*(M) \geq t-1$. Therefore, as $n = r(M) + r^*(M)$, we also have that $n \geq s+t-2$. □

Note that the bound in Lemma 2.3.2 is tight. In particular, for any positive integers $s, t \geq 2$, the uniform matroid $U_{s-1,s+t-2}$ is nearly (s,t)-cyclic. In fact, $U_{s-1,s+t-2}$ is (s,t)-cyclic.

Lemma 2.3.3. *Let* M *be an* (s,t)*-cyclic matroid on* n *elements. If* $n > s+t-2$*, then*

(i) n *is even, and*

(ii) $s \equiv t \bmod 2$.

Proof. Suppose $n > s+t-2$. To prove (i), assume that n is odd. Let $\sigma = (e_1, e_2, \ldots, e_n)$ be an (s,t)-cyclic ordering of M, and let $\sigma(i, i+s-1)$ be a circuit of M. Then, for all even k, the set $\sigma(i+k, i+s-1+k)$ is a circuit of M. In particular, taking $k = n-1$, the set $\sigma(i-1, i+s-2)$ is a circuit of M. But, by Lemma 2.3.1, the set $\sigma(i-t, i-1)$ is a cocircuit of M, and, since $n > s+t-2$, this cocircuit intersects the circuit $\sigma(i-1, i+s-2)$ in one element. This contradiction implies n is even.

For the proof of (ii), assume that $s \not\equiv t \bmod 2$. Let $\sigma = (e_1, e_2, \ldots, e_n)$ be an (s,t)-cyclic ordering of M, and let $\sigma(i, i+s-1)$ be a circuit of M. By Lemma 2.3.1, the set $\sigma(i-t, i-1)$ is a cocircuit of M. By the assumption, $s+t-1$ is even and so, as $(i-t)+(s+t-1) = i+s-1$, the set $\sigma(i+s-1, i+s+t-2)$ is a cocircuit. But this cocircuit intersects $\sigma(i, i+s-1)$ in precisely one element, contradicting orthogonality. Therefore, $s \equiv t \bmod 2$, completing the proof of (ii). □

The bound in Lemma 2.3.3 is tight. For example, choosing one of s and t to be even and the other to be odd, the uniform matroid $U_{s-1,s+t-2}$ is an (s,t)-cyclic matroid on $s+t-2$ elements. However, Lemma 2.3.3 shows that there is no (s,t)-cyclic matroid with more elements.

Generalising [4, Lemma 4.3, Lemma 5.1, Lemma 5.3], the next four lemmas concern the independent sets, closure operator, and rank function of (s,t)-cyclic matroids. A consequence of the first of these lemmas is that if $s=t$ and s is even, then the s-element circuits and s-element cocircuits in an (s,s)-cyclic ordering of a matroid coincide. On the other hand, if $s=t$ and s is odd, then the s-element circuits and s-element cocircuits in an (s,s)-cyclic ordering of a matroid behave like the 3-element circuits and 3-element cocircuits in $(3,3)$-cyclic orderings of wheels and whirls.

Lemma 2.3.4. *Let M be an (s,t)-cyclic matroid on n elements, where $n>s+t-2$, and let $\sigma=(e_1,e_2,\ldots,e_n)$ be an (s,t)-cyclic ordering of M. Suppose that $\sigma(i,i+s-1)$ is a circuit of M. If s and t are even, then*

(i) $\sigma(i,i+t-1)$ *is a cocircuit,*

(ii) $\sigma(i+1,i+s)$ *is independent, and*

(iii) $\sigma(i+1,i+t)$ *is coindependent.*

Furthermore, if s and t are odd, then

(iv) $\sigma(i+1,i+t)$ *is a cocircuit,*

(v) $\sigma(i+1,i+s)$ *is independent, and*

(vi) $\sigma(i,i+t-1)$ *is coindependent.*

Proof. By Lemma 2.3.1, the set $\sigma(i-t,i-1)$ is a cocircuit of M. If t is even, this implies $\sigma(i,i+t-1)$ is a cocircuit; otherwise, t is odd and $\sigma(i+1,i+t)$ is a cocircuit.

We next show that $\sigma(i+1,i+s)$ is independent. Suppose this is not the case. Then $\sigma(i+1,i+s)$ contains a circuit, call it C. By Lemma 2.3.1, the set $\sigma(i+s,i+s+t-1)$ is a cocircuit of M. Therefore, if $e_{i+s}\in C$, then C intersects $\sigma(i+s,i+s+t-1)$ in exactly one element, a contradiction. But if $e_{i+s}\notin C$, then C is properly contained in the circuit $\sigma(i,i+s-1)$, another contradiction. Thus, no such circuit C exists, and so $\sigma(i+1,i+s)$ is independent. We have shown that, if $\sigma(i,i+s-1)$ is a circuit, then $\sigma(i+1,i+s)$ is independent. Since M^* is a (t,s)-cyclic matroid, this implies that if $\sigma(j,j+t-1)$ is a cocircuit, then $\sigma(j+1,j+t)$ is coindependent. This is sufficient to show (iii) and (vi) and complete the proof. $\square$

Lemma 2.3.5. *Let M be an (s,t)-cyclic matroid on n elements, where $n > s+t-2$, and let $\sigma = (e_1, e_2, \dots, e_n)$ be an (s,t)-cyclic ordering of M. Then, for all $i \in [n]$ and $s-1 \le k \le n-t$,*

(i) $e_{i+k} \in \mathrm{cl}(\sigma(i, i+k-1))$ *if and only if* $\sigma(i+k-s+1, i+k)$ *is a circuit, and*

(ii) $e_{i-1} \in \mathrm{cl}(\sigma(i, i+k-1))$ *if and only if* $\sigma(i-1, i+s-2)$ *is a circuit.*

Proof. We will prove (i). Then (ii) follows from the fact that reversing the order of σ gives another (s,t)-cyclic ordering of M. Since $k \ge s-1$, if $\sigma(i+k-s+1, i+k)$ is a circuit, then $e_{i+k} \in \mathrm{cl}(\sigma(i, i+k-1))$. Conversely, suppose $e_{i+k} \in \mathrm{cl}(\sigma(i, i+k-1))$. Then there exists a circuit C contained in $\sigma(i, i+k)$ such that C contains e_{i+k}. Assume $\sigma(i+k-s+1, i+k)$ is not a circuit. If s and t are even, then, by Lemma 2.3.4, the set $\sigma(i+k-s, i+k-s+t-1)$ is a cocircuit and so, as s is even, the set $\sigma(i+k, i+k+t-1)$ is also a cocircuit. Since $k \le n-t$, this last cocircuit intersects C only in the element e_{i+k}, a contradiction. Therefore, $\sigma(i+k-s+1, i+k)$ is a circuit. Similarly, if s and t are odd, then, by Lemma 2.3.4, the set $\sigma(i+k-s+1, i+k-s+t)$ is a cocircuit, which means $\sigma(i+k, i+k+t-1)$ is also a cocircuit. Again, this contradicts orthogonality with C, showing that $\sigma(i+k-s+1, i+k)$ is a circuit, and completing the proof of the lemma. $\square$

Lemma 2.3.6. *Let M be an (s,t)-cyclic matroid on n elements, where $n > s+t-2$, and let $\sigma = (e_1, e_2, \dots, e_n)$ be an (s,t)-cyclic ordering of M. Then, for all $i \in [n]$ and $1 \le k \le n-t+1$,*

$$r(\sigma(i, i+k-1)) = \begin{cases} k, & \text{if } k < s; \\ \left\lfloor \frac{s+k-1}{2} \right\rfloor, & \text{if } k \ge s \text{ and } \sigma(i, i+s-1) \text{ is a circuit}; \\ \left\lceil \frac{s+k-1}{2} \right\rceil, & \text{if } k \ge s \text{ and } \sigma(i, i+s-1) \text{ is not a circuit.} \end{cases}$$

Proof. The proof is by induction on k. If $k < s$, then $\sigma(i, i+k-1)$ is a proper subset of an s-element circuit, so it is independent. Therefore, $r(\sigma(i, i+k-1)) = k$. Now suppose $k = s$. If $\sigma(i, i+s-1)$ is a circuit, then

$$r(\sigma(i, i+s-1)) = s-1 = \left\lfloor \tfrac{s+s-1}{2} \right\rfloor,$$

while if $\sigma(i, i+s-1)$ is not a circuit, then, by Lemma 2.3.5,

$$r(\sigma(i, i+s-1)) = s = \left\lceil \tfrac{s+s-1}{2} \right\rceil.$$

Thus the lemma holds for all $1 \le k \le s$.

Now suppose that $s+1 \le k \le n-t+1$, and the lemma holds for the set $\sigma(i, i+k-2)$. Consider $\sigma(i, i+k-1)$. First assume that $\sigma(i, i+s-1)$ is a circuit. If $s+k$ is odd, then

$k-s$ is odd, and it follows by Lemma 2.3.4(ii) and (v) that $\sigma(i+k-s, i+k-1)$ is not a circuit. Therefore, by Lemma 2.3.5, $e_{i+k-1} \notin \mathrm{cl}(\sigma(i, i+k-2))$, and so, by the induction assumption,

$$\begin{aligned} r(\sigma(i, i+k-1)) &= r(\sigma(i, i+k-2)) + 1 \\ &= \left\lfloor \tfrac{s+k-2}{2} \right\rfloor + 1 = \left\lfloor \tfrac{s+k}{2} \right\rfloor = \left\lfloor \tfrac{s+k-1}{2} \right\rfloor \end{aligned}$$

as $s+k$ is odd. If $s+k$ is even, then $\sigma(i+k-s, i+k-1)$ is a circuit, and so $e_{i+k-1} \in \mathrm{cl}(\sigma(i, i+k-2))$. Therefore

$$\begin{aligned} r(\sigma(i, i+k-1)) &= r(\sigma(i, i+k-2)) \\ &= \left\lfloor \tfrac{s+k-2}{2} \right\rfloor = \left\lfloor \tfrac{s+k-1}{2} \right\rfloor \end{aligned}$$

as $s+k$ is even.

Now assume that $\sigma(i, i+s-1)$ is not a circuit. If $s+k$ is odd, then $\sigma(i+k-s, i+k-1)$ is a circuit, and so, by the induction assumption and Lemma 2.3.5,

$$\begin{aligned} r(\sigma(i, i+k-1)) &= r(\sigma(i, i+k-2)) \\ &= \left\lceil \tfrac{s+k-2}{2} \right\rceil = \left\lceil \tfrac{s+k-1}{2} \right\rceil \end{aligned}$$

as $s+k$ is odd. If $s+k$ is even, then $\sigma(i+k-s, i+k-1)$ is not a circuit, and so, by Lemma 2.3.5 and the induction assumption,

$$\begin{aligned} r(\sigma(i, i+k-1)) &= r(\sigma(i, i+k-2)) + 1 \\ &= \left\lceil \tfrac{s+k-2}{2} \right\rceil + 1 = \left\lceil \tfrac{s+k}{2} \right\rceil = \left\lceil \tfrac{s+k-1}{2} \right\rceil \end{aligned}$$

as $s+k$ is even. This completes the proof of the lemma. □

The next lemma shows that the rank of an (s,t)-cyclic matroid on n elements is invariant under s, t, and n.

Lemma 2.3.7. *Let M be an (s,t)-cyclic matroid on n elements. Then $r(M) = \frac{n+s-t}{2}$ and $r^*(M) = \frac{n-s+t}{2}$.*

Proof. By Lemma 2.3.2, the matroid M has at least $s+t-2$ elements. Since M has an s-element circuit and a t-element cocircuit, $r(M) \geq s-1$ and $r^*(M) \geq t-1$. Therefore, if $n = s+t-2$, then

$$r(M) = s-1 = \tfrac{(s+t-2)+s-t}{2}$$

and

$$r^*(M) = t-1 = \tfrac{(s+t-2)-s+t}{2}.$$

Otherwise, by Lemma 2.3.6, the set $\{e_1, e_2, \ldots, e_{n-t+1}\}$ either has rank $\lfloor \frac{n+s-t}{2} \rfloor$ or rank $\lceil \frac{n+s-t}{2} \rceil$. By Lemma 2.3.3, we have that $n+s-t$ is even, so

$$r(\sigma(1, n-t+1)) = \frac{n+s-t}{2}.$$

Therefore, $r(M) \geq \frac{n+s-t}{2}$. Similarly, by Lemmas 2.3.3 and 2.3.6, we get that

$$r^*(\sigma(1, n-s+1)) = \frac{n-s+t}{2},$$

and so $r^*(M) \geq \frac{n-s+t}{2}$. Since $\frac{n+s-t}{2} + \frac{n-s+t}{2} = n$, it follows that $r(M) = \frac{n+s-t}{2}$ and $r^*(M) = \frac{n-s+t}{2}$. □

The last lemma in this section will be used to prove Theorem 1.4.1 in the next section; we include it here as it may be of independent interest.

Lemma 2.3.8. *Let s and t be positive integers exceeding one, and let $\sigma = (e_1, e_2, \ldots, e_n)$ be a nearly (s,t)-cyclic ordering of a matroid M, where $n \geq s+t$. If $\sigma(i, i+s-1)$ is a circuit for all odd $i \in [n]$, then σ is an (s,t)-cyclic ordering of M.*

Proof. It is sufficient to prove that, for all odd $i \in [n]$, the set $\sigma(i-t+2, i+1)$ is a cocircuit. Consider the set $\sigma(i-t+2, i)$. This set contains $t-1$ consecutive elements of σ, so must be contained in a t-element cocircuit C^*. Let e_j be the unique element of C^* not contained in $\sigma(i-t+2, i)$. If $e_j \notin \sigma(i+1, i+s-1)$, then C^* intersects the circuit $\sigma(i, i+s-1)$ in exactly one element, contradicting orthogonality. Furthermore, if $e_j \in \sigma(i+2, i+s-1)$, then, as $n \geq s+t$, the cocircuit C^* intersects the circuit $\sigma(i+2, i+s+1)$ in exactly one element. This last contradiction implies that $e_j = e_{i+1}$, completing the proof of the lemma. □

2.4 Proof of Theorem 1.4.1

This section consists of the proof of Theorem 1.4.1. Throughout the section, let M be a nearly (s,t)-cyclic matroid, where $s, t \geq 3$, and let $\sigma = (e_1, e_2, \ldots, e_n)$ be a nearly (s,t)-cyclic ordering of M. We shall prove that, provided n is sufficiently large, σ is an (s,t)-cyclic ordering of M.

For all $i \in [n]$, let C_i be an arbitrarily chosen circuit of size s containing $\sigma(i, i+s-2)$ and let C_i^* be an arbitrarily chosen cocircuit of size t containing $\sigma(i, i+t-2)$. There is a unique element of C_i not contained in $\sigma(i, i+s-2)$; call this element c_i. Likewise, let c_i^* be the unique element of C_i^* not contained in $\sigma(i, i+t-2)$.

Lemma 2.4.1. *If $n \geq s + 2t - 4$, then $c_i \neq c_{i+1}$ for all $i \in [n]$.*

Proof. Suppose $n \geq s+2t-4$ and $c_i = c_{i+1}$ for some $i \in [n]$. Then $C_i = \sigma(i, i+s-2) \cup \{c_i\}$ and $C_{i+1} = \sigma(i+1, i+s-1) \cup \{c_i\}$. By circuit elimination, there is a circuit, say C, of M contained in $\sigma(i, i+s-1)$. If C does not contain e_i, then C is properly contained in the circuit C_{i+1}, a contradiction. Similarly, if C does not contain e_{i+s-1}, then C is properly contained in C_i. Therefore, C contains both e_i and e_{i+s-1}.

Since $t \geq 3$, we have that $n \geq s + t - 2$. Therefore,

$$\sigma(i+s-1, i+s+t-3) \cap C = \{e_{i+s-1}\}$$

Since $\sigma(i+s-1, i+s+t-3) \cup \{c^*_{i+s-1}\}$ is a cocircuit, orthogonality implies that $c^*_{i+s-1} \in C - \{e_{i+s-1}\} \subseteq \sigma(i, i+s-2)$. This means that C^*_{i+s-1} and $\sigma(i, i+s-2)$ also intersect in exactly one element. Applying orthogonality again, we have that $c_i \in \sigma(i+s-1, i+s+t-3)$.

Similarly, the $(t-1)$-element set $\sigma(i-t+2, i)$ intersects C in only the element e_i. Therefore, orthogonality between C^*_{i-t+2} and C implies that $c^*_{i-t+2} \in C - \{e_i\} \subseteq \sigma(i+1, i+s-1)$. Applying orthogonality again, this time between C^*_{i-t+2} and C_{i+1}, shows that $c_{i+1} \in \sigma(i-t+2, i)$. But $c_i = c_{i+1}$, and so c_i is contained in both $\sigma(i-t+2, i)$ and $\sigma(i+s-1, i+s+t-3)$, two sets which are disjoint since $n \geq s + 2t - 4$. This contradiction implies that $c_i \neq c_{i+1}$ and completes the proof. □

The next lemma is used several times in the proof of Lemma 2.4.3.

Lemma 2.4.2. *Suppose there exists $d_i \neq c_i$ such that $D_i = \sigma(i, i+s-2) \cup \{d_i\}$ is a circuit of M. Let $j \in [n]$ such that $|\sigma(j, j+t-2) \cap \{c_i, d_i\}| = 1$. Then $\sigma(j, j+t-2)$ intersects $\sigma(i, i+s-2)$.*

Proof. Without loss of generality, we may assume that $c_i \in \sigma(j, j+t-2)$ and $d_i \notin \sigma(j, j+t-2)$. Suppose $\sigma(j, j+t-2)$ does not intersect $\sigma(i, i+s-2)$. Then $\sigma(j, j+t-2)$ intersects C_i in one element. Therefore, by orthogonality, $c^*_j \in \sigma(i, i+s-2)$. But now $c^*_j \in D_i$, so C^*_j and D_i intersect in one element. This contradiction to orthogonality implies that $\sigma(j, j+t-2)$ intersects $\sigma(i, i+s-2)$, and completes the proof. □

Lemma 2.4.3. *If $n \geq s + 2t - 4$, then, for all $i \in [n]$, there is a unique circuit of size s containing $\sigma(i, i+s-2)$.*

Proof. We know C_i is an s-element circuit containing $\sigma(i, i+s-2)$. Suppose that there is a second such circuit. This means that there is an element d_i, distinct from c_i, such that $\sigma(i, i+s-2) \cup \{d_i\}$ is a circuit. Call this circuit D_i.

Now, for some $j \in [n]$, we have $c_i = e_j$. Consider the $(t-1)$-element subsets $\sigma(j-t+2, j)$ and $\sigma(j, j+t-2)$. Since $c_i \neq d_i$, at least one of these sets does not contain d_i. Up to symmetry, we may assume that $d_i \notin \sigma(j-t+2, j)$. Now, $|\sigma(j-t+2, j) \cap \{c_i, d_i\}| = 1$ and so, by Lemma 2.4.2, the set $\sigma(j-t+2, j)$ intersects $\sigma(i, i+s-2)$. Since $n \geq s+2t-5$, this implies that $\sigma(j, j+t-2)$ does not intersect $\sigma(i, i+s-2)$. Applying Lemma 2.4.2 again, we see that $|\sigma(j, j+t-2) \cap \{c_i, d_i\}| \neq 1$, so $d_i \in \sigma(j, j+t-2)$. Therefore, $\sigma(j+1, j+t-1)$ contains d_i but does not contain c_i. However, since $n \geq s+2t-4$ and $\sigma(j-t+2, j)$ intersects $\sigma(i, i+s-2)$, we also have that $\sigma(j+1, j+t-1)$ is disjoint from $\sigma(i, i+s-2)$. This contradiction to Lemma 2.4.2 shows that no such d_i exists, thereby completing the proof. □

Lemma 2.4.4. *Let $i, j \in [n]$ such that $c_i \in \sigma(j+1, j+t-2)$, and suppose that $n \geq 2s+t-4$. Then each of the following holds:*

(i) *If $\sigma(j, j+t-1)$ does not intersect $\sigma(i, i+s-1)$, then $c_{i+1} \in \sigma(j, j+t-1)$.*

(ii) *If $\sigma(j, j+t-1)$ does not intersect $\sigma(i-1, i+s-2)$, then $c_{i-1} \in \sigma(j, j+t-1)$.*

Proof. We prove (i). Then (ii) follows by reversing the order of σ. Suppose that $\sigma(j, j+t-1)$ does not intersect $\sigma(i, i+s-1)$. Assume that $c_{i+1} \notin \sigma(j, j+t-1)$, and consider the $(t-1)$-element sets $\sigma(j, j+t-2)$ and $\sigma(j+1, j+t-1)$. Each of these sets contains c_i and does not contain c_{i+1}. Furthermore, since $\sigma(j, j+t-1)$ and $\sigma(i, i+s-1)$ are disjoint, each of $\sigma(j, j+t-2)$ and $\sigma(j+1, j+t-1)$ intersects C_i in exactly one element and does not intersect C_{i+1}. Therefore, by orthogonality, c_j^* and c_{j+1}^* are both contained in C_i, but not contained in C_{i+1}. The only possibility is $c_j^* = c_{j+1}^* = e_i$. However, this contradicts Lemma 2.4.1 when applied to M^*. Therefore, $c_{i+1} \in \sigma(j, j+t-1)$. □

Lemma 2.4.5. *Let $i \in [n]$, and suppose that $c_i = e_j$. If $n \geq s+2t-2$ and $n \geq 2s+t-4$, then at least one of the following holds:*

(i) *c_i and c_{i+1} are both contained in $\sigma(i-1, i+s)$;*

(ii) *$c_{i+1} = e_{j+1}$; or*

(iii) *$c_{i+1} = e_{j-1}$.*

Proof. Suppose (i) does not hold, that is, at least one of c_i and c_{i+1} is not contained in $\sigma(i-1, i+s)$. Choose $k \in [n]$ such that $e_k \in \{c_i, c_{i+1}\}$ and $e_k \notin \sigma(i-1, i+s)$. Let $e_{k'}$ be the other element of c_i and c_{i+1}. We establish the lemma by proving that either $k' = k+1$ or $k' = k-1$, which we shall do using Lemma 2.4.4.

First assume that $e_k \notin \sigma(i-t+2, i+s+t-3)$. This means that neither $\sigma(k-1, k+t-2)$ nor $\sigma(k-t+2, k+1)$ intersect $\sigma(i, i+s-1)$. So, by Lemma 2.4.4 (using part (i) if $e_k = c_i$

or part (ii) if $e_k = c_{i+1}$), we have that $e_{k'} \in \sigma(k-1, k+t-2) \cap \sigma(k-t+2, k+1)$. Now,

$$\sigma(k-1, k+t-2) \cap \sigma(k-t+2, k+1) = \{e_{k-1}, e_k, e_{k+1}\}$$

and, by Lemma 2.4.1, $e_{k'} \neq e_k$. Therefore, either $e_{k'} = e_{k-1}$ or $e_{k'} = e_{k+1}$, the desired result.

Now assume that $e_k \in \sigma(i-t+2, i+s+t-3)$. Then, as $e_k \notin \sigma(i-1, i+s)$, either $e_k \in \sigma(i+s+1, i+s+t-3)$ or $e_k \in \sigma(i-t+2, i-2)$. We consider only the former case; the analysis for the latter case is symmetrical. Thus, suppose $e_k \in \sigma(i+s+1, i+s+t-3)$. Now, $\sigma(k-1, k+t-2)$ does not intersect $\sigma(i, i+s-1)$, as k is at most $i+s+t-3$ and $n \geq s+2t-2$. Therefore, by Lemma 2.4.4, we have that $e_{k'} \in \sigma(k-1, k+t-2)$. If $e_{k'} \neq e_{k-1}$ and $e_{k'} \neq e_{k+1}$, then $e_{k'} \in \sigma(k+2, k+t-2)$. Furthermore, since $n \geq s+2t-2$, the sets $\sigma(i, i+s-1)$ and $\sigma(k+1, k+t)$ do not intersect. However, $e_k \notin \sigma(k+1, k+t)$, contradicting Lemma 2.4.4. Thus either $e_{k'} = e_{k-1}$ or $e_{k'} = e_{k+1}$, thereby completing the proof of the lemma. □

Lemma 2.4.6. *If $n \geq s+2t-1$ and $n \geq 2s+t-4$, then $c_i \neq c_{i+2}$ for all $i \in [n]$.*

Proof. Suppose $c_i = c_{i+2}$ for some $i \in [n]$. Then $C_i = \sigma(i, i+s-2) \cup \{c_i\}$ and $C_{i+2} = \sigma(i+2, i+s) \cup \{c_i\}$. By circuit elimination, there is also a circuit, say C, of M contained in $\sigma(i, i+s)$. If C contains neither e_{i+s-1} nor e_{i+s}, then C is contained in $\sigma(i, i+s-2)$, and thus properly contained in C_i, a contradiction. So C contains at least one of e_{i+s-1} and e_{i+s}. We next show that c_i is contained in $\sigma(i+s+1, i+s+t-1)$.

First, if e_{i+s} is not contained in C, then $e_{i+s-1} \in C$, in which case the $(t-1)$-element set $\sigma(i+s-1, i+s+t-3)$ intersects C in one element. Therefore, by orthogonality, $c^*_{i+s-1} \in \sigma(i, i+s-2)$. Now, orthogonality between C_i and C^*_{i+s-1} implies $c_i \in \sigma(i+s-1, i+s+t-3)$. Furthermore, c_i can be neither e_{i+s-1} nor e_{i+s} since these elements are contained in $\sigma(i+2, i+s)$ and $c_i = c_{i+2}$, so $c_i \in \sigma(i+s+1, i+s+t-3)$.

Now assume that $e_{i+s} \in C$. Orthogonality with C^*_{i+s} implies that $c^*_{i+s} \in \sigma(i, i+s-1)$, so either $c^*_{i+s} = e_{i+s-1}$ or $c^*_{i+s} \in \sigma(i, i+s-2)$. In the latter case, orthogonality with C_i implies that $c_i \in \sigma(i+s+1, i+s+t-2)$. Thus, we may assume that $c^*_{i+s} = e_{i+s-1}$. Now, C^*_{i+s} intersects $\sigma(i+1, i+s-1)$ in one element, so $c_{i+1} \in \sigma(i+s, i+s+t-2)$. Either $c_{i+1} = e_{i+s}$, or $c_{i+1} \in \sigma(i+s+1, i+s+t-2)$. Say $c_{i+1} = e_{i+s}$. Then both $\sigma(i+1, i+s)$ and $\sigma(i+2, i+s) \cup \{c_i\}$ are circuits of M (noting that $c_i \neq e_{i+1}$ because $e_{i+1} \in \sigma(i, i+s-2)$). This contradicts Lemma 2.4.3, so $c_{i+1} \in \sigma(i+s+1, i+s+t-2)$. Since $c_{i+1} \notin \sigma(i-1, i+s)$, and $n \geq s+2t-1$ and $n \geq 2s+t-4$, it follows by Lemma 2.4.5 that the elements c_i and c_{i+1} are consecutive, so $c_i \in \sigma(i+s+1, i+s+t-1)$.

We have now shown that, in all cases, $c_i \in \sigma(i+s+1, i+s+t-1)$. But, using a symmetrical argument and comparing C and C_{i+2}, we can show that $c_{i+2} \in \sigma(i-t+1, i-1)$. Now,

$c_{i+2} = c_i$, so $c_i \in \sigma(i-t+1, i-1)$ and $c_i \in \sigma(i+s+1, i+s+t-1)$. But, since $n \geq s+2t-1$, these two sets are disjoint. This contradiction completes the proof of the lemma. $\square$

Lemma 2.4.7. *Let $n \geq s+2t-1$ and $t \geq s$. If there exists $i \in [n]$ such that $\sigma(i, i+s-1)$ is a circuit of M, then M is (s,t)-cyclic.*

Proof. Let $i \in [n]$ such that $\sigma(i, i+s-1)$ is a circuit of M. We will show that $\sigma(i+2, i+s+1)$ is also a circuit. It then follows that $\sigma(i+2k, i+2k+s-1)$ is a circuit for all $k \geq 1$ and so, by Lemma 2.3.8, M is (s,t)-cyclic.

Since $\sigma(i, i+s-1)$ is a circuit, it follows by Lemma 2.4.3 that $c_i = e_{i+s-1}$ and $c_{i+1} = e_i$. By Lemma 2.4.5, either $c_{i+2} \in \sigma(i, i+s+1)$ or $c_{i+2} = e_{i-1}$ or $c_{i+2} = e_{i+1}$. Therefore, $c_{i+2} \in \{e_{i-1}, e_i, e_{i+1}, e_{i+s+1}\}$. If $c_{i+2} = e_{i+s+1}$, then $\sigma(i+2, i+s+1)$ is a circuit, and we have the desired result.

Furthermore, if $c_{i+2} = e_i$, then $c_{i+2} = c_{i+1}$, contradicting Lemma 2.4.1. If $c_{i+2} = e_{i+1}$, then both $\sigma(i, i+s-1)$ and $\sigma(i+1, i+s)$ are circuits containing $\sigma(i+1, i+s-1)$, contradicting Lemma 2.4.3. Therefore we may assume that $c_{i+2} = e_{i-1}$.

Now consider c_{i+3}. Since c_{i+2} is not contained in $\sigma(i+1, i+s+2)$, it follows by Lemma 2.4.5 that either $c_{i+3} = e_{i-2}$ or $c_{i+3} = e_i$. But $c_{i+1} = e_i$, so $c_{i+3} \neq e_i$ by Lemma 2.4.6. Therefore, $c_{i+3} = e_{i-2}$. More generally, suppose that $c_{i+k-2} = e_{i-k+3}$ and $c_{i+k-1} = e_{i-k+2}$, for some $k \geq 4$. If $n \geq 2k+s-2$, then $c_{i+k-1} \notin \sigma(i+k-2, i+k+s-1)$, and we can apply Lemma 2.4.5 to show that $c_{i+k} \in \{e_{i-k+1}, e_{i-k+3}\}$. But $c_{i+k-2} = e_{i-k+3}$, so $c_{i+k} = e_{i-k+1}$ by Lemma 2.4.6.

By induction, we deduce, for all $k \geq 2$ satisfying $n \geq 2k+s-2$, that $c_{i+k} = e_{i-k+1}$. Suppose $t = s$. Taking $k = s$, we have that $n \geq 3s-2$, and so $c_{i+s} = e_{i-s+1}$. Therefore, assuming $t > s$, we have that $c_{i+s} = e_{i-s+1} \in \sigma(i-t+2, i)$. This means that the $(t-1)$-element set $\sigma(i-t+2, i)$ intersects each of C_i and C_{i+s} in one element, and so $c^*_{i-t+2} \in C_i \cap C_{i+s}$. But C_i and C_{i+s} are disjoint, a contradiction. Thus, we may assume that $s = t$.

We apply Lemma 2.4.5 to c_{i-1} with the aim of showing that $c_{i-1} = e_{i+s}$. Suppose $c_{i-1} = e_j$. If $c_{i-1} \notin \sigma(i-2, i+s-1)$, then either $c_i = e_{j-1}$ or $c_i = e_{j+1}$. Since $c_i = e_{i+s-1}$, it follows that either $c_{i-1} \in \sigma(i-2, i+s-1)$ or $c_{i-1} = e_{i+s}$. Now consider the $(t-1)$-element set $\sigma(i+s, i+s+t-2)$. This intersects $C_{i+2} = \sigma(i+2, i+s) \cup \{e_{i-1}\}$ in one element. So, either $c^*_{i+s} \in \sigma(i+2, i+s-1)$ or $c^*_{i+s} = e_{i-1}$. In the former case, C^*_{i+s} intersects $\sigma(i, i+s-1)$ in one element, contradicting orthogonality. So $c^*_{i+s} = e_{i-1}$. But then $\sigma(i-1, i+s-3)$ intersects C^*_{i+s} in one element, and so $c_{i-1} \in \sigma(i+s, i+s+t-2)$. Therefore, $c_{i-1} \notin \sigma(i-2, i+s-1)$, and so $c_{i-1} = e_{i+s}$.

Consider c_{i-2}. Since $c_{i-1} \notin \sigma(i-3, i+s-2)$, it follows by Lemma 2.4.5 that either

$c_{i-2} = e_{i+s-1}$ or $c_{i-2} = e_{i+s+1}$. But $c_i = e_{i+s-1}$ and so, by Lemma 2.4.6, $c_{i-2} = e_{i+s+1}$. More generally, suppose $c_{i-k+3} = e_{i+s+k-4}$ and $c_{i-k+2} = e_{i+s+k-3}$, for some $k \geq 4$. If $n \geq 2k+s-2$, then $c_{i-k+2} \notin \sigma(i-k, i-k+s+1)$, and we can apply Lemma 2.4.5 to show that $c_{i-k+1} \in \{e_{i+s+k-4}, e_{i+s+k-2}\}$. But $c_{i-k+3} = e_{i+s+k-4}$, so $c_{i-k+1} = e_{i+s+k-2}$.

Therefore, by induction, for all $k \geq 2$ satisfying $n \geq 2k+s-2$, we have $c_{i+k} = e_{i-k+1}$ and $c_{i-k+1} = e_{i+s+k-2}$. If $s = t = 3$, we have $c_{i+2} = e_{i-1}$ and $c_{i-1} = e_{i+3}$. By orthogonality between C_i^* and C_{i-1}, we have that either $c_i^* = e_{i-1}$ or $c_i^* = e_{i+3}$. For either possibility, C_i^* intersects C_{i+2} in one element, a contradiction. Now assume that $s = t \geq 4$, and consider the $(t-1)$-element set $\sigma(i, i+t-2)$. This set intersects each of $\sigma(i-s+2, i)$ and $\sigma(i+t-2, i+s+t-4)$ in exactly one element. Now, since $n \geq 3s-4$, we have that $c_{i-s+2} = e_{i+2s-3}$ and, since $n \geq s+2t-6$, we have that $c_{i+t-2} = e_{i-t+3} = e_{i-s+3}$. Neither c_{i-s+2} nor c_{i+t-2} are contained in $\sigma(i, i+t-2)$, and so $c_i^* \in C_{i-s+2} \cap C_{i+t-2} = \{e_{i-s+3}\}$. But now, since $c_{i-s+1} = e_{i+2s-2}$, we have that $C_{i-s+1} = \sigma(i-s+1, i-1) \cup \{e_{i+2s-2}\}$, which intersects C_i^* in one element. This contradiction to orthogonality completes the proof of the lemma. □

Lemma 2.4.8. *Let $n \geq s+2t-1$, and suppose that $t \geq s$. If $c_i = e_{i+s}$, then $c_{i+1} = e_{i+s+1}$.*

Proof. As $t \geq s$, it follows by Lemma 2.4.5 that either $c_{i+1} \in \sigma(i-1, i+s)$, or $c_{i+1} = e_{i+s+1}$. Therefore, $c_{i+1} \in \{e_{i-1}, e_i, e_{i+s}, e_{i+s+1}\}$. By Lemma 2.4.1, $c_{i+1} \neq e_{i+s}$. Also, if $c_{i+1} = e_i$, then both $\sigma(i, i+s-1)$ and $\sigma(i, i+s-2) \cup \{e_{i+s}\}$ are circuits containing $\sigma(i, i+s-2)$, contradicting Lemma 2.4.3.

Suppose $c_{i+1} = e_{i-1}$, and consider the $(t-1)$-element set $\sigma(i-t+1, i-1)$. As $n \geq s+2t-1$, this set intersects C_{i+1} in exactly one element, but does not intersect C_i. Therefore, $c_{i-t+1}^* \in C_{i+1}$, but not in $c_{i-t+1}^* \notin C_i$; the only possibility is $c_{i-t+1}^* = e_{i+s-1}$.

Now consider the $(t-1)$-element set $\sigma(i+s, i+s+t-2)$. As $n \geq s+2t-1$, this set intersects C_i in exactly one element, and does not intersect C_{i+1}. Therefore, $c_{i+s}^* = e_i$. Finally, consider the $(s-1)$-element set $\sigma(i+2, i+s)$. This last set intersects each of C_{i+s}^* and C_{i-t+1}^* in exactly one element. But C_{i+s}^* and C_{i-t+1}^* are disjoint, a contradiction. Therefore, $c_{i+1} = e_{i+s+1}$. □

Lemma 2.4.9. *Let $n \geq s+2t-1$ and $t \geq s$. If $c_i = e_{i+s-1+k}$ for some $1 \leq k < n-s$, then $c_{i+1} = e_{i+s+k}$.*

Proof. The proof is by induction on k. If $k = 1$, then the result follows immediately from Lemma 2.4.8. Suppose $k = 2$, so that, $c_i = e_{i+s+1}$. By Lemma 2.4.5, either $c_{i+1} = e_{i+s}$ or $c_{i+1} = e_{i+s+2}$. If $c_{i+1} = e_{i+s}$, then $\sigma(i+1, i+s)$ is a circuit. But, by Lemma 2.4.7, this implies M is (s,t)-cyclic, which, by Lemma 2.4.3, contradicts the uniqueness of the circuit containing $\sigma(i, i+s-2)$. So $c_{i+1} = e_{i+s+2}$, and the lemma holds for $k = 2$.

Now let $k \geq 3$, and suppose that, for all $i' \in [n]$, if $c_{i'} = e_{i'+s-1+(k-2)}$, then $c_{i'+1} = e_{i'+s+(k-2)}$. We shall complete the proof by proving that the lemma holds for k. So, let $c_i = e_{i+s-1+k}$. Then, by Lemma 2.4.5, either $c_{i+1} = e_{i+s-2+k}$ or $c_{i+1} = e_{i+s+k}$. If $c_{i+1} = e_{i+s-2+k}$, then, by the induction assumption, $c_{i+2} = e_{i+s-1+k}$. But now $c_{i+2} = c_i$. This contradiction to Lemma 2.4.6 shows that $c_{i+1} = e_{i+s+k}$, and completes the proof of the lemma. □

At last we are ready to prove Theorem 1.4.1.

Proof of Theorem 1.4.1. Since σ is an (s,t)-cyclic ordering of M if and only if σ is a (t,s)-cyclic ordering of M^*, we may assume, without loss of generality, that $t \geq s$. For the purposes of obtaining a contradiction, suppose there is no $j \in [n]$ such that $\sigma(j, j+s-1)$ is a circuit of M. Since σ is a nearly (s,t)-cyclic ordering of M, it follows by Lemma 2.4.9 that there exists $1 \leq k < n-s$ such that, for all $i \in [n]$, the set $\sigma(i, i+s-2) \cup \{e_{i+s-1+k}\}$ is a circuit. In particular, by Lemma 2.4.3, $C_i = \sigma(i, i+s-2) \cup \{e_{i+s-1+k}\}$. Take one such i, and consider the $(t-1)$-element set $\sigma(i, i+t-2)$. As $n \geq 2s+t-3$, the $(s-1)$-element sets $\sigma(i-s+1, i-1)$ and $\sigma(i+t-1, i+s+t-3)$ are disjoint, so at least one of these two sets does not contain c_i^*.

We will establish a contradiction for when $c_i^* \notin \sigma(i-s+1, i-1)$. A symmetrical argument applies when $c_i^* \notin \sigma(i+t-1, i+s+t-3)$. So suppose $c_i^* \notin \sigma(i-s+1, i-1)$. Then $\sigma(i-s+2, i)$ intersects C_i^* in exactly one element. Therefore, either $c_{i-s+2} = c_i^*$ or $c_{i-s+2} \in \sigma(i+1, i+t-2)$.

First assume that $c_{i-s+2} \in \sigma(i+1, i+t-2)$. We know that $c_{i-s+2} \neq e_{i+1}$, for otherwise $\sigma(i-s+2, i+1)$ is a circuit. So $c_{i-s+2} \in \sigma(i+2, i+t-2)$. But now, by Lemma 2.4.9, $c_{i-s+1} \in \sigma(i+1, i+t-3)$, and so C_{i-s+1} and C_i^* intersect in exactly one element, a contradiction.

Now assume that $c_{i-s+2} = c_i^*$. Consider the $(s-1)$-element set $\sigma\,(i+t-2, i+s+t-4)$. Suppose $c_i^* \notin \sigma(i+t-1, i+s+t-3)$. Then, by orthogonality, either $c_{i+t-2} = c_i^*$ or $c_{i+t-2} \in \sigma(i, i+t-3)$. But $c_{i+t-2} \neq e_{i+t-3}$, since then $\sigma(i+t-3, i+s+t-4)$ is a circuit, and $c_{i+t-2} \notin \sigma(i, i+t-4)$ since then C_{i+t-1} and C_i^* intersect in exactly one element, by Lemma 2.4.9. Furthermore, $c_{i+t-2} \neq c_i^*$, since then $c_{i+t-2} = c_{i-s+2}$, contradicting Lemmas 2.4.3 and 2.4.9. Therefore, $c_i^* \in \sigma(i+t-1, i+s+t-3)$.

It now follows that $c_{i-s+2} = e_{i+t-2+\ell}$ for some $1 \leq \ell \leq s-1$. Therefore, by Lemma 2.4.9, $c_{i-s+2-\ell} = e_{i+t-2}$. Furthermore, as $n \geq 3s+t-5$, the $(s-1)$-element set $\sigma(i-s+2-\ell, i-\ell)$

does not contain $c_i^* = e_{i+t-2+\ell}$ and does not intersect $\sigma(i, i+t-2)$. So $C_{i-s+2-\ell}$ and C_i^* intersect in exactly one element. This contradiction to orthogonality establishes that M must contain a circuit $\sigma(j, j+s-1)$ for some $j \in [n]$, and so, by Lemma 2.4.7, σ is an (s,t)-cyclic ordering of M. This completes the proof of the theorem. □

2.5 Proof of Theorem 1.4.3

In this section, we prove Theorem 1.4.3. We begin by defining a class of matroids that contains, for all positive integers s exceeding one and all positive even integers n, the matroid Ψ_s^n. The proof of Theorem 1.4.3 is a consequence of a more general weak-map result, namely Theorem 2.5.4, that we establish for this class.

Recall that for a vertex v of a graph G, we denote the set of vertices of G adjacent to v, that is, the *neighbours of* v, by $N(v)$. More generally, for a subset U of vertices of G, the *neighbours of* U, denoted $N(U)$, is

$$\bigcup_{v \in U} N(v).$$

We next define a multi-path matroid. Let E be a set of n elements, and suppose that $\sigma = (e_1, e_2, \ldots, e_n)$ is a cyclic ordering of E. Let m be a positive integer exceeding one. Choose distinct elements $x_1, x_2, \ldots, x_m \in [n]$ and distinct elements $y_1, y_2, \ldots, y_m \in [n]$ such that $e_{x_i} \in \sigma(x_{i-1}, x_{i+1})$ and $e_{y_i} \in \sigma(y_{i-1}, y_{i+1})$ for all $i \in [m]$, where subscripts of x and y are interpreted modulo m, and, furthermore, the intervals $\sigma(x_i, y_i)$ form an antichain of σ, that is, there is no $i, i' \in [m]$ such that $\sigma(x_i, y_i) \subseteq \sigma(x_{i'}, y_{i'})$. Let G denote the bipartite graph with parts E and $[m]$, and whose set of edges satisfy $N(i) = \sigma(x_i, y_i)$ for all $i \in [m]$. The transversal matroid on ground set E with presentation

$$\mathcal{I} = (N(1), N(2), \ldots, N(m))$$

is called a *multi-path matroid* and is denoted by $M[\mathcal{I}]$. Let $M^*[\mathcal{I}]$ denote the dual of $M[\mathcal{I}]$, and observe that, for all $i \in [m]$, the set $\sigma(x_i, y_i)$ is a circuit of $M^*[\mathcal{I}]$. Multi-path matroids were introduced in [2].

As an example, let s be a positive integer exceeding one and let n be a positive even integer, and suppose that $\sigma = (e_1, e_2, \ldots, e_n)$ is a cyclic ordering of E and $m = \frac{n}{2}$. By choosing $x_i = 2i - 1$ and $y_i = 2i + s - 2$ for all $i \in \left[\frac{n}{2}\right]$, we have that $G \cong G_s^n$, the bipartite graph defined in the introduction, and $M^*[\mathcal{I}] \cong \Psi_s^n$.

The initial goal of this section is to establish Theorem 2.5.4 which says that, up to isomorphism, $M^*[\mathcal{I}]$ is at least as free as any matroid on the same ground set satisfying a certain rank condition; that is, up to isomorphism, every such matroid is a weak-map image of $M^*[\mathcal{J}]$.

A subset $X \subseteq E$ is independent in $M^*[\mathcal{J}]$ if and only if $E - X$ is cospanning. In other words, X is independent in $M^*[\mathcal{J}]$ if and only if there is a complete matching from $[m]$ into $E - X$. By Hall's Theorem [14], this is true precisely if, for all subsets J of $[m]$, we have that $|N(J) - X| \geq |J|$. We repeatedly use this fact in the proofs in this section. To ease reading, in the statements of these lemmas and theorem, the multi-path matroid $M[\mathcal{J}]$ has ground set E and is constructed as above.

Lemma 2.5.1. *$r(M^*[\mathcal{J}]) = |E| - m$.*

Proof. It is sufficient to prove that $r(M[\mathcal{J}]) = m$. Let $X \subseteq E$ be a set of $m+1$ elements. Clearly there is no matching of X into $[m]$, so X is dependent. Therefore, $r(M[\mathcal{J}]) \leq m$. For all $i \in [m]$, we have that $\{i, e_{x_i}\}$ is an edge of the bipartite graph G. Therefore, $\{\{1, e_{x_1}\}, \{2, e_{x_2}\}, \ldots, \{m, e_{x_m}\}\}$ is a matching of G. Hence $r(M[\mathcal{J}]) \geq m$, so $r(M[\mathcal{J}]) = m$, completing the proof. □

Lemma 2.5.2. *Let C be a circuit of $M^*[\mathcal{J}]$. Let $J \subseteq [m]$ such that $|N(J) - C| < |J|$. Then C is a subset of $N(J)$ containing $|N(J)| - |J| + 1$ elements.*

Proof. If C is not a subset of $N(J)$, then there exists an element e of C such that $e \notin N(J)$. Then

$$|N(J) - (C - \{e\})| = |N(J) - C| < |J|.$$

But this implies that $C - \{e\}$ is dependent, a contradiction. Thus C is a subset of $N(J)$.

To see that C contains $|N(J)| - |J| + 1$ elements, suppose that $|N(J) - C| < |J| - 1$, and let $e \in C$. Then, as C is a subset of $N(J)$, we have

$$|N(J) - (C - \{e\})| = |N(J) - C| + 1 < |J|.$$

Again, this implies $C - \{e\}$ is dependent, a contradiction. Thus

$$|N(J)| - |C| = |N(J) - C| = |J| - 1.$$

Rearranging this last equation gives $|C| = |N(J)| - |J| + 1$, thereby completing the proof of the lemma. □

Lemma 2.5.3. *Let C be a circuit of $M^*[\mathcal{J}]$. Then either C has $|E| - m + 1$ elements or there exist $i, j \in [m]$ such that each of the following hold:*

(i) $N([i, j]) = \sigma(x_i, y_j)$,

(ii) *C is a subset of $N([i,j])$ containing $|N([i,j])| - |[i,j]| + 1$ elements,*

(iii) *either $i = j$, or $N([i,j]) - N([i+1,j]) \subseteq C$,*

(iv) *either $i = j$, or $N([i,j]) - N([i,j-1]) \subseteq C$, and*

(v) *$\sigma(x_i, y_j) \subseteq \mathrm{cl}(C)$,*

Proof. Since C is dependent, there exists $J \subseteq [m]$ such that $|N(J) - C| < |J|$. If $N(J) = E$, then $N([m]) = E$, so $|N([m]) - C| = |E - C| < |J| \leq m$. Therefore, by Lemma 2.5.2, C has $|E| - m + 1$ elements. So suppose that $N(J) \neq E$.

We next show that we may assume that J has the property that $N(J) = \sigma(x_i, y_j)$ for some $i, j \in [m]$. If J does not satisfy this property, then partition J into maximal subsets with disjoint, consecutive neighbourhoods. More formally, since

$$N(J) = \bigcup_{i_0 \in J} \sigma(x_{i_0}, y_{i_0}),$$

we may partition J into sets $J_1, J_2, \ldots, J_k$ such that, for all $\ell \in [k]$, there exist $i_\ell, j_\ell \in [m]$ with $N(J_\ell) = \sigma(x_{i_\ell}, y_{j_\ell})$. Furthermore, we may choose such a partition in which, for all distinct $\ell, \ell' \in [k]$, the sets $\sigma(x_{i_\ell}, y_{j_\ell})$ and $\sigma(x_{i'_\ell}, y_{j'_\ell})$ are disjoint. Now,

$$\begin{aligned} |N(J_1) - C| + |N(J_2) - C| + \cdots + |N(J_k) - C| &= |N(J) - C| \\ &< |J| \\ &= |J_1| + |J_2| + \cdots + |J_k|. \end{aligned}$$

It follows that there exists $\ell \in [k]$ such that $|N(J_\ell) - C| < |J_\ell|$, in which case replace J with J_ℓ.

We have chosen $J \subseteq [m]$ such that $|N(J) - C| < |J|$ and $N(J) = \sigma(x_i, y_j)$ for some $i, j \in [m]$. It follows from the definition of the bipartite graph G that $J \subseteq [i,j]$. Furthermore, $N([i,j]) \subseteq \sigma(x_i, y_j)$, so $N([i,j]) = \sigma(x_i, y_j)$, that is, (i) holds. Therefore,

$$|N([i,j]) - C| = |N(J) - C| < |J| \leq |[i,j]|\,.$$

Hence, by Lemma 2.5.2, C is a subset of $N([i,j])$ containing $|N([i,j])| - |[i,j]| + 1$ elements, so (ii) holds.

We next show that we may choose $i' \in [m]$ such that the pair i', j satisfies (i), (ii), and (iii). Initially, choose $i' = i$, and suppose i' and j do not satisfy (iii). Then $i' \neq j$, and there exists $f \in N([i',j]) - N([i'+1,j])$ with $f \notin C$. First, assume $N([i',j]) - N([i'+1,j]) = \{f\}$.

Then C is a subset of $N([i'+1,j])$ and

$$\begin{aligned}|C| &= |N([i',j])| - |[i',j]| + 1\\ &= (|N([i'+1,j])| + 1) - (|[i'+1,j]| + 1) + 1\\ &= |N([i'+1,j])| - |[i'+1,j]| + 1,\end{aligned}$$

so $i'+1, j$ satisfies (ii). Furthermore, it follows from the definition of the bipartite graph G that, since $N([i',j]) = \sigma(x_{i'}, y_j)$, we have that $N([i'+1,j]) = \sigma(x_{i'+1}, y_j)$. Thus, $i'+1, j$ satisfies (i) and (ii), and we may replace i' in the pair i', j with $i'+1$.

Hence, we may assume there exists $f' \in N([i',j]) - N([i'+1,j])$ with $f' \neq f$. First assume $f' \in C$. Then, by (ii),

$$\begin{aligned}|N([i'+1,j]) - (C - \{f'\})| &= |N([i'+1,j]) - C|\\ &< |N([i',j]) - C|\\ &= \left|[i',j]\right| - 1\\ &= \left|[i'+1,j]\right|.\end{aligned}$$

Therefore, $C - \{f'\}$ is dependent, a contradiction. Now assume $f' \notin C$. Since $f, f' \notin C$,

$$|N([i'+1,j]) - C| < |N([i',j]) - C| - 1.$$

Let $x \in C$. Then, by (ii),

$$\begin{aligned}|N([i'+1,j]) - (C - \{x\})| &\leq |N([i'+1,j]) - C| + 1\\ &< |N([i',j]) - C| = \left|[i',j]\right| - 1 = \left|[i'+1,j]\right|.\end{aligned}$$

But this implies that $C - \{x\}$ is dependent, and thus the pair i', j satisfies (i), (ii) and (iii). A symmetrical argument shows that we may choose $j' \in [m]$ such that the pair i', j' satisfies (i)-(iv).

It remains to show (v). Let $e \in C$, and let $e' \in \sigma(x_{i'}, y_{j'}) - C$. Then

$$|N([i',j']) - ((C - \{e\}) \cup \{e'\})| = |N([i',j']) - C| < |[i',j']|.$$

Therefore, $(C - \{e\}) \cup \{e'\}$ is dependent, so contains a circuit C'. The circuit C' contains the element e', as otherwise C' is a proper subset of C. Therefore, $e' \in \mathrm{cl}(C)$, completing the proof of the lemma. □

Theorem 2.5.4. *Let M be a matroid on ground set E such that, for all $i \in [m]$ and $1 \leq k \leq m$, we have*

$$\begin{aligned}&r_M\big(\sigma(x_i,y_i) \cup \sigma(x_{i+1},y_{i+1}) \cup \cdots \cup \sigma(x_{i+k-1},y_{i+k-1})\big)\\ &\qquad \leq r_{M^*[\mathcal{I}]}\big(\sigma(x_i,y_i) \cup (x_{i+1},y_{i+1}) \cup \cdots \cup \sigma(x_{i+k-1},y_{i+k-1})\big).\end{aligned}$$

If $M[\mathcal{I}]$ has no loops, then, under the identity map, M is a weak-map image of $M^*[\mathcal{I}]$.

Proof. Let φ denote the identity map from the ground set E of $M^*[\mathcal{I}]$ to the ground set E of M. To prove the theorem, we will show that if C is a circuit of $M^*[\mathcal{I}]$, then $\varphi(C)$ contains a circuit of M. Let C be a circuit of $M^*[\mathcal{I}]$. Now, as $M[\mathcal{I}]$ has no loops, every element of E is in $N(i) = \sigma(x_i, y_i)$ for some $i \in [m]$. Therefore, $\sigma(x_1, y_1) \cup \sigma(x_2, y_2) \cup \cdots \cup \sigma(x_m, y_m) = E$. Thus, by Lemma 2.5.1

$$\begin{aligned}|E| - m &= r(M^*[\mathcal{I}]) \\ &= r_{M^*[\mathcal{I}]}\big(\sigma(x_1, y_1) \cup \sigma(x_2, y_2) \cup \cdots \cup \sigma(x_m, y_m)\big) \\ &\geq r_M\big(\sigma(x_1, y_1) \cup \sigma(x_2, y_2) \cup \cdots \cup \sigma(x_m, y_m)\big) \\ &= r(M).\end{aligned}$$

Therefore, if C contains $|E| - m + 1$ elements, then $\varphi(C)$ contains a circuit of M.

Otherwise, by Lemma 2.5.3, there exist $i, j \in [m]$ such that C is a subset of $\sigma(x_i, y_j)$ containing $|\sigma(x_i, y_j)| - |[i, j]| + 1$ elements. Furthermore, by Lemma 2.5.3(i), we have that

$$N([i, j]) = \sigma(x_i, y_i) \cup \sigma(x_{i+1}, y_{i+1}) \cup \cdots \cup \sigma(x_j, y_j) = \sigma(x_i, y_j)$$

and so $r_M(\sigma(x_i, y_j)) \leq r_{M^*[\mathcal{I}]}(\sigma(x_i, y_j))$. By Lemma 2.5.3(v), we have that $\sigma(x_i, y_j) \subseteq \mathrm{cl}(C)$, so $r_{M^*[\mathcal{I}]}(\sigma(x_i, y_j)) = r_{M^*[\mathcal{I}]}(C) = |C| - 1$. Thus,

$$r_M(C) \leq r_M(\sigma(x_i, y_j)) \leq r_{M^*[\mathcal{I}]}(\sigma(x_i, y_j)) = |C| - 1.$$

Therefore, $\varphi(C)$ contains a circuit of M. □

The previous results in this section apply for any multi-path matroid $M^*[\mathcal{I}]$. We now turn our attention to the case where $M^*[\mathcal{I}] \cong \Psi_s^n$, towards proving Theorem 1.4.3. We first show that Ψ_s^n is self-dual.

Lemma 2.5.5. *Let s be an integer exceeding two, and let $\phi_s : E \to E$ be the identity map if s is even, or the map $\phi_s(e_i) = e_{i+1}$ if s is odd. Then Ψ_s^n is self-dual under the map ϕ_s.*

Proof. Let B be a basis of Ψ_s^n. We show that $\phi_s^{-1}(E - B)$ is also a basis of Ψ_s^n. By Lemma 2.5.1, we have that $\left|\phi_s^{-1}(E - B)\right| = r(\Psi_s^n) = \frac{n}{2}$. Furthermore, by Lemma 2.5.3, a circuit of Ψ_s^n is either a set of $\frac{n}{2} + 1$ elements, or a subset of $\sigma(x_i, y_{i+k}) = \sigma(2i - 1, 2i + 2k + s - 2)$ containing $|\sigma(2i - 1, 2i + 2k + s - 2)| - (k + 1) + 1 = s + k$ elements, for some $i \in [m]$ and $k \leq \frac{n}{2} - s$. Hence, to show that $\phi_s^{-1}(E - B)$ contains no circuits, and is therefore a basis, it suffices to show that, for all odd $i \in [n]$ and $k \leq \frac{n}{2} - s$, we have that $\left|\phi_s^{-1}(E - B) \cap \sigma(i, i + s - 1 + 2k)\right| < s + k$.

First, suppose s is even. Then

$$\begin{aligned}\phi_s(E-\sigma(i,i+s-1+2k)) &= \sigma(i+s+2k,i-1)\\ &= \sigma\left(i+s+2k,i+s+2k+s-1+2\left(\tfrac{n}{2}-k-s\right)\right).\end{aligned}$$

Therefore, since $i+s+2k$ is odd, there exists $j\in\left[\frac{n}{2}\right]$ such that

$$N\left(\left[j,j+\left(\tfrac{n}{2}-k-s\right)\right]\right)=\phi_s(E-\sigma(i,i+s-1+2k)).$$

Now, B is independent, so

$$\begin{aligned}\left|N\left(\left[j,j+\left(\tfrac{n}{2}-k-s\right)\right]\right)-B\right| &= |\phi_s(E-\sigma(i,i+s-1+2k))-B|\\ &\geq \tfrac{n}{2}-k-s+1.\end{aligned}$$

It follows that

$$|B\cap\phi_s\left(E-\sigma(i,i+s-1+2k)\right)|<\tfrac{n}{2}-k.$$

On the other hand, if s is odd, then

$$\begin{aligned}\phi_s(E-\sigma(i,i+s-1+2k)) &= \phi_s(\sigma(i+s+2k,i-1))\\ &= \sigma(i+s+2k+1,i)\\ &= \sigma\left(i+s+2k+1,i+s+2k+1+s-1+2\left(\tfrac{n}{2}-k-s\right)\right).\end{aligned}$$

Since $i+s+2k+1$ is odd, there exists $j\in\left[\frac{n}{2}\right]$ such that

$$N\left(\left[j,j+\left(\tfrac{n}{2}-k-s\right)\right]\right)=\phi_s(E-\sigma(i,i+s-1+2k)).$$

As before, since B is independent, it follows that

$$|B\cap\phi_s\left(E-\sigma(i,i+s-1+2k)\right)|<\tfrac{n}{2}-k.$$

In both cases,

$$\left|\phi_s^{-1}(B)\cap\left(E-\sigma(i,i+s-1+2k)\right)\right|<\tfrac{n}{2}-k$$

and so

$$\left|\phi_s^{-1}(B)\cap\sigma(i,i+s-1+2k)\right|>k.$$

Therefore,

$$\begin{aligned}\left|\phi_s^{-1}(E-B)\cap\sigma(i,i+s-1+2k)\right| &< |\sigma(i,i+s-1+2k)|-k\\ &= s+2k-k=s+k\end{aligned}$$

as required. $\square$

Lemma 2.5.6. *Let s and t be positive integers exceeding one, such that $t \geq s$. If n is a positive even integer with $n \geq s+t-2$ and $s \equiv t \bmod 2$, then $T^{\frac{t-s}{2}}(\Psi_s^n)$ is an (s,t)-cyclic matroid.*

Proof. Without loss of generality, we may assume that the ground set $\{e_1, e_2, \ldots, e_n\}$ of Ψ_s^n is consistent with the bipartite graph G_s^n associated with the dual of Ψ_s^n as described in the introduction. In particular, G_s^n has vertex parts $\{e_1, e_2, \ldots, e_n\}$ and $[\frac{n}{2}]$ and, for all $i \in \{1, 2, \ldots, \frac{n}{2}\}$, we have

$$N(i) = \{e_{2i-1}, e_{2i}, \ldots, e_{2i+s-2}\}.$$

The proof is by induction on t. Suppose that $t = s$, and consider $T^0(\Psi_s^n) = \Psi_s^n$. It is easily checked that, for all odd $i \in [n]$, the set $\{e_i, e_{i+1}, \ldots, e_{i+s-1}\}$ is an s-element circuit of Ψ_s^n. By Lemma 2.5.5, the set $\{e_j, e_{j+1}, \ldots, e_{j+s-1}\}$ is an s-element cocircuit of Ψ_s^n for all odd $j \in [n]$ if s is even, or for all even $j \in [n]$ if s is odd. Therefore Ψ_s^n is (s,s)-cyclic, and the lemma holds if $t = s$.

Now suppose that $t > s$ and that the matroid $T^{\frac{(t-2)-s}{2}}(\Psi_s^n)$ is $(s, t-2)$-cyclic. Consider

$$T^{\frac{t-s}{2}}(\Psi_s^n) = T\left(T^{\frac{(t-2)-s}{2}}(\Psi_s^n)\right).$$

It follows from Lemma 2.2.1 that each non-spanning circuit of $T^{\frac{(t-2)-s}{2}}(\Psi_s^n)$ is a circuit of $T^{\frac{t-s}{2}}(\Psi_s^n)$. Now, by Lemma 2.3.7,

$$\begin{aligned} r\left(T^{\frac{(t-2)-s}{2}}(\Psi_s^n)\right) &= \tfrac{n+s-(t-2)}{2} \\ &\geq \tfrac{(s+t-2)+s-t+2}{2} \\ &= s. \end{aligned}$$

Therefore, for all odd $i \in [n]$, we have that $\{e_i, e_{i+1}, \ldots, e_{i+s-1}\}$ is a non-spanning circuit of $T^{\frac{(t-2)-s}{2}}(\Psi_s^n)$, so is also an s-element circuit of $T^{\frac{t-s}{2}}(\Psi_s^n)$. Furthermore, for all $j \in [n]$, if $\{e_j, e_{j+1}, \ldots, e_{j+t-3}\}$ and $\{e_{j+2}, e_{j+3}, \ldots, e_{j+t-1}\}$ are $(t-2)$-element cocircuits of $T^{\frac{(t-2)-s}{2}}(\Psi_s^n)$, then $\{e_j, e_{j+1}, \ldots, e_{j+t-1}\}$ is a t-element cocircuit of $T^{\frac{t-s}{2}}(\Psi_s^n)$. To see this, if f is the element freely added to $T^{\frac{(t-2)-s}{2}}(\Psi_s^n)$, then it is easily checked that

$$\left(E\left(T^{\frac{(t-2)-s}{2}}(\Psi_s^n)\right) - \{e_j, e_{j+1}, \ldots, e_{j+t-1}\}\right) \cup \{f\}$$

is a hyperplane of the resulting matroid. Therefore

$$E\left(T^{\frac{t-s}{2}}(\Psi_s^n)\right) - \{e_j, e_{j+1}, \ldots, e_{j+t-1}\}$$

is a hyperplane of $T^{\frac{t-s}{2}}(\Psi_s^n)$, so $\{e_j, e_{j+1}, \dots, e_{j+t-1}\}$ is a t-element cocircuit of $T^{\frac{t-s}{2}}(\Psi_s^n)$. Hence, by induction, $T^{\frac{t-s}{2}}(\Psi_s^n)$ is (s,t)-cyclic. □

Proof of Theorem 1.4.3. Let M be an (s,t)-cyclic matroid on n elements, where $n \geq s+t-1$ and $t \geq s$. Then, by Lemma 2.3.3, n is even, and $s \equiv t \bmod 2$. Let $\sigma = (e_1, e_2, \dots, e_n)$ be an (s,t)-cyclic ordering of $E(M)$. Without loss of generality, we may assume that, for all odd $i \in [n]$, the set $\sigma(i, i+s-1)$ is an s-element circuit of M. Now consider Ψ_s^n. To ease reading, we may assume that $E(M) = E(\Psi_s^n)$ and $\sigma = (e_1, e_2, \dots, e_n)$ is an (s,s)-cyclic ordering of Ψ_s^n, where $\sigma(i, i+s-1)$ is an s-element circuit of Ψ_s^n for all odd $i \in [n]$. Note that the dual of Ψ_s^n has no loops.

First suppose that $t = s$. By Lemma 2.5.6, both M and Ψ_s^n are (s,s)-cyclic matroids with n elements. Therefore, by Lemma 2.3.6, for all $i \in [\frac{n}{2}]$ and k such that $1 \leq k \leq m$, we have that

$$\begin{aligned} r_M\big(\sigma(x_i, y_i) &\cup \sigma(x_{i+1}, y_{i+1}) \cup \dots \cup \sigma(x_{i+k-1}, y_{i+k-1})\big) \\ &= r_{M^*[\mathcal{I}]}\big(\sigma(x_i, y_i) \cup \sigma(x_{i+1}, y_{i+1}) \cup \dots \cup \sigma(x_{i+k-1}, y_{i+k-1})\big), \end{aligned}$$

where $x_i = e_{2i-1}$ and $y_i = e_{2i+s-2}$ for all $i \in \{1, 2, \dots, \frac{n}{2}\}$. Hence, by Theorem 2.5.4, under the identity map, M is a weak-map image of Ψ_s^n.

Now suppose $t > s$. By Lemma 2.5.6, the matroid $T^{\frac{t-s}{2}}(\Psi_s^n)$ is an (s,t)-cyclic matroid. It remains to show that M is a weak-map image of $T^{\frac{t-s}{2}}(\Psi_s^n)$. Let I be an independent set in M. By Theorem 2.5.4, under the identity map, M is a weak-map image of Ψ_s^n, and so I is an independent set in Ψ_s^n. From Lemma 2.3.7, we have that

$$r(M) = \tfrac{n+s-t}{2} = \tfrac{n}{2} - \left(\tfrac{t-s}{2}\right) = r(\Psi_s^n) - \left(\tfrac{t-s}{2}\right),$$

Therefore, $|I| \leq r(\Psi_s^n) - \left(\frac{t-s}{2}\right)$. Therefore, as $T^{\frac{t-s}{2}}(\Psi_s^n)$ is the $\left(\frac{t-s}{2}\right)$-th truncation of Ψ_s^n, it follows by Lemma 2.2.1 that I is independent in $T^{\frac{t-s}{2}}(\Psi_s^n)$. In particular, under the identity map, M is a weak-map image of $T^{\frac{t-s}{2}}(\Psi_s^n)$. This completes the proof of Theorem 1.4.3. □

2.6 Counterexample

In this section, we give a counterexample to a conjecture posed in [4]. Let s be an integer exceeding two, and let M be an (s,s)-cyclic matroid such that $|E(M)| \geq 2s+2$. A matroid N is an *inflation* of M if N can be obtained from M by first taking an elementary quotient in which none of the s-element cocircuits corresponding to consecutive elements in the cyclic ordering are preserved, which produces an $(s, s+2)$-cyclic matroid, and then taking an elementary lift in which none of the s-element circuits corresponding to consecutive elements in the cyclic ordering are preserved. The resulting matroid N is $(s+2, s+2)$-cyclic. The conjecture in [4, Conjecture 6.1] is the following:

Conjecture 2.6.1. *Let s be an integer exceeding two, and let M be an (s,s)-cyclic matroid.*

(i) *If s is even, then M can be obtained from a spike or a swirl by a sequence of inflations.*

(ii) *If s is odd, then M can be obtained from a wheel or a whirl by a sequence of inflations.*

Now consider the matroid Ψ_s^n, where $s \geq 5$. If Ψ_s^n can be obtained from a spike, swirl, wheel, or whirl by a sequence of inflations, then Ψ_s^n is an elementary lift of some $(s-2, s)$-cyclic matroid, or, equivalently, using Lemma 2.5.5, $(\Psi_s^n)^* \cong \Psi_s^n$ is the elementary quotient of some $(s, s-2)$-cyclic matroid. We shall establish a counterexample to Conjecture 2.6.1 by showing that no such $(s, s-2)$-cyclic matroid exists; in fact, we prove a more general result.

Theorem 1.4.2. *Let $s \geq 3$, and let $n \geq 4s-8$ be even. Let M be a matroid on n elements with cyclic ordering $\sigma = (e_1, e_2, \dots, e_n)$ such that, for all odd $i \in \{1, 2, \dots, n\}$, the set $\{e_i, e_{i+1}, \dots, e_{i+s-1}\}$ is an s-element circuit. Then Ψ_s^n is not a quotient of M.*

For the remainder of this section, let M' be a rank-$(\frac{n}{2}+1)$ matroid in which there is a cyclic ordering $\sigma = (e_1, e_2, \dots, e_n)$ of its ground set such that $\{e_i, e_{i+1}, \dots, e_{i+s-1}\}$ is a circuit of M for all odd $i \in [n]$. Further assume that σ is also an (s,s)-cyclic ordering of Ψ_s^n such that $\{e_i, e_{i+1}, \dots, e_{i+s-1}\}$ is a circuit of Ψ_s^n for all odd $i \in [n]$. The following results show that Ψ_s^n is not a quotient of M'. For the next lemma see, for example, [21, Proposition 7.3.6].

Lemma 2.6.2. *Let M_1 and M_2 be matroids on the same ground set. Then M_2 is a quotient of M_1 if and only if every circuit of M_1 is a union of circuits of M_2.*

Key to the counterexample shall be the following sets. Let M be a matroid on n elements and let s be an integer exceeding three. Suppose that $\sigma = (e_1, e_2, \dots, e_n)$ is a cyclic ordering of $E(M)$ such that, for all odd $i \in [n]$, the set $\{e_i, e_{i+1}, \dots, e_{i+s-1}\}$ is an s-element circuit of M. For all odd $i \in [n]$, and for all integers k and ℓ such that $2 \leq k, \ell \leq s-1$ and $s-1 \leq k+\ell \leq 2s-4$, define

$$C_{i,k,\ell} = \sigma(i, i+k-1) \cup \sigma(i+2k+\ell-s+2, i+2k+2\ell-s+1).$$

Informally, starting at e_i, there are k consecutive elements of σ in $C_{i,k,\ell}$, followed by $k+\ell-(s-2)$ consecutive elements of σ not in $C_{i,k,\ell}$, followed by ℓ consecutive elements of σ in $C_{i,k,\ell}$.

The next lemma establishes that certain subsets of the ground set of Ψ_s^n containing $C_{i,k,\ell}$ are circuits of Ψ_s^n. The subsequent lemma shows that these subsets are also circuits of M'. We will eventually combine these two lemmas to show that Ψ_s^n is not a quotient of M'.

Lemma 2.6.3. *Suppose that $n \geq 4s-8$. Then, for all odd $i \in [n]$, and for all k and ℓ such that $2 \leq k, \ell \leq s-1$ and $s-1 \leq k+\ell \leq 2s-4$, the set $C_{i,k,\ell} \cup \{x\}$ is a circuit of Ψ_s^n, where $x \in \sigma(i+k, i+2k+\ell-s+1)$, and*

$$x \neq \begin{cases} e_{i+k} & \text{if } k = s-1; \\ e_{i+2k+\ell-s+1} & \text{if } \ell = s-1. \end{cases}$$

Proof. Recall the bipartite graph G_s^n whose vertex parts are $E = \{e_1, e_2, \dots, e_n\}$ and $\{1, 2, \dots, \frac{n}{2}\}$ and, for all $i \in \{1, 2, \dots, \frac{n}{2}\}$, the set of neighbours of i is

$$N(i) = \{e_{2i-1}, e_{2i}, \dots, e_{2i+s-2}\},$$

where subscripts are interpreted modulo n. Let $i_0 = \frac{i+1}{2}$ and $j_0 = \frac{i+2(k+\ell-s)+3}{2}$. Observe that

$$N(i_0) = \{e_i, e_{i+1}, \dots, e_{i+k-1}, \dots, e_{i+s-1}\}$$

and

$$N(j_0) = \{e_{i+2(k+\ell-s)+2}, e_{i+2(k+\ell-s)+3}, \dots, e_{i+2k+\ell-s+2}, \dots, e_{i+2(k+\ell)-s+1}\}.$$

In particular, $N(i_0) \cup N(j_0)$ contains $C_{i,k,\ell}$. Also recall that Ψ_s^n is the dual of the transversal matroid on E in which

$$(N(1), N(2), \dots, N(\tfrac{n}{2}))$$

is a presentation.

We first show that $C_{i,k,\ell} \cup \{x\}$ is dependent in Ψ_s^n by showing that $E - (C_{i,k,\ell} \cup \{x\})$ is not cospanning in Ψ_s^n. Consider G_s^n and the subset $[i_0, j_0]$ of $[\frac{n}{2}]$. Since $n \geq 4s-8$, we have that $N([i_0, j_0]) \neq E$, and so

$$|N([i_0, j_0])| = 2k + 2\ell - s + 2.$$

Therefore, as $C_{i,k,\ell} \cup \{x\} \subseteq N([i_0, j_0])$ and $|C_{i,k,\ell} \cup \{x\}| = k + \ell + 1$, it follows that

$$\begin{aligned}
&\Big| N([i_0, j_0]) - (C_{i,k,\ell} \cup \{x\}) \Big| \\
&\qquad = |N([i_0, j_0])| - |C_{i,k,\ell} \cup \{x\}| \\
&\qquad = (2k + 2\ell - s + 2) - (k + \ell + 1) \\
&\qquad = k + \ell - s + 1 \\
&\qquad < k + \ell - s + 2 \\
&\qquad = |[i_0, j_0]| .
\end{aligned}$$

Hence, by Hall's Theorem [14], $E - (C_{i,k,\ell} \cup \{x\})$ is not cospanning in Ψ_s^n. Thus $C_{i,k,\ell} \cup \{x\}$ is dependent in Ψ_s^n.

Since $C_{i,k,\ell} \cup \{x\}$ is dependent, $C_{i,k,\ell} \cup \{x\}$ contains a circuit C of Ψ_s^n. If $|C| = |E| - \frac{n}{2} + 1 = \frac{n}{2} + 1$, then, as $n \geq 4s - 8$, it follows that $|C| \geq 2s - 3$. Furthermore, $|C| \leq |C_{i,k,\ell} \cup \{x\}| = k + \ell + 1 \leq 2s - 3$. Thus $C = C_{i,k,\ell} \cup \{x\}$, and so $C_{i,k,\ell} \cup \{x\}$ is a circuit of Ψ_s^n. Therefore, by Lemma 2.5.3, we may assume that there are $i_1, j_1 \in [\frac{n}{2}]$ satisfying (i)–(v) of that lemma. If $i_1 = j_1$, then, by Lemma 2.5.3(ii), $C = N(i_1)$ for some $i_1 \in [\frac{n}{2}]$. Now, C, and thus $C_{i,k,\ell} \cup \{x\}$, contains s consecutive elements of σ. But if $C_{i,k,\ell} \cup \{x\}$ contains s consecutive elements, then $k + \ell = s - 1$, in which case $C_{i,k,\ell} \cup \{x\}$ is a circuit, and we are done. Therefore $i_1 \neq j_1$, and, by Lemma 2.5.3(iii) and (iv),

$$N([i_1, j_1]) - N([i_1 + 1, j_1]) = \{e_{2i_1 - 1}, e_{2i_1}\} \subseteq C \tag{2.1}$$

and

$$N([i_1, j_1]) - N([i_1, j_1 - 1]) = \{e_{2j_1 + s - 3}, e_{2j_1 + s - 2}\} \subseteq C. \tag{2.2}$$

Suppose $e_{2i_1 - 1} \notin \sigma(i, i + k - 1)$. Then, by (2.1),

$$\begin{aligned}
C &\subseteq (C_{i,k,\ell} \cup \{x\}) - \sigma(i, i + k - 1) \\
&= \sigma(i + 2k + \ell - s + 2, i + 2k + 2\ell - s + 1) \cup \{x\}.
\end{aligned}$$

However, since $i_1 \neq j_1$, we have that $|C| \geq s + 1$, while

$$|\sigma(i + 2k + \ell - s + 2, i + 2k + 2\ell - s + 1) \cup \{x\}| = \ell + 1 \leq s.$$

This contradiction implies that $e_{2i_1 - 1} \in \sigma(i, i + k - 1)$. Symmetrically,

$$e_{2j_1 + s - 2} \in \sigma(i + 2k + \ell - s + 2, i + 2k + 2\ell - s + 1)$$

and so $j_1 = j_0 - j'$ for some $0 \leq j' \leq \lceil \frac{\ell}{2} \rceil$.

By Lemma 2.5.3(ii), C is a subset of $N([i_1, j_1])$ containing $|N([i_1, j_1])| - |[i_1, j_1]| + 1$ elements. Now,

$$|N([i_1, j_1])| = |N([i_0, j_0])| - 2(i' + j') = 2k + 2\ell - s + 2 - 2(i' + j'),$$

and

$$|[i_1, j_1]| = |[i_0, j_0]| - (i' + j') = k + \ell - s + 2 - (i' + j')$$

so

$$|C| = k + \ell + 1 - (i' + j'). \tag{2.3}$$

On the other hand,

$$|C| \leq |(C_{i,k,\ell} \cup \{x\}) \cap N([i_1, j_1])| = k + \ell + 1 - 2(i' + j'). \tag{2.4}$$

Therefore, since both (2.3) and (2.4) hold, we have that $i' = j' = 0$, that is, $i_0 = i_1$ and $j_0 = j_1$, and that $|C| = |C_{i,k,\ell} \cup \{x\}|$. Hence, $C = C_{i,k,\ell} \cup \{x\}$, completing the proof of the lemma. □

Lemma 2.6.4. *Let $n \geq 4s - 8$, and suppose that Ψ_s^n is a quotient of M'. Then, for all odd $i \in [n]$, and for all k and ℓ such that $2 \leq k, \ell \leq s - 1$ and $s - 1 \leq k + \ell \leq 2s - 4$, the set $C_{i,k,\ell} \cup \{x\}$ is a circuit of M', where $x \in \sigma(i + k, i + 2k + \ell - s + 1)$, and*

$$x \neq \begin{cases} e_{i+k} & \text{if } k = s - 1; \\ e_{i+2k+\ell-s+1} & \text{if } \ell = s - 1. \end{cases}$$

Proof. Since Ψ_s^n is a quotient of M', it follows by Lemma 2.6.2 that every circuit of M' is a union of circuits of Ψ_s^n. Now, by Lemma 2.6.3, $C_{i,k,\ell} \cup \{x\}$ is a circuit of Ψ_s^n. Therefore, to prove the lemma, it suffices to show that M' has a circuit contained in $C_{i,k,\ell} \cup \{x\}$. The proof is by induction on $k + \ell$.

If $k + \ell = s - 1$, then

$$C_{i,k,\ell} = \sigma(i, i + k - 1) \cup \sigma(i + k + 1, i + s - 1).$$

Therefore, $x = e_{i+k}$, and $C_{i,k,\ell} \cup \{x\} = \sigma(i, i + s - 1)$ which is a circuit of M'. Furthermore, if $k + \ell = s$, then

$$C_{i,k,\ell} = \sigma(i, i + k - 1) \cup \sigma(i + k + 2, i + s + 1),$$

so $C_{i,k,\ell} \cup \{x\} = \sigma(i, i + s + 1) - \{y\}$, where y is the element of $\{e_{i+k}, e_{i+k+1}\}$ which is not equal to x. Since $y \in \sigma(i, i + s - 1) \cap \sigma(i + 2, i + s + 1)$, it follows by circuit elimination that M' has a circuit contained in $C_{i,k,\ell} \cup \{x\}$, as desired.

Now suppose that the lemma holds for all $2 \leq k', \ell' \leq s-1$ and $s-1 \leq k'+\ell' \leq 2s-4$ such that $k'+\ell' = k+\ell-1$. First assume that either k or ℓ is equal to $s-1$. If $k = s-1$, then $x \neq e_{i+s-1}$. Therefore, by the induction assumption,

$$C_{i+2,k-1,\ell} \cup \{x\} = \sigma(i+2, i+s-1) \cup \{x\} \cup \sigma(i+\ell+s, i+2\ell+s-1)$$

is a circuit of M'. Thus, by circuit elimination between $C_{i+2,k-1,\ell} \cup \{x\}$ and $\sigma(i, i+s-1)$ on e_{i+s-1}, the matroid M' has a circuit contained in

$$\begin{aligned} \sigma(i, i+s-2) \cup \{x\} \cup \sigma(i+\ell+s, i+2\ell+s-1) &= C_{i,s-1,\ell} \cup \{x\} \\ &= C_{i,k,\ell} \cup \{x\} \end{aligned}$$

as desired. A similar argument shows the lemma holds if $\ell = s-1$.

We may now assume that neither k nor ℓ is equal to $s-1$. Furthermore, since $k+\ell \geq s+1$, we have that $k \neq 2$ and $\ell \neq 2$. Assume $k = \ell = 3$. This implies that $s = 5$, so

$$C_{i,k,\ell} = C_{i,3,3} = \{e_i, e_{i+1}, e_{i+2}, e_{i+6}, e_{i+7}, e_{i+8}\}.$$

By the induction assumption, if $x \in \{e_{i+4}, e_{i+5}\}$, then the desired result follows from circuit elimination between

$$C_{i,3,2} \cup \{e_{i+4}\} = \{e_i, e_{i+1}, e_{i+2}, e_{i+4}, e_{i+5}, e_{i+6}\}$$

and $\{e_{i+4}, e_{i+5}, e_{i+6}, e_{i+7}, e_{i+8}\}$. If $x = e_{i+3}$, then the result follows from circuit elimination between

$$C_{i+2,2,3} \cup \{e_{i+4}\} = \{e_{i+2}, e_{i+3}, e_{i+4}, e_{i+6}, e_{i+7}, e_{i+8}\}$$

and $\{e_i, e_{i+1}, e_{i+2}, e_{i+3}, e_{i+4}\}$.

Lastly, assume that either $k \geq 4$ or $\ell \geq 4$, which implies $s \geq 6$. We establish that the lemma holds when $k \geq 4$. The proof of the lemma when $\ell \geq 4$ is similar and omitted. Assume $k \geq 4$. Suppose $x \neq e_{i+2k+\ell-s+1}$, that is $x \in \sigma(i+k, i+2k+\ell-s)$. Then, by the induction assumption, the set

$$C_{i,k,\ell-1} \cup \{x\} = \sigma(i, i+k-1) \cup \{x\} \cup \sigma(i+2k+\ell-s+1, i+2k+2\ell-s-1)$$

is a circuit. If $\ell = s-2$ and $x = e_{i+2k+\ell-s}$, then the set

$$\sigma(i+2k+\ell-s, i+2k+2\ell-s+1) = \sigma(i+2k-2, i+2k+s-3)$$

is an s-element circuit of M'. Hence, circuit elimination between this circuit and $C_{i,k,\ell-1} \cup \{x\}$ on the element $e_{i+2k+\ell-s+1}$ gives a ciruit of M' contained in

$$\sigma(i, i+k-1) \cup \{e_{i+2k+\ell-s}\} \cup \sigma(i+2k+\ell-s+2, i+2k+2\ell-s+1) = C_{i,k,\ell} \cup \{x\}$$

as desired. Otherwise, since $k \geq 4$, the set

$$C_{i+2,k-2,\ell+1} \cup \{x\} = \sigma(i+2, i+k-1) \cup \{x\} \cup \sigma(i+2k+\ell-s+1, i+2k+2\ell-s+1)$$

is a circuit. Again, circuit elimination between this circuit and $C_{i,k,\ell-1} \cup \{x\}$ on the element $e_{i+2k+\ell-s+1}$ implies that M' has a circuit contained in

$$\sigma(i, i+k-1) \cup \{x\} \cup \sigma(i+2k+\ell-s+2, i+2k+2\ell-s+1) = C_{i,k,\ell} \cup \{x\}$$

as desired.

The final case to consider is when $x = e_{i+2k+\ell-s+1}$. By the induction assumption, and since $k \neq s-1$, the set

$$C_{i,k,\ell-1} \cup \{e_{i+k}\} = \sigma(i, i+k-1) \cup \{e_{i+k}\} \cup \sigma(i+2k+\ell-s+1, i+2k+2\ell-s-1)$$

is a circuit of M'. Additionally, since $k \geq 4$, the set

$$C_{i+2,k-2,\ell+1} \cup \{e_{i+k}\} = \sigma(i+2, i+k-1) \cup \{e_{i+k}\} \cup \sigma(i+2k+\ell-s+1, i+2k+2\ell-s+1)$$

is a circuit of M'. Circuit elimination between these circuits on the element e_{i+k} implies that M' has a circuit contained in

$$\sigma(i, i+k-1) \cup \sigma(i+2k+\ell-s+1, i+2k+2\ell-s+1) = C_{i,k,\ell} \cup \{e_{i+2k+\ell-s+1}\}.$$

This completes the proof of the case when $k \geq 4$, and thus completes the proof of the lemma. □

Proposition 2.6.5. *Let $n \geq 4s-8$, where s is an integer exceeding three. Then Ψ_s^n is not a quotient of M'.*

Proof. Suppose Ψ_s^n is a quotient of M'. We establish a contradiction by showing that $r(M') \leq \frac{n}{2}$. By definition of M', $\{e_1, e_2, \ldots, e_s\}$ is a circuit with rank $s-1$. The element e_{s+1} may or may not be in the closure of $\{e_1, e_2, \ldots, e_s\}$, so $r(\{e_1, e_2, \ldots, e_{s+1}\}) \leq s$. Since $\{e_3, e_4, \ldots, e_{s+2}\}$ is a circuit, $e_{s+2} \in \mathrm{cl}(\{e_1, e_2, \ldots, e_{s+1}\})$, that is, $r(\{e_1, e_2, \ldots, e_{s+2}\}) \leq s$. Repeating this process, we see that $r(\{e_1, e_2, \ldots, e_{s+2u}\}) \leq s-1+u$ for all $u \leq \frac{n-s}{2}$. In particular, when $u = \frac{n}{2} - s + 1$, we have that $r(\{e_1, e_2, \ldots, e_{n-s+2}\}) \leq \frac{n}{2}$. However, by Lemma 2.6.4 with $i = n-2s+5$ and $k = \ell = s-2$, the set

$$\{e_{n-2s+5}, e_{n-2s+6}, \ldots, e_{n-s+2}\} \cup \{x\} \cup \{e_1, e_2, e_3, \ldots, e_{s-2}\}$$

is a circuit for all $x \in \{e_{n-s+3}, e_{n-s+4}, \ldots, e_{n-1}, e_n\}$, and so $\{e_1, e_2, \ldots, e_{n-s+2}\}$ is spanning. This implies $r(M') \leq \frac{n}{2}$, a contradiction. □

chapter 3

Detachable Pairs

3.1 Introduction

This chapter is concerned with finding a generalisation of Tutte's Wheels-and-Whirls Theorem in which, rather than single elements, 2-element subsets are deleted or contracted. In particular, for a 3-connected matroid M and a pair $\{x, y\} \subseteq E(M)$, then we say $\{x, y\}$ is a *detachable pair* if either $M\backslash x\backslash y$ is 3-connected or $M/x/y$ is 3-connected. We prove Theorem 1.6.2, which describes precisely the 3-connected matroids which have no detachable pairs.

This builds on a result of Williams [34].

Theorem 1.6.1. *Let M be a 3-connected matroid with $|E(M)| \geq 13$. Then either*

(i) *M has a detachable pair,*

(ii) *there exists a matroid M' such that M' can be constructed by performing a single Δ-Y or Y-Δ exchange on M and M' has a detachable pair, or*

(iii) *M is a spike.*

It follows from Theorem 1.6.1 that if M is a 3-connected matroid with no detachable pairs and no 3-element circuits or cocircuits, then M is a spike. It remains to consider 3-connected matroids with at least one 3-element circuit or cocircuit. The organisation of this chapter is as follows. Section 3.2 contains some preliminaries. In Section 3.3, we formally define the 3-connected matroids with no detachable pairs. The remainder of this chapter is the proof of Theorem 1.6.2, the structure of which is described at the end of Section 3.3.

3.2 Preliminaries

3.2.1 Connectivity

Let M be a matroid with ground set E. Let $X, Y \subseteq E$. The *local connectivity* of subsets $X, Y \subseteq E(M)$ is defined as

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

The *connectivity* of X in M is defined

$$\lambda_M(X) = \sqcap(X, E - X) = r_M(X) + r_M(E - X) - r(M)$$

or, equivalently,

$$\lambda_M(X) = r_M(X) + r^*_M(X) - |X|.$$

It follows from the above definitions that $\lambda_M(X) = \lambda_M(E - X)$ and $\lambda_{M^*}(X) = \lambda_M(X)$. When it is apparent which matroid we are referring to, we will often write $\lambda(X) = \lambda_M(X)$.

The next two lemmas follow straightforwardly from the definition (see, for example, [21, Corollary 8.2.6, Proposition 8.2.14]). They will be applied frequently throughout the proof of Theorem 1.6.2, often without statement of their application.

Lemma 3.2.1. *Let M be a matroid, and let $X \subseteq E(M)$ and $e \in E(M) - X$. Then*

$$\lambda_{M/e}(X) = \begin{cases} \lambda_M(X) - 1 & \textit{if } e \in \mathrm{cl}(X), \\ \lambda_M(X) & \textit{if } e \notin \mathrm{cl}(X). \end{cases}$$

Dually,

$$\lambda_{M\backslash e}(X) = \begin{cases} \lambda_M(X) - 1 & \textit{if } e \in \mathrm{cl}^*(X), \\ \lambda_M(X) & \textit{if } e \notin \mathrm{cl}^*(X). \end{cases}$$

Lemma 3.2.2. *Let $X \subseteq E(M)$ and let $e \in E(M) - X$. Then*

$$\lambda(X \cup \{e\}) = \begin{cases} \lambda(X) - 1 & \textit{if } e \in \mathrm{cl}(X) \textit{ and } e \in \mathrm{cl}^*(X), \\ \lambda(X) & \textit{if } e \in \mathrm{cl}(X) \textit{ and } e \notin \mathrm{cl}^*(X), \\ \lambda(X) & \textit{if } e \notin \mathrm{cl}(X) \textit{ and } e \in \mathrm{cl}^*(X), \\ \lambda(X) + 1 & \textit{if } e \notin \mathrm{cl}(X) \textit{ and } e \notin \mathrm{cl}^*(X). \end{cases}$$

We say that X is a *k-separation* if $\lambda(X) = k - 1$ and $|X| \geq k$ and $|E(M) - X| \geq k$. A matroid is k-connected if it contains no k'-separations, for all $k' < k$.

Small 3-separations will be important for this work. If M is 3-connected, then a 3-separation of M with 3 elements is either a 3-element circuit, called a *triangle*, or a 3-element cocircuit, called a *triad*. Contracting an element in a triangle produces a 2-element circuit, and thus the resulting matroid is not 3-connected. Dually, the matroid produced by deleting an element from a triad is not 3-connected. A 3-separation of M with 4 elements either contains a triangle or a triad, or it is a *quad*, which is a 4-element set that is both a circuit and a cocircuit.

The next two well-known lemmas are useful for identifying elements which may be deleted or contracted while retaining 3-connectivity. The first, which is commonly referred to as Bixby's Lemma, states that if an element e is not contained in a triangle or a triad, then either $M\backslash e$ or M/e is 3-connected [1, Theorem 1].

Lemma 3.2.3. *Let M be a 3-connected matroid and let $e \in E(M)$. Then either $\mathrm{si}(M/e)$ is 3-connected or $\mathrm{co}(M\backslash e)$ is 3-connected.*

The next lemma is called Tutte's Triangle Lemma [32, 7.2].

Lemma 3.2.4. *Let M be a 3-connected matroid, and let T be a triangle of M. Let e, e' be distinct elements of T. If there is no triad of M containing e and e', then either $M\backslash e$ is 3-connected or $M\backslash e'$ is 3-connected.*

Applying Tutte's Triangle Lemma to M^* rather than M gives the following corollary, which we also refer to as Tutte's Triangle Lemma.

Corollary 3.2.5. *Let M be a 3-connected matroid, and let T^* be a triad of M. Let e, e' be distinct elements of T^*. If there is no triangle of M containing e and e', then either M/e is 3-connected or M/e' is 3-connected.*

There are two consequences of Tutte's Triangle Lemma which are particularly useful for us. Firstly, if T is a triangle of M which does not intersect a triad, then there are at least two elements of T which can be deleted while retaining 3-connectivity. Dually, a triad which does not intersect a triangle contains at least two elements which can be contracted while retaining 3-connectivity. Secondly, let $\{e_1, e_2, e_3, e_4\}$ be a set such that $\{e_1, e_2, e_3\}$ is a triangle, and $\{e_2, e_3, e_4\}$ is a triad. If e_1 is not contained in a triad, then $M\backslash e_1$ is 3-connected, and if e_4 is not contained in a triangle, then M/e_4 is 3-connected.

3.2.2 Fans

The set $\{e_1, e_2, e_3, e_4\}$ in the previous paragraph is an example of a fan. In general, a *fan* is a set F with ordering $(e_1, e_2, \ldots, e_{|F|})$ such that $\{e_1, e_2, e_3\}$ is either a triangle or a triad, and, for all $i \in \{1, 2, \ldots, |F| - 3\}$, if $\{e_i, e_{i+1}, e_{i+2}\}$ is a triangle then $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a triad, and if $\{e_i, e_{i+1}, e_{i+2}\}$ is a triad, then $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a triangle. Differing from [21], we shall also say that any set containing fewer than three elements is a fan.

The *length* of a fan is the number of elements it contains. Let F be a fan of length $k \geq 3$ and ordering $(e_1, e_2, \ldots, e_k)$. Note that if k is even, then either $\{e_1, e_2, e_3\}$ is a triangle and $\{e_{k-2}, e_{k-1}, e_k\}$ is a triad, or $\{e_1, e_2, e_3\}$ is a triad and $\{e_{k-2}, e_{k-1}, e_k\}$ is a triangle. Similarly, if k is odd, then $\{e_1, e_2, e_3\}$ and $\{e_{k-2}, e_{k-1}, e_k\}$ are either both triangles or both triads. We say that the elements e_1 and e_k are the *ends* of the fan F. Furthermore, F is *maximal* if there is no fan F' such that $F \subset F'$. Every fan is contained in a maximal fan.

If F is a fan with ordering $(e_1, e_2, \ldots, e_{|F|})$, then $(e_{|F|}, e_{|F|-1}, \ldots, e_1)$ is also an ordering of F satisfying the fan properties. When exploiting this symmetry, we use the phrase "up to reversing the ordering of F". For example, if C is a circuit containing one of e_1 and $e_{|F|}$, we might say "up to reversing the ordering of F, we may assume that $e_1 \in C$". If F has length at least five, then these are the only two orderings of F. However, this is not true for fans of length three and four. If F has length four then (e_1, e_2, e_3, e_4) and (e_1, e_3, e_2, e_4) are both orderings of F which satisfy the fan properties. This means that we may choose the order of the elements e_2 and e_3 arbitrarily "up to the ordering of F". Similarly, if F has length three, then the ordering of F is arbitrary. Although the same fan can have multiple orderings, it is often convenient to refer to a fan as an ordering and not as a set. For example, we might say "$(e_1, e_2, \ldots, e_{|F|})$ is a fan" as shorthand for "the set $\{e_1, e_2, \ldots, e_{|F|}\}$ is a fan with ordering $(e_1, e_2, \ldots, e_{|F|})$".

We now prove a number of basic lemmas concerning the structure of fans in 3-connected matroids. These results are known, but we provide proofs for completeness.

Lemma 3.2.6. *Let M be a 3-connected matroid, and let $F = (e_1, e_2, \ldots, e_{|F|})$ be a fan of M such that $F \geq 2$ and $|E(M)| \geq |F| + 2$. Then*

$$r(F) = \begin{cases} \left\lfloor \frac{|F|}{2} \right\rfloor + 1, & \textit{if } \{e_1, e_2, e_3\} \textit{ is a triangle;} \\ \left\lceil \frac{|F|}{2} \right\rceil + 1, & \textit{if } \{e_1, e_2, e_3\} \textit{ is a triad,} \end{cases}$$

$$r^*(F) = \begin{cases} \left\lceil \frac{|F|}{2} \right\rceil + 1, & \textit{if } \{e_1, e_2, e_3\} \textit{ is a triangle;} \\ \left\lfloor \frac{|F|}{2} \right\rfloor + 1, & \textit{if } \{e_1, e_2, e_3\} \textit{ is a triad, and} \end{cases}$$

$$\lambda(F) = 2$$

Proof. If $|F| = 2$, then $r(F) = r^*(F) = 2 = \frac{|F|}{2} + 1$ and $\lambda(F) = 2$, so the result holds. Suppose $|F| > 2$, and the result holds for all fans F' with $|F'| = |F| - 1$. In particular, the result holds for the fan $F - \{e_1\}$ which has ordering $(e_2, e_3, \ldots, e_{|F|})$. If $\{e_1, e_2, e_3\}$ is a triangle, then $\{e_2, e_3, e_4\}$ is a triad. Therefore,

$$r(F - \{e_1\}) = \left\lceil \frac{|F| - 1}{2} \right\rceil + 1 = \left\lfloor \frac{|F|}{2} \right\rfloor + 1$$

and

$$r^*(F - \{e_1\}) = \left\lfloor \frac{|F| - 1}{2} \right\rfloor + 1 = \left\lceil \frac{|F|}{2} \right\rceil.$$

Now, $e_1 \in \mathrm{cl}(F - \{e_1\})$, and thus

$$r(F) = r(F - \{e_1\}) = \left\lfloor \frac{|F|}{2} \right\rfloor + 1.$$

If $e_1 \in \mathrm{cl}^*(F - \{e_1\})$, then $\lambda(F) = \lambda(F - \{e_1\}) - 1 = 1$, by Lemma 3.2.2. This contradicts the 3-connectivity of M since $|E(M)| \geq |F| + 2$. Thus, $e_1 \notin \mathrm{cl}^*(F - \{e_1\})$, so $\lambda(F) = \lambda(F - \{e_1\}) = 2$, and

$$r^*(F) = r^*(F - \{e_1\}) + 1 = \left\lceil \frac{|F|}{2} \right\rceil + 1.$$

This completes the proof in the case where $\{e_1, e_2, e_3\}$ is a triangle. The argument for when $\{e_1, e_2, e_3\}$ is a triad is symmetrical and omitted. □

Lemma 3.2.7. *Let M be a 3-connected matroid, and let F be a fan of M such that $|F| \geq 4$. Then either M is a wheel or a whirl, or $|E(M)| \geq |F| + 2$.*

Proof. Suppose that M is not a wheel or a whirl. Let F^+ be a maximal fan of M containing F, and let $(e_1, e_2, \ldots, e_{|F^+|})$ be an ordering of F. We shall show that $|E(M)| \geq |F^+| + 2 \geq |F| + 2$.

Suppose $|E(M)| = |F^+| + 1$, that is, $E(M) = F^+ \cup \{x\}$ for some $x \notin F^+$. Now, $(e_3, e_4, \ldots, e_{|F^+|})$ is a fan, so $\lambda(\{e_3, e_4, \ldots, e_{|F^+|}\}) = 2$. This means that

$$\lambda(E(M) - \{e_3, e_4, \ldots, e_{|F^+|}\}) = \lambda(\{e_1, e_2, x\}) = 2.$$

Hence, $\{e_1, e_2, x\}$ is either a triangle or a triad. The sets $\{e_1, e_2, x\}$ and $\{e_2, e_3, e_4\}$ intersect in one element. Thus, by orthogonality, if $\{e_2, e_3, e_4\}$ is a triangle then $\{e_1, e_2, x\}$ is a triad, and if $\{e_2, e_3, e_4\}$ is a triad then $\{e_1, e_2, x\}$ is a triangle. But this implies $(x, e_1, e_2, \ldots, e_{|F^+|})$ is a fan of M, contradicting the maximality of F^+.

Suppose $E(M) = F^+$. If $|F^+| = 4$, then F^+ contains a triad, and the complement of this triad is a hyperplane with one element. In other words, $r(M) = 2$, so $M \cong U_{2,4}$, which is the rank-2 whirl. Otherwise, $|F^+| \geq 5$. Then $(e_2, e_3, \ldots, e_{|F^+|-2})$ is a fan of length at least two, so $\lambda(\{e_2, e_3, \ldots, e_{|F^+|-2}\}) = 2$. This implies that $\lambda(\{e_{|F^+|-1}, e_{|F^+|}, e_1\}) = 2$. Similarly, $\lambda(\{e_3, e_4, \ldots, e_{|F^+|-1}\}) = 2$, so $\lambda(\{e_{|F^+|}, e_1, e_2\}) = 2$. Therefore, $\{e_{|F^+|-1}, e_{|F^+|}, e_1\}$ and $\{e_{|F^+|}, e_1, e_2\}$ are each a triangle or a triad, from which it follows that M is a wheel or a whirl. Thus, $|E(M)| \geq |F^+| + 2$, completing the proof. □

Lemma 3.2.8. *Let M be a 3-connected matroid, and let $F = (e_1, e_2, \ldots, e_{|F|})$ be a maximal fan of M such that $|F| \geq 3$. Then either M is a wheel or a whirl, or all of the following hold:*

(i) *if $\{e_1, e_2, e_3\}$ is a triad, then e_1 is not contained in a triangle,*

(ii) *if $\{e_1, e_2, e_3\}$ is a triangle, then e_1 is not contained in a triad,*

(iii) *if $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$ is a triad, then $e_{|F|}$ is not contained in a triangle, and*

(iv) *if $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$ is a triangle, then $e_{|F|}$ is not contained in a triad.*

Proof. We prove (i). Then (ii) follows by applying (i) to M^*, and (iii) and (iv) follow by reversing the ordering of the fan F. Suppose $\{e_1, e_2, e_3\}$ is a triad, and that e_1 is contained in some triangle T of M. By orthogonality, T contains a second element of the triad $\{e_1, e_2, e_3\}$. First, suppose $|T \cap F| = 2$. Then there exists $x \in T$ with $x \notin F$. If $T = \{e_1, e_2, x\}$, then $(x, e_1, e_2, \ldots, e_|F|)$ is a fan of M, which contradicts the maximality of F. On the other hand, suppose $T = \{e_1, e_3, x\}$. If $|F| \geq 5$, then T intersects the triad $\{e_3, e_4, e_5\}$ in one element, contradicting orthogonality. Therefore, $|F| \leq 4$. But now $F \cup \{x\}$ is a fan of M with ordering (x, e_1, e_3, e_2) if $|F| = 3$, or with ordering (x, e_1, e_3, e_2, e_4) if $|F| = 4$. This contradicts the maximality of F, and thus $T \subseteq F$.

Therefore, $e_1 \in \mathrm{cl}(F - \{e_1\})$ and $e_1 \in \mathrm{cl}^*(F - \{e_1\})$, so $\lambda(F) = \lambda(F - \{e_1\}) - 1 = 1$. Since M is 3-connected, this implies that $|E(M)| \leq |F| + 1$. Lemma 3.2.7 implies that M is a wheel or a whirl and completes the proof. □

A consequence of Lemma 3.2.8 is that, if $F = (e_1, e_2, \ldots, e_{|F|})$ is a maximal fan of a 3-connected matroid M such that $|F| \geq 4$, then either e_1 is contained in a triangle and not contained in a triad, or e_1 is contained in a triad and not contained in a triangle. In the former case, Tutte's Triangle Lemma implies that $M \backslash e_1$ is 3-connected, and in the latter case M/e_1 is 3-connected. Similarly, either $M \backslash e_{|F|}$ or $M/e_{|F|}$ is 3-connected.

Lemma 3.2.9. *Let M be a 3-connected matroid, and let $F = (e_1, e_2, \ldots, e_{|F|})$ be a maximal fan of M with length at least four. Then either M is a wheel or whirl, or, for all $i \in \{1, 2, \ldots, |F| - 1\}$, both of the followng hold:*

(i) *if $\{e_i, e_{i+1}\}$ is contained in a triangle T, then either $T = \{e_{i-1}, e_i, e_{i+1}\}$ or $T = \{e_i, e_{i+1}, e_{i+2}\}$, and*

(i) *if* $\{e_i, e_{i+1}\}$ *is contained in a triad* T^**, then either* $T^* = \{e_{i-1}, e_i, e_{i+1}\}$ *or* $T^* = \{e_i, e_{i+1}, e_{i+2}\}$.

Proof. Suppose M is not a wheel or a whirl. We prove (i), from which (ii) follows by applying (i) to M^*. First, suppose $i = 1$. Assume $\{e_1, e_2\}$ is contained in a triangle $T = \{e_1, e_2, x\}$ with $x \neq e_3$. Since e_1 is contained in a triangle, Lemma 3.2.8 implies that $\{e_1, e_2, e_3\}$ is a triangle. By orthogonality with the triad $\{e_2, e_3, e_4\}$, we have that $x = e_4$, so $\{e_1, e_2, e_4\}$ is a triangle. But now $\lambda(\{e_1, e_2, e_3, e_4\}) = 1$, and, since M is not a wheel or a whirl, $|E(M)| \geq |F| + 2$. This is a contradiction to the 3-connectivity of M. Hence, if $\{e_1, e_2\}$ is contained in a triangle, then this triangle is $\{e_1, e_2, e_3\}$. Similarly, by reversing the ordering of F, we see that if $\{e_{|F|-1}, e_{|F|}\}$ is contained in a triangle, then this triangle is $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$. Hence, the result holds for $i = 1$ and $i = |F| - 1$.

It remains prove the result for $i \in \{2, 3, \ldots, |F| - 2\}$. Suppose $T = \{e_i, e_{i+1}, x\}$ is a triangle with $x \notin \{e_{i-1}, e_{i+2}\}$. The set $\{e_i, e_{i+1}\}$ is contained in both a triangle and a triad of F — either $\{e_{i-1}, e_i, e_{i+1}\}$ is a triangle and $\{e_i, e_{i+1}, e_{i+2}\}$ is a triad, or $\{e_{i-1}, e_i, e_{i+1}\}$ is a triad and $\{e_i, e_{i+1}, e_{i+2}\}$ is a triangle. Up to reversing the ordering of the fan, we may assume the former. By circuit elimination with $\{e_{i-1}, e_i, e_{i+1}\}$, the set $\{e_{i-1}, e_i, x\}$ is a triangle of M. But $x \notin \{e_{i+1}, e_{i+2}\}$, so this contradicts orthogonality with the triad $\{e_i, e_{i+1}, e_{i+2}\}$ and completes the proof. □

Lemma 3.2.10. *Let* M *be a* 3*-connected matroid and let* $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ *and* $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ *be distinct maximal fans of* M *such that* $|F_1| \geq 4$ *and* $|F_2| \geq 3$*. Let* $e \in F_1 \cap F_2$*. Then* $e \in \{e_2, e_3, \ldots, e_{|F_1|-1}\}$ *if and only if* $|F_2| \geq 4$ *and* $e \in \{f_2, f_3, \ldots, f_{|F_2|-1}\}$.

Proof. First, assume $e \in \{e_2, e_3, \ldots, e_{|F_1|-1}\}$. Thus, e is contained in both a triangle and a triad of F_1. Noting that M has two distinct, maximal fans, and is therefore not a wheel or a whirl, Lemma 3.2.8 implies that $e \notin \{f_1, f_{|F_2|}\}$. Furthermore, if $|F_2| = 3$, then F_2 intersects both a triangle and a triad. This implies that F_2 is contained in a 4-element fan, contradicting the maximality of F_2. Thus, $|F_2| \geq 4$ and $e \in \{f_2, f_3, \ldots, f_{|F_2|-1}\}$, as desired. Conversely, if $|F_2| \geq 4$ and $e \in \{f_2, f_3, \ldots, f_{|F_2|-1}\}$, then e is contained in a triangle and a triad, so Lemma 3.2.8 implies that $e \notin \{e_1, e_{|F_1|}\}$ and completes the proof. □

The next lemma shows that two maximal fans intersect in only their end elements, unless they form a so-called $M(K_4)$*-separator*. An $M(K_4)$-separator of a matroid M, pictured in Figure 3.1, is a set $\{a, b, c, x, y, z\}$ such that $\{x, y, z\}$ is a triad, and $\{a, b, c\}$, $\{a, x, y\}$, $\{b, x, z\}$, and $\{c, y, z\}$ are all triangles.

Lemma 3.2.11. *Let* M *be a* 3*-connected matroid such that* $|E(M)| \geq 8$*. Let* $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ *and* $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ *be distinct maximal fans of* M *such that*

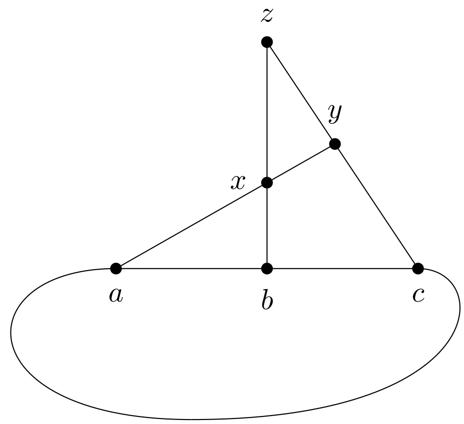


Figure 3.1: $M(K_4)$-separator.

$|F_1| \geq 4$ and $|F_2| \geq 3$. Then either $F_1 \cap F_2 \subseteq \{e_1, e_{|F_1|}\}$, or $F_1 \cup F_2$ is an $M(K_4)$-separator in either M or M^*.

Proof. Suppose $F_1 \cap F_2 \not\subseteq \{e_1, e_{|F_1|}\}$, that is, there exists $e_i \in F_1 \cap F_2$ such that $i \in \{2, 3, \ldots, |F_1| - 1\}$. Since F_1 and F_2 are distinct, F_1 has an element which is not an element of F_2. This means there exists such an i for which either $e_{i+1} \notin F_2$ or $e_{i-1} \notin F_2$. Up to reversing the ordering of F_1, we may assume that $e_{i-1} \notin F_2$. The set $\{e_{i-1}, e_i, e_{i+1}\}$ is either a triangle or a triad. Up to duality, we may assume that $\{e_{i-1}, e_i, e_{i+1}\}$ is a triangle. By Lemma 3.2.10, we have that $|F_2| \geq 4$ and $e_i \in \{f_2, f_3, \ldots, f_{|F_2|-1}\}$. Let $e_i = f_j$ with $j \in \{2, 3, \ldots, |F_2| - 1\}$. Now, e_i is contained in a triad of F_2, and this triad is either $\{f_{j-2}, f_{j-1}, e_i\}$ or $\{f_{j-1}, e_i, f_{j+1}\}$ or $\{e_i, f_{j+1}, f_{j+2}\}$.

First, suppose $\{f_{j-1}, e_i, f_{j+1}\}$ is a triad. Then, by orthogonality with the triangle $\{e_{i-1}, e_i, e_{i+1}\}$, and since $e_{i-1} \notin F_2$, we have that $e_{i+1} \in \{f_{j-1}, f_{j+1}\}$. Now, e_{i+1} is contained in both a triangle and a triad, which implies, by Lemma 3.2.8, that $e_{i+1} \notin \{f_1, f_{|F_2|}\}$. Therefore, if $e_{i+1} = f_{j-1}$, then $\{f_{j-2}, f_{j-1}, f_j\}$ is a triangle containing both e_i and e_{i+1}, and if $e_{i+1} = f_{j+1}$, then $\{f_j, f_{j+1}, f_{j+2}\}$ is a triangle containing both e_i and e_{i+1}. But since $e_{i-1} \notin F_2$, this triangle is distinct from the triangle $\{e_{i-1}, e_i, e_{i+1}\}$, contradicting Lemma 3.2.9.

Therefore, either $\{f_{j-2}, f_{j-1}, e_i\}$ or $\{e_i, f_{j+1}, f_{j+2}\}$ is a triad. Up to reversing the ordering of F_2, we may assume that $\{e_i, f_{j+1}, f_{j+2}\}$ is a triad. By orthogonality with $\{e_{i-1}, e_i, e_{i+1}\}$, this triad contains e_{i+1}. Suppose $e_{i+1} = f_{j+1}$. Since $e_i \notin \{f_1, f_{|F_2|}\}$, we have that $\{f_{j-1}, f_j, f_{j+1}\}$ is a triangle containing both e_i and e_{i+1}. This contradiction to Lemma 3.2.9 implies that $e_{i+1} = f_{j+2}$. Now, e_{i+1} is contained in both a triangle and a triad, so $e_{i+1} \notin \{f_1, f_{|F_2|}\}$. Therefore, M has triangles $\{f_{j-1}, e_i, f_{j+1}\}$ and $\{f_{j+1}, e_{i+1}, f_{j+3}\}$. Similarly, $e_{i+1} \notin \{e_1, e_{|F_1|}\}$, so M has a triad $\{e_i, e_{i+1}, e_{i+2}\}$. By orthogonality, $e_{i+2} = f_{j+1}$. Furthermore, e_{i+2} is contained in both a triangle and a triad, so $e_{i+2} \notin \{e_1, e_{|F_1|}\}$, which means $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ is a triangle. Now, $\{f_{j+1}, f_{j+2}, f_{j+3}\}$ is also a triangle containing $\{e_{i+1}, e_{i+2}\}$. Lemma 3.2.9 implies that these are the same triangle, so $e_{i+3} = f_{j+3}$.

We label these elements in the following way: $a = e_{i-1}$, $b = f_{j-1}$, $c = e_{i+3} = f_{j+3}$, $x = e_i = f_j$, $y = e_{i+1} = f_{j+2}$, $z = e_{i+2} = f_{j+1}$. Now, $\{x, y, z\}$ is a triad, and $\{a, x, y\}$, $\{b, x, z\}$, $\{c, y, z\}$ are all triangles. We show that $F_1 \cup F_2$ is an $M(K_4)$-separator in M by showing that none of a, b, or c are contained in triads, so that $F_1 \cup F_2 = \{a, b, c, x, y, z\}$, and that $\{a, b, c\}$ is a triangle.

First, assume that one of a, b, or c is contained in a triad T^*. Orthogonality with the triangles $\{a, x, y\}$, $\{b, x, z\}$, and $\{c, y, z\}$ implies that $T^* \subseteq \{a, b, c, x, y, z\}$. But now $\lambda(\{a, b, c, x, y, z\}) \leq 1$, a contradiction since $|E(M)| \geq 8$. Hence, no such triad exists, so $F_1 \cup F_2 = \{a, b, c, x, y, z\}$. Now, we show that $\{a, b, c\}$ is a triangle. By orthogonality with the triad $\{x, y, z\}$, we have that $x \notin \text{cl}(\{a, b, c\})$, so $r(\{a, b, c, x\}) = r(\{a, b, c\}) + 1$. Therefore, since $\{y, z\} \subseteq \text{cl}(\{a, b, c, x\})$, we have that $r(F_1 \cup F_2) = r(\{a, b, c\}) + 1$. But conversely, $\{a, b, c\} \subseteq \text{cl}(\{x, y, z\})$, so $r(F_1 \cup F_2) = r(\{x, y, z\}) = 3$. Therefore, $r(\{a, b, c\}) = 2$, and M has an $M(K_4)$-separator. □

3.2.3 Vertical and cyclic separations

Let M be a matroid. A *vertical k-separation* of M is a partition $(X, \{e\}, Y)$ of $E(M)$ such that $\lambda(X) = k-1$ and $\lambda(Y) = k-1$, and $e \in \text{cl}(X) \cap \text{cl}(Y)$, and $r(X) \geq k$ and $r(Y) \geq k$. A *cyclic k-separation* of M is a partition $(X, \{e\}, Y)$ such that $\lambda(X) = k-1$ and $\lambda(Y) = k-1$, and $e \in \text{cl}^*(X) \cap \text{cl}^*(Y)$, and $r^*(X) \geq k$ and $r^*(Y) \geq k$. Note that a vertical k-separation of M is a cyclic k-separation of M^*. The importance of vertical and cyclic 3-separations to this work is illustrated by the following lemma (see [25, Lemma 3.1]).

Lemma 3.2.12. *Let e be an element of a 3-connected matroid M. If $\text{si}(M/e)$ is not 3-connected, then there exists a vertical 3-separation $(X, \{e\}, Y)$ of M. Similarly, if $\text{co}(M \backslash e)$ is not 3-connected, then there exists a cyclic 3-separation $(X', \{e\}, Y')$ of M.*

The following lemmas about vertical and cyclic separations will be useful.

Lemma 3.2.13. *Let M be a 3-connected matroid, and let $(X, \{e\}, Y)$ be a vertical 3-separation of M. Let $y \in Y$. If $y \in \text{cl}(X)$, then $(X \cup \{y\}, \{e\}, Y - \{y\})$ is a vertical 3-separation of M. Furthermore, if $y \in \text{cl}^*(X)$ and e is not contained in a triangle of M, then $(X \cup \{y\}, \{e\}, Y - \{y\})$ is a vertical 3-separation of M.*

Proof. First, we suppose that either $y \in \mathrm{cl}(X)$ or $y \in \mathrm{cl}^*(X)$ and show that

$$\lambda(X \cup \{y\}) = \lambda(Y - \{y\}) = 2.$$

Since $y \in \mathrm{cl}(X) \cup \mathrm{cl}^*(X)$, we have that $\lambda(X \cup \{y\}) \leq 2$. Furthermore, $r(Y) \geq 3$, so $|Y - \{y\}| \geq 2$, and thus, since M is 3-connected, $\lambda(X \cup \{y\}) \geq 2$. Hence, $\lambda(X \cup \{y\}) = 2$. Now, $e \in \mathrm{cl}(X)$, which implies $\lambda(X \cup \{e, y\}) = 2$. Thus, $\lambda(E(M) - (X \cup \{e, y\})) = \lambda(Y - \{y\}) = 2$.

Next, we show that $e \in \mathrm{cl}(X \cup \{y\})$ and $e \in \mathrm{cl}(Y - \{y\})$. Since $e \in \mathrm{cl}(X)$, clearly $e \in \mathrm{cl}(X \cup \{y\})$. Now, $\lambda((Y - \{y\}) \cup \{e\}) = \lambda(X \cup \{y\}) = 2$, so either $e \in \mathrm{cl}(Y - \{y\})$ or $e \in \mathrm{cl}^*(Y - \{y\})$. Since $e \in \mathrm{cl}(X \cup \{y\})$, orthogonality implies that $e \in \mathrm{cl}(Y - \{y\})$.

To show that $(X \cup \{y\}, \{e\}, Y - \{y\})$ is a vertical 3-separation, it remains to show that $r(X \cup \{y\}) \geq 3$ and $r(Y - \{y\}) \geq 3$. Clearly, $r(X \cup \{y\}) \geq r(X) \geq 3$. To complete the proof, we show that, if $r(Y - \{y\}) = 2$, then $y \in \mathrm{cl}^*(X)$ and e is contained in a triangle of M. Since $r(Y) \geq 3$, then, if $r(Y - \{y\}) = 2$, we have that $y \notin \mathrm{cl}(Y - \{y\})$. But $\lambda(Y) = \lambda(Y - \{y\})$, so $y \in \mathrm{cl}^*(Y - \{y\})$. Therefore, by orthogonality, $y \notin \mathrm{cl}(X)$, so $y \in \mathrm{cl}^*(X)$. Now, $e \in \mathrm{cl}(Y - \{y\})$, so, since $r(Y - \{y\}) = 2$, we have that e is contained in a rank-2 circuit, that is, e is contained in a triangle. This completes the proof. □

Lemma 3.2.14. *Let M be a 3-connected matroid, and let F be a fan of M such that $|F| \geq 3$. Let $(X, \{e\}, Y)$ be a vertical 3-separation of M such that $e \notin F$ and e is not contained in a triangle. Then M has a vertical 3-separation $(X', \{e\}, Y')$ such that $F \subseteq X'$.*

Proof. Suppose F has ordering $(e_1, e_2, \ldots, e_{|F|})$. Consider the set $\{e_1, e_2, e_3\}$, which is a triangle or a triad. Either $|\{e_1, e_2, e_3\} \cap X| \geq 2$ or $|\{e_1, e_2, e_3\} \cap Y| \geq 2$. Without loss of generality, we may assume the former. Hence, if $\{e_1, e_2, e_3\}$ is a triangle, then $\{e_1, e_2, e_3\} \subseteq \mathrm{cl}(X)$, and if $\{e_1, e_2, e_3\}$ is a triad, then $\{e_1, e_2, e_3\} \subseteq \mathrm{cl}^*(X)$. Since e is not contained in a triangle, Lemma 3.2.13 implies that M has a vertical 3-separation $(X_3, \{e\}, Y_3)$, with $X_3 = X \cup \{e_1, e_2, e_3\}$ and $Y_3 = Y - \{e_1, e_2, e_3\}$. Now, $\{e_2, e_3, e_4\}$ is a triangle or a triad, so either $e_4 \in \mathrm{cl}(X_3)$ or $e_4 \in \mathrm{cl}^*(X_3)$. Thus, M has a vertical 3-separation $(X_4, \{e\}, Y_4)$, with $X_4 = X \cup \{e_1, e_2, e_3, e_4\}$ and $Y_4 = Y - \{e_1, e_2, e_3, e_4\}$. Repeating in this way, we see that M has a vertical 3-separation $(X \cup F, \{e\}, Y - F)$, completing the proof. □

Naturally, applying Lemmas 3.2.13 and 3.2.14 to M^* gives dual results concerning cyclic 3-separations.

Corollary 3.2.15. *Let M be a 3-connected matroid, and let $(X, \{e\}, Y)$ be a cyclic 3-separation of M. Let $y \in Y$. If $y \in \mathrm{cl}^*(X)$, then $(X \cup \{y\}, \{e\}, Y - \{y\})$ is a cyclic*

3-separation of M. Furthermore, if $y \in \mathrm{cl}(X)$ and e is not contained in a triad of M, then $(X \cup \{y\}, \{e\}, Y - \{y\})$ is a cyclic 3-separation of M.

Corollary 3.2.16. *Let M be a 3-connected matroid, and let F be a fan of M such that $|F| \geq 3$. Let $(X, \{e\}, Y)$ be a cyclic 3-separation of M such that $e \notin F$ and e is not contained in a triad. Then M has a cyclic 3-separation $(X', \{e\}, Y')$ such that $F \subseteq X'$.*

3.3 Matroids With No Detachable Pairs

In this section, we formally define the 3-connected matroids with no detachable pairs. Let M be a 3-connected matroid. Following [23], we say that $\Phi = (P_1, P_2, \ldots, P_m)$, with $m \geq 2$, is a *flower* in M if Φ is a partition of M such that, for all $i \in [m]$, we have that $|P_i| \geq 2$, and $\lambda(P_i) = 2$, and $\lambda(P_i \cup P_{i+1}) = 2$, where subscripts are interpreted modulo m. The sets P_i are called *petals* of Φ. The flower Φ is an *anemone* if, for all proper non-empty subsets J of $[m]$, we have that $\lambda(\bigcup_{j \in J} P_j) = 2$. Furthermore, the anemone Φ is

(i) a *paddle* if $\sqcap(P_i, P_j) = 2$ for all distinct $i, j \in [m]$, and

(ii) *spike-like* if $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in [m]$.

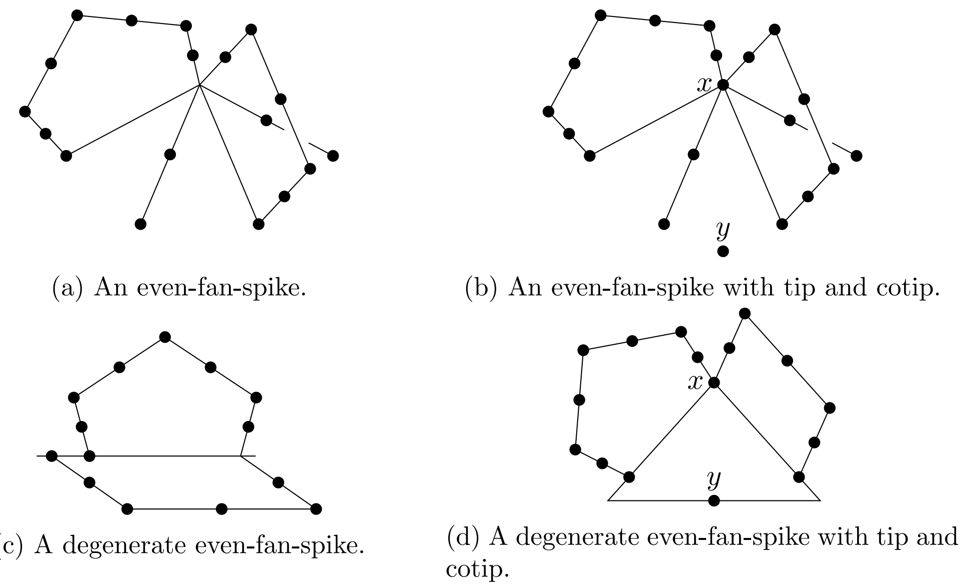
3.3.1 Spike-like anemones with no detachable pairs

We say that M is an *even-fan-spike* if there is a spike-like anemone $\Phi = (P_1, P_2, \ldots, P_m)$ in M such that $m \geq 3$ and P_i is a fan with even length at least two for every $i \in [m]$, and $\bigcap_{i \in [m]} \mathrm{cl}(P_i) = \bigcap_{i \in [m]} \mathrm{cl}^*(P_i) = \emptyset$. If $|P_i| = 2$ for each $i \in [m]$, then M is a (tipless) spike. An example of an even-fan-spike is show in Figure 3.2a.

We say that M is an *even-fan-spike with tip and cotip* if there is a spike-like anemone $\Phi = (P_1, P_2, \ldots, P_m)$ in M where $m \geq 3$ and there exists distinct elements $x, y \in E(M)$ such that, for all $i \in [m]$, the set $P_i \cup \{x, y\}$ is a fan with even length at least four and ends x and y, and $\bigcap_{i \in [m]} \mathrm{cl}(P_i) = \{x\}$ and $\bigcap_{i \in [m]} \mathrm{cl}^*(P_i) = \{y\}$. If $|P_i \cup \{x, y\}| = 4$ for all $i \in [m]$, then M is a spike with tip x and cotip y. An example of an even-fan-spike with tip and cotip is shown in Figure 3.2b.

We now consider the degenerate cases where the anemone Φ has two petals. We say that M is a *degenerate even-fan-spike* if $E(M) = P_1 \cup P_2$ such that $P_1 = (p^1_1, p^1_2, \ldots, p^1_{|P_1|})$ and $P_2 = (p^2_1, p^2_2, \ldots, p^2_{|P_2|})$ are disjoint fans with even length at least four such that

(i) $\{p^1_1, p^1_2, p^1_3\}$ and $\{p^2_1, p^2_2, p^2_3\}$ are triads,



(a) An even-fan-spike.

(b) An even-fan-spike with tip and cotip.

(c) A degenerate even-fan-spike.

(d) A degenerate even-fan-spike with tip and cotip.

Figure 3.2: Spike-like anemones with no detachable pairs.

(ii) $\{p^1_{|P_1|-2}, p^1_{|P_1|-1}, p^1_{|P_1|}\}$ and $\{p^2_{|P_2|-2}, p^2_{|P_2|-1}, p^2_{|P_2|}\}$ are triangles,

(iii) $\{p^1_1, p^1_2, p^2_1, p^2_2\}$ is a circuit, and

(iv) $\{p^1_{|P_1|-1}, p^1_{|P_1|}, p^2_{|P_2|-1}, p^2_{|P_2|}\}$ is a cocircuit.

Additionally, M is a *degenerate even-fan-spike with tip and cotip* if $E(M) = P_1 \cup P_2 \cup \{x, y\}$ such that $P_1 \cup \{x, y\}$ and $P_2 \cup \{x, y\}$ are each even fans of length at least four with ends x and y, and $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) = \{x\}$ and $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) = \{y\}$. A degenerate even-fan-spike and a degenerate even-fan-spike with tip and cotip are shown in Figures 3.2c and 3.2d.

3.3.2 Accordions

Let M be a 3-connected matroid, and let $F = (e_1, e_2, \ldots, e_{|F|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$ is a triangle and $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$ is a triad. Let $G \subseteq E(M) - F$.

We say that G is a *fan-type accordion end with F* if $G \cup \{e_1\}$ is a maximal fan of length 5 with ordering $(e_1, g_2, g_3, g_4, g_5)$ such that $\{e_1, g_2, g_3\}$ and $\{g_3, g_4, g_5\}$ are both triangles, and $\{e_1, e_2, g_3, g_5\}$ is a cocircuit, and $\sqcap(\{g_2, g_4\}, E(M) - (F \cup G)) = 1$. Dually, G is a *co-fan-type accordion end with F* if $G \cup \{e_{|F|}\}$ is a maximal fan of length 5 with ordering $(e_{|F|}, h_2, h_3, h_4, h_5)$ such that $\{e_{|F|}, h_2, h_3\}$ and $\{h_3, h_4, h_5\}$ are both triads, and

$\{e_{|F|-1}, e_{|F|}, h_3, h_5\}$ is a circuit, and $\sqcap(\{h_4, h_5\}, E(M) - (F \cup G)) = 1$.

Next, G is a *K_4-type accordion end with F* if $G = \{a_1, a_2, b_1, b_2\}$ such that $\{e_1, a_1, a_2\}$ and $\{e_1, b_1, b_2\}$ are triangles, and $\{e_1, e_2, a_1, b_1\}$ and $\{e_1, e_2, a_2, b_2\}$ are cocircuits, and $\sqcap(\{a_1, b_1\}, E(M) - (F \cup G)) = \sqcap(\{a_2, b_2\}, E(M) - (F \cup G)) = 1$. Dually, G is a *co-K_4-type accordion end with F* if $G = \{c_1, c_2, d_1, d_2\}$ such that $\{e_{|F|}, c_1, c_2\}$ and $\{e_{|F|}, d_1, d_2\}$ are triads, and $\{e_{|F|-1}, e_{|F|}, c_1, d_1\}$ and $\{e_{|F|-1}, e_{|F|}, c_2, d_2\}$ are circuits, and $\sqcap(\{c_1, c_2\}, E(M) - (F \cup G)) = \sqcap(\{d_1, d_2\}, E(M) - (F \cup G)) = 1$.

Finally, G is a *triangle-type accordion end with F* if $G \cup \{e_1\}$ is a triangle and $\sqcap(G, E(M) - (F \cup G)) = 1$, and G is a *triad-type accordion end with F* if $G \cup \{e_{|F|}\}$ is a triad, and $\sqcap(G, E(M) - (F \cup G)) = 1$.

The matroid M is an *accordion* if $E(M)$ has a partition (G, F, H) of $E(M)$ such that F is a maximal fan with even length at least four, and G is a fan-type, K_4-type, or triangle-type accordion end with F, and H is a co-fan-type, co-K_4-type, or triad-type accordion end with F. The nine types of accordion are illustrated in Figure 3.3.

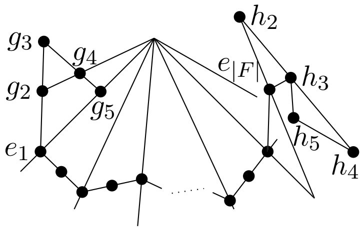
3.3.3 Paddles with no detachable pairs.

Suppose M has a paddle $\Phi = (P_1, P_2, \ldots, P_m)$, with $m \geq 3$, and an element $x \in E(M)$ such that, for all $i \in [m]$, the set $P_i \cup \{x\}$ is an even fan of length at least four with ordering $(p_1^i, p_2^i, \ldots, p_{|P_i|}^i, x)$. We say that M is an *even-fan-paddle* if $\bigcap_{i \in [m]} \mathrm{cl}(P_i) = \{x\}$ and $\bigcap_{i \in [m]} \mathrm{cl}^*(P_i) = \emptyset$, and, for all $i, j \in [m]$, the set $\{p_1^i, p_2^i, p_1^j, p_2^j\}$ is a circuit. An even-fan-paddle is shown in Figure 3.4a.

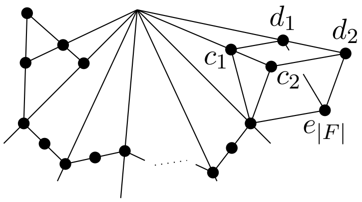
As with even-fan-spikes, we consider a degenerate case where Φ has two petals. The matroid M is a *degenerate even-fan-paddle* if $E(M) = P_1 \cup P_2 \cup \{x, y\}$ such that $P_1 \cup \{x\}$ is an even fan of length at least four with ordering $(p_1^1, p_2^1, \ldots, p_{|P_1|}^1, x)$, and $P_2 \cup \{x\}$ is an even fan of length at least four with ordering $(p_1^2, p_2^2, \ldots, p_{|P_2|}^2, x)$, and $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) = \{x, y\}$ and $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) = \emptyset$, and $\{p_1^1, p_2^1, p_1^2, p_2^2\}$ is a circuit. A degenerate even-fan-paddle is shown in Figure 3.4b.

The matroid $M(K_{3,m})$, shown in Figure 3.5a for when $m = 3$, has no detachable pairs for all $m \geq 3$. Note that $M(K_{3,m})$ has a paddle $(P_1, P_2, \ldots, P_m)$ such that P_i is a triad for all $i \in [m]$. In the remainder of this section, we define types of petals which may be attached to $M(K_{3,m})$ to produce a matroid with no detachable pairs (see Figure 3.5).

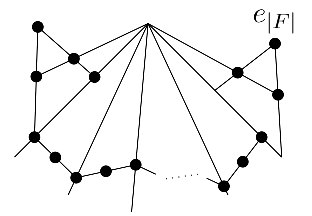
Let T^* be a triad of M, and let $X \subseteq E(M)$ disjoint from T^*. Then X is a *4-element-fan-petal relative to T^** if X is a fan of length four with ordering (e_1, e_2, e_3, e_4) such that



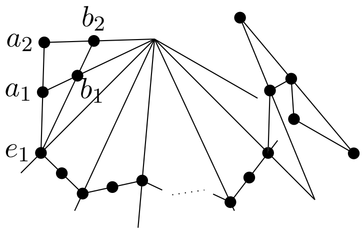
(a) An accordion with a fan-type and a co-fan-type accordion end.



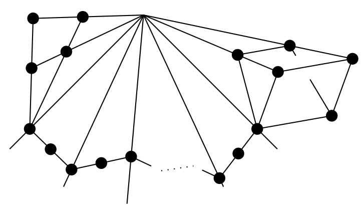
(b) An accordion with a fan-type and a co-K_4-type accordion end.



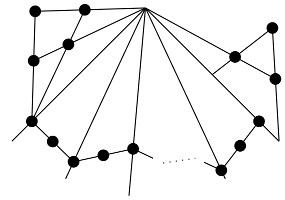
(c) An accordion with a fan-type and a triad-type accordion end.



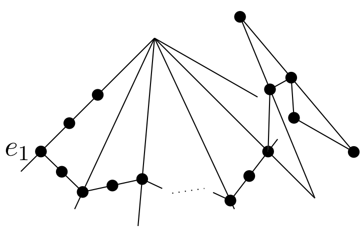
(d) An accordion with a K_4-type and a co-fan-type accordion end.



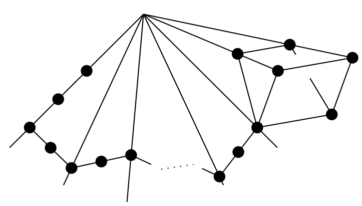
(e) An accordion with a K_4-type and a co-K_4-type accordion end.



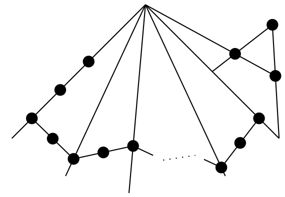
(f) An accordion with a K_4-type and a triad-type accordion end.



(g) An accordion with a triangle-type and a co-fan-type accordion end.

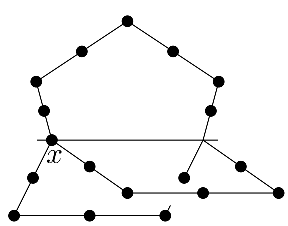


(h) An accordion with a triangle-type and a co-K_4-type accordion end.

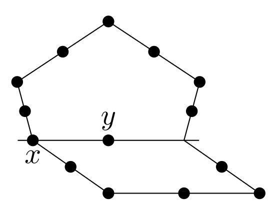


(i) An accordion with a triangle-type and a triad-type accordion end.

Figure 3.3: The nine types of accordion.

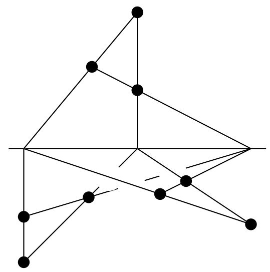


(a) An even-fan-paddle.

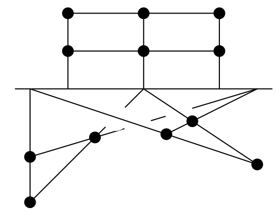


(b) A degenerate even-fan-paddle.

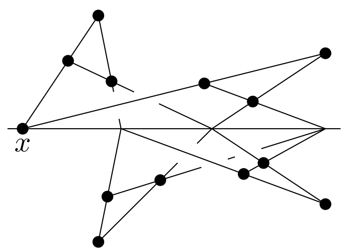
Figure 3.4: Examples of even-fan-paddles.



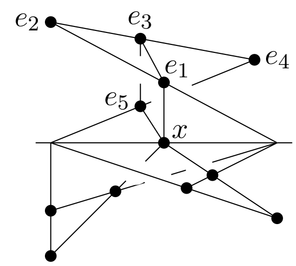
(a) $M(K_{3,3})$.



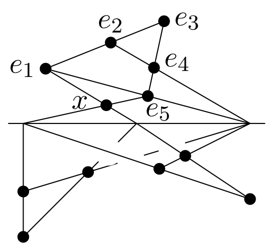
(b) $M(K_{3,2})$ with a $M^*(K_{3,2})$ petal.



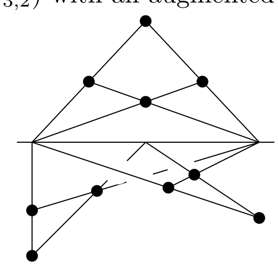
(c) $M(K_{3,2})$ with two 4-element-fan-petals.



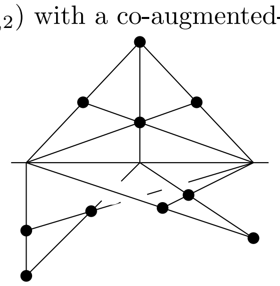
(d) $M(K_{3,2})$ with an augmented-fan-petal.



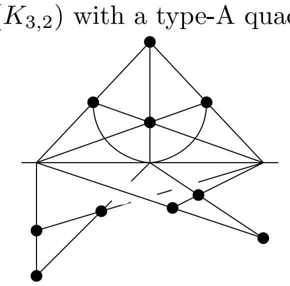
(e) $M(K_{3,2})$ with a co-augmented-fan-petal.



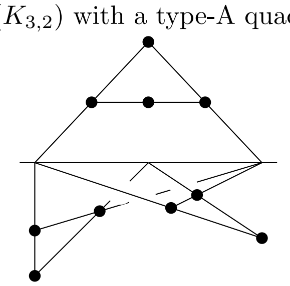
(f) $M(K_{3,2})$ with a type-A quad-petal.



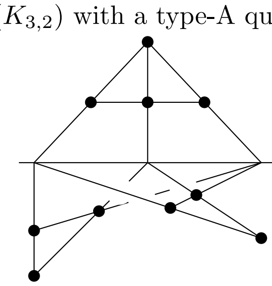
(g) $M(K_{3,2})$ with a type-A quad-petal.



(h) $M(K_{3,2})$ with a type-A quad-petal.



(i) $M(K_{3,2})$ with a type-B quad-petal.



(j) $M(K_{3,2})$ with a type-B quad-petal.

Figure 3.5: Examples of paddles with no detachable pairs.

$\{e_1, e_2, e_3\}$ is a triad, $\{e_2, e_3, e_4\}$ is a triangle, $e_4 \in \mathrm{cl}(T^*)$, and $\{e_1, e_2, e_3\} = \{x, x', x''\}$ such that $\sqcap(\{x, x'\}, T^*) = \sqcap(\{x, x''\}, T^*) = 1$.

Suppose M has a paddle $(P_1, P_2, \ldots, P_m)$, and there exists $x \in E(M)$ and $0 \leq t \leq m$ such that $M \backslash (\{x\} \cup \bigcup_{i=1}^{t} P_i) \cong M(K_{3,m-t})$ and, for all $i \in [m]$, the set $P_i - \{x\}$ is a triad and $x \in \mathrm{cl}(P_i - \{x\})$, and for all $j \in [t]$, distinct from i, the set $P_j \cup \{x\}$ is a 4-element-fan-petal relative to $P_i - \{x\}$. Then M has no detachable pairs. To illustrate this matroid, we note that it can be constructed as follows. Start with $U_{2,4}$ on ground set $\{x, y, z, w\}$. Repeatedly attach $M(K_4)$ along a three-element subset of $\{a, b, c, d\}$ by generalised parallel connection. Finally, delete y, z, and w.

Now, we suppose the matroid M has a paddle $(P_1, P_2, \ldots, P_m)$ such that $M \backslash P_1 \cong M(K_{3,m})$ and P_i is a triad for all $i \in \{2, 3, \ldots, m\}$. We find the possibilities for P_1 such that M has no detachable pairs. In most cases, the petal P_1 will have to attach to $P_2 \cup P_3 \cup \cdots \cup P_m$ in a certain way. However, if $M|P_1 \cong M^*(K_{3,m})$, then M has no detachable pairs regardless.

Let $X \subseteq E(M)$. The set X is an *augmented-fan-petal relative to* T^* if $X = F \cup \{x\}$ such that F is a fan of length five with ordering $(e_1, e_2, e_3, e_4, e_5)$ where $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triads, and $\{e_1, e_3, e_5, x\}$ is a circuit, and $T^* \cup \{x\}$ is a 4-element fan, and $\sqcap(\{e_1, e_2\}, T^*) = \sqcap(\{e_4, e_5\}, T^*) = 1$.

We say X is a *co-augmented-fan-petal relative to* T^* if $X = F \cup \{x\}$ such that F is a fan of length five with ordering $(e_1, e_2, e_3, e_4, e_5)$ where $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triangles, and $\{e_1, e_3, e_5, x\}$ is a cocircuit, and $\sqcap(\{e_1, x\}, T^*) = \sqcap(\{e_5, x\}, T^*) = \sqcap(\{e_2, e_4\}, T^*) = 1$.

We say X is a *type-A quad-petal relative to* T^* if $|X| = 4$, and X is a circuit and a cocircuit, and, for all $x \in X$, there exists distinct $x', x'' \in X - \{x\}$ such that $\sqcap(\{x, x'\}, T^*) = \sqcap(\{x, x''\}, T^*) = 1$. Finally, we say X is a *type-B quad-petal relative to* T^* if X is a cocircuit, and $X = T \cup \{x\}$ such that T is a triangle, and there exists distinct $x', x'' \in T$ such that $\sqcap(\{x, x'\}, T^*) = \sqcap(\{x, x''\}, T^*) = 1$. It is not difficult to verify that there are three different type-A quad-petals and two different type-B quad-petals, shown in Figure 3.5f-j.

If, for all $i \in \{2, 3, \ldots, m\}$, the petal P_1 is an augmented-fan-petal, a co-augmented-fan-petal, or a quad-petal (type-A or type-B) relative to P_i, then M has no detachable pairs.

3.3.4 Proof Strategy

We now prove that the matroids in this section and their duals are the only 3-connected matroids with no detachable pairs.

Theorem 1.6.2. *Let M be a 3-connected matroid with $|E(M)| \geq 13$. Then one of the* *following holds:*

(i) *M has a detachable pair,*

(ii) *M is a wheel or a whirl,*

(iii) *M is an accordion,*

(iv) *M is an even-fan-spike or a degenerate even-fan-spike,*

(v) *M is an even-fan-spike with tip and cotip or a degenerate even-fan-spike with tip and cotip,*

(vi) *M or M^* is a degenerate even-fan-paddle, or*

(vii) *M' has a paddle $(P_1, P_2, \ldots, P_m)$ for some $M' \in \{M, M^*\}$ and $m \geq 3$, and either*

  (a) *M' is an even-fan-paddle,*

  (b) *$M' \cong M(K_{3,m})$,*

  (c) *there exists $x \in E(M)$ and $0 \leq t \leq m$ such that $M' \backslash (\{x\} \cup \bigcup_{i=1}^{t} P_i) \cong M(K_{3,m-t})$ and, for all $i \in \{1, 2, \ldots, m\}$, the set $P_i - \{x\}$ is a triad and $x \in \mathrm{cl}(P_i - \{x\})$, and for all $j \in \{1, 2, \ldots, t\}$, distinct from i, the set $P_j \cup \{x\}$ is a 4-element-fan-petal relative to $P_i - \{x\}$, or*

  (d) *$M' \backslash P_1 \cong M(K_{3,m-1})$, and, for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and either*

    (I) *$M'|P_1 \cong M(K_{3,t})$ for some $t \geq 2$,*

    (II) *P_1 is an augmented-fan-petal relative to P_i,*

    (III) *P_1 is a co-augmented-fan-petal relative to P_i, or*

    (IV) *P_1 is a quad-petal relative to P_i.*

An outline of the proof is as follows. Let M be a 3-connected matroid with no detachable pairs. In Sections 3.5 and 3.6, we assume that M has distinct maximal fans $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ such that $|F_1| \geq 4$ and $|F_2| \geq 3$. First, in Section 3.5, we consider the case where F_1 and F_2 are disjoint and either $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads or $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads. Under these assumptions, we prove that one of (iv), (vii)(c) (with $1 \leq t < m$), and (vii)(d)(II) holds. In Section 3.6, we assume that $F_1 \cap F_2 \neq \emptyset$, and prove that, if the assumptions of Section 3.5 do not hold, then M is one of the matroids described in (iii), (iv), (v), (vi), (vii)(a), and (vii)(c) (with $t = m$). Next, in Section 3.7, we suppose that M has a 4-element fan, but the conditions of Sections 3.5 and 3.6 do not hold. We show that one of (ii), (iv), and (vii)(d)(III) holds. The remaining cases are when M has no 4-element fans whatsoever, and this is what we consider in Section 3.8. If M has no triangles or triads, then Theorem 1.6.1 implies that M is a spike and outcome (iv) of Theorem 1.6.2 holds. Otherwise, we assume that M has a triangle or a triad, and prove that one of (vii)(b), (vii)(c) (with $t = 0$), (vii)(d)(I), and (vii)(d)(IV) holds.

3.4 Further Preliminary Lemmas

In this section, we prove lemmas which will be useful throughout the proof of Theorem 1.6.2. Note that each lemma may be applied to the matroid M^* rather than M to obtain a dual result. These dual results are often not explicitly stated, but are used frequently.

Lemma 3.4.1. *Let M be a 3-connected matroid. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$ and $|X| \geq 3$ and $|E(M)| \geq |X|+3$. Let $e \in E(M)-X$. If $e \in \mathrm{cl}(X)$, then either e is contained in a triad or $M\backslash e$ is 3-connected.*

Proof. First, assume that both $r(X) > 2$ and $r(E(M)-(X\cup\{e\})) > 2$. Then $\lambda_{M/e}(X) = 1$, and $|E(M/e)| \geq |X| + 2$, so M/e is not 3-connected. Furthermore, $\lambda_{\mathrm{si}(M/e)}(X) = 1$, and, since $r(X) > 2$ and $r(E(M) - (X \cup \{e\})) > 2$, there are at least two elements of X and two elements of $E(M) - (X \cup \{e\})$ remaining in $\mathrm{si}(M/e)$. Therefore, $\mathrm{si}(M/e)$ is not 3-connected, and so, by Bixby's Lemma, $\mathrm{co}(M\backslash e)$ is 3-connected. It follows that either $M\backslash e$ is 3-connected, or e is contained in a triad.

Now assume $r(X) = 2$. Suppose e is not contained in a triad and $M\backslash e$ is not 3-connected. Then $\mathrm{co}(M\backslash e)$ is not 3-connected, so M has a cyclic 3-separation $(P, \{e\}, Q)$. Since $|X| > 2$ and $r(X) = 2$, we have that $X \subseteq \mathrm{cl}(T)$ for some triangle $T \subseteq X$. Now, by Corollary 3.2.16, we may assume that $T \subseteq P$ and, by Corollary 3.2.15, we may assume that $X \subseteq P$. But now $e \in \mathrm{cl}(P)$ and $e \in \mathrm{cl}^*(P)$. This means that $\lambda(P \cup \{e\}) = 1$, a contradiction.

Finally, assume $r(E(M) - (X \cup \{e\})) = 2$. Note that

$$\lambda(E(M) - X) = \lambda(X) = 2$$

and

$$\lambda(E(M) - (X \cup \{e\})) = \lambda(X \cup \{e\}) = 2.$$

Hence, either $e \in \mathrm{cl}(E(M) - (X \cup \{e\}))$ or $e \in \mathrm{cl}^*(E(M) - (X \cup \{e\}))$. Orthogonality implies that $e \in \mathrm{cl}(E(M) - (X \cup \{e\}))$. This means that we can apply the argument in the previous paragraph to the set $E(M) - (X \cup \{e\})$ rather than X, which completes the proof. □

Lemma 3.4.2. *Let M be a 3-connected matroid, and let $X \subseteq E(M)$ be a quad. Let $e \in X$. If e is not contained in a triad, then $M\backslash e$ is 3-connected. Similarly, if e is not contained in a triangle, then M/e is 3-connected.*

Proof. Suppose e is not contained in a triad and $M\backslash e$ is not 3-connected. Then $\mathrm{co}(M\backslash e)$ is also not 3-connected. Thus, M has a cyclic 3-separation $(P, \{e\}, Q)$. Either $|X \cap P| \geq 2$

or $|X \cap Q| \geq 2$. Without loss of generality, assume the former. If $|X \cap P| = 3$, then $e \in \mathrm{cl}(P) \cap \mathrm{cl}^*(P)$, a contradiction. Otherwise, $|X \cap P| = 2$, and $|X \cap Q| = 1$. Let f be the unique element of $X \cap Q$. Then $f \in \mathrm{cl}(P \cup \{e\})$ and $f \in \mathrm{cl}^*(P \cup \{e\})$. Again, this contradicts the 3-connectivity of M since $|Q - \{e\}| \geq 2$. Thus, if e is not contained in a triad, then $M \backslash e$ is 3-connected. A dual argument shows that if e is not contained in a triangle, then M/e is 3-connected and completes the proof. □

Lemma 3.4.3. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $|X| \geq 2$ and $|E(M)| \geq |X| + 4$. Let $e \in E(M) - X$ such that either $\lambda(X) = 2$ or $\lambda(X \cup \{e\}) = 2$, and $M \backslash e$ is 3-connected. Furthermore, let $f \in E(M) - X$ such that $f \in \mathrm{cl}(X)$ and f is not contained in a triad of M. Then M has a 4-element cocircuit $C^* = \{e, f, g, h\}$ such that $g \in X$ and $h \notin X$.*

Proof. We first prove that there is a triad of $M \backslash e$ containing f. Suppose this is not the case. Since $M \backslash e$ is 3-connected and $|X| \geq 2$ and $|E(M)| \geq |X| + 4$, we have that $\lambda_{M \backslash e}(X) \geq 2$. Therefore, if $\lambda_M(X) = 2$, Lemma 3.2.1 implies that $\lambda_{M \backslash e}(X) = 2$. If $\lambda_M(X) \neq 2$, then $\lambda_M(X \cup \{e\}) = 2$. This implies that $\lambda_M(X) = 3$ and $e \in \mathrm{cl}^*(X)$. Again, Lemma 3.2.1 implies that $\lambda_{M \backslash e}(X) = 2$. If $|X| \geq 3$, then Lemma 3.4.1 implies that $M \backslash e \backslash f$ is 3-connected, so M has a detachable pair, a contradiction. Otherwise, $|X| = 2$. Since $f \in \mathrm{cl}(X)$, the set $X \cup \{f\}$ is a triangle. If $X \cup \{f\}$ is not contained in a 4-element fan of $M \backslash e$, then Tutte's Triangle Lemma implies that there exists $x \in X \cup \{f\}$ such that $M \backslash e \backslash x$ is 3-connected, a contradiction. Therefore, there is a 4-element fan of $M \backslash e$ containing the triangle $X \cup \{f\}$. But f is not contained in a triad of $M \backslash f$, so f is an end of this fan. Thus, $M \backslash e \backslash f$ is 3-connected.

In all cases, if f is not contained in a triad of $M \backslash e$, then M has a detachable pair. Hence, f is contained in a triad T^* of $M \backslash e$. Since $f \in \mathrm{cl}(X)$, orthogonality implies that there exists $g \in T^*$ such that $g \in X$. Furthermore, if $T^* \subseteq X \cup \{f\}$, then $f \in \mathrm{cl}(X)$ and $f \in \mathrm{cl}^*(X)$. This implies $\lambda(X \cup \{f\}) < 2$, a contradiction to the 3-connectivity of $M \backslash f$, since $|E(M \backslash e)| \geq |X| + 3$. Thus, there exists $h \in T^*$ with $h \notin X$. Since f is not contained in a triad of M, we have that $T^* \cup \{e\} = \{e, f, g, h\}$ is a cocircuit of M. □

Lemma 3.4.4. *Let M be a 3-connected matroid. Let $C = \{e, f, g, h\}$ be a 4-element circuit of M such that $\{g, h\}$ is contained in a triad T^* of M. If $\mathrm{si}(M/f)$ is 3-connected and e is not contained in a triad, then $M \backslash e$ is 3-connected.*

Proof. Suppose $M \backslash e$ is not 3-connected. Then M has a cyclic 3-separation $(P, \{e\}, Q)$. By Corollary 3.2.16, we may assume that $T^* \subseteq P$. If $f \in P$, then $C - \{e\} \subseteq P$. This means that $e \in \mathrm{cl}(P) \cap \mathrm{cl}^*(P)$, a contradiction. So $f \in Q$. But now $f \in \mathrm{cl}(P \cup \{e\})$. This contradicts the fact that $\mathrm{si}(M/f)$ is 3-connected, and completes the proof. □

Lemma 3.4.5. *Let M be a 3-connected matroid with no detachable pairs. Let $C = \{e, f, g, h\}$ be a 4-element circuit of M such that $\{g, h\}$ is contained in a triad of M, and neither e nor f is contained in a triad of M. Let $x \in E(M) - C$ such that $M \backslash x$ is 3-connected. Then M has a 4-element cocircuit C^* containing x and either e or f.*

Proof. Suppose neither e nor f is contained in a triad of $M \backslash x$. Since $M \backslash x \backslash e$ is not 3-connected, Lemma 3.4.4 implies that $\mathrm{si}(M \backslash x/f)$ is not 3-connected. Hence, by Bixby's Lemma, $\mathrm{co}(M \backslash x \backslash f)$ is 3-connected. But f is not contained in a triad of $M \backslash x$, so $M \backslash x \backslash f$ is 3-connected, and M has a detachable pair. This contradiction implies that $M \backslash x$ has a triad T^* containing either e or f. Since neither e nor f is contained in a triad of M, this means that $T^* \cup \{x\}$ is a 4-element cocircuit of M, completing the proof. □

Lemma 3.4.6. *Let M be a 3-connected matroid, and let e and f be distinct elements of $E(M)$ such that $M/e \backslash f$ is 3-connected. Then either $M \backslash f$ is 3-connected, or $\{e, f\}$ is contained in a triad of M.*

Proof. Suppose $M \backslash f$ is not 3-connected. This implies there exists a 2-separation X of $M \backslash f$ such that $X - \{e\}$ is not a 2-separation of $M \backslash f/e$. But $\lambda_{M \backslash f/e}(X - \{e\}) \leq \lambda_{M \backslash f}(X) \leq 1$. Therefore, either $|X - \{e\}| = 1$ or $|(E(M) - X) - \{e\}| = 1$. Without loss of generality, we may assume the former, which implies that $e \in X$ and $|X| = 2$. Since $\lambda_{M \backslash f}(X) \leq 1$ and $\lambda_M(X) \geq 2$, we have that $f \in \mathrm{cl}^*(X)$, and thus $X \cup \{f\}$ is a triad of M containing both e and f. □

Lemma 3.4.7. *Let M be a 3-connected matroid with no detachable pairs. Let C be a 4-element circuit of M, and let $e \in C$ such that M/e is 3-connected and not a wheel or a whirl. Then there is a maximal fan of M/e containing $C - \{e\}$ with end elements e^- and e^+ such that either $\{e^-, e\}$ is contained in a triad of M or $M \backslash e^-$ is 3-connected, and either $\{e^+, e\}$ is contained in a triad of M or $M \backslash e^+$ is 3-connected.*

Proof. In M/e, the set $C - \{e\}$ is a triangle. If $C - \{e\}$ is not contained in a 4-element fan of M/e, then Tutte's Triangle Lemma implies that there exists $e^-, e^+ \in C - \{e\}$ such that $M/e \backslash e^-$ and $M/e \backslash e^+$ are 3-connected. By Lemma 3.4.6, either $\{e^-, e\}$ is contained in a triad of M, or $M \backslash e^-$ is 3-connected. Similarly, either $\{e^+, e\}$ is contained in a triad of M, or $M \backslash e^+$ is 3-connected. Thus, the result holds.

Otherwise, M/e has a maximal fan of length at least four containing $C - \{e\}$. Let e^- and e^+ be the end elements of this fan. Since, M/e is not a wheel or a whirl, we have that either e^- is contained in a triad and not a triangle, in which case $M/e/e^-$ is 3-connected, or e^- is contained in a triangle and not a triad, in which case $M/e \backslash e^-$ is 3-connected. Since M has no detachable pairs, we have that $M/e/e^-$ is not 3-connected, so $M/e\backslash e^-$ is 3-connected. Similarly, $M/e\backslash e^+$ is 3-connected. Again, the result follows from Lemma 3.4.6. □

The next lemma will be used frequently throughout the proof of Theorem 1.6.2 to identify subsets which contain every deletable element of a 3-connected matroid. We introduce the following notation. A *deletable collection* of a matroid M is a collection of subsets $(\{e\}, X_1, X_2, \dots, X_k)$ of $E(M)$, with $k \geq 2$, such that

(i) $e \notin X_1 \cup X_2 \cdots \cup X_k$,

(ii) $X_1 \cap X_2 \cap \cdots \cap X_k = \emptyset$,

(iii) either $\lambda(X_1) = 2$, or $X_1 \cup \{e\}$ is a quad,

(iv) $e \in \mathrm{cl}(X_i)$ for all $i \in [k]$, and

(v) e is not contained in a triad.

Lemma 3.4.8. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$, and $|E(M)| \geq |X| + 3$. If X contains a deletable collection, then, for all $x \in E(M) - X$, the matroid $M\backslash x$ is not 3-connected.*

Proof. Suppose there exists $x \in E(M) - X$ such that $M\backslash x$ is 3-connected. Let $(\{e\}, X_1, X_2, \dots, X_k)$ be a deletable collection contained in X. First, suppose $\lambda(X_1) = 2$. Now, $|E(M)| \geq |X| + 3 \geq |X_1| + 4$, so Lemma 3.4.3 implies that the matroid M has a 4-element cocircuit containing $\{e, x\}$. Otherwise, suppose $X_1 \cup \{e\}$ is both a quad, in which case $X_1 \cup \{e\}$ is still a quad in $M\backslash x$. Since $M\backslash x\backslash e$ is not 3-connected, Lemma 3.4.2 implies that $M\backslash x$ has a triad containing e, and so M has a 4-element cocircuit containing $\{e, x\}$. In either case, the matroid M has a 4-element cocircuit C^* containing $\{e, x\}$. Since $e \in \mathrm{cl}(X_1)$, orthogonality implies that there exists $f \in C^*$ with $f \in X_1$. But $X_1 \cap X_2 \cap \cdots \cap X_k = \emptyset$, so there exists $i \in [k]$ such that $f \notin X_i$. Now, orthogonality implies that C^* contains an element of X_i, so $C^* = \{x, e, f, g\}$ with $f \in X_1$ and $g \in X_i$. But now $x \in \mathrm{cl}^*(X)$, so $\lambda_{M\backslash x}(X) \leq 1$. Since $|E(M\backslash x)| \geq |X| + 2$, this is a contradiction to the 3-connectivity of $M\backslash x$, and completes the proof. □

Lemma 3.4.9. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$ and $|E(M)| \geq |X| + 3$. If X contains a deletable collection, then for all $x \in E(M) - X$, if x is contained in a triangle, then x is contained in a triad.*

Proof. Suppose there exists $e \in E(M) - X$ such that e is contained in a triangle T, but e is not contained in a triad. If T is contained in a 4-element fan, then e is an end of this fan since e is not contained in a triad. This implies $M\backslash e$ is 3-connected, contradicting Lemma 3.4.8. Thus, T is not contained in a 4-element fan, and Tutte's Triangle Lemma implies that there exists $x, y \in T$ such that $M\backslash x$ and $M\backslash y$ are both 3-connected. Thus, $x, y \in X$. But now $e \in \mathrm{cl}(X)$, which implies, by Lemma 3.4.1, that $M\backslash e$ is 3-connected. This is a contradiction which completes the proof. □

Lemma 3.4.10. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$, and $|E(M)| \geq |X| + 3$, and X contains a deletable collection. Suppose there exists $Y \subseteq X$ and $y \in X - Y$ such that $\lambda(Y) = 2$, and $y \in \mathrm{cl}^*(Y)$, and y is not contained in a triangle of M. Furthermore, suppose, for all $y' \in Y \cup \{y\}$, that $y' \in \mathrm{cl}(X - \{y'\})$. Then every element of $E(M) - X$ is contained in a triad.*

Proof. Suppose there exists $e \notin X$ such that e is not contained in a triad. By Lemma 3.4.9, the element e is also not contained in a triangle. Now, Bixby's Lemma implies that either M/e or $M\backslash e$ is 3-connected. Since $M\backslash e$ is not 3-connected, we have that M/e is 3-connected. By Lemma 3.4.3, there is a 4-element circuit C of M containing $\{e, y\}$. Furthermore, $C = \{e, f, y, z\}$ such that $f \notin X \cup \{e\}$ and $z \in Y$.

Suppose f is not contained in a triad of M. Neither e nor f is contained in a triad, which implies that, in M/y, the set $C - \{y\}$ does not intersect a triad. By Lemma 3.4.7, there exists distinct $e^-, e^+ \in C - \{e\}$ such that either $\{e^-, y\}$ is contained in a triad of M or $M\backslash e^-$ is 3-connected, and either $\{e^+, y\}$ is contained in a triad of M or $M\backslash e^+$ is 3-connected. Now, $|(C - \{y\}) \cap X| = 1$, so either $e^- \in \{e, f\}$ or $e^+ \in \{e, f\}$. Without loss of generality, assume the former. Neither e nor f is contained in a triad, which implies $M\backslash e^-$ is 3-connected. But $e^- \notin X$, a contradiction.

So f is contained in a triad T^* of M. By orthogonality, T^* contains a second element of C. Now, e is not contained in a triad, so T^* contains either y or z. We have that $y \in \mathrm{cl}(X - \{y\})$ and $z \in \mathrm{cl}(X - \{z\})$, so orthogonality implies that $f \in \mathrm{cl}^*(X)$, and thus $\lambda(X \cup \{f\}) = 2$. Now, $e \in \mathrm{cl}(X \cup \{f\})$, and M/e is 3-connected, which implies that $|E(M/e)| \leq |X \cup \{f\}| + 1$, that is, $|E(M)| = |X| + 3$. But $\lambda(E(M) - X) = \lambda(X) = 2$, so $E(M) - X$ is either a triangle or a triad containing e, a contradiction. □

We can apply Lemmas 3.4.8, 3.4.9 and 3.4.10 to M^*. A *contractable collection* of a matroid M is a collection of sets $(\{e\}, X_1, X_2, \ldots, X_k)$, with $k \geq 2$, such that

(i) $e \notin X_1 \cup X_2 \cdots \cup X_k$,

(ii) $X_1 \cap X_2 \cap \cdots \cap X_k = \emptyset$,

(iii) either $\lambda(X_1) = 2$, or $X_1 \cup \{e\}$ is a quad,

(iv) $e \in \mathrm{cl}^*(X_i)$ for all $i \in [k]$, and

(v) e is not contained in a triangle.

Corollary 3.4.11. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$, and $|E(M)| \geq |X| + 3$. If X contains a contractable collection, then, for all $x \in E(M) - X$, the matroid M/x is not 3-connected.*

Corollary 3.4.12. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$ and $|E(M)| \geq |X| + 3$. If X contains a contractable collection, then, for all $x \in E(M) - X$, if x is contained in a triad, then x is contained in a triangle.*

Corollary 3.4.13. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$, and $|E(M)| \geq |X|+3$, and X contains a contractable collection. Suppose there exists $Y \subseteq X$ and $y \in X - Y$ such that $\lambda(Y) = 2$, and $y \in \mathrm{cl}(Y)$, and y is not contained in a triad of M. Furthermore, suppose, for all $y' \in Y \cup \{y\}$, that $y' \in \mathrm{cl}^*(X - \{y'\})$. Then every element of $E(M) - X$ is contained in a triangle.*

Lemma 3.4.14. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$, and $|E(M)| \geq |X| + 3$, and X contains a deletable collection. Let $Y \subseteq E(M)$ such that $\lambda(Y) = 2$, and $|E(M)| \geq |Y| + 3$, and Y contains a contractable collection. Then every element of $E(M) - (X \cup Y)$ is contained in a maximal fan of length at least four with end elements in $X \cup Y$.*

Proof. Let $e \notin X \cup Y$. To show the result, it is sufficient to prove that e is contained in both a triangle and a triad. If e is contained in neither a triangle nor a triad, then Bixby's Lemma implies that either $M\backslash e$ or M/e is 3-connected, contradicting either Lemma 3.4.8 or Corollary 3.4.11. By Lemma 3.4.9, if e is contained in a triangle then e is also contained in a triad. Dually, by Corollary 3.4.12, if e is contained in a triad, then e is also contained in a triangle. This completes the proof. □

We now consider specific structures which may arise in 3-connected matroids with no detachable pairs.

Lemma 3.4.15. *Let M be a 3-connected matroid with no detachable pairs. Let $X \subseteq E(M)$ such that $\lambda(X) = 2$, and $|X| \geq 3$, and $|E(M)| \geq |X| + 7$, and, for all $x \in X$, we have that $x \in \mathrm{cl}^*(X - \{x\})$. Suppose there exists distinct $a, b, c \in E(M) - X$ such that $\{a, b, c\} \subseteq \mathrm{cl}(X)$ and none of a, b, or c are contained in a triad. Then there exists distinct $d, e, f \in E(M) - (X \cup \{a, b, c\})$ such that $\{d, e, f\} \subseteq \mathrm{cl}^*(X \cup \{a, b, c\})$ and none of d, e, or f are contained in a triangle.*

Proof. By Lemma 3.4.1, each of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected. Hence, by Lemma 3.4.3, there is a 4-element cocircuit $C_1^* = \{a, b, d, x\}$ of M, where $x \in X$ and $d \notin X \cup \{a, b, c\}$. Similarly, M has 4-element cocircuits $\{a, c, e, y\}$ and $\{b, c, f, z\}$ with $y, z \in X$ and $e, f \notin X \cup \{a, b, c\}$. Note that these cocircuits are all distinct.

If $d = e$, then cocircuit elimination implies that M has a 4-element cocircuit C^* contained in $\{a, b, c, x, y\}$. The cocircuit C^* contains one of a, b, and c — assume, without loss of generality, that $a \in C^*$. Then $a \in \mathrm{cl}(X \cup \{b, c\})$ and $a \in \mathrm{cl}^*(X \cup \{b, c\})$, a contradiction. In a similar way, we see that all of d, e, and f are distinct. Furthermore, $\{d, e, f\} \subseteq \mathrm{cl}^*(X \cup \{a, b, c\})$.

To complete the proof, we show that none of d, e, and f are contained in a triangle. Suppose M has a triangle T containing d. By orthogonality, T contains an element of $\{a, b, x\}$. If $x \in T$, then, since $x \in \mathrm{cl}^*(X - \{x\})$, orthogonality implies that T contains a second element of X. But now $d \in \mathrm{cl}(X)$ and $d \in \mathrm{cl}^*(X \cup \{a, b\})$, a contradiction. If $a \in T$, then orthogonality with $\{a, c, e, y\}$ implies that T contains one of $\{c, e, y\}$, so $d \in \mathrm{cl}(X \cup \{a, b, c, e\})$ and $d \in \mathrm{cl}^*(X \cup \{a, b, c, e\})$. This is a contradiction since $|E(M)| \geq |X \cup \{a, b, c, d, e\}| + 2$. Finally, if $b \in T$, then T contains one of $\{c, f, z\}$, so $d \in \mathrm{cl}(X \cup \{a, b, c, f\}) \cap \mathrm{cl}^*(X \cup \{a, b, c, f\})$. This contradiction shows that d is not contained in a triangle, and similarly e and f are not contained in triangles. □

Lemma 3.4.16. *Let M be a 3-connected matroid such that $|E(M)| \geq 11$. Suppose there exists distinct $a, b, c, d \in E(M)$ such that $r(\{a, b, c, d\}) = 2$. Then M has a detachable pair.*

Proof. Assume, with the aim of reaching a contradiction, that M does not contain a detachable pair. To begin, suppose M has a triad T^* which intersects $\{a, b, c, d\}$. Orthogonality implies that $T^* \subseteq \{a, b, c, d\}$. But now the set T^* is both a triangle and a triad, contradicting the 3-connectivity of M. Thus, $\{a, b, c, d\}$ does not intersect a triad. It follows that $(\{a\}, \{b, c\}, \{b, d\}, \{c, d\})$ is a deletable collection. We shall find an element $z \notin \{a, b, c, d\}$ such that $M \backslash z$ is 3-connected. Since $\lambda(\{a, b, c, d\}) = 2$ and $|E(M)| \geq 7$, this will contradict Lemma 3.4.8 and complete the proof.

Let $x, y \in \{a, b, c, d\}$. By Lemma 3.4.1, we have that $M \backslash x$ is 3-connected, and $y \in \mathrm{cl}(\{a, b, c, d\} - \{y\})$. Thus, by Lemma 3.4.3, there is a 4-element cocircuit of M containing $\{x, y\}$. Furthermore, this cocircuit contains another element of $\{a, b, c, d\}$, and an element which is not an element of $\{a, b, c, d\}$.

So M has a 4-element cocircuit C_1^* containing a and b. Without loss of generality, let $C_1^* = \{a, b, c, e\}$ with $e \notin \{a, b, c, d\}$. Similarly, M has a 4-element cocircuit containing a and d, which we may assume is $C_2^* = \{a, b, d, f\}$ with $f \notin \{a, b, c, d\}$. If $e = f$, then cocircuit elimination implies M has a cocircuit contained in $\{a, b, c, d\}$, a contradiction to the 3-connectivity of M. So $e \neq f$. Similarly, M has a 4-element cocircuit containing c and d, which we may take to be $C_3^* = \{a, c, d, g\}$ with $g \notin \{a, b, c, d, e, f\}$.

We apply the dual of Lemma 3.4.15 with $X = \{a, b, c, d\}$. Certainly, $\{e, f, g\} \subseteq \mathrm{cl}^*(\{a, b, c, d\})$ and, for all $x \in \{a, b, c, d\}$, we have that $x \in \mathrm{cl}(\{a, b, c, d\} - \{x\})$. Suppose e is contained in a triangle T of M. Then, by orthogonality with C_1^*, the triangle T contains one of $\{a, b, c\}$. In turn, orthogonality with either C_2^* or C_3^* implies that T contains a second element of $\{a, b, c, d, f, g\}$. But now $e \in \text{cl}(\{a, b, c, d, f, g\})$ and $e \in \text{cl}^*(\{a, b, c, d, f, g\})$, a contradiction. Hence, the element e, and symmetrically f and g, is not contained in a triangle. Thus, Lemma 3.4.15 implies that M has elements h, i, j such that $\{h, i, j\} \subseteq \text{cl}(\{a, b, c, d, e, f, g\})$ and none of h, i, and j are contained in a triad. In particular, $M\backslash h$ is 3-connected, a contradiction which completes the proof. □

Lemma 3.4.17. *Let M be a 3-connected matroid with no detachable pairs. Let $F = (e_1, e_2, \ldots, e_{|F|})$ be a maximal fan with odd length at least five such that $\{e_1, e_2, e_3\}$ is a triangle. Then $|F| = 5$, and there exists $z \in E(M) - F$ such that $\{e_1, e_3, e_5, z\}$ is a cocircuit.*

Proof. Since $|F|$ is odd, the set $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$ is also a triangle. Therefore, $M\backslash e_{|F|}$ is 3-connected, and $e_1 \in \text{cl}(\{e_2, e_3\})$. By Lemma 3.2.7, and observing that M is not a wheel or a whirl since M has a maximal fan of odd length, we have that $|E(M)| \geq |F| + 2 \geq |\{e_2, e_3\}| + 4$. Thus, by Lemma 3.4.3, there is a 4-element cocircuit C^* of M containing $\{e_1, e_{|F|}\}$ and an element $z \notin F$. Furthermore, by orthogonality, C^* contains one element of $\{e_2, e_3\}$ and one element of $\{e_{|F|-2}, e_{|F|-1}\}$. The only possibility is $|F| = 5$ and $e_3 \in C^*$, which completes the proof. □

Lemma 3.4.18. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 8$. Let $F = (e_1, e_2, \ldots, e_{|F|})$ be a maximal fan of M with such that $|F| \geq 3$ and $\{e_1, e_2, e_3\}$ is a triad. Let T^* be a triad of M which is not contained in a 4-element fan. Then one of the following holds.*

(i) *$|F| = 3$ and $|F \cap T^*| > 0$,*
(ii) *$e_1 \in T^*$,*
(iii) *F is a 4-element-fan-petal relative to T^*, or*
(iv) *$M|(F \cup T^*) \cong M(K_{3,2})$.*

Proof. Suppose neither *(i)* nor *(ii)* holds. Note that this implies, by Lemma 3.2.11, that the triads $\{e_1, e_2, e_3\}$ and T^* are disjoint. Let $x \in T^*$. If $|F| \geq 4$, then M/e_1 is 3-connected. If $|F| = 3$, then F is a triad not contained in a 4-element fan, and Tutte's Triangle Lemma implies that at least two of M/e_1, M/e_2, and M/e_3 are 3-connected. Thus, without loss of generality, we assume that M/e_1 is 3-connected. In either case, $x \in \text{cl}^*(T^* - \{x\})$, so Lemma 3.4.3 implies that there is a 4-element circuit C_1 of M containing $\{x, e_1\}$. By orthogonality, C_1 contains another element of T^* and another element of $\{e_1, e_2, e_3\}$. Hence, $C_1 = \{e_1, e_i, x, y\}$, with $i \in \{2, 3\}$ and $y \in T^*$. Let z be the unique element of $T^* - \{x, y\}$. Lemma 3.4.3 again implies that there is a 4-element circuit C_2 of M containing $\{e_1, z\}$, and another element of T^*, and another element of $\{e_1, e_2, e_3\}$. Without loss of generality, let $C_2 = \{e_1, e_j, x, z\}$ with $j \in \{2, 3\}$.

Suppose $i = j$. Circuit elimination implies that M has a circuit C contained in $\{x, y, z, e_1\}$. By orthogonality with $\{e_1, e_2, e_3\}$, we have that $e_1 \notin C$. Therefore, the triad $\{e_1, e_2, e_3\}$ contains a circuit, a contradiction to the 3-connectivity of M. Hence, $i \neq j$, and so, without loss of generality, $C_1 = \{x, y, e_1, e_2\}$ and $C_2 = \{x, z, e_1, e_3\}$.

If $|F| \geq 5$, then C_2 intersects the triad $\{e_3, e_4, e_5\}$ in one element, a contradiction. Therefore, $|F| \leq 4$. Suppose $|F| = 4$. We show that F is a 4-element-fan-petal relative to T^*. Now,

$$\sqcap(\{e_1, e_2\}, T^*) = r(\{e_1, e_2\}) + r(T^*) - r(T^* \cup \{e_1, e_2\}) = 2 + 3 - 4 = 1$$

Similarly, $\sqcap(\{e_1, e_3\}, T^*) = 1$. To show that F is a 4-element-fan-petal relative to T^*, it remains to show that $e_4 \in \mathrm{cl}(T^*)$. We have that $r(F) = 3$, so $r(F \cup \{x\}) = 4$, by orthogonality with T^*. Now, $y, z \in \mathrm{cl}(F \cup \{x\})$, and thus $r(F \cup T^*) = 4$. By orthogonality with the triad $\{e_1, e_2, e_3\}$, we have that $r(F \cup T^*) > r(T^* \cup \{e_4\})$. Thus, $r(T^* \cup \{e_4\}) = r(T^*) = 3$, so $e_4 \in \mathrm{cl}(T^*)$, and F is a 4-element-fan-petal relative to T^*.

Finally, suppose $|F| = 3$. Either M/e_2 or M/e_3 is 3-connected. Without loss of generality, we assume M/e_2 is 3-connected. Then M has a 4-element circuit C_3 containing $\{e_2, z\}$, and one of e_1 and e_3, and one of x and y. If $e_1 \in C_3$, then circuit elimination with C_1 implies that M has a circuit contained in $T^* \cup \{e_2\}$, and orthogonality with $\{e_1, e_2, e_3\}$ implies that M has a circuit contained in T^*, a contradiction. Similarly, if $x \in C_3$, then circuit elimination with C_2 and orthogonality implies that M has a circuit in $\{e_1, e_2, e_3\}$. Therefore, $C_3 = \{e_2, e_3, y, z\}$, which implies that $M|(F \cup T^*) \cong M(K_{3,2})$, completing the proof. □

Lemma 3.4.19. *Let M be a 3-connected matroid. Suppose $E(M)$ can be partitioned into $P_1, P_2, \ldots, P_m$, with $m \geq 2$, such that $|P_1| \geq 2$, and, for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and $r(P_i \cup P_j) = r(P_j) + 1$, for all $j \in [m] - \{i\}$. Then $(P_1, P_2, \ldots, P_m)$ is a paddle of M.*

Proof. First, let i, j be distinct elements of $[m]$ and suppose, without loss of generality, that $i \neq 1$. Then

$$\begin{aligned}\sqcap(P_i, P_j) &= r(P_i) + r(P_j) - r(P_i \cup P_j)\\ &= 3 + r(P_j) - (r(P_j) + 1) = 2.\end{aligned}$$

Next, let J be a proper non-empty subset of $[m]$, and let $X = \bigcup_{i \in J} P_i$. To complete the proof, we show that $\lambda(X) = 2$. First, assume that $1 \notin J$. If $|J| = 1$, then X is a triad, so

$\lambda(X) = 2$. Otherwise, let $i \in J$, and assume that $\lambda(X - P_i) = 2$. Now, $r(X) \le r(X - P_i) + 1$, and, since P_i is a triad, $r^*(X) \le r^*(X - P_i) + 2$. Thus,

$$\lambda(X) \le (r(X - P_i) + 1) + (r^*(X - P_i) + 2) - (|X| + 3) = 2$$

Thus, $\lambda(X) = 2$, as desired. Finally, if $1 \in J$, then $1 \notin [m] - J$. Hence, $\lambda(X) = \lambda(\bigcup_{i \in [m]-J} P_i) = 2$, which completes the proof. □

3.5 Disjoint Fans

Armed with the lemmas from the previous sections, we begin the proof in earnest. In this section, we prove the following:

Theorem 3.5.1. *Let M be a 3-connected matroid such that $|E(M)| \ge 13$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be disjoint, maximal fans of M such that $|F_1| \ge 4$ and $|F_2| \ge 3$. If $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads, then one of the following holds:*

(i) *M has a detachable pair,*

(ii) *M is an even-fan-spike or a degenerate even-fan-spike, or*

(iii) *M has a paddle $(P_1, P_2, \ldots, P_m)$ such that either*

   (a) *there exists $x \in E(M)$ and $1 \le t < m$ such that $M' \backslash (\{x\} \cup \bigcup_{i=1}^{t} P_i) \cong M(K_{3,m-t})$ and for all $i \in \{1, 2, \ldots, m\}$ the set $P_i - \{x\}$ is a triad, and for all $j \in \{1, 2, \ldots, t\}$, distinct from i, the set $P_j \cup \{x\}$ is a 4-element-fan-petal relative to $P_i - \{x\}$,*

   (b) *$M \backslash P_1 \cong M(K_{3,m-1})$ and, for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and P_1 is an augmented-fan-petal relative to P_1.*

3.5.1 $\mathbf{F_2}$ has length three

First, we consider the case where $|F_2| = 3$, and show that either M has a detachable pair, or has the structure described in Theorem 3.5.1(iii)(a).

Lemma 3.5.2. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \ge 10$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M such that $|F_1| \ge 4$ and $\{e_1, e_2, e_3\}$ is a triad. Let F_2 be a triad of M which is disjoint from F_1 and not contained in a 4-element fan. Then both of the following hold:*

(i) *F_1 is a 4-element-fan-petal relative to F_2, and*

(ii) *every element of $E(M) - (F_1 \cup F_2)$ is contained in a triad.*

Proof. Since F_1 and F_2 are disjoint, we have that Lemma 3.4.18(i) and (ii) do not hold. Furthermore, $|F_1| \geq 4$, which means Lemma 3.4.18(iv) does not hold. Therefore, Lemma 3.4.18(iii) holds, and F_1 is a 4-element-fan-petal relative to F_2.

Note that $e_4 \in \text{cl}(F_1 - \{e_4\})$ and $e_4 \in \text{cl}(F_2)$. Furthermore, $\lambda(F_1 - \{e_4\}) = 2$, and e_4 is not contained in a triad by Lemma 3.2.8. Therefore, $(\{e_4\}, F_1 - \{e_4\}, F_2)$ is a deletable collection. Also, $\lambda(F_1 \cup F_2) = 2$ and $|E(M)| \geq |F_1 \cup F_2| + 3 = 10$. Furthermore, $e_1 \in \text{cl}^*(F_1 - \{e_1\})$ and, for all $i \in \{1, 2, \ldots, |F_1|\}$, we have that $e_i \in \text{cl}(F_1 \cup F_2)$. Thus, by Lemma 3.4.10, every element of $E(M) - (F_1 \cup F_2)$ is contained in a triad and completes the proof. □

Lemma 3.5.3. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 10$. Let $F_1 = (e_1, e_2, e_3, e_4)$ be a maximal fan of M such that $\{e_1, e_2, e_3\}$ is a triad. Let F_2 be a triad of M which is disjoint from F_1 and not contained in a 4-element fan. Furthermore, let F_3 be a maximal fan of M, distinct from F_1 and F_2, such that $|F_3| \geq 4$. Then $e_4 \in F_3$, and F_3 is a 4-element-fan-petal relative to $\{e_1, e_2, e_3\}$ and F_2.*

Proof. By Lemma 3.5.2, we have that F_1 is a 4-element-fan-petal relative to F_2. So let $F_2 = \{f_1, f_2, f_3\}$ such that $\{e_1, e_2, f_1, f_2\}$ and $\{e_1, e_3, f_1, f_3\}$ are circuits. Also let $(g_1, g_2, \ldots, g_{|F_3|})$ be an ordering of F_3.

If $\{g_1, g_2, g_3\}$ is a triangle, then g_1 is not contained in a triad, and so, Lemma 3.5.2 implies that $g_1 \in F_1 \cup F_2$. The only element of $F_1 \cup F_2$ which is not contained in a triad is e_4, so $g_1 = e_4$. Similarly, if $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$ is a triangle, then $g_{|F_3|} = e_4$. Therefore, either $\{g_1, g_2, g_3\}$ is a triad, or $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$ is a triad. Without loss of generality, assume the former.

Now, $|F_3| \geq 4$, so Lemma 3.4.18(i) and (iv) do not hold. Suppose $g_1 \in F_1 \cup F_2$. Lemma 3.2.8 implies that $g_1 \neq e_4$, and Lemma 3.2.11 implies that $g_2, g_3 \notin F_1 \cup F_2$. Therefore, the triad $\{g_1, g_2, g_3\}$ intersects either the circuit $\{e_1, e_2, f_1, f_2\}$ or the circuit $\{e_1, e_3, f_1, f_3\}$ in one element. This contradiction to orthogonality implies that $g_1 \notin F_1 \cup F_2$, so Lemma 3.4.18(ii) does not hold. Hence, F_3 is a 4-element-fan-petal relative to F_2. This means that $|F_3| = 4$, so g_4 is not contained in a triad, and thus $g_4 = e_4$. Now, Lemma 3.4.3 implies that M has a circuit containing $\{e_1, g_1\}$ and, by orthogonality, one of $\{e_2, e_3\}$ and one of $\{g_2, g_3\}$. This implies that either $\sqcap(\{g_1, g_2\}, \{e_1, e_2, e_3\}) = 1$ or $\sqcap(\{g_1, g_3\}, \{e_1, e_2, e_3\}) = 1$, and the triangles $\{e_2, e_3, e_4\}$ and $\{g_2, g_3, e_4\}$ imply that $\sqcap(\{g_2, g_3\}, \{e_1, e_2, e_3\}) = 1$. Hence, F_3 is a 4-element-fan-petal relative to $\{e_1, e_2, e_3\}$, completing the proof. □

Lemma 3.5.4. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 11$. Let $F_1 = (e_1, e_2, e_3, e_4)$ be a maximal fan of M such that $\{e_1, e_2, e_3\}$ is a triad. Let F_2 be a triad of M which is disjoint from F_1 and not contained in a 4-element fan. Furthermore, let $F_3 \not\subseteq F_1 \cup F_2$ be a triad of M which is not contained in a 4-element fan. Then F_1 is a 4-element-fan-petal relative to F_3 and $M|(F_2 \cup F_3) \cong M(K_{3,2})$.*

Proof. Let $F_2 = \{f_1, f_2, f_3\}$ such that $\{e_1, e_2, f_1, f_2\}$ and $\{e_1, e_3, f_1, f_3\}$ are circuits. Suppose F_1 and F_3 are disjoint. Then Lemma 3.5.2 implies that F_1 is a 4-element-fan-petal relative to F_3. Furthermore, orthogonality with the circuits $\{e_1, e_2, f_1, f_2\}$ and $\{e_1, e_3, f_1, f_3\}$ implies that F_2 and F_3 are disjoint. Therefore, by Lemma 3.4.18, we have that $M|(F_2 \cup F_3) \cong M(K_{3,2})$, and the result holds.

Otherwise, suppose that $F_1 \cap F_3 \neq \emptyset$. This implies, by Lemma 3.2.11 and Lemma 3.2.8, that $e_1 \in F_3$. If $|F_2 \cap F_3| = 2$, then $r^*(F_2 \cup F_3) = 2$, contradicting Lemma 3.4.16. Thus, orthogonality with $\{e_1, e_2, f_1, f_2\}$ and $\{e_1, e_3, f_1, f_3\}$ implies that $T^* = \{e_1, f_1, e\}$, for some $e \notin F_1 \cup F_2$. Now, $(\{e_1\}, F_1 - \{e_1\}, \{f_1, e\})$ is a contractable collection. Thus, $F_1 \cup F_2 \cup \{e\}$ contains both a deletable collection and a contractable collection.

Let $g \notin F_1 \cup F_2 \cup \{e\}$. By Lemma 3.4.14, the element g is contained in a maximal fan G. Lemma 3.5.3 implies that G is a 4-element-fan-petal relative to F_2, so G has an ordering (g_1, g_2, g_3, e_4) such that $\{g_1, g_2, g_3\}$ is a triad. Furthermore, $g_1 \in F_1 \cup F_2 \cup \{e\}$. But $g_1 \notin F_1 \cup F_2$, by orthogonality, so $g_1 = e$. Note that since G is a 4-element-fan-petal relative to F_2, there is a circuit C of M containing $\{e, g_2\}$ and two elements of F_2.

Since $|E(M)| \geq 11$, there exists $h \notin F_1 \cup F_2 \cup G$. As before, h is contained in a maximal fan H with ordering (e, h_2, h_3, e_4) such that $\{e, h_2, h_3\}$. But this triad intersects the circuit C in one element, a contradiction which completes the proof. □

Lemma 3.5.5. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 11$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M such that $|F_1| \geq 4$ and $\{e_1, e_2, e_3\}$ is a triad. Let F_2 be a triad of M which is disjoint from F_1 and not contained in a 4-element fan. Then M has a paddle $(P_1, P_2, \ldots, P_m)$ and an element $x \in E(M)$ and $1 \leq t < m$ such that $M' \backslash (\{x\} \cup \bigcup_{i=1}^{t} P_i) \cong M(K_{3,m-t})$ and for all $i \in \{1, 2, \ldots, m\}$ the set $P_i - \{x\}$ is a triad, and for all $j \in \{1, 2, \ldots, t\}$, distinct from i, the set $P_j \cup \{x\}$ is a 4-element-fan-petal relative to $P_i - \{x\}$.*

Proof. By Lemma 3.5.2, we have that F_1 is a 4-element-fan-petal relative to F_2. Let $e \notin F_1 \cup F_2$. By Lemma 3.5.2, the element e is contained in a triad T^*. If T^* is contained in a 4-element fan, then Lemma 3.5.3 implies that $T^* \cup \{e_4\}$ is a 4-element-fan-petal relative to $F_1 - \{e_4\}$ and F_2. Otherwise, Lemma 3.5.4 implies that F_1 is a 4-element-fan-petal relative to T^* and $M|(F_2 \cup T^*) \cong M(K_{3,2})$.

It follows that $E(M)$ can be partitioned into $P_1, P_2, \ldots, P_m$, with $m \geq 3$ and $1 \leq t < m$ such that $P_1 = F_1$ and P_i is a triad for all $i \in \{2, 3, \ldots, m\}$. Furthermore, $M\backslash(\bigcup_{i=1}^{t} P_i) \cong M(K_{3,m-t})$ and for all distinct $i \in \{1, 2, \ldots, t\}$ and $j \in \{1, 2, \ldots, m\}$, the set $P_i \cup \{e_4\}$ is a 4-element-fan-petal relative to P_j. By Lemma 3.4.19, we have that $(P_1, P_2, \ldots, P_m)$ is a paddle of M, completing the proof. □

3.5.2 $\mathbf{F_2}$ has odd length

Next, we consider the case where F_2 is odd and has length at least five, and show that either M has a detachable pair, or M has the structure described in Theorem 3.5.1(iii)(b).

Lemma 3.5.6. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 12$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be disjoint, maximal fans of M such that $|F_1| \geq 4$, and $|F_2| \geq 5$ and odd. If $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads, then both of the following hold:*

- (i) *$|F_1| = 4$ and $|F_2| = 5$ and $F_2 \cup \{e_4\}$ is an augmented-fan-petal relative to the triad $\{e_1, e_2, e_3\}$, and*
- (ii) *every element of $E(M) - (F_1 \cup F_2)$ is contained in a triad.*

Proof. Lemma 3.4.17 implies that $|F_2| = 5$. By Lemma 3.4.3, there is a 4-element circuit C_1 of M containing $\{e_1, f_1\}$. Orthogonality with the triads $\{f_1, f_2, f_3\}$ and $\{f_3, f_4, f_5\}$ implies that $f_2 \in C_1$, and orthogonality with $\{e_1, e_2, e_3\}$ implies that either $e_2 \in C_1$ or $e_3 \in C_1$. Hence, $C_1 = \{e_1, e_i, f_1, f_2\}$ with $i \in \{2, 3\}$. Similarly, M has a 4-element circuit $C_2 = \{e_1, e_j, f_4, f_5\}$ with $j \in \{2, 3\}$. If $i = j$, then circuit elimination implies M has a circuit contained in $\{f_1, f_2, f_4, f_5, e_1\}$, and e_1 is not contained in this circuit by orthogonality with $\{e_1, e_2, e_3\}$. But now M has a circuit contained in $\{f_1, f_2, f_4, f_5\}$, which means $\lambda(F_2) \leq 1$, a contradiction. This means that either $i = 3$ or $j = 3$, which contradicts orthogonality with $\{e_3, e_4, e_5\}$ if $|F_1| \geq 5$. Hence, $|F_1| = 4$.

Note that $\lambda(F_1 \cup F_2) = 2$. Furthermore, $r(F_1 \cup F_2) = r(F_2) + 1$, and by orthogonality with the triad $\{e_1, e_2, e_3\}$, we have that $e_1, e_2, e_3 \notin \mathrm{cl}(F_2 \cup \{e_4\})$. It follows that $e_4 \in \mathrm{cl}(F_2)$, so $(\{e_4\}, F_1 - \{e_4\}, F_2)$ is a deletable collection. Since $e_1 \in \mathrm{cl}^*(F_1 - \{e_1\})$ and $|E(M)| \geq |F_1 \cup F_2| + 3 = 12$, Lemma 3.4.10 implies that every element of $E(M) - (F_1 \cup F_2)$ is contained in a triad.

Now, $\sqcap(\{f_1, f_2\}, \{e_1, e_2, e_3\}) = \sqcap(\{f_4, f_5\}, \{e_1, e_2, e_3\}) = 1$. To show that $F_2 \cup \{e_4\}$ is an augmented-fan-petal relative to $\{e_1, e_2, e_3\}$, it remains to show that $\{f_1, f_3, f_5, e_4\}$ is a circuit of M. By Lemma 3.4.17, there is a 4-element circuit $\{f_1, f_3, f_5, z\}$ of M. Assume, with the aim of reaching a contradiction, that $z \neq f_4$. It follows, by orthogonality, that $z \notin$

$F_1 \cup F_2$, and thus z is contained in a triad T^*. Orthogonality with the circuit $\{f_1, f_3, f_5, z\}$ implies that either $f_1 \in T^*$ or $f_5 \in T^*$. Furthermore, orthogonality with either C_1 or C_2 implies that $e_1 \in T^*$. But now $z \in \mathrm{cl}(F_2)$ and $z \in \mathrm{cl}^*(F_1 \cup F_2)$, so $\lambda(F_1 \cup F_2 \cup \{z\}) \leq 1$. This is a contradiction, since $|E(M)| \geq |F_1 \cup F_2 \cup \{z\}| + 2 = 12$, which completes the proof. □

Lemma 3.5.7. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 12$. Let $F_1 = (e_1, e_2, e_3, e_4)$ and $F_2 = (f_1, f_2, f_3, f_4, f_5)$ be disjoint maximal fans of M such that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads. Let $e \in E(M) - (F_1 \cup F_2)$. Then e is contained in a triad T^* such that $F_2 \cup \{e_4\}$ is a augmented-fan-petal relative to T^* and $M|(T^* \cup \{e_1, e_2, e_3\}) \cong M(K_{3,2})$.*

Proof. By Lemma 3.5.6, $F_2 \cup \{e_4\}$ is an augmented-fan-petal relative to $\{e_1, e_2, e_3\}$. Furthermore, the element e is contained in a triad T^*. Suppose T^* is not contained in a 4-element fan. Since $|F_2| = 5$, Lemma 3.4.18(i), (iii), and (iv) do not hold. Thus, $f_1 \in T^*$. Furthermore, by reversing the ordering of F_2, we see that $f_5 \in T^*$. Hence, $T^* = \{f_1, f_5, e\}$. But now F_1 and T^* are disjoint, which contradicts Lemma 3.5.5.

So T^* is contained in a 4-element fan. Let F_3 be the maximal fan containing T^*, and let $(g_1, g_2, \ldots, g_{|F_3|})$ be an ordering of F_3. Suppose $g_1 \in F_2$, which means that $g_1 \in \{f_1, f_5\}$. Lemma 3.2.8 implies that $\{g_1, g_2, g_3\}$ is a triad. Since F_2 is an augmented-fan-petal relative to F_1, orthogonality implies that $F_2 \cup F_3$ is not a $M(K_4)$-separator in M^*. Thus, Lemma 3.2.11 implies that $g_2, g_3 \notin F_2$, and Lemma 3.2.8 implies that $e_4 \notin \{g_1, g_2, g_3\}$. But now the triad $\{g_1, g_2, g_3\}$ intersects the circuit $\{f_1, f_3, f_5, e_4\}$ in one element, a contradiction. Similarly, $g_{|F_3|} \notin F_2$, which implies F_2 and F_3 are disjoint. If $\{g_1, g_2, g_3\}$ is a triangle, then $g_1 \in F_1 \cup F_2$, so $g_1 = e_4$. Similarly, if $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$ is a triangle, then $g_{|F_3|} = e_4$. Therefore, either $\{g_1, g_2, g_3\}$ or $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$ is a triad, so we may assume that $\{g_1, g_2, g_3\}$ is a triad. Thus, by Lemma 3.5.6, we have that $|F_3| = 4$ and $F_2 \cup \{g_4\}$ is an augmented-fan-petal relative to $T^* = \{g_1, g_2, g_3\}$. Also, g_4 is not contained in a triad, so $g_4 = e_4$. Finally, since $F_2 \cup \{e_4\}$ is an augmented-fan-petal relative to both $\{e_1, e_2, e_3\}$ and T^*, circuit elimination implies that $M|(T^* \cup \{e_1, e_2, e_3\}) \cong M(K_{3,2})$. This completes the proof. □

Lemma 3.5.8. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 12$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be disjoint, maximal fans of M such that $|F_1| \geq 4$, and $|F_2| \geq 5$ and odd. If $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads, then M has a paddle $(P_1, P_2, \ldots, P_m)$ where $M \backslash P_1 \cong M(K_{3,m-1})$, and, for all $i \in \{2, 3, \ldots, m\}$, the petal P_i is a triad, and P_1 is an augmented-fan-petal relative to P_i.*

Proof. By Lemma 3.5.6, we have that $|F_1| = 4$ and $F_2 \cup \{e_4\}$ is an augmented-fan-petal relative to $\{e_1, e_2, e_3\}$. Let $e \notin F_1 \cup F_2$. By Lemma 3.5.7, there exists a triad T^* of M containing

e such that $F_2 \cup \{e_4\}$ is an augmented-fan-petal relative to T^* and $M|(\{e_1, e_2, e_3\} \cup T^*) \cong M(K_{3,2})$. It follows that $E(M)$ can be partitioned into $P_1, P_2, \dots, P_m \subseteq E(M)$ such that $P_1 = F_2 \cup \{e_4\}$ and $M \backslash P_1 \cong M(K_{3,m-1})$ and, for all $i \in \{2, 3, \dots, m\}$, the set P_i is a triad and P_1 is an augmented-fan-petal relative to P_i. By Lemma 3.4.19, we have that $(P_1, P_2, \dots, P_m)$ is a paddle of M, completing the proof. □

3.5.3 $\mathbf{F_1}$ and $\mathbf{F_2}$ have even length

Finally, we consider the case where both F_1 and F_2 are even, and show that M is an even-fan-spike or a degenerate even-fan-spike. Notice that in this section we are assuming that M has two disjoint even fans of length at least four, but certain lemmas apply when one of the fans has length two — these lemmas will be useful again later on.

Lemma 3.5.9. *Let M be a 3-connected matroid with no detachable pairs. Let F_1 and F_2 be disjoint maximal fans of M with even length at least four. Then there exists orderings $(e_1, e_2, \dots, e_{|F_1|})$ and $(f_1, f_2, \dots, f_{|F_2|})$ of F_1 and F_2 respectively such that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triads, and $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$ and $\{f_{|F_1|-2}, f_{|F_1|-1}, f_{|F_1|}\}$ are triangles, and either $\{e_1, e_2, f_1, f_2\}$ is a circuit and $\{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$ is a cocircuit, or $|F_1| = |F_2| = 4$ and $\{e_1, e_2, f_1, f_2\}$ is a circuit and $\{e_2, e_4, f_2, f_4\}$ is a cocircuit.*

Proof. Let $(e_1, e_2, \dots, e_{|F_1|})$ and $(f_1, f_2, \dots, f_{|F_2|})$ be orderings of F_1 and F_2 respectively such that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triads, and $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$ and $\{f_{|F_1|-2}, f_{|F_1|-1}, f_{|F_1|}\}$ are triangles. By Lemma 3.4.3, there is a 4-element circuit C of M containing $\{e_1, f_1\}$. Orthogonality implies that C contains either e_2 or e_3. If $|F_1| > 4$, then orthogonality with $\{e_3, e_4, e_5\}$ implies that $e_2 \in C$. Furthermore, if $|F_1| = 4$, then, up to the ordering of F_1, we may assume that $e_2 \in C$. Similarly, orthogonality implies that either $f_2 \in C$ or $f_3 \in C$, so we may assume that $f_2 \in C$. Thus, $C = \{e_1, e_2, f_1, f_2\}$.

By Lemma 3.4.3, there is a 4-element cocircuit C^* of M containing $\{e_{|F_1|}, f_{|F_2|}\}$, and either $e_{|F_1|-2}$ or $e_{|F_1|-1}$, and either $f_{|F_2|-2}$ or $f_{|F_2|-1}$. If $C^* = \{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$ then the result holds. Otherwise, either $e_{|F_1|-2} \in C^*$ or $f_{|F_2|-2} \in C^*$. Without loss of generality, assume the former. If $|F_1| > 4$, then C^* intersects the triangle $\{e_{|F_1|-4}, e_{|F_1|-3}, e_{|F_1|-2}\}$ in one element, so $|F_1| = 4$. Now, $e_2 \in C \cap C^*$, so orthogonality implies that $f_2 \in C^*$. Thus, $|F_1| = |F_2| = 4$ and $\{e_2, e_4, f_2, f_4\}$ is a cocircuit, completing the proof. □

Lemma 3.5.10. *Let M be a 3-connected matroid. Let $F = (e_1, e_2, \dots, e_{|F|})$ be a maximal fan of M with length at least two such that either $|F| = 2$ or $\{e_1, e_2, e_3\}$ is a triad. Suppose there exists a 4-element circuit $C = \{e_1, e_i, a, b\}$ of M with $i \in \{2, 3\}$ and $a, b \notin F$. Then for all $x \in E(M) - (F \cup C)$, we have that $x \notin \mathrm{cl}^*(F)$.*

Proof. Suppose, to the contrary, that there exists $e \in E(M)-(F\cup X)$ such that $e \in \mathrm{cl}^*(F)$. If $|F| = 2$, then $F \cup \{e\}$ is a triad, which contradicts the maximality of F. So we may assume that $|F| \geq 3$. Since $e_1 \in \mathrm{cl}^*(F - \{e_1\})$, we also have that $e \in \mathrm{cl}^*(F - \{e_1\})$, so $\lambda((F - \{e_1\}) \cup \{e\}) = 2$. The circuit C implies that $e_i \in \mathrm{cl}(E(M) - ((F - \{e_1\}) \cup \{e\}))$, so $\lambda(E(M) - ((F - \{e_1, e_i\}) \cup \{e\})) = 2$. In turn, the element of $\{e_2, e_3\}$ which isn't e_i is contained in $\mathrm{cl}^*(E(M)-((F-\{e_1, e_i\})\cup\{e\}))$, so $\lambda(E(M)-((F-\{e_1, e_2, e_3\})\cup\{e\})) = 2$. Repeating in this way, we see that $\lambda(E(M) - \{e_{|F|-1}, e_{|F|}, e\}) = 2$, so $\{e_{|F|-1}, e_{|F|}, e\}$ is either a triangle or a triad. Since $e \in \mathrm{cl}^*(F)$, we have that $\{e_{|F|-1}, e_{|F|}, e\}$ is a triad. If $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$ is a triangle, then the fan F is not maximal, a contradiction. Hence, $\{e_{|F|-2}, e_{|F|-1}, e_{|F|}\}$ is a triad. Orthogonality implies that $|F| = 3$, but now the triad $\{e_{|F|-1}, e_{|F|}, e\}$ intersects the circuit C in one element, a contradiction. □

Lemma 3.5.11. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 9$. Let $P_1, P_2, \dots, P_m$ be disjoint subsets of $E(M)$, with $m \geq 2$, such that, for all $i \in [m]$, the set $P_i = (p^i_1, p^i_2, \dots, p^i_{|P_i|})$ is a maximal fan with even length at least two, where either $|P_i| = 2$, or $\{p^i_1, p^i_2, p^i_3\}$ is a triad. Furthermore, for all distinct $i, j \in [m]$, suppose there is a 4-element circuit $C_{i,j}$ containing $\{p^i_1, p^j_1\}$ such that $|C_{i,j} \cap P_i| = |C_{i,j} \cap P_j| = 2$, and a 4-element cocircuit $C^*_{i,j}$ containing $\{p^i_{|P_i|}, p^i_{|P_j|}\}$ such that $|C^*_{i,j} \cap P_i| = |C^*_{i,j} \cap P_j| = 2$. If $|E(M)| \leq |P_1 \cup P_2 \cup \cdots \cup P_m| + 2$, then M is a degenerate even-fan-spike or an even-fan-spike.*

Proof. First, suppose $m = 2$ and $E(M) = P_1 \cup P_2$. We show that M is a degenerate even-fan-spike. Suppose $|P_1| = 2$. Since $\lambda(P_2 - \{p^2_1\}) = 2$, we also have that $\lambda(P_1 \cup \{p^2_1\}) = 2$. But now $P_1 \cup \{p^2_1\}$ is either a triangle or a triad, contradicting the maximality of P_1. Thus, $|P_1| \geq 4$ and, similarly, $|P_2| \geq 4$. Since $|E(M)| \geq 9$, Lemma 3.5.9 implies that $\{p^1_1, p^1_2, p^2_1, p^2_2\}$ is a circuit and $\{p^1_{|P_1|-1}, p^1_{|P_1|}, p^2_{|P_2|-1}, p^2_{|P_2|}\}$ is a cocircuit, and thus M is a degenerate even-fan-spike.

Otherwise, let J be a proper, non-empty subset of $[m]$, and let $X = \bigcup_{i\in J} P_i$. We show that $\lambda(X) = 2$. If $|J| = 1$, then X is a fan, so $\lambda(X) = 2$. Otherwise, let $j \in J$, and suppose that $\lambda(X - P_j) = 2$. Now, for some $i \in J - \{j\}$, the circuit $C_{i,j}$ implies that $p^j_1 \in \mathrm{cl}(X - \{p^j_1\})$. But $p^j_1 \notin \mathrm{cl}(P_j - \{p^j_1\})$, and so $r(X) \leq r(X - P_j) + r(P_j) - 1$. Similarly, $r^*(X) \leq r^*(X - P_j) + r^*(P_j) - 1$. Therefore,

$$\begin{aligned}\lambda(X) &\leq (r(X - P_j) + r(P_j) - 1) + (r^*(X - P_j) + r^*(P_j) - 1) - (|X - P_j| + |P_j|)\\ &= \lambda(X - \{P_j\}) + \lambda(P_j) - 2 = 2\end{aligned}$$

This shows that $\lambda(X) = 2$. In particular, for all distinct $i, j \in [m]$, we have that $\lambda(P_i \cup P_j) = 2$. It follows that $r(P_i \cup P_j) = r(P_i) + r(P_j) - 1$, and so $\sqcap(P_i, P_j) = 1$. Thus, if $E(M) = P_1 \cup P_2 \cup \cdots \cup P_m$, then $(P_1, P_2, \dots, P_m)$ is a spike-like anemone, so M is an even-fan-spike.

Now, suppose $E(M) = P_1 \cup P_2 \cup \cdots \cup P_m \cup \{x\}$. Since $\lambda(P_1 \cup P_2 \cup \cdots \cup P_{m-1}) = 2$, we have that $\lambda(P_m \cup \{x\}) = 2$. This implies that either $x \in \mathrm{cl}(P_m)$ or $x \in \mathrm{cl}^*(P_m)$, which contradicts either Lemma 3.5.10 or its dual.

The last case to consider is when $E(M) = P_1 \cup P_2 \cup \cdots \cup P_m \cup \{x, y\}$. For all proper, non-empty subsets J of $[m]$, we have that $\lambda(\bigcup_{i \in [m]-J} P_i) = 2$, so $\lambda(\{x, y\} \cup \bigcup_{i \in J} P_i) = 2$. This shows that $(P_1, P_2, \ldots, P_m, \{x, y\})$ is an anemone. Also, for all $i \in [m]$, we have that $x \notin \mathrm{cl}(P_i)$ and $x \notin \mathrm{cl}^*(P_i)$, by Lemma 3.5.10. Since $\lambda(P_i \cup \{x, y\}) = 2$, this implies that $y \in \mathrm{cl}(P_i \cup \{x\}) \cap \mathrm{cl}^*(P_i \cup \{x\})$. Therefore, $\sqcap(P_i, \{x, y\}) = r(P_i) + 2 - (r(P_i) + 1) = 1$. Hence, $(P_1, P_2, \ldots, P_m, \{x, y\})$ is a spike-like anemone, so M is an even-fan-spike, completing the proof. $\square$

Lemma 3.5.12. *Let M be a 3-connected matroid with no detachable pairs. Let F_1 and F_2 be maximal disjoint fans of M with even length at least four such that $|E(M)| \geq |F_1 \cup F_2| + 3$. Let $e \notin F_1 \cup F_2$. If e is contained in a triangle or a triad, then e is contained in a 4-element fan.*

Proof. Let $(e_1, e_2, \ldots, e_{|F_1|})$ and $(f_1, f_2, \ldots, f_{|F_2|})$ be orderings of F_1 and F_2 respectively such that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads, and M has a circuit C containing $\{e_1, f_1\}$ and a cocircuit C^* containing $\{e_{|F_1|}, f_{|F_2|}\}$. Suppose there exists $e \notin F_1 \cup F_2$ such that e is contained in a triad T^* and not contained in a 4-element fan. If T^* is disjoint from F_1, then Lemma 3.5.5 contradicts the existence of two disjoint maximal fans with even length in M. Thus, $T^* \cap F_1 \neq \emptyset$, and, similarly, $T^* \cap F_2 \neq \emptyset$. Lemma 3.2.11 implies that $T^* = \{e_1, f_1, e\}$.

If $|E(M)| = |F_1 \cup F_2| + 3$, then, as $\lambda(E(M) - (F_1 \cup F_2)) = 2$, we have that $E(M) - (F_1 \cup F_2)$ is either a triangle or a triad disjoint from F_1 and F_2. By orthogonality with the circuit C and the cocircuit C^*, we have that $E(M) - (F_1 \cup F_2)$ is not contained in a 4-element fan. But $E(M) - (F_1 \cup F_2)$ is disjoint from F_1 and F_2, contradicting Lemma 3.5.5. Otherwise, $|E(M)| \geq |F_1 \cup F_2| + 4$, and $e \in \mathrm{cl}^*(F_1 \cup F_2)$. Thus, Lemma 3.4.3 implies that M has a 4-element circuit C' containing $\{e, e_1\}$, and one of $\{e_2, e_3\}$, and an element f with $f \notin F_1 \cup F_2 \cup \{e\}$. Suppose f is contained in a triad T_2^*. If T_2^* is not contained in a 4-element fan, then $T_2^* = \{e_1, f_1, f\}$. But now $r^*(\{e_1, f_1, e, f\}) = 2$, contradicting Lemma 3.4.16. Thus, T_2^* is contained in a 4-element fan. By assumption, e is not contained in a 4-element fan, so $e \notin T_2^*$. Hence, orthogonality with $\{e_1, f_1, e, f\}$ implies that either $e_1 \in T_2^*$ or $f_1 \in T_2^*$, and, furthermore, this is the only element of $T_2^* \cap (F_1 \cup F_2)$ by Lemma 3.2.11. But this contradicts orthogonality with the circuit C, so f is not contained in a triad.

Now, $f \in \mathrm{cl}(F_1 \cup F_2 \cup \{e\})$, so $M \backslash f$ is 3-connected by Lemma 3.4.1. Thus, M has a 4-element cocircuit containing $\{f, f_{|F_2|}\}$, and one of $\{f_{|F_2|-2}, f_{|F_2|-1}\}$, and one of $\{e, e_1, e_2\}$. But now $f \in \mathrm{cl}^*(F_1 \cup F_2 \cup \{e\})$, a contradiction.

Hence, if e is contained in a triad, then e is contained in a 4-element fan. A dual argument shows that if e is contained in a triangle, then e is contained in a 4-element fan, completing the proof. □

Lemma 3.5.13. *Let M be a 3-connected matroid such that $|E(M)| \geq 13$. Let $F_1 = (e_1, e_2, e_3, e_4)$ and $F_2 = (f_1, f_2, f_3, f_4)$ be disjoint maximal fans of M. Suppose that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triads and $\{e_1, e_2, f_1, f_2\}$ is a circuit and $\{e_2, e_4, f_2, f_4\}$ is a cocircuit. Then M has a detachable pair.*

Proof. Suppose, to the contrary, that M has no detachable pairs. First, assume there exists $e \notin F_1 \cup F_2$ such that e is contained in a triangle or triad. Then Lemma 3.5.12 implies that there is a 4-element fan of M which contains e. Let F_3 be a maximal fan containing e. Orthogonality with the circuit $\{e_1, e_2, f_1, f_2\}$ and the cocircuit $\{e_2, e_4, f_2, f_4\}$ implies that F_3 is disjoint from F_1 and F_2. Furthermore, by Lemma 3.5.8, we have that $|F_3|$ is not odd. Thus, let $(g_1, g_2, \ldots, g_{|F_3|})$ be an ordering of F_3 such that $\{g_1, g_2, g_3\}$ is a triad and $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$ is a triangle.

There is a 4-element circuit C of M containing $\{e_1, g_1\}$, and one of $\{e_2, e_3\}$, and one of $\{g_2, g_3\}$. Orthogonality with $\{e_2, e_4, f_2, f_4\}$ implies that $e_3 \in C$. Furthermore, if $|F_3| \geq 5$, then orthogonality implies that $g_2 \in C$, and if $|F_3| = 4$, then we may assume that $g_2 \in C$ up to the ordering of F_3. Thus, $C = \{e_1, e_3, g_1, g_2\}$. By Lemma 3.5.9, either $\{e_2, e_4, g_{|F_3|-1}, g_{|F_3|}\}$ is a cocircuit, or $|F_3| = 4$ and $\{e_3, e_4, g_2, g_4\}$ is a cocircuit. The former case contradicts orthogonality with the circuit $\{e_1, e_2, f_1, f_2\}$, so the latter holds. Similarly, M has a 4-element circuit containing $\{f_1, g_1\}$, and, by orthogonality with $\{e_2, e_4, f_2, f_4\}$ and $\{e_3, e_4, g_2, g_4\}$, this circuit is $\{f_1, f_3, g_1, g_3\}$. But now $\lambda(F_1 \cup F_2 \cup F_3) \leq 1$, which implies $E(M) \leq |F_1 \cup F_2 \cup F_3| + 1$. Lemma 3.5.9 implies that $E(M) = F_1 \cup F_2 \cup F_3$, so that $|E(M)| = 12$, a contradiction.

Otherwise, for all $x \notin F_1 \cup F_2$, we have that x is not contained in a triangle or a triad. Let $f \notin F_1 \cup F_2$. Bixby's Lemma implies that either M/f or $M\backslash f$ is 3-connected. Up to duality, we may assume the former. Then M has a 4-element circuit C_1 containing $\{e_1, e\}$, one of $\{e_2, e_3\}$, and an element $g \notin F_1 \cup F_2$. By orthogonality with $\{e_2, e_4, f_2, f_4\}$, we have that $C_1 = \{e_1, e_3, f, g\}$. Also, M has a 4-element circuit $C_2 = \{f_1, f_3, f, g'\}$, for $g' \notin F_1 \cup F_2$.

Suppose $g = g'$. Then circuit elimination implies that M has a circuit contained in $\{e_1, e_3, f_1, f_3, f\}$. But $f \notin \mathrm{cl}(F_1 \cup F_2)$ which means that $\{e_1, e_3, f_1, f_3\}$ is a circuit of M. This implies that $\lambda(F_1 \cup F_2) \leq 1$, a contradiction. Now, g is not contained in a triad, so Lemma 3.4.4 implies that $M\backslash g$ is 3-connected. Hence, M has a 4-element cocircuit containing $\{f_4, g\}$, one of $\{f_2, f_3\}$, and an element which is not an element of $F_1 \cup F_2$. Thus, by orthogonality, M has a cocircuit $\{f_3, f_4, f, g\}$. Similarly, M has a cocircuit $\{e_3, e_4, f, g'\}$.

But now $\lambda(F_1 \cup F_2 \cup \{f, g, g'\}) \leq 1$, a contradiction since $|E(M)| \geq 13$. □

Lemma 3.5.14. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 13$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of even length at least four such that $\{e_1, e_2, e_3\}$ is a triad, and let $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be a maximal fan, disjoint from F_1, with even length at least two such that either $|F_2| = 2$ or $\{f_1, f_2, f_3\}$ is a triad. If $\{e_1, e_2, f_1, f_2\}$ is a circuit and $\{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$ is a cocircuit, then M is a degenerate even-fan-spike or an even-fan-spike.*

Proof. By the assumptions of the lemma, we may choose disjoint subsets $P_1, P_2, \ldots, P_m$ of M such that, for all $i \in [m]$, the set $P_i = (p^i_1, p^i_2, \ldots, p^i_{|P_i|})$ is a maximal fan with even length at least two and ordering such that either $|P_i| = 2$ or $\{p^i_1, p^i_2, p^i_3\}$ is a triad, and, for all $j \in [m] - \{i\}$, the set $C_{i,j} = \{p^i_1, p^i_2, p^j_1, p^j_2\}$ is a circuit, and the set $C^*_{i,j} = \{p^i_{|P_i|-1}, p^i_{|P_i|}, p^j_{|P_j|-1}, p^j_{|P_j|}\}$ is a cocircuit. Furthermore, we may assume that $|P_1| \geq 4$.

If $|E(M)| \leq |P_1 \cup P_2 \cup \cdots \cup P_m| + 2$, then the result follows from Lemma 3.5.11. Otherwise, let $e \notin P_1 \cup P_2 \cup \cdots \cup P_m$ such that e is contained in a triangle or a triad. By Lemma 3.5.12, the element e is contained in a 4-element fan. Let P' be a maximal fan containing e. By orthogonality with the circuits $C_{i,j}$ and the cocircuits $C^*_{i,j}$, the fan P' is disjoint from each of the P_i. Furthermore, by Lemma 3.5.8, we have that $|P'|$ is not odd. By Lemma 3.5.9 and Lemma 3.5.13, there exists an ordering $(p'_1, p'_2, \ldots, p'_{|P'|})$ of P' such that $\{p'_1, p'_2, p'_3\}$ is a triad and $\{p'_1, p'_2, p^1_1, p^1_2\}$ is a circuit and $\{p'_{|P'|-1}, p'_{|P'|}, p^1_{|P_1|-1}, p^1_{|P_1|}\}$ is a cocircuit. For all $i \in [m]$, circuit elimination with $C_{1,i}$ implies that $\{p'_1, p'_2, p^i_1, p^i_2\}$ is a circuit, and cocircuit elimination with $C^*_{1,i}$ implies that $\{p'_{|P'|-1}, p'_{|P'|}, p^i_{|P_i|-1}, p^i_{|P_i|}\}$ is a cocircuit.

Set $P_{m+1} = P'$, and repeat the above process for every element of M which is contained in a triangle or a triad. This means we can assume that every element of M which is not an element of $P_1 \cup P_2 \cup \cdots \cup P_m$ is contained in neither a triangle nor a triad. So let $e \notin P_1 \cup P_2 \cup \cdots \cup P_m$. By Bixby's Lemma, either M/e or $M\backslash e$ is 3-connected. Without loss of generality, assume the former. Then M has a 4-element circuit C containing $\{e, p^1_1\}$ and one of $\{p^1_2, p^1_3\}$ and, by orthogonality, an element $e' \notin P_1 \cup P_2 \cup \cdots \cup P_m$. Furthermore, $p^1_3 \notin C$ by orthogonality with $\{p^1_3, p^1_4, p^1_5\}$ if $|P_1| \geq 5$, or by orthogonality with $\{p^1_3, p^1_4, p^2_3, p^2_4\}$ if $|P_1| = 4$. Thus, $\{e, e', p^1_1, p^1_2\}$ is a circuit.

Now, e' is not contained in a triangle or a triad. Lemma 3.4.4 implies that $M\backslash e'$ is 3-connected. Therefore, M has a 4-element cocircuit C^* containing $\{e', p^1_{|P_1|}\}$, and either $p^1_{|P_1|-2}$ or $p^1_{|P_1|-1}$, and an element which is not an element of $P_1 \cup P_2 \cup \cdots \cup P_m$. As before, orthogonality with $\{p^1_{|P_1|-4}, p^1_{|P_1|-3}, p^1_{|P_1|-2}\}$ if $|P_1| \geq 5$, or with $\{p^1_1, p^1_2, p^2_1, p^2_2\}$ if $|P_1| = 4$, implies that $p^1_{|P_1|-1} \in C^*$. Orthogonality with C implies that $e \in C^*$, so $C^* = \{e, e', p^1_{|P_1|-1}, p^1_{|P_1|}\}$.

Now, $\{e, e'\}$ is a maximal 2-element fan, and, for all $i \in [m]$, circuit and cocircuit elimination with $C_{1,i}$ and $C^*_{1,i}$ implies that $\{e, e', p^i_1, p^i_2\}$ is a circuit and $\{e, e', p^i_{|P_i|-1}, p^i_{|P_i|}\}$ is a cocircuit. Set $P_{m+1} = \{e, e'\}$, and repeat in this way. This completes the proof. □

3.5.4 Putting it together

We now combine the lemmas in this section to prove Theorem 3.5.1.

Proof of Theorem 3.5.1. Suppose M does not have a detachable pair. If $|F_2| = 3$, then Lemma 3.5.5 implies that (iii)(a) holds. Otherwise, $|F_2| \geq 4$. If either $|F_1|$ or $|F_2|$ is odd, then Lemma 3.5.8 implies that (iii)(b) holds. Finally, if both $|F_1|$ and $|F_2|$ are even, then Lemmas 3.5.9, 3.5.13, and 3.5.14 combine to show that (ii) holds and completes the proof. □

3.6 Intersecting Fans

For the remainder of the proof of Theorem 1.6.2, we may assume that M has no disjoint maximal fans $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ such that $|F_1| \geq 4$, and $|F_2| \geq 3$, and $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triads. Similarly, if $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are both triangles, then M^* is one of the matroids described in Theorem 3.5.1, so we may assume that this is not the case either. As shorthand for this assumption, we shall say M has no *disjoint fans with like ends*. This section concerns 3-connected matroids which have two intersecting fans. In particular, we prove the following.

Theorem 3.6.1. *Let M be a 3-connected matroid such that $|E(M)| \geq 13$ and M has no disjoint fans with like ends. Let F_1 and F_2 be distinct, maximal fans of M such that $|F_1| \geq 4$ and $|F_2| \geq 3$, and $|F_1 \cap F_2| \neq \emptyset$. Then one of the following holds.*

(i) *M has a detachable pair,*

(ii) *M is an even-fan-spike with three petals,*

(iii) *M is an even-fan-spike with tip and cotip,*

(iv) *M is an accordion,*

(v) *M or M^* is a degenerate even-fan-paddle, or*

(vi) *M' has a paddle $(P_1, P_2, \ldots, P_m)$ for some $M' \in \{M, M^*\}$ and $m \geq 3$, and either*

  (a) *M' is an even-fan-paddle, or*

(b) *there exists* $x \in E(M)$ *such that, for all distinct* $i, j \in \{1, 2, \ldots, m\}$, *the set* $P_j - \{x\}$ *is a triad, and* $P_i \cup \{x\}$ *is a 4-element-fan-petal relative to* $P_j - \{x\}$.

3.6.1 $\mathbf{F_1}$ and $\mathbf{F_2}$ have odd length

First, we consider the case where both F_1 and F_2 have odd length. By Lemma 3.4.17, we only need to consider fans with length three or five.

Lemma 3.6.2. *Let* M *be a* 3*-connected matroid such that* $|E(M)| \geq 13$. *Let* $F_1 = (e_1, e_2, e_3, e_4, e_5)$ *be a maximal fan of* M, *and suppose there exists* $e \in E(M) - F_1$ *such that* $\{e_1, e_5, e\}$ *is a triangle. Then* M *has a detachable pair.*

Proof. Suppose, to the contrary, that M has no detachable pairs. Note that e_1 and e_5 are contained in the triangle $\{e_1, e_5, e\}$, so, by Lemma 3.2.8, we have that $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triangles. Therefore, $e_1 \in \mathrm{cl}(\{e_2, e_3, e_4\})$ and $e_1 \in \mathrm{cl}(\{e_5, e\})$. Furthermore, e_1 is not contained in a triad. Hence, $(\{e_1\}, \{e_2, e_3, e_4\}, \{e_5, e\})$ is a deletable collection, and $\lambda(F \cup \{e\}) = 2$. We complete the proof of the lemma by finding $x \notin F \cup \{e\}$ such that $M\backslash x$ is 3-connected, a contradiction to Lemma 3.4.8.

Now, $\{e_1, e_5, e\} \subseteq \mathrm{cl}(\{e_2, e_3, e_4\})$. Furthermore, e_1 and e_5 are not contained in a triad, and e not contained in a triad, since orthogonality with $\{e_1, e_5, e\}$ implies that this triad contains either e_1 or e_5. Now, $|E(M)| \geq |\{e_2, e_3, e_4\}| + 7 = 10$, so Lemma 3.4.15 implies that M has elements $f, f', f'' \notin F \cup \{e\}$ such that $\{f, f', f''\} \subseteq \mathrm{cl}^*(F \cup \{e\})$ and none of f, f', and f'' are contained in a triangle. Additionally, for all $x \in F \cup \{e\}$, we have that $x \in \mathrm{cl}((F \cup \{e\}) - \{x\})$, and that $|E(M)| \geq |F \cup \{e\}| + 7 = 13$. Hence, by the dual of Lemma 3.4.15, there exists $g, g', g'' \in \mathrm{cl}(F \cup \{e, f, f', f''\})$ which are not contained in triads. In particular, $M\backslash g$ is 3-connected, a contradiction. $\square$

A consequence of the above lemma is the following, which implies that a matroid with no detachable pairs has no $M(K_4)$-separators. In particular, by Lemma 3.2.11, maximal fans intersect in only the end elements.

Corollary 3.6.3. *Let* M *be a* 3*-connected matroid such that* $|E(M)| \geq 13$. *If* M *has an* $M(K_4)$*-separator, then* M *has a detachable pair.*

Lemma 3.6.4. *Let* M *be a* 3*-connected matroid such that* $|E(M)| \geq 13$. *Let* F_1 *be a maximal fan of* M *with ordering* $(e_1, e_2, e_3, e_4, e_5)$ *such that* $\{e_1, e_2, e_3\}$ *and* $\{e_3, e_4, e_5\}$ *are triangles. If* M *has a triangle* T *which is not contained in a* 4*-element fan, then* M *has a detachable pair.*

Proof. Suppose M has no detachable pairs, and consider the dual of Lemma 3.4.18. Since $|F_1| = 5$, we have that F_1 and T do not satisfy Lemma 3.4.18(ii), (iii), or (iv). Hence, $e_1 \in T$. Furthermore, by reversing the ordering of F_1, Lemma 3.4.18 implies that $e_5 \in T$. Thus, $T = \{e_1, e_5, e\}$, for some $e \notin F$, which contradicts Lemma 3.6.2. □

Lemma 3.6.5. *Let M be a 3-connected matroid such that $|E(M)| \geq 13$. Let $F_1 = (e_1, e_2, e_3, e_4, e_5)$ and $F_2 = (f_1, f_2, f_3, f_4, f_5)$ be distinct, maximal fans of M such that $e_1 = f_1$. Then M has a detachable pair.*

Proof. Up to duality, we may assume that $\{e_1, e_2, e_3\}$ is a triangle. Since $e_1 = f_1$, Lemma 3.2.8 implies that $\{f_1, f_2, f_3\}$ is also a triangle. Now assume, with the aim of reaching a contradiction, that M does not have a detachable pair. Corollary 3.6.3 implies that $F_1 \cup F_2$ is not an $M(K_4)$-separator, so either $F_1 \cap F_2 = \{e_1\} = \{f_1\}$ or $F_1 \cap F_2 = \{e_1, e_5\} = \{f_1, f_5\}$. By Lemma 3.4.17, there exist $z, z' \in E(M)$ such that $\{e_1, e_3, e_5, z\}$ and $\{f_1, f_3, f_5, z'\}$ are cocircuits. By orthogonality with $\{f_1, f_2, f_3\}$, we have that $z \in \{f_2, f_3\}$, and by orthogonality with $\{e_1, e_2, e_3\}$, we have that $z' \in \{e_2, e_3\}$.

First, suppose $F_1 \cap F_2 = \{e_1\}$. Now, $\lambda(F_1 \cup F_2 - \{f_5\}) = 2$. But $f_5 \in \text{cl}(F_1 \cup F_2 - \{f_5\})$ and $f_5 \in \text{cl}^*(F_1 \cup F_2 - \{f_5\})$. Thus, $\lambda(F_1 \cup F_2) \leq 1$, a contradiction. Otherwise, if $F_1 \cap F_2 = \{e_1, e_5\}$, then $\lambda(F_1 \cup F_2 - \{f_4\}) = 2$ and $f_4 \in \text{cl}(F_1 \cup F_2 - \{f_4\}) \cap \text{cl}^*(F_1 \cup F_2 - \{f_4\})$. Again, $\lambda(F_1 \cup F_2) \leq 1$, which completes the proof. □

3.6.2 $\mathbf{F_1}$ and $\mathbf{F_2}$ intersect at both ends

Now, we may assume that at least one of F_1 and F_2 are even. We first consider the case where both F_1 and F_2 are even and, in particular, when F_1 and F_2 intersect in both end elements.

Lemma 3.6.6. *Let M be a 3-connected matroid with no detachable pairs. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be distinct, maximal fans of M with even length at least four. If $e_1 = f_1$ and $e_{|F_1|} = f_{|F_2|}$, then every element of M is contained in a maximal fan of length at least four with ends e_1 and $e_{|F_1|}$.*

Proof. Without loss of generality, assume that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triangles, and $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$ and $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$ are triads. Clearly, the result holds if $E(M) = F_1 \cup F_2$.

Suppose $E(M) = F_1 \cup F_2 \cup \{x\}$. Note that $\lambda(F_1 \cup F_2 - \{e_1, e_{|F_1|}\}) = 2$. But this implies that $\lambda(\{e_1, e_{|F_1|}, x\}) = 2$, so $\{e_1, e_{|F_1|}, x\}$ is either a triangle or a triad. This is a contradiction to orthogonality, since e_1 is contained in a triangle and $e_{|F_1|}$ is contained in a triad.

Next, suppose $E(M) = F_1 \cup F_2 \cup \{x, y\}$. Since $\lambda(F_1 \cup F_2 - \{e_1\}) = 2$, we have that $\lambda(\{e_1, x, y\}) = 2$. Thus, $\{e_1, x, y\}$ is a triangle. Similarly, $\lambda(\{e_{|F_1|}, x, y\}) = 2$, so $\{e_{|F_1|}, x, y\}$ is a triad. Thus, M has a maximal fan with ordering $(e_1, x, y, e_{|F_1|})$ and the result holds.

Finally, suppose $|E(M)| \geq |F_1 \cup F_2| + 3$. Note that $\lambda(F_1 \cup F_2) = 2$, and $(\{e_1\}, F_1 - \{e_1\}, F_2 - \{e_1\})$ is a deletable collection, and $(\{e_{|F_1|}\}, F_1 - \{e_{|F_1|}\}, F_2 - \{e_{|F_1|}\})$ is a contractable collection. Let $e \notin F_1 \cup F_2$. Lemma 3.4.14 implies that e is contained in a maximal fan F_3 of length at least four with ends in $F_1 \cup F_2$. Lemma 3.2.8 implies that the ends of F_3 are e_1 and $e_{|F_1|}$, completing the proof. □

Lemma 3.6.7. *Let M be a 3-connected matroid with no detachable pairs. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be distinct, maximal fans of M with even length at least four. If $e_1 = f_1$ and $e_{|F_1|} = f_{|F_2|}$, then M is a degenerate even-fan-spike with tip and cotip, or M is an even-fan-spike with tip and cotip.*

Proof. Suppose, without loss of generality, that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triangles, and $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$ and $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$ are triads. If $E(M) = F_1 \cup F_2$, then M is a degenerate even-fan-spike with tip e_1 and cotip $e_{|F_1|}$. Otherwise, choose disjoint subsets $P_1, P_2, \ldots, P_m$, with $m \geq 2$, such that

(i) for all $i \in [m]$, the set $P_i \cup \{e_1, e_{|F_1|}\}$ is an even fan,

(ii) for all proper, non-empty subsets J of $[m]$, we have that $\lambda(\bigcup_{i \in J} P_i) = 2$, and

(iii) for all distinct $i, j \in [m]$, we have that $\sqcap(P_i, P_j) = 2$.

If $E(M) = P_1 \cup P_2 \cup \cdots \cup P_m$, then $(P_1, P_2, \ldots, P_m)$ is a spike-like anemone. This implies that M is an even-fan-spike with tip and cotip, and completes the proof.

Otherwise, let $e \in E(M) - (P_1 \cup P_2 \cup \cdots \cup P_m)$. By Lemma 3.6.6, the element e is contained in a maximal fan F_3 with length at least four, ends e_1 and $e_{|F_1|}$, and ordering $(e_1, g_1, g_2, \ldots, g_{|F_3|-2}, e_{|F_1|})$. Let $P' = F_3 - \{e_1, e_{|F_1|}\}$. Let J be a proper, non-empty subset of $[m]$, and let $X = \bigcup_{i \in J} P_i$. Then $g_1 \in \mathrm{cl}(X \cup (P' - \{g_1\}))$ but $g_1 \notin \mathrm{cl}(P' - \{g_1\})$. This implies that $r(X \cup P') \leq r(X) + r(P') - 1$. Similarly, $r^*(X \cup P') \leq r^*(X) + r^*(P') - 1$. It follows that $\lambda(X \cup P') = \lambda(X) = 2$. Furthermore, for all $i \in [m]$, we have that $r(P_i \cup P') = r(P_i) + r(P') - 1$, and so $\sqcap(P_i, P') = 1$.

Therefore, if $E(M) = P_1 \cup P_2 \cup \cdots \cup P_m \cup P'$, then $(P_1, P_2, \ldots, P_m, P')$ is a spike-like anemone of M, completing the proof. Otherwise, repeating in the above manner establishes the result. □

3.6.3 $\mathbf{F_1}$ and $\mathbf{F_2}$ intersect in one end

Next, we consider the case where F_1 and F_2 both have even length and intersect in exactly one element.

Lemma 3.6.8. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends. Let $F_1 = (e_1, e_2, \dots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \dots, f_{|F_2|})$ be distinct, maximal fans of M with even length at least four such that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triangles. If $e_1 = f_1$ and $e_{|F_1|} \neq f_{|F_2|}$ and $|E(M)| \leq |F_1 \cup F_2| + 2$, then M is a degenerate even-fan-paddle.*

Proof. Lemma 3.4.3 and orthogonality implies that there is a 4-element circuit C of M containing $\{e_{|F_1|}, f_{|F_2|}\}$, and one of $\{e_{|F_1|-2}, e_{|F_1|-1}\}$, and one of $\{f_{|F_2|-2}, f_{|F_2|-1}\}$.

First, assume that $E(M) = F_1 \cup F_2$. Since M is 3-connected, the set $\{e_{|F_1|}, f_{|F_2|}\}$ is coindependent. Therefore, $E(M) - \{e_{|F_1|}, e_{|F_2|}\}$ is spanning, so

$$e_{|F_1|} \in \mathrm{cl}(F_1 \cup F_2 - \{e_{|F_1|}, f_{|F_2|}\}).$$

Now, $f_{|F_2|-1} \in \mathrm{cl}(F_2 - \{f_{|F_2|-1}, f_{|F_2|}\})$, so we have that $e_{|F_1|} \in \mathrm{cl}(F_1 \cup F_2 - \{e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\})$. Orthogonality with the triad $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$ implies that

$$e_{|F_1|} \in \mathrm{cl}(F_1 \cup F_2 - \{e_{|F_1|}, f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}).$$

Repeating in this way, we see that $e_{|F_1|} \in \mathrm{cl}(F_1 - \{e_{|F_1|}\})$. But this means that $\lambda(F_1) = 1$, a contradiction.

Next, suppose $E(M) = F_1 \cup F_2 \cup \{x\}$ with $x \notin F_1 \cup F_2$. Since $\lambda(F_1 - \{e_1\}) = 2$, we also have that $\lambda(F_2 \cup \{x\}) = 2$. Thus, either $x \in \mathrm{cl}(F_2)$ or $x \in \mathrm{cl}^*(F_2)$. Lemma 3.5.10 implies that $x \notin \mathrm{cl}^*(F_2)$, so $x \in \mathrm{cl}(F_2)$. Similarly, $x \in \mathrm{cl}(F_1)$, which implies that M is a degenerate even-fan-paddle.

Finally, suppose $E(M) = F_1 \cup F_2 \cup \{x, y\}$. We have that $\lambda(F_1 \cup F_2 - \{e_1\}) = 2$, so $\lambda(\{e_1, x, y\}) = 2$. It follows that $\{e_1, x, y\}$ is a triangle. If $\{x, y\}$ is contained in a triad, then this triad contains either $e_{|F_1|}$ or $f_{|F_2|}$, which contradicts orthogonality with the circuit C. Hence, $\{x, y\}$ is not contained in a triad, so $\{e_1, x, y\}$ is not contained in a 4-element fan. By Tutte's Triangle Lemma, either $M\backslash x$ or $M\backslash y$ is 3-connected. Without loss of generality, assume the former. Lemma 3.4.3 implies that M has a 4-element cocircuit C^* containing $\{e_1, x\}$, and one of $\{e_2, e_3\}$ and one of $\{f_2, f_3\}$. If $|F_1| > 4$, then orthogonality implies that $e_2 \in C^*$, and if $|F_1| = 4$, then we may assume $e_2 \in C^*$ up to the ordering of F_1. Similarly, we may assume $f_2 \in C^*$, so that $C^* = \{e_2, e_1, f_2, x\}$. Now, $M\backslash x$ has a fan $(e_{|F_1|}, e_{|F_1|-1}, \dots, e_2, e_1, f_2, f_3, \dots, f_{|F_2|})$. Since $|E(M\backslash x)| = |F_1 \cup F_2| + 1$, Lemma 3.2.7

implies that $M\backslash x$ is a wheel or a whirl. But $e_{|F_1|}$ is not contained in a triangle of M, so is also not contained in a triangle of $M\backslash x$. This is a contradiction which completes the proof. □

Lemma 3.6.9. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be distinct, maximal fans of M with even length at least four such that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triangles. Suppose $e_1 = f_1$ and $e_{|F_1|} \neq f_{|F_2|}$ and $|E(M)| \geq |F_1 \cup F_2| + 3$. Then, for all $x \notin F_1 \cup F_2$, the element x is contained in a maximal fan of even length at least four with ends e_1 and x^+ such that $x^+ \notin F_1 \cup F_2$.*

Proof. Lemma 3.4.3 implies that M has a 4-element circuit C containing $\{e_{|F_1|}, f_{|F_2|}\}$, which we may assume is $\{e_{|F_1|-1}, e_{|F_1|}, f_{|F_2|-1}, f_{|F_2|}\}$. Notice that $(\{e_1\}, F_1 - \{e_1\}, F_2 - \{e_1\})$ is a deletable collection and $\lambda(F_1 \cup F_2) = 2$. Furthermore, $e_{|F_1|} \in \mathrm{cl}^*(F_1 - \{e_{|F_1|}\})$ and, for each $i \in \{1, 2, \ldots, |F_1|\}$, we have that $e_i \in \mathrm{cl}(F_1 \cup F_2 - \{e_i\})$. Hence, by Lemma 3.4.10, every element of $E(M) - (F_1 \cup F_2)$ is contained in a triad.

Let $F_3 = (g_1, g_2, \ldots, g_{|F_3|})$ be a maximal fan of M, distinct from F_1 and F_2, such that $|F_3| \geq 4$. Since M has no disjoint fans with like ends, we have that $F_1 \cap F_3 \neq \emptyset$ and $F_2 \cap F_3 \neq \emptyset$. Furthermore, orthogonality with the circuit C implies that $e_{|F_1|} \notin F_3$ and $f_{|F_2|} \notin F_3$. Therefore, $e_1 \in F_3$, so, without loss of generality, $e_1 = g_1$ and $\{g_1, g_2, g_3\}$ is a triangle. Furthermore, $g_{|F_3|} \notin F_1 \cup F_2$. This implies that $g_{|F_3|}$ is contained in a triad, so $\{g_{|F_3|-2}, g_{|F_3|-1}, g_{|F_3|}\}$ is a triad. Hence, F_3 has even length.

Let $e \notin F_1 \cup F_2$. To complete the proof, it remains to show that e is contained in a 4-element fan. Suppose this is not the case. Now, e is contained in a triad T^* which is not contained in a 4-element fan. Since M has no disjoint fans with like ends, we have that $F_1 \cap T^* \neq \emptyset$ and $F_2 \cap T^* \neq \emptyset$. Hence, $T^* = \{e, e_{|F_1|}, f_{|F_2|}\}$. Now, let $f \notin F_1 \cup F_2 \cup \{e\}$. The element f is contained in a triad T_2^*. If T_2^* is not contained in a 4-element fan, then $T_2^* = \{f, e_{|F_1|}, f_{|F_2|}\}$. But this means that $r^*(\{e, f, e_{|F_1|}, f_{|F_2|}\}) = 2$, a contradiction to Lemma 3.4.16. So there is a maximal fan F with length at least four containing T_2^*. Now, F has even length and ends e_1 and f^+, say, with $f^+ \notin F_1 \cup F_2$. Furthermore, e is not contained in a 4-element fan, so $e \notin F$. But now $F \cap T^* = \emptyset$, and M has a pair of disjoint fans with like ends. This contradiction completes the proof. □

Lemma 3.6.10. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ be distinct maximal fans of M with even length at least four such that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are triangles. Suppose $e_1 = f_1$ and $e_{|F_1|} \neq f_{|F_2|}$. Then either*

(i) *M is a degenerate even-fan-paddle,*

(ii) *M is an even-fan-paddle, or*

(iii) *M has a paddle $(P_1, P_2, \dots, P_m)$ with $m \geq 3$ and $x \in E(M)$ such that, for all distinct $i, j \in \{1, 2, \dots, m\}$, the set $P_j - \{x\}$ is a triad, and $P_i \cup \{x\}$ is a 4-element-fan-petal relative to $P_j - \{x\}$.*

Proof. If $|E(M)| \leq |F_1 \cup F_2| + 2$, then M is a degenerate even-fan-paddle by Lemma 3.6.8. Otherwise, $|E(M)| \geq |F_1 \cup F_2| + 3$, and we may choose disjoint subsets $P_1, P_2, \dots, P_m \subseteq E(M)$, with $m \geq 2$, such that

(i) for all $i \in [m]$, the set $P_i \cup \{e_1\}$ is a maximal fan with even length at least four and ordering $(p^i_1, p^i_2, \dots, p^i_{|P_i|}, e_1)$,

(ii) for all proper, non-empty subsets J of $[m]$, we have that $\lambda(\bigcup_{i \in J} P_i) = 2$, and

(iii) for all distinct $i, j \in [m]$, we have that $\sqcap(P_i, P_j) = 2$.

Furthermore, for distinct $i, j \in \{1, 2, \dots, m\}$, Lemma 3.4.3 implies M has a circuit $C_{i,j}$ containing $\{p^i_1, p^j_1\}$, and one of $\{p^i_2, p^i_3\}$, and one of $\{p^j_2, p^j_3\}$.

Let $e \notin E(M) - (P_1 \cup P_2 \cup \cdots \cup P_m)$. By Lemma 3.6.9, the element e is contained in a set P' such that $P' \cup \{e_1\}$ is a maximal fan with even length at least four and ordering $(p'_1, p'_2, \dots, p'_{|P'|}, e_1)$, and, for all $i \in [m]$, there is a circuit C of M containing $\{p'_1, p^i_1\}$. Let I be a proper, non-empty subset of $[m]$, and let $X = \bigcup_{i \in I} P_i$. Now, $p'_1 \in \mathrm{cl}(X \cup P' - \{p'_1\})$ and $p'_1 \notin \mathrm{cl}(P' - \{p'_1\})$, and $p'_{|P'|} \in \mathrm{cl}(X \cup P' - \{p'_{|P'|}\})$ and $p'_{|P'|} \notin \mathrm{cl}(P' - \{p'_{|P'|}\})$. Thus, $r(X \cup P') \leq r(X) + r(P') - 2$. Since $\lambda(X \cup P') \geq 2$, the only possibility is $\lambda(X \cup P') = 2$ and $r(X \cup P') = r(X) + r(P') - 2$. Also, for all $i \in [m]$, we have that $r(P_i \cup P') = r(P_i) + r(P') - 2$, so $\sqcap(P_i, P') = 2$.

Set $P_{m+1} = P'$, and repeat in this way until we can partition $E(M)$ into such sets $P_1, P_2, \dots, P_m$. Then $(P_1, P_2, \dots, P_m)$ is a paddle. Assume $|P_i \cup \{e_4\}| = 4$, for all $i \in [m]$. For all distinct $i, j \in [m]$, we have that $\sqcap(\{p^i_2, p^i_3\}, P_j - \{e_4\}) = 1$, and the circuit $C_{i,j}$ implies that either $\sqcap(\{p^i_1, p^i_2\}, P_j - \{e_4\}) = 1$ or $\sqcap(\{p^i_1, p^i_3\}, P_j - \{e_4\}) = 1$. Hence, $P_i \cup \{e_4\}$ is a 4-element-fan-petal relative to $P_j - \{e_4\}$, and (iii) holds.

Otherwise, suppose, without loss of generality, that $|P_1 \cup \{e_4\}| > 4$. Orthogonality with $\{p^1_3, p^1_4, p^1_5\}$ implies that, for all $i \in \{2, 3, \dots, m\}$, the circuit $C_{1,i}$ contains p^1_2. Furthermore, either $|P_i \cup \{e_4\}| > 4$ and $C_{1,i}$ contains p^i_2, or $|P_i \cup \{e_4\}| = 4$ and we may choose the ordering of $P_i \cup \{e_4\}$ such that $p^i_2 \in C_{1,i}$. Now, for any other $j \in [m]$, circuit elimination between $C_{1,i}$ and $C_{1,j}$ implies that $\{p^i_1, p^i_2, p^j_1, p^j_2\}$ is a circuit. Hence, M is an even-fan-paddle, completing the proof. $\square$

3.6.4 One of $\mathbf{F_1}$ and $\mathbf{F_2}$ is odd

Finally, we consider the case where exactly one of F_1 and F_2 is odd, and show that the resulting matroid is either an accordion or an even-fan-spike with three petals.

Lemma 3.6.11. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends such that $|E(M)| \geq 8$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$ is a triangle, and let $F_2 = (f_1, f_2, f_3, f_4, f_5)$ be a maximal fan of M such that $e_1 = f_1$. Then $|E(M)| \geq |F_1 \cup F_2| + 2$, and $F_2 - \{e_1\}$ is a fan-type accordion end with F_1 in M, and a co-fan-type accordion end with F_1 in M^*.*

Proof. By Lemma 3.2.8, the set $\{f_1, f_2, f_3\}$ is a triangle. Lemma 3.4.17 implies that there exists $z \notin F_2$ such that $\{f_1, f_3, f_5, z\}$ is a cocircuit. By orthogonality, $z \in \{e_2, e_3\}$, and so, up to the ordering of F_1 if $|F_1| = 4$, we have that $z = e_2$. It follows that $F_1 \cup F_2 - \{f_5\}$ is a fan of $M \backslash f_5$. The element $e_{|F_1|}$ is not contained in a triangle of M, so is also not contained in a triangle of $M \backslash f_5$. Thus, $M \backslash f_5$ is not a wheel or a whirl, so Lemma 3.2.7 implies that $|E(M \backslash f_5)| \geq |F_1 \cup F_2 - \{f_5\}| + 2$, and thus $|E(M)| \geq |F_1 \cup F_2| + 2$.

To show that $F_2 - \{e_1\}$ is a fan-type accordion end with F_1, it remains to show that $\sqcap(\{f_2, f_4\}, E(M) - (F_1 \cup F_2)) = 1$. First, note that $f_5 \in \text{cl}^*(F_1 \cup \{f_3\})$, so $\lambda(F_1 \cup \{f_3, f_5\}) \leq 3$. Hence,

$$\lambda(E(M) - (F_1 \cup \{f_3, f_5\})) = \lambda((E(M) - (F_1 \cup F_2)) \cup \{f_2, f_4\}) \leq 3.$$

By orthogonality with the triangle $\{f_1, f_2, f_3\}$ and the triad $\{f_2, f_3, f_4\}$ we have that $f_2 \notin \text{cl}(E(M) - (F_1 \cup F_2))$ and $f_2 \notin \text{cl}^*(E(M) - (F_1 \cup F_2))$. Thus, $\lambda((E(M) - (F_1 \cup F_2)) \cup \{f_2\}) = 3$. Now, $f_4 \notin \text{cl}^*((E(M) - (F_1 \cup F_2)) \cup \{f_2\})$, by orthogonality with $\{f_3, f_4, f_5\}$, so $f_4 \in \text{cl}((E(M) - (F_1 \cup F_2)) \cup \{f_2\})$. It follows that

$$r((E(M) - (F_1 \cup F_2)) \cup \{f_2, f_4\}) = r(E(M) - (F_1 \cup F_2)) + 1$$

so $\sqcap(\{f_2, f_4\}, E(M) - (F_1 \cup F_2)) = 1$, as desired.

To complete the proof, we show that $F_2 - \{e_1\}$ is a co-fan-type accordion end with F_1 in M^* by showing that $\sqcap^*(\{f_4, f_5\}, E(M) - (F_1 \cup F_2)) = 1$. Orthogonality with the cocircuits $\{f_2, f_3, f_4\}$ and $\{e_2, e_1, f_3, f_5\}$ implies that

$$r((E(M) - (F_1 \cup F_2)) \cup \{f_4, f_5\}) = r(E(M) - (F_1 \cup F_2)) + 2$$

and orthogonality with $\{e_3, e_4, e_5\}$ implies that

$$r^*((E(M) - (F_1 \cup F_2)) \cup \{f_4, f_5\}) \geq r^*(E(M) - (F_1 \cup F_2)) + 1.$$

But
$$\lambda((E(M) - (F_1 \cup F_2)) \cup \{f_4, f_5\}) = \lambda(F_1 \cup \{f_2, f_3\}) \leq 3$$
so
$$r^*((E(M) - (F_1 \cup F_2)) \cup \{f_4, f_5\}) = r^*(E(M) - (F_1 \cup F_2)) + 1$$
which means that $\sqcap^*(\{f_4, f_5\}, E(M) - (F_1 \cup F_2)) = 1$. □

Lemma 3.6.12. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends. Let $F_1 = (e_1, e_2, \dots, e_{|F_1|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$ is a triangle, and let $\{e_1, f_2, f_3\}$ be a triangle of M which is not contained in a 4-element fan such that $\{e_1, e_2, f_2, f_3\}$ is a cocircuit. Then $|E(M)| \geq |F_1 \cup \{f_2, f_3\}| + 2$ and $\{f_2, f_3\}$ is a triangle-type accordion end with F_1 in M, and a triad-type accordion end with F_1 in M^*.*

Proof. By Tutte's Triangle Lemma, wither $M\backslash f_2$ or $M\backslash f_3$ is 3-connected. Without loss of generality, we assume the latter. The matroid $M\backslash f_3$ has a fan $F_1 \cup \{f_2\}$. Furthermore, $M\backslash f_3$ is not a wheel or a whirl, since $e_{|F_1|}$ is not contained in a triangle. Thus, by Lemma 3.2.7, we have that $|E(M)| \geq |F_1 \cup \{f_2, f_3\}| + 2$. By orthogonality with the triangle $\{e_1, f_2, f_3\}$ and the cocircuit $\{e_1, e_2, f_2, f_3\}$, we have that $f_2 \notin \mathrm{cl}(E(M) - (F_1 \cup \{f_2, f_3\}))$ and $f_2 \notin \mathrm{cl}^*(E(M) - (F_1 \cup \{f_2, f_3\}))$. Since $\lambda(E(M) - F_1) = 2$, it follows that $f_3 \in \mathrm{cl}(E(M) - (F_1 \cup \{f_3\}))$ and $f_3 \in \mathrm{cl}^*(E(M) - (F_1 \cup \{f_3\}))$. Thus,
$$r(E(M) - F_1) = r(E(M) - (F_1 \cup \{f_2, f_3\})) + 1$$
and
$$r^*(E(M) - F_1) = r^*(E(M) - (F_1 \cup \{f_2, f_3\})) + 1.$$
It follows that $\sqcap(\{f_2, f_3\}, E(M) - (F_1 \cup \{f_2, f_3\})) = 1$, so $\{f_2, f_3\}$ is a triangle-type accordion end with F_1 in M, and $\sqcap^*(\{f_2, f_3\}, E(M) - (F_1 \cup \{f_2, f_3\})) = 1$, so $\{f_2, f_3\}$ is a triad-type accordion end with F_1 in M^*. □

Naturally, in the next lemma, we aim to show that if M is a 3-connected matroid with a maximal fan F_1 of even length at least four with ordering $(e_1, e_2, \dots, e_{|F_1|})$ and distinct triangles $\{e_1, f_2, f_3\}$ and $\{e_1, f_2, f_3\}$, then $\{f_2, f_3, g_2, g_3\}$ is a K_4-type accordion end with F_1. However, there is one problematic case we need to consider: if $|F_1| = 4$, and $\{e_1, e_2, f_2, g_2\}$ and $\{e_1, e_3, f_3, g_3\}$ are cocircuits, and
$$\sqcap(\{f_2, g_2\}, E(M) - (F_1 \cup \{f_2, f_3, g_2, g_3\})) = \sqcap(\{f_3, g_3\}, E(M) - (F_1 \cup \{f_2, f_3, g_2, g_3\})) = 1$$
then we say $\{f_2, f_3, g_2, g_3\}$ is an *almost-K_4-type accordion end with F_1*. Similarly, if $\{e_1, e_2, f_2, g_2\}$ and $\{e_1, e_3, f_3, g_3\}$ are circuits, and
$$\sqcap^*(\{f_2, f_3\}, E(M) - (F_1 \cup \{f_2, f_3, g_2, g_3\})) = \sqcap^*(\{g_2, g_3\}, E(M) - (F_1 \cup \{f_2, f_3, g_2, g_3\})) = 1$$
then we say $\{f_2, f_3, g_2, g_3\}$ is an *almost-co-K_4-type accordion end with F_1*.

Lemma 3.6.13. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends such that $|E(M)| \geq 10$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M with even length at least four. Let $\{e_1, f_2, f_3\}$ and $\{e_1, g_2, g_3\}$ be triangles of M which are not contained in 4-element fans such that $\{e_1, e_2, f_2, g_2\}$ is a cocircuit. Then either M is an even-fan-spike with three petals, or all of the following hold.*

(i) $|E(M)| \geq |F_1 \cup \{f_2, f_3, g_2, g_3\}| + 2,$

(ii) $\{f_2, f_3, g_2, g_3\}$ *is a K_4-type or an almost-K_4-type accordion end with F_1 in M, and*

(iii) $\{f_2, f_3, g_2, g_3\}$ *is a co-K_4-type or an almost-co-K_4-type accordion end with F_1 in M^*.*

Proof. We show that $M\backslash f_3$ is 3-connected. Suppose, to the contrary, that $M\backslash f_3$ is not 3-connected. The element f_3 is not contained in a triad, so M has a cyclic 3-separation $(X, \{f_3\}, Y)$. By Corollary 3.2.16, we may assume that $F_1 \subseteq X$. If $f_2 \in X$, then $f_3 \in \mathrm{cl}(X)$, a contradiction. Furthermore, by Corollary 3.2.15, we have that $f_2 \notin \mathrm{cl}(X)$ and $f_2 \notin \mathrm{cl}^*(X)$. This implies that $g_2 \in Y$. In turn, $g_3 \in Y$, since $g_2 \notin \mathrm{cl}(X)$. But now $e_1 \in \mathrm{cl}(Y)$, so M has a cyclic-3-separation $(X - \{e_1\}, \{f_3\}, Y \cup \{e_1\})$. But $f_3 \in \mathrm{cl}(Y \cup \{e_1\})$, a contradiction. Thus, $M\backslash f_3$ is 3-connected. So M has a 4-element cocircuit C^* containing $\{e_1, f_3\}$, and one of $\{e_2, e_3\}$, and one of $\{g_2, g_3\}$. If $g_2 \in C^*$, then, by cocircuit elimination, M has a cocircuit contained in $\{e_1, e_2, e_3, f_2, f_3\}$. But now $\lambda(\{e_1, e_2, e_3, f_2, f_3\}) = 2$, and $(\{e_1\}, \{e_2, e_3\}, \{f_2, f_3\})$ is a deletable collection. This contradicts Lemma 3.4.8 since either $M\backslash g_2$ or $M\backslash g_3$ is 3-connected. Hence, $g_3 \in C^*$. Furthermore, if $e_3 \in C^*$, then $|F_1| = 4$.

If $E(M) = F_1 \cup \{f_2, f_3, g_2, g_3\}$ then $(F_1, \{f_2, f_3\}, \{g_2, g_3\})$ is a spike-like anemone, so M is a even-fan-spike with three petals. Next, suppose $E(M) = F_1 \cup \{f_2, f_3, g_2, g_3, x\}$. We have that $\lambda(\{e_1, e_2, e_3, f_2, f_3, g_2, g_3\}) = 2$. Furthermore, $|F_1| > 4$, since $|E(M)| \geq 10$, so $\lambda((F_1 - \{e_{|F_1|-1}, e_{|F_1|}\}) \cup \{f_2, f_3, g_2, g_3\}) = 2$. Therefore, $\lambda(\{e_{|F_1|-1}, e_{|F_1|}, x\}) = 2$, so $\{e_{|F_1|-1}, e_{|F_1|}, x\}$ is either a triangle or a triad. But this contradicts the maximality of F_1. Hence, $|E(M)| \geq |F_1 \cup \{f_2, f_3, g_2, g_3\}| + 2$.

Now, let $H = E(M) - (F_1 \cup \{f_2, f_3, g_2, g_3\})$. We show that $\sqcap(\{f_2, g_2\}, H) = \sqcap(\{f_3, g_3\}, H) = 1$ and $\sqcap^*(\{f_2, f_3\}, H) = \sqcap^*(\{g_2, g_3\}, H) = 1$. This means that, if $C^* = \{e_1, e_2, f_3, g_3\}$, then $\{f_2, f_3, g_2, g_3\}$ is a K_4-type accordion end with F_1 in M and a co-K_4-type accordion end with F_1 in M^*, while if $C^* = \{e_1, e_3, f_3, g_3\}$, then $\{f_2, f_3, g_2, g_3\}$ is an almost-K_4-type accordion end with F_1 in M and a almost-co-K_4-type accordion end with F_1 in M^*. Orthogonality with $\{e_1, f_2, f_3\}$ and $\{e_1, g_2, g_3\}$ implies that $r^*(H \cup \{f_2, g_2\}) = r^*(H) + 2$. Orthogonality with C^* implies that $r(H \cup \{f_2, g_2\}) \geq r(H) + 1$. But

$$\lambda(H \cup \{f_2, g_2\}) = \lambda(F_1 \cup \{f_3, g_3\}) \leq 3.$$

Thus, $r(H \cup \{f_2\}) = r(H) + 1$, so $\sqcap(\{f_2, g_2\}, H) = 1$. In the same way, $\sqcap(\{f_3, g_3\}, H) = 1$.

Similarly, orthogonality with the cocircuits C_1^* and C_2^* and with the triangle $\{e_1, f_2, f_3\}$ imply that $r(H \cup \{f_2, f_3\}) = r(H)+2$ and $r^*(H \cup \{f_2, f_3\}) \geq r^*(H)+1$. Since $\lambda(H \cup \{f_2, f_3\}) \leq 3$, we have that $r^*(H \cup \{f_2, f_3\}) = r^*(H) + 1$, so $\sqcap^*(\{f_2, f_3\}, H) = 1$. Symmetrically, $\sqcap^*(\{g_2, g_3\}, H) = 1$, completing the proof. □

Lemma 3.6.14. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends such that $|E(M)| \geq 11$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$ is a triangle. Let G be a fan-type, triangle-type, K_4-type, or almost-K_4-type accordion end with F_1 such that $|E(M)| \geq |F_1 \cup G| + 2$. Then M is an accordion.*

Proof. First, observe that $(\{e_1\}, \{e_2, e_3\}, G)$ is a deletable collection, and $\lambda(G \cup \{e_1, e_2, e_3\}) = 2$. Suppose $|E(M)| = |F_1 \cup G| + 2$, and let $H = E(M) - (F_1 \cup G)$. Now, $\lambda(F_1 \cup G - \{e_{|F_1|}\}) = 2$, which implies that $H \cup \{e_{|F_1|}\}$ is a triad. Furthermore, $|E(M)| \geq 11$, which means that $|F_1| > 4$. Now, $\lambda(F_1 \cup G - \{e_{|F_1|-1}, e_{|F_1|}\}) = 2$, and so $\lambda(H \cup \{e_{|F_1|-1}, e_{|F_1|}\}) = 2$. Thus, either $e_{|F_1|-1} \in \mathrm{cl}(H \cup \{e_{|F_1|}\})$ or $e_{|F_1|-1} \in \mathrm{cl}^*(H \cup \{e_{|F_1|}\})$. In the latter case, $r^*(H \cup \{e_{|F_1|-1}, e_{|F_1|}\}) = 2$, contradicting Lemma 3.4.16. Hence, since $e_{|F_1|}$ is not contained in a triangle, it follows that $H \cup \{e_{|F_1|-1}, e_{|F_1|}\}$ is a circuit. By Lemma 3.6.11, the set H is a co-fan-type accordion end with F_1. Since $|F_1| > 4$, we have that G is not an almost-K_4-type accordion end, so M is an accordion. Hence, we may assume that $|E(M)| \geq |F_1 \cup G| + 3$.

We show that there is a triad of M which is not contained in $F_1 \cup G$. Suppose this is not the case, that is, no element of $E(M) - (F_1 \cup G)$ is contained in a triad. Let $e \in E(M) - (F_1 \cup G)$. By Lemma 3.4.9, the element e is also not contained in a triangle. Furthermore, $M \backslash e$ is not 3-connected. Thus, by Bixby's Lemma, M/e is 3-connected. If $|E(M)| = |F_1 \cup G| + 3$, then, since $\lambda(E(M) - (F_1 \cup G)) = 2$, we have that $E(M) - (F_1 \cup G)$ is either a triangle or a triad. But no element of $E(M) - (F_1 \cup G)$ is contained in a triad, so $E(M) - (F_1 \cup G)$ is a triangle which is not contained in a 4-element fan and is disjoint from F_1. This is a contradiction, so $|E(M)| \geq |F_1 \cup G| + 4$. Thus, Lemma 3.4.3 implies that M has a 4-element circuit C containing $\{e, e_{|F_1|}\}$, and one of $\{e_{|F_1|-1}, e_{|F_1|-2}\}$, and an element $f \notin F_1 \cup G$. But f is not contained in a triad, so Lemma 3.4.4 implies that $M \backslash f$ is 3-connected. This contradiction implies that M has a triad $T^* \not\subseteq F_1 \cup G$.

Suppose T^* is contained in a 4-element fan, and let F_2 be a maximal fan containing T^*. Since M has no disjoint fans with like ends, we have that $F_1 \cap F_2 \neq \emptyset$. By either Lemma 3.6.7 or Lemma 3.6.10, the fan F_2 does not have even length, so $|F_2|$ is odd, and thus $|F_2| = 5$. If F_2 intersects $G \cup \{e_1\}$, then F_2 is an odd fan intersecting either a triangle or a 5-element fan, contradicting Lemma 3.6.4 or Lemma 3.6.5. Thus, $e_1 \notin F_2$, so $e_{|F_1|} \in F_2$. Now, Lemma 3.6.11 implies that $H = F_2 - \{e_{|F_1|}\}$ is a co-fan-type accordion end with F_1. Next, we consider the cases where T^* is not contained in a 4-element fan, and show that M has a triad-type or co-K_4-type accordion end.

We have that $F_1 \cap T^* \neq \emptyset$, which implies that $e_{|F_1|} \in T^*$. Without loss of generality, $T^* = \{e_{|F_1|}, f_2, f_3\}$ such that M/f_2 is 3-connected. Now, M has a 4-element circuit C containing $\{f_2, e_{|F_1|}\}$, one of $\{e_{|F_1|-1}, e_{|F_1|-2}\}$, and an element $e \notin F_1 \cup G$. If $|F_1| > 4$, then orthogonality with $\{e_{|F_1|-4}, e_{|F_1|-3}, e_{|F_1|-2}\}$ implies that $e_{|F_2|-1} \in C$. If $|F_1| = 4$, then, regardless of which type of accordion end G is, we have that $e_{|F_1|-2} = e_2 \in \mathrm{cl}^*(G \cup \{e_1\})$. Again, $e_{|F_2|-1} \in C$. If $e = f_3$, then Lemma 3.6.12 implies that $H = \{f_2, f_3\}$ is a triad-type accordion end with F_1. Otherwise, suppose $e \neq f_3$. If e is not contained in a triad, then Lemma 3.4.4 implies that $M \backslash e$ is 3-connected, a contradiction. Thus, there is a triad T_2^* of M containing e. Furthermore, $T_2^* \cap F_1 \neq \emptyset$, so $e_{|F_1|} \in T_2^*$. Now, Lemma 3.6.13 implies that $H = T^* \cup T_2^* - \{e_{|F_1|}\}$ is a co-K_4-type accordion end or an almost-co-K_4-type accordion end with F_1.

In all cases, we have a set H which is either a co-fan-type, triad-type, co-K_4-type, or almost-co-K_4-type accordion end with F_1. Also note that there is a circuit of M containing $\{e_{|F_1|-1}, e_{|F_1|}\}$ and two elements of H. By orthogonality, G is not an almost-K_4-type accordion end, and similarly H is not an almost-co-K_4-type accordion end. Now, M has a contractable collection $(\{e_{|F_1|}\}, \{e_{|F_1|-1}, e_{|F_1|-2}\}, H)$ and $\lambda(H \cup \{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}) = 2$. Combined with the deletable collection $(\{e_1\}, \{e_2, e_3\}, G)$, Lemma 3.4.14 implies that every element of $E(M) - (F_1 \cup G \cup H)$ is contained in a 4-element fan. If F is a maximal fan of M with length at least four which is not contained in $F_1 \cup G \cup H$, then F has odd length and contains either e_1 or $e_{|F_1|}$. But now F intersects a maximal fan of odd length in either G or H. This is a contradiction, and shows that $E(M) = F_1 \cup G \cup H$. Thus, M is an accordion. $\square$

Lemma 3.6.15. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$ is a triangle. Let $T = \{e_1, f_2, f_3\}$ be a triangle of M and $T^* = \{e_{|F_1|}, g_2, g_3\}$ be a triad of M, and let $e \notin F_1 \cup \{f_2, f_3, g_2, g_3\}$ such that $\{e_1, e_2, f_2, e\}$ is a cocircuit. If M has an element $x \neq e$ such that x is not contained in a triangle or a triad and $M \backslash x$ is 3-connected, then $x \in \mathrm{cl}^*(F_1 \cup \{f_2, f_3\})$ and $x \in \mathrm{cl}(F_1 \cup \{g_2, g_3\})$.*

Proof. Lemma 3.4.3 implies that M has a 4-element cocircuit C^* containing $\{e_1, x\}$, and either e_2 or e_3, and either f_2 or f_3, so $x \in \mathrm{cl}^*(F_1 \cup \{f_2, f_3\})$. Now, suppose M/x is not 3-connected. Then M has a vertical 3-separation $(X, \{x\}, Y)$, and we may assume that $F_1 \subseteq X$. If $\{f_2, f_3\} \subseteq X$, then $x \in \mathrm{cl}^*(X)$, a contradiction. This implies that $f_2 \notin \mathrm{cl}(X)$ and $f_2 \notin \mathrm{cl}^*(X)$, from which it follows that $\{e, f_2, f_3\} \subseteq Y$. But now $e_1 \in \mathrm{cl}(Y)$, and $e_2 \in \mathrm{cl}^*(Y \cup \{e_1\})$, and, repeating in this way, we see that $(X - F_1, \{x\}, Y \cup F_1)$ is a vertical 3-separation of M. However, $x \in \mathrm{cl}^*(Y \cup F_1)$, a contradiction. Hence, M/x is 3-connected, so M has a 4-element circuit containing $\{e_{|F_1|}, x\}$, and either $e_{|F_1|-2}$ or $e_{|F_1|-1}$, and either

g_2 or g_3. Thus, $x \in \mathrm{cl}(F_1 \cup \{g_2, g_3\})$, as desired. □

Lemma 3.6.16. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends such that $|E(M)| \geq 13$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$ is a triangle and M has no other 4-element fans. Let $\{e_1, f_2, f_3\}$ be a triangle of M such that $M \backslash f_2$ is 3-connected. Furthermore, let $e \in E(M) - (F_1 \cup \{f_2, f_3\})$ such that $\{e_1, e_2, f_2, e\}$ is a cocircuit and e is not contained in a triangle. Then M is an even-fan-spike with three petals.*

Proof. Since $M \backslash f_2$ is 3-connected, Lemma 3.4.4 implies that M/e is 3-connected. Therefore, M has a 4-element circuit C containing $\{e_{|F_1|}, e\}$, and one of $e_{|F_1|-2}$ and $e_{|F_1|-1}$, and an element $f \notin F_1 \cup \{e\}$. If T^* is a triad of M which is not contained in F_1, then, since T^* is not contained in a 4-element fan and M has no disjoint fans with like ends, we have that $T^* \cap F_1 \neq \emptyset$. Hence, $e_{|F_1|} \in T^*$, and, by orthogonality with C, either $e \in T^*$ or $f \in T^*$. Now, every triad of M which is not contained in F_1 contains either $\{e_4, e\}$ or $\{e_4, f\}$. It follows, by Lemma 3.4.16, that there are at most two elements of $E(M) - (F_1 \cup \{f_2, f_3, e, f\})$ contained in triads.

The strategy for this proof is to find a set X with $F_1 \cup \{f_2, f_3, e, f\} \subseteq X$ and $\lambda(X) = 2$. Then $(\{e_1\}, F_1 - \{e_1\}, \{f_2, f_3\})$ is a deletable collection, and $e_{|F_1|} \in \mathrm{cl}^*(F_1 - \{e_{|F_1|}\})$, and, for all $i \in \{1, 2, \ldots, |F_1|\}$, we have that $e_i \in \mathrm{cl}(X)$. Hence, if $|E(M)| \geq |X| + 3$, Lemma 3.4.10 implies that every element of $E(M) - X$ is contained in a triad. But $E(M) - X$ has at most two elements contained in triads. This is a contradiction, so $|E(M)| \leq |X| + 2$.

We set about finding such a set X. First, suppose $f \neq f_2$. Orthogonality with the cocircuit $\{e_1, e_2, f_2, e\}$ implies that $e_2 \in C$, so $|F_1| = 4$. Suppose f is not contained in a triad. If $f = f_3$, then $\lambda(F_1 \cup \{f_2, f_3, e\}) = 2$. Thus, $|E(M)| \leq |F_1 \cup \{f_2, f_3, e\}| + 2 = 9$, a contradiction. Otherwise, since f is not contained in a triad, Lemma 3.4.4 implies that $M \backslash f$ is 3-connected. Hence, M has a 4-element cocircuit containing $\{e_1, f\}$, and one of $\{e_2, e_3\}$, and one of $\{f_2, f_3\}$. Now, $\lambda(F_1 \cup \{f_2, f_3, e, f\}) = 2$, again a contradiction.

Next, assume that f is contained in a triad T^*. The triad T^* also contains e_4. Let $g \notin F_1 \cup T^* \cup \{f_2, f_3, e\}$ such that g is not contained in a triad. Such an element must exist, since $E(M) - (F_1 \cup \{f_2, f_3, e, f\})$ has at most two elements contained in triads. If g is contained in a triangle T, then T contains e_1, since M has no disjoint fans with like ends. But $e \notin T$ since e is not contained in a triangle, and $f_2 \notin T$ as otherwise $r(\{e_1, f_2, f_3, g\}) = 2$. This contradicts orthogonality with the cocicircuit $\{e_1, e_2, f_2, e\}$, so g is not contained in a triangle or a triad. By Bixby's Lemma, either $M \backslash g$ or M/g is 3-connected. If $M \backslash g$ is 3-connected, then Lemma 3.6.15 implies that $g \in \mathrm{cl}(F_1 \cup T^* \cup \{f_2, f_3\})$ and $g \in \mathrm{cl}^*(F_1 \cup T^* \cup \{f_2, f_3\})$. If M/g is 3-connected, then the dual of Lemma 3.6.15 implies that $g \in \mathrm{cl}(F_1 \cup T^* \cup \{f_2, f_3\})$ and $g \in \mathrm{cl}^*(F_1 \cup T^* \cup \{f_2, f_3\})$. In either case,

$\lambda(F_1 \cup T^* \cup \{f_2, f_3, e, g\}) = 2$, a contradiction since $|E(M)| \geq |F_1 \cup T^* \cup \{f_2, f_3, e, g\}| + 3 = 13$.

Thus, $f = f_2$. This means that $\lambda(F_1 \cup \{f_2, f_3, e\}) = 2$, and so $|E(M)| \leq |F_1 \cup \{f_2, f_3, e\}| + 2$. Suppose $|E(M)| = |F_1 \cup \{f_2, f_3, e\}| + 2$. We have that $\lambda(F_1 \cup \{e, f_2\}) = 2$, so $\lambda(E(M) - (F_1 \cup \{e, f_2\})) = 2$. Hence, $E(M) - (F_1 \cup \{e, f_2\})$ is a triangle or a triad disjoint from F_1, a contradiction. Now, $|E(M)| \leq |F_1 \cup \{e, f_2\}| + 2$, so M is an even-fan-spike by Lemma 3.5.11. □

Lemma 3.6.17. *Let M be a 3-connected matroid with no detachable pairs and no disjoint fans with like ends such that $|E(M)| \geq 13$. Let $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$ is a triangle. Let $F_2 = (f_1, f_2, \ldots, |F_1|)$ be a maximal fan of M with odd length at least three such that $f_1 = e_1$. Then M is either an even-fan-spike with three petals or an accordion.*

Proof. First, suppose M has a maximal fan F, distinct from F_1, with length at least four. Then $F \cap F_1 \neq \emptyset$ and F has odd length, so $|F| = 5$. Up to reversing the ordering of F_1 and duality, we have that $e_1 \in F$ and $\{e_1, e_2, e_3\}$ is a triangle. By Lemma 3.6.11, the set $G = F_2 - \{e_1\}$ is a fan-type accordion end with F_1, and $|E(M)| \geq |F_1 \cup G| + 2$. Thus, M is an accordion by Lemma 3.6.14.

Otherwise, M has no 4-element fans outside of F_1 whatsoever, so $|F_2| = 3$. Without loss of generality, assume that $M \backslash f_2$ is 3-connected, so M has a 4-element cocircuit $C^* = \{e_1, e_2, f_2, e\}$, where $e \notin F_1$. If $e = f_3$, then $\{f_2, f_3\}$ is a triangle-type accordion end and $|E(M)| \geq |F_1 \cup \{f_2, f_3\}| + 2$, by Lemma 3.6.12. Again, M is an accordion by Lemma 3.6.14.

Finally, suppose $e \neq f_3$. If e is not contained in a triangle, then M is an even-fan-spike with three petals by Lemma 3.6.16. Otherwise, e is contained in a triangle T, which contains e_1. We apply Lemma 3.6.13. Either M is an even-fan-spike with three petals, or $F_2 \cup T - \{e_1\}$ is a K_4-type or almost-K_4-type accordion end with F_1 and $|E(M)| \geq |F_1 \cup F_2 \cup T| + 2$. Therefore, M is an accordion, by Lemma 3.6.14. □

3.6.5 Putting it together

Proof of Theorem 3.6.1. Let $(e_1, e_2, \ldots, e_{|F_1|})$ and $(f_1, f_2, \ldots, f_{|F_2|})$ be orderings of F_1 and F_2 respectively. By Lemma 3.2.11, we may assume that $e_1 = f_1$, and up to duality, suppose that $\{e_1, e_2, e_3\}$ is a triangle. By Lemma 3.2.8, the set $\{f_1, f_2, f_3\}$ is also a triangle. Suppose F_1 and F_2 are both odd. If $F_1 \cup F_2$ is an $M(K_4)$-separator, then M has a detachable pair by Corollary 3.6.3. Otherwise, if $|F_2| = 3$, then Lemma 3.6.4 implies that M has a detachable pair, and if $|F_2| > 3$, then Lemma 3.6.5 implies that M has a detachable pair. Next, suppose both F_1 and F_2 are even. If $F_1 \cap F_2 = \{e_1, e_{|F_1|}\}$, then M is a degenerate even-fan-spike with tip and cotip or an even-fan-spike with tip and cotip by Lemma 3.6.7. If $F_1 \cap F_2 = \{e_1\}$, then either (iv) or (v) holds by Lemma 3.6.10. Finally, if F_1 is odd and F_2 is even or vice versa, then Lemma 3.6.17 implies that either M is an even-fan-spike with three petals or M is an accordion. □

3.7 Remaining Fan Cases

We may now assume that M has no disjoint fans with like ends, and no intersecting fans. This means that, if $F_1 = (e_1, e_2, \ldots, e_{|F_1|})$ and $F_2 = (f_1, f_2, \ldots, f_{|F_2|})$ are distinct maximal fans such that $|F_1| \geq 4$ and $|F_1| \geq 3$, then F_1 and F_2 are disjoint and either $\{e_1, e_2, e_3\}$ and $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$ are both triangles and $\{f_1, f_2, f_3\}$ and $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$ are both triads, or vice versa. We say that M has *no distinct fans with like ends.* The goal of this section is to consider the case in which M has a 4-element fan, but no distinct fans with like ends, and prove the following.

Theorem 3.7.1. *Let M be a 3-connected matroid with no distinct fans with like ends such that $|E(M)| \geq 13$. Let F be a maximal fan of M with length at least four. Then either*

(i) *M has a detachable pair,*

(ii) *M is a wheel or a whirl,*

(iii) *M is an even-fan-spike,*

(iv) *M' has a paddle $(P_1, P_2, \ldots, P_m)$ for some $M' \in \{M, M^*\}$ and $m \geq 3$ such that $M' \backslash P_1 \cong M(K_{3,m-1})$, and, for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and P_1 is a co-augmented-fan-petal relative to P_i.*

3.7.1 Two Fans

First, we consider the case where M has two distinct maximal fans with length at least four.

Lemma 3.7.2. *Let M be a 3-connected matroid with no distinct fans with like ends such that $|E(M)| \geq 13$. Let F_1 and F_2 be distinct, maximal fans of M with length at least four. Then M has a detachable pair.*

Proof. Suppose, with the aim of reaching a contradiction, that M does not have a detachable pair. Let $(e_1, e_2, \ldots, e_{|F_1|})$ be an ordering of F_1, and $(f_1, f_2, \ldots, f_{|F_2|})$ be an ordering of F_2. Since M has no distinct fans with like ends, we may assume that $\{e_1, e_2, e_3\}$ and $\{e_{|F_1|-2}, e_{|F_1|-1}, e_{|F_1|}\}$ are triangles, and $\{f_1, f_2, f_3\}$ and $\{f_{|F_2|-2}, f_{|F_2|-1}, f_{|F_2|}\}$ are triads. This implies that F_1 and F_2 have odd length, so, by Lemma 3.4.17, we have that $|F_1| = 5$ and $|F_2| = 5$. Furthermore, there exists $z \notin F_1$ such that $\{e_1, e_3, e_5, z\}$ is a cocircuit, and there exists $z' \notin F_2$ such that $\{f_1, f_3, f_5, z'\}$ is a circuit.

Since M has no distinct fans with like ends, there are no triangles or triads of M outside of F_1 and F_2. This means that z is not contained in a triangle. Since $z \in \mathrm{cl}^*(F_1)$, we have that M/z is 3-connected. Similarly, $M\backslash z'$ is 3-connected. We show that $z \in \{f_1, f_5\}$. Suppose this is not the case. Lemma 3.4.3 implies that M has a 4-element circuit C_1 containing $\{z, f_1\}$. By orthogonality with $\{f_1, f_2, f_3\}$ and $\{e_1, e_3, e_5, z\}$, the circuit C_1 contains f_2 and either e_1 or e_5. Without loss of generality, assume that $C_1 = \{z, e_1, f_1, f_2\}$. Also, M has a 4-element circuit C_2 containing $\{z, f_4, f_5\}$ and either e_1 or e_5. If $e_1 \in C_2$, then circuit elimination implies M has a circuit contained in $\{f_1, f_2, f_4, f_5\}$, a contradiction. So $C_2 = \{z, e_5, f_4, f_5\}$.

Also, orthogonality with $\{e_1, e_3, e_5, z\}$ implies that $z' \notin \{e_1, e_5\}$. Hence, Lemma 3.4.3 and orthogonality implies that M has cocircuits $C_1^* = \{z', f_1, e_1, e_2\}$ and $C_2^* = \{z', f_5, e_4, e_5\}$. But now $\lambda(F_1 \cup F_2 \cup \{z, z'\}) \leq 1$, so $|E(M)| \leq |F_1 \cup F_2 \cup \{z, z'\}| + 1 = 13$. But $|E(M)| \geq 13$, so $E(M) = F_1 \cup F_2 \cup \{z, z', x\}$, for some $x \notin F_1 \cup F_2 \cup \{z, z'\}$. Now, $\lambda(F_1 \cup \{z\}) = 2$ and $\lambda(F_2 \cup \{z'\}) = 2$, which implies that either $x \in \mathrm{cl}(F_1 \cup \{z\})$ and $x \in \mathrm{cl}(F_2 \cup \{z'\})$, or $x \in \mathrm{cl}^*(F_1 \cup \{z\})$ and $x \in \mathrm{cl}^*(F_2 \cup \{z'\})$. Up to duality, we may assume the former — in particular, $x \in \mathrm{cl}(F_2 \cup \{z'\})$. But $z' \in \mathrm{cl}(F_2)$, so $x \in \mathrm{cl}(F_2)$, and $\lambda(F_2 \cup \{x\}) = 2$. Now, $\lambda(F_1 \cup \{z, z'\}) = 2$. The cocircuits C_1^* and C_2^* imply that $\lambda(F_1 \cup \{z, z', f_1, f_5\}) = 2$, and the circuit $\{f_1, f_3, f_5, z'\}$ implies that $\lambda(F_1 \cup \{z, z', f_1, f_3, f_5\}) = 2$. Thus, $\lambda(\{f_2, f_4, x\}) = 2$, which implies by orthogonality that $\{f_2, f_4, x\}$ is a triad. But now $x \in \mathrm{cl}(F_2) \cap \mathrm{cl}^*(F_2)$, a contradiction.

Thus, $z \in \{f_1, f_5\}$. Dually, $z' \in \{e_1, e_5\}$. Then $(\{z\}, F_1, F_2 - \{z\})$ is a contractable triple and $(\{z'\}, F_1 - \{z'\}, F_2)$ is a deletable triple and $\lambda(F_1 \cup F_2) = 2$. Since $|E(M)| \geq 13$, Lemma 3.4.14 implies that every element of M which is not contained in $F_1 \cup F_2$ is contained in a 4-element fan. But M has no distinct fans with like ends, so M has no other 4-element fans, a contradiction which completes the proof. □

3.7.2 Even fan

Next, we consider the case where M has an even fan, and show that M is an even-fan-spike. We may also assume that M has no other triangles or triads.

Lemma 3.7.3. *Let M be a 3-connected matroid with no detachable pairs. Let $F = (e_1, e_2, \ldots, e_{|F|})$ be a maximal fan of M with even length at least four such that $\{e_1, e_2, e_3\}$*

is a triangle. Suppose M has no triangles or triads outside of F. Let $e \notin F$ such that $M\backslash e$ is 3-connected. Then either M is an even-fan-spike, or $|F| = 4$ and there exists $f, g, h \in E(M) - F$ (not necessarily distinct) for which

(i) *there exists $i \in \{2, 3\}$ such that $\{e_1, e_i, e, f\}$ is a cocircuit and $\{e_i, e_4, f, g\}$ is a circuit, and*

(ii) *$\lambda(F \cup \{e, f, g, h\}) = 2$.*

Proof. By Lemma 3.4.3 and orthogonality with $\{e_1, e_2, e_3\}$, there exists $f \notin F \cup \{e\}$ such that $\{e_1, e_i, e, f\}$ is a cocircuit of M for some $i \in \{2, 3\}$. Now, f is not contained in a triangle, so Lemma 3.4.4 implies that M/f is 3-connected. Thus, M has a 4-element circuit $C = \{e_{|F|}, e_j, f, g\}$ for some $g \notin F \cup \{f\}$ and $j \in \{|F| - 2, |F| - 1\}$. If $e_j \neq e_i$, then orthogonality with C^* implies that $g = e$. Furthermore, either $|F| > 4$ and orthogonality implies that $C^* = \{e_1, e_2, e, f\}$ and $C = \{e_{|F|-1}, e_{|F|}, e, f\}$, or $|F| = 4$ and we may choose an ordering of F such that $C^* = \{e_1, e_2, e, f\}$ and $C = \{e_3, e_4, e, f\}$. In either case, Lemma 3.5.14 implies that M is an even-fan-spike, as desired. Hence, $e_j = e_i$, which implies that $|F| = 4$.

If $g = e$, then $\lambda(F \cup \{e, f\}) = 2$, and the result holds. Otherwise, Lemma 3.4.4 implies that $M\backslash g$ is 3-connected. Thus, M has a 4-element cocircuit C_2^* containing $\{e_1, g\}$ and one of $\{e_2, e_3\}$ and an element $h \notin F \cup \{g\}$. If $h \in \{e, f\}$, then $\lambda(F \cup \{e, f, g\}) = 2$, as desired. Otherwise, orthogonality with C implies that $C_2^* = \{e_1, e_i, g, h\}$, and M/h is 3-connected. Now, M has a 4-element circuit C_2 containing $\{e_4, h\}$ and either e_2 or e_3. If $e_i \in C_2$, then orthogonality with C^* implies that either $e \in C_2$ or $f \in C_2$, and if $e_i \notin C_2$, then orthogonality with C_2^* implies that $g \in C_2$. In either case, $\lambda(F \cup \{e, f, g, h\}) = 2$, completing the proof. □

Lemma 3.7.4. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 13$. Let F be a maximal fan of M with even length at least four. If M has no triangles or triads outside of F, then M is either a wheel, a whirl or an even-fan-spike.*

Proof. If $E(M) = F$, then M is a wheel or a whirl by Lemma 3.2.7. Otherwise, let $e \in E(M) - F$, and suppose that M is not an even-fan-spike. Either $M\backslash e$ or M/e is 3-connected, so, up to duality, we may assume that $M\backslash e$ is 3-connected. By Lemma 3.7.3, there exists $f, g, h \notin F$ and an ordering (e_1, e_2, e_3, e_4) of F such that $\{e_1, e_2, e, f\}$ is a cocircuit and $\{e_2, e_4, f, g\}$ is a circuit, and $\lambda(F \cup \{e, f, g, h\}) = 2$.

Now, let $e' \notin F \cup \{e, f, g, h\}$. Again, up to duality we may assume that $M\backslash e'$ is 3-connected, so there exists $f', g', h' \notin F$ and $i \in \{2, 3\}$ such that $\{e_1, e_i, e', f'\}$ is a cocircuit and $\{e_i, e_4, f', g'\}$ is a circuit and $\lambda(F \cup \{e', f', g', h'\}) = 2$. Furthermore, if $i = 2$, then orthogonality implies that $f' \in \{f, g\}$. But now $e' \in \text{cl}^*(F \cup \{e, f, g, h\})$, which contradicts the fact that $M \backslash e'$ is 3-connected. So $i = 3$.

Since $|E(M)| \geq 13$, there exists $e'' \notin F \cup \{e, f, g, h, e', f', g', h'\}$. Again assume, without loss of generality, that $M \backslash e''$ is 3-connected. Thus, M has a 4-element cocircuit C^* containing $\{e_1, e''\}$, and either e_2 or e_3. But if $e_2 \in C^*$, then $e'' \in \text{cl}^*(F \cup \{e, f, g, h\})$ and if $e_3 \in C^*$, then $e'' \in \text{cl}^*(F \cup \{e', f', g', h'\})$. Either case is a contradiction, and completes the proof. □

3.7.3 Odd fan

Finally, we consider the case where M has an odd fan. The next lemma is similar to Lemma 3.4.4 and will be useful in this section.

Lemma 3.7.5. *Let M be a 3-connected matroid. Let $F = (e_1, e_2, e_3, e_4, e_5)$ be a maximal fan of M such that $\{e_1, e_2, e_3\}$ is a triangle, and let $z \in E(M) - F$ such that $\{e_1, e_3, e_5, z\}$ is a cocircuit. If M has a circuit $\{e_1, z, e, f\}$ such that M/e is 3-connected and f is not contained in a triad, then $M \backslash f$ is 3-connected.*

Proof. Suppose $M \backslash f$ is not 3-connected. Since f is not contained in a triad, M has a cyclic 3-separation $(X, \{f\}, Y)$ such that $F \subseteq X$. Now, $z \in \text{cl}^*(F)$, so we may assume that $z \in X$. If $e \in X$, then $f \in \text{cl}(X)$, a contradiction. Otherwise, $e \in Y$, and $e \in \text{cl}(X \cup \{f\})$, which contradicts the fact that M/e is 3-connected. □

Lemma 3.7.6. *Let M be a 3-connected matroid such that $|E(M)| \geq 11$. Let $F = (e_1, e_2, e_3, e_4, e_5)$ be a maximal fan of M such that $\{e_1, e_2, e_3\}$ is a triangle. Suppose M has no triangles or triads outside of F. Then M has a detachable pair.*

Proof. Suppose, to the contrary, that M has no detachable pairs. By Lemma 3.4.17, there exists $z \in E(M) - F$ such that $\{e_1, e_3, e_5, z\}$ is a cocircuit, so $z \in cl^*(F)$. Let $e \notin F \cup \{z\}$. Suppose M/e is 3-connected. Then M has a 4-element circuit C containing $\{e, z\}$, and either e_1 or e_5, and an element $f \notin F \cup \{z\}$. Suppose, without loss of generality, that $C = \{e_1, z, e, f\}$. Note that $(\{e_1\}, F - \{e_1\}, \{z, e, f\})$ is a deletable collection.

By Lemma 3.7.5, the matroid $M \backslash f$ is 3-connected. Now, M has a 4-element cocircuit C^* containing $\{e_5, f\}$, and, by orthogonality with C, either $C^* = \{e_4, e_5, z, f\}$ or $C^* = \{e_4, e_5, e, f\}$. In either case, $\lambda(F \cup \{z, e, f\}) = 2$. Furthermore, $z \in \text{cl}^*(F)$ and, for all $x \in F \cup \{z\}$, we have that $x \in \text{cl}(F \cup \{z, e, f\})$. Since $|E(M)| \geq 11$, Lemma 3.4.10 implies that every element of $E(M) - (F \cup \{z, e, f\})$ is contained in a triad, a contradiction.

Thus, $M\backslash e$ is 3-connected, and, furthermore, for all $x \in E(M)-(F\cup\{z\})$, the matroid M/x is not 3-connected. Now, M has a 4-element cocircuit $\{e_1, e_2, e, f'\}$ where $f' \notin F \cup \{z\}$. But Lemma 3.4.4 implies that M/f' is 3-connected, a contradiction which completes the proof. □

Lemma 3.7.7. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 13$. Let $F = (e_1, e_2, e_3, e_4, e_5)$ be a maximal fan of M such that $\{e_1, e_2, e_3\}$ is a triangle, and M has no triangles outside of F. Let $z \in E(M) - F$ such that $\{e_1, e_3, e_5, z\}$ is a cocircuit. Then M has a triad which is disjoint from $F \cup \{z\}$.*

Proof. Suppose that every triad of M intersects $F \cup \{z\}$. By Lemma 3.7.6, the matroid M has a triad T^* outside of F. Now, $T^* \cap (F \cup \{z\}) \neq \emptyset$. By Lemma 3.2.8 and Lemma 3.2.11, we have that T^* and F are disjoint. Thus, $z \in T^*$, so let $T^* = \{z, e, f\}$. Note that $(\{z\}, F, \{e, f\})$ is a contractable collection. T^* is not contained in a 4-element fan, so Tutte's Triangle Lemma implies that either M/e or M/f is 3-connected. We may assume that M/e is 3-connected. By Lemma 3.4.3, there is a 4-element circuit $C = \{e_i, z, e, g\}$ of M, for some $i \in \{1, 5\}$ and $g \notin F \cup \{e, z\}$. Assume, without loss of generality, that $i = 1$. Now, $(\{e_1\}, F - \{e_1\}, \{z, e, g\})$ is a deletable collection. If $g = f$, then $\lambda(F \cup \{z, e, f\}) = 2$ and $F \cup \{z, e, f\}$ contains both a deletable and a contractable triple. This contradicts Lemma 3.4.14. Hence, $g \neq f$.

Suppose g is not contained in a triad. Lemma 3.7.5 implies that $M\backslash g$ is 3-connected. Thus, M has a 4-element cocircuit C^* containing $\{e_4, e_5, g\}$ and one of $\{e, z, e_1\}$. Now, $\lambda(F \cup \{z, e, f, g\}) = 2$, again contradicting Lemma 3.4.14.

Otherwise, g is contained in a triad of M. This triad contains z, so M has a triad $\{z, g, h\}$, for some $h \notin F \cup \{z, e, f, g\}$. Lemma 3.4.3 implies that M has a 4-element circuit C' containing $\{e, h\}$. By orthogonality, C' contains one of $\{z, f\}$ and one of $\{z, g\}$. Furthermore, if $z \in C'$, then C also contains one of e_1 and e_5. But now $\lambda(F \cup \{z, e, f, g, h\}) = 2$, a contradiction which completes the proof. □

Lemma 3.7.8. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 12$. Let $F = (e_1, e_2, e_3, e_4, e_5)$ be a maximal fan of M such that $\{e_1, e_2, e_3\}$ is a triangle and M has no triangles outside of F. Let $z \in E(M) - F$ such that $\{e_1, e_3, e_5, z\}$ is a cocircuit, and let T^* be a triad of M which does not contain z. Then*

(i) *$T^* = \{a, b, c\}$ such that $\{e_1, z, a, b\}$ and $\{e_5, z, b, c\}$ are circuits, and*

(ii) *every element of $E(M) - (F \cup \{z\} \cup T^*)$ is contained in a triad.*

Proof. By Lemma 3.4.3 and orthogonality, we may label $T^* = \{a, b, c\}$ such that $C_1 = \{e_i, z, a, b\}$ and $C_2 = \{e_j, z, b, c\}$, for some $i, j \in \{1, 5\}$. If $i = j$, then circuit elimination and orthogonality with $\{e_1, e_3, e_5, z\}$ implies that M has a circuit contained in T^*. This is a contradiction, so $i \neq j$, proving (i).

Now, $(\{e_1\}, F - \{e_1\}, T^* \cup \{z\})$ is a deletable collection. Furthermore, $\lambda(F \cup T^* \cup \{z\}) = 2$, and $z \in \mathrm{cl}^*(F)$, and, for all $x \in F \cup \{z\}$, we have that $z \in \mathrm{cl}(F \cup T^* \cup \{z\})$. Hence, by Lemma 3.4.10, every element of $E(M) - (F \cup T^* \cup \{z\})$ is contained in a triad. □

Lemma 3.7.9. *Let M be a 3-connected matroid with no detachable pairs such that $|E(M)| \geq 13$. Let $F = (e_1, e_2, \ldots, e_{|F|})$ be a maximal fan of M with odd length at least five such that $\{e_1, e_2, e_3\}$ is a triangle and M has no triangles outside of F. Then M has a paddle $(P_1, P_2, \ldots, P_m)$ such that $M \backslash P_1 \cong M(K_{3,m-1})$ and, for all $i \in \{2, 3, \ldots, m\}$, the petal P_i is a triad and P_1 is a co-augmented-fan-petal relative to P_i.*

Proof. By Lemma 3.4.17, we have that $|F| = 5$ and there exists $z \notin F$ such that $\{e_1, e_3, e_5, z\}$ is a cocircuit. By Lemma 3.7.7, there exists a triad T_1^* disjoint from $F \cup \{z\}$, and by Lemma 3.7.8(i), we have that $T_1^* = \{a^1, b^1, c^1\}$ such that $\{e_1, z, a^1, b^1\}$ and $\{e_5, z, b^1, c^1\}$ are circuits. Let $e \notin F \cup \{z\} \cup T_1^*$. By Lemma 3.7.8, there is a triad T_2^* containing e. Furthermore, orthogonality implies that T_2^* is disjoint from $F \cup \{z\} \cup T_1^*$, so $T_2^* = \{a^2, b^2, c^2\}$ such that $\{e_1, z, a^2, b^2\}$ and $\{e_5, z, b^2, c^2\}$ are circuits. Furthermore, Lemma 3.4.18 implies that $M|(T_1^* \cup T_2^*) \cong M(K_{3,2})$.

It follows that we may partition $E(M)$ into $P_1, P_2, \ldots, P_m$, with $m \geq 3$, such that $P_1 = F \cup \{z\}$ and $M \backslash P_1 \cong M(K_{3,m-1})$ and, for all $i \in \{2, 3, \ldots, m\}$, the set $P_i = \{a^i, b^i, c^i\}$ is a triad such that $\{e_1, z, a^i, b^i\}$ and $\{e_5, z, b^i, c^i\}$ are circuits. By Lemma 3.4.19, we have that $(P_1, P_2, \ldots, P_m)$ is a paddle of M.

To complete the proof, we show that P_1 is a co-augmented-fan-petal relative to P_i, for all $i \in \{2, 3, \ldots, m\}$. The circuits $\{e_1, z, a^i, b^i\}$ and $\{e_5, z, b^i, c^i\}$ imply that $\sqcap(\{e_1, z\}, P_i) = 1$ and $\sqcap(\{e_5, z\}, P_i) = 1$. We show that $\sqcap(\{e_2, e_4\}, P_i) = 1$. Observe that

$$\lambda(P_i \cup \{e_2, e_4\}) = \lambda((P_2 \cup P_3 \cup \cdots \cup P_m - P_i) \cup \{e_1, e_3, e_5, z\}) \leq \lambda(P_2 \cup P_3 \cup \cdots \cup P_m - P_i) + 1 = 3.$$

Orthogonality with the triangles $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ imply that $r^*(P_i \cup \{e_2, e_3\}) = r^*(P_i) + 2$, and orthogonality with $\{e_2, e_3, e_4\}$ implies that $r(P_i \cup \{e_2, e_4\}) \geq r(P_i) + 1$. Therefore,

$$\lambda(P_i \cup \{e_2, e_4\}) \geq (r(P_i) + 1) + (r^*(P_i) + 2) - (|P_i| + 2) = \lambda(P_i) + 1 = 3$$

Therefore, $\lambda(P_i \cup \{e_2, e_4\}) = 2$, so $r(P_i \cup \{e_2, e_4\}) = r(P_i) + 1$, completing the proof. □

3.7.4 Putting it together

Proof of Theorem 3.7.1. Suppose M has a maximal fan G, distinct from F, with length at least four. Lemma 3.7.2 implies that M has a detachable pair. Otherwise, M has no 4-element fans outside of F. If F has even length, then M has no triangles or triads outside of F, and Lemma 3.7.4 implies that M is a wheel, a whirl, or an even-fan-spike. Otherwise, F has odd length. Up to duality, we may assume that the end elements of F are contained in triangles. This means that M has no triangles outside of F, so the result follows from Lemma 3.7.9. □

3.8 No Fans

Now we may assume that M has no 4-element fans whatsoever.

Theorem 3.8.1. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 13$. Then one of the following holds.*

(i) *M has a detachable pair,*

(ii) *M is a spike, or*

(iii) *M' has a paddle $(P_1, P_2, \ldots, P_m)$ for some $M' \in \{M, M^*\}$ and $m \geq 3$, and either*

  (a) *$M' \cong M(K_{3,m})$, where P_i is a triad for each $i \in [m]$,*

  (b) *there exists $x \in E(M)$ such that $M' \backslash x \cong M(K_{3,m})$ and, for all $i \in \{1, 2, \ldots, m\}$, the set $P_i - \{x\}$ is a triad and $x \in \mathrm{cl}(P_i - \{x\})$, or*

  (c) *$M' \backslash P_1 \cong M(K_{3,m-1})$, and, for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and either*

    (I) *$M'|P_1 \cong M^*(K_{3,t})$ for some $t \geq 2$, or*

    (II) *P_1 is a quad-petal relative to P_i.*

3.8.1 Intersecting triads

First, we consider the case where M has two intersecting triads, working towards showing that M has a detachable pair.

Lemma 3.8.2. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 9$. Let $T_1^* = \{t, a_1, a_2\}$ and $T_2^* = \{t, b_1, b_2\}$ be triads of M. Let $e \notin T_1^* \cup T_2^*$ such that M/e is 3-connected. Then M has a detachable pair.*

Proof. Suppose, with the aim of reaching a contradiction, that M has no detachable pairs. Note that $(\{t\}, \{a_1, a_2\}, \{b_1, b_2\})$ is a contractable collection. Since $e \notin T_1^* \cup T_2^*$ and M/e is 3-connected, Corollary 3.4.11 implies that $\lambda(T_1^* \cup T_2^*) > 2$. In particular, this means that $T_1^* \cup T_2^*$ is independent.

By Lemma 3.4.3, there is a 4-element circuit C of M containing $\{e, t\}$. By orthogonality, and without loss of generality, we have that $C = \{e, t, a_1, b_1\}$. Additionally, M has a 4-element circuit C_1 containing $\{e, a_2\}$. If C_1 contains three elements of $T_1^* \cup T_2^*$, then circuit elimination with C on the element e implies that M has a circuit contained in $T_1^* \cup T_2^*$, a contradiction. Thus, $C_1 = \{a_1, a_2, e, f\}$, for some $f \notin T_1^* \cup T_2^* \cup \{e\}$. Similarly, M has a circuit $C_2 = \{b_1, b_2, e, g\}$, with $g \notin T_1^* \cup T_2^* \cup \{e\}$. Furthermore, note that if $f = g$, then $T_1^* \cup T_2^*$ is not independent, so $f \neq g$.

Now, we show that M/a_2 is 3-connected. If this is not the case, then M has a vertical 3-separation $(X, \{a_2\}, Y)$. Without loss of generality, assume that $T_2^* \subseteq X$. Then $a_1 \in Y$, as otherwise $a_2 \in \mathrm{cl}^*(X)$. This further implies that $e \in Y$, as otherwise $a_1 \in \mathrm{cl}(X)$. Now, $\lambda(X \cup \{a_1, a_2\}) = 2$. But $e \in \mathrm{cl}(X \cup \{a_1, a_2\})$, which contradicts the fact that M/e is 3-connected, since $|Y - \{a_1, a_2\}| \geq 2$. Hence, M/a_2 is 3-connected. This implies, by Lemma 3.4.3, that M has a 4-element circuit C' containing $\{a_2, b_2\}$. Orthogonality implies that M contains one of $\{t, a_1\}$, one of $\{t, b_1\}$, one of $\{a_1, e, f\}$, and one of $\{b_1, e, g\}$. Furthermore, $C' \nsubseteq T_1^* \cup T_2^*$, so the only possibility is $C' = \{e, t, a_2, b_2\}$. But now circuit elimination between C and C' implies that M has a circuit contained in $T_1^* \cup T_2^*$. This contradiction completes the proof. □

Lemma 3.8.3. *Let M be a 3-connected matroid with no detachable pairs and no 4-element fans such that $|E(M)| \geq 9$. Let T_1^* and T_2^* be triads of M such that $|T_1^* \cap T_2^*| = 1$. Then M has no other triads.*

Proof. Let T_3^* be a triad of M distinct from T_1^* and T_2^*. If $|T_3^* - (T_1^* \cap T_2^*)| \geq 2$, then Tutte's Triangle Lemma implies that there exists $x \in T_3^* - (T_1^* \cap T_2^*)$ such that M/x is 3-connected, a contradiction to Lemma 3.8.2. Otherwise, $|T_3^* - (T_1^* \cap T_2^*)| = 1$. By Lemma 3.4.16, we have that $|T_1^* \cap T_2^*| = 1$ and $|T_1^* \cap T_3^*| = 1$. This means that we can label the elements of T_1^*, T_2^*, and T_3^* such that $T_1^* = \{a_1, b_1, a_2\}$, and $T_2^* = \{a_2, b_2, a_3\}$, and $T_3^* = \{a_3, b_3, a_1\}$. By Lemma 3.8.2, none of M/b_1, M/b_2, and M/b_3 are 3-connected, so M has a vertical 3-separation $(X, \{b_3\}, Y)$, Without loss of generality, $T_1^* \subseteq X$. This implies $a_3 \in Y$, as otherwise $b_3 \in \mathrm{cl}^*(X)$, and, in turn, $b_2 \in Y$, since $a_3 \notin \mathrm{cl}^*(X)$. But now $\lambda(X \cup \{b_3, a_3\}) = 2$, and $b_2 \in \mathrm{cl}^*(X \cup \{b_3, a_3\})$. Since $|Y - \{b_2, a_3\}| \geq 2$, this implies that M/b_2 is 3-connected, a contradiction. □

The next lemma will be useful in a few different places in this section.

Lemma 3.8.4. *Let M be a 3-connected matroid with no detachable pairs. Suppose that, for all $x \in E(M)$, if x is not contained in a triad, then M/x is not 3-connected. If there exists distinct $e, f \in E(M)$ such that neither e nor f is contained in a triangle or a triad, then M has a 4-element cocircuit C^* such that $e, f \in C^*$, and a triad T^* such that $T^* \cap C^* = \{g\}$ and M/g is 3-connected.*

Proof. Suppose $\{e, f\}$ is not contained in a 4-element cocircuit of M. Neither M/e nor M/f are 3-connected, so Bixby's Lemma implies that $M\backslash e$ and $M\backslash f$ are 3-connected. Since f is not contained in a triangle of M, we also have that f is not contained in triangle of $M\backslash e$. Also, f is not contained in a triad of M and $\{e, f\}$ is not contained in a 4-element cocircuit of M, so f is not contained in a triad of $M\backslash e$. Hence, as $M\backslash e\backslash f$ is not 3-connected, we have that $M\backslash e/f$ is 3-connected. But Lemma 3.4.6 implies that M/f is 3-connected, a contradiction. Therefore, M has a 4-element cocircuit C^* containing $\{e, f\}$.

Now, M/e is not 3-connected, so M has a vertical 3-separation $(X, \{e\}, Y)$. If $|C^* \cap X| = 3$, then $e \in \mathrm{cl}^*(X)$, a contradiction. Likewise, $|C^* \cap Y| \neq 3$. Hence, without loss of generality, we may assume that $|C^* \cap X| = 2$ and $|C^* \cap Y| = 1$. Let g be the unique element of $C^* \cap Y$. Then $g \in \mathrm{cl}^*(X \cup \{e\})$, so $\mathrm{co}(M\backslash g)$ is not 3-connected. Thus, $\mathrm{si}(M/g)$ is 3-connected. Suppose g is contained in a triangle T. By orthogonality, T contains a second element of C^*. But now Lemma 3.4.4 implies that M/f is 3-connected, a contradiction. This means that M/g is 3-connected, and so g is contained in a triad T^*.

Suppose T^* contains a second element of C^*, so that $|T^* \cap X| \geq 1$. If $g \in \mathrm{cl}^*(X)$, then $(X \cup \{g\}, \{e\}, Y - \{g\})$ is a vertical 3-separation of M, and $e \in \mathrm{cl}^*(X \cup \{g\})$. Thus, $g \notin \mathrm{cl}^*(X)$, so $|T^* \cap Y| = 2$ and $|T^* \cap X| = 1$. But this means that $(X - T^*, \{e\}, Y \cup T^*)$ is a vertical 3-separation. However, $C^* - \{e, f\} \subseteq T^*$, which implies $f \in \mathrm{cl}^*(Y \cup T^* \cup \{e\})$. This means that $M\backslash f$ is not 3-connected, a contradiction which completes the proof. □

Lemma 3.8.5. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 9$. Suppose that, for all $x \in E(M)$, if x is not contained in a triad, then M/x is not 3-connected. Let $T_1^* = \{t, a_1, a_2\}$ and $T_2^* = \{t, b_1, b_2\}$ be triads of M, and let e and f be distinct elements of $E(M)$ which are not contained in a triangle or a triad. Then M has a detachable pair.*

Proof. By Lemma 3.8.4, there exists a 4-element cocircuit C^* containing $\{e, f\}$, and a triad T^* such that $C^* \cap T^* = \{g\}$ where M/g is 3-connected. By Lemma 3.8.3, we have that $T^* = T_1^*$ or $T^* = T_2^*$. Take $T^* = T_1^*$. If $g = t$, then, since either M/a_1 or M/a_2 is 3-connected, Lemma 3.4.3 implies that M has a 4-element circuit containing t and either a_1 or a_2. Otherwise, since M/g is 3-connected, Lemma 3.4.3 implies that M has a 4-element circuit containing $\{g, t\}$. In either case, M has a 4-element circuit $C = \{a_i, b_j, t, h\}$, for some element $h \in E(M)$ and $i, j \in \{1,2\}$ such that $g \in C$. Since $g \in C \cap C^*$, orthogonlaity implies that $|C \cap C^*| \geq 2$.

Neither M/e nor M/f is 3-connected, so M has a vertical 3-separation $(X, \{e\}, Y)$. We may assume that $h \neq e$, for if $h = e$, then apply identical logic to a vertical 3-separation $(X', \{f\}, Y')$. We show that there is such a vertical 3-separation in which $T_1^* \cup T_2^* \cup \{h\} \subseteq X$. Suppose, without loss of generality, that $T_1^* \subseteq X$. If $\{b_1, b_2\} \subseteq X$, then $h \in \mathrm{cl}(X)$, and the result follows. Otherwise, assume $\{b_1, b_2\} \subseteq Y$. If $h \in X$, then $b_j \in \mathrm{cl}(X)$ and we may assume that $T_1^* \cup T_2^* \cup \{h\} \subseteq X$, and if $h \in Y$, then $t \in \mathrm{cl}^*(Y)$ and $a_i \in \mathrm{cl}(Y \cup \{t\})$, so we may assume that $T_1^* \cup T_2^* \cup \{h\} \subseteq Y$. Thus, we take $T_1^* \cup T_2^* \cup \{h\} \subseteq X$.

Now, $C \subseteq X$, so $|C^* \cap X| \geq 2$. If $|C^* \cap X| = 3$, then $e \in \mathrm{cl}^*(X)$, a contradiction. So $|C^* \cap X| = 2$, and there exists a unique element y of $C^* \cap Y$. But $y \in \mathrm{cl}^*(X)$, and y is not contained in a triangle since such a triangle would contain a second element of C^* and none of e, f, or g are contained in a triangle. This means that M/y is 3-connected. However, now y is contained in a triad distinct from T_1^* and T_2^*, a contradiction to Lemma 3.8.3 which completes the proof. □

3.8.2 Disjoint triads

We move onto the case where M has two disjoint triads.

Lemma 3.8.6. *Let M be a 3-connected matroid. Let $T^* = \{a_1, a_2, a_3\}$ be a triad of M, and let e, f, g, h be distinct elements of $E(M) - T^*$ such that $\{a_1, a_2, e, f\}$ and $\{a_2, a_3, e, g\}$ are circuits, and $\{e, f, g, h\}$ is a cocircuit, and h is not contained in a triangle. Then M/h is 3-connected.*

Proof. Suppose M/h is not 3-connected. Then M has a vertical 3-separation $(X, \{h\}, Y)$ such that $T^* \subseteq X$. If $\{e, f, g\} \cap X \neq \emptyset$, then $\{e, f, g\} \subseteq \mathrm{cl}(X)$, so $(X \cup \{e, f, g\}, \{h\}, Y - \{e, f, g\})$ is a vertical 3-separation. However, $h \in \mathrm{cl}^*(X \cup \{e, f, g\})$, a contradiction. Otherwise, $\{e, f, g\} \subseteq Y$, which means that $h \in \mathrm{cl}^*(Y)$, another contradiction. Therefore, M/h is 3-connected. □

Lemma 3.8.7. *Let M be a 3-connected matroid with no detachable pairs and no 4-element fans such that $|E(M)| \geq 13$. Let T_1^* and T_2^* be disjoint triads of M, and let e be an element of M such that e is not contained in a triangle or a triad and M/e is 3-connected. Then*

(i) *there exists $X \subseteq E(M)$ such that $e \in X$ and X is a quad-petal relative to T_1^* and T_2^*, and*

(ii) *every element of $E(M) - X$ is contained in a triad.*

Proof. By Lemma 3.4.18, we have that $T_1^* = \{a_1, a_2, a_3\}$ and $T_2^* = \{b_1, b_2, b_3\}$ such that, for all distinct $i, j \in \{1, 2, 3\}$, the set $\{a_i, a_j, b_i, b_j\}$ is a circuit. Lemma 3.4.3 implies that M has a 4-element circuit C_1 containing $\{e, a_1\}$. Without loss of generality, let $C_1 = \{a_1, a_2, e, f\}$, for some $f \notin T_1^* \cup T_2^*$. Similarly, M has a 4-element circuit $C_2 = \{a_2, a_3, e, g\}$, for some $g \notin T_2^*$. Note that $f \neq g$, as then $e \in \mathrm{cl}(T_1^*)$, which contradicts the fact that M/e is 3-connected.

Suppose f is contained in a triad. Since e is not contained in a triad, orthogonality implies this triad is $\{a_1, b_1, f\}$. But this contradicts Lemma 3.8.3. Thus, f, and similarly g, is not contained in a triad of M. Now, $M\backslash f$ is 3-connected by Lemma 3.4.4. By Lemma 3.4.5, there is a 4-element cocircuit C^* of M containing either $\{e, f\}$ or $\{f, g\}$. We prove that there is a 4-element cocircuit of M containing $\{f, g\}$, so first suppose that $\{e, f\} \subseteq C^*$. If C^* also contains g, then we have the desired result. Otherwise, orthogonality with C_2 implies that either $a_2 \in C^*$ or $a_3 \in C^*$. It follows that $C^* = \{a_i, b_i, e, f\}$, with $i \in \{2, 3\}$, so $\lambda(T_1^* \cup T_2^* \cup \{e, f\}) = 2$. In particular, $\lambda_{M\backslash f}(T_1^* \cup T_2^* \cup \{e\}) = 2$ and $g \in \mathrm{cl}(T_1^* \cup T_2^* \cup \{e\})$. Thus, since $M\backslash f\backslash g$ is not 3-connected, the element g is contained in a triad of $M\backslash f$, and thus $\{f, g\}$ is contained in a cocircuit of M.

In all cases, there is a 4-element cocircuit of M containing $\{f, g\}$. Suppose e is not contained in this cocircuit. Then orthogonality with C_1 and C_2 implies that M has a cocircuit $\{a_2, b_2, f, g\}$. But now $\lambda(T_1^* \cup T_2^* \cup \{f, g\}) = 2$ and $(\{a_2\}, \{a_1, a_3\}, \{b_2, f, g\})$ is a contractable collection. This is a contradiction, since M/e is 3-connected. It follows that M has a cocircuit $\{e, f, g, h\}$ with $h \notin T_1^* \cup T_2^* \cup \{e, f, g\}$.

Suppose h is contained in a triangle T. By orthogonality, T contains one of $\{e, f, g\}$, and $e \notin T$ since e is not contained in a triangle. Suppose T contains exactly one of f and g. Say $f \in T$ but $g \notin T$, so that $T = \{f, h, x\}$ for some $x \notin T_1^* \cup T_2^* \cup \{e, f, g, h\}$. Then Lemma 3.4.3 implies that M has a 4-element cocircuit D^* containing $\{g, x\}$. The cocircuit D^* contains one of $\{f, h\}$ and, by orthogonality with C_2, we have that $e \in D^*$. Thus, either $D^* = \{e, f, g, x\}$ or $D^* = \{e, g, h, x\}$. But cocircuit elimination with $\{e, f, g, h\}$ implies that M has a cocircuit contained in $\{f, g, h, x\}$, which is a contradiction to orthogonality. Thus, $T = \{f, g, h\}$. Furthermore, $\sqcap(\{e, f\}, T_1^*) = \sqcap(\{e, g\}, T_1^*) = 1$, so $\{e, f, g, h\}$ is a type-B quad-petal relative to T_1^* and, similarly, $\{e, f, g, h\}$ is a type-B quad-petal relative to T_2^*. Now, $(\{f\}, \{g, h\}, \{e, a_1, a_2\})$ is a deletable collection, and $\lambda(T_1^* \cup \{e, f, g, h\}) = 2$. Additionally, $e \in \mathrm{cl}^*(\{f, g, h\})$ and thus, by Lemma 3.4.10, every element of $E(M) - (T_1^* \cup \{e, f, g, h\})$ is contained in a triad.

Otherwise, h is not contained in a triangle. By Lemma 3.8.6, the matroid M/h is 3-connected, which means that M has a 4-element circuit containing $\{a_1, h\}$, and one of $\{a_2, a_3\}$ and one of $\{e, f, g\}$. Circuit elimination with either $\{a_1, a_2, e, f\}$ or $\{a_1, a_3, e, g\}$ implies that $\{e, f, g, h\}$ is a quad.

Now, for all $x \in \{e, f, g, h\}$, we have that M/x is 3-connected, by Lemma 3.4.2. Hence, M has a 4-element circuit C' containing $\{a_1, x\}$, and one of $\{a_2, a_3\}$, and an element $x' \in \{e, f, g, h\} - \{x\}$. Similarly, M has a 4-element circuit containing x and the unique element of $T_1^* - C'$, and another element of T_1^*, and an element $x'' \in \{e, f, g, h\} - \{x\}$. Note that $x' \neq x''$ since $x \notin \mathrm{cl}(T_1^*)$. Thus, $\sqcap(\{x, x'\}, T_1^*) = \sqcap(\{x, x''\}, T_1^*) = 1$, which implies that $\{e, f, g, h\}$ is a type-A quad-petal relative to T_1^*, and, similarly, $\{e, f, g, h\}$ is a type-A quad-petal relative to T_2^*. Now, $(\{e\}, \{f, g, h\}, \{f, a_1, a_2\}, \{g, a_2, a_3\})$ is a deletable collection, and $a_1 \in \mathrm{cl}^*(\{a_2, a_3\})$. By Lemma 3.4.10, every element of $E(M) - (T_1^* \cup \{e, f, g, h\})$ is contained in a triad. □

Lemma 3.8.8. *Let M be a 3-connected matroid with no detachable pairs and no 4-element fans such that $|E(M)| \geq 13$. Let T_1^* and T_2^* be disjoint triads of M, and let e be an element of M such that e is not contained in a triangle or a triad and M/e is 3-connected. Then M has a paddle $(P_1, P_2, \ldots, P_m)$ such that $M \backslash P_1 \cong M(K_{3,m-1})$ and for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and P_1 is a quad-petal relative to P_i.*

Proof. By Lemma 3.8.7, there exists $X \subseteq E(M)$ such that X is a quad-petal relative to T_1^* and T_2^*. Furthermore, for all $x \notin X \cup T_1^* \cup T_2^*$, the element x is contained in a triad T^*. By orthogonality, T^* is disjoint from $X \cup T_1^* \cup T_2^*$. Hence, Lemma 3.8.7 implies that X is a quad-petal relative to T^*, and Lemma 3.4.18 implies that $M|(T_1^* \cup T_2^* \cup T^*) \cong M(K_{3,3})$. It follows that $E(M)$ can be partitioned into $P_1, P_2, P_3, \ldots, P_m$ such that $P_1 = X$ and $M \backslash P_1 \cong M(K_{3,m-1})$ and, for all $i \in \{2, 3, \ldots, m\}$, the set P_i is a triad and X is a quad-petal relative to P_i. Lemma 3.4.19 implies that $(P_1, P_2, \ldots, P_m)$ is a paddle of M, and completes the proof. □

Lemma 3.8.9. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 11$. Suppose that, for all $x \in E(M)$, if x is not contained in a triad, then M/x is not 3-connected. Let T_1^* and T_2^* be disjoint triads of M, and let e and f be distinct elements of $E(M)$ which are not contained in a triangle or a triad. Then M has a detachable pair.*

Proof. By Lemma 3.4.18, we have that $T_1^* = \{a_1, a_2, a_3\}$ and $T_2^* = \{b_1, b_2, b_3\}$ such that, for all distinct $i, j \in \{1, 2, 3\}$, the set $\{a_i, a_j, b_i, b_j\}$ is a circuit. By Lemma 3.8.4, there exists a 4-element cocircuit C^* containing $\{e, f\}$ and a triad T^* such that $C^* \cap T^* = \{g\}$, where M/g is 3-connected.

Suppose that C^* and $T_1^* \cup T_2^*$ are disjoint. This means that $g \notin T_1^* \cup T_2^*$, so $T^* \neq T_1^*$ and $T^* \neq T_2^*$. Therefore, Lemma 3.8.3 implies that T^* is disjoint from T_1^* and T_2^*, so $M|(T^* \cup T_1^* \cup T_2^*) \cong M(K_{3,3})$. In particular, M has a 4-element C containing g, another element of T^*, and two elements of T_1^*. But $|C^* \cap T^*| = 1$ and $|C^* \cap T_1^*| = 0$, so C and C^* intersect in one element, a contradiction to orthogonality.

Otherwise, $C^* \cap (T_1^* \cup T_2^*) \neq \emptyset$. Orthogonality implies that $C^* = \{a_i, b_i, e, f\}$, for some $i \in \{1, 2, 3\}$. Now, M/e is not 3-connected, so M has a vertical 3-separation $(X, \{e\}, Y)$. Without loss of generality, assume that $T_1^* \subseteq X$. If $T_2^* \subseteq X$, then either $f \in X$, which means $e \in \mathrm{cl}^*(X)$, and so $M \backslash e$ is not 3-connected, or $f \in Y$, which means $f \in \mathrm{cl}^*(X \cup \{e\})$, and so $M \backslash f$ is not 3-connected. Either case is a contradiction, so $T_2^* \subseteq Y$. Now, either $f \in X$ or $f \in Y$. We may assume, without loss of generality, the former. It follows that $\lambda(X \cup \{e\} \cup T_2^*) < 2$, which implies that $|Y - T_2^*| = 1$. Hence, $Y = T_2^* \cup \{z\}$, for some element z. Since $\lambda(Y) = 2$, either $z \in \mathrm{cl}(T_2^*)$ or $z \in \mathrm{cl}^*(T_2^*)$. If $z \in \mathrm{cl}(T_2^*)$, then $T_2^* \cup \{z\}$ is a 4-element circuit, which contradicts orthogonality with C^*. Otherwise, $r^*(T_2^* \cup \{z\}) = 2$, contradicting Lemma 3.4.16 and completing the proof. □

Lemma 3.8.10. *Let M be a 3-connected matroid with no detachable pairs and no 4-element fans such that $|E(M)| \geq 9$. Let T_1^* and T_2^* be disjoint triads of M, let T be a triangle of M, and let e be an element of $E(M)$ which is not contained in a triangle or a triad. Then there exists $f \in E(M)$ such that f is not contained in a triangle or a triad and M/f is 3-connected.*

Proof. If M/e is 3-connected, then clearly the result holds. Otherwise, $M \backslash e$ is 3-connected. By Lemma 3.4.3, there is a 4-element cocircuit $C^* = \{e, f, g, h\}$ such that $\{g, h\} \subseteq T$ and $f \notin T$.

Suppose f is contained in a triangle T'. By orthogonality, T' contains one of $\{e, f, g\}$. Furthermore, e is not contained in a triangle, so $T \cap T' \neq \emptyset$. But $M \backslash e$ is 3-connected, which contradicts Lemma 3.8.2. Next, suppose f is contained in a triad T^*. If T^* intersects T_1^*, then, by Lemma 3.8.3, we have that $T^* = T_1^*$. Similarly, if T^* intersects T_2^*, then $T^* = T_2^*$. This means that T^* is disjoint from at least one of T_1^* and T_2^*. By Lemma 3.4.18, there is a 4-element circuit C of M containing f and another element of T^* and two elements of either T_1^* or T_2^*. But e is not contained in a triad, and f and g are not contained in triads since M has no 4-element fans. Therefore, C intersects C^* in one element, a contradiction which implies that f is contained in neither a triangle nor a triad. Lemma 3.4.4 implies that M/f is 3-connected, as desired. □

Lemma 3.8.11. *Let M be a 3-connected matroid with no detachable pairs and no 4-element fans such that $|E(M)| \geq 11$. Suppose that, for all $x \in E(M)$, if x is not contained in a triad, then M/x is not 3-connected. Let T_1^* and T_2^* be disjoint triads of M, and let e be an element of M which is not contained in a triangle or a triad. Then M has a paddle $(P_1, P_2, \ldots, P_m)$ such that $M \backslash e \cong M(K_{3,m})$ and for all $i \in \{1, 2, \ldots, m\}$ the set $P_i - \{e\}$ is a triad and $e \in \mathrm{cl}(P_i - \{e\})$.*

Proof. By Lemma 3.8.10, the matroid M has no triangles. If there exists $f \neq e$ such that f is not contained in a triangle or a triad, then Lemma 3.8.9 implies that M has a detachable pair. So every element of $E(M)-\{e\}$ is contained in a triad. Furthermore, by Lemma 3.8.3, there are no intersecting triads of M. Therefore, $E(M)-\{e\}$ can be partitioned into $P_1, P_2, \ldots, P_m$ such that $M\backslash e \cong M(K_{3,m})$ and, for all $i \in \{1, 2, \ldots, m\}$, the set P_i is a triad. Additionally, $M\backslash(P_i \cup \{e\}) \cong M(K_{3,m-1})$. Therefore, $\lambda(E(M)-(P_i \cup \{e\})) = 2$, so $\lambda(P_i \cup \{e\}) = 2$. By Lemma 3.4.16, we have that $e \notin \mathrm{cl}^*(P_i)$, so $e \in \mathrm{cl}(P_i)$. This completes the proof. □

Lemma 3.8.12. *Let M be a 3-connected matroid with no detachable pairs and no 4-element fans such that $|E(M)| \geq 10$. Let T_1^* and T_2^* be disjoint triads of M, and suppose that every element of M is contained in a triangle or a triad. Then either $M \cong M(K_{3,m})$, or there exists $X \subseteq M$ and $s, t \geq 2$ such that $M|X \cong M(K_{3,s})$ and $M\backslash X \cong M^*(K_{3,t})$.*

Proof. By Lemma 3.8.3, there is no pair of intersecting triads of M. We have that $T_1^* = \{a_1, a_2, a_3\}$ and $T_2^* = \{b_1, b_2, b_3\}$ such that, for all distinct $i, j \in \{1, 2, 3\}$, the set $\{a_i, a_j, b_i, b_j\}$ is a circuit. Moreover, if X is the set of elements of M which are contained in triads, then $M|X \cong M(K_{3,s})$, for some $s \geq 2$. If $E(M) = X$, then $M \cong M(K_{3,m})$ and the result holds. Otherwise, there exists a triangle T disjoint from X.

We consider the case where $E(M) = X \cup T$. Now, $\lambda(X - T_1^*) = 2$, and so $\lambda(T_1^* \cup T) = 2$. Suppose there exists $x \in T$ such that $x \in \mathrm{cl}^*((T_1^* \cup T) - \{x\})$. Then there is a cocircuit C^* of M contained in $T_1^* \cup T$ which contains x and an element of T_1^*. But orthogonality with the circuits $\{a_i, a_j, b_i, b_j\}$ implies that $T_1^* \subseteq C^*$, a contradiction. Since $\lambda(T_1^* \cup T) = 2$, it follows that $x \in \mathrm{cl}(T_1^*)$, for all $x \in T$. In particular, for all distinct $x, y \in T$, Lemma 3.4.3 implies there is a 4-element cocircuit of M containing x and y. But this cocircuit contains at least one element of X. Since $|E(M)| \geq 10$, the set X contains at least three triads, so this is a contradiction to orthogonality.

Therefore, M has triangle T' distinct from T. Suppose T and T' are not disjoint. By Lemma 3.8.3, there are no other elements of M contained in a triangle, and so $E(M) = X \cup T \cup T'$. Since $|T \cap T'| = 1$, Tutte's Triangle Lemma implies that there exists $x \in T - T'$ such that $M\backslash x$ is 3-connected. So, for $y \in T'$, Lemma 3.4.3 implies that there is a 4-element cocircuit C_1^* of M containing $\{x, y\}$. Orthogonality implies that $C_1^* \subseteq T \cup T'$. Also note that either C_1^* contains a triangle or C_1^* is a quad. Hence, $\lambda(C_1^*) = 2$. Let z be the unique element of $T \cup T' - C_1^*$. Now, $z \in \mathrm{cl}(C_1^*)$, so $M\backslash z$ is 3-connected. Again, Lemma 3.4.3 implies that M has a 4-element cocircuit C_2^* containing z, and $C_2^* \subseteq T \cup T'$. But now $z \in \mathrm{cl}(C_1^*)$ and $z \in \mathrm{cl}^*(C_1^*)$, a contradiction.

So M has no intersecting triangles. This means that $E(M) - X$ can be partitioned into disjoint triangles, and thus, by Lemma 3.4.18, we have that $M\backslash X \cong M^*(K_{3,t})$, for some $t \geq 2$. □

3.8.3 Triad and no triangles

The final case we need to consider is when M has exactly one triad and at most one triangle.

Lemma 3.8.13. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 11$. Let T^* be a triad of M and let T be a triangle of M such that M has no other triads or triangles. Let $e \in E(M) - (T \cup T^*)$ such that M/e is 3-connected. Then $T^* = \{a_1, a_2, a_3\}$ and $T = \{b_1, b_2, b_3\}$ such that $\{a_1, a_2, e, b_1\}$ and $\{a_2, a_3, e, b_3\}$ are circuits.*

Proof. By Lemma 3.4.3, we have that $T^* = \{a_1, a_2, a_3\}$ such that $\{a_1, a_2, e, f\}$ and $\{a_2, a_3, e, g\}$ are circuits for some $f, g \notin T^* \cup \{e\}$. Now, f and g are not contained in triads, so, by Lemma 3.4.4, we have that $M\backslash f$ and $M\backslash g$ are both 3-connected. Suppose $f \notin T$. Then $T = \{b_1, b_2, b_3\}$ such that M has a 4-element cocircuit C_1^* containing $\{b_1, b_2, f\}$ and a 4-element cocircuit C_2^* containing $\{b_2, b_3, f\}$. Orthogonality implies that C_1^* and C_2^* each contain an element of $\{a_1, a_2, e\}$. If $g \in T$, then $\lambda(T^* \cup T \cup \{e, f\}) = 2$ and $(\{g\}, T - \{g\}, \{a_2, a_3, e\})$ is a deletable collection. But $a_1 \in \mathrm{cl}^*(\{a_2, a_3\})$, and, for all $i \in \{1, 2, 3\}$, we have that $a_i \in \mathrm{cl}((T^* - \{a_i\}) \cup T \cup \{e, f\})$. This contradicts Lemma 3.4.10. Otherwise, $g \notin T$, and orthogonality with $\{a_2, a_3, e, g\}$ implies that M has cocircuits $C_1^* = \{b_1, b_2, f, a_1\}$ and $C_2^* = \{b_2, b_3, f, a_1\}$ are cocircuits. Cocircuit elimination implies that M has a cocircuit contained in $\{b_1, b_2, b_3, f\}$ and so, by orthogonality, M has a cocircuit contained in $\{b_1, b_2, b_3\}$. This is a contradiction which implies that $f \in T$ and, similarly, $g \in T$. $\square$

Lemma 3.8.14. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 11$. Let T^* be a triad of M and let T be a triangle of M such that M has no other triangles or triads. Then M has a detachable pair.*

Proof. Let $e \notin T \cup T^*$. Either M/e or $M\backslash e$ is 3-connected. Up to duality, we may assume the former. By Lemma 3.8.13, we have that $T^* = \{a_1, a_2, a_3\}$ and $T = \{b_1, b_2, b_3\}$ such that $\{a_1, a_2, e, b_1\}$ and $\{a_2, a_3, e, b_3\}$ are circuits. Note that $(\{b_1\}, \{b_2, b_3\}, T^* \cup \{e\})$ is a deletable collection.

Let $f \notin T \cup T^* \cup \{e\}$. Suppose $M\backslash f$ is 3-connected. The dual of Lemma 3.8.13, and orthogonality, implies that M has cocircuits $\{b_1, b_2, f, a_1\}$ and $\{b_2, b_3, f, a_3\}$. But now $\lambda(T \cup T^* \cup \{e, f\}) = 2$ and $(\{a_1\}, \{a_2, a_3\}, T \cup \{f\})$ is a contractable collection. This contradicts Lemma 3.4.14.

Otherwise, $M\backslash f$ is not 3-connected, so M has a cyclic 3-separation $(X, \{f\}, Y)$ such that $T^* \subseteq X$. Furthermore, Bixby's Lemma implies that M/f is 3-connected, and Lemma 3.8.13 implies that $T \subseteq \mathrm{cl}(T^* \cup \{f\})$. Also, $e \in \mathrm{cl}(T^* \cup T)$, so $e \in \mathrm{cl}(T^* \cup \{f\})$. If $T \subseteq X$, then

$f \in \mathrm{cl}(X)$, a contradiction. Therefore, $T \subseteq Y$. Since $T \not\subseteq \mathrm{cl}(X)$, we have that $e \in Y$. But now $e \in \mathrm{cl}(X \cup \{f\})$, which contradicts the fact that M/e is 3-connected. □

Lemma 3.8.15. *Let M be a 3-connected matroid. Let $T^* = \{a_1, a_2, a_3\}$ be a triad of M such that M has no other triangles or triads, and let e, f be distinct elements of $E(M) - T^*$ such that $\{a_1, a_2, e, f\}$ is a circuit. Suppose there exists X such that $T^* \cup \{e, f\} \subseteq X$ and $\lambda(X) = 2$ and X contains a contractable collection and $|E(M)| \geq |X| + 3$. Then M has a detachable pair.*

Proof. Note that $|E(M)| \geq |X| + 4$, as otherwise $E(M) - X$ is a triangle or a triad. Now, suppose M does not contain a detachable pair, and let $x \notin X$. By Corollary 3.4.11, the matroid M/x is not 3-connected, so $M \backslash x$ is 3-connected. Lemma 3.4.5 implies that M has a 4-element cocircuit C^* containing x and either e or f. Since $x \notin \mathrm{cl}^*(X)$, there exists $y \in C^*$ with $y \notin X$. In $M \backslash x$, we have that $y \in \mathrm{cl}^*(X)$ and $|E(M \backslash x)| \geq |X| + 3$. Therefore, $M \backslash x/y$ is 3-connected. But Lemma 3.4.6 implies that M/y is 3-connected. This is a contradiction which completes the proof. □

Lemma 3.8.16. *Let M be a 3-connected matroid such that $|E(M)| \geq 10$. Let $T^* = \{a_1, a_2, a_3\}$ be a triad of M such that M has no other triangles or triads, and let e, f, g be distinct elements of $E(M) - T^*$ such that $\{a_1, a_2, e, f\}$ and $\{a_2, a_3, e, g\}$ are circuits, and $\{e, f, g\}$ is contained in a 4-element cocircuit C^*. Then M has a detachable pair.*

Proof. Suppose M does not have a detachable pair. If $C^* \subseteq T^* \cup \{e, f, g\}$, then $\lambda(T^* \cup \{e, f, g\}) = 2$. Furthermore, $|C^* \cap T^*| = 1$, so $(C^* \cap T^*, T^* - C^*, C^* - T^*)$ is a contractable collection. But this contradicts Lemma 3.8.15. Otherwise, there exists $h \notin T^* \cup \{e, f, g\}$ such that $C^* = \{e, f, g, h\}$. By Lemma 3.8.6, the matroid M/h is 3-connected. Therefore, there is a 4-element circuit C of M containing $\{a_2, h\}$, which contains one of $\{a_1, a_3\}$, and one of $\{e, f, g\}$. Now, $\lambda(T^* \cup \{e, f, g, h\}) = 2$. Furthermore, by circuit elimination with $\{a_1, a_2, e, f\}$ if $a_1 \in C$ or with $\{a_2, a_3, e, g\}$ if $a_3 \in C$, there is a circuit of M contained in $\{e, f, g, h\}$. This implies that $\{e, f, g, h\}$ is a quad, so $(\{e\}, \{f, g, h\}, \{a_1, a_2, f\}, \{a_2, a_3, g\})$ is a deletable collection. But $a_1 \in \mathrm{cl}^*(\{a_2, a_3\})$ and, for all $i \in \{1, 2, 3\}$, we have that $a_i \in \mathrm{cl}(T^* \cup \{e, f, g, h\})$. But this contradicts Lemma 3.4.10 and completes the proof. □

Lemma 3.8.17. *Let M be a 3-connected matroid such that $|E(M)| \geq 13$. Let $T^* = \{a_1, a_2, a_3\}$ be a triad of M such that M has no other triangles or triads, and let e, f, g, h be distinct elements of $E(M) - T^*$ such that $\{a_1, a_2, e, f\}$ and $\{a_2, a_3, e, g\}$ are circuits, and M has a cocircuit C^* such that $h \in C^*$ and $|C^* \cap \{e, f, g\}| = 2$ and $|C^* \cap T^*| = 1$. Then M has a detachable pair.*

Proof. Suppose M does not have a detachable pair. Let a_i be the unique element of $C^* \cap T^*$. Then $(\{a_i\}, T^* - \{a_i\}, C^* - \{a_i\})$ is a contractable collection. To begin, we show that M/h

is 3-connected. If this is not the case, then M has a vertical 3-separation $(X, \{h\}, Y)$, and, without loss of generality, $T^* \subseteq X$. Suppose $|\{e,f,g\} \cap X| \geq 1$. Then $\{e,f,g\} \subseteq \mathrm{cl}(X)$, so we may assume that $\{e,f,g\} \subseteq X$. This implies that $h \in \mathrm{cl}^*(X)$, a contradiction. Thus, $\{e,f,g\} \subseteq Y$. But $a_i \in \mathrm{cl}^*(Y \cup \{h\})$ so $\lambda(Y \cup \{h, a_i\}) = 2$. The circuits $\{a_1, a_2, e, f\}$ and $\{a_2, a_3, e, g\}$ imply that $\lambda(Y \cup \{h\} \cup T^*) < 2$, and so $|Y \cup T^*| = 1$. Let z be the unique element of $Y - T^*$. Then either $z \in \mathrm{cl}(T^*)$ or $z \in \mathrm{cl}^*(T^*)$. But the former case implies that $T^* \cup \{z\}$ is a circuit, which contradicts orthogonality with C^*, and the latter case implies that $r^*(T^* \cup \{z\}) = 2$, which contradicts Lemma 3.4.16. Thus, M/h is 3-connected.

Lemma 3.4.3 implies that $T^* = \{a_i, a_j, a_k\}$ such that $\{a_i, a_j, h, f'\}$ and $\{a_i, a_j, h, g'\}$ are circuits. Furthermore, if $f' \in T^* \cup \{e,f,g\}$ or $g' \in T^* \cup \{e,f,g\}$ or $f' = g'$, then $h \in \mathrm{cl}(T^* \cup \{e,f,g\})$. This means that $\lambda(T^* \cup \{e,f,g,h\}) = 2$, which contradicts Lemma 3.8.15. Thus, $f', g' \notin T^* \cup \{e,f,g\}$. Lemma 3.4.4 implies that $M \backslash f'$ is 3-connected, and Lemma 3.4.5 implies that M has a 4-element cocircuit D^* containg either $\{f', h\}$ or $\{f', g'\}$. By Lemma 3.8.16, the cocircuit D^* does not contain $\{f', g', h\}$, so orthogonality with $\{a_i, a_j, h, f'\}$ and $\{a_i, a_j, h, g'\}$ implies that D^* contains an element of T^*. Now, orthogonality with $\{a_1, a_2, e, f\}$ and $\{a_2, a_3, e, g\}$ implies that D^* contains another element of $T^* \cup \{e,f,g\}$. But this means that $\lambda(T^* \cup \{e,f,g,h,f',g'\}) = 2$, again contradicting Lemma 3.8.15. □

Lemma 3.8.18. *Let M be a 3-connected matroid such that $|E(M)| \geq 12$. Let T^* be a triad of M, and suppose M has no other triangles or triads. Let $e \notin T^*$ such that M/e is 3-connected. Then M has a detachable pair.*

Proof. Suppose M does not have a detachable pair. By Lemma 3.4.3, we have that $T^* = \{a_1, a_2, a_3\}$ such that $\{a_1, a_2, e, f\}$ and $\{a_2, a_3, e, g\}$ are circuits for some $f, g \notin T^* \cup \{e, f\}$. Furthermore, note that $f \neq g$, as $e \notin \mathrm{cl}(T^*)$. By Lemma 3.4.4, we have that $M \backslash f$ is 3-connected. By Lemma 3.4.5, there is a 4-element cocircuit C^* of M containing either $\{e,f\}$ or $\{f,g\}$.

By Lemma 3.8.16, we have that $\{e,f,g\} \not\subseteq C^*$. Therefore, orthogonality implies that C^* contains an element of T^*. Now, Lemma 3.8.17 implies that $|C^* \cap T^*| \neq 1$. Therefore, $|C^* \cap T^*| = 2$. If $\{f,g\} \subseteq C^*$, then $\lambda(T^* \cup \{f,g\}) = 2$. But $e \in \mathrm{cl}(T^* \cup \{f,g\})$, which contradicts the fact that M/e is 3-connected. Otherwise, $\{e,f\} \subseteq C^*$. Lemma 3.4.4 implies that $M \backslash g$ is 3-connected, so, by Lemma 3.4.5, there is a 4-element cocircuit D^* of M containing g and either e or f. Again, Lemma 3.8.16 and Lemma 3.8.17 imply that $|D^* \cap T^*| = 2$. If $C^* \cap T^* = D^* \cap T^*$, then cocircuit elimination implies that $\{e,f,g\}$ is a triad, a contradiction. Otherwise, there is a unique element a_i which is contained in both $C^* \cap T^*$ and $D^* \cap T^*$. Thus, $(\{a_i\}, T^* - \{a_i\}, C^* - \{a_i\}, D^* - \{a_i\})$ is a contractable collection and $\lambda(T^* \cup \{e,f,g\}) = 2$. This contradicts Lemma 3.8.15, and completes the proof. □

Lemma 3.8.19. *Let M be a 3-connected matroid such that $|E(M)| \geq 12$. Let T^* be a triad of M and suppose M has no other triangles or triads. Then M has a detachable pair.*

Proof. Suppose M does not have a detachable pair. By Lemma 3.8.18, for all $x \notin T^*$, we have that M/x is not 3-connected. So let $e \notin T^*$. There is a vertical 3-separation $(X, \{e\}, Y)$ of M such that $T^* \subseteq X$. Choose an element $f \in X - T^*$. By Lemma 3.8.4, there is a 4-element cocircuit C^* of M containing $\{e, f\}$ and exactly one element of T^*. But now $|C^* \cap X| \geq 2$. If $|C^* \cap X| = 3$, then $e \in \mathrm{cl}^*(X)$, a contradiction. Otherwise, $|C^* \cap X| = 2$, so there is a unique element g of $C^* \cap Y$. But $g \in \mathrm{cl}^*(X \cup \{e\})$, so M/g is 3-connected, another contradiction. □

3.8.4 Putting it together

We complete the proof of Theorem 3.8.1.

Proof of Theorem 3.8.1. Suppose M does not have a detachable pair. If M has no triangles or triads whatsoever, then M is a spike by Theorem 1.6.1. If M has exactly one triad and no triangles, then M has a detachable pair by Lemma 3.8.19, and if M has exactly one triangle and no triads, then M has a detachable pair by the dual of Lemma 3.8.19. If M has exactly one triangle and exactly one triad, then M has a detachable pair by Lemma 3.8.14. Thus, for the remainder of the proof, M either has two triads or two triangles. Up to duality, assume that M has two distinct triads T_1^* and T_2^*. Suppose that T_1^* and T_2^* are disjoint. Let e be an element of $E(M)$ such that e is not contained in a triangle or a triad and M/e is 3-connected. Lemma 3.8.8 implies that (iii)(c)(II) holds. Otherwise, no such element e exists, and thus, for all $x \in E(M)$, if x is not contained in a triad, then M/x is not 3-connected. If M has an element which is not contained in a triangle or a triad, then (iii)(b) holds by Lemma 3.8.11. If every element of M is contained in a triangle or a triad, then, by Lemma 3.8.12, either $M \cong M(K_{3,m})$ or (iii)(c)(I) holds.

Thus, the result holds if M has a pair of disjoint triads, and similarly if M has a pair of disjoint triangles. Now, suppose that T_1^* and T_2^* intersect. By Lemma 3.8.3, there are no other triads of M. Thus, M has exactly five elements which are contained in a triad. Lemma 3.8.2 implies that, for all $x \in E(M)$, if x is not contained in a triad then M/x is not 3-connected. Now, Lemma 3.8.5 implies that M has at most one element which is not contained in a triangle or a triad. Since M has no pair of disjoint triangles, Lemma 3.8.3 implies that there are at most five elements of M which are contained in a triangle. But now $|E(M)| \leq 11$, a contradiction which completes the proof. □

And that's it! Theorem 1.6.2 follows from Theorems 3.5.1, 3.6.1, 3.7.1, and 3.8.1.

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