WILD TRIANGLES IN 3-CONNECTED MATROIDS

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Abstract. Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M. In this paper, we describe the structure of M relative to $\{a, b, c\}$ when, for all t in $\{a, b, c\}$, either $M\backslash t$ is not 3-connected, or $M\backslash t$ has a 3-separation that is not equivalent to one induced by M.

1. Introduction

In this paper, we consider an extension of Tutte's Triangle Lemma but, rather than focus on a *particular* connectivity notion, we consider a more general question. *Given a triangle T in a 3-connected matroid M, when is it impossible to delete an element from T without either losing 3-connectivity or creating new unwanted 3-separations?* The main result of this paper answers this question by describing the structure of the matroid relative to such a triangle. The consequences of this theorem include the triangle theorems for internally 4-connected matroids [3, Theorem 6.1] and for k-coherent matroids [4]. Indeed, the proof of the latter uses the result of this paper. Moreover, the latter forms part of the connectivity theory that leads to a proof of Kahn's Conjecture [5] for 4-connected matroids [4].

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* if $|X|, |E - X| \geq k$. A k-separating set X, or a k-separating partition $(X, E-X)$, or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. The *full closure* of X, denoted $\mathrm{fcl}(X)$, is the intersection of all fully closed sets that contain X. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. If $\mathrm{fcl}(A_1)$ or $\mathrm{fcl}(B_1)$ is $E(M)$, then (A_1, B_1) is *sequential*.

Let e be an element of a matroid M such that both M and $M\backslash e$ are 3-connected. A 3-separation (X, Y) of $M\backslash e$ is *well blocked* by e if, for all exactly 3-separating partitions (X', Y') equivalent to (X, Y), neither $(X' \cup e, Y')$ nor $(X', Y' \cup e)$ is exactly 3-separating in M. An element f of M *exposes* a 3-separation (U, V) if (U, V) is a 3-separation of $M\backslash f$ that is

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well blocked by f. Although (U,V) is actually a 3-separation of $M\backslash f$, we often say that f *exposes a* 3*-separation* (U,V) *in* M. Evidently, if e exposes an exactly 3-separating partition (E_1,E_2), then e exposes all exactly 3-separating partitions (E_1',E_2') that are equivalent to (E_1,E_2).

A triangle T of a 3-connected matroid M is *wild* if, for all t in T, either $M\backslash t$ is not 3-connected, or $M\backslash t$ is 3-connected and t exposes a 3-separation in M. The task of this paper is to characterize wild triangles.

We begin with some examples. An ordered partition $(P_1,P_2,\dots,P_n)$ of the ground set of a 3-connected matroid M is a *flower* [7, 8] if $\lambda_M(P_i)=2=\lambda_M(P_i\cup P_{i+1})$ for all i in $\{1,2,\dots,n\}$, where all subscripts are interpreted modulo n. A *quad* is a 4-element set in M that is both a circuit and a cocircuit. In particular, a quad is 3-separating. In describing these examples, we shall use some technical language for flowers, which is recalled from [7] in Section 2. In the matroid M illustrated in Figure 1, $M\backslash a,b,c$ has a tight swirl-like flower $(P_1,P_2,\dots,P_6)$. Moreover, $a\in \mathrm{cl}(P_1\cup P_2)\cap \mathrm{cl}(P_3\cup P_4\cup P_5\cup P_6)$, and b and c are symmetrically placed. Other wild triangles can be obtained by modifying this situation. For example, the underlying flower $(P_1,P_2,\dots,P_6)$ need not be swirl-like, but may be spike-like or a copaddle (but never a paddle). Moreover, one can replace certain elements of the flower $(P_1,P_2,\dots,P_6)$ by series classes, but only in a controlled way. A wild triangle of one of the types described above is a *standard* wild triangle. A precise definition is given in Section 2.

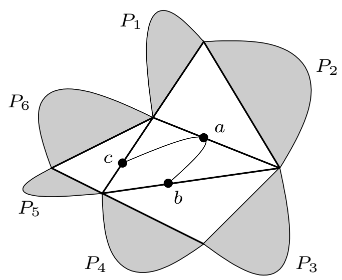

FIGURE 1. A standard wild triangle $\{a,b,c\}$.

Via a $\Delta-Y$ exchange, we can obtain another type of wild triangle. Let T be a standard wild triangle of the matroid M, and let M' be the matroid obtained by performing a $\Delta-Y$ exchange on the triangle T and then taking the dual. Then the triangle corresponding to T in M' is wild. We call such a wild triangle *costandard*. An illustration is given in Figure 2.

Let M be a matroid with a non-sequential 3-separation (X,Y). Then X is a *trident* in M if $|X|=7$, say $X=\{a,b,c,t,s,u,v\}$, and $\{a,b,c\}$ is a triangle, while $\{t,s,u,b\}$, $\{t,u,v,c\}$, and $\{t,s,v,a\}$ are quads exposed in $M\backslash a$, $M\backslash b$, $M\backslash c$, respectively (see Figure 3). Evidently, the triangle $\{a,b,c\}$ is wild but is neither standard nor costandard.

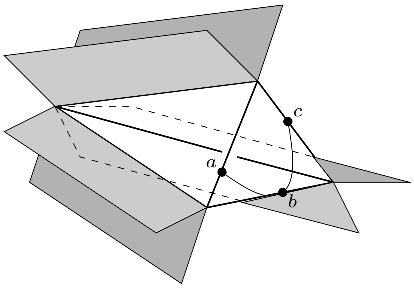

FIGURE 2. A costandard wild triangle $\{a, b, c\}$.

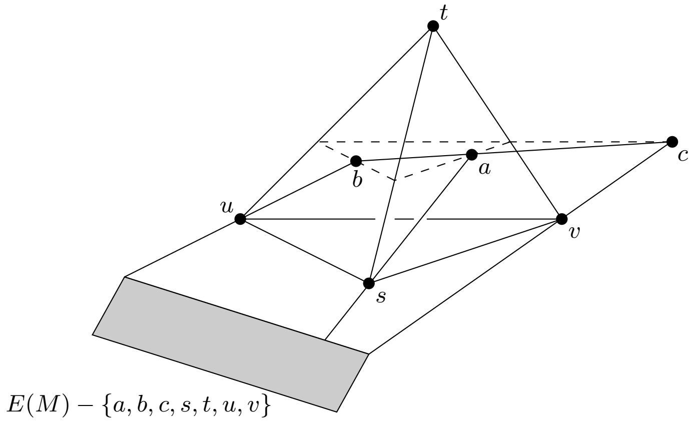

FIGURE 3. A trident.

In a 3-connected matroid M, if F is a fan with at least four elements and T is an *internal triangle* in F, that is, one containing neither end of F, then T is another type of wild triangle. At last, we can state our main theorem.

Theorem 1.1. *Let T be a wild triangle of a 3-connected matroid M with at least twelve elements. Then T is a standard or costandard wild triangle, a triangle in a trident of M, or an internal triangle of a fan of M.*

Let M be a 3-connected matroid. If M has no 3-separations (X, Y) with $|X|, |Y| \geq 4$, then M is *internally* 4*-connected*; M is *sequentially* 4*-connected* if it has no non-sequential 3-separations. It is easily seen that no matroid with at least 12 elements and a wild triangle is internally 4-connected, so an immediate consequence of Theorem 1.1 is the following result, which establishes the substantial part of [3, Theorem 6.1].

Corollary 1.2. *Let T be a triangle of an internally* 4*-connected matroid M where $|E(M)| \geq 12$. Then there is an element t in T such that $M\backslash t$ is sequentially* 4*-connected.*

The next section presents some basic preliminaries. In Section 3, we give precise definitions of the types of wild triangles. Then we state Theorem 3.1, a strengthening of Theorem 1.1, along with Corollary 3.2, which gives more detailed information about the structure around a wild triangle. Section 4 proves an extension of Tutte's Triangle Lemma thereby splitting the proof of Theorem 1.1 into two cases and settling the first. The rest of the paper is devoted to settling the second. Section 5 begins the proof of this second case, and Section 6 gives an overview of the rest of the proof, dividing it into six cases, (A)-(F). Section 7 shows that, in each of cases (A)-(D), the triangle $\{a, b, c\}$ is in a trident in M. In Section 8, we consider case (F) and show that, by performing a $\Delta - Y$ exchange on M and dualizing, we can reduce to the subcase of case (E) in which we have symmetry between a, b, and c. In Section 9, we show that, when case (E) occurs, either one of cases (A)-(D) occurs, or we are in the subcase of (E) in which we have symmetry between a, b, and c. That section also completes the proofs of Theorems 1.1 and 3.1. Finally, Section 10 proves Corollary 3.2.

2. Preliminaries

Our terminology will follow Oxley [6] except that the simplification and cosimplification of a matroid N will be denoted by $\mathrm{si}(N)$ and $\mathrm{co}(N)$, respectively. We write $x \in \mathrm{cl}^{(*)}(Y)$ to mean that $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. The set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$.

Let X be an exactly 3-separating set in a matroid M. If there is an ordering $(x_1, x_2, \ldots, x_n)$ of X such that $\{x_1, x_2, \ldots, x_i\}$ is 3-separating for all i in $[n]$, then X is *sequential*. An exactly 3-separating partition (X, Y) of M is *sequential* if X or Y is a sequential 3-separating set.

The connectivity function λ_M of a matroid M has many attractive properties. In particular, $\lambda_M = \lambda_{M^*}$. Moreover, $\lambda_M(X) = \lambda_M(E - X)$. We often abbreviate λ_M as λ. This function is submodular, that is, $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E(M)$. The next two lemmas are consequences of this. We make frequent use of the first and write *by uncrossing* to mean "by an application of Lemma 2.1."

Lemma 2.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

Lemma 2.2. *Let M be a 2-connected matroid, and let X and Y be subsets of $E(M)$ with $\lambda(X) = 2$ and $\lambda(Y) = 1$. If neither $X \cap Y$ nor $E - (X \cup Y)$ is empty, then $\lambda(X \cap Y) = 1$ or $\lambda(X \cup Y) = 1$.*

The connectivity function is also monotone under taking minors.

Lemma 2.3. *Let X be a set in a matroid M. If N is a minor of M, then*

$$\lambda_N(X \cap E(N)) \leq \lambda_M(X).$$

Let Δ be a triangle $\{a, b, c\}$ of a matroid M and take a copy of $M(K_4)$ having Δ as a triangle and $\{a', b', c'\}$ as the complementary triad, where e' is the element of $M(K_4)$ that is not in a triangle with e. Let $P_{\Delta}(M(K_4), M)$ be the *generalized parallel connection* of $M(K_4)$ and M. We write ΔM for $P_{\Delta}(M(K_4), M)\backslash\Delta$ and say that ΔM is obtained from M by a $\Delta - Y$ *exchange on* Δ. Note that ΔM has ground set $(E(M)-\{a, b, c\})\cup\{a', b', c'\}$. It is common to relabel a', b', and c' as a, b, and c so that M and ΔM have the same ground set, and we do this in the next section. But, in Section 8, when we are proving various properties of ΔM, we keep the original labelling.

A subset S a 3-connected matroid M is a *fan* in M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads beginning with either. We call $(s_1, s_2, \ldots, s_n)$ a *fan ordering* of S. If $n \geq 4$, then s_1 and s_n, which are the only elements of S that are not in both a triangle and a triad contained in S, are the *ends* of the fan. The remaining elements of S are the *internal elements* of the fan. An *internal triangle* of S is a triangle all of whose elements are internal elements of S.

Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ in a 3-connected matroid M. The sets $P_1, P_2, \ldots, P_n$ are the *petals* of Φ. Each has at least two elements. It is shown in [7, Theorem 4.1] that every flower in a 3-connected matroid is either an *anemone* or a *daisy*. In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering $(P_1, P_2, \ldots, P_n)$.

Let Φ_1 and Φ_2 be flowers in a matroid M. A natural quasi ordering on the set of flowers of M is obtained by setting $\Phi_1 \preceq \Phi_2$ if every non-sequential 3-separation displayed by Φ_1 is equivalent to one displayed by Φ_2. If $\Phi_1 \preceq \Phi_2$ and $\Phi_2 \preceq \Phi_1$, then Φ_1 and Φ_2 are *equivalent* flowers. Such flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations of M. Let Φ be a flower of M. The *order* of Φ is the minimum number of petals in a flower equivalent to Φ. An element e of M is *loose* in Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ; otherwise e is *tight*. A petal P_i is *loose* if all its elements are loose; and P_i is *tight* otherwise.

The classes of anemones and daisies can be further refined using the following concept. For sets X and Y in a matroid M, the *local connectivity* $\sqcap(X, Y)$ between X and Y is given by $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$.

Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ with $n \geq 3$. If Φ is an anemone, then $\sqcap(P_i, P_j)$ takes a fixed value k in $\{0, 1, 2\}$ for all distinct $i, j \in [n]$. We call Φ a *paddle* if $k = 2$, a *copaddle* if $k = 0$, and a *spike-like* flower if $k = 1$ and $n \geq 4$. Similarly, if Φ is a daisy, then $\sqcap(P_i, P_j) = 1$ for all consecutive i and j. We say Φ is *swirl-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and Φ is *Vámos-like* if $n = 4$ and $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

If (P_1, P_2, P_3) is a flower Φ and $\sqcap(P_i, P_j) = 1$ for all distinct i and j, we call Φ *ambiguous* if it has no loose elements, *spike-like* if there is an element in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$ or $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$, and *swirl-like* otherwise. Every flower with at least three petals is of one of these six *types*: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous [7].

We conclude this section with seven connectivity lemmas. We have omitted the more routine of the proofs.

Lemma 2.4. *Let M be a 3-connected matroid. If f exposes a 3-separation (U,V) in M, then (U,V) is non-sequential. In particular, $|U|,|V| \geq 4$. Moreover, if $|V|=4$, then V is a quad of $M\backslash f$.*

Proof. If (U,V) is sequential, then, without loss of generality, $(U,V) \cong (U',\{v_1,v_2\})$, an exactly 3-separating partition of $M\backslash f$. Since $(U' \cup f,\{v_1,v_2\})$ is an exactly 3-separating partition of M, we deduce that (U,V) is not well blocked by f, so f does not expose (U,V).

Now suppose that $|V|=4$. Since U does not span V in M or M^*, we have $r(V), r^*(V) \geq 3$. As $r(V)+r^*(V)-|V|=2$, we deduce that $r(V)=r^*(V)=3$. If V contains a triangle, then V is sequential. Hence V is a circuit. By duality, we conclude that V is a quad. □

Lemma 2.5. *Let (X,Y) be an exactly 3-separating partition of a 3-connected matroid M. Suppose $|X| \geq 3$ and $x \in X$. Then*

(i) *$x \in \mathrm{cl}^{(*)}(X-x)$, that is, $x \in \mathrm{cl}(X-x)$ or $x \in \mathrm{cl}^*(X-x)$; and*
(ii) *$(X-x, Y \cup x)$ is exactly 3-separating if and only if x is in exactly one of $\mathrm{cl}(X-x)\cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X-x)\cap\mathrm{cl}^*(Y)$.*

Lemma 2.6. *For a matroid M, let (X,Y) be a k-separation of $M\backslash T$ and $\{T_X,T_Y\}$ be a partition of T into possibly empty sets. If $T_X \subseteq \mathrm{cl}(X)$ and $T_Y \subseteq \mathrm{cl}(Y)$, then $(X\cup T_X, Y\cup T_Y)$ is a k-separation of M.*

Lemma 2.7. *Let e be an element of a matroid M and X be a subset of $E(M)-e$. If $\lambda(X)=k$ and $\lambda(X\cup e)\leq k-1$, then $e\in\mathrm{cl}(X)$ and $e \in \mathrm{cl}^*(X)$.*

Proof. We have $k=r(X)+r^*(X)-|X|$ and $k-1 \geq r(X\cup e)+r^*(X\cup e)-|X\cup e|$. Hence $r(X\cup e)+r^*(X\cup e)-|X| \leq k = r(X)+r^*(X)-|X|$, so $r(X\cup e)=r(X)$ and $r^*(X\cup e)=r^*(X)$. Thus $e\in\mathrm{cl}(X)$ and $e\in\mathrm{cl}^*(X)$. □

Lemma 2.8. *Let $\{a,b,c\}$ be a triangle of a matroid M and suppose that M and $M\backslash a$ are 3-connected. Let (A_1,A_2) be a 3-separation of $M\backslash a$ that is exposed by a. Then*

(i) *neither A_1 nor A_2 contains $\{b,c\}$; and*
(ii) *if $b\in A_1$, then $b\in \mathrm{cl}_{M\backslash a}(A_1-b)$.*

Proof. If $\{b,c\}\subseteq A_1$, then $a\in\mathrm{cl}(A_1)$, so $(A_1\cup e, A_2)$ is a 3-separation of M; a contradiction. Hence, by symmetry, (i) holds. Now suppose that $b \in A_1$. By Lemma 2.4, $|A_1|,|A_2|\geq 4$. Assume $b\notin \mathrm{cl}_{M\backslash a}(A_1-b)$. Then $r(A_1-b)+r(A_2\cup b)\leq r(A_1)+r(A_2)$. Since $|A_1-b|\geq 3$, it follows that $(A_1-b, A_2\cup b)$ is a 3-separating partition equivalent to (A_1,A_2). But $a\in\mathrm{cl}(A_2\cup b)$ so (A_1,A_2) is not well blocked by a. □

Lemma 2.9. *Let Q be a quad in a 3-connected matroid M. If $e\in Q$, then $\mathrm{si}(M/e)$ is 3-connected.*

Lemma 2.10. *Let $(P_1, P_2, \dots, P_k)$ be a flower in a 3-connected matroid. If P_2 is loose, then $P_2 \subseteq \mathrm{fcl}(P_1)$.*

Proof. Suppose first that $P_2 = \{x, y\}$ and x is in $\mathrm{cl}^{(*)}(P_i) - P_i$ for some $i \neq 2$. If $i = 1$, then Lemma 5.2 of [7] implies that $y \in \mathrm{cl}^{(*)}(P_1 \cup x)$, so $P_2 \subseteq \mathrm{fcl}(P_1)$, as required. If $i \neq 1$, then $P_3 \cup P_4 \cup \cdots \cup P_k \cup x$ is 3-separating. Hence so is $P_1 \cup y$, and Lemma 5.2 of [7] again implies that $P_2 \subseteq \mathrm{fcl}(P_1)$.

Now assume the result holds for $|P_2| < n$ and let $|P_2| = n \geq 3$. As P_2 is loose, it has an element x such that $x \in \mathrm{cl}^{(*)}(P_i) - P_i$ for some $i \neq 2$. If $i = 1$, then $(P_1 \cup x, P_2 - x, P_3 \dots, P_k)$ is a flower in which $P_2 - x$ is loose so, by the induction assumption, $P_2 - x \subseteq \mathrm{fcl}(P_1 \cup x)$. Hence $P_2 \subseteq \mathrm{fcl}(P_1)$ as $x \in \mathrm{fcl}(P_1)$. Now suppose $i \neq 1$. Then $(P_1, P_2 - x, P_3 \dots, P_i \cup x, \dots, P_k)$ is a flower in which $P_2 - x$ is loose. Hence, by the induction assumption, $P_2 - x \subseteq \mathrm{fcl}(P_1)$. Moreover, as both $P_2 - x$ and P_2 are 3-separating, $x \in \mathrm{cl}^{(*)}(P_2 - x)$. Hence $x \in \mathrm{fcl}(P_1)$ and so $P_2 \subseteq \mathrm{fcl}(P_1)$. The lemma follows by induction. $\square$

3. Wild Triangles

In this section, we give precise definitions of the types of wild triangles. We then state a strengthening of Theorem 1.1 that gives additional information about the structure of a matroid M around a wild triangle. Finally, we state three corollaries that give still more details of this structure.

Let $\{a, b, c\}$ be a triangle of a 3-connected matroid M. Then $\{a, b, c\}$ is a *standard* wild triangle if there is a partition $\mathbf{P} = (P_1, P_2, \dots, P_6)$ of $E(M) - \{a, b, c\}$ such that $|P_i| \geq 2$ for all i and the following hold:

(i) $M \backslash a$, $M \backslash b$, and $M \backslash c$ are 3-connected, $M \backslash a, b, c$ is connected, and $\mathrm{co}(M \backslash a, b, c)$ is 3-connected.
(ii) $(P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup c)$ is a flower in M.
(iii) $(P_2 \cup P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup P_1 \cup c)$, $(P_4 \cup P_5 \cup P_6 \cup c, P_1 \cup P_2 \cup P_3 \cup a)$, and $(P_6 \cup P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup P_5 \cup b)$ are 3-separations exposed in M by a, b, and c, respectively.

A partition $\mathbf{P}$ satisfying these conditions is a partition *associated* to $\{a, b, c\}$. Such a partition need not be unique, even up to equivalence.

Let Δ be a triangle $\{a, b, c\}$ of a 3-connected matroid M and let ΔM denote the matroid obtained by performing a $\Delta - Y$ exchange on Δ. We assume that the ground sets of M and ΔM are equal by labelling the latter in the natural way. Then Δ is a *costandard* wild triangle in M if Δ is a standard wild triangle in $(\Delta M)^*$. Let $\mathbf{P} = (P_1, P_2, \dots, P_6)$ be a partition of $E(M) - \{a, b, c\}$. Then $\mathbf{P}$ is *associated* to the costandard wild triangle Δ in M if $\mathbf{P}$ is associated to the standard wild triangle Δ in $(\Delta M)^*$.

Let R be a 3-separating set $\{a, b, c, s, t, u, v\}$ in a 3-connected matroid M, where $\{a, b, c\}$ is a triangle. Then R is a *trident* with wild triangle $\{a, b, c\}$ if $\{t, s, u, b\}$, $\{t, u, v, c\}$, and $\{t, s, v, a\}$ are exposed quads in $M \backslash a$, $M \backslash b$, and $M \backslash c$, respectively. These quads need not be the only 3-separations exposed by a, b, or c (see Section 7). Observe that $(M/t)|(R - t) \cong M(K_4)$.

Theorem 3.1. *Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M, where $|E(M)| \neq 11$, and suppose that $\{a, b, c\}$ is not an internal triangle of a fan of M. Then $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected. Moreover, if $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2) are 3-separations exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1$, $b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$, then exactly one of the following holds:*

- (i) *$\{a, b, c\}$ is a wild triangle in a trident;*
- (ii) *$\{a, b, c\}$ is a standard wild triangle and $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2) can be replaced by equivalent 3-separations such that*
  - (a) *$(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ is a partition associated to $\{a, b, c\}$;*
  - (b) *every 2-element cocircuit of $M\backslash a, b, c$ meets exactly two of $A_2 \cap B_1, B_2 \cap C_1$, and $C_2 \cap A_1$; and*
  - (c) *in $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$, every union of consecutive sets is exactly 3-separating in $M\backslash a, b, c$;*
- (iii) *$\{a, b, c\}$ is a costandard wild triangle; more particularly, if M' is the matroid that is obtained from M by performing a $\Delta - Y$ exchange on $\{a, b, c\}$ in M and then taking the dual of the result, then M' is 3-connected and $((A_2 - c) \cup b, (A_1 - b) \cup c), ((B_2 - a) \cup c, (B_1 - c) \cup a)$, and $((C_2 - b) \cup a, (C_1 - a) \cup b)$ are 3-separations in M' exposed by a, b, and c, respectively. Moreover, (ii) holds when $(M, A_1, A_2, B_1, B_2, C_1, C_2)$ is replaced by $(M', (A_2 - c) \cup b, (A_1 - b) \cup c, (B_2 - a) \cup c, (B_1 - c) \cup a, (C_2 - b) \cup a, (C_1 - a) \cup b)$.*

The next corollary gives a more detailed description of the structure associated with a standard wild triangle. The detailed structure of costandard wild triangles can be obtained straightforwardly from this. Let $(P_1, P_2, \dots, P_n)$ be a partition $\mathbf{P}$ of a set E and let A be a subset of E. Then the partition of Z *induced* by $\mathbf{P}$ is the partition $(P_1 \cap Z, P_2 \cap Z, \dots, P_n \cap Z)$.

Corollary 3.2. *Let $\{a, b, c\}$ be a standard wild triangle of a 3-connected matroid M with an associated partition $\mathbf{P}$. Let $N = \mathrm{co}(M\backslash a, b, c)$ and let $\mathbf{Q}$ be the partition $(Q_1, Q_2, \dots, Q_6)$ of $E(N)$ induced by $\mathbf{P}$. Then $\mathbf{Q}$ is a tight flower in N that is swirl-like, spike-like, or a copaddle. Moreover:*

- (i) *If $\mathbf{Q}$ is swirl-like, then the non-trivial series classes of $M\backslash a, b, c$ have size exactly 2 and there are at most three such series pairs. An element of $E(N)$ corresponding to a series pair of $M\backslash a, b, c$ is in one of $\mathrm{cl}^*(Q_2) \cap \mathrm{cl}^*(Q_3)$, $\mathrm{cl}^*(Q_4) \cap \mathrm{cl}^*(Q_5)$, or $\mathrm{cl}^*(Q_6) \cap \mathrm{cl}^*(Q_1)$.*
- (ii) *If $\mathbf{Q}$ is spike-like, then there is at most one non-trivial series class in $M\backslash a, b, c$. This non-trivial series class has size at most 3 and the element of $E(N)$ corresponding to it is the unique element that is in $\mathrm{cl}^*(Q_i)$ for all i in $\{1, 2, \dots, 6\}$.*
- (iii) *If $\mathbf{Q}$ is a copaddle, then all non-trivial series classes have size at most 3. Elements of $E(N)$ corresponding to such series classes are in $\mathrm{cl}^*(Q_i)$ for all i in $\{1, 2, \dots, 6\}$.*

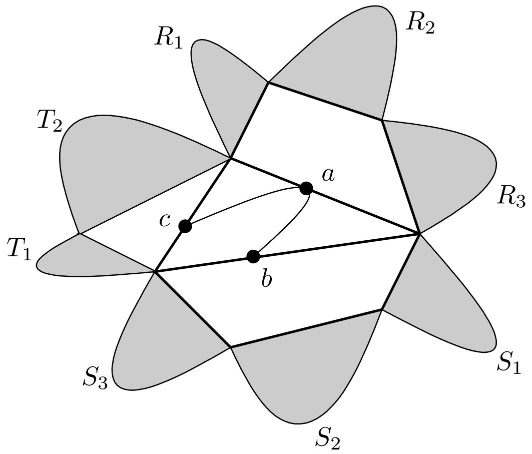

FIGURE 4. Inequivalent 3-separations are exposed by each of a and b.

From the last result, the reader may be tempted to think that, up to equivalence, all 3-separations exposed by a, b, or c can be seen from the flower $\mathbf{Q}$. The diagram in Figure 4 indicates that this is not the case.

Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M. Evidently $M\backslash a$ is or is not 3-connected. In the former case, by Bixby's Lemma [1], $\mathrm{co}(M\backslash a, b)$ or $\mathrm{si}(M\backslash a/b)$ is 3-connected. The final result of this section indicates precisely how to distinguish the different types of wild triangles.

Corollary 3.3. *Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M. Then $\{a, b, c\}$ is an internal triangle of a fan if and only if $M\backslash a$ is not 3-connected. Moreover, when $M\backslash a$ is 3-connected,*

(i) *$\{a, b, c\}$ is in a trident if and only if both $\mathrm{co}(M\backslash a, b)$ and $\mathrm{si}(M\backslash a/b)$ are 3-connected;*
(ii) *$\{a, b, c\}$ is a standard wild triangle if and only if $\mathrm{co}(M\backslash a, b)$ is 3-connected but $\mathrm{si}(M\backslash a/b)$ is not; and*
(iii) *$\{a, b, c\}$ is a costandard wild triangle if and only if $\mathrm{si}(M\backslash a/b)$ is 3-connected but $\mathrm{co}(M\backslash a, b)$ is not.*

4. AN EXTENSION OF TUTTE'S TRIANGLE LEMMA

The main theorem of the paper notes that one way in which a wild triangle can occur in a 3-connected matroid is as an internal triangle of a fan. In this section, we identify precisely when such wild triangles arise.

The next result is Tutte's Triangle Lemma [10], an important tool in matroid structure theory that is used, for example, in the proofs of Tutte's Wheels-and-Whirls Theorem [10] and Seymour's Splitter Theorem [9].

Lemma 4.1. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M. Suppose that $M\backslash b$ is not 3-connected, that no fan of M has b as an internal element, and that $|E(M)| \geq 4$. Then both $M\backslash a$ and $M\backslash c$ are 3-connected.*

The next theorem, the main result of this section, is an obvious strengthening of the last lemma. As such, it is of independent interest. Moreover, it has, as a straightforward consequence, Corollary 4.3, which splits wild triangles into two types and completely describes the first type.

Theorem 4.2. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M. Suppose that $M\backslash b$ is not 3-connected, that no fan of M has b as an internal element, and that $|E(M)| \geq 4$. Then both $M\backslash a$ and $M\backslash c$ are 3-connected and neither a nor c exposes a 3-separation in M.*

Proof. By Tutte's Triangle Lemma, $M\backslash a$ and $M\backslash c$ are 3-connected. Let (A, C) be a 2-separation of $M\backslash b$. As M is 3-connected, $\{a, c\}$ is not contained in A or C, so we may assume $a \in A$ and $c \in C$. Observe that

4.2.1. $|A|, |C| > 2$.

If $|A| = 2$, then A is a series pair in $M\backslash b$ so $A \cup b$ is a triad of M. It follows that $\{a, b, c\}$ is contained in a fan of M with at least four elements; a contradiction. Hence $|A| > 2$, and (4.2.1) follows by symmetry.

4.2.2. $a \in \mathrm{cl}(A - a)$ *and* $c \in \mathrm{cl}(C - c)$.

If $a \notin \mathrm{cl}(A - a)$, then $(A - a, C \cup a)$ is a 2-separation of $M\backslash b$ with $\{a, c\} \subseteq C \cup a$. This contradiction and symmetry imply (4.2.2).

4.2.3. $\lambda_{M\backslash a}(A - a) = 2$ *and* $\lambda_{M\backslash a}(C) = 2$.

We have $\lambda_{M\backslash b}(A) = 1$ and $a \in \mathrm{cl}(A - a)$, so $\lambda_{M\backslash b,a}(A - a) = 1$. Hence $\lambda_{M\backslash a}(A-a) \leq 2$. But $|A-a| \geq 2$ and $M\backslash a$ is 3-connected, so $\lambda_{M\backslash a}(A-a) = 2$. A similar, but easier, argument gives that $\lambda_{M\backslash a}(C) = 2$, so (4.2.3) holds.

Now assume that a exposes a 3-separation (R, G). Then, without loss of generality, $b \in G$ and $c \in R$. Next we show that

4.2.4. $(A - a) \cap G \neq \emptyset \neq (A - a) \cap R$.

If $(A - a) \cap G = \emptyset$, then $A - a \subseteq R$ so, by (4.2.2), $a \in \mathrm{cl}(R)$ contradicting the fact that a exposes (R, G). By symmetry, we conclude that (4.2.4) holds.

4.2.5. $(C - c) \cap R \neq \emptyset$.

If not, then $C - c \subseteq G$ so, by (4.2.2), $c \in \mathrm{cl}(G)$. Also $b \in G$. Hence $a \in \mathrm{cl}(G)$; a contradiction. Thus (4.2.5) holds.

4.2.6. $\lambda_{M\backslash a}(G \cap A) \leq 2$.

We have $\lambda_{M\backslash a}(G) = 2$ and $\lambda_{M\backslash a}(A-a) = 2$. Also, $|E-(G\cup(A-a))| \geq 2$ so, by uncrossing, $\lambda_{M\backslash a}(G \cap (A - a)) \leq 2$ and (4.2.6) follows.

4.2.7. $\lambda_{M\backslash a}(C \cap R) = 2$.

Since $\lambda_{M\backslash a}(R) = 2 = \lambda_{M\backslash a}(C)$, and $|(E - a) - (C \cup R)| \geq 2$ so, by uncrossing, $\lambda_{M\backslash a}(C \cap R) \leq 2$. But $|C \cap R| \geq 2$, so $\lambda_{M\backslash a}(C \cap R) = 2$.

4.2.8. $C \cap G \neq \emptyset$.

Assume $C \subseteq R$. Since $b \notin \mathrm{cl}(A)$, we have $b \notin \mathrm{cl}_{M\backslash a}(A-a)$, so $b \in \mathrm{cl}^*{}_{M\backslash a}(C)$. Hence $b \in \mathrm{cl}^*{}_{M\backslash a}(R)$. Thus, in $M\backslash a$, we have $(R,G) \cong (R \cup b, G-b)$. But $\{b,c\} \subseteq R \cup b$, so $a \in \mathrm{cl}(R \cup b)$. Hence a does not expose (R,G). This contradiction establishes (4.2.8).

4.2.9. $|C \cap G| \geq 2$.

Assume that $C \cap G = \{g\}$. If $C-g$ spans g, then $(R,G) \cong (R \cup g, G-g)$. By Lemma 2.4, $|G-g| \geq 3$ so $(R \cup g, G-g)$ is a 3-separation of $M\backslash a$ that is exposed by a. Replacing (R,G) by $(R\cup g, G-g)$ gives a contradiction to (4.2.8). Hence $g \notin \mathrm{cl}(C-g)$, so $g \in \mathrm{cl}^*(A \cup b)$. But $b \in \mathrm{cl}^*(A)$, so $g \in \mathrm{cl}^*(A)$. Hence $g \in \mathrm{cl}^*{}_{M\backslash b}(A)$ and so $(A \cup g, C-g)$ is a 2-separation of $M\backslash b$ with $a \in A \cup g$ and $c \in C-g$. Replacing (A,C) by $(A\cup g, C-g)$ gives a contradiction to (4.2.8). Hence (4.2.9) holds.

4.2.10. $\lambda_{M\backslash a}(A \cap R) \leq 2$.

We have $\lambda_{M\backslash a}(A-a) = 2 = \lambda_{M\backslash a}(R)$ and $|(E-a)-(A\cap R)| \geq 2$ by (4.2.9) so, by uncrossing, $\lambda_{M\backslash a}((A-a)\cap R) \leq 2$, that is, $\lambda_{M\backslash a}(A \cap R) \leq 2$.

4.2.11. $|A \cap R| > 1$ *or* $|A \cap G| > 1$.

Suppose that $|A \cap R| = 1 = |A \cap G|$. Then $|A| = 3$. Let $A \cap R = \{x\}$. Then $r(A) + r(C) = r(M\backslash b) + 1 = r(M) + 1$. But $a \in \mathrm{cl}(A-a)$ and so $r(A-a) + r(C) = r(M\backslash a) + 1$. Hence, as $M\backslash a$ is 3-connected, $r((A-a)\cup b) + r(C) = r(M\backslash a) + 2$, so $(A-a)\cup b$ is an independent triad of $M\backslash a$. Thus $x \in \mathrm{cl}^*{}_{M\backslash a}(((A-a)\cup b)-x)$. But $((A-a)\cup b)-x \subseteq G$, so $(R,G) \cong (R-x, G\cup x)$. As $(R-x, G\cup x)$ is a 3-separation of $M\backslash a$ exposed by a, we can replace (R,G) by $(R-x, G \cup x)$ to get a contradiction to (4.2.4). Thus (4.2.11) holds.

4.2.12. $|A \cap G| \neq 1 \neq |A \cap R|$.

Let $\{X,Y\} = \{R,G\}$ and assume that $A \cap X = \{x\}$. Then, by (4.2.11), $|A \cap Y| \geq 2$. If $x \in \mathrm{cl}(A \cap Y)$, then, as $a \in \mathrm{cl}(A-a)$, we deduce that $a \in \mathrm{cl}(Y)$, a contradiction. Thus $x \notin \mathrm{cl}(A \cap Y)$.

Now $A-a$ and Y are 3-separating in $M\backslash a$ and, since $|C\cap R|, |C \cap G| \geq 2$, the union $(A-a)\cup Y$ avoids at least two elements of $M\backslash a$. Hence, by uncrossing, $(A-a)\cap Y$, which equals $A \cap Y$, is 3-separating in $M\backslash a$. As $(A\cap Y)\cup x = A-a$, a 3-separating set in $M\backslash a$, we deduce that $x \in \mathrm{cl}_{M\backslash a}(A\cap Y)$ or $x \in \mathrm{cl}^*{}_{M\backslash a}(A \cap Y)$. The first possibility was eliminated above. Thus $x \in \mathrm{cl}^*{}_{M\backslash a}(A\cap Y) \subseteq \mathrm{cl}^*{}_{M\backslash a}(Y)$. Hence $(X,Y) \cong (X-x, Y \cup x)$. Replacing (X,Y) by $(X-x, Y\cup x)$, we get a contradiction to (4.2.4) since $(X-x)\cap(A-a) = \emptyset$. We conclude that (4.2.12) holds.

By (4.2.4) and (4.2.12), we have $|A \cap R| \geq 2$ and $|A \cap G| \geq 2$. Hence, by (4.2.6) and (4.2.10), $\lambda_{M\backslash a}(A\cap R) = 2 = \lambda_{M\backslash a}(A \cap G)$. Since $\lambda_{M\backslash a}(A-a) = 2$, we have $\lambda_{M\backslash a}(C \cup b) = 2$. Moreover, by uncrossing, $\lambda_{M\backslash a}((C\cap G)\cup b) = 2$. Thus $(A\cap R, C\cap R, (C\cap G)\cup b, A \cap G)$ is a flower Φ in $M\backslash a$. Since $b \notin \mathrm{cl}(C)$, we have $b \notin \mathrm{cl}_{M\backslash a}(C)$, so $b \in \mathrm{cl}^*{}_{M\backslash a}(A-a)$. Hence $b \in \mathrm{cl}^*{}_{M\backslash a}((A\cap R)\cup(A\cap$

$G))-[(A\cap R)\cup(A\cap G)]$. Thus, by [7, Lemma 5.5(i)], $b \in \mathrm{cl}^*{}_{M\backslash a}(A\cap G)$. Hence Φ is equivalent to $(A\cap R, C\cap R, C\cap G, (A\cap G)\cup b)$. Now $a \in \mathrm{cl}(A-a)$ and so $c \in \mathrm{cl}((A-a)\cup b)$, that is, $c \in \mathrm{cl}((A\cap R)\cup(A\cap G)\cup b)-[(A\cap R)\cup(A\cap G)\cup b]$. Hence, by [7, Lemma 5.5(i)] again, $c \in \mathrm{cl}(A\cap R)$. Thus $\{a, c\} \subseteq \mathrm{cl}(A)$, so $b \in \mathrm{cl}(A)$ and $(A \cup b, C)$ is a 2-separation of M; a contradiction. □

Corollary 4.3. *Let $\{a, b, c\}$ be a wild triangle in a 3-connected matroid M. Then either*

(i) *none of $M\backslash a$, $M\backslash b$, and $M\backslash c$ is 3-connected, and M has a fan in which $\{a, b, c\}$ is an internal triangle; or*
(ii) *all of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected, and each of a, b, and c exposes a 3-separation of M.*

Proof. If all of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected, then (ii) holds. Hence we may assume that $M\backslash b$ is not 3-connected. Then, by Theorem 4.2, M has a fan having b as an internal element. Thus b is in a triad T^*. Now $M \not\cong U_{2,4}$ so, by orthogonality, we may assume that $T^* = \{b, c, d\}$ where $d \neq a$. Then $\{a, b, c, d\}$ is a fan of M. Hence $M\backslash c$ is not 3-connected. If $M\backslash a$ is not 3-connected, then, by Lemma 4.1, a is an internal element of a fan of M. Thus a is in a triad of M and (z, a, b, c, d) is a fan ordering of a fan in M. In this case, (i) holds.

We may now assume that $M\backslash a$ is 3-connected. We shall show that a does not expose a 3-separation of M. Suppose that M has a 3-separation (R, G) that is exposed by a. Then $|R|, |G| \geq 4$ and, by Lemma 2.8(i), we may assume that $b \in R$ and $c \in G$. Without loss of generality, $d \in R$. Then $R \supseteq \{b, d\}$ so $c \in \mathrm{cl}^*{}_{M\backslash a}(R)$. Hence $(R \cup c, G - c)$ is an exactly 3-separating partition of $M\backslash a$ that is equivalent to (R, G). But $\{b, c\} \subseteq R \cup c$, so (R, G) is not well blocked by a; a contradiction. □

5. Towards the Main Result

The proof of the main result is long and essentially occupies the rest of the paper. In view of Corollary 4.3, we can make the following assumptions:

- M is a 3-connected matroid having $\{a, b, c\}$ as a triangle;
- all of $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected; and
- a, b, and c expose 3-separations in M.

These assumptions will remain in effect for the rest of the paper.

We shall take $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ to be arbitrary 3-separations, $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2), in M exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1, b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$. The symmetries revealed here are summarized in Table 1. These symmetries will be constantly exploited. This section contains a number of observations about how the sets A_1, A_2, B_1, B_2, C_1, and C_2 interact. This leads into the following section, which contains an overview of the logic of the proof of the main result.

By Lemma 2.4, we have

5.0.1. $|A_1|, |A_2|, |B_1|, |B_2|, |C_1|, |C_2| \geq 4$.

a	b	c
B_2	C_2	A_2
C_1	A_1	B_1

TABLE 1. Location of the elements of $\{a, b, c\}$.

Next we show that

5.0.2. $a \in \mathrm{cl}(B_2 - a) \cap \mathrm{cl}(C_1 - a), b \in \mathrm{cl}(C_2 - b) \cap \mathrm{cl}(A_1 - b), c \in \mathrm{cl}(A_2 - c) \cap \mathrm{cl}(B_1 - c)$.

By symmetry, it suffices to prove that $a \in \mathrm{cl}(B_2 - a)$. Assume not. Then $a \notin \mathrm{cl}_{M\backslash b}(B_2 - a)$ so, by duality, $a \in \mathrm{cl}^*{}_{M\backslash b}(B_1)$. Hence $(B_1 \cup a, B_2 - a) \cong (B_1, B_2)$. But $\{a, c\} \subseteq B_1 \cup a$, so (B_1, B_2) is not well blocked by b; a contradiction. Thus (5.0.2) holds.

5.0.3. $A_1 \cap B_1 \neq \emptyset$.

Since $b \in \mathrm{cl}(A_1 - b)$, if $A_1 \cap B_1 = \emptyset$, then $b \in \mathrm{cl}(B_2)$, a contradiction.

5.0.4. $|A_1 \cap B_1| \geq 2$.

Suppose $A_1 \cap B_1 = \{x\}$. If $x \in \mathrm{cl}(A_2 \cap B_1)$, then $x \in \mathrm{cl}(A_2)$. Hence $(A_1, A_2) \cong (A_1 - x, A_2 \cup x)$ and replacing (A_1, A_2) by $(A_1 - x, A_2 \cup x)$ gives a contradiction to (5.0.3). Thus $x \notin \mathrm{cl}(A_2 \cap B_1)$, that is, $x \notin \mathrm{cl}(B_1 - x)$. Hence $(B_1 - x, B_2 \cup x)$ is a 3-separation of $M\backslash b$ that is equivalent to (B_1, B_2). Replacing (B_1, B_2) by $(B_1 - x, B_2 \cup x)$ gives a contradiction to (5.0.3).

By symmetry, we deduce

5.0.5. $|A_i \cap B_i|, |B_i \cap C_i|, |C_i \cap A_i| \geq 2$ *for each* i *in* $\{1, 2\}$.

5.0.6. *If* $X \subseteq \{a, b, c\}$, *then*

$$\lambda_{M\backslash X}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1) = 2.$$

Since $r(M) = r(M\backslash a, b, c)$, it suffices to show that $\lambda_{M\backslash a,b,c}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1) = 2$. By (5.0.2), we have $c \in \mathrm{cl}(A_2 - c)$ and $a \in \mathrm{cl}(B_2 - a)$. Moreover, $\{a, c\}$ spans b. Thus $(A_2 \cup B_2) - \{a, c\}$ spans $\{a, b, c\}$ and so $\lambda_{M\backslash a,b,c}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1)$. Hence $\lambda_{M\backslash X}(A_1 \cap B_1) = \lambda_M(A_1 \cap B_1)$.

As $a \in \mathrm{cl}(B_2 - a)$, we have $\lambda_{M\backslash a,b}(B_2 - a) = \lambda_{M\backslash b}(B_2) = 2$. By symmetry, $\lambda_{M\backslash a,b}(A_1 - b) = 2$. But $A_2 = E - \{a, b\} - (A_1 - b)$, so $\lambda_{M\backslash a,b}(A_2) = 2$. Now, from the last paragraph, $\lambda_{M\backslash a,b}(A_1 \cap B_1) = \lambda_{M\backslash b}(A_1 \cap B_1)$. Since $M\backslash b$ is 3-connected, it follows by (5.0.5) that $\lambda_{M\backslash a,b}(A_1 \cap B_1) \geq 2$. By symmetry, $\lambda_{M\backslash a}(A_2 \cap B_2) = \lambda_{M\backslash a,b}(A_2 \cap B_2) \geq 2$. Hence

$$\begin{aligned}
2 + 2 &\leq \lambda_{M\backslash a,b}(A_1 \cap B_1) + \lambda_{M\backslash a,b}(A_2 \cap B_2) \\
&= \lambda_{M\backslash a,b}(A_1 \cap B_1) + \lambda_{M\backslash a,b}(E - \{a, b\} - (A_2 \cap B_2)) \\
&= \lambda_{M\backslash a,b}((A_1 - b) \cap B_1) + \lambda_{M\backslash a,b}((A_1 - b) \cup B_1) \\
&\leq \lambda_{M\backslash a,b}(A_1 - b) + \lambda_{M\backslash a,b}(B_1) \\
&= 2 + 2.
\end{aligned}$$

Thus equality holds throughout, so $\lambda_{M\backslash a,b}(A_1 \cap B_1) = 2$ and (5.0.6) holds.

By symmetry with (5.0.6), we have the following.

5.0.7. *Suppose $X \subseteq \{a, b, c\}$. If J and K are distinct members of $\{A, B, C\}$ and $i \in \{1, 2\}$, then $\lambda_{M\backslash X}(J_i \cap K_i) = \lambda_M(J_i \cap K_i) = 2$.*

5.0.8. $\lambda_M(A_2 \cap B_1) = \lambda_{M\backslash a}(A_2 \cap B_1) = \lambda_{M\backslash b}(A_2 \cap B_1) = \lambda_{M\backslash a,b}(A_2 \cap B_1)$;
$\lambda_M(B_2 \cap C_1) = \lambda_{M\backslash b}(B_2 \cap C_1) = \lambda_{M\backslash c}(B_2 \cap C_1) = \lambda_{M\backslash b,c}(B_2 \cap C_1)$;
$\lambda_M(C_2 \cap A_1) = \lambda_{M\backslash c}(C_2 \cap A_1) = \lambda_{M\backslash a}(C_2 \cap A_1) = \lambda_{M\backslash c,a}(C_2 \cap A_1)$.

Since $a \in \mathrm{cl}(B_2 - a)$ and $b \in \mathrm{cl}(A_1 - b)$, the first line holds; the second and third lines hold by symmetry.

5.0.9. $|(A_2 \cap B_1) - c| \geq 1, |(B_2 \cap C_1) - a| \geq 1,$ *and* $|(C_2 \cap A_1) - b| \geq 1$.

Suppose that $A_2 \cap B_1 = \{c\}$. Then, since $\lambda_{M\backslash a}(A_2 \cap B_2) = 2 = \lambda_{M\backslash a}(A_2)$, we have $(A_1, A_2) \cong (A_1 \cup c, A_2 - c)$ in $M\backslash a$. But $\{b, c\} \subseteq A_1 \cup c$, so (A_1, A_2) is not exposed by a; a contradiction. We conclude that (5.0.9) holds.

5.0.10. *None of $A_1 \cap B_2, B_1 \cap C_2,$ and $C_1 \cap A_2$ is empty.*

Suppose $A_1 \cap B_2 = \emptyset$. As $b \in \mathrm{cl}(A_1 - b) = \mathrm{cl}(A_1 \cap B_1)$, we deduce that $b \in \mathrm{cl}(B_1)$. This contradiction establishes that (5.0.9) holds.

5.0.11. $\lambda_M(A_2 \cap B_1) \in \{2, 3\}$ *and* $\lambda_{M\backslash a,b}(A_1 \cap B_2) \in \{1, 2\}$*; if* $\lambda_M(A_2 \cap B_1) = 3$*, then* $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$*.*

Since $b \in \mathrm{cl}(A_1 - b)$, we deduce that $\lambda_{M\backslash a,b}(A_2) = \lambda_{M\backslash a}(A_2) = 2$. By symmetry, $\lambda_{M\backslash a,b}(B_1) = 2$. Thus $\lambda_{M\backslash a,b}(A_2 \cap B_1) + \lambda_{M\backslash a,b}(A_2 \cup B_1) \leq 4$. Since $|(E - \{a, b\} - (A_2 \cup B_1))| = |A_1 \cap B_2| \geq 1$, we have $\lambda_{M\backslash a,b}(A_2 \cup B_1) \geq 1$, so $\lambda_{M\backslash a,b}(A_2 \cap B_1) \leq 3$. But $a \in \mathrm{cl}(B_2 - a)$ and $b \in \mathrm{cl}(A_1 - b)$, so $\lambda_M(A_2 \cap B_1) = \lambda_{M\backslash a,b}(A_2 \cap B_1) \leq 3$. By (5.0.9), $\lambda_M(A_2 \cap B_1) \geq 2$. Hence $\lambda_M(A_2 \cap B_1) \in \{2, 3\}$. Since $\lambda_{M\backslash a,b}(A_2 \cup B_1) = \lambda_{M\backslash a,b}(A_1 \cap B_2)$, we deduce that $\lambda_{M\backslash a,b}(A_1 \cap B_2) \in \{1, 2\}$. Moreover, if $\lambda_M(A_2 \cap B_1) = 3$, then $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$. We conclude that (5.0.11) holds.

6. Overview

This section gives an overview of the logic of the argument to follow. The division of cases is based on the cardinality and connectivity of the sets $A_1 \cap B_2$ and $A_2 \cap B_1$. By (5.0.10) and (5.0.9), we know that $|A_1 \cap B_2| \geq 1$ and $|A_2 \cap B_1| \geq 2$. Moreover, by (5.0.11), $\lambda_{M\backslash a,b}(A_1 \cap B_2) \in \{1, 2\}$. The argument will distinguish the following six cases:

(A) $|A_1 \cap B_2| = 1$;
(B) $|A_2 \cap B_1| = 2$;
(C) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| = 3$;
(D) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $|A_1 \cap B_2| = 2$;
(E) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| \geq 4$; and
(F) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $|A_1 \cap B_2| \geq 3$.

In case (A), Lemma 7.3 identifies three types of special structures that can arise after possibly replacing $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2) by equivalent 3-separations. We call these structures pretridents of type I, II, and III. From a pretrident of type I, we immediately obtain a trident in M. In case (B), we show in Lemma 7.4 that $|B_1 \cap C_2| = 1$ or $|C_1 \cap A_2| = 1$ so, by symmetry, we have reduced to case (A) and again we find that $\{a, b, c\}$ is in a pretrident. In case (C), we show, in Lemma 7.5, that either $|E(M)| = 11$, or we can reduce to case (B) and hence to case (A). In case (D), Lemma 7.7 shows that $|E(M)| = 11$ or we can reduce to an earlier case. In case (E), which we shall treat last, we show that either a symmetric case to case (C) occurs, or the two sets of symmetric conditions to (E) also hold and outcome (ii) of Theorem 3.1 arises. Finally, in case (F), we show, in Lemma 8.4, that case (E) and its symmetric counterparts hold in the matroid M' that is obtained from M by performing a $\Delta - Y$ exchange in M on the triangle $\{a, b, c\}$ and then taking the dual of the result. Thus outcome (iii) of Theorem 3.1 arises.

Pretridents of type II and III appear in neither of Theorems 1.1 and 3.1. The next section starts by describing the structure of M around $\{a, b, c\}$ relative to the 3-separations with which we begin, only allowing replacement of these 3-separations by equivalent ones. Then Lemma 7.9 shows that, when $\{a, b, c\}$ is in a pretrident of type II or III, we can find a pretrident of type I containing $\{a, b, c\}$ by altering the choice of 3-separations exposed by a, b, and c to ones that need not be equivalent to those with which we began.

7. Tridents

In this section, we begin the treatment of the six cases noted in the preceding section. Specifically, we deal with cases (A)-(D) here. We begin with an elementary lemma. Recall that the assumptions noted at the outset of Section 5 are still in effect and that **A**, **B**, and **C** are arbitrary 3-separations, $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2), exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1, b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$.

Lemma 7.1. *If* $|A_2 \cap B_1 \cap C_2| \geq 2$*, then*

$$\lambda_M(A_2 \cap B_1 \cap C_2) = 2 = \lambda_{M\backslash b,c}(A_2 \cap B_1 \cap C_2).$$

Proof. By (5.0.7), $2 = \lambda_{M\backslash b}(A_2 \cap C_2)$. Since $\lambda_{M\backslash b}(B_1) = 2$, we have, by uncrossing, that $\lambda_{M\backslash b}(A_2 \cap B_1 \cap C_2) = 2$. Hence $\lambda_M(A_2 \cap B_1 \cap C_2) = 2$ as $b \in \mathrm{cl}(A_1 - b)$. Since $a \in B_2$, we have $c \in \mathrm{cl}((A_1 - b) \cup B_2)$. Hence $\lambda_{M\backslash b,c}(A_2 \cap B_1 \cap C_2) = 2$. □

The next lemma begins the treatment of case (A).

Lemma 7.2. *Suppose that* $|A_1 \cap B_2| = 1$ *and* $A_1 \cap B_2 \subseteq C_1$*. Then*

$$|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = 1.$$

Proof. Let $A_1 \cap B_2 \cap C_1 = \{r_{12}\}$. Since $|A_1 \cap C_1| \geq 2$, we have $|A_1 \cap B_1 \cap C_1| \geq 1$. Suppose that $|A_1 \cap B_1 \cap C_1| \geq 2$. By (5.0.7) that $A_1 \cap C_1$ and $A_1 \cap B_1$ are 3-separating in M and so in $M\backslash b$. Their intersection has at least two elements, so their union is 3-separating in $M\backslash b$. Hence $\lambda_{M\backslash b}((A_1\cap B_1)\cup r_{12}) = 2$. Thus, by Lemma 2.5, $r_{12} \in \mathrm{cl}^{(*)}_{M\backslash b}(A_1 \cap B_1)$. Hence $(B_1, B_2) \cong (B_1 \cup r_{12}, B_2 - r_{12})$ in $M\backslash b$. But $b \in \mathrm{cl}(A_1 - b)$ and $A_1 - b \subseteq B_1 \cup r_{12}$ so (B_1, B_2) is not exposed by b. Thus $|A_1 \cap B_1 \cap C_1| = 1$, say $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$.

Now $|A_1| \geq 4$, so $|A_1 \cap B_1 \cap C_2| \geq 1$. Suppose $|A_1 \cap B_1 \cap C_2| \geq 2$. By (5.0.7), $\lambda_{M\backslash c}(A_1\cap B_1) = 2$ and so, by Lemma 2.5, $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_1\cap B_1\cap C_2)$. Thus $(C_1, C_2) \cong (C_1 - r_{11}, C_2 \cup r_{11})$ in $M\backslash c$ and $|A_1 \cap (C_1 - r_{11})| = 1$; a contradiction to (5.0.5). Hence $|A_1\cap B_1\cap C_2| = 1$, say $A_1\cap B_1\cap C_1 = \{g_{11}\}$.

Next we show that $|A_2 \cap B_1 \cap C_1| = 1$. We have $|B_1 \cap C_1| \geq 2$, so $|A_2 \cap B_1 \cap C_1| \geq 1$. Assume that $|A_2 \cap B_1 \cap C_1| \geq 2$. We have $\lambda_{M\backslash a}(A_2) = 2 = \lambda_{M\backslash a}(B_1 \cap C_1)$. Thus, by uncrossing, $\lambda_{M\backslash a}(A_2 \cap B_1 \cap C_1) = 2$. Since $\lambda_M((A_2 \cap B_1 \cap C_1) \cup r_{11}) = 2 = \lambda_{M\backslash a}((A_2 \cap B_1 \cap C_1) \cup r_{11})$, we have $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_2\cap B_1\cap C_1)$ so $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_2)$. Hence $(A_1, A_2) \cong (A_1-r_{11}, A_2\cup r_{11})$ in $M\backslash a$. But, replacing (A_1, A_2) by $(A_1 - r_{11}, A_2 \cup r_{11})$ gives a contradiction to (5.0.5) since $|(A_1 - r_{11}) \cap B_1| = 1$. Thus $|A_2 \cap B_1 \cap C_1| = 1$. $\square$

Lemma 7.3. *Suppose that $|A_1 \cap B_2| = 1$. Then, after*

(i) *the possible replacement of $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2) by equivalent 3-separations;*
(ii) *a possible relabelling of $(A_1, A_2, a, B_1, B_2, b, C_1, C_2, c)$ by $(B_2, B_1, b, A_2, A_1, a, C_2, C_1, c)$; and*
(iii) *a possible rotation of the labels on the triples $(A_1, A_2, a), (B_1, B_2, b)$, and (C_1, C_2, c);*

the following hold:

(a) $|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = 1$*; and*
(b) *A_1 is a quad of $M\backslash a$ and $A_1 \cup a$ is a cocircuit of M.*

In addition,

(I) *$|A_2 \cap B_1 \cap C_2| = 0$ and $A_2 \cap B_2 \subseteq C_2$; or*
(II) *$|A_2 \cap B_1 \cap C_2| = 0$ and $\lambda_M(A_2 \cap B_2 \cap C_2) = 2 = \lambda_{M\backslash a,c}(A_2 \cap C_1) = \lambda_M(A_2 \cap B_2 \cap C_1)$; or*
(III) *$\lambda_M(A_2 \cap B_2 \cap C_2) = 2 = \lambda_{M\backslash a,c}(A_2 \cap C_1) = \lambda_M(A_2 \cap B_2 \cap C_1)$ and $\lambda_M(A_2 \cap B_1 \cap C_2) = 2 = \lambda_{M\backslash b,c}(B_1 \cap C_2)$.*

The situations corresponding to (I), (II), and (III) are shown in Figure 5. Each of the parts of the diagram should be interpreted as basically a Venn diagram. The elements of C_1 correspond to black points while those in C_2 are shaded gray. Regions that are shaded indicate the presence of at least two elements. The placement of a and b is to indicate that their deletion from M exposes the 3-separations (A_1, A_2) and (B_1, B_2).

To achieve outcomes (I)-(III) of Lemma 7.3, we allow equivalence moves and relabelling as described in (i)-(iii) of the lemma. When we can manipulate $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in this way so that (I), (II), or (III) in Figure 5 occurs, we shall say that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a *pretrident of type I, type II, or type III*, respectively, or that $\{a, b, c\}$ *occurs in a pretrident with respect to* $\mathbf{A}$, $\mathbf{B}$, *and*

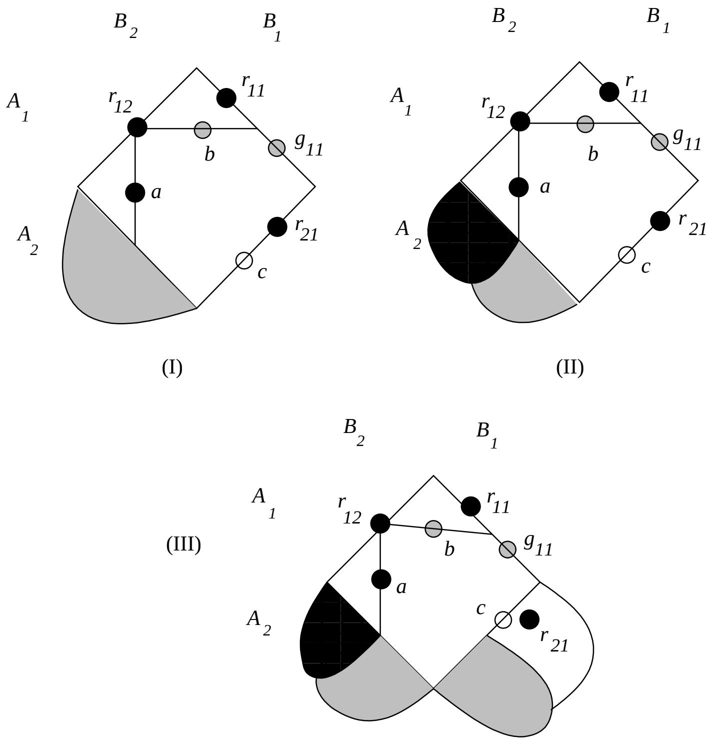

FIGURE 5. The three types of pretrident.

C. We observe that, when $\{a, b, c\}$ occurs in a pretrident of type I, each of A_1, B_1, and C_1 has exactly four elements and so, by Lemma 2.4, these sets are quads of $M\backslash a$, $M\backslash b$, and $M\backslash c$, respectively. Thus $A_1 \cup B_1 \cup C_1$ is a trident in M containing $\{a, b, c\}$. In fact, Lemma 7.9 shows that, when $\{a, b, c\}$ occurs in any of the three types of pretridents, $\{r_{11}, r_{12}, r_{21}, g_{11}, a, b, c\}$ is a trident in M, where elements are labelled as in Figure 5.

Proof of Lemma 7.3. Without loss of generality, we may assume that $A_1 \cap B_2 \subseteq C_1$. Thus $B_2 \cap C_2 = A_2 \cap B_2 \cap C_2$. Moreover, by Lemma 7.2,

$$|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = 1.$$

We denote the elements of these three sets by r_{11}, g_{11}, and r_{21}, respectively. Let $A_1 \cap B_2 \cap C_1 = \{r_{12}\}$. Since $|A_1| = 4$, by Lemma 2.4, we must have that A_1 is a quad of $M\backslash a$. As $b \in A_1$, it follows by orthogonality with the triangle $\{a, b, c\}$ that $A_1 \cup a$ is a cocircuit of M.

Since $|B_2 \cap C_2| \geq 2$, we have $|A_2 \cap B_2 \cap C_2| \geq 2$. Suppose that $|A_2 \cap B_2 \cap C_1| = 1$, say $A_2 \cap B_2 \cap C_1 = \{r_{22}\}$. Then $r_{22} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_2 \cap B_2 \cap C_2)$. Thus, by replacing (C_1, C_2) by $(C_1 - r_{22}, C_2 \cup r_{22})$, an equivalent 3-separating partition of $M\backslash c$, we reduce to the case when $|A_2 \cap B_2 \cap C_1| = 0$. Thus we may assume that either

(i) $|A_2 \cap B_2 \cap C_1| = 0$; or
(ii) $|A_2 \cap B_2 \cap C_1| \geq 2$.

In case (ii), $\lambda_M(A_2 \cap B_2 \cap C_1) \geq 2$. Now, by (5.0.11), $\lambda_{M\backslash a,c}(A_2 \cap C_1) \in \{1, 2\}$. Moreover, $\lambda_M(A_2 \cap B_2 \cap C_1) = \lambda_{M\backslash a,c}(A_2 \cap B_2 \cap C_1)$. If $\lambda_{M\backslash a,c}(A_2 \cap B_2 \cap C_1) > \lambda_{M\backslash a,c}(C_1 \cap A_2)$, then, by Lemma 2.7, $r_{21} \in \mathrm{cl}(A_2 \cap B_2 \cap C_1)$. Thus we can replace (B_1, B_2) by the equivalent 3-separating partition $(B_1 - r_{21}, B_2 \cup r_{21})$ to get a contradiction to (5.0.5). Thus, in case (ii), $\lambda_M(A_2 \cap B_2 \cap C_1) = 2$ and $\lambda_{M\backslash a,c}(C_1 \cap A_2) = 2$. Hence our two cases become:

(i) $|A_2 \cap B_2 \cap C_1| = 0$; or
(ii) $|A_2 \cap B_2 \cap C_1| \geq 2$ and $\lambda_M(A_2 \cap B_2 \cap C_1) = 2 = \lambda_{M\backslash a,c}(C_1 \cap A_2)$.

Now consider $|A_2 \cap B_1|$. If this is 2, then case (I) or (II) holds depending on which of (i) and (ii) holds. If $|A_2 \cap B_1| = 3$, then $A_2 \cap B_1 \cap C_2 = \{g_{21}\}$, say. Since both $B_2 \cap C_2$ and $(B_2 \cap C_2) \cup g_{21} = A_2 \cap C_2$ are exactly 3-separating set in $M\backslash b$, we deduce that $g_{21} \in \mathrm{cl}^{(*)}_{M\backslash b}(B_2 \cap C_2)$. Thus, after replacing (B_1, B_2) by the equivalent 3-separation $(B_1 - g_{21}, B_2 \cup g_{21})$, we have reduced to the case when $|A_2 \cap B_1| = 2$. Again case (I) or (II) holds.

We may now assume that $|A_2 \cap B_1| \geq 4$, so $|A_2 \cap B_1 \cap C_2| \geq 2$. Then, by a symmetric argument to that given in the penultimate paragraph, we deduce that $\lambda_M(A_2 \cap B_1 \cap C_2) = 2 = \lambda_{M\backslash b,c}(B_1 \cap C_2)$. If case (i) occurs, then $|C_1 \cap A_2| = 1$. By rotating the labels on the triples $(A_1, A_2, a), (B_1, B_2, b)$, and (C_1, C_2, c), we obtain that case (I) or case (II) holds. If case (ii) occurs, then we have that case (III) holds. □

Next we show if (B) arises, then a symmetric relabelling gives (A).

Lemma 7.4. *If $|A_2 \cap B_1| = 2$, then either*

(i) $|A_1 \cap B_1 \cap C_1| = |A_1 \cap B_1 \cap C_2| = |A_2 \cap B_1 \cap C_1| = |B_1 \cap C_2| = 1$ *and* $|A_2 \cap B_1 \cap C_2| = 0$*; or*
(ii) $|A_2 \cap B_2 \cap C_1| = |A_2 \cap B_2 \cap C_2| = |A_2 \cap B_1 \cap C_2| = |C_1 \cap A_2| = 1$ *and* $|A_2 \cap B_1 \cap C_1| = 0$.

In each case, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ *is a pretrident.*

Proof. Suppose $(A_2 \cap B_1) - c \subseteq C_1$, say $(A_2 \cap B_1) - c = \{r_{21}\}$. Since $|B_1 \cap C_1| \geq 2$, we have $|A_1 \cap B_1 \cap C_1| \geq 1$. If $|A_1 \cap B_1 \cap C_1| \geq 2$, then, since $B_1 \cap C_1$ and $A_1 \cap B_1$ are exactly 3-separating in $M\backslash a$, so too is their union. Hence $r_{21} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_1 \cap B_1)$, so $r_{21} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_1)$. Replacing (A_1, A_2) by the equivalent 3-separating partition $(A_1 \cup r_{21}, A_2 - r_{21})$ gives a contradiction to (5.0.9). Hence $|A_1 \cap B_1 \cap C_1| = 1$, say $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$.

If $|A_1 \cap B_1 \cap C_2| \geq 2$, then, as $A_1 \cap B_1$ is exactly 3-separating in $M\backslash c$, it follows that $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_1 \cap B_1 \cap C_2)$. Thus $(C_1, C_2) \cong (C_1 - r_{11}, C_2 \cup r_{11})$ in $M\backslash c$. But $|B_1 \cap (C_1 - r_{11})| = 1$ so we have contradicted (5.0.5). Hence $|A_1 \cap B_1 \cap C_2| = 1$. Thus $|B_1 \cap C_2| = 1$. Hence if $(A_2 \cap B_1) - c \subseteq C_1$, then (i) holds. By symmetry, if $(A_2 \cap B_1) - c \subseteq C_2$, then (ii) holds.

In cases (i) and (ii), we have $|B_1 \cap C_2| = 1$ and $|C_1 \cap A_2| = 1$, respectively. These are symmetric to the case $|A_1 \cap B_2| = 1$ so, by Lemma 7.3, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident. □

The next lemma treats case (C).

Lemma 7.5. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| = 3$, then*

(i) *$A_2 \cap B_1$ is a triangle; and*

(ii) *either $|E(M)| = 11$, or $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident.*

Proof. By (5.0.11), $\lambda_M(A_2 \cap B_1) = 2$. Since $A_2 \cap B_1$ has three elements, it is a triangle or a triad of M. The triangle $\{a, b, c\}$ implies that $A_2 \cap B_1$ is not a triad, so it is a triangle. Since $c \notin \mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)$, we deduce that $|A_2 \cap B_1 \cap C_1| = 1 = |A_2 \cap B_1 \cap C_2|$. Let $A_2 \cap B_1 \cap C_2 = \{g_{21}\}$. Since $|A_2 \cap C_2| \geq 2$, we have $|A_2 \cap B_2 \cap C_2| \geq 1$. Assume $|A_2 \cap B_2 \cap C_2| \geq 2$. Then, as $\lambda_M(A_2 \cap B_2) = 2$ and $\lambda_M(A_2 \cap C_2) = 2$, it follows by uncrossing that $\lambda_M((A_2 \cap B_2) \cup g_{21}) = 2$. Hence $g_{21} \in \mathrm{cl}^{(*)}_M(A_2 \cap B_2)$. As $A_2 \cap B_1$ is a triangle, it follows that $g_{21} \in \mathrm{cl}(A_2 \cap B_2) \subseteq \mathrm{cl}(B_2)$. Thus $(B_1, B_2) \cong (B_1 - g_{21}, B_2 \cup g_{21})$ in $M\backslash c$. Replacing (B_1, B_2) by $(B_1 - g_{21}, B_2 \cup g_{21})$, we get that $|A_2 \cap (B_1 - g_{21})| = 2$. Hence, by Lemma 7.4, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident. We may now assume that $|A_2 \cap B_2 \cap C_2| = 1$, say $A_2 \cap B_2 \cap C_2 = \{g_{22}\}$.

Since $|A_2 \cap B_2| \geq 2$, we have $|A_2 \cap B_2 \cap C_1| \geq 1$. Assume $|A_2 \cap B_2 \cap C_1| \geq 2$. Now $\lambda_{M\backslash c}(A_2 \cap B_2) = 2 = \lambda_{M\backslash c}(C_1)$. Thus, by uncrossing, $\lambda_{M\backslash c}(A_2 \cap B_2 \cap C_1) = 2$. Hence $g_{22} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_2 \cap B_2 \cap C_1) \subseteq \mathrm{cl}^{(*)}_{M\backslash c}(C_1)$. Hence $(C_1, C_2) \cong (C_1 \cup g_{22}, C_2 - g_{22})$ in $M\backslash c$. But $|(C_2 - g_{22}) \cap A_2| = 1$; a contradiction to (5.0.5). Hence $|A_2 \cap B_2 \cap C_1| = 1$.

By the symmetry between $A_2 \cap B_2$ and $A_1 \cap B_1$, we deduce that $|A_1 \cap B_1 \cap C_1| = 1 = |A_1 \cap B_1 \cap C_2|$. Now $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$. Thus, either $|A_1 \cap B_2| = 2$ and so $|E(M)| = 11$, or $|A_1 \cap B_2| > 2$. Since we have assumed that $|E(M)| \neq 11$, we deduce that $|A_1 \cap B_2| > 2$. Thus, without loss of generality, $|A_1 \cap B_2 \cap C_2| \geq 2$. Thus $\lambda_M(A_1 \cap B_2 \cap C_2) \geq 2 = \lambda_M((A_1 \cap B_2 \cap C_2) \cup g_{22})$. Hence, by Lemma 2.5 or 2.7, we have $g_{22} \in \mathrm{cl}^{(*)}(A_1 \cap B_2 \cap C_2)$, so $g_{22} \in \mathrm{cl}^{(*)}_{M\backslash a}(A_1 \cap B_2 \cap C_2) \subseteq \mathrm{cl}^{(*)}_{M\backslash a}(A_1)$. Hence $(A_1, A_2) \cong (A_1 \cup g_{22}, A_2 - g_{22})$ in $M\backslash a$. But $|(A_2 - g_{22}) \cap B_2| = 1$, contradicting (5.0.5). □

In combination with the last lemma, the next lemma guarantees that, when case (E) arises but none of cases (A)-(C) arise, we can assume that we have symmetry between (A_1, A_2), (B_1, B_2), and (C_1, C_2).

Lemma 7.6. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| > 3$, then $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 2$ and $\lambda_{M\backslash c,a}(C_1 \cap A_2) = 2$.*

Proof. By symmetry, it suffices to prove the first equation. By (5.0.11), we may assume that $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1$. We have $2 = \lambda_{M\backslash b}(B_1)$ and $c \in \mathrm{cl}(B_1 - c)$, so $2 = \lambda_{M\backslash b}(B_1 - c) = \lambda_{M\backslash b,c}(B_1 - c)$. Since $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$, by (5.0.11) we have $\lambda_{M\backslash a,b}(A_2 \cap B_1) = 2$. Then, since $a \in \mathrm{cl}(B_2 - a)$ and $b \in \mathrm{cl}(A_1 - b)$, we have $\lambda_M(A_2 \cap B_1) = 2$.

Now $|A_2 \cap B_1| > 3$. Thus $c \in \mathrm{cl}^{(*)}((A_2 \cap B_1) - c)$. But $\{a, b, c\}$ is a triangle, so $c \notin \mathrm{cl}^*((A_2 \cap B_1) - c)$. Hence $c \in \mathrm{cl}((A_2 \cap B_1) - c)$. Thus $\lambda_{M\backslash c}((A_2 \cap B_1) - c) = 2$, so $\lambda_{M\backslash b,c}((A_2 \cap B_1) - c) = 2$ since $b \in \mathrm{cl}(A_1 - b)$.

Since $c \in \mathrm{cl}((A_2 \cap B_1) - c)$ but $c \notin \mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)$, we deduce that both $B_1 \cap C_2 \cap A_2$ and $B_1 \cap C_1 \cap A_2$ are nonempty. Since $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1$ and $\lambda_{M\backslash b,c}((A_2 \cap B_1) - c) = 2$, we have, by Lemma 2.2, that $\lambda_{M\backslash b,c}(B_1 \cap C_2 \cap A_2) = 1$ or $\lambda_{M\backslash b,c}(((A_2 \cap B_1) - c) \cup (B_1 \cap C_2)) = 1$. But the latter implies that $\lambda_{M\backslash b}((A_2 \cap B_1) \cup (B_1 \cap C_2)) = 1$; a contradiction since $M\backslash b$ is 3-connected. Thus $\lambda_{M\backslash b,c}(B_1 \cap C_2 \cap A_2) = 1$ so, by Lemma 7.1, $|B_1 \cap C_2 \cap A_2| = 1$. Let $B_1 \cap C_2 \cap A_2 = \{g_{21}\}$.

Since $|A_2 \cap B_1| > 3$, we have $|A_2 \cap B_1 \cap C_1| \geq 2$. Since $\lambda_M(A_2 \cap B_1) = 2$ and $\lambda_M(B_1 \cap C_1) = 2$, it follows by uncrossing that $\lambda_M(A_2 \cap B_1 \cap C_1) = 2$ and hence $\lambda_{M\backslash c}(A_2 \cap B_1 \cap C_1) = 2$. Thus, by Lemma 2.5, $g_{21} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_2 \cap B_1 \cap C_1)$, so $g_{21} \in \mathrm{cl}^{(*)}_{M\backslash c}(C_1)$. Therefore $(C_1, C_2) \cong (C_1 \cup g_{21}, C_2 - g_{21})$ in $M\backslash c$. But $c \in \mathrm{cl}(C_1 \cup g_{21})$; a contradiction. $\square$

The next lemma treats case (D).

Lemma 7.7. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $|A_1 \cap B_2| = 2$, then $|E(M)| = 11$ or $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident.*

Proof. Assume that $|E(M)| \neq 11$ and that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident. Suppose that $A_1 \cap B_2 \subseteq C_1$. We have $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ and $b \in \mathrm{cl}(C_2 - b)$. Hence $\lambda_{M\backslash a}(A_1 \cap B_2) = 1$; a contradiction. Thus $|A_1 \cap B_2 \cap C_2| \geq 1$. By symmetry, $|A_1 \cap B_2 \cap C_1| \geq 1$. Hence $|A_1 \cap B_2 \cap C_2| = 1 = |A_1 \cap B_2 \cap C_1|$, say $A_1 \cap B_2 \cap C_1 = \{r_{12}\}$ and $A_1 \cap B_2 \cap C_2 = \{g_{12}\}$.

If $|A_1 \cap B_1 \cap C_1| \geq 2$, then, as $A_1 \cap B_1$ and $B_1 \cap C_1$ are both exactly 3-separating in $M\backslash b$, so is their intersection. Since $(A_1 \cap B_1 \cap C_1) \cup r_{12}$ is also exactly 3-separating in $M\backslash b$, we deduce that $r_{12} \in \mathrm{cl}^{(*)}_{M\backslash b}(A_1 \cap B_1 \cap C_1)$. Hence $(B_1, B_2) \cong (B_1 \cup r_{12}, B_2 - r_{12})$. But $|A_1 \cap (B_2 - r_{12})| = 1$ and so, by Lemma 7.3, $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident; a contradiction. Hence we may assume that $|A_1 \cap B_1 \cap C_1| \leq 1$. Since $|A_1 \cap C_1| \geq 2$, we get $|A_1 \cap B_1 \cap C_1| = 1$. By symmetry, $|A_2 \cap B_2 \cap C_2| = 1$. Let $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$.

As $|A_1 \cap B_1| \geq 2$, we have $|A_1 \cap B_1 \cap C_2| \geq 1$. If $|A_1 \cap B_1 \cap C_2| \geq 2$, then $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_1 \cap B_1 \cap C_2)$ so $(C_1, C_2) \cong (C_1 - r_{11}, C_2 \cup r_{11})$. Since $|A_1 \cap (C_1 - r_{11})| = 1$, we have a contradiction to (5.0.5). Hence $|A_1 \cap B_1 \cap C_2| = 1$ and, by symmetry, $|A_2 \cap B_2 \cap C_1| = 1$.

As $|A_2 \cap C_2|, |B_1 \cap C_1| \geq 2$, we deduce that $|A_2 \cap B_1 \cap C_2| \geq 1$ and $|A_2 \cap B_1 \cap C_1| \geq 1$. If equality holds in both, then $|E(M)| = 11$; a contradiction. Hence we may assume that $|A_2 \cap B_1 \cap C_2| \geq 2$. Then, by Lemma 7.1, $\lambda_{M\backslash b,c}(A_2 \cap B_1 \cap C_2) = 2$. Now $|B_2 \cap C_1| > 2$. Suppose that $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 2$. If $|B_2 \cap C_1| = 3$, then, by Lemma 7.5, $|E(M)| = 11$ or $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident; a contradiction. Hence $|B_2 \cap C_1| > 3$ and, by Lemma 7.6, $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$; a contradiction. We may now assume that $\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1$. Then $\lambda_{M\backslash b,c}((A_2 \cap B_1 \cap C_2) \cup g_{11}) = 1$ where $A_1 \cap B_1 \cap C_2 = \{g_{11}\}$.

By Lemma 2.7, $g_{11} \in \mathrm{cl}_{M\backslash b,c}(A_2 \cap B_1 \cap C_2)$. Hence $(A_1, A_2) \cong (A_1 - g_{11}, A_2 \cup g_{11})$. But $|(A_1 - g_{11}) \cap B_1| = 1$, a contradiction to (5.0.5). $\square$

The next result summarizes the lemmas to date in this section. It notes that when any of cases (A)-(D) occurs, by replacing (A_1, A_2), (B_1, B_2), and (C_1, C_2) by equivalent 3-separations and performing a symmetric relabelling, we get one of the three outcomes shown in Figure 5.

Corollary 7.8. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M, where $|E(M)| \neq 11$. Suppose that all of $M\backslash a, M\backslash b$, and $M\backslash c$ are 3-connected and that (A_1, A_2), (B_1, B_2), and (C_1, C_2) are 3-separations exposed by a, b, and c, respectively, with $a \in B_2 \cap C_1, b \in C_2 \cap A_1$, and $c \in A_2 \cap B_1$. If*

(A) $|A_1 \cap B_2| = 1$, *or*
(B) $|A_2 \cap B_1| = 2$, *or*
(C) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ *and* $|A_2 \cap B_1| = 3$, *or*
(D) $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$ *and* $|A_1 \cap B_2| = 2$,

then $\{a, b, c\}$ is in a pretrident with respect to (A_1, A_2), (B_1, B_2), and (C_1, C_2).

We show next that, when $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident, there are potentially different 3-separations $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ exposed by b and c so that $(a, b, c, \mathbf{A}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ is a pretrident of type I, hence $\{a, b, c\}$ is in a trident. When we choose $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ so $\hat{B}_1$ and $\hat{C}_1$ are quads of $M\backslash b$ and $M\backslash c$, we have no guarantee that these new 3-separations are equivalent to the original ones.

Lemma 7.9. *Suppose that $|A_1 \cap B_2| = 1$ and $A_1 \cap B_2 \subseteq C_1$. Then A_1 is a quad of $M\backslash a$. Moreover, there are 3-separations $(\hat{B}_1, \hat{B}_2)$ and $(\hat{C}_1, \hat{C}_2)$ that are exposed by b and c, respectively, such that $\hat{B}_1$ and $\hat{C}_1$ are quads in $M\backslash b$ and $M\backslash c$. In particular, $A_1 \cup \hat{B}_1 \cup \hat{C}_1$ is a trident in M.*

Proof. From Lemma 7.3, A_1 is a quad of $M\backslash a$ and $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident of type I, II, or III. If this pretrident has type I, then the lemma holds with $(\hat{B}_1, \hat{B}_2) = (B_1, B_2)$, and $(\hat{C}_1, \hat{C}_2) = (C_1, C_2)$. Thus assume that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident of type II or III. We shall maintain the same labelling of elements as before (see Figure 5).

To complete the proof of the lemma, we shall show that

(i) $\{r_{11}, r_{12}, r_{21}, a\}$ is a quad of $M\backslash c$ and $(\{r_{11}, r_{12}, r_{21}, a\}, E - c - \{r_{11}, r_{12}, r_{21}, a\})$ is exposed by c; and
(ii) $\{r_{11}, r_{21}, g_{11}, c\}$ is a quad of $M\backslash b$ and $(\{r_{11}, r_{21}, g_{11}, c\}, E - b - \{r_{11}, r_{21}, g_{11}, c\})$ is exposed by b.

First observe that $\lambda_{M\backslash c}((A_1 \cup B_1) - c) = 2 = \lambda_{M\backslash c}(C_1)$. Hence, by uncrossing, $\lambda_{M\backslash c}(C_1 \cap (A_1 \cup B_1)) = 2$, that is, $\{r_{11}, r_{12}, r_{21}, a\}$ is 3-separating in $M\backslash c$. We show next that $\{r_{11}, r_{12}, r_{21}, a\}$ is a circuit of M. Assume not. Then it contains a triangle. But $a \notin \mathrm{cl}(A_1)$, so $\{r_{11}, r_{12}, a\}$ is not a triangle. If $\{r_{11}, r_{21}, a\}$ is a triangle, then $\{a, c\} \subseteq \mathrm{cl}(B_1)$, so $b \in \mathrm{cl}(B_1)$; a contradiction. If $\{r_{12}, r_{21}, a\}$ is a triangle, then $(B_1, B_2) \cong (B_1 - r_{21}, B_2 \cup$

$r_{21})$. But $|(B_1 - r_{21}) \cap C_1| = 1$; a contradiction to (5.0.5). Finally, if $\{r_{11}, r_{12}, r_{21}\}$ is a triangle, then $(B_1, B_2) \cong (B_1 \cup r_{12}, B_2 - r_{12})$. But $A_1 \cap (B_2 - r_{12}) = \emptyset$; a contradiction to (5.0.10). Hence $\{r_{11}, r_{12}, r_{21}, a\}$ is a circuit.

Next we show that $\{r_{11}, r_{12}, r_{21}, a\}$ is a cocircuit of $M\backslash c$. Assume not. Then this set contains a triad of $M\backslash c$. If $\{r_{11}, r_{12}, r_{21}\}$ is a triad of $M\backslash c$, then, by orthogonality, it is a triad of M and hence of $M\backslash b$. Thus $(B_1, B_2) \cong (B_1 \cup r_{12}, B_2 - r_{12})$. As $A_1 \cap (B_2 - r_{12}) = \emptyset$, we contradict (5.0.10). If $\{r_{11}, r_{12}, a\}$ is a triad of $M\backslash c$, then, by orthogonality, $\{r_{11}, r_{12}, a, c\}$ is a cocircuit of M, so $\{r_{11}, r_{12}, c\}$ is a triad of $M\backslash a$. Then $(A_1, A_2) \cong (A_1 \cup c, A_2 - c)$. But $a \in \mathrm{cl}(A_1 \cup c)$; a contradiction. If $\{r_{11}, r_{21}, a\}$ is a triad of $M\backslash c$, then $\{r_{11}, r_{21}, c\}$ is a triad of $M\backslash a$. Thus $(A_1, A_2) \cong (A_1 - r_{11}, A_2 \cup r_{11})$. But $|(A_1 - r_{11}) \cap B_1| = 1$, contradicting (5.0.5). Finally, if $\{r_{12}, r_{21}, a\}$ is a triad of $M\backslash c$, then $\{r_{12}, r_{21}, c\}$ is a triad of $M\backslash a$, so $(A_1, A_2) \cong (A_1 - r_{12}, A_2 \cup r_{12})$ and we get a contradiction since $(A_1 - r_{12}) \cap B_2 = \emptyset$. Thus $\{r_{11}, r_{12}, r_{21}, a\}$ is a cocircuit of $M\backslash c$, so $\{r_{11}, r_{12}, r_{21}, a, c\}$ is a cocircuit of M.

We show next that $(\{r_{11}, r_{12}, r_{21}, a\}, E - c - \{r_{11}, r_{12}, r_{21}, a\})$ is exposed by c. First observe that $c \notin \mathrm{cl}(E - c - \{r_{11}, r_{12}, r_{21}, a\})$ since $\{r_{11}, r_{12}, r_{21}, a, c\}$ is a cocircuit of M. Because $E - c - \{r_{11}, r_{12}, r_{21}, a\}$ is fully closed in $M\backslash c$, we need only consider the full closure of $\{r_{11}, r_{12}, r_{21}, a\}$ in $M\backslash c$. Since $\{r_{11}, r_{12}, r_{21}, a\} \subseteq C_1$, it follows that $\mathrm{fcl}_{M\backslash c}(\{r_{11}, r_{12}, r_{21}, a\}) \subseteq \mathrm{fcl}_{M\backslash c}(C_1)$. Now $c \notin \mathrm{cl}(\mathrm{fcl}_{M\backslash c}(C_1))$ so $c \notin \mathrm{cl}(\mathrm{fcl}_{M\backslash c}(\{r_{11}, r_{12}, r_{21}, a\})$. We conclude that (i) holds. Let $\hat{\mathbf{C}} = (\hat{C}_1, \hat{C}_2) = (\{r_{11}, r_{12}, r_{21}, a\}, E - c - \{r_{11}, r_{12}, r_{21}, a\})$.

If $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident of type II, then, since $\hat{\mathbf{C}}$ is exposed by c, we see that $(a, b, c, \mathbf{A}, \mathbf{B}, \hat{\mathbf{C}})$ is a pretrident of type I. Finally, let $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ be a pretrident of type III. Then, by redrawing the figure, we see that $(c, a, b, \hat{\mathbf{C}}, \mathbf{A}, \mathbf{B})$ is a pretrident of type II. After this move, $\{r_{11}, r_{21}, g_{11}, c\}$ is in a symmetric position to that of $\{r_{11}, r_{12}, r_{21}, a\}$ before the move. Thus (ii) holds. In particular, letting $\hat{\mathbf{B}} = (\hat{B}_1, \hat{B}_2) = (\{r_{11}, r_{21}, g_{11}, c\}, E - b - \{r_{11}, r_{21}, g_{11}, c\})$, we have that $\hat{\mathbf{B}}$ is a 3-separation exposed by b. Thus $(c, a, b, \hat{\mathbf{C}}, \mathbf{A}, \hat{\mathbf{B}})$ is a pretrident of type I. Redrawing again, we find that $(a, b, c, \mathbf{A}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ is a pretrident of type I. $\square$

8. A Delta-Wye Exchange

In this section, we show that if case (F) occurs in M, then, after performing a $\Delta - Y$ exchange and taking the dual of the result, we get a matroid in which case (E) and the two sets of symmetric conditions occur.

Lemma 8.1. *Assume that $|E(M)| \neq 11$, that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident, and that $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$. Then*

(i) *$|A_2 \cap B_1| \geq 3$, $|B_2 \cap C_1| \geq 3$, $|C_2 \cap A_1| \geq 3$;*
(ii) *$\lambda_{M\backslash b,c}(B_1 \cap C_2) = 1 = \lambda_{M\backslash c,a}(C_1 \cap A_2)$; and*
(iii) *$|A_1 \cap B_2| \geq 3$, $|B_1 \cap C_2| \geq 3$, $|C_1 \cap A_2| \geq 3$.*

Proof. Part (i) follows from Lemma 7.4; part (ii) follows using part (i), symmetry, and Lemmas 7.5 and 7.6; and part (iii) follows from (ii) by Lemma 7.7 and symmetry. □

The next two lemmas introduce the matroid M' that appears in outcome (iii) of Theorem 3.1 and then prove that M' satisfies the conditions imposed on M at the start of Section 5.

Lemma 8.2. *Let Δ be the triangle $\{a,b,c\}$ of M and consider a copy of $M(K_4)$ that has Δ as a triangle and has $\{a',b',c'\}$ as the complementary triad, where e' is the element of $M(K_4)$ that is not in a triangle with e. Let $\Delta M = P_\Delta(M(K_4),M)\backslash\Delta$, that is, ΔM is obtained from M by a $\Delta - Y$ exchange on Δ. Then*

(i) *ΔM is 3-connected;*
(ii) *for all $\{x,y,z\} = \{a,b,c\}$, the matroid $\Delta M/x'$ can be obtained from $M\backslash x$ by relabelling y and z by z' and y', respectively; and*
(iii) *each of $\Delta M/a'$, $\Delta M/b'$, and $\Delta M/c'$ is 3-connected.*

Proof. We know that each of $M\backslash a$, $M\backslash b$, and $M\backslash c$ is 3-connected. Hence $M\backslash a,b$ is connected and, by considering circuits, it is straightforward to check that ΔM is connected. Suppose ΔM has a 2-separation (X_1,X_2). Without loss of generality, we may assume that $\{a',b'\} \subseteq X_1$. If $\{a',b',c'\} \subseteq X_1$, then we can add Δ to X_1 without raising the rank, so we get a 2-separation of $P_\Delta(M(K_4),M)$; a contradiction. Thus we may assume that $c' \in X_2$. But $c' \in \mathrm{cl}^*_{\Delta M}(X_1)$. Hence, provided $|X_2| > 2$, we get that $(X_1 \cup c', X_2 - c')$ is a 2-separation of ΔM and $\{a',b',c'\} \subseteq X_1 \cup c'$, so we have a contradiction, as above. Hence we may assume that $|X_2| > 2$. In that case, X_2 is a series pair $\{c',x\}$ in $P_\Delta(M(K_4),M)\backslash\{a,b,c\}$. The triangle $\{a',b',c\}$ implies that $\{c',x\}$ is also a cocircuit of $P_\Delta(M(K_4),M)\backslash\{a,b\}$. It follows, by orthogonality, that $\{c',x,a,b\}$ is a cocircuit of $P_\Delta(M(K_4),M)$. Thus $\{x,a,b\}$ is a triad of M; a contradiction since $\{a,b,c\}$ is not in a fan with four or more elements. Hence ΔM is 3-connected. Evidently, $\Delta M/a'$ can be obtained from $M\backslash a$ by relabelling b and c by c' and b'. Thus $\Delta M/a'$ is 3-connected and, by symmetry, (ii) and (iii) hold. □

Under the relabelling described in (ii) of the last lemma, the 3-separations $(A_1,A_2),(B_1,B_2)$, and (C_1,C_2) of $M\backslash a$, $M\backslash b$, and $M\backslash c$ map to the 3-separations $((A_1-b)\cup c',(A_2-c)\cup b'),((B_1-c)\cup a',(B_2-a)\cup c')$, and $((C_1-a)\cup b',(C_2-b)\cup a')$ of $\Delta M/a'$, $\Delta M/b'$, and $\Delta M/c'$, respectively. These 3-separations are also 3-separations of the dual matroids. We shall denote these 3-separations by $(A_1',A_2'),(B_1',B_2')$, and (C_1',C_2'). The following table summarizes the inclusions we have.

We write M' for $(\Delta M)^*$. To obtain the matroid M' in (iii) of Theorem 3.1, we need to relabel a',b', and c' as a,b, and c. However, for clarity in the remaining proofs in this section, we shall not do this relabelling yet.

a'	b'	c'
B'_1	C'_1	A'_1
C'_2	A'_2	B'_2

TABLE 2. Location of the elements of $\{a', b', c'\}$.

Lemma 8.3. *Each of M', $M'\backslash a'$, $M'\backslash b'$, and $M'\backslash c'$ is 3-connected. Moreover, $(A'_1, A'_2), (B'_1, B'_2)$, and (C'_1, C'_2) are 3-separations in M' that are exposed by a', b', and c', respectively.*

Proof. The first sentence is an immediate consequence of the last lemma. Now suppose that $(A'_1, A'_2) \cong (A''_1, A''_2)$ in $M'\backslash a'$ and that $(A''_i \cup a', A''_j)$ is an exactly 3-separating partition of M', for some $\{i, j\} = \{1, 2\}$. Then (A'_1, A'_2) and (A''_1, A''_2) are equivalent exactly 3-separating partitions of $(M')^*/a'$, that is, of $\Delta M/a'$; and $(A''_i \cup a', A''_j)$ is an exactly 3-separating partition of ΔM. Recall that $\Delta M/a'$ is $M\backslash a$ with b and c relabelled as c' and b', respectively. Hence $M\backslash a$ has an exactly 3-separating partition (X_i, X_j) that is equivalent to (A_i, A_j) and corresponds to (A''_i, A''_j) under this relabelling. Since $a \notin \mathrm{cl}(X_i) \cup \mathrm{cl}(X_j)$, it follows that neither X_i nor X_j contains $\{b, c\}$. It follows that one of b' and c' is in A''_i and the other is in A''_j.

By (5.0.1), $|A''_j| > 3$. Now $\Delta M\backslash(A''_i \cup a')$ contains exactly one of b' and c' so this element is a coloop of $\Delta M\backslash(A''_i \cup a')$. Thus $(A''_i \cup a' \cup \{b', c'\}, A''_j - \{b', c'\})$ is an exactly 3-separating partition of ΔM and so $(A''_i \cup \{b', c'\}, A''_j - \{b', c'\})$ is an exactly 3-separating partition of $\Delta M/a'$. But the last exactly 3-separating partition is equivalent to (A''_i, A''_j) as $|A''_i \cup \{b', c'\}| = |A''_i| + 1$, so we have reduced to the case in which $\{b', c'\} \subseteq A''_i$, which we have already eliminated. We conclude that the 3-separation (A'_1, A'_2) of $M'\backslash a'$ is, indeed, exposed by a' and the rest of the lemma follows by symmetry. $\square$

Lemma 8.4. *Assume that $|E(M)| \neq 11$, that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident, and that $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 1$. Then*

(i) *$\lambda_{M'\backslash a',c'}(A'_1 \cap C'_2) = 2$ and $|A'_2 \cap C'_1| \geq 4$;*
(ii) *$\lambda_{M'\backslash b',a'}(B'_1 \cap A'_2) = 2$ and $|B'_2 \cap A'_1| \geq 4$; and*
(iii) *$\lambda_{M'\backslash c',b'}(C'_1 \cap B'_2) = 2$ and $|C'_2 \cap B'_1| \geq 4$.*

Proof. First observe that $A'_2 \cap C'_1 = (A_2 \cap C_1) \cup b'$ and $A'_1 \cap C'_2 = (A_1 \cap C_2) - b$. Since, by Lemma 8.1, $|C_2 \cap A_1| \geq 3$ and $|C_1 \cap A_2| \geq 3$, we deduce that $|A'_2 \cap C'_1| \geq 4$ and $|A'_1 \cap C'_2| \geq 2$. Similarly, $|B'_2 \cap A'_1| \geq 4$ and $|C'_2 \cap B'_1| \geq 4$.

Now assume none of (i)–(iii) holds. As $M\backslash a$ is 3-connected and $A_1 \cap B_2$ is a non-minimal 3-separation of $M\backslash a, b$, it follows by Bixby's Lemma (see [1] or [6, Proposition 8.4.6]) that $M\backslash a/b$ is 3-connected up to parallel pairs. Hence $(M\backslash a/b)^*$ is 3-connected up to series pairs. But $\Delta M/a', c'$ is $M\backslash a/b$ with c relabelled as b'. Thus $M'\backslash a', c'$ is 3-connected up to series pairs.

Now $\lambda_{M'\backslash c'}(C'_2) = \lambda_{\Delta M/c'}((C_2 - b) \cup a') = \lambda_{M\backslash c}(C_2) = 2$. Likewise, $\lambda_{M'\backslash a'}(A'_1) = 2$. Thus, since $|C'_2| \geq 4$ and $M'\backslash a', c'$ is 3-connected up to series pairs, $2 \leq \lambda_{M'\backslash a',c'}(C'_2 - a') \leq 2$ so $\lambda_{M'\backslash a',c'}(C'_2 - a') = 2$. Similarly, $\lambda_{M'\backslash a',c'}(A'_1 - c') = 2$.

By the submodularity of the connectivity function,

$$4 \geq \lambda_{M'\backslash a',c'}((A'_1 - c') \cup (C'_2 - a')) + \lambda_{M'\backslash a',c'}(A'_1 \cap C'_2),$$

so $4 \geq \lambda_{M'\backslash a',c'}(A'_2 \cap C'_1) + \lambda_{M'\backslash a',c'}(A'_1 \cap C'_2)$. As $M'\backslash a', c'$ is 3-connected up to series pairs and $|A'_1 \cap C'_2| \geq 2$, we deduce that either $\lambda_{M'\backslash a',c'}(A'_1 \cap C'_2) = 2$, or $A'_1 \cap C'_2$ is a series pair of $M'\backslash a', c'$. In the first case, since $|A'_2 \cap C'_1| \geq 4$, we have that (i) holds; a contradiction. Thus $|A'_1 \cap C'_2| = 2$.

By Lemma 8.1 and symmetry, since neither (ii) nor (iii) holds, each of $|B'_1 \cap A'_2|$ and $|C'_1 \cap B'_2|$ is 2. Hence each of $|A_1 \cap C_2|$, $|B_1 \cap A_2|$, and $|C_1 \cap B_2|$ is 3. But $|A_1 \cap B_2| \geq 3$ so, by symmetry, we may assume that $|A_1 \cap B_2 \cap C_2| \geq 2$. Since $b \in C_2$, it follows that $A_1 \cap B_1 \subseteq C_1$. Now $|B_1 \cap C_2| \geq 3$, so $|B_1 \cap C_2 \cap A_2| \geq 3$. Since $A_2 \cap B_1$ also contains c, we deduce that $|A_2 \cap B_1| \geq 4$, contradicting the fact that $|A_2 \cap B_1| = 3$.

We conclude that at least one of (i), (ii), and (iii) holds. But, by applying Lemma 7.6 to M', we conclude that all of (i), (ii), and (iii) hold. □

To end this section, we prove three lemmas that will be used in the proof of Corollary 3.3. We also note that a trident in M yields a trident in M'.

Lemma 8.5. *Let $\{a, b, c\}$ be a wild triangle that is in a trident in a 3-connected matroid M. Then both* $\mathrm{co}(M\backslash a, b)$ *and* $\mathrm{si}(M\backslash a/b)$ *are 3-connected.*

Proof. We know that $M\backslash a$ has a quad Q containing b. By applying Lemma 2.9 to $M\backslash a$, we deduce that $\mathrm{si}(M\backslash a/b)$ is 3-connected. Now Q is also a quad of $(M\backslash a)^*$, that is, of M^*/a. By applying Lemma 2.9 to the last matroid, we deduce that $\mathrm{si}(M^*/a/b)$ is 3-connected. Thus $(\mathrm{co}(M\backslash a, b))^*$ and hence $\mathrm{co}(M\backslash a, b)$ is 3-connected. □

Lemma 8.6. *Let $\{a, b, c\}$ be a standard wild triangle in a 3-connected matroid M. Then*

(i) $\mathrm{si}(M\backslash a/b)$ *is not 3-connected; and*
(ii) $\mathrm{co}(M\backslash a, b)$ *is 3-connected.*

Proof. Let $(P_1, P_2, \dots, P_6)$ be a partition associated to $\{a, b, c\}$. Then $(P_1 \cup P_2 \cup a, P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup c)$ is a flower. Thus $(P_3 \cup P_4 \cup b, P_5 \cup P_6 \cup P_1 \cup P_2 \cup \{a, c\})$ is a 3-separation of M. Moreover, $b \in \mathrm{cl}(P_3 \cup P_4) \cap \mathrm{cl}(P_5 \cup P_6 \cup P_1 \cup P_2 \cup \{a, c\})$ and $|P_3|, |P_4| \geq 2$. Thus $(P_3 \cup P_4, P_5 \cup P_6 \cup P_1 \cup P_2 \cup c)$ is a vertical 2-separation of $M/b\backslash a$ unless $r_M(P_3 \cup P_4 \cup b) = 2$. But, in the exceptional case, $b \in \mathrm{cl}(P_3)$, so $(P_4 \cup P_5 \cup P_6 \cup c, P_1 \cup P_2 \cup P_3 \cup a)$ is not exposed by b; a contradiction. Thus $\mathrm{si}(M\backslash a/b)$ is not 3-connected. As $M\backslash a$ is 3-connected, Bixby's Lemma implies that $\mathrm{co}(M\backslash a, b)$ is 3-connected. □

Lemma 8.7. *Let $\{a, b, c\}$ be a costandard wild triangle in a 3-connected matroid M. Then*

(i) $\mathrm{co}(M\backslash a, b)$ *is not 3-connected; and*
(ii) $\mathrm{si}(M\backslash a/b)$ *is 3-connected.*

Proof. Retaining the labelling we have been using in this section for ΔM, we have that $\{a', b', c'\}$ is a standard wild triangle of $(\Delta M)^*$. By Lemma 8.6 and symmetry, $\mathrm{si}((\Delta M)^*/c'\backslash b')$ is not 3-connected. Thus $\mathrm{co}(\Delta M\backslash c'/b')$ is not 3-connected. But $\Delta M\backslash c'/b'$ is $M\backslash a, b$ with c relabelled as a', so $\mathrm{co}(M\backslash a, b)$ is not 3-connected. Also, by Bixby's Lemma, $\mathrm{si}(M\backslash a/b)$ is 3-connected. □

Lemma 8.8. *If $\{a, b, c\}$ is a wild triangle that is in a trident R in a 3-connected matroid M, then $(R - \{a, b, c\}) \cup \{a', b', c'\}$ is a trident in M'.*

Proof. Let $R = \{a, b, c, s, t, u, v\}$ where $\{t, s, u, b\}, \{t, u, v, c\}$, and $\{t, s, v, a\}$ are exposed quads in $M\backslash a, M\backslash b$, and $M\backslash c$, respectively. Then one easily checks that $\{t, s, u, a', c'\}, \{t, u, v, a', b'\}$, and $\{t, s, v, b', c'\}$ are circuits of ΔM; and $\{t, s, u, c'\}, \{t, u, v, a'\}$, and $\{t, s, v, b'\}$ are cocircuits of ΔM. The result follows since M' is $(\Delta M)^*$. □

9. The Target

In this section, we treat case (E). We begin by noting the following immediate consequence of Lemmas 7.5 and 7.6.

Corollary 9.1. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| > 3$, then either*

(i) *$|E(M)| = 11$ or $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident; or*
(ii) *$\lambda_{M\backslash b,c}(B_1 \cap C_2) = 2$ and $|B_2 \cap C_1| > 3$, and $\lambda_{M\backslash c,a}(C_1 \cap A_2) = 2$ and $|C_2 \cap A_1| > 3$.*

In view of this, many lemmas in this section will assume not only that (E) occurs but also that the symmetric conditions listed in (ii) above hold.

Lemma 9.2. *If $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$ and $|A_2 \cap B_1| > 2$, then*

(i) *$c \in \mathrm{cl}((A_2 \cap B_1) - c)$ and $c \notin \mathrm{cl}^*((A_2 \cap B_1) - c)$; and*
(ii) *$|A_2 \cap B_1 \cap C_1| > 0$ and $|A_2 \cap B_1 \cap C_2| > 0$.*

Proof. By (5.0.11), since $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$, we have $\lambda_M(A_2 \cap B_1) = 2$. As $|A_2 \cap B_1| > 2$ and $c \in A_2 \cap B_1$, Lemma 2.5 implies that $c \in \mathrm{cl}^{(*)}((A_2 \cap B_1) - c)$. But $\{a, b, c\}$ is a triangle of M so, by orthogonality, $c \notin \mathrm{cl}^*((A_2 \cap B_1) - c)$. Hence $c \in \mathrm{cl}((A_2 \cap B_1) - c)$. Thus (i) holds. As $c \notin \mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)$, it follows that $(A_2 \cap B_1) - c \nsubseteq C_1$, and $(A_2 \cap B_1) - c \nsubseteq C_2$, so (ii) holds. □

The next lemma draws heavily on results from [7].

Lemma 9.3. *Let P be a 2-element petal of a tight flower Φ of order at least three in a 3-connected matroid N. Let (R, G) be a 3-separation of N with $|R|, |G| \geq 4$. If both $R \cap P$ and $G \cap P$ are non-empty, then Φ has order three, and the union of P with one of the other petals is a quad.*

Proof. Let $\Phi = (P_1, P_2, \dots, P_n)$ and $P_2 = P$. Because Φ has order at least three, all the petals of Φ are tight. Each element of P is tight since P must contain at least one tight element but, by [7, Lemma 5.8], P cannot contain exactly one tight element. Let $P_2 \cap R = \{r_2\}$ and $P_2 \cap G = \{g_2\}$.

Suppose that $|P_1 \cap G| \geq 2$ and $|(P_3 \cup P_4 \cup \cdots \cup P_n) \cap R| \geq 2$. Then $r_2 \cup (P_3 \cup P_4 \cup \cdots \cup P_n)$, which is the union of the 3-separating sets $P_3 \cup P_4 \cup \cdots \cup P_n$ and $R \cap (P_2 \cup P_3 \cup \cdots \cup P_n)$, is also 3-separating. Hence $r_2 \in \mathrm{fcl}(P_3 \cup P_4 \cup \cdots \cup P_n)$ so, by [7, Lemma 5.9], r_2 is loose, a contradiction. Thus $|P_1 \cap G| \leq 1$ or $|(P_3 \cup P_4 \cup \cdots \cup P_n) \cap R| \leq 1$. By symmetry, $|P_1 \cap R| \leq 1$ or $|(P_3 \cup P_4 \cup \cdots \cup P_n) \cap G| \leq 1$. Because $|G| \geq 4$ and $|R \cap P_2|, |G \cap P_2| = 1$, at most one of $|P_1 \cap G| \leq 1$ and $|(P_3 \cup P_4 \cup \cdots \cup P_n) \cap G| \leq 1$ holds. By the symmetry between R and G, we deduce that either $|P_1 \cap G| \leq 1$ and $|P_1 \cap R| \leq 1$; or $|(P_3 \cup P_4 \cup \cdots \cup P_n) \cap G| \leq 1$ and $|(P_3 \cup P_4 \cup \cdots \cup P_n) \cap R| \leq 1$. By reflective symmetry in Φ, we also have that either $|P_3 \cap G| \leq 1$ and $|P_3 \cap R| \leq 1$; or $|(P_4 \cup P_5 \cup \cdots \cup P_n \cup P_1) \cap G| \leq 1$ and $|(P_4 \cup P_5 \cup \cdots \cup P_n \cup P_1) \cap R| \leq 1$. Since $|R|, |G| \geq 4$, it follows that $n = 3$ and $|P_1|$ or $|P_3|$ is 2.

Suppose that $|P_1| = 2$. Then $P_1 \cup P_2$ is 3-separating in N having exactly four elements. If $P_1 \cup P_2$ properly contains a circuit or a cocircuit, then P_1 or P_2 is not tight. Hence $P_1 \cup P_2$ is a quad. $\square$

Lemma 9.4. *Assume that* $\lambda_{M\backslash a,b}(A_1 \cap B_2) = \lambda_{M\backslash b,c}(B_1 \cap C_2) = \lambda_{M\backslash c,a}(C_1 \cap A_2) = 2$ *and* $|A_2 \cap B_1|, |B_2 \cap C_1|, |C_2 \cap A_1| > 3$. *Then* (C_1, C_2) *can be replaced by an equivalent* 3*-separating partition exposed by* c *such that*

$$A_1 \cap B_1 \cap C_1 = \emptyset = A_2 \cap B_2 \cap C_2.$$

Proof. First observe that, by (5.0.11), $\lambda_M(A_2 \cap B_1) = 2$. Now, by Lemma 9.2, $A_2 \cap B_1 \cap C_1$ and $A_2 \cap B_1 \cap C_2$ are non-empty. Suppose that $A_2 \cap B_1 \cap C_1 = \{r_{21}\}$. Then $|(A_2 \cap B_1) - c| \geq 3$ and $\lambda_{M\backslash c}((A_2 \cap B_1) - c) = 2$, so $r_{21} \in \mathrm{cl}^{(*)}_{M\backslash c}(A_2 \cap B_1 \cap C_2)$. Hence $(C_1 - r_{21}, C_2 \cup r_{21}) \cong (C_1, C_2)$. But $c \in \mathrm{cl}((A_2 \cap B_1 \cap C_2) \cup r_{21}) \subseteq \mathrm{cl}(C_2 \cup r_{21})$ contradicting the fact that c exposes (C_1, C_2). Thus

$$|A_2 \cap B_1 \cap C_1| \geq 2.$$

A symmetric argument to that just given establishes that

$$|A_2 \cap B_1 \cap C_2| \geq 2.$$

We now prove an observation that simplifies the argument to follow.

9.4.1. *If* $|A_1 \cap B_1 \cap C_2| = 1$, *then* $|A_1 \cap B_1 \cap C_1| = 1$.

Assume that $|A_1 \cap B_1 \cap C_1| \geq 2$ and let $A_1 \cap B_1 \cap C_2 = \{g_{11}\}$. As $\lambda_{M\backslash c}(A_1 \cap B_1) = 2$, we deduce that $g_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}((A_1 \cap B_1) - g_{11})$ so we can replace (C_1, C_2) by the equivalent $(C_1 \cup g_{11}, C_2 - g_{11})$. After this is done, $A_1 \cap B_1 \subseteq C_1$. But, by Lemma 9.2 and symmetry, $b \in \mathrm{cl}((A_1 \cap C_2) - b)$. Hence $b \in \mathrm{cl}(B_2)$; a contradiction. Thus (9.4.1) holds.

Most of the rest of the proof of the lemma will be occupied with proving the following assertion from which the lemma will follow straightforwardly.

9.4.2. (C_1, C_2) *can be replaced by an equivalent* 3*-separation in which* $A_1 \cap B_1 \cap C_1 = \emptyset$ *and* $A_2 \cap B_2 \cap C_2$ *is unchanged.*

Suppose not and assume that $A_1 \cap B_1 \cap C_1 = \{r_{11}\}$. If $|A_1 \cap B_1 \cap C_2| \geq 2$, then $r_{11} \in \mathrm{cl}^{(*)}_{M\backslash c}((A_1 \cap B_1) - r_{11})$ and we can replace (C_1, C_2) by $(C_1 - r_{11}, C_2 \cup r_{11})$ to obtain that $A_1 \cap B_1 \cap C_1 = \emptyset$. Since this change has no effect on $A_2 \cap B_2 \cap C_2$, we have a contradiction. Hence if $|A_1 \cap B_1 \cap C_1| = 1$, then we may assume that $|A_1 \cap B_1 \cap C_2| = 1$. On the other hand, if $|A_1 \cap B_1 \cap C_1| \geq 2$, then, by (9.4.1) and the consequence of Lemma 9.2 that $A_1 \cap B_1 \cap C_2$ is non-empty, we deduce that $|A_1 \cap B_1 \cap C_2| \geq 2$. We show next that

9.4.3. $|A_1 \cap B_1 \cap C_1| \geq 2$ *and* $|A_2 \cap B_2 \cap C_2| \geq 2$.

From above, we know that if (9.4.3) fails, then we may assume that $|A_1 \cap B_1 \cap C_1| = 1 = |A_1 \cap B_1 \cap C_2|$, so $|A_1 \cap B_1| = 2$.

Consider the flower $(B_2, A_1 \cap B_1, A_2 \cap B_1)$ in $M\backslash b$. We show next that this flower, Φ_b, is tight. Assume not. The petal B_2 is not loose otherwise $B_2 \subseteq \mathrm{fcl}_{M\backslash b}(B_1)$ and (B_1, B_2) is sequential; a contradiction. If $A_2 \cap B_1$ is loose, then, by Lemma 2.10, $\mathrm{fcl}_{M\backslash b}(A_1 \cap B_1) \supseteq A_2 \cap B_1$. But $|A_1 \cap B_1| = 2$ and so B_1 is sequential; a contradiction. Hence $A_2 \cap B_1$ is not loose.

Now suppose that $A_1 \cap B_1$ is loose. By Lemma 2.10, $A_1 \cap B_1 \subseteq \mathrm{fcl}(B_2)$ so (B_1, B_2) can be replaced by an equivalent 3-separating partition in which $|A_1 \cap B_1| < 2$; a contradiction. Thus $A_1 \cap B_1$ is not loose, so Φ_b is a tight flower. Therefore, by Lemma 9.3, $|A_2 \cap B_1| = 2$ or $|B_2| = 2$. Neither of these holds, and this contradiction completes the proof that (9.4.3) holds.

To complete the proof of (9.4.2), we shall apply Lemma 8.2 of [7] to the flower Φ_b using the 3-separation $(B_1 \cap C_1, E - b - (B_1 \cap C_1))$ as (R, G) of that lemma. By that lemma, there is a flower Φ'_b that refines Φ_b and displays $(B_1 \cap C_1, E - b - (B_1 \cap C_1))$, namely $(B_2, A_1 \cap B_1 \cap C_2, A_1 \cap B_1 \cap C_1, A_2 \cap B_1 \cap C_1, (A_2 \cap B_1) - C_1)$. Let $Z = (E - b) - (A_2 \cap B_1)$. Then $a \in B_2 \subseteq Z$. Since $A_1 - b \subseteq Z$, we deduce that $b \in \mathrm{cl}(Z)$. Hence $\{a, b\} \subseteq \mathrm{cl}(Z)$, so $c \in \mathrm{cl}(Z)$. By applying [7, Lemma 5.5(ii)] to Φ'_b, we deduce that $c \in \mathrm{cl}(B_2)$ or $c \in \mathrm{cl}(A_1 \cap B_1 \cap C_1)$. But $a \in B_2$. If $c \in \mathrm{cl}(B_2)$, then $b \in \mathrm{cl}(B_2)$; a contradiction. Hence $c \in \mathrm{cl}(A_1 \cap B_1 \cap C_1)$, so $c \in \mathrm{cl}(C_1)$; a contradiction.

We conclude that (9.4.2) holds. By a symmetric argument, we can modify (C_1, C_2) again to get $A_2 \cap B_2 \cap C_2 = \emptyset$ while maintaining $A_1 \cap B_1 \cap C_1 = \emptyset$. □

The next lemma completes the treatment of case (E) by showing that when (E) and the two sets of symmetric conditions hold, $\{a, b, c\}$ is a standard wild triangle in M and $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ is an associated partition.

Lemma 9.5. *Assume that* $\lambda_{M\backslash a,b}(A_1 \cap B_2) = \lambda_{M\backslash b,c}(B_1 \cap C_2) = \lambda_{M\backslash c,a}(C_1 \cap A_2) = 2$ *and* $|A_2 \cap B_1|, |B_2 \cap C_1|, |C_2 \cap A_1| > 3$. *If* $A_1 \cap B_1 \cap C_1 = \emptyset = A_2 \cap B_2 \cap C_2$, *then*

(i) $(A_2 \cap B_1, B_2 \cap C_1, C_2 \cap A_1)$ *is a flower in* M;

(ii) $M\backslash a, b, c$ *is connected,* $\mathrm{co}(M\backslash a, b, c)$ *is* 3*-connected, and every* 2*-element cocircuit of* $M\backslash a, b, c$ *meets exactly two of* $A_2 \cap B_1, B_2 \cap C_1$, *and* $C_2 \cap A_1$; *and*

(iii) $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ *partitions the ground set of* $M\backslash a, b, c$ *and every union of consecutive sets is exactly 3-separating in* $M\backslash a, b, c$.

Proof. Since $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2$, by (5.0.11), $\lambda_M(A_2 \cap B_1) = 2$. By symmetry, $\lambda_M(B_2 \cap C_1) = \lambda_M(C_2 \cap A_1) = 2$. Hence $(A_2 \cap B_1, B_2 \cap C_1, C_2 \cap A_1)$ is a flower Ψ in M. Certainly $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ partitions $E(M\backslash a, b, c)$. Moreover, by (5.0.6) and (5.0.7), all six of the sets in the partition are exactly 3-separating in $M\backslash a, b, c$. The unions of the first and second sets and of the second and third sets are $B_2 \cap C_1$ and $A_1 \cap B_2$. Both of the last two sets are exactly 3-separating in $M\backslash a, b, c$ since $\lambda_M(B_2 \cap C_1) = 2 = \lambda_{M\backslash a,b,c}(B_2 \cap C_1)$ and $\lambda_{M\backslash a,b}(A_1 \cap B_2) = 2 = \lambda_{M\backslash a,b,c}(A_1 \cap B_2)$. By symmetry, we deduce that the union of every two consecutive sets in the distinguished partition is exactly 3-separating. The union of the first three sets is $B_2 - a$, which is exactly 3-separating in $M\backslash a, b, c$. Using symmetry and complements gives that every union of consecutive sets in the distinguished partition is exactly 3-separating.

9.5.1. $M\backslash a, b, c$ *is connected.*

Since each of $M\backslash a$, $M\backslash b$, and $M\backslash c$ is 3-connected, each of $M\backslash a, b$, $M\backslash b, c$ and $M\backslash c, a$ is connected. Suppose that (X, Y) is a 1-separation in $M\backslash a, b, c$ and consider the flower Ψ of M. If $X \supseteq (A_1 \cap C_2) - b$, then $(X \cup b, Y)$ is a 1-separation of $M\backslash a, c$; a contradiction. Thus, for every 1-separation (X, Y) of $M\backslash a, b, c$, each of X and Y meets every petal of Ψ. Now each such petal has at least four elements. Without loss of generality, $X \cap (A_1 \cap C_2)$ has at least two elements. Since this set is contained in $A_1 \cap C_2$, we have $\lambda_{M\backslash a,b,c}(X \cap A_1 \cap C_2) = \lambda_M(X \cap A_1 \cap C_2) \geq 2$. Now

$$\begin{aligned} 2 &= \lambda_{M\backslash a,b,c}(X) + \lambda_{M\backslash a,b,c}((A_1 \cap C_2) - b) \\ &\geq \lambda_{M\backslash a,b,c}(X \cap A_1 \cap C_2) + \lambda_{M\backslash a,b,c}(X \cup ((A_1 \cap C_2) - b)) \\ &\geq 2 + \lambda_{M\backslash a,b,c}(X \cup ((A_1 \cap C_2) - b)). \end{aligned}$$

Hence $\lambda_{M\backslash a,b,c}(X \cup ((A_1 \cap C_2) - b)) = 0$, so $(X \cup ((A_1 \cap C_2) - b), Y - ((A_1 \cap C_2) - b))$ is a 1-separation of $M\backslash a, b, c$ in which $Y - ((A_1 \cap C_2) - b)$ avoids some petal of Ψ; a contradiction. Thus (9.5.1) holds.

Now let (X, Y) be a 2-separation of $M\backslash a, b, c$. We show next that:

9.5.2. *If* $X \supseteq (A_1 \cap C_2) - b$, *then* $|Y \cap A_2 \cap B_1| = 1 = |Y \cap B_2 \cap C_1|$ *so* $|Y| = 2$.

Suppose that $X \supseteq (A_1 \cap C_2) - b$. Then $(X \cup b, Y)$ is a 2-separation of $M\backslash a, c$. As $M\backslash a$ and $M\backslash c$ are 3-connected, neither X nor Y contains $(B_2 \cap C_1) - a$ or $(A_2 \cap B_1) - c$. Now $\lambda_{M\backslash a,c}(X \cup b) = 1$ and $\lambda_{M\backslash a,c}((B_2 \cap C_1) - a) = 2$. Thus, by the submodularity of the connectivity function,

$$\begin{aligned} 3 &\geq \lambda_{M\backslash a,c}(X \cup b \cup ((B_2 \cap C_1) - a)) + \lambda_{M\backslash a,c}((X \cup b) \cap ((B_2 \cap C_1) - a)) \\ &= \lambda_{M\backslash c}(X \cup b \cup (B_2 \cap C_1)) + \lambda_{M\backslash a}(X \cap B_2 \cap C_1) \\ &= \lambda_{M\backslash c}(Y \cap A_2 \cap B_1) + \lambda_{M\backslash a}(X \cap B_2 \cap C_1). \end{aligned}$$

Since both $M\backslash c$ and $M\backslash a$ are 3-connected and both $Y \cap A_2 \cap B_1$ and $X \cap B_2 \cap C_1$ are nonempty, we deduce that

$$|Y \cap A_2 \cap B_1| = 1 \text{ or } |X \cap B_2 \cap C_1| = 1.$$

By symmetry,

$$|Y \cap B_2 \cap C_1| = 1 \text{ or } |X \cap A_2 \cap B_1| = 1.$$

Since both $|(A_2 \cap B_1) \cap (X \cup Y)|$ and $|(B_2 \cap C_1) \cap (X \cup Y)|$ exceed two, we deduce that either $|Y \cap A_2 \cap B_1| = 1 = |Y \cap B_2 \cap C_1|$, or $|X \cap A_2 \cap B_1| = 1 = |X \cap B_2 \cap C_1|$. In the first case, the required result holds so assume that the second case occurs letting x be the unique element of $B_2 \cap C_1 \cap X$. We have, by submodularity, that

$$\begin{aligned}
3 &= \lambda_{M\backslash a,b,c}(X) + \lambda_{M\backslash a,b,c}(E - \{a,b,c\} - (B_2 \cap C_1)) \\
&\geq \lambda_{M\backslash a,b,c}(X - x) + \lambda_{M\backslash a,b,c}((E - \{a,b,c\} - (B_2 \cap C_1)) \cup x) \\
&= \lambda_{M\backslash a,b,c}(X - x) + \lambda_{M\backslash a}((B_2 \cap C_1) - x - a) \\
&\geq \lambda_{M\backslash a,b,c}(X - x) + 2,
\end{aligned}$$

where the last inequality holds because $|(B_2 \cap C_1) - x - a| \geq 2$. Hence $\lambda_{M\backslash a,b,c}(X - x) \leq 1$. But $X - x$ spans b and $Y \cup x$ spans a. Therefore $((X - x) \cup b, Y \cup x \cup a)$ is a 1-separation of $M\backslash c$. This contradiction completes the proof of (9.5.2).

Next we establish the following:

9.5.3. *If $|X| \geq |Y|$, then either*

(i) *$|Y| = 2$ and Y meets exactly two of $A_1 \cap C_2, B_1 \cap A_2$, and $C_1 \cap B_2$; or*

(ii) *$|Y| = 3$ and Y meets each of $A_1 \cap C_2, B_1 \cap A_2$, and $C_1 \cap B_2$.*

If $|Y| = 2$ and $Y \subseteq A_1 \cap C_2$, then $(X \cup \{a,c\}, Y)$ is a 2-separation of $M\backslash b$; a contradiction. It follows by symmetry that if $|Y| = 2$, then (i) holds. Now suppose that $|Y| \geq 3$ but (ii) does not hold. Then, by (9.5.2), both X and Y meet each of $A_1 \cap C_2, B_1 \cap A_2$, and $C_1 \cap B_2$.

Since (ii) fails, we may assume, by symmetry, that $|X \cap B_2 \cap C_1| \geq 2$. Now $\lambda_{M\backslash a,b,c}(X) = 1$ and $\lambda_{M\backslash a,b,c}((B_2 \cap C_1) - a) = 2$. Thus, by Lemma 2.2, $\lambda_{M\backslash a,b,c}(X \cap B_2 \cap C_1) = 1$, or $\lambda_{M\backslash a,b,c}(X \cup ((B_2 \cap C_1) - a)) = 1$. In each case, we have a new 2-separation of $M\backslash a,b,c$ to which we can apply one of the symmetric versions of (9.5.2). In the first case, because $|E - \{a,b,c\} - (X \cap B_2 \cap C_1)| \geq 3$, we get an immediate contradiction to (9.5.2). In the second case, (9.5.2) implies that Y contains exactly one element of each of $A_2 \cap B_1$ and $C_2 \cap A_1$. Hence $|X \cap A_2 \cap B_1| \geq 2$ so, by a symmetric argument to that just given, we get that Y contains exactly one element of each of $B_2 \cap C_1$ and $C_2 \cap A_1$. Thus (ii) holds; a contradiction. Hence (9.5.3) holds.

By (9.5.3), to complete the proof of the lemma, we need to show that if $|Y| = 3$, then Y is a series class of $M\backslash a,b,c$. Let $Y = \{y_1, y_2, y_3\}$ and suppose that Y is not a series class of $M\backslash a,b,c$. Since $r_{M\backslash a,b,c}(Y) + r^*_{M\backslash a,b,c}(Y) - |Y| =$

1, it follows that Y is a triangle of M. Moreover, Y contains a unique cocircuit of $M\backslash a, b, c$. Without loss of generality, we may assume that this cocircuit is either Y or $\{y_1, y_2\}$. We may also assume that $y_1 \in C_1 \cap B_2$, $y_2 \in A_1 \cap C_2$, and $y_3 \in B_1 \cap A_2$. We now think in terms of the familiar Venn diagram involving A_1, A_2, B_1, and B_2. Without loss of generality, $y_3 \in C_2$.

We show next that

9.5.4. $y_1 \in A_1 \cap B_2$.

We have $\lambda_{M\backslash a,b,c}(Y) = 1$ and $\lambda_{M\backslash a,b,c}(A_2 - c) = 1$. Thus, by Lemma 2.2, either $\lambda_{M\backslash a,b,c}(Y \cap (A_2 - c)) = 1$, or $\lambda_{M\backslash a,b,c}(Y \cup (A_2 - c)) = 1$. In the first case, since $Y \cap (A_2 - c)$ is y_3 or $\{y_1, y_3\}$, we deduce that $y_1 \in A_1 \cap B_2$ or $\{y_1, y_3\}$ is a cocircuit of $M\backslash a, b, c$. But the last possibility does not arise so, in the first case, $y_1 \in A_1 \cap B_2$. In the second case, $\lambda_{M\backslash a,b,c}(A_1 - b - Y) = 1$. Since $Y \cup (A_2 - c) \supseteq (A_2 \cap B_1) - c$, by (9.5.2) and symmetry, $|A_1 - b - Y| = 2$. But $|A_1 \cap B_2|, |A_1 \cap B_1| \geq 2$. Hence, again, $y_1 \in A_1 \cap B_2$. Thus (9.5.4) holds.

As y_1 and y_2 are in A_1, we can move y_3 from $A_2 \cap B_1$ into $A_1 \cap B_1$ maintaining the fact that $A_1 \cap B_1 \cap C_1 = \emptyset$ but changing $|A_2 \cap B_1 \cap Y|$ to 0. This gives a contradiction to (9.5.3) provided we are still in case (E), that is, provided this move maintains the fact that $|A_2 \cap B_1| = 4$. Thus suppose that, before the move, $|(A_2 \cap B_1) - y_3| = 3$. Then, by Lemma 7.5, $(A_2 \cap B_1) - y_3$ is a triangle. Since $\lambda_M(A_1 \cup B_2) = \lambda_M(A_1 \cup B_2 \cup y_3)$ and $y_3 \in \mathrm{cl}(A_1)$, we deduce that $y_3 \in \mathrm{cl}((A_2 \cap B_1) - y_3)$. Therefore $A_2 \cap B_1$ is a 4-point line containing c. Since C_1 or C_2 must contain at least two elements of this line, C_1 or C_2 spans c; a contradiction. □

We are now ready to complete the proofs of Theorems 3.1 and 1.1 and of Corollary 3.3. It should be noted here that the proof that we give for the first of these results also proves the variant of that result in which outcome (i) is replaced by the outcome that $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident in M. The reason for our interest in this alternative statement is that it indicates what can be said about the structure around the wild triangle $\{a, b, c\}$ when we begin with a certain collection of 3-separations exposed by a, b, and c and only allow ourselves to move to equivalent 3-separations.

Proof of Theorem 3.1. By Corollary 4.3, $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected. In Corollary 7.8, we showed that if one of cases (A)-(D) arises, then $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident in M. Corollary 9.1 gives that when case (E) occurs, either $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident, or we have symmetry between $(A_1, A_2), (B_1, B_2)$, and (C_1, C_2). In the latter case, Lemma 9.5 gives that outcome (ii) of the theorem holds. If case (F) occurs and $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not a pretrident, then, by Lemma 8.4, case (E) and the symmetric conditions hold in M'. Hence outcome (iii) of the theorem arises.

Next, we note that, from Lemma 7.9, if $(a, b, c, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is a pretrident, then $\{a, b, c\}$ is in a trident in M, so outcome (i) of the theorem occurs.

We now know that $\{a, b, c\}$ is in a trident, $\{a, b, c\}$ is a standard wild triangle, or $\{a, b, c\}$ is a costandard wild triangle. By combining Lemmas 8.5, 8.6, and 8.7, we obtain that these three possibilities are mutually exclusive. We conclude that exactly one of outcomes (i)-(iii) of the theorem occurs. □

Proof of Theorem 1.1. This theorem follows immediately by combining Corollary 4.3 with Theorem 3.1. □

Proof of Corollary 3.3. By Corollary 4.3, $\{a, b, c\}$ is an internal triangle of a fan if and only if $M\backslash a$ is not 3-connected. If $M\backslash a$ is 3-connected, then the result follows by combining Theorem 3.1 with Lemmas 8.5, 8.6, and 8.7. □

10. Finishing Off

In this section we prove Corollary 3.2 relying heavily on [7]. We begin with four lemmas on flowers, omitting the straightforward proof of the first.

Lemma 10.1. *Let $(P_1, P_2, \dots, P_n)$ be a flower Φ in a 3-connected matroid M, where $n \geq 3$, and let x be an element of P_1. Let M' be obtained by adding a nonempty set X of elements in parallel to x. Assume that there is a partition $(X_2, X_3, \dots, X_n)$ of X, where some subsets may be empty, such that each union of consecutive pairs in the partition $(P_1, P_2 \cup X_2, \dots, P_n \cup X_n)$ of $E(M')$ is 3-separating. Then the following hold.*

(i) *If $X_i \neq \emptyset$, then $x \in \mathrm{cl}(P_i)$ for all i in $[n]$.*
(ii) *Φ is not a copaddle.*
(iii) *If Φ is swirl-like, then, up to relabelling, $X = X_2$ and x is a loose element in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2)$.*
(iv) *If Φ is spike-like, then x is the unique element of M that is in $\mathrm{cl}(P_i)$ for all i in $[n]$.*
(v) *If Φ is a paddle, then $x \in \mathrm{cl}(P_i)$ for all i in $[n]$.*

Lemma 10.2. *Let Φ be a tight flower (P_1, P_2, P_3, P_4) of a 3-connected matroid M. Then there is a tight flower (Q_1, Q_2, Q_3, Q_4) equivalent to Φ such that $Q_1 \cup Q_2$, Q_2, and $Q_2 \cup Q_3$ are fully closed.*

Proof. Begin by considering the flower $\Phi' = (P_1', P_2', P_3', P_4') = (P_1 - \mathrm{fcl}(P_2), \mathrm{fcl}(P_2), P_3 - \mathrm{fcl}(P_2), P_4 - \mathrm{fcl}(P_2))$. By [7, Corollary 5.12 and Theorem 6.5], Φ' is a tight flower equivalent to and therefore of the same type as Φ. By [7, Lemma 5.9], $\mathrm{fcl}(P_1' \cup P_2') - (P_1' \cup P_2') \subseteq (\mathrm{fcl}(P_1') - P_1') \cup (\mathrm{fcl}(P_2') - P_2')$. But P_2' is fully closed, so $\mathrm{fcl}(P_1' \cup P_2') - (P_1' \cup P_2') \subseteq (\mathrm{fcl}(P_1') - P_1')$. Using this fact and symmetry, we deduce by [7, Theorems 6.1 and 7.1] that if Φ' is Vámos-like or is an anemone, all loose elements of Φ' are contained in P_2', so Φ' is the required flower (Q_1, Q_2, Q_3, Q_4). If Φ' is swirl-like, then, by [7, Theorem 7.4], the elements in $\mathrm{fcl}(P_1') - (P_1' \cup P_2')$ and $\mathrm{fcl}(P_3') - (P_3' \cup P_2')$ form disjoint subsets of P_4'. By moving these subsets of P_4' into P_1' and P_3', respectively, we obtain the required flower (Q_1, Q_2, Q_3, Q_4). □

Lemma 10.3. *Let Φ be a tight flower (P_1, P_2, P_3, P_4) of a 3-connected matroid M and let Ψ be a partition (Q_1, Q_2, Q_3, Q_4) of $E(M)$ where $Q_1 \cup Q_2$ and $Q_2 \cup Q_3$ are 3-separating sets equivalent to $P_1 \cup P_2$ and $P_2 \cup P_3$, respectively. Then Ψ is a tight flower equivalent to Φ.*

Proof. By Lemma 10.2, there is a tight flower (P'_1, P'_2, P'_3, P'_4) equivalent to Φ such that $P'_1 \cup P'_2$ and $P'_2 \cup P'_3$ are fully closed. Thus

$$\begin{aligned} Q_1 \cup Q_2 \cup Q_3 &\subseteq \mathrm{fcl}(Q_1 \cup Q_2) \cup \mathrm{fcl}(Q_2 \cup Q_3) \\ &= \mathrm{fcl}(P_1 \cup P_2) \cup \mathrm{fcl}(P_2 \cup P_3) \\ &= \mathrm{fcl}(P'_1 \cup P'_2) \cup \mathrm{fcl}(P'_2 \cup P'_3) \\ &= P'_1 \cup P'_2 \cup P'_3 \end{aligned}$$

Since $|P'_4| \geq 2$, we see that $|Q_4| \geq 2$. Now $Q_3 \cup Q_4$ and $Q_4 \cup Q_1$, as the complements of $Q_1 \cup Q_2$ and $Q_2 \cup Q_3$, are also 3-separating and are equivalent to $P_3 \cup P_4$ and $P_4 \cup P_1$. Hence, by a symmetric argument to the above, we deduce that $|Q_2| \geq 2$ and, similarly, $|Q_1|, |Q_3| \geq 2$. It now follows by uncrossing that Ψ is a flower.

By Lemma 10.2 again, there is a flower (Q'_1, Q'_2, Q'_3, Q'_4) equivalent to Ψ such that $Q'_1 \cup Q'_2$ and $Q'_2 \cup Q'_3$ are fully closed. This means that $Q'_1 \cup Q'_2 = P'_1 \cup P'_2$ and $Q'_2 \cup Q'_3 = P'_2 \cup P'_3$. Hence $(Q'_1, Q'_2, Q'_3, Q'_4) = (P'_1, P'_2, P'_3, P'_4)$. Thus Ψ is equivalent to Φ. □

The last lemma proves the base case of the following more general result.

Lemma 10.4. *Let Φ be a tight flower $(P_1, P_2, \ldots, P_k)$ in a 3-connected matroid M, let Ψ be a partition of $(Q_1, Q_2, \ldots, Q_k)$ of $E(M)$, and let t be an integer with $2 \leq t \leq k-2$. Assume that, for all j in $[k]$, the set $Q_{j+1} \cup Q_{j+2} \cup \cdots \cup Q_{j+t}$ is an exactly 3-separating set equivalent to $P_{j+1} \cup P_{j+2} \cup \cdots \cup P_{j+t}$. Then Ψ is a flower equivalent to Φ.*

Proof. Let $C(t)$ be the specified condition on the sets $Q_{j+1} \cup Q_{j+2} \cup \cdots \cup Q_{j+t}$. Evidently, if $C(s)$ holds, then so does $C(k-s)$. Now suppose that $C(s)$ holds for some s with $2 < s \leq k-2$. By [7, Lemma 5.9], the flower $(P_1, P_2 \cup \cdots \cup P_s, P_{s+1}, P_{s+2} \cup \cdots \cup P_k)$ is tight. By applying Lemma 10.3 to this flower and the partition $(Q_1, Q_2 \cup \cdots \cup Q_s, Q_{s+1}, Q_{s+2} \cup \cdots \cup Q_k)$, we deduce that $Q_2 \cup \cdots \cup Q_s$ is an exactly 3-separating set equivalent to $P_2 \cup \cdots \cup P_s$. Hence, by symmetry, if $C(s)$ holds, then so does $C(s-1)$. Since $C(t)$ holds, we deduce that $C(s)$ holds for all s in $\{2, 3, \ldots, k-2\}$. Since $C(2)$ holds, Ψ is a flower.

Consider the flowers $\Phi, (P_1, P_2, P_3, P_4 \cup \cdots \cup P_k), (Q_1, Q_2, Q_3, Q_4 \cup \cdots \cup Q_k)$, and Ψ. By Lemma 10.3, the second and third are equivalent. Moreover, if one is an anemone, they all are. Assume Φ is a daisy. Then so is Ψ. As $C(s)$ holds for all s in $\{2, 3, \ldots, k-2\}$, it follows that Ψ is equivalent to Φ.

Finally, suppose Φ is an anemone. Then the tight flower $(P_1, P_3, P_2, P_4, P_5, \ldots, P_k)$ and the partition $(Q_1, Q_3, Q_2, Q_4, Q_5, \ldots, Q_k)$ obey $C(2)$ and so obey $C(s)$ for all s in $\{2, 3, \ldots, k-2\}$. As every permutation of $(1, 2, \ldots, k)$ can be obtained as a product of transpositions, it follows that the anemones Φ and Ψ are equivalent. □

Proof of Corollary 3.2. By the definition of an associated partition, $\mathbf{P}$ is associated with 3-separations (A'_1, A'_2), (B'_1, B'_2), and (C'_1, C'_2) exposed by a, b, and c. By Theorem 3.1, there are equivalent exposed 3-separations (A_1, A_2), (B_1, B_2), and (C_1, C_2) that satisfy (a)-(c) of part (ii) of that theorem. Recall that $N = \mathrm{co}(M\backslash a, b, c)$ and that $\mathbf{Q}$ is the partition $(Q_1, Q_2, \ldots, Q_6)$ of $E(N)$ induced by $\mathbf{P}$. Let $(R_1, R_2, \ldots, R_6)$ be the partition $\mathbf{R}$ of $E(N)$ induced by $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$.

10.4.1. $\mathbf{R}$ *is a tight flower in* N.

From Theorem 3.1(ii)(c), all unions of consecutive sets of $(A_2 \cap B_2, C_1 \cap A_1, B_2 \cap C_2, A_1 \cap B_1, C_2 \cap A_2, B_1 \cap C_1)$ are 3-separating in $M\backslash a, b, c$. Hence all unions of consecutive sets of $\mathbf{R}$ are 3-separating in N as, by Lemma 2.3, the connectivity of a set cannot increase in a minor. To show that $\mathbf{R}$ is a tight flower in N, it suffices, by Lemma 2.10 and symmetry, to show that R_1 contains at least two elements that are not in the full closure of R_6. Recall that $R_1 = (A_2 \cap B_2) \cap E(N)$.

In $M\backslash a$, as $A_2 \cap B_2$, $A_1 \cap C_1$, and their union are 3-separating, $(A_2 \cap B_2, A_1 \cap C_1, B_1 \cup C_2)$ is a flower. We show next that this flower is tight. If not, then, by Lemma 2.10 and symmetry, either $B_1 \cup C_2 \subseteq \mathrm{fcl}_{M\backslash a}(A_2 \cap B_2)$, or $A_2 \cap B_2 \subseteq \mathrm{fcl}_{M\backslash a}(A_1 \cap C_1)$. In the first case, $A_2 \cap B_1 \subseteq \mathrm{fcl}_{M\backslash a}(A_2 \cap B_2)$, so the elements of $A_2 \cap B_1$ can be moved from B_1 into B_2 to make $A_2 \cap B_1$ empty, contradicting (5.0.9). In the second case, we can move the elements of $A_2 \cap B_2$ into A_1 to reduce $A_2 \cap B_2$ to an empty set, contradicting (5.0.5). Thus $(A_2 \cap B_2, A_1 \cap C_1, B_1 \cup C_2)$ is, indeed, a tight flower in $M\backslash a$.

Since $A_2 \cap B_2$ contains at least two tight elements of this flower, to complete the proof of (10.4.1), it suffices to show that if x is such an element, then $x \notin \mathrm{fcl}_N(R_6)$. Evidently, $x \notin \mathrm{fcl}_{M\backslash a}(B_1 \cup C_2)$. Since $c \in \mathrm{cl}(B_1 - c)$ and $b \in \mathrm{cl}(C_2 - b)$, we deduce that $x \notin \mathrm{fcl}_{M\backslash a,b,c}((B_1 \cup C_2) - \{b, c\})$. By Theorem 3.1(ii)(b), $\mathrm{fcl}_{M\backslash a,b,c}((B_1 \cup C_2) - \{b, c\})$ contains all non-trivial series classes of $M\backslash a, b, c$, so $\mathrm{fcl}_{M\backslash a,b,c}((B_1 \cup C_2) - \{b, c\})$ contains all the elements of $M\backslash a, b, c$ that are removed to obtain N. Hence $x \in E(N)$. Moreover, as $R_3 \cup R_4 \cup R_5 \cup R_6 = [(B_1 \cup C_2) - \{b, c\}] \cap E(N)$, it follows that $x \notin \mathrm{fcl}_N(R_3 \cup R_4 \cup \cdots \cup R_6)$. In particular, $x \notin \mathrm{fcl}_N(R_6)$ and 10.4.1 follows.

Next we consider the type of $\mathbf{R}$. First suppose $N = M\backslash a, b, c$. If $\mathbf{R}$ is a paddle, then so is $(A_2 \cap B_2, (C_1 \cap A_1) \cup (B_2 \cap C_2), A_1 \cap B_1, (C_2 \cap A_2) \cup (B_1 \cap C_1))$. Now $(C_2 \cap A_2) \cup (B_1 \cap C_1) = (A_2 \cap B_1) - c$, and c is in the closure of both this set and its complement in $E(M\backslash a, b, c)$. As $\mathbf{R}$ is a paddle, $c \in \mathrm{cl}(A_1 \cap B_1)$ so $\{b, c\} \subseteq \mathrm{cl}(A_1)$ and $a \in \mathrm{cl}(A_1)$; a contradiction. Thus $\mathbf{R}$ is not a paddle, so $\mathbf{R}$ is swirl-like, spike-like, or a copaddle, and (i)–(iii) hold for $\mathbf{R}$.

Now suppose that $N \neq M\backslash a, b, c$. Then, by combining Theorem 3.1(ii)(b) with the dual of Lemma 10.1, it is straightforward to show that $\mathbf{R}$ is not a paddle and that it satisfies (i)–(iii) of the corollary.

10.4.2. *The partition* $(Q_2 \cup Q_3 \cup Q_4, Q_5 \cup Q_6 \cup Q_1)$ *of* $E(N)$ *induced by* (A'_1, A'_2) *is a* 3*-separation equivalent to* $(R_2 \cup R_3 \cup R_4, R_5 \cup R_6 \cup R_1)$.

To see this, consider the definition of equivalence. By that, there is a sequence $(S_1, S_2, \ldots, S_k)$ of sets in $M\backslash a$ with $A_1 = S_1$ and $S_k = A'_1$ such that, for all i in $\{2, 3, \ldots, k\}$, we have $\lambda_{M\backslash a}(S_i) = 2$ and $|S_i - S_{i-1}| = 1$. Each member of the corresponding sequence in N is certainly 3-separating, and, after ignoring possible equal members of this sequence, we obtain a sequence of 3-separating sets in N that guarantees the truth of 10.4.2.

Using 10.4.2 and complements, we deduce from Lemma 10.4 that $\mathbf{Q}$ is a tight flower equivalent to $\mathbf{R}$. Because $\mathbf{R}$ satisfies (i)–(iii) of the corollary, the assertions in the first sentences of (i)–(iii) hold for $\mathbf{Q}$. The assertions in the second sentences follow by applying the dual of Lemma 10.1 to $\mathbf{Q}$. $\square$

References

[1] Bixby, R. E., A simple theorem on 3-connectivity, *Linear Algebra Appl.* **45** (1982), 123–126.

[2] Cunningham, W. H. and Edmonds, J., A combinatorial decomposition theory, *Canad. J. Math.* **32** (1980), 734–765.

[3] Geelen, J. and Whittle, G., Matroid 4–connectivity: A deletion–contraction theorem, *J. Combin. Theory Ser. B* **83** (2001), 15–37.

[4] Geelen, J., Gerards, B., and Whittle, G., Inequivalent representations of matroids I-V, in preparation.

[5] Kahn, J., On the uniqueness of matroid representation over $GF(4)$, *Bull. London Math. Soc.* **20** (1988), 5–10.

[6] Oxley, J. G., *Matroid Theory*, Oxford University Press, New York, 1992.

[7] Oxley, J., Semple, C., and Whittle, G., The structure of the 3-separations of 3-connected matroids, *J. Combin. Theory Ser. B* **92** (2004), 257–293.

[8] Oxley, J., Semple, C., and Whittle, G., The structure of the 3-separations of 3-connected matroids II, *European J. Combin.* **28** (2007), 1239-1261.

[9] Seymour, P.D., Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.

[10] Tutte, W. T., Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.

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