ON 4–DIMENSIONAL MINKOWSKI PLANES WITH 7–DIMENSIONAL AUTOMORPHISM GROUP

by

GÜNTER F. STEINKE
Department of Mathematics, University of Canterbury,
Christchurch, New Zealand

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Abstract. This paper concerns 4–dimensional (topological locally compact connected) Minkowski planes that admit a 7–dimensional automorphism group Σ. It is shown that such a plane is either classical or has a distinguished point that is fixed by the connected component of Σ and that the derived affine plane at this point is a 4–dimensional translation plane with a 7–dimensional collineation group.

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ABSTRACT. This paper concerns 4-dimensional (topological locally compact connected) Minkowski planes that admit a 7-dimensional automorphism group Σ. It is shown that such a plane is either classical or has a distinguished point that is fixed by the connected component of Σ and that the derived affine plane at this point is a 4-dimensional translation plane with a 7-dimensional collineation group.

A Minkowski plane $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ consists of a set of points P, a set of at least two circles $\mathcal{K}$ (considered as subsets of P) and two equivalence relations $\|_+$ and $\|_-$ on P (parallelisms) such that three pairwise non-parallel points (that is, neither (+)-parallel nor (−)-parallel) can be joined by a unique circle, such that the circles which touch a fixed circle K at $p \in K$ partition $P \setminus |p|$ (here $|p| = |p|_+ \cup |p|_-$ denotes the union of the two parallel classes of p), such that each parallel class meets each circle in a unique point (parallel projection), such that each (+)-parallel class and each (−)-parallel class intersect in a unique point, and such that there is a circle that contains at least three points (compare [15]). A topological Minkowski plane is a Minkowski plane in which the point set P and the set of circles $\mathcal{K}$ carry topologies such that the geometric operations of joining, touching, the parallel projections, intersecting parallel classes of different type, and intersecting circles are continuous operations on their domains of definition (see [15]). A topological Minkowski plane is called (locally) compact, connected, or finite-dimensional if the point space has the respective topological property. For brevity, a locally compact connected finite-dimensional topological Minkowski plane will be called a finite-dimensional Minkowski plane. According to [9, 2.3] a finite-dimensional Minkowski plane can only be of dimension 2 or 4. In these cases the automorphism group of $\mathcal{M}$ is a Lie group with respect to the compact-open topology of dimension at most 6 and 12, respectively, see [16]. The classical model of a 2- or 4-dimensional Minkowski

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plane is obtained as the geometry of non-trivial plane sections of a ruled quadric in the real or complex projective 3-dimensional space respectively. In these cases the topologies on the point set and the set of circles are induced from the surrounding projective 3-space (the set of planes in the projective 3-space carries a natural topology which can be obtained by duality from the topology on the point set in the 3-space).

Whereas there are many models of non-classical 2-dimensional Minkowski planes (see [15], [8], and [17]) no non-classical 4-dimensional Minkowski planes are known yet. Furthermore, it was shown in [19] that a 4-dimensional Minkowski plane that admits an 8-dimensional automorphism group must be classical. Here we investigate 4-dimensional Minkowski planes that admit a 7-dimensional group of automorphism Σ. We prove the following

Theorem. *If a 4-dimensional Minkowski plane* $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ *admits a closed connected 7-dimensional group of automorphisms* Σ*, then* $\mathcal{M}$ *is classical or* Σ *fixes precisely one point* $p \in P$ *and the derived plane at* p *is a 4-dimensional translation plane with a 7-dimensional collineation group.*

1. Notation and preliminaries

We maintain the notation of [19]. In the remainder of this paper $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ always denotes a 4-dimensional Minkowski plane. The $(+)$- and $(-)$-parallel class of a point x is denoted by $|x|_+$ and $|x|_-$ respectively. Furthermore, let $\Pi^\pm$ be the collection of $(\pm)$-parallel classes. $\Pi^\pm$ endowed with the quotient topology becomes homeomorphic to the 2-sphere $\mathbb{S}^2$.

For every point p of $\mathcal{M}$, there is an associated incidence structure, called the derived affine plane $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ at p, whose point set A_p consists of all points of $\mathcal{M}$ that are not parallel to p and whose set of lines $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{M}$ passing through p and of all parallel classes not passing through p. Indeed, $\mathcal{M}$ is a Minkowski plane if and only if all incidence structures $\mathcal{A}_p$ are affine planes. It can easily be seen that derived affine planes of the classical Minkowski plane are even topological locally compact connected affine planes. More generally, it was shown in [18] that this is true for each finite-dimensional Minkowski plane. Furthermore, the projective extension $\mathcal{P}_p$ of $\mathcal{A}_p$ can be made to a topological locally compact connected projective plane.

According to [9, 2.5] there is a characterization of the classical complex Minkowski plane in terms of a single derivation. This is due to a remarkable result of T. Buchanan [5] which says that the only topological ovals in the Desarguesian projective complex plane are conics.

1.1 Theorem. *A 4-dimensional Minkowski plane is isomorphic to the classical complex Minkowski plane if and only if at least one derived affine plane is Desarguesian.*

The group $\Gamma = Aut(\mathcal{M})$ of all continuous automorphisms of $\mathcal{M}$ carries the compact-open topology. Γ is a Lie group of dimension at most 12 in this topology; cf. [16]. This upper bound follows from the *dimension formula* (cf. [6])

$$dim\ \Delta = dim\ \Delta_x + dim\ \Delta(x)$$

which relates the topological dimensions of the locally compact transformation group Δ operating on a manifold X, the stabilizer Δ_x of a point $x \in X$, and the orbit $\Delta(x)$ of this point and from the following

1.2 Rigidity Lemma. *The stabilizer $\Gamma_{x,y,z}$ of three pairwise non-parallel points x, y, z is 0-dimensional.*

Γ has two distinguished closed normal subgroups

$$T^{\pm} = \{\gamma \in \Gamma |\ x \parallel_{\pm} \gamma(x) \text{ for all } x \in P\}.$$

Obviously $T^+ \cap T^- = \{id\}$. Since an automorphism maps parallel points to parallel points, the connected component Γ^1 of the identity in Γ operates on $\Pi^{\pm}$. The normal subgroups $T^{\pm}$ are just the kernels of this action. We denote the canonical projection onto the factor goup $\Gamma/T^{\pm}$ by

$$\pi^{\pm} : \Gamma \to \Gamma/T^{\pm}.$$

The group $\pi^{\pm}(\Gamma^1)$ operates effectively on $\Pi^{\pm} \approx \mathbb{S}^2$. Since T^+ and T^- intersect trivially, $T^{\pm} \cong \pi^{\mp}(T^{\pm})$ can be identified with a normal subgroup of $\pi^{\mp}(\Gamma^1)$. We define $T^{\pm}_{\Sigma} = \Sigma \cap T^{\pm}$ and $\Sigma^{\pm} = \pi^{\pm}(\Sigma)$ for a closed connected subgroup Σ.

The classical Minkowski plane can be characterized by the size of these normal subgroups, see [19, 3.2].

1.3 Theorem. *A 4-dimensional Minkowski plane is isomorphic to the classical complex Minkowski plane if and only if one of the normal subgroups $T^{\pm}$ is at least 4-dimensional.*

The connected component $\Delta = (\Gamma_p)^1$ of the stabilizer of a point p operates in the derived affine plane $\mathcal{A}_p$ at p as a group of collineations of $\mathcal{A}_p$. The group Δ can even be considered as a group of collineations of $\mathcal{P}_p$. There, Δ fixes the infinite line L_∞ (with respect to $\mathcal{A}_p$); furthermore, Δ fixes the infinite points $\omega_+, \omega_- \in L_\infty$ of lines induced by (+)- and (−)-parallel classes respectively. The Rigidity Lemma 1.2 is an immediate consequence of this fact.

2. Proof of the theorem

In this section Σ denotes a 7-dimensional closed connected subgroup of the automorphism group $\Gamma = Aut(\mathcal{M})$ of a 4-dimensional Minkowski plane $\mathcal{M}$. Thus $dim\ \Sigma^{\pm} + dim\ T^{\pm}_{\Sigma} = 7$.

2.1 Proposition. *Σ fixes at least one parallel class. If Σ acts fixed-point-free, then $\mathcal{M}$ is classical.*

Proof. Since there are no semi-simple Lie groups of dimension 7, the group Σ must fix a parallel class according to [19, 2.12]. Let $|x|_+$ be such a fixed parallel class.

Suppose that Σ acts transitively on $|x|_+$. Then Σ^- acts transitively on Π^- and we have $\Sigma^- \cong T_\Sigma^+$ by [19, 5.2] (as Σ^+ is not transitive on Π^+). Thus $T_\Sigma^+ \cong PSL(2,\mathbb{C})$ or $T_\Sigma^+ \cong SO(3,\mathbb{R})$. In the former case T_Σ^+ is 6-dimensional and in the latter case T_Σ^- is 4-dimensional. Hence the Minkowski plane $\mathcal{M}$ is classical by Theorem 1.3.

Suppose now that Σ neither acts transitively on $|x|_+$ nor does it fix a point. Then Σ has an orbit B homeomorphic to the 1-sphere $\mathbb{S}^1$ (see [7]). We further may assume that $B = \Sigma(x)$ is the orbit of x. Let Φ denote the kernel of the action of Σ on B. Thus Σ/Φ acts transitively and effectively on $B \approx \mathbb{S}^1$. It follows from the classification of such groups (cf. [11, 3.18]) that $dim\ \Sigma/\Phi \leq 3$, i.e. $dim\ \Phi \geq 4$. Let $\mathcal{K}_x$ be the set of circles passing through x and let $K \in \mathcal{K}_x$. Since Φ_K fixes $K \cap |B|_-$ pointwise, we obtain $dim\ \Phi_K = 0$ by the Rigidity Lemma 1.2. This yields $dim\ \Phi \leq 4$, because $\mathcal{K}_x$ is 4-dimensional. Together with the previous inequality we find $dim\ \Phi = 4$. In particular, the connected component Ψ of the identity of Φ operates transitively on $\mathcal{K}_x$ as $\mathcal{K}_x$ is connected.

We choose $y, z \in B$ such that x, y, z are pairwise disjoint and we choose $a_i, b_i \in |y|_- \setminus \{y\}$ for $i = 1,2$ with $a_i \neq b_i$. Define $b_i' = |b_i|_+ \cap |z|_-$ so that x, a_i, b_i' are pairwise non-parallel. By the transitivity of Ψ on $\mathcal{K}_x$, there is an automorphism $\psi \in \Psi$ that maps the circle through x, a_1, b_1' onto the circle through x, a_2, b_2'. Then ψ maps a_1 to a_2 and b_1 to b_2. This shows that Ψ is 2-transitive on $|y|_- \setminus \{y\} \approx \mathbb{R}^2$; obviously, Ψ is effective on $|y|_- \setminus \{y\}$. According to the classification of such groups (compare [21, pp. 222–223] and [22]), Ψ is isomorphic and acts equivalently to $L_2(\mathbb{C}) = \mathbb{C} \rtimes \mathbb{C} \setminus \{0\} = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}$. Since Φ and T_Σ^+ intersect trivially, we can identify Φ with a normal subgroup of Σ^+. Thus Σ^+ too operates transitively and effectively on $\Pi^+ \setminus |x|_+ \approx \mathbb{R}^2$ and Σ^+ is isomorphic and acts equivalently to $L_2(\mathbb{C})$, $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$, or $GL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ in its standard action on $\mathbb{R}^2$; cf. [21, pp. 222–223] and [22].

Let $w \not\parallel_+ x$. Then $\Phi' = \Phi_{|w|_+}$ is a 2-dimensional normal subgroup of $\Sigma' = \Sigma^+_{|w|_+}$ and the latter stabilizer is isomorphic to $\mathbb{C} \setminus \{0\}$, $SL(2,\mathbb{R})$, or $GL^+(2,\mathbb{R})$. In the last two cases Σ' contains a subgroup Θ of codimension at most 1 that is isomorphic to $SL(2,\mathbb{R})$. Moreover, $\Phi' \cap \Theta$ is a normal subgroup of Θ of dimension at least 1. However, $\Theta \cong SL(2,\mathbb{R})$ is simple and thus $\Theta \subseteq \Phi'$ contrary to $dim\ \Phi' = 2$. Hence $\Sigma^+ \cong \Psi \cong L_2(C)$ and T_Σ^+ is 3-dimensional.

It follows from the Rigidity Lemma 1.2 that on one hand each non-trivial automorphism $\tau \in T_\Sigma^+$ has at most two fixed points in $|x|_+$. On the other hand, for every $u \in |x|_+$ there is a non-trivial $\tau \in T_\Sigma^+$ such that τ fixes u (because $dim\ T_\Sigma^+ = 3$). Since Φ and T_Σ^+ intersect trivially, τ commutes with Φ elementwise. Thus u is a universal fixed point of Φ. As u was arbitrary this shows that $\Phi = T_\Sigma^-$. Hence $\mathcal{M}$ is classical by Theorem 1.3. $\square$

2.2 Proposition. *Assume that Σ fixes a point p. Then p is the unique fixed point of Σ, the derived affine plane $\mathcal{A}_p$ at p is a 4-dimensional translation plane (with translation line being the infinite line), and $\mathcal{M}$ is classical or $\mathcal{A}_p$ has a 7-dimensional collineation group Δ that fixes two points on the infinite line (i.e. the connected component of Δ is induced by Σ).*

Proof. Let p be a fixed point of Σ and assume that Σ has a second fixed point q. We choose a point r parallel to q but not parallel to p. Then the stabilizer $\Sigma_{p,r}$ is at least 5-dimensional. This implies that the stabilizer $\Sigma_{p,r,s}$ where s is a third point parallel to neither p nor r is at least 1-dimensional contrary to the Rigidity Lemma 1.2. Hence p must be the only fixed point of Σ.

Let $\mathcal{P}$ be the projective extension of the derived affine plane $\mathcal{A}_p$ at p. The group Σ induces a 7-dimensional group of collineations of $\mathcal{P}$ that fixes the infinite line L_∞ and the two infinite points ω_+ and ω_- of lines induced by parallel classes. Hence, according to [3, Satz 3], $\mathcal{P}$ is a 4-dimensional translation plane with translation line being the infinite line L_∞.

Let Ψ denote the connected component of the identity of the full collineation group of $\mathcal{P}$. We assume that Ψ is at least 8-dimensional and show that the classical Minkowski plane results. Let $\mathcal{K}_p$ be the set of all circles passing through the fixed point p. If Σ does not act transitively on $\mathcal{K}_p$, then there is a circle $K \in \mathcal{K}_p$ whose stabilizer Σ_K is at least 4-dimensional and so the stabilizer $\Sigma_{K,q}$ of a point $q \in K \setminus \{p\}$ is at least 2-dimensional. This means for the projective plane $\mathcal{P}$ that there is a group of collineations of dimension at least two which fixes the affine point q and three lines passing through q (namely the lines induced by $|q|_+$, $|q|_-$, and K). According to [1, Lemma 6] the translation plane $\mathcal{P}$ is Desarguesian. Hence $\mathcal{M}$ is classical by Theorem 1.1.

We finally assume that Σ operates transitively on $\mathcal{K}_p$. Thus Σ operates transitively on $L_\infty \setminus \{\omega_+, \omega_-\}$ in $\mathcal{P}$. According to [1, Lemma 5] the translation plane $\mathcal{P}$ is Desarguesian or Ψ fixes L_∞. If $\mathit{dim}\ \Psi \geq 9$, then $\mathcal{P}$ is Desarguesian according to [12] and [13]. So let us assume that $\mathit{dim}\ \Psi = 8$ and that L_∞ is fixed under Ψ. In this case Σ is a subgroup of Ψ of codimension 1. Thus ω_+ and ω_- have orbits of dimension at most 1. Because Σ is transitive on $L_\infty \setminus \{\omega_+, \omega_-\}$ each such point has a 2-dimensional orbit. This implies that ω_+ and ω_- are fixed by Ψ. Therefore $\mathcal{P}$ is Desarguesian by [14, §1]. In both cases $\mathcal{M}$ is classical by Theorem 1.1.

If Ψ is 7-dimensional, then $\Psi = \Sigma$ and Ψ fixes ω_+ and ω_-. $\square$

2.3 Remark. If $\mathcal{M}$ is not classical, then the normal subgroups $T^\pm$ are at most 3-dimensional and $T = T^+ \times T^-$ is at most 6-dimensional by Theorem 1.3. We shall see below that in the situation of Proposition 2.2 the respective groups of a non-classical Minkowski plane attain those upper bounds.

2.4. 4-dimensional translation planes with a 7-dimensional collineation group that fixes two ponts on the translation line were classified in [2]. Those planes fall into four different families and each family in turn can be parametrized by at most three continuous or discrete parameters. To begin with, a 4-dimensional translation plane can be represented in the following form : affine points are the points of $\mathbb{R}^4$ and lines are of the form $\{(c,w) \mid w \in \mathbb{R}^2\}$ for $c \in \mathbb{R}^2$ or of the form $\{(z, Bz+t) \mid z \in \mathbb{R}^2\}$ where $t \in \mathbb{R}^2$ and B is a 2×2 matrix. The subspaces corresponding to these matrices B together with the vertical subspace $S = \{(0,w) \mid w \in \mathbb{R}^2\}$ form a collection $\mathcal{B}$ of 2-dimensional subspaces of $\mathbb{R}^4$ that partition $\mathbb{R}^4$ except 0 (spread). One can always assume that $W = \{(z,0) \mid z \in \mathbb{R}^2\}$ belongs to $\mathcal{B}$. Each map $(z,w) \mapsto (rz+a, rw+b)$ for $a,b \in \mathbb{R}^2$ and $r \in \mathbb{R}\setminus\{0\}$ is a collineation of this plane and the full collineation group is the semi-direct product of the translation group (the above maps with $r=1$) and the stabilizer of $(0,0)$.

In the following we list the connected component Δ of the stabilizer of $(0,0)$ and the partition $\mathcal{B}$ for each of the four families of translation planes as given in [2, Satz 1–4]. As noted in [4, pp. 189–191] one must have $p=0$ in the first two families in order to obtain topological planes. Furthermore, in the last two families one can assume $c=0$ up to isomorphism. (An isomorphism can be chosen of the form $x \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\psi & \sin\psi \\ & & -\sin\psi & \cos\psi \end{pmatrix}$ for suitable ψ; this matrix centralizes the corresponding group Δ, so that both planes have the same group of collineations Δ.) The two fixed points on the infinite line L_∞ are the infinite points of the vertical line S and of the horizontal line W respectively. Thus all matrices in Δ have block structure $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with some 2×2 matrices A, B. For brevity, we denote such a matrix built from A and B by $\delta_{A,B}$. Given a 2×2 matrix M representing a 2-dimensional subspace in $\mathcal{B}$, we use the notation $M^{\delta_{A,B}} = BMA^{-1}$ and $M^\Delta = \{M^\delta \mid \delta \in \Delta\}$. Moreover, we represent the 2-dimensional subspaces $\neq W, S$ in $\mathcal{B}$ by their describing matrices. The four families then are:

$$(1)\quad \Delta_w = \left\{ \begin{pmatrix} r\cos\phi & -r\sin\phi & & \\ r\sin\phi & r\cos\phi & & \\ & & rs & \\ & & & rs^w \end{pmatrix} \;\middle|\; r>0, s>0,\ 0 \le \phi < 2\pi \right\},$$

$$\mathcal{B}_{w,c} = \{W,S\} \cup \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}^{\Delta_w},$$

where $0 < w < 1$ and $0 \le c \le \frac{2\sqrt{w}}{1-w}$,

$$(2)\quad \Delta = \left\{ \begin{pmatrix} r\cos\phi & -r\sin\phi & & \\ r\sin\phi & r\cos\phi & & \\ & & re^t & \\ & & rte^t & re^t \end{pmatrix} \;\middle|\; r>0,\ t \in \mathbb{R},\ 0 \le \phi < 2\pi \right\},$$

$$\mathcal{B}_d = \{W,S\} \cup \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^{\Delta},$$

where $d \ge \frac{1}{2}$,

(3) $\Delta_{p,q} = \{\begin{pmatrix} ae^{ps}\cos s & ae^{ps}\sin s & & \\ -ae^{ps}\sin s & ae^{ps}\cos s & & \\ & & ae^{qt}\cos t & ae^{qt}\sin t \\ & & -ae^{qt}\sin t & ae^{qt}\cos t \end{pmatrix} \mid \begin{matrix} a > 0, \\ s, t \in \mathbb{R} \end{matrix}\},$

$\mathcal{B}_{p,q,d} = \{W, S\} \cup \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^{\Delta_{p,q}},$

where $-1 < d < 0$, $p = q \geq \frac{d+1}{2\sqrt{-d}}$, or

$d > 0$, $d \neq 1$, $q > 0$, $p = \frac{k-1}{k+1}q$, k a positive integer,

$(d^2-1)^2 + (q+p)^2 d(d-1)^2 - (q-p)^2 d(d+1)^2 \leq 0,$

(4) $\Delta_{m,n} = \{\begin{pmatrix} a\cos m\phi & a\sin m\phi & & \\ -a\sin m\phi & a\cos m\phi & & \\ & & b\cos n\phi & b\sin n\phi \\ & & -b\sin n\phi & b\cos n\phi \end{pmatrix} \mid \begin{matrix} a, b > 0, \\ 0 \leq \phi < 2\pi \end{matrix}\},$

$\mathcal{B}_{m,n,d} = \{W, S\} \cup \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^{\Delta_{m,n}},$

where m, n are integers, $0 < m \leq n$, such that

$$\begin{cases} m = n = 1, \ -1 < d < 0, \quad \text{or} \\ \frac{m}{n} \leq d \leq \frac{n}{m}, \ d \neq 1, \ n = m+1 \text{ or } n = m+2, \text{ and } m \text{ odd.} \end{cases}$$

In all four cases Δ is 3-dimensional and abelian. Moreover, if such a translation plane occurs as the derived affine plane of a Minkowski plane, we may assume that two points $(x, y, u, v), (x', y', u', v') \in \mathbb{R}^4$ are (+)- or (−)-parallel if and only if $x = x'$, $y = y'$ or $u = u'$, $v = v'$ respectively and that the point of derivation has the coordinates (∞, ∞).

$T^{\pm}$ comprises all translations in (±)-direction and thus $T^{\pm}$ is at least 2-dimensional. From the list above one further sees that $r = 1$, $\phi = 0$ in families (1) and (2), $a = 1$, $s = 0$ in family (3), and $a = 1$, $\phi = 0$ in family (4) yields an additional 1-dimensional subgroup of T^+. Similarly, $r = 1$, $s = 1$ in family (1), $r = 1$, $t = 0$ in family (2), $a = 1$, $t = 0$ in family (3), and $b = 1$, $\phi = 0$ in family (4) yields an additional 1-dimensional subgroup of T^-. Thus $T^{\pm}$ is 3-dimensional and $T = T^+ \times T^-$ is 6-dimensional.

Using this classification of 4-dimensional translation planes and coherence properties of 4-dimensional Minkowski planes it is, in thoery, possible to determine all 4-dimensional Minkowski plane with a 7-dimensional automorphism group, but it is difficult to verify the axioms of a Minkowski plane for the resulting incidence structures. The verification of the continuity of the geometric operations is facilitated by [20]; for Minkowski planes in standard representation it suffices to prove that joining is continuous.

Circles with a 1-dimensional compact subgroup in its stabilizer can only occur in one particular situation.

2.5 Proposition. *Assume that $\mathcal{M}$ is non-classical and that there is a circle K not passing through the unique fixed point p of Σ such that its stabilizer Σ_K in Σ contains a 1-dimensional compact subgroup. Then this subgroup fixes each circle* *through $K \cap |p|_+$ and $K \cap |p|_-$. Every stabilizer of a circle not passing through p contains a 1-dimensional compact subgroup. Moreover, the derived affine plane $\mathcal{A}_p$ at p is one of the planes described in* [2, Satz 4] *(i.e. family (4) above) with parameters $c = 0$ and $m = n = 1$. In particular, this is true, if Σ_K is 2-dimensional.*

If no such circle exists, then Σ operates transitively on the set of circles not passing through p and the connected component of each stabilizer of a circle is isomorphic to $\mathbb{R}$.

Proof. Let p be the unique fixed point of the 7-dimensional group of automorphisms Σ. We use the description of the derived affine plane $\mathcal{A}$ at p as given in 2.4.

Let K be a circle not passing through p. The stabilizer Σ_K of K fixes the points $p_+ = K \cap |p|_+$ and $p_- = K \cap |p|_-$ and also $q = |p_+|_- \cap |p_-|_+$. By the Rigidity Lemma 1.2 we have $dim\ \Sigma_{K,z} \leq dim\ \Sigma_{p_+,p_-,z} = 0$ for each point $z \in K \setminus \{p_+, p_-\}$. Thus $dim\ \Sigma_K \leq 2$, because K is 2-dimensional. Since the pencil of circles through p_+ and p_- is 2-dimensional and as $dim\ \Sigma_{p_+,p_-} = dim\ \Sigma_q = 3$, the stabilizer Σ_K is at least 1-dimensional. Furthermore, $\Sigma_K \cap T^{\pm} = \{id\}$.

Let Φ be the connected component of Σ_K. We may assume that q is the point $(0,0,0,0) \in \mathbb{R}^4$; thus Φ is a subgroup of one of the groups Δ listed in 2.4. From the identifications made above it follows that $\Delta \cap T^+$ and $\Delta \cap T^-$ consists of all those matrices $\delta_{A,B} \in \Delta$ with $A = I$, the 2×2 identity matrix, and $B = I$ respectively.

Suppose that Σ_K contains a 1-dimensional compact subgroup, that is, the connected component Θ of this subgroup is isomorphic to $SO(2,\mathbb{R})$. Then $\Theta^- \cong \Theta \cong SO(2,\mathbb{R})$ can be identified with a subgroup of $\{B|\ \delta_{A,B} \in \Delta\}$. This excludes the families (1) and (2). In order to find Θ for the families (3) and (4) note that the simply connected covering group of Δ is $\mathbb{R}^3$ and that a 1-dimensional subgroup of $\mathbb{R}^3$ can be represented as a line through the origin. Θ is then found by projecting onto Δ where the parameters of the line have to be chosen such that this image becomes compact. In this way it is easy to see that for the family (3) the subgroup Θ must be

$$\Theta = \{ \begin{pmatrix} \cos\tilde{q}\phi & \sin\tilde{q}\phi & & \\ -\sin\tilde{q}\phi & \cos\tilde{q}\phi & & \\ & & \cos\tilde{p}\phi & \sin\tilde{p}\phi \\ & & -\sin\tilde{p}\phi & \cos\tilde{p}\phi \end{pmatrix} \mid \phi \in \mathbb{R}\}$$

where we write $\tilde{p}$ and $\tilde{q}$ for the parameter p and q of the plane to distinguish them form the points p and q as defined above. In order that this group is compact one must have $\tilde{p} = 0$ or $\tilde{p} = \tilde{q} > 0$. In the former case however, Θ is contained in T^- contrary to $\Sigma_K \cap T^- = \{id\}$. We now assume that $\tilde{p} = \tilde{q}$. Let

$$S = \{ \begin{pmatrix} e^{\tilde{p}s}\cos s & e^{\tilde{p}s}\sin s & & \\ -e^{\tilde{p}s}\sin s & e^{\tilde{p}s}\cos s & & \\ & & e^{\tilde{q}t}\cos t & e^{\tilde{q}t}\sin t \\ & & -e^{\tilde{q}t}\sin t & e^{\tilde{q}t}\cos t \end{pmatrix} \mid s,t \in \mathbb{R}\}$$

i.e. $S = \Delta \cap (T^+ \times T^-)$; then $S \cap \Theta = \{id\}$ and we can identify Θ with a subgroup of $\Delta/S \cong \mathbb{R}$ – a contradiction. In family (4) the subgroup Θ must be

$$\Theta = \{ \begin{pmatrix} \cos m\phi & \sin m\phi & & \\ -\sin m\phi & \cos m\phi & & \\ & & \cos n\phi & \sin n\phi \\ & & -\sin n\phi & \cos n\phi \end{pmatrix} \mid 0 \leq \phi < 2\pi \}.$$

If $1 \leq m < n$, then $\phi = \frac{2\pi}{n} < 2\pi$; for this ϕ one finds that the corresponding automorphism is in $\Sigma_K \cap T^- = \{id\}$. Thus $\cos m\phi = 1$ and $\sin m\phi = 0$ and $\frac{m}{n}$ must be an integer contrary to $0 < \frac{m}{n} < 1$. This shows that $m = n = 1$ in family (4).

Θ also operates on the set of circles through p_+ and p_-. Extending this set by $|q|$ and $|p|$ one obtains a larger set which is homeomorphic to $\mathbb{S}^2$ by the coherence axioms in a 4-dimensional Minkowski plane (for the definition of coherence see [15, 2.1]). Furthermore, the action of Θ extends to $\mathbb{S}^2$ where $\Theta \cong SO(2,\mathbb{R})$ possesses at least three fixed elements (namely $|q|$, $|p|$, and K). Hence, the group Θ must act trivially on $\mathbb{S}^2$ according to [10, VI.6.7.1]. This proves that each stabilizer of a circle not passing through p contains a 1-dimensional compact subgroup because, by the transitivity of Σ on affine points, such a stabilizer is conjugate to a stabilizer of a circle passing through p_+ and p_-.

We now suppose that $dim\ \Sigma_K = 2$; thus Φ is 2-dimensional too. Since $dim\ \Phi_z \leq dim\ \Sigma_{p_+,p_-,z} = 0$ for each point $z \in K \setminus \{p_+, p_-\}$ and because Φ is connected, we deduce that Φ acts transitively and effectively on $K \setminus \{p_+, p_-\}$. As $\Phi \leq \Delta$ is abelian, Φ and $K \setminus \{p_+, p_-\}$ are homeomorphic. But a 2-dimensional abelian connected group is isomorphic to $\mathbb{R}^2$, $\mathbb{R} \times SO(2,\mathbb{R})$ or $SO(2,\mathbb{R}) \times SO(2,\mathbb{R})$. In the first and third case however, Φ cannot be homeomorphic to $K \setminus \{p_+, p_-\} \approx \mathbb{R} \times SO(2,\mathbb{R})$. This implies that $\Phi \cong \mathbb{R} \times SO(2,\mathbb{R})$. Hence Φ contains a 1-dimensional compact subgroup.

If no stabilizer contains a 1-dimensional compact subgroup, $dim\ \Sigma_L = 1$ for all circles $L \in \mathcal{K} \setminus \mathcal{K}_p$ not passing through p. Hence, each such circle has a 6-dimensional orbit. As the set of all these circles is connected and 6-dimensional, Σ operates transitively on $\mathcal{K} \setminus \mathcal{K}_p$. The connected component of Σ_L must obviously be isomorphic to $\mathbb{R}$. $\square$

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