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CHRISTCHURCH NEW ZEALAND

Summer Research Project

Picture-Proofs and Greek Construction

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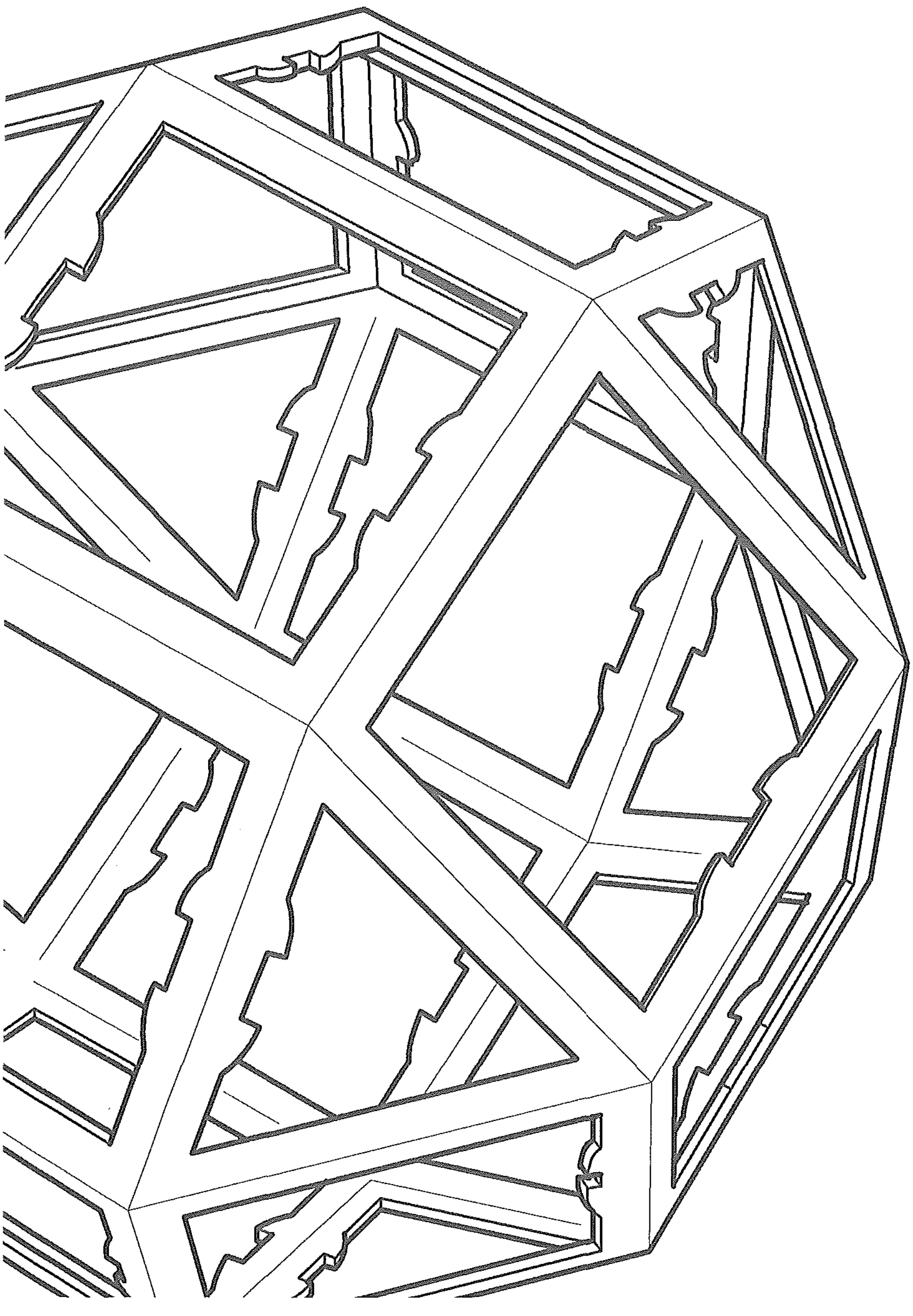

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Picture-Proofs and Greek Construction

0. Introduction

Justification in mathematics is a very particular thing; no other field has such high standards placed on the notion of 'proof'. There is justification and there is evidence. That non-deductive methods have long been seen as belonging to the latter category is uncontroversial. One contrarian to this view is Brown and his belief in a kind of non-deductive justification he calls 'picture-proofs'. Brown's most recent and thorough treatment of picture-proofs is found in *Philosophy of Mathematics: A Contemporary Introduction to the World of Proofs and Pictures* published in 2008; the discussions contained therein will form the basis for this paper. This belief in picture-proofs will be placed in the context of the highly visual culture of Greek mathematics and through this, conclusions will be drawn about the validity and usefulness of these picture-proofs. Sections 1-3 will be devoted to the discussion of picture-proofs and Brown's argument for their existence. In this discussion, further arguments will be provided to supplement Brown's own with the view to establishing the strongest possible account of picture-proofs. Section 4 will consider the role of pictures in Greek mathematics and will introduce the question of whether Brown's interpretation of pictures has a place within Greek mathematics. Section 5 will clearly state what it means to assess Brown's view in the context of Greek mathematics. Following this, an attempt will be made to argue that Brown's views are consistent with Greek mathematics. Section 6 will attempt to bring these ideas together in an example from Euclid. Finally, Section 7 will discuss the state of picture-proofs given the conclusions so far drawn from the paper.

1. Brown's Picture-Proofs

Brown (2008) begins with this controversial statement: "Pictures can prove theorems" (p. 26). In order to show this, he makes a subtle but significant change to the ordinary view of things. If a picture[1] is used in a proof, then the standard view of its purpose is to supply the evidence on which to base a conjecture. A symbolic proof[2] is then supplied and the conjecture is now a theorem. Hence, the symbolic proof is seen to have established a heretofore uncertain theorem. The picture was simply an aid to understanding or a nudge in the right direction. Brown disagrees. He claims instead that the picture is a reliable method of proof and establishes the theorem in its own right[3]. Instead of the mathematician formalising what can be grasped from the picture, the mathematician is simply finding an alternative route (namely, the traditional symbolic method) to proving the theorem. Hence, in this instance, the already established theorem is confirming that the mathematician's symbolic proof was correct[4]. That is, the theorem is proved by the picture, and so the "independently-known-to-be-true" (Brown, 2008, p. 30) theorem confirms a symbolic proof because

[1]Throughout this paper, 'picture' will be used in place of 'diagram' as Brown uses this terminology almost exclusively. The significance of this use of terminology will later be discussed.

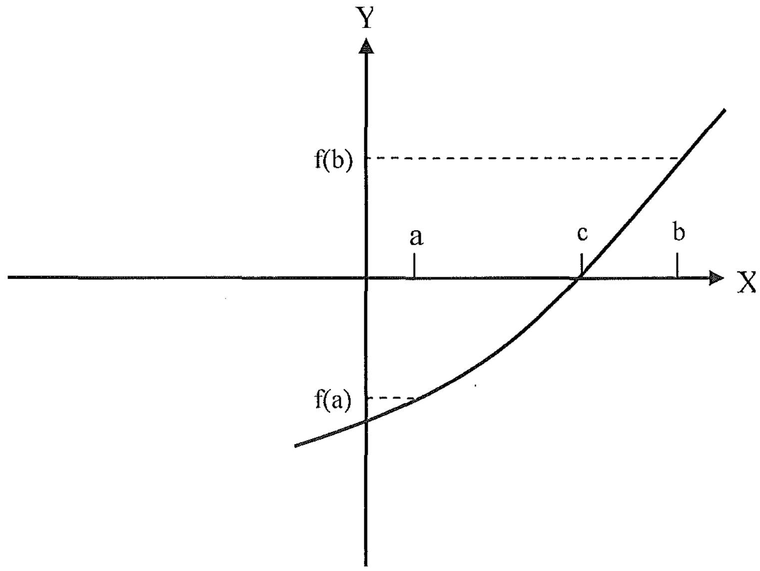
[2] All that is meant by a symbolic proof is the rigourous verbal, symbolic and logical apparatus through which the mathematician establishes the truth of a conjecture.

[3] This does not mean, however, that all pictures are capable of proving theorems; only that some *can*.

[4] By this, Brown (2008) means "The other (the analytic proof) is questionable, but our confidence in it as a technique is greatly enhanced by the fact that it agrees with the reliable method [the picture]" (p. 166).

the symbolic proof arrives at something true; namely, the theorem. The picture-proof is not simply an aid to understanding, but as good as the symbolic proof itself.

A symbolic proof is the means of establishing a theorem by excluding all other logical possibilities arising from the premises of the proof. That is, by its very nature, the symbolic proof establishes the generality of a result by excluding the mere possibility of an alternative. The picture-proof, by contrast, must have a different answer for generality. A mathematical picture is by necessity a picture of a single case; a picture cannot at once represent all possible cases and so prove a result. Brown asks the reader to consider the following example, that is, the Intermediate Value Theorem (or more specifically, its corollary the Intermediate Zero Theorem):



This picture represents a single function; that is, a specific case of the Intermediate Zero Theorem. Clearly, for this continuous function over the interval [a,b], it holds that f changed from negative to positive and there is some c between a and b such that f(c)=0. This is evidence for the conjecture that, if for some continuous function f on the interval [a,b] and if f changes from negative to positive (or vice versa), then there is a c between a and b such that f(c)=0. The Intermediate Zero Theorem, first proved symbolically by Bolzano, states that this is always the case. Brown claims that the above picture is, in fact, a picture-proof; the picture does not merely provide grounds for the conjecture. In terms of Brown's distinction mentioned above, the picture already proved the theorem and Bolzano's proof was confirmed by the already established theorem. Bolzano and many other mathematicians would instead claim that the picture was simply evidence for the conjecture, but Bolzano supplied the first rigourous proof. Bolzano's proof, by its very nature, proved the generality of the theorem, that is, that the theorem holds for all continuous functions f. The above picture shows a specific function f, yet Brown claims it is a picture-proof of the Intermediate Zero Theorem and so it must establish the generality of that theorem. That is, the picture, while only showing a specific function f, must prove the result for all functions. Brown's solution to this will be discussed in the following two sections.

A possible objection here is to claim that Brown and Bolzano are using different notions of continuity: Bolzano is using the epsilon-delta definition and Brown is using the idea of pencil continuity, that is, a line being drawn without the pencil leaving the page. There are at least two possible responses here: firstly, Brown's (2008) response is to claim that "it would be a mistake to infer that the results of the two proofs are *incommensurable*" (p. 30). He argues that if the two notions of continuity were not somehow deeply related, then illustrating Bolzano's theorem with a picture or applying Bolzano's results to situations in geometry would be nonsensical in the first place. A second response would be to argue that Bolzano's notion of continuity is a formalisation of Brown's geometric notion. In this sense, the accuracy of Bolzano's proof would be established by it agreeing with the pictorial notion. Through either argument, it appears difficult to claim that the different notions of continuity are too different to say anything worthwhile about the relationship between symbolic and picture proof.

2. Platonism & Mathematical Objects

To Brown (2008), the solution to the problem of generality is that "some 'pictures' are not really pictures, but rather are windows to Plato's heaven" (p. 40). Before explaining Brown's position more fully, it is first necessary to consider platonism[5]. To Brown (2008), "The essential ingredient [of contemporary platonism] is *the existence and accessibility of the forms themselves, in particular the mathematical forms*" (p. 10). So, he is not concerned with any literal interpretation of Plato's heaven, but is concerned with contemporary platonism, which "is the view that there exist such things as abstract objects — where an abstract object is an object that does not exist in space or time and which is therefore entirely non-physical and non-mental" (Balageur, 2009, introduction section, para. 1). The mathematical forms mentioned in the above quote from Brown are these abstract objects. A platonic form is the common thread between objects. That is, an apple is an apple because it is the instantiation of the form of an apple. The form of the apple is something like a perfect blueprint for an apple and all physical apples are simply imperfect deviations. So, two apples which are not entirely similar, but still seem to have common properties, are called the same because they instantiate the platonic form of an apple. This holds true for mathematical objects. A circle drawn on a blackboard is the instantiation of the perfect circle form. The circle on the blackboard is fine to be getting along with, but it is the perfect circle form which is being considered in any calculations or mathematical uses of the circle.

This contemporary brand of platonism is by no means universally accepted. An obvious objection is to claim that the leap to abstract objects is a dubious one. After all, abstract objects by definition can never be evidenced, so why should anyone suppose their existence? A common approach to this, which Brown (2008) adopts, is to claim that "We have mathematical knowledge and we need to explain it; the best explanation is that there are mathematical objects and that we can 'see' them" (p. 15). Brown expands little on this idea, but a further argument is needed to avoid simply assuming that what is true and what is the best explanation are the same.

[5] Throughout this paper, contemporary versions of the doctrine of Platonism will be spelt with a lower case 'p'; platonism. This is to avoid the equivocation of Platonism as Plato himself would have understood it and platonism in a more modern sense.

To expand upon Brown's argument above, it will be useful here to defer to the notion of 'the inference to the best explanation'[6]; a form of abductive reasoning[7]. The idea is this: it is reasonable to assume the existence of an entity for the explanation of phenomena if and only if the supposed entity only has properties attributed to it which are of explanatory relevance to the stated phenomena. In terms of Brown's argument, the question becomes: do platonic mathematical objects only have properties attributed to them of explanatory relevance to mathematical knowledge? The answer is quite clearly yes. Platonic mathematical objects are, by definition, the abstraction from mathematical knowledge. That is, the platonic mathematical object of a circle is simply the abstraction of the blackboard circle. So, the mathematical object has all of the *necessary* properties of a circle as if it exhibited any non-necessary properties, the object would be of a specific, rather than general, nature. It is a mathematical object, so it cannot be representing a specific case. Therefore, the mathematical object only has mathematical properties ascribed to it which are necessary in order for it to be the abstraction of its instantiation. Furthermore, it is necessary that mathematical objects are non-mental, non-physical and outside of space and time (that is, they have always existed and always will) because otherwise mathematics would not be certain; there would be no mathematical truths because all of the above are finite. If mathematical objects were mental constructions, then when humans die so too the mathematics they have created. If the mathematical objects were physical, existing in space and time, they would also be finite. If mathematical truth is to be certain, this cannot be the case. Finally, the mathematical objects must be causally inert because it is impossible to interact in any way with an object defined as non-mental, non-physical and outside of space and time. It is worth considering what is meant by mathematical objects being 'causally inert'. It is often supposed that in order to know something, the person doing the knowing must have some causal relationship with that something. For example, "If I know that Mary is wearing a red shirt, it is because I am in causal contact: photons from Mary enter my eyes, and so on" (Brown, 2008, p. 17). As mathematical objects are non-mental, non-physical and exist outside of space and time, it is impossible that this sort of causal relationship could take place. There is simply no way for the mathematician to form such a relationship with these abstract objects.

The above discussion leaves a choice between abstract mathematical objects or the uncertainty of mathematical choice. Most mathematicians would be more comfortable having abstract mathematical objects than giving up the view that mathematics is certain, so the above arguments for the relevance of the properties of mathematical objects appears to be acceptable. Hence, the conditions of the 'inference to the best explanation' have been satisfied. Having formalised Brown's reasoning somewhat, it appears then that the leap from mathematical knowledge to mathematical objects is at least reasonable.

The above does not deal with the more important problem: how can the mathematician know something without being causally related to that something? That is, mathematicians somehow 'use' mathematical objects, but how can they

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[6] See Harman.

[7] Abductive reasoning is a form reasoning which infers an 'economical' explanation from an incomplete set of information. By 'economical', it is meant that all parts of the explanation are *necessary* to explain the set of information.

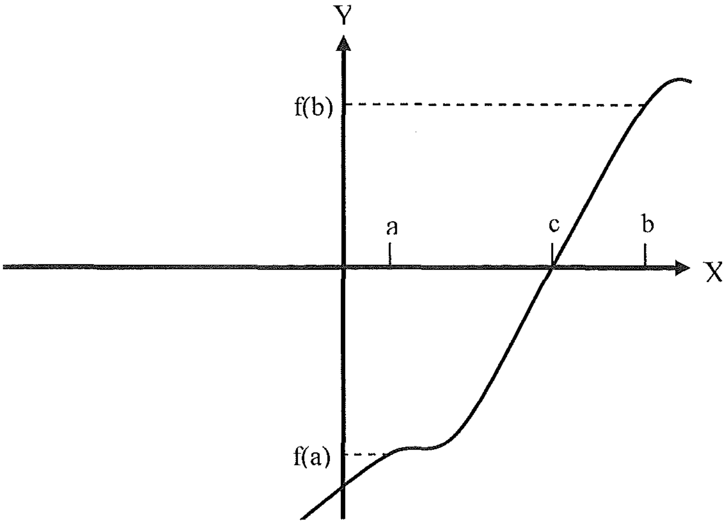
gather information from this process without somehow being causally related to the mathematical object? Brown attempts to deal with this by citing an example from quantum physics in which an experiment is conducted and information can be gained without a causal relationship. This is an unconventional approach and relies perhaps too heavily on current theories. As Brown (2008) himself notes: "[. . .] arguments like this are problematic. It relies on significant assumptions about the physical world" (p. 18) and "[. . .] the assumptions that go into the physics of this situation are at least as plausible, or even more plausible, than the assumptions involved in the causal theory of knowledge. It seems perfectly sensible to dump the latter" (p.19). So, in light of this uncertainty, it seems necessary to consider other responses to the problem of causal accessibility.

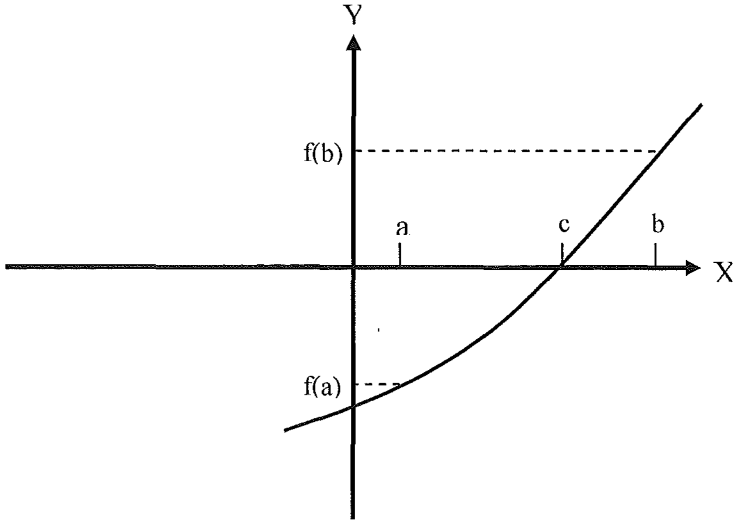
A proposed solution to the problem of accessibility which, it will be argued, appears to tie in well with pictures is that of Balageur (1998). Balaguer believes that all possible abstract mathematical objects exist. From this, he argues that any mathematical theory which is internally consistent is then 'picking out' the mathematical objects to deal with. As a result, the mathematician does not need to have a causal relationship with mathematical objects; instead, the mathematician must formulate internally consistent mathematical theories which pick out the existing mathematical objects. In this way, "we can acquire knowledge of abstract mathematical objects *without* the aid of any sort of contact with such objects" (Balaguer, 1998, p.49).

In order to apply this to pictures, it is first necessary to note that an imperfect circle drawn on a blackboard cannot be an abstract object. Only the perfect platonic forms exist as abstract objects. Consider: An imperfect circle is drawn on a blackboard. There is nothing internally inconsistent about this. If, however, the imperfect circle is supposed to pick out an existing and imperfect abstract object of a circle, then ordinary uses and theorems about circles would no longer be consistent (e.g., radius=circumference/2*pi). This is worth noting because otherwise, on Balaguer's view, the mathematician could use pictures to pick out imperfect mathematical objects. Such a use of mathematical objects would result in mathematical systems being internally inconsistent and thus pictures could never access Plato's heaven. As a result of this, visual pictures would seem to 'pick out' their Platonic form. It appears then that Balaguer's view is open to pictures accessing their platonic forms and furthermore, provides a solution to the problem of accessing these platonic forms.

3. The Generality of Picture-Proofs

Consider two pictures related to the Intermediate Zero Theorem:





For each picture, the theorem quite clearly holds; that is, the function f must at some point c between a and b equal zero. According to Brown, the platonic form which is the common bond between these pictures is the proof of generality. The line of reasoning is this: each picture is a proof and each picture is an instantiation of the same platonic form. Through each picture, the mathematician can 'grasp' the platonic form. So, if the platonic form can be grasped through the picture, it follows that the result (Intermediate Zero Theorem) proved by the picture holds for the platonic form. Hence, if the platonic form is something like a blueprint for all of its instantiations, then the pictures instantiated from this blueprint must consist of the same properties. So, through either of the above pictures, the platonic form can be grasped. The Intermediate Zero Theorem holds for the pictures and so it holds for the platonic form. The platonic form is the abstraction for all possible functions under the conditions of the proof, so the Intermediate Zero Theorem must hold for all possible functions under the conditions of the proof. Hence, the platonic form has established the generality of the result.

It is worth considering what Brown means when he uses vague terms like 'grasp'. To Brown (2008), "[. . .] it [the picture-proof] works in a different way, more like an instrument. [. . .] As telescopes help the unaided eye, so some diagrams are instruments rather than representations which help the unaided mind's eye" (p. 40). Furthermore, "We can intuit mathematical objects and grasp mathematics truths. Mathematical entities can be 'seen' or 'grasped' with 'the mind's eye'" (p. 14). So it appears then that seeing a picture with the unaided eye is parallel to grasping the platonic form with the mind's eye. This might seem incorrect; seeing a picture with the unaided eye is a physical process which is understood. Brown here replies that, though the physiology of the process is well understood, the rest is not. The mind-body problem, that is, the relationship between the physical input (what is seen) and the mental reaction is by no means solved. Brown (2008) concludes: "[. . .] our ignorance in the mathematical case is no worse than our ignorance in the case of everyday objects" (p. 16).

It is possible to raise certain objections about Brown's view of picture proofs working like an instrument. Folina (1999) has replied to Brown, saying: "telescopes are not *themselves* justificatory: it is not the telescope which is cited as the primary justification for an astronomical claim" (p. 426). Folina then goes on to argue that, despite Brown claiming to show that pictures can prove things, he is instead showing that only the abstract objects can do the proving. That is, "it seems that although he wants to call them 'proofs', visual pictures do not justify general mathematical claims; only mental, or abstract, 'pictures' can prove such appropriate evidence" (Folina, 1999, p. 426). In this sense, the visual picture is heuristic; it helps the mathematician to understand or access both the abstract picture and the symbolic proof. This objection will be returned to and expanded upon in section 7.

4. The Role of Pictures in Greek Mathematics

If Brown is correct in his view of pictures and if pictures play a vital role in Greek mathematics, then this motivates the question: Does Brown's interpretation of pictures have a place within Greek mathematics and if not, what are the consequences of this for Brown's view? The rest of the paper will deal with this question. Firstly, the role of pictures in Greek mathematics must be investigated and defined; that is the goal of this section.

An important part of the historical work of Netz[8] (2005) on Greek mathematics is to show that picture and proof exist together in such a way that "Rather than one of them governing the other, the text and diagram present, let us say, a *cohabitation*" (p. 171). That is to say, the picture and proof are logically dependent upon one another. Netz establishes this through a detailed survey of the lettering of a diagram. Surviving Greek proofs refer to the corresponding diagram by means of referring to letters added to the diagram at useful places, such as the intersection of two lines. Netz distinguishes four kinds of specification through which the diagram is given its letters. Firstly, *full specification* is a specification of letters such that the text perfectly determines the letters. For example, a statement like, "Let A be the centre of a circle" is a full specification because there can be only one possible centre. Secondly, *under specification* is the specification of letters such that the text is not sufficient to be certain of its meaning; it is necessary to also inspect the picture. For example, "Let there be a circle so that BC is the radius of the circle" is under specified because the statement alone is not sufficient to establish whether B or C is the centre of the circle. Thirdly, *un-specification* is the lack of specification of letters. For example, "Let there be a circle so that BC is the radius of the circle. I say that BD is twice BC" is a case of *un-specification* because D is not specified at all. Lastly, *pre specification* is a specification of letters such that the specification is gradually defined. That is, the specification may begin as un-specified and become under or fully specified or the specification may begin as under specified and become fully specified. All of these kinds of specifications except full specification are instances where the picture is relied upon, so that, "the reader went beyond what was given in the text, and related the letters to the points as they appear in the configuration of the diagram" (Netz, 2005, p. 168). In this sense, the picture precedes the proof and the proof assumes the picture. Furthermore, through his survey of Greek texts, Netz claims that most letters are not completely specified. As a result, Netz (1998) concludes that "Without diagrams, objects lose their reference; so, obviously, assertions lose their truth-value. Ergo, part of the content is supplied by the diagram, and not solely by the text. The diagram is not just a pedagogic aid, it is a necessary logical component" (p.34).

The subject of generality is a more difficult one. Despite the attempt to apply Brown's views to Greek mathematics, rather than to the kind of thinking done by Greek mathematicians, it is still worth considering how the Greeks may have considered the problem of generality. The explanation of Netz is built upon work done by Mueller, so the view established here is very much shared by the two. The problem, as Mueller (1981) states it, is this: "How can one move from an argument based upon a particular example to a general conclusion, from an argument about a straight line AB to a conclusion about any straight line?" (p. 13). The core of the solution in Greek mathematics to this question, so Netz and Mueller argue, is repeatability. The argument is this: the Greek proof is constructed in such a way that the structure is clearly and easily extendable. That is, the Greek mathematician writes the proof in such a way that the mathematical intelligence of the reader can mentally repeat the steps for similar cases. Netz (1999) uses the metaphor of a shadow of a proof: "The *homoios* [statements of the form 'similarly we will prove...'] demands one to glance,

[8] Netz uses 'diagram', rather than 'picture'. The significance of this will be dealt with in section 7.

briefly, at the shadow of a proof. That shadow being removed, the result would no longer be valid. It is only because we can imagine ourselves going through the proof that we allow ourselves to assume its result without that proof" (p. 244). More formally, the result will often take the form of a conditional statement. That is, $C(x)$ is the set of assumptions, the construction and $P(x)$ is the implication, so that it is hoped to prove that $C(x) \rightarrow P(x)$. The actual Greek proof is only proving a specific instance, a specific $C(a) \rightarrow P(a)$. This, however, is not only a proof that $C(a)$ leads to $P(a)$, but it is a proof of the provability of $P(a)$ on grounds of $C(a)$. This shows that if the particular result P derives from the construction C, then the extension from $C(a) \rightarrow P(a)$ to $C(x) \rightarrow P(x)$ is justified. Hence, "extendibility refers to the extendibility of the particular proof, rather than of the particular result" (Netz, 1999, p. 257). It should be emphasised that this method of 'proof' is perhaps more a case of convincing the reader than the actual justification involved in a proof. That is, a leap based on mathematical intelligence may be evidence, but it is not necessarily a proof. It is an intuitive notion, as Netz (1999) notes: "Nothing in the practices of Greek geometry would suggest that a proof of repeatability is either possible or necessary. Everything in the practices of Greek geometry prepares one to accept the intuition of repeatability as a substitute for its proof" (p. 269). So, generality is at least partially accounted for in the sense that a Greek proof is constructed in such a way to allow easy repeatability. Hence, the importance of pictures in Greek mathematics and a notion of generality in Greek mathematics have both been accounted for. These are the two main areas of Brown's view, so this is sufficient to proceed with.

5. Picture-Proofs and Greek Mathematics

As has been established, pictures played an essential, logical role in Greek mathematical proofs. Given, then, that the Greeks were among the few to treat pictures as having a legitimate role, it is worth considering how this role relates to Brown's view of picture-proofs. If it can be shown that Brown's view is consistent with Greek mathematics, then something can be said as to the accuracy and scope of Brown's theory. If, on the other hand, Brown's view is shown to be inconsistent with Greek mathematics, then something can be said as to why his theory fails to include the Greek picture. By 'consistent with Greek mathematics', it is not meant that Greek mathematicians would agree, sympathise or make use of Brown's picture-proofs. It is simply meant that the mathematical results and means of proving them are not inconsistent with Brown's view. In order to perform this application, it is first necessary to clearly state Brown's view and the important points behind it. Brown believes pictures can prove things. The two most important points behind this claim are:

(i) *Reassigning the burden of proof.* The picture-proof is the first proof of the theorem available to the mathematician. An attempt at a written proof to a corresponding picture-proof is the mathematician having a written proof confirmed by the *prima facie* reliable picture-proof.

(ii) *Platonic answer to generality.* The picture-proof attains its generality through being a window into Plato's heaven. That is, the picture acts as an instrument, like a telescope, for the mathematician to grasp the Platonic form of the picture and so prove the generality of the result.

To consider the whether Greek mathematics is consistent with picture-proofs is to consider whether the above two points are consistent with Greek mathematics. To begin with (i), it is clear that the Greek mathematician began with the picture. This is the first point in Brown's reassignment; the written proof arises from the picture. The evidence for this is from Netz's survey of the specification of letters in a picture. This should be uncontroversial. The following point needing to be established in order for (i) to be consistent with Greek mathematics is rather more subtle. Before beginning, however, it is worth noting that the below discussion is dealing only with cases where the picture acts as a picture-proof. It is sufficient for Brown to be consistent with Greek mathematics that only the picture-proof cases hold. There are four ways of considering the relationship between diagram and text:

(a) *the text does the proving,*

(b) *the picture and text are dependant on one another to prove anything,*

(c) *the text is dependent on the picture to prove anything and finally,*

(d) *the text or picture can do the proving independent of one another.*

In order for (i) to be consistent with Greek mathematics, it will be shown that it is necessary for Brown to reject both (a) and (b).

Option (a) is the first Brown can reject. He can do so for two reasons. Firstly, Brown aims to show that it is pictures that can do the proving. If this is the case, and Brown must take it to be so, then option (a) cannot hope to demonstrate that (i) is consistent with Greek mathematics because it explicitly denies Brown's main claim that pictures can prove theorems. Secondly, from the above, Netz argues convincingly that the specification of letters in pictures demonstrates that the text, for the most part, relies on the diagram. So, (a) can be rejected.

Netz's (2005) claims: "Rather than one of them governing the other, the text and diagram present, let us say, a cohabitation" (p 171). So, clearly, Netz is in the (b)-camp. It is worth considering more carefully the approach of Netz. He argues from the assumption that pictures cannot prove, but aims to show nonetheless that pictures play an essential, logical role in the proving. This implies that the text is dependent on the picture and that the picture relies on the text to prove the content of the picture. The picture, unable to prove anything by itself, is dependent on the text. This is inconsistent with Brown's view. That is, Brown's reassignment is only correct if the diagram is a proof *independently* of any other evidence. This does not, however, mean that Brown's views are inconsistent with Greek mathematics. Netz begins with the (implicit) assumption that pictures cannot prove and argues from there. As a result of this, though Netz shows that the text is dependent upon the picture, he does not show that the picture is dependent upon the text; he assumes this. Because Netz assumes, rather than proves, it is possible to dispute this assumption. As Brown's views dispute Netz's assumption, Brown does not need to allow that the picture is dependent in any way upon the text. As a result of this, in trying to establish that (i) is consistent with Greek mathematics, (b) can be rejected[9].

It appears, then, that (c) and (d) are both consistent with (i). It is worth considering the implication of being consistent with two of the options. Firstly, having both options open to Brown implies that his views are consistent with two kinds of Greek proofs: those in which the text is dependent on the picture and those in which the text and picture are independent of one another. If the text is dependent upon a picture because of a certain specification of letters in the picture, then even if there can be found Greek proofs in which the specification of letters imply that the text *does not* depend on the picture, Brown can account for these kinds of proofs. That is, if there are Greek proofs where the text is independent from the picture, Brown can still claim that the picture is doing the proving and his reassignment holds. This is because the acceptance of (d) allows Brown to account for situations of *independence*. That (i) is consistent with Greek mathematics has thus been shown. A final note: there has been no explicitly positive reasons give here as to why this reassignment is justified. The only reason perhaps suggested by the above is that the scope of Brown's views covers most of the kind of pictures in Greek mathematics. This does not necessarily imply that Brown is correct in supposing this reassignment, but the above does suggest that he is, at least, supposing nothing inconsistent in doing so.

To show that (ii) is consistent with Greek mathematics is a matter altogether more complicated than the previous. This is not an attempt to argue that the Greek mathematician's answer to generality was through a Platonist account, but rather an attempt to show that the pictures in Greek mathematics operate in a way consistent with Brown's picture-proofs. This is a difficult endeavour for two reasons: firstly, Brown gives no express criteria by which to judge whether a picture is indeed a picture-proof and secondly, by what means Greek proofs attempt to attain generality is a matter not entirely settled. That Brown gives no criteria is something which will be referred to again later, but it will be useful now to consider the second point. It was earlier argued, through Netz and Mueller, that repeatability is the key to how the Greek mathematician accounted for generality and this notion will be used in considering the platonist account of generality offered by Brown.

In order for Greek Mathematics to be consistent with Brown, then the pictures in Greek Mathematics which work as 'picture proofs' must be in no way inhibited by the practices of Greek Mathematics. So, the question becomes: Is there anything in Greek mathematics which inhibits a platonist account of generality? That is to say, is there anything in Greek Mathematics which prevents the picture being a representation of its platonic form? There appear to be two possible means through which Greek Mathematics could prevent a picture representing its platonic form. Firstly, the picture could have no form to represent; that is, there is no platonic form corresponding to the picture or secondly, there is something 'closed' about the picture which does not allow representation. To begin with the former, a picture having no platonic form implies that the picture is unique in the sense that it is not a special case of a general result[10]. This means that the argument that Greek Mathematics is inconsistent with a platonist account of generality can be rejected because no such unique pictures can

[9] This is not to say that Netz is wrong. This is simply a strategy by which the supporter of Brown may avoid the difficulties which arise from Netz's claim of cohabitation.

[10] This does not mean, however, that a picture being unique implies that it has no platonic form.

exist in Greek Mathematics. This is because Greek Mathematics attained its generality through repeatability and a picture cannot be unique if it is repeatable. The second argument, that pictures in Greek Mathematics could be in some way 'closed', is rejected as a result of repeatability as well. This is because a proof in Greek Mathematics was constructed in such a way that it could easily be repeated for other situations of a similar kind. Hence, in this way, the mathematical intelligence of the reader could somehow fill in the gaps. It is as a result of this that pictures in Greek Mathematics can be seen as being open and inspectable which further implies that these pictures cannot be 'closed' and so can represent their platonic form.

If Greek mathematicians rely on this mathematical intelligence to somehow account for repeatability, is this not similar to having the mathematical intelligence to see how a picture is the instantiation of a platonic form? Of course, none of this mathematical intelligence business is justificatory. A proof, in order to be a proof, must involve justification. Despite this, the notion of mathematical intelligence seems to suggest that, in general, there is something open and inspectable about Greek pictures. This, in turn, suggests that Greek pictures are not inconsistent with being the instantiations of a platonic form. A leap has been made here from mathematical intelligence applying to the repeatability of a proof to mathematical intelligence applying to the repeatability of the picture corresponding to that same proof. It is necessary to justify this leap.

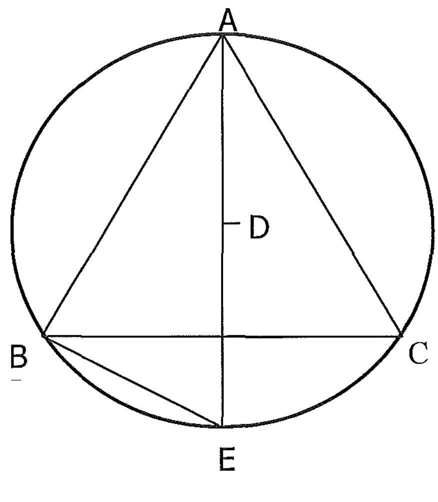
Mathematical intelligence is the means by which the reader of Greek mathematics can extend a Greek proof, rather than extending the result. That is, the proof is constructed in such a way as to allow the mathematical intelligence of the reader to easily grasp how the proof could be constructed to account for different cases. The 'cognitive background' of this mathematical intelligence is explored by Netz. Netz argues that, instead of seeing the variability of the picture as infinite, the Greeks viewed the variability of a picture as a finite number of relevant discrete cases. That is, the cases in which the possibility of some problems can be supposed and so these cases are relevant to understanding the picture and its variability. If the Greeks had the mathematical intelligence to inspect a diagram for the relevant variability, then it seems permissible to allow them the capacity to consider the 'irrelevant' cases as well. For example, suppose there is drawn a picture in which there is a straight line bisected by another straight line. If a random point is to be taken along the original line, it is worth considering the cases which appear significant: point taken at the point of intersection, point taken to one side of the bisection and point taken to the other side of the bisection. If the Greeks could realise this kind of variability, then simply varying the random point within the relevant cases so that there is an infinite number would seem doable. As a result of this, the leap above from one application of mathematical intelligence to another is justified. So, if mathematical intelligence is important to the generality of Greek proofs, it seems as though this suggests that there is something open and inspectable about pictures in Greek Mathematics. As a result of this, it cannot be argued that the pictures are 'closed' in the sense that they cannot represent their platonic form. This implies that neither of the above means—no platonic form or 'closed' picture—through which Greek Mathematics could reject a platonic account of generality show that picture-proofs are inconsistent with Greek Mathematics. Hence, this demonstrates that Brown's view of pictures is consistent with the Greek view of proofs. If Greek mathematics is to be at all consistent, then it must be allowed that Brown's views correspond to this consistency.

Despite the above appearance of consistency, it is worth noting a slight awkwardness when comparing the generality of the proof to the generality of a picture. It has been established that the construction of the proof is important in repeatability and therefore generality. Brown's view of picture-proofs, however, cannot involve this notion of construction. Through the picture, the mathematician 'grasps' the Platonic form and so the generality of the picture. This does not hint at the mathematician mentally or physically constructing the picture. It is more a case of the mathematician receiving information than constructing it. There is no obvious means by which to reconcile these notions. This point is not an inconsistency between Greek mathematics and Brown, but it is worth nothing for later discussion.

6. An Example

At this stage, it is worth considering the application of all of the above in a concrete example. That is, through an example from Euclid, the following will be discussed: (i) The logical necessity of the picture as deduced from the letter-specifications of Netz, (ii) repeatability and generality as deduced from the construction of the proof and (iii) the interpretation from the view of picture-proofs as addressed in the preceding section. To this end, the following proof and picture are introduced[11]:

Book XIII, Proposition 14



If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius of the circle.

Let ABC be a circle, and let the equilateral triangle ABC be inscribed in it;

I say that the square on one side of the triangle ABC is triple of the square on the radius of the circle.

[11] (Heath, 1908, p. 466-7, vol. 3). Items in the square brackets are references to earlier results.

For let the centre D of the circle ABC be taken, let AD be joined and carried through to E, and let BE be joined.
Then, since the triangle ABC is equilateral, therefore the circumference BE is a sixth part of the circumference of the circle ABC.
Therefore the circumference BE is a sixth part of the circumference of the circle; therefore the straight line BE belongs to a hexagon; therefore it is equal to the radius DE. [IV. 15, Por.]
And, since AE is double of DE, the square on AE is quadruple of the square on ED, that is, of the square on BE.
But the square on AE is equal to the squares on AB, BE [III. 31, 1. 47]; therefore the squares on AB, BE are quadruple of the square on BE.
But BE is equal to DE; therefore the square on AB is triple of the square on DE.
Therefore the square on the side of the triangle is triple of the square of the radius.

Q.E.D.

To begin with (i), it will be useful to restate the classifications Netz proposes: fully specified, under specified, pre-specified and no specification. The proof begins with the assertion "let ABC be a circle". This is a case of pre-specification as, although the three points necessary to define a circle have been stated, there is no reason to put them anywhere in particular on the circle. At this stage in the proof, A, B and C are all under specified. Soon after, however, Euclid states, "let the equilateral triangle ABC be inscribed in it". At this point, then, the letters A, B and C have become fully specified. It is worth noting that this is not because they must be at certain points on the circle independent of one another, but rather that they must be at certain points on the circle relative to one another. The assertion "let the centre D of the circle ABC be taken" is a clear case of full specification. That is, there is only one possible point at which D may exist and so no ambiguity is possible. The case of the letter E is slightly more complex. Netz (2005) notes from an observation of a proof in Apollonius that, in this kind of example, the picture makes it difficult to realise the specification and that "visual information compels itself in an unobtrusive, almost unnoticed way" (p. 169). It is first necessary to unlearn or unsee where E is placed in the diagram. Then, when Euclid asserts, "let AD be joined and carried through to E", it is possible to note that this does not assert that the point E is at the intersection with the circle. E, then, is under specified; the text asserts a line of possible points, not a point at the intersection. If this is the case, how then is it possible to reconstruct the picture, or at least know that a reconstruction is correct? It is possible to use the entire proposition to more accurately reconstruct the picture. This, however, does not mean that, by some later point in the proposition, E will be fully specified. That would not be a legitimate reading of the text because the latter stages of the proof are meant to derive, not define. E is then not a convoluted case of pre-specification, but a genuine case of under specification.

As a result of the above classification of points, it is possible to deduce the logical importance of the picture. The process through which A, B and C move from being under specified to fully specified in the proof is important to consider. The reader of the proof has two possible reactions to an under specified letter: to look at the picture or to suspend belief in the hope that the text will later fully specify it. If it is the former, then this demonstrates the reliance of text on picture. The latter case seems difficult to justify. That the letter E is never fully specified only re-enforces this point further. Without the picture, E is not specified so that the proof's "invalidity consists not in contradictions, but in non sequiturs" (Netz, 2005, p. 170). Without the picture, the proof leads nowhere.

Secondly, (ii), that generality and repeatability derive from the construction of the proof, is perhaps more obvious than the above. It is first worth breaking down Euclid's above proof into its logical components. The first assertion that "If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius of the circle" is the *protasis*, the statement of the form $C(x) \rightarrow P(x)$. The second assertion "Let *ABC* be a circle, and let the equilateral triangle *ABC* be inscribed in it" is the *ekthesis*, the statement of the form C(a). The following assertion "I say that the square on one side of the triangle *ABC* is triple the square on the radius of the circle" is the *diorismos*, the statement of the form P(a). All the following assertions, except the very last, are the *kataskeue* and *apodeixis*. The *kataskeue* are the construction statements and the *apodeixis* are the predicate statements. These assertions are of the form: C(b), ..., C(n), P(b), ..., P(a). The final assertion, the *sumperasma*, is simply a repeat of the *protasis*. It is the confirmation of what has been proved of the form $C(x) \rightarrow P(x)$.

The construction formulae, the *kataskeue*, are those of the type, "let AD <u>be joined</u> and <u>carried through</u> E, and let BE <u>be joined</u>". The predicate formulae, the *apodeixis*, are the relations drawn from the construction formula, such as, "therefore the straight line BE <u>belongs to a hexagon</u>, therefore it is <u>equal to the radius DE</u>". The generality of the result is derived because the *kataskeue* and *apodeixis* prove P(a) from C(a). This proves, not only that $C(a) \rightarrow P(a)$, but it proves the provability of P(a) from C(a). In doing so, it shows that the $C(x) \rightarrow P(x)$ is justified if the construction formulae are situationally independent[12]. That is, the mathematical intelligence of the reader allows for the extension from $C(a) \rightarrow P(a)$ to $C(x) \rightarrow P(x)$. More concretely, it can be easily seen from Euclid's above proof how the text could be altered to account for a different case. The essence of this easy extendibility derives from the 'construction to predicate' structure of the proof.

Lastly, (iii), the interpretation of Brown, in this case is perhaps obvious: the picture is the proof and the text is confirmed by the already known to be true proof of the picture. There is more subtlety to this than is at first apparent. It had already been alluded to that picture-proofs and construction do not go together particularly well. In this example from Euclid, it is possible to see why. The construction formulae, the *kataskeue* take the form of statements which require some action; e.g., that a line be drawn. This type of construction underlies the importance of interaction with a picture in order to properly grasp it. A picture into Plato's heaven, however, involves no interaction whatsoever; it is simply the receiving of information: "I should add that the way the picture works is much like direction perception; it is not some sort of encoded argument" (Brown, 2008, p. 30). A more lengthy discussion of this, and the implications for Brown's view, will be found below.

7. The State of Picture-Proofs

[12] That is, the construction formulae are proposed in such a way that they are independent of a special case; they apply to the general case.

Having considered picture-proofs in the context of Greek mathematics, it is now possible to draw some conclusions regarding Brown's view. In this analysis of Brown so far, there have been no major inconsistencies, either internally or with regards to Greek mathematics. Instead, Brown's view has seemed to be one of explanation. Brown supports contemporary platonism in the belief that it offers the best explanation for mathematical objects. Additionally, Brown's views could be seen as explaining much of Greek mathematics. This may not be justification for his views, but is it is nevertheless evidence for their scope. The consistency of Brown's view is perhaps not what is at stake. Instead, it is their usefulness, both as means of explanation and as a means of discovering further mathematics. Through all of the above, the former has been established, but the latter remains doubtful.

Until now, the usefulness of picture-proofs has only been discussed implicitly. Thus far, the problems of construction and Folina's view (as discussed in Section 3)—that according to Brown, it ought to be the Platonic form, not the visual picture, doing the proving—have been hinted at. One final discussion is necessary to bring these points together and highlight the problem of usefulness they cause for Brown. Throughout this paper, 'picture' has been used as a substitute for 'diagram' as this is Brown's terminology. It is worth reflecting on this choice of words as it underlies the common thread between the problems of construction and the problems of Folina. Though 'picture' and 'diagram' both suggest the representation of visual information, there is a subtle difference between them. This difference is manifest in the use of these words as verbs. The verb 'to picture' means to form a mental image of something, to conceive of it in the mind's eye. Given Brown's earlier pronouncements about the reception of the platonic form through the mind's eye, it seems reasonable to assert that this was the sense in which he used 'picture'. The verb 'to diagram', by contrast, suggests a much more physical, active process; to make a diagram of something. This difference underlies an important point: namely, that while the normal use of 'diagram' suggests taking an active role in representation, the use of 'picture' suggests the reception of information through the mind's eye. So, Brown must ignore the suggestion of having to interact with a 'diagram' in order to understand it. The implication for Brown is that any idea of 'picturing' the Platonic form becomes even more abstract. It cannot involve any interaction and so cannot be harnessed in any useful fashion. All of this highlights the issue, which Folina draws attention to, that the visual picture is of no justificatory use if the Platonic form is the source of generality. The picture ceases to be in any way useful and all important responsibilities are transferred to the Platonic form. So, although picture-proofs may offer truth, it is an isolated and unhelpful truth. That is, picture-proofs do little to advance the cause of non-deductive methods of proof.

There are, perhaps, ways to remedy this over-abstraction. Firstly, Brown needs to establish an extended definition of 'proof' in order to include non-deductive evidence. It is not sufficient to point out that pictures offer convincing evidence, it is necessary to formally define 'proof' in such a way that pictorial evidence constitutes 'proof'. Having done so, this would allow Brown to establish a set of conditions for judging whether a picture is indeed a picture-proof. Without these additions, the notion of picture-proofs is irrelevant. This is because even if a picture is a picture-proof, there is no rigorous way of knowing this. If there is no way of knowing this, then a symbolic proof will have to be provided to properly confirm the picture. Having to have a picture confirmed by a proof is against everything which Brown is about, so these conditions are absolutely necessary. Without them, pictures are reduced to the use which Brown so ardently opposed: to simply aid understanding.

Even if these improvements were made, and even if the improvements were reasonable, the question still remains: is it possible to grasp a picture without interaction? That is, is it possible to receive through the mind's eye the information from Plato's heaven? If it is accepted that it is, as Folina claims, the platonic form which does the justification, then the above improvements suggest that there will be a place for the visual picture. If there is a systematic means for judging whether a picture is a picture-proof, then the means of judging must involve the visual picture. That is, it is difficult to imagine a 'test' for picture-proofs attempting to consider objects in Plato's heaven, so the visual picture must play some role. In this sense, it seems, that although the platonic form is doing the proving, the visual picture is indispensable. This would, to some extent, give the visual picture a place in Brown's views where Folina had argued the visual picture would have no significance.

8. Conclusion

This paper began by putting forward the strongest possible version of Brown's views before considering them in the context of Greek mathematics. Through this, it has been established that Brown's views in their current incarnation do well in explaining but lack in usefulness. As a result of this, it has been argued that Brown must establish an expanded formal definition of 'proof' and the conditions by which a picture-proof can be judged as being a picture-proof. If this is possible, then picture-proofs will offer both explanation and usefulness. If it is impossible, as may be the case, then there is little to do except acknowledge that picture-proofs may cover a great many cases, but do nothing to aid the progress of mathematics.

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