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by

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No. 28 November, 1982

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Oblique wave groups consist of waves whose straight parallel lines of constant phase are oblique to the straight parallel lines of constant phase of the group. Numerical solutions for periodic oblique wave groups with envelopes of permanent shape are calculated from the equations for irrotational three dimensional deep water motion with nonlinear upper free surface conditions. It is shown that some analytical solutions for oblique wave groups calculated from the two (horizontal) dimensional nonlinear Schrödinger equation are in error because they ignore the resonant forcing of certain harmonics in two dimensions. The numerical method casts doubts also on the physical relevance of solutions of the nonlinear Schrödinger equation for which the envelope of the group passes through zero. Particular attention is given to oblique wave groups whose group to wave angle is in the neighbourhood of the critical angle $\tan^{-1}(1/\sqrt{2})$, corresponding to waves on the boundary wedge of the Kelvin ship-wave pattern.

1. INTRODUCTION

When waves are generated over a range of horizontal directions, it must be expected that the lines of constant wave phase in a locally sinusoidal group of waves are oblique to the lines of constant phase of the envelope enclosing the group. One well known example is that of the waves along the boundary wedge of the Kelvin ship-wave pattern (Lighthill (1978), figures 70,71). Another example results from the instability of finite amplitude deep water gravity waves to disturbances in two horizontal dimensions. An oblique unstable modulation of a length large compared with the wavelength causes a regular wave train to grow into an oblique wave group structure. Such oblique instabilities dominate parallel instabilities for moderate and large wave steepnesses (Mclean *et al* (1981)).

A simple description of a periodic oblique wave group is given by the superposition of two sinusoidal waves differing slightly in wave-number components along one horizontal direction. Their water surface displacement may be represented by

$$\eta(x_1,x_2,t) = a\cos\{k_1x_1+k_2x_2-\omega t\} + a\cos\{(k_1+\delta k_1)x_1+k_2x_2 - (\omega+\delta\omega)t\} \tag{1.1a}$$

$$= 2a\cos\{(\delta k_1/2)x_1-(\delta\omega/2)t\}\cos\{(k_1+\delta k_1/2)x_1 + k_2x_2-(\omega+\delta\omega/2)t\} \tag{1.1b}$$

where $\omega = \{g(k_1^2+k_2^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$ is the frequency of waves in deep water. This superposition describes a slowly varying wave train propagating at angle θ to the x_1-direction, where $\tan\theta = k_2/k_1$ approximately, whose group envelope propagates in the x_1-direction with a velocity approximately

$$\partial\omega/\partial k_1 = \frac{1}{2}(g/k)^{\frac{1}{2}}\cos\theta\ , \tag{1.2}$$

with $k = (k_1^2+k_2^2)^{\frac{1}{2}}$. A generalisation of this description is

$$\eta(x_1,x_2,t) = \sum_{k_1} a(k_1)\cos\{k_1x_1+k_2x_2-\omega t\}\ , \tag{1.3}$$

whose spectrum lies in a narrow waveband centred on wavenumber (k_0,k_2) for fixed k_2, and ω is dependent on the group amplitude as well as k_1 and k_2.

Hui & Hamilton (1979) calculated oblique wave group solutions from the nonlinear Schrödinger equation in two horizontal dimensions. This equation, derived originally by Zakharov (1968), assumes that the spectrum of surface waves rises to a narrow central peak in two dimensional wavenumber space. The wave frequencies are expanded in a Taylor series about the central wavenumber, with the leading terms of the series contributing the linear terms to the nonlinear Schrödinger equation. If resonances cause further significant peaks in the spectrum of water waves, the Taylor series expansion is not valid, and the nonlinear Schrödinger equation fails as a model equation for the wave group structure. It will be shown that resonances are either absent or insignificant for oblique wave groups whose group to wave angle is less than about $\tan^{-1}(1/2)$, but that for greater angles, including the important critical angle $\tan^{-1}(1/\sqrt{2})$, resonances can be significant even to the extent at some angles that they dominate the oblique wave group structure.

The nonlinear Schrödinger equation has apparent oblique wave group solutions for which the envelope of the group passes through zero. Hui & Hamilton (1979) calculated these solutions and sketched them in their figures 2 and 3. If A denotes the functional form of such a solution of the nonlinear Schrödinger equation, the physical group envelope is given by $|A|$. This is not a solution of the nonlinear Schrodinger equation if A passes through zero, because the derivatives of $|A|$ do not exist at such a point. Also, the wavelengths of waves moving through a zero of the group envelope differ from those in the centre of the group, for kinematic reasons, rendering the nonlinear Schrödinger equation of doubtful validity as a model of wave group evolution near an envelope zero. Despite exhaustive attempts, no numerical solutions of the full equations could be found to correspond to the apparent oblique wave group solutions of the nonlinear Schrödinger equation which have envelope zeros.

2. OBLIQUE WAVE GROUPS

The set of equations governing gravity waves in inviscid irrotational motion on the surface of deep water is

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \ , \qquad z < \varepsilon\eta(x,y,t) \ , \tag{2.1a}$$

$$\phi_x, \phi_y, \phi_z \to 0 \ , \quad z \to -\infty \ , \tag{2.1b}$$

$$\eta_t - \phi_z + \varepsilon\eta_x\phi_x + \varepsilon\eta_y\phi_y = 0 \ , \quad z = \varepsilon\eta(x,y,t) \ , \tag{2.1c}$$

$$\eta + \phi_t + \tfrac{1}{2}\varepsilon(\phi_x^2 + \phi_z^2) = 0 \ , \qquad z = \varepsilon\eta(x,y,t) \ . \tag{2.1d}$$

The dimensional variables are the surface displacement $a\eta$, the velocity potential $(g\ell)^{\frac{1}{2}}a\phi$, and $\ell x, \ell y, \ell z$, $(\ell/g)^{\frac{1}{2}}t$, where a is a measure of water wave amplitude, $2\pi\ell$ is a typical wavelength, and $\varepsilon = a/\ell$ is a measure of wave steepness. The origin of coordinates lies in the mean water surface with the z-axis vertically upwards.

The nondimensional equivalent of the simple periodic oblique group with an envelope of permanent shape, equation (1.3), is

$$\eta = \sum_k a_k \cos\{(k/k_0)x\cos\theta + y\sin\theta - \omega t\} \tag{2.2a}$$

$$= \sum_k a_k \cos\{(k-k_0)/k_0\left(x-\frac{1}{2}\cos\theta\, t\right)\cos\theta + x\cos\theta + y\sin\theta - (1+\alpha)t\}, \tag{2.2b}$$

which describes a slowly varying wave train of typical wavelength $2\pi\ell$ propagating at angle θ to the x-direction, whose group envelope propagates with velocity $\frac{1}{2}\cos\theta(g\ell)^{\frac{1}{2}}$ in the x-direction. The dimensional wavenumber components in the x-direction are k/L, where $2\pi L$ is the group length. The central wavenumber component in this direction is k_0/L, such that

$$k_0 = L\cos\theta/\ell \tag{2.3}$$

is the number of wavelengths in one group length in the x-direction (k_0 is not required to be an integer, although k takes only integer values). Since the velocity of the group is chosen to be the linear group velocity on deep water, the nonlinear amplitude dependence is reflected in a non-dimensional frequency contribution α, an unknown function of ε, k_0, and θ.

A complete representation of an oblique periodic group including wavebands centred on jk_0, $j = 0,1,2,...,$ where $j = 1$ denotes the dominant waveband, is given by

$$\eta = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k_2(j)} a_{jk} \cos\left\{(k-jk_0)/k_0\left(x-\frac{1}{2}\cos\theta\, t\right)\cos\theta + j(x\cos\theta + y\sin\theta - (1+\alpha)t)\right\}. \tag{2.4a}$$

The bounds of the summations are determined numerically by trial and error so that the set of amplitudes a_{jk} includes all those amplitudes greater in magnitude than some small prescribed value. Since η is chosen to have a zero mean and the argument is symmetric in k when j = 0, the lower bound $k_1(0)$ may be set equal to 1 without loss of generality. Other lower bounds $k_1(j)$, j > 0, may be negative. The associated solution of Laplace's equation (2.1a) is

$$\phi = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k_2(j)} b_{jk}\, e^{\kappa_{jk} z} \sin\left\{(k - jk_0)/k_0 \left(x - \tfrac{1}{2}\cos\theta\, t\right)\cos\theta + j(x\cos\theta + y\sin\theta - (1+\alpha)t)\right\}, \qquad (2.4b)$$

where $\kappa_{jk} = \{(k\cos\theta/k_0)^2 + (j\sin\theta)^2\}^{\frac{1}{2}}$.

When equations (2.4a,b) are substituted into equations (2.1c,d), with c_{jk} denoting the cosine in equation (2.4a) and s_{jk} the sine in equation (2.4b), the resulting expressions may be written

$$F = \sum_j \sum_k \left\{\left[j\left(1+\alpha-\tfrac{1}{2}\cos^2\theta\right) + \tfrac{1}{2}k\cos^2\theta/k_0\right] a_{jk}s_{jk} - \kappa_{jk}n_{jk}e^{\epsilon\kappa_{jk}\eta} s_{jk}\right\}$$
$$- \epsilon\left(\sum_j \sum_k k\cos\theta/k_0\, a_{jk}s_{jk}\right) \times \left(\sum_j \sum_k k\cos\theta/k_0\, b_{jk}e^{\epsilon\kappa_{jk}\eta} c_{jk}\right)$$
$$- \epsilon\left(\sum_j \sum_k j\sin\theta\, a_{jk}s_{jk}\right) \times \left(\sum_j \sum_k j\sin\theta\, b_{jk}e^{\epsilon\kappa_{jk}\eta} c_{jk}\right) = 0, \qquad (2.5a)$$

$$G = \sum_j \sum_k \left\{a_{jk}c_{jk} - \left[j\left(1+\alpha-\tfrac{1}{2}\cos^2\theta\right) + \tfrac{1}{2}k\cos^2\theta/k_0\right]b_{jk}e^{\epsilon\kappa_{jk}\eta} c_{jk}\right\}$$
$$+ \tfrac{1}{2}\epsilon\left(\sum_j \sum_k k\cos\theta/k_0\, b_{jk}e^{\epsilon\kappa_{jk}\eta} c_{jk}\right)^2 + \tfrac{1}{2}\epsilon\left(\sum_j \sum_k j\sin\theta\, b_{jk}e^{\epsilon\kappa_{jk}\eta} c_{jk}\right)^2$$
$$+ \tfrac{1}{2}\epsilon\left(\sum_j \sum_k \kappa_{jk}b_{jk}e^{\epsilon\kappa_{jk}\eta} s_{jk}\right)^2 = 0, \qquad (2.5b)$$

where $e^{\epsilon\kappa_{jk}\eta} = \exp\left\{\epsilon\kappa_{jk} \sum_p \sum_q a_{pq}c_{pq}\right\}$. If the measure of amplitude, a, is taken to be the water surface displacement $\eta(0,0,0)$, then

$$H = \sum_j \sum_k a_{jk} = 1\,. \qquad (2.5c)$$

Equations (2.5a,b) may be transformed numerically to

$$F = \sum_m \sum_n F_{mn}s_{mn} = 0, \qquad (2.6a)$$

$$G = \sum_m \sum_n G_{mn}c_{mn} = 0, \qquad (2.6b)$$

from which

$$F_{mn} = G_{mn} = 0 \ , \quad \text{all } m,n. \tag{2.7}$$

The Fourier coefficients F_{mn}, G_{mn} are nonlinear functions of a_{jk}, b_{jk}, and α for given ε, k_0, and θ. Equations (2.7) are solved numerically by Newton's method, which for F is given by

$$\sum_j \sum_k \left(\frac{\partial F}{\partial a_{jk}}\right)_{mn} (a_{jk} - a'_{jk}) + \sum_j \sum_k \left(\frac{\partial F}{\partial b_{jk}}\right)_{mn} (b_{jk} - b'_{jk}) + \left(\frac{\partial F}{\partial \alpha}\right)_{mn} (\alpha - \alpha') = F_{mn} \tag{2.8}$$

for all m,n. Each coefficient on the left of equations (2.8) is an m,n Fourier coefficient of a partial derivative of equation (2.5a), and the prime denotes the new value of each variable. The coefficients and the right of equations (2.8) are evaluated at the old values of the variables. There is a similar set of equations derived from G and a single equation derived from H. The complete set of linear equations is solved numerically for $a_{jk} - a'_{jk}$, $b_{jk} - b'_{jk}$, $\alpha - \alpha'$, the new values of the variables are calculated, and the procedure is repeated until the differences are less than some small arbitrary number (10^{-8} in the examples following). This method of solution is the same in principle as that used previously for the calculation of periodic wave group solutions in one horizontal dimension for which the envelope length equals the wavelength (Bryant, 1983).

A useful feature of the method is that the Fourier coefficients F_{mn}, G_{mn} may be found for wavebands m and wavenumbers n outside those included in the calculation. These coefficients then show which wavebands and wavenumbers should be added to the calculation to improve the precision with which equations (2.6a,b) are satisfied over the complete range of x,y, and t. All calculations were performed in double precision on a Prime 750 computer, with subroutines adapted from the Harwell Subroutine Library.

3. RESONANCE

When the wavenumber and frequency of one of the forced wave harmonics in the oblique wave group (equation 2.4a)) satisfy the linear dispersion relation, a resonant peak occurs near this wavenumber in the spectrum of the group in two dimensional wavenumber space. This resonance criterion for deep water gravity waves is

$$j\left(1+\alpha-\frac{1}{2}\cos^2\theta\right)+\frac{1}{2}k\cos^2\theta/k_0 = \{(k\cos\theta/k_0)^2+(j\sin\theta)^2\}^{1/4}. \tag{3.1}$$

The values of k/k_0 at which linear resonance occurs are plotted as a function of θ for the dominant waveband $j = 1$ in figure 1. The frequency correction α makes little contribution to this curve, the points being calculated for oblique groups with $\varepsilon = 0.05$, $k_0 = 10$, when α increases from 0.0006 to 0.0008 over the range of θ shown. Resonances in other wavebands ($j = 0,2,3,\ldots$) do not occur for values of k/k_0 at which harmonics contribute to the oblique wave group structure.

The points on the resonance curve corresponding to integer values of k indicate values of θ near which oblique wave groups are distorted or may not exist because of the resonant generation of one of their harmonics. As θ approaches such a value, the resonating harmonic increases in magnitude relative to the other harmonics, and as the value is crossed the harmonic and the Jacobian in Newton's method change sign. This effect is slight for small values of k/k_0, but as k/k_0 increases towards 1 the effect becomes significant. The curve in figure 1 indicates that, if k_0 is 10, a_6 is resonant at $\theta = 0.553$ and a_7 is resonant at $\theta = 0.592$. The amplitudes of these two harmonics are sketched in figure 2 as functions of θ for a range including resonance. (For convenience, amplitudes a_{1k} in the dominant waveband are denoted by a_k.)

No solutions for oblique wave groups could be found when $0.550 < \theta < 0.555$, where a_6 is resonant. As θ approaches 0.550 from below, the spectral peaks at $(j=1,\ k=6)$ and $(j=1,\ k=k_0=10)$ are of equal magnitude at $\theta = 0.544$, and the resonant peak dominates the central peak for $\theta > 0.544$. The harmonic a_6 at $\theta = 0.550$ has a magnitude 5 times that of the central harmonic a_{10}. The oblique wave group at $\theta = 0.555$, on the other side of resonance, is discussed and sketched in §6.

The harmonic a_7 rises more strongly towards resonance than does a_6, with the spectral peaks at ($j = 1$, $k = 7$) and ($j = 1$, $k = k_0 = 10$) being equal at $\theta = 0.572$. The resonant peak dominates the central peak as θ increases, until at $\theta = 0.596$ the harmonic a_7 has a magnitude 32 times that of the central harmonic a_{10}. No solutions could be found when $0.596 < \theta < 0.598$. Nonlinear modification due to the large amplitude is the probable reason for this discontinuity in a_7 occurring beyond the value of linear resonance deduced from equation (3.1).

Resonances for lower harmonics when k_0 is 10 occur at $\theta = 0.522$ for a_5 and at $\theta = 0.490$ for a_4. The spectral peaks associated with these and other lower resonances decrease in significance as k decreases. Contributions from spectral peaks other than the central peak are insignificant for $\theta < \tan^{-1}(\frac{1}{2}) = 0.464$, where oblique wave group solutions of the nonlinear Schrödinger equation without envelope zeros are all in good agreement with the full solutions at wave steepness $\varepsilon = 0.05$. At larger values of θ up to the critical value $\theta = \tan^{-1}(1/\sqrt{2}) = 0.6155$, the assumption of a single spectral peak, which is essential for the nonlinear Schrödinger equation, is valid for limited intervals of θ only.

4. NONLINEAR SCHRÖDINGER EQUATION

The nonlinear Schrödinger equation describing wave propagation in the X-direction on deep water, with the same dimensional scaling as in §2, is

$$i\left(A_t + \frac{1}{2} A_X\right) - \frac{1}{8} A_{XX} + \frac{1}{4} A_{YY} - \frac{1}{2}\varepsilon^2 |A|^2 A = 0, \tag{4.1}$$

where the nondimensional surface displacement is

$$\eta = \Re\{A(X,Y,t)\exp i(X-t)\}, \tag{4.2}$$

and A is a slowly varying function of X,Y, and t. Oblique wave group solutions of the form described by equations (2.2) require the rotation of axes

$$x = X\cos\theta - Y\sin\theta$$

$$y = X\sin\theta + Y\cos\theta$$

followed by substitution in equation (4.1) of

$$A = F\left(x - \frac{1}{2}\cos\theta\; t\right)e^{-i\alpha t}, \tag{4.3}$$

when F is found to satisfy

$$(\cos^2\theta - 2\sin^2\theta)F'' - 8\alpha F + 4\varepsilon^2 F^3 = 0. \tag{4.4}$$

This is the equation whose analytical solutions are derived by Hui & Hamilton (1979). It should be noted that the envelope of the wave train is given by $|A(X,Y,t)|$, which is a solution of equation (4.1) only if $A(X,Y,t)$ does not pass through zero.

Equation (4.4) is solved numerically by a simplified version of the method described in §2. The Fourier series expansion for F which is consistent with equation (2.2b) is

$$F = \sum_k a_k \exp i\left\{(k-k_0)/k_0\left(x - \frac{1}{2}\cos\theta\; t\right)\cos\theta\right\}, \tag{4.5}$$

where a_k is symmetric about $k = k_0$. When equation (4.5) is substituted into equation (4.4), the result may be transformed numerically to

$$G = \sum_m G_m \exp i\left\{(m-k_0)/k_0\left(x - \frac{1}{2}\cos\theta\; t\right)\cos\theta\right\} = 0, \tag{4.6}$$

where G_m is symmetric about $m = k_0$, and is a nonlinear function of a_k and α for given ε, k_0, and θ. The Fourier transforms of the partial derivatives $(\partial G/\partial a_k)_m$, $(\partial G/\partial\alpha)_m$ are calculated, and Newton's method based on equations similar to equations (2.8) is used with the same procedure as was described in §2. All the types of periodic solution derived by Hui & Hamilton (1979) can be calculated by this method.

The set of amplitudes a_k calculated from the nonlinear Schrödinger equation is equivalent to the set of surface displacement amplitudes a_{1k} in the general description (equation (2.4a)) of periodic oblique wave groups. The amplitudes a_k, with suitably chosen velocity potential amplitudes b_{1k}, can be used therefore as starting values for the calculation of general solutions for η and ϕ. Hui & Hamilton (1979, figure 2) found two forms of periodic solution when $\theta < \tan^{-1}(1/\sqrt{2})$. The dn solution has no zero crossings, and when used to provide starting values it converged to a general solution of the same form except for distortion in some examples due to the resonances described in §3. The cn solution has zero crossings, and when used to provide starting values it either diverged numerically or converged to a dn form of solution with half the group length. Exhaustive attempts to find any form of general solution corresponding to the cn solution of the nonlinear Schrödinger equation were all unsuccessful.

The only periodic oblique wave group solution found by Hui & Hamilton (1979, figure 3) when $\theta > \tan^{-1}(1/\sqrt{2})$ is of sn form with zero crossings. No general solutions could be calculated from this form either. The numerical evidence is that only those solutions $A(X,Y,t)$ of the nonlinear Schrödinger equation (4.1) which do not have zero crossings are applicable to water waves. This implies that there are no applicable solutions when $\theta > \tan^{-1}(1/\sqrt{2})$.

5. CRITICAL ANGLE

The boundary wedge of the Kelvin ship-wave pattern is a caustic with oblique wave group structure, according to the linear theory, for which the wave crest to group crest angle is the critical angle $\tan^{-1}(1/\sqrt{2})$. The water surface displacement on the boundary wedge dominates asymptotically the surface displacement within the wedge. The linear theory is summarised by Hui & Hamilton (1979, §4) and is illustrated by Lighthill (1978, figures 70, 71).

The numerical solutions exhibit an interesting behaviour as θ increases through the critical value. As θ approaches this value from below, the central wavenumber in the dominant waveband $(j = 1)$ increases from the value k_0 set by the scaling. Apparent numerical solutions exist for θ greater than the critical value, but because the central wavenumber exceeds k_0 by a significant ratio, the solutions need rescaling to interpret them correctly. If $2\pi\ell'$ denotes the correct typical wavelength of the wave train, and θ' is the correct angle it makes with the x-direction, the velocity of the group envelope is

$$\frac{1}{2}(g\ell)^{\frac{1}{2}}\cos\theta = \frac{1}{2}(g\ell')^{\frac{1}{2}}\cos\theta' \ , \tag{5.1}$$

and the wavelength in the y-direction is

$$2\pi\ell/\sin\theta = 2\pi\ell'/\sin\theta' \ . \tag{5.2}$$

Equation (2.3) becomes

$$k_0' = L\cos\theta'/\ell' \ ,$$

giving the correct number of wavelengths per group length in the x-direction. When the ratio ℓ'/ℓ is eliminated between equations (5.1) and (5.2), and the solution $\theta' = \theta$ discarded, θ and θ' are related by

$$\sin^2\theta' + \sin\theta'\sin\theta + \sin^2\theta = 1. \tag{5.3}$$

Knowing that the critical angle is $\sin^{-1}(1/\sqrt{3})$, equation (5.3) may be rewritten

$$(1-3\sin^2\theta')(3\sin^2\theta - 1) = (3\sin\theta\sin\theta' + 2)(\sin\theta - \sin\theta')^2 \ . \tag{5.4}$$

This equation shows that if an apparent solution exists for $\sin\theta > 1/\sqrt{3}$, the correct angle θ' made by the wave crests with the group crests satisfies $\sin\theta' < 1/\sqrt{3}$. The solution at the critical angle $\sin^{-1}(1/\sqrt{3})$ rescales to itself. No numerical solutions have been found, using step by step changes in θ through the critical angle, for which the correct angle between the wave crests and the group crests exceeds the critical angle.

The oblique wave group solution when θ is set equal to the critical angle with $\varepsilon = 0.05$, $k_0 = 10$, is sketched in §6, and the harmonics with magnitudes exceeding 10^{-5} are tabulated in the Appendix. The spectrum of the group has two equal peaks at $(j = 1, k = 8)$ and $(j = 1, k = 11)$. The perspective sketch of the group in figure 6 shows 8 wavelengths per group length in the x-direction, consistent with the resonance associated with the first peak (equation 3.1). The central peak displays a shift in central wavenumber as the numerical solution avoids a group whose correct angle between wave crests and group crests is the critical angle. The existence of two equal peaks in the spectrum confirms that the non-linear Schrödinger equation is not a valid model equation for oblique wave groups near the critical angle.

It is not clear from calculations with θ set equal to the critical value whether the wave height outside the central group structure tends towards zero as k_0 is increased, as would be expected if the oblique wave group is to tend towards a solitary oblique wave group structure. The central group structure in figure 6 contains 3 of the 8 wavelengths in one group length, and the waves outside the central structure have a height which is almost constant at 0.23 of the maximum wave height. Linear resonance at the critical angle occurs when $k/k_0 = 0.739$ (Equation 3.1). The harmonics a_k for which k lies near this value have larger amplitudes than other harmonics at all calculated values of k_0, and contribute to the wave motion both inside and outside the central group structure. It appears that the boundary wedge of the Kelvin ship-wave pattern does not correspond to oblique wave groups in the solitary group limit, except in a qualitative sense, but should be calculated as a direct solution of equations (2.1).

6. EXAMPLES

Three representative examples of periodic oblique wave groups are presented, all with $\varepsilon = 0.05$, $k_0 = 10$. The first example, at $\theta = 0.5$, with $\alpha = 0.00058$, lies between the resonant angles for a_4 and a_5 (from figure 1). Neither harmonic contributes significantly to the wave group, which is evident in figure 3(a) where the envelope for the full solution coincides with the envelope calculated from the nonlinear Schrödinger equation. A perspective view of the oblique wave group at an instant is sketched in figure 4. This example is typical of those which may be calculated either from the full equations or from the nonlinear Schrödinger equation. The full solution contains 115 harmonics (231 variables) in 7 wavebands $0 \leqslant j \leqslant 6$, the wavenumber range being $1 \leqslant k \leqslant 64$. The maximum Fourier coefficient F_{mn}, G_{mn} not included in the calculation has magnitude 1×10^{-6}. The maximum magnitude of F and G over the 256×16 points used in the final calculation is 2.6×10^{-4} with a root mean square deviation of F and G from zero of 5.5×10^{-5}. (A computer listing of the harmonics for the three examples may be obtained from the author).

The second example, at $\theta = 0.555$, with $\alpha = 0.00082$, lies immediately above the resonant angle for a_6 (figure 2). The envelope for the full solution in figure 3(b) differs from the envelope calculated from the nonlinear Schrödinger equation because of the nearness of a_6 to resonance. A perspective view of the oblique wave group at an instant is sketched in figure 5. The presence of two significant peaks in the spectrum of the water surface displacement causes the wave crests to be less regular than in the first example. This solution contains 140 harmonics (281 variables) in 7 wavebands $0 \leqslant j \leqslant 6$, the wavenumber range being $-1 \leqslant k \leqslant 65$. The maximum Fourier coefficient F_{mn}, G_{mn} not included has magnitude 1×10^{-6}, the maximum magnitude of F and G over the 256×16 points used in the final calculation is 3.1×10^{-4}, with a root mean square deviation of F and G from zero of 6.6×10^{-5}.

The third example, at $\theta = \tan^{-1}(1/\sqrt{2}) = 0.6155$, with $\alpha = 0.00081$, is the oblique wave group that results when the wave crest to group crest angle is set equal to the critical angle in the calculation of the full solution. There is no equivalent solution to the nonlinear Schrödinger equation at this angle. The envelope of the wave group is drawn in figure 3(c), and a perspective view of the wave group at an instant is sketched in figure 6. The full solution contains 145 harmonics (291 variables) in 7 wavebands $0 \leqslant j \leqslant 6$, the wavenumber range being $1 \leqslant k \leqslant 67$. The maximum Fourier coefficient F_{mn}, G_{mn} not included has magnitude 1×10^{-6}, the maximum magnitude of F and G over the 256×16 points used in the final calculation is 3.0×10^{-4}, with a root mean square deviation of F and G from zero of 6.0×10^{-5}. The harmonics with magnitudes exceeding 10^{-5} are listed in the Appendix.

7. DISCUSSION

The calculations of oblique wave groups reveal serious deficiencies in the applicability of the nonlinear Schrödinger equation to water wave motion in two horizontal dimensions. The fault lies with the assumption in this equation that the water wave spectrum has a narrow peak in wave-number space. The possibility exists in two horizontal dimensions that the resonant forcing of certain wave harmonics generates a second or further spectral peaks, a property demonstrated originally by Phillips (1960). Solutions calculated from the nonlinear Schrödinger equation which are to be applied to water wave motion must be tested to determine whether any of the significant harmonics in the solution satisfy the linear dispersion relation. If there are harmonics with this property, such a solution is not applicable.

It has been assumed that solutions of the nonlinear Schrödinger equation which pass through zero are applicable to water waves (Hui &

Hamilton (1979), Peregrine (1983)). The numerical evidence from the present calculations of oblique wave groups suggests that this assumption is not valid, probably because the envelope is described by $|A|$ not by A in equation (4.2). The numerical calculations indicate not only that solutions with envelope zeros are not applicable, but neither do new applicable solutions exist with geometries in the neighbourhood of them.

On the positive side, this investigation has demonstrated a straightforward method for the numerical calculation of solutions of the equations for irrotational gravity wave motion in deep water. The method has generalisations to water of finite depth, to short waves with significant surface tension, and to other forms of water wave motion. Properties may be found for the full nonlinear theory without needing recourse to model nonlinear equations such as the Schrödinger equation or the Zakharov equation.

APPENDIX

Table of harmonics with magnitudes exceeding 10^{-5} for the oblique wave group example whose angle is set equal to the critical angle:

a_{jk}

$j=0$, $k=$ 1 to 9

-0.00019	-0.00030	-0.00031	-0.00022	-0.00011	-0.00005	-0.00002
-0.00001	-0.00000					

$j=1$, $k=$ 4 to 23

-0.00018	-0.00155	-0.00974	-0.09222	0.20724	0.16395	0.19112
0.20014	0.15273	0.08715	0.04269	0.01941	0.00840	0.00350
0.00141	0.00056	0.00022	0.00008	0.00003	0.00001	

$j=2$, $k=$ 13 to 32

0.00002	0.00008	-0.00088	0.00020	0.00066	0.00157	0.00280
0.00370	0.00397	0.00374	0.00315	0.00237	0.00161	0.00100
0.00058	0.00032	0.00017	0.00008	0.00004	0.00002	

$j=3$, $k=$ 27 to 40

0.00002	0.00004	0.00007	0.00010	0.00012	0.00013	0.00012
0.00011	0.00009	0.00007	0.00005	0.00003	0.00002	0.00001

b_{jk}

j=0, k= 1 to 9

-0.00579	-0.00487	-0.00344	-0.00187	-0.00077	-0.00028	-0.00010
-0.00003	-0.00001					

j=1, k= 4 to 23

-0.00022	-0.00184	-0.01121	-0.10233	0.22193	0.16957	0.19109
0.19365	0.14318	0.07926	0.03770	0.01667	0.00703	0.00285
0.00112	0.00043	0.00016	0.00006	0.00002	0.00001	

j=2, k= 13 to 32

0.00000	0.00000	0.00001	0.00001	0.00002	0.00002	0.00001
-0.00001	-0.00002	-0.00002	-0.00001	-0.00001	-0.00001	-0.00000
-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	-0.00000	

j=3, k= 27 to 40

0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

The complete listing of the harmonics for this and the other examples may be obtained from the author.

REFERENCES

Bryant, P.J. 1983 Cyclic gravity waves in deep water. To appear in *J. Aust. Math. Soc(B)*, 25 (Waves and Fluids issue).

Hui, W.H. & Hamilton, J. 1979 Exact solutions of a three-dimensional nonlinear Schrödinger equation applied to gravity waves. *J. Fluid Mech.* 93, 117-133.

Lighthill, James, 1978 *Waves in Fluids*. C.U.P.

McLean, J.W., Ma, Y.C., Martin, D.U., Saffman, P.G. & Yuen, H.C. 1981 Three-dimensional instability of finite-amplitude water waves. *Phys. Rev. Lett.* 46, 817-820.

Peregrine, D.H. 1983 Water waves, nonlinear Schrödinger equations and their solutions. To appear in *J. Aust. Math. Soc(B)*, 25 (Waves and Fluids issue).

Phillips, O.M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part 1. *J. Fluid Mech.* 9, 193-217.

Zakharov, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zh. Prikl. Mekh. Tekh. Fiz.* 9, 86-94 (*J. Appl. Mech. Tech. Phys.* 2, 190-4).

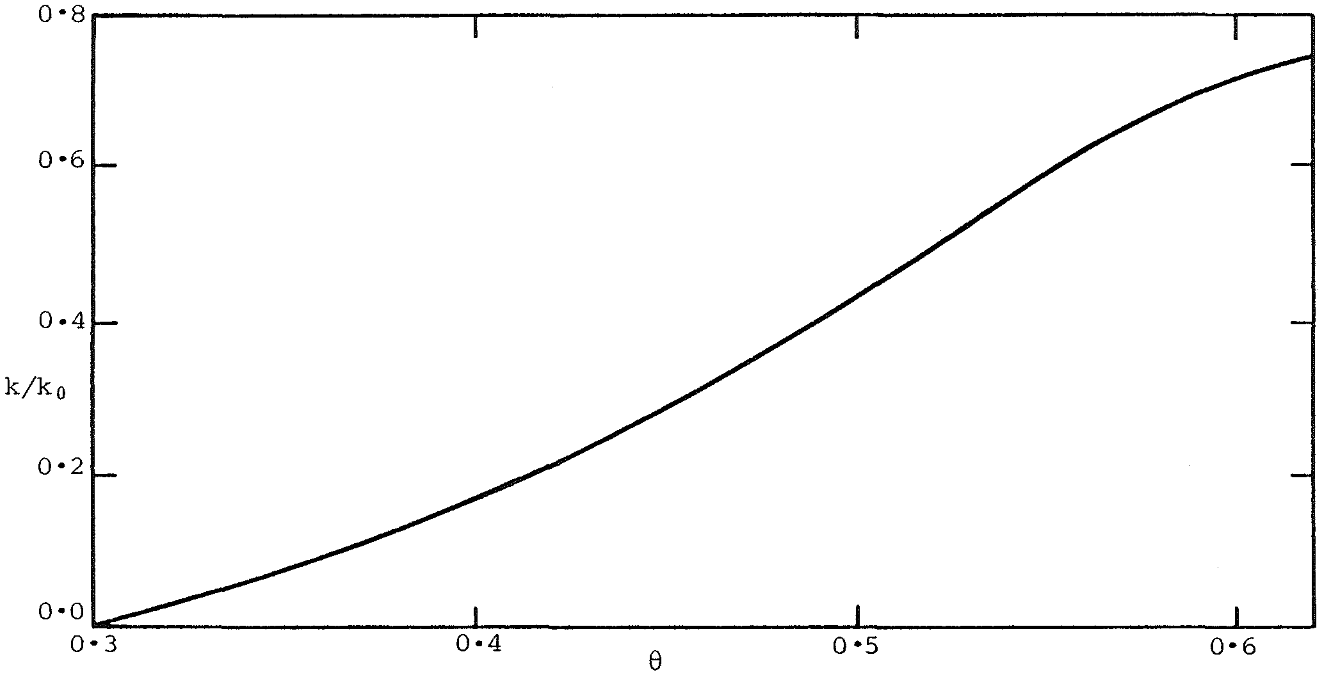

Figure 1   Wavenumber ratio for resonance in the dominant waveband.

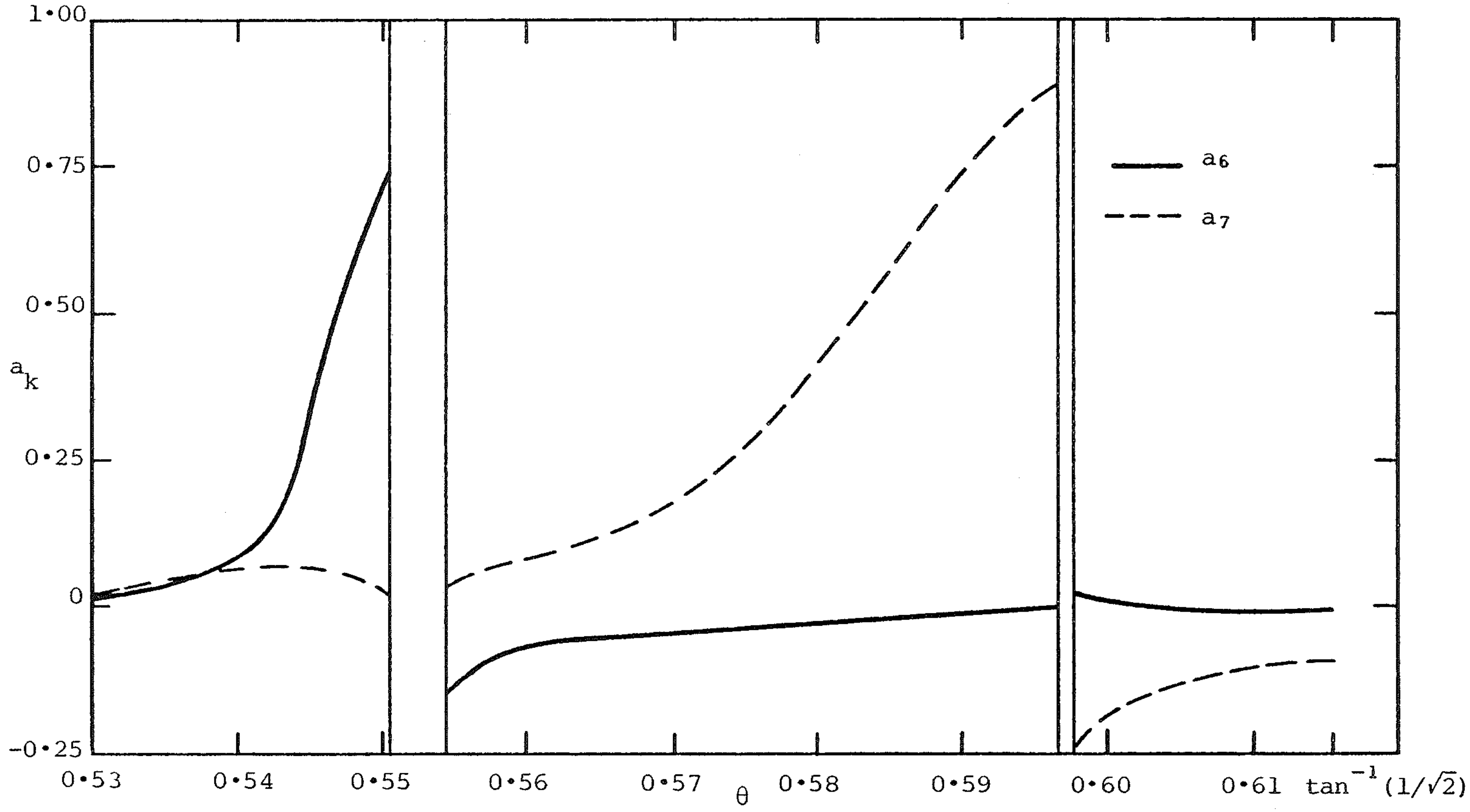

Figure 2 Resonant behaviour of oblique wave group harmonics a_6, a_7 when $\varepsilon = 0{\cdot}05$, $k_0 = 10$.

(a) $\theta = 0{\cdot}5$

(b) $\theta = 0{\cdot}555$

(c) $\theta = \tan^{-1}(1/\sqrt{2})$

Figure 3 Envelopes of the oblique wave group examples, horizontal contraction $2{\cdot}5\pi$. The solid curves are the full solutions, and the dashed curves are solutions of the non-linear Schrödinger equation.

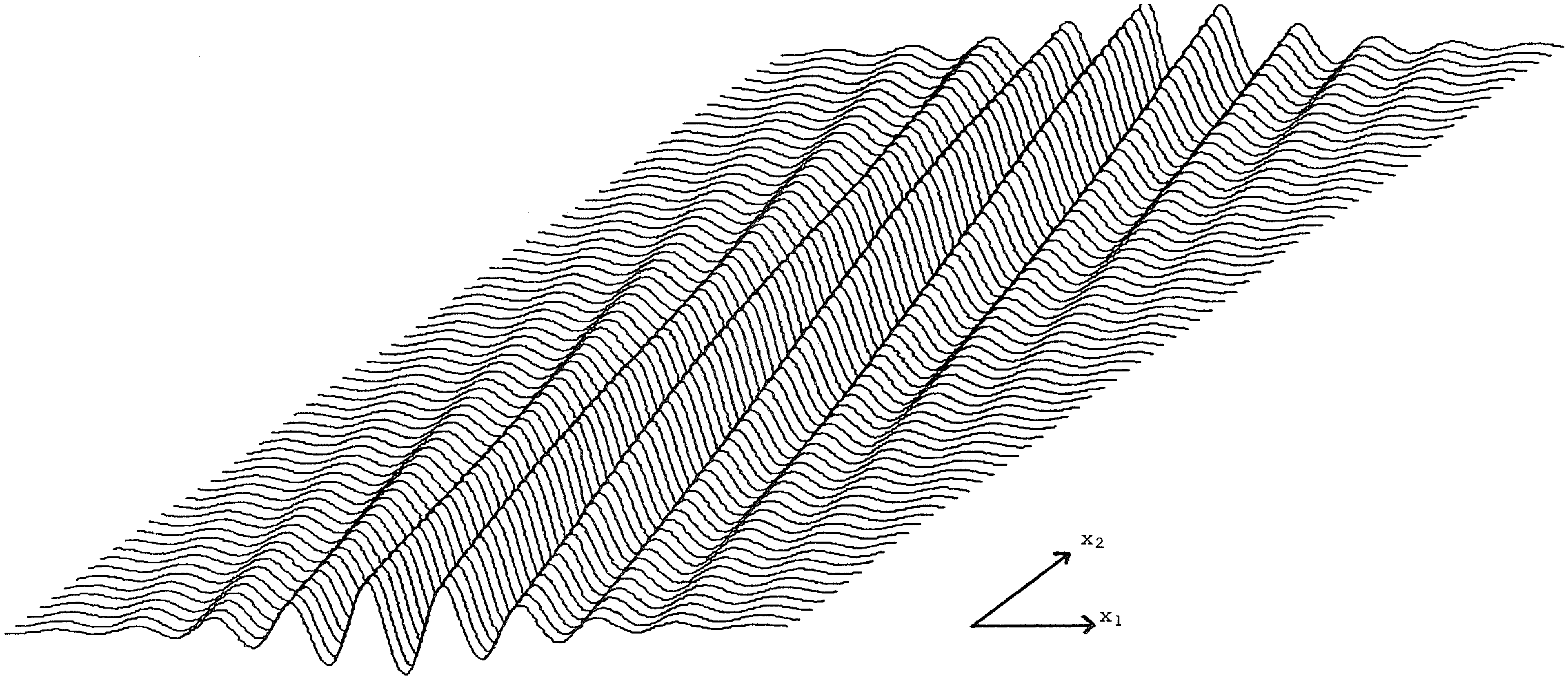

Figure 4 Perspective view of one group length,

horizontal contraction $2 \cdot 5\pi$, when $\varepsilon = 0 \cdot 05$, $k_0 = 10$, $\theta = 0 \cdot 5$.

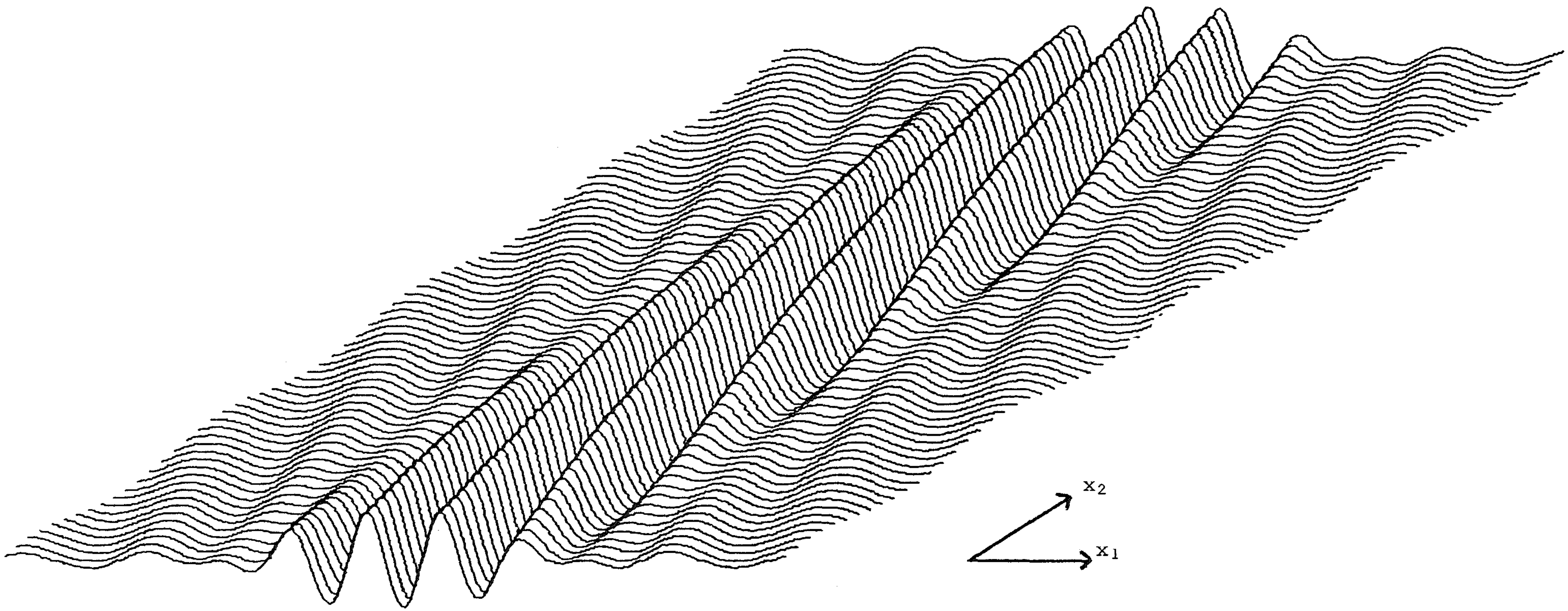

Figure 5 Perspective view of one group length, horizontal contraction $2 \cdot 5\pi$, when $\varepsilon = 0 \cdot 05$, $k_0 = 10$, $\theta = 0 \cdot 555$.

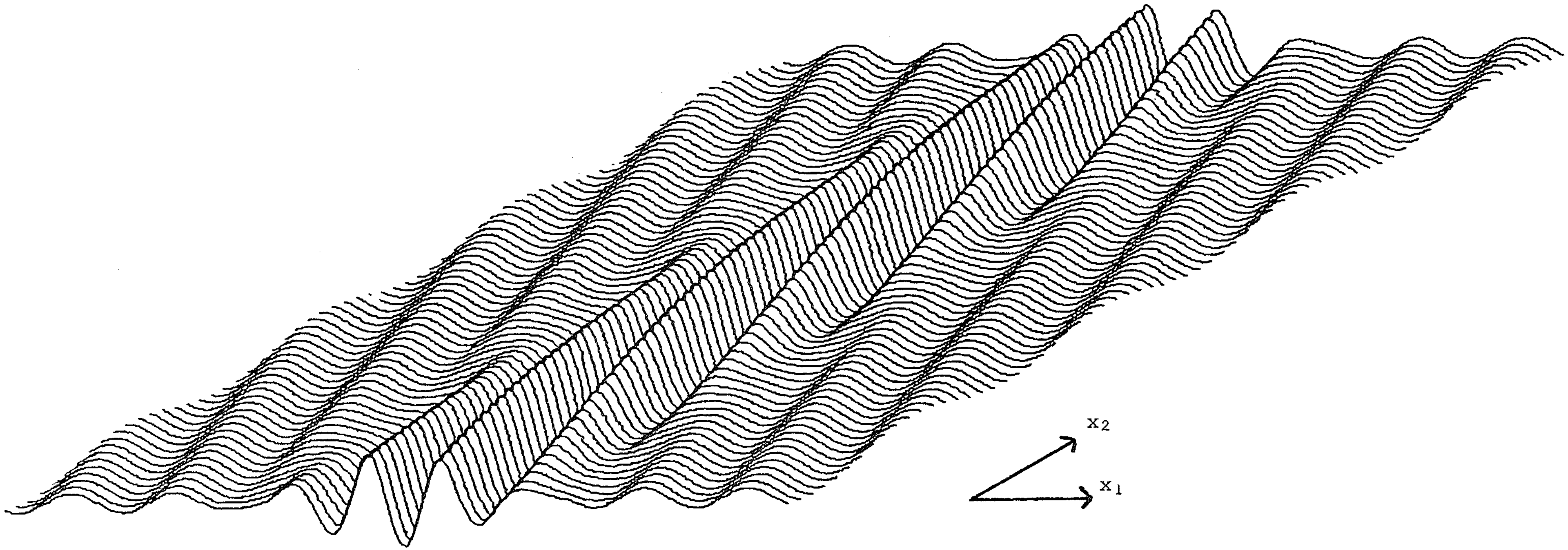

Figure 6   Perspective view of one group length,

horizontal contraction $2{\cdot}5\pi$, when $\varepsilon = 0{\cdot}05$, $k_0 = 10$, $\theta = \tan^{-1}(1/\sqrt{2})$.