Generalized Quadrangles and Projective Axes of Symmetry *

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Abstract

We investigate generalized quadrangles Γ that admit at least two projective axes of symmetry. We show that if there are three such axes incident with a common point x, then x is a translation point of Γ. In case that Γ is moreover a compact connected quadrangle with topological parameters (p,p), $p \in \mathbb{N}$, then Γ is a topological translation generalized quadrangle. We further investigate the case of two opposite projective axes of symmetry and obtain a characterization of the dual of the symplectic quadrangle over $\mathbb{R}$ or $\mathbb{C}$ among compact connected quadrangles with equal topological parameters.

1 Introduction

Symmetries about a line L of a generalized quadrangle Γ, that is, collineations of Γ that fix each line of Γ meeting L, play a prominent role in the investigation of generalized quadrangles and other geometries and the structure of a quadrangle is well understood if it admits sufficiently many symmetries. An axis of symmetry of a generalized quadrangle Γ is a line L of Γ for which the group of all symmetries with axis L acts regularly on the set of points of any line meeting L minus its common point with L. If all lines of a generalized quadrangle are axes of symmetry, then the quadrangle is Moufang, In translation generalized quadrangles with translation point x every line trough x is an axis of symmetry.

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A projective line in a generalized quadrangle Γ is a regular line L for which the associated dual net Γ_L^* is a dual affine plane (see below for precise definitions). In the finite and topological case, projectivity of a regular line is equivalent with equality of the parameters. Hence, the assumption of projectivity of an axis of symmetry in the general case is a way to study infinite quadrangles which behave roughly like finite quadrangles of order (s,s), or topological quadrangles with topological parameters (p,p). It is our intention to show that we can thus generalize results from the finite case, obtain new results for the topological case, via intermediate but weaker results in the general case.

In this paper we investigate generalized quadrangles Γ that admit at least two projective axes of symmetry. Two axes of symmetry either have a point in common or are opposite. Since there are plentiful examples of generalized quadrangles that are not translation generalized quadrangles but admit one or two concurrent axes of symmetry we assume that in the case of concurrent axes of symmetry there are at least three such lines. We show that, in fact, it suffices to require that there are three concurrent projective axes of symmetry in order to obtain a translation generalized quadrangle. For topological quadrangles the assumption on projectivity of the axes involved can be replaced by the topological condition that the two topological parameters of the compact quadrangle are equal. Using the correspondence between antiregular compact connected generalized quadrangles and 2- and 4-dimensional Laguerre planes we translate this result into one for 2- and 4-dimensional Laguerre planes, thus strengthening a description of one of the possible types in Ruth Kleinewillinghöfer's classification of Laguerre planes with respect to G-translations.

In the finite case, the corresponding results can be found in [26].

The existence of two opposite projective axes of symmetry turns out to be rather more restrictive. Under certain additional natural assumptions one only obtains duals of symplectic quadrangles over fields of characteristic not equal to 2. In the topological situation the additional assumptions made in the general case are guaranteed. This allows us to characterize the duals of the symplectic quadrangles over $\mathbb{R}$ or $\mathbb{C}$ among all compact connected quadrangles with equal topological parameters by the existence of two opposite projective axes of symmetry. This generalizes and extends a result of Koen Thas [25] in the finite case.

In section 2 we review basic definitions of abstract generalized quadrangles and associated Laguerre and Minkowski planes in a topological context. The next section deals with three concurrent projective axes of symmetry and in section 4 we investigate the case of two opposite projective axes of symmetry.

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2 Preliminaries

2.1 Abstract Generalized Quadrangles

A *generalized quadrangle* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consists of a point set $\mathcal{P}$, a line set $\mathcal{L}$ (disjoint from $\mathcal{P}$) and a symmetric incidence relation $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$ such that

(PL) no pair of points is incident with a pair of lines and every element is incident with at least two elements;

(GQ) for every point x and every line L not incident with x, there exists a unique point y and a unique line M such that $x\mathtt{I}M\mathtt{I}y\mathtt{I}L$.

If every element is incident with at least three elements, then the generalized quadrangle is called *thick*. As a consequence, any two lines have the same cardinality and dually there is a constant number of lines through a point. In the finite case we say that Γ has order (s, t) if there are $s + 1$ points on each line and $t + 1$ lines through a point. Note that we adopt common linguistic expressions such as *points lie on a line, lines go through points* to describe incidence. We will also use the notions of *collinear points* and *concurrent lines* for points that are incident with a common line and lines that go through a common point, respectively. Non-collinear points and non-concurrent lines will also be called *opposite*. If $x \in \mathcal{P} \cup \mathcal{L}$ is collinear or concurrent with $y \in \mathcal{P} \cup \mathcal{L}$, then we write $x \sim y$. An incident point-line pair is called a *flag*.

Note that, if $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized quadrangle, then also $(\mathcal{L}, \mathcal{P}, \mathtt{I})$ is a generalized quadrangle. We will denote the latter by Γ^D and call it the *dual* of Γ. The *duality principle* states that in every definition and statement, one may interchange the words 'point' and 'line' to obtain a new definition or statement.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized quadrangle and let x be an arbitrary point. The set of points of Γ collinear with x will be denoted by $x^\perp$. For a set $X \subseteq \mathcal{P}$, we denote by $X^\perp$ the set of points collinear to all points of X, and we abbreviate $(X^\perp)^\perp$ by $X^{\perp\perp}$. If $x \neq y$ are collinear points, then $\{x, y\}^\perp$ is just the point row of the line through x and y, which we will also denote by xy. If y is a point opposite x, then $\{x, y\}^\perp$ is called the *trace* of the pair (x, y). The *span* of the pair (x, y) is the set $\{x, y\}^{\perp\perp}$. If every span containing x is also a trace (of a different pair of points, needless to say), then the point x is called *regular.* Dually one defines *regular lines.* If x is a regular point, then the geometry $\Gamma_x^* = (x^\perp \setminus \{x\}, \{\{x,y\}^\perp : y \not\sim x\}, \in \text{ or } \ni)$ is a *dual net* (associated to x), i.e., it has the property that for every point $z \in x^\perp \setminus \{x\}$ and every *block* $B = \{x,y\}^\perp$, with y opposite x not containing z, there is a unique point $z' \in B$ not collinear with z (collinearity in Γ_x^*). If Γ_x^* is a dual affine plane (that is, a projective plane with one point deleted), then we call x a *projective point.* The motivation for this terminology is that the geometry $\Gamma_x = (x^\perp, \{\{x,y\}^\perp : y \in \mathcal{P}\}, \in \text{ or } \ni)$ is then a projective plane, called the *perp-plane* in x. Projective points have nice properties. For instance, one can easily check that x is a projective point if and only if the geometry $(\mathcal{P} \setminus x^\perp, (\mathcal{L} \setminus \{xy : y \in x^\perp, y \neq x\}) \cup \{\{x,y\}^{\perp\perp} : y \not\sim x\}, \mathtt{I} \text{ or } \in \text{ or } \ni)$ is a generalized quadrangle if and only if every pair of distinct traces contained in $x^\perp$ meet in a unique point.

Projective points can also be approached with triads. A *triad* is a triplet of pairwise opposite points. A centre of a triad $\{x,y,z\}$ is an element of $\{x,y,z\}^\perp$. A triad is called *(uni)centric* if it has a (unique) centre. Now, a regular point x is projective if and only if every triad containing x is centric. If every triad is centric, then the quadrangle ic called centric.

A point x of a generalized quadrangle Γ is called *antiregular*, if the centre of every triad containing x either is empty or contains precisely two points. A point p is called *strongly antiregular* if each triad containing p or contained in $p^\perp \setminus \{p\}$ is antiregular. Finally, a generalized quadrangle is *antiregular*, if every point is antiregular, that is, $|T^\perp| \in \{0,2\}$ for every triad. If x is a strongly antiregular point of Γ, then the geometry $(x^\perp \setminus \{x\}, \{x,y\}^\perp : y \not\sim x\}, \sim)$ is a Laguerre plane; see [5], Theorem 3.1. More generally, a *Laguerre plane* $\Delta = (P, \mathcal{C}, \|)$ is a geometry consisting of a set P of points, a set $\mathcal{C}$ of at least two circles and an equivalence relation $\|$ on P, called *parallelism*, such that three mutually non-parallel points can be uniquely joined by a circle (joining); such that to every circle $C \in \mathcal{C}$ and any two non-parallel points p, q, where $p \in C$ and $q \notin C$, there is precisely one circle L passing through q which touches C at p, i.e., $C \cap D = \{p\}$ (touching); such that parallel classes and circles intersect in a unique point (parallel projection); and such that each circle contains at least three points (richness); cf. [2, 14]. In fact, if Δ satisfies the so-called oval tangent condition at infinity, then its Lie geometry is a thick generalized quadrangle that admits a strongly antiregular point, see Theorem 3.4 in [20] and the remark following that Theorem.

Finally we introduce some notions concerning symmetry in generalized quadrangles. In general, a *collineation* of a generalized quadrangle is a permutation of the points and of the lines preserving the incidence relation. A point x of a generalized quadrangle is called a *centre of symmetry* if it is regular and if the group of collineations fixing $x^\perp$ pointwise acts transitively on the set $\{x, y\}^{\perp\perp} \setminus \{x\}$, for some, and hence for every, point y opposite x. The dual notion is called an *axis of symmetry*.

2.2 The Symplectic Quadrangles

The prototype class of examples of generalized quadrangles is the class of *symplectic quadrangles*, which are defined as follows. Let ρ be a symplectic polarity in a 3-dimensional projective space $\mathsf{PG}(3, \mathbb{K})$ over a field $\mathbb{K}$. If $\mathcal{P}$ is the point set of $\mathsf{PG}(3, \mathbb{K})$, if $\mathcal{L}$ is the set of lines of $\mathsf{PG}(3, \mathbb{K})$ fixed under ρ, and if $\mathtt{I}$ denotes the incidence relation in $\mathsf{PG}(3, \mathbb{K})$, then $\mathsf{W}(\mathbb{K}) = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized quadrangle, called the *symplectic quadrangle (over* $\mathbb{K}$*)*. All the points of $\mathsf{W}(\mathbb{K})$ are regular, even projective.

If the characteristic of $\mathbb{K}$ is not equal to 2, then all lines of $\mathsf{W}(\mathbb{K})$ are antiregular.

The symplectic quadrangle has a lot of symmetry. All points of $\mathsf{W}(\mathbb{K})$ are centres of symmetry.

2.3 Topology

In the symplectic quadrangles over $\mathbb{R}$ or $\mathbb{C}$, the set of points and the set of lines carry topologies that are induced from the surrounding 3-dimensional real or complex (topological) projective space. Since the operations of joining two points of the projective space by a line and intersecting a plane by a line are continuous, one obtains the continuity of the geometric operations in the generalized quadrangle. that is, one has a topological generalized quadrangle in the following sense. A *topological generalized quadrangle* is a (thick) generalized quadrangle where the point set and the set of lines carry Hausdorff topologies such that the mapping that takes an anti-flag (x, L) to the unique flag (y, M) where $x \in M$ and $y \in L$ becomes continuous.

One requires 'good' topologies in order to obtain better results on the geometry of a generalized quadrangle. A topological generalized quadrangle is called (locally) compact, connected, or finite-dimensional, if the point space has the respective topological property where the dimension refers to the topological (covering) dimension of a space. A compact connected finite-dimensional generalized quadrangle Γ has topological parameters (p, q) if each point row is p-dimensional and each line pencil is q-dimensional. In this case, each point row is homotopy equivalent to a p-sphere $\mathbb{S}_p$ and each line pencil is homotopy equivalent to a q-sphere $\mathbb{S}_q$. The point space, the line space, and the flag space are generalized manifolds of dimension $2p+q$, $p+2q$ and $2(p+q)$, respectively. Furthermore, if $p, q > 1$, then $p+q$ is odd or $p = q \in \{2, 4\}$; compare [8], [9] or [32], section 9.5 and the references given there. The group of all continuous automorphisms of Γ, endowed with the compact-open topology, is a locally compact, second countable topological transformation group and a Lie group if $p = q < 4$.

In a similar fashion, a *topological Laguerre plane* is a Laguerre plane where the set of points and the set of circles carry non-indiscrete topologies (hence there are proper non-empty open subsets) such that the geometric operations of joining, touching, parallel projection, and intersecting distinct circles are continuous on their domains of definition, cf. [2, 14]. A topological Laguerre plane is called (locally) compact, connected, or finite-dimensional, if the point space has the respective topological property. For brevity, an n-dimensional Laguerre plane is a locally compact topological Laguerre plane whose point space is n-dimensional. Note that such a plane is connected if n is positive. A connected finite-dimensional Laguerre plane is of dimension 2 or 4; see Rainer Löwen [12], 2.3. Circles in a $2n$-dimensional Laguerre plane, $n = 1, 2$, are homeomorphic to $\mathbb{S}_n$. There are no disjoint circles in 4-dimensional planes.

For every $2n$-dimensional Laguerre plane Δ, $n = 1, 2$, the associated Lie geometry (whose points are the points and circles of Δ plus one additional point at infinity, denoted by ∞, and whose lines are the extended parallel classes, that is, the parallel classes of Δ to which the point ∞ is added, and the extended tangent pencils, that is, the collections of all circles that touch a given circle at a point p together with p and incidence being the natural one) with respect to a suitable topology on the point set and the topology on the line set induced by the Hausdorff metric is a compact antiregular topological generalized quadrangle with topological parameters (n, n). Conversely, each derivation of a compact topological antiregular generalized quadrangle with topological parameters (n, n) is a $2n$-dimensional Laguerre plane. A remarkable result of Andreas Schroth [20] even shows that every compact topological generalized quadrangle with topological parameters (n, n) can be constructed from a $2n$-dimensional Laguerre plane: either the derivation at every point of the quadrangle, or the derivation at every point of the dual quadrangle, yields a $2n$-dimensional Laguerre plane. Equivalently, every compact topological generalized quadrangle with topological parameters (n, n) is antiregular up to duality.

2.4 Moufang Sets and 2-transitive Groups

Let X be a set and let $(U_x)_{x \in X}$ be a family of permutation groups acting on X such that U_x fixes x, acts sharply transitively on $X \setminus \{x\}$ and permutes the U_y, $y \in X$, by conjugation. Then $(X, (U_x)_{x\in X})$ is called a *Moufang set* and the groups U_x are called *root groups* of the Moufang set. We will encounter Moufang sets in the following way. Let G be a permutation group acting on a set X and suppose the stabilizer G_x of $x \in X$ has a normal subgroup U_x acting sharply transitively on $X \setminus \{x\}$ such that, for $x, y \in X$, U_x and U_y are conjugate in G. Then $(X, (U_x)_{x\in X})$ is a Moufang set. The group generated by all U_x is called the *little projective group* of the Moufang set.

It is easy to see that the little projective group of any Moufang set is a 2-transitive group. In case of Lie groups, there is an explicit classification of 2-transitive Lie groups, originally due to Jacques Tits [27], [28]. A more recent proof by Linus Kramer [10] also includes the determination of the Moufang sets. The next result follows directly from Theorem 3.3 and Lemmas 7.1 and 7.2 of [10].

Lemma 2.1 *If G is the little projective group of a Moufang set and a connected Lie group acting on a sphere of dimension 1, 2 or 4, then $G \equiv \mathsf{PSL}_2(\mathbb{K})$, with $\mathbb{K}$ one of the skew fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The dimensions of the groups are 3, 6 and 15, respectively.*

The dimensions of the groups are listed in Section 94.33 of [17].

3 Generalized Quadrangles with three Concurrent Projective Axes of Symmetry

We begin with some general assumptions and notations which we will keep through this section. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized quadrangle having three concurrent *projective* axes of symmetry L_1, L_2, L_3, all passing through the point x. Let G_i be the symmetry group corresponding with L_i, $i = 1, 2, 3$. Then $[G_i, G_j]$ is trivial, for $i, j \in \{1, 2, 3\}$, $i \neq j$. Hence the product $G := G_1G_2G_3$ is a group of collineations of Γ. Notice that the product map $G_1 \times G_2 \times G_3 \longrightarrow G : (g_1, g_2, g_3) \mapsto g_1g_2g_3$ is a surjective homomorphism of groups since $G_i \cap G_j = \{\mathrm{id}\}$, for $i, j \in \{1, 2, 3\}$, $i \neq j$.

Lemma 3.1 *The group G acts sharply transitively on the set $\mathcal{P} \setminus x^\perp$ of points opposite x.*

Proof. Let z and z' be two arbitrary points opposite x. Then there is a unique subquadrangle Γ' with parameters $(p, 0)$ containing L_1, L_2 and z (denoting the unique line through

z concurrent with L_1 by M_1, Γ' is defined by the line set $\{L_2, M_1\}^\perp \cup \{L_2, M_1\}^{\perp\perp}$). Denote by M_3 the unique line through z' concurrent with L_3. Since L_1 is a projective line, there is a line in the intersection $\{L_1, M_3\}^\perp \cap \{L_2, M_1\}^{\perp\perp}$ and hence we see that the line M_3 meets Γ' in a unique point z_2. Let z_1 be the unique point incident with M_1 and collinear with z_2; compare Figure 1. Then the line z_1z_2 belongs to Γ' and meets L_2. Hence there are unique collineations $g_i \in G_i$, $i = 1, 2, 3$, such that $zg_1 = z_1$, $z_1g_2 = z_2$ and $z_2g_3 = z'$. The collineation $g := g_1g_2g_3$ belongs to G and maps z to z'.

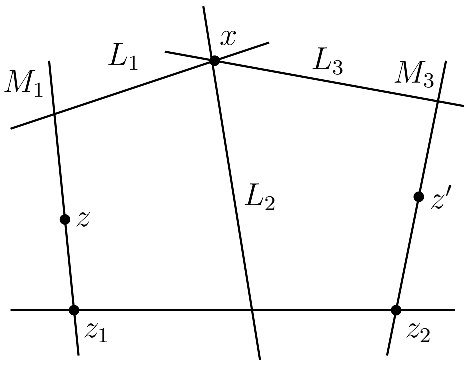

Figure 1: Three concurrent axes of symmetry

Suppose $h \in G$ fixes z. Put $h = h_1h_2h_3$, with $h_i \in G_i$, $i = 1, 2, 3$. Since h, h_1, h_2 preserve Γ', so does h_3, implying $h_3 = \mathrm{id}$. Since h and h_1 fix M_1, so does h_2, implying $h_2 = \mathrm{id}$.

The lemma is proved. □

Lemma 3.2 *All the G_i, $i = 1, 2, 3$, are abelian. Hence G is abelian.*

Proof. In the dual affine plane $\mathcal{A}^*_{L_3}$ corresponding to the projective line L_3, the symmetry groups G_1 and G_2 induce full elation groups with common axis (the set of parallel points corresponding to the lines of Γ through x) and distinct centres (the points corresponding to the lines L_1 and L_2, respectively). Hence this common axis is a translation line of $\mathcal{A}^*_{L_3}$ and consequently, the translation group is abelian. This translation group acts as a permutation group on $L_3^\perp$ as G_1G_2 restricted to $L_3^\perp$. Therefore, if $g_1, g_1' \in G_1$, then g_1g_1' coincides with $g_1'g_1$ on $L_3^\perp$. Hence the commutator $[g_1, g_1']$ belongs to $G_1 \cap G_3$, implying $[g_1, g_1'] = \mathrm{id}$. This shows that G_1 is abelian.

Similarly, also G_2 and G_3 are abelian. Thus $G_1 \times G_2 \times G_3$ is abelian and so is G as a homomorphic image. □

We can now prove our main general aim of this section.

Theorem 3.3 *If a generalized quadrangle has three concurrent projective axes of symmetry, all incident with the point x, then every line incident with x is an axis of symmetry and x is a translation point in Γ.*

Proof. We use the above notation. Let L be any line incident with x and let z, z' be two collinear points such that the line zz' meets L. It suffices to show that there exists some symmetry with axis L mapping z to z'.

Let $g \in G$ be the unique element of G mapping z to z'. Since G fixes every line through x, the line L is fixed by g. Furthermore, if $zz' \cap L = \{y\}$, then $y = L \cap z^\perp = L \cap (z')^\perp$. Therefore $yg = (L \cap z^\perp)g = Lg \cap (zg)^\perp = L \cap (z')^\perp = y$ so that y and thus zz' is fixed by g. Since G is abelian, it centralizes g and thus g fixes all lines in the orbit $(zz')G$ and all points in the orbit yG. Hence g fixes all lines concurrent with zz' using the transitivity of G implied by Lemma 3.1. But G acts transitively on the points of $L \setminus \{x\}$ so that g fixes L pointwise. This shows that g is a symmetry with axis L.. □

We now have the following corollaries.

Corollary 3.4 *let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a compact connected quadrangle with topological parameters (p, p), $p \in \mathbb{N}$. Suppose that Γ has three concurrent axes of symmetry, L_1, L_2, L_3, all meeting in the point x. Then Γ is a topological translation generalized quadrangle with translation point x.*

Proof. By the foregoing theorem, it suffices to show that each of L_1, L_2, L_3 is projective. But this follows directly from [19], see also [11]. □

An *axis* in a Laguerre plane Δ is a parallel class A such that the collection of all automorphisms in the kernel of Δ (i.e., all automorphisms of Δ that fix each parallel class globally) that fix precisely the points of A plus the identity is transitive on the set of points of each parallel class different from A. An elation Laguerre plane is a Laguerre plane that admits a subgroup in the kernel that acts sharply transitively on the set of circles. In an elation Laguerre plane every parallel class is an axis.

Corollary 3.5 *Let $\Delta = (\mathcal{P}, \mathcal{C}, \|)$ be a locally compact connected finite-dimensional Laguerre plane with at least three axes, then all parallel classes are axes and Δ is an elation Laguerre plane.*

Proof. In the canonical construction of the associated quadrangle Γ as the Lie geometry of Δ, an axis of the Laguerre plane is an axis of symmetry of the quadrangle, and vice versa. Also, Γ has topological parameters (p, p), where $p \in \{1, 2\}$ is the dimension of a circle in Δ. The result now follows directly from the previous corollary. Finally, the group generated by all the collineations to the three axes is a subgroup in the kernel that acts sharply transitively on the set of circles by Lemma 3.1 and thus Δ is an elation Laguerre plane. □

Ruth Kleinewillinghöfer [6] classified Laguerre planes with respect to *central automorphisms*. With respect to Laguerre translations Kleinewillinghöfer obtained 11 types of Laguerre planes; see [6] Satz 3.3, or [7] Satz 2. In type D the set $\mathcal{E} \subseteq \Pi$ of all parallel classes G for which the automorphism group Γ of $\mathcal{L}$ is linearly transitive with respect to G-translations, that is, G is an axis, contains at least 3 parallel classes. In the topological case we can say that $\mathcal{E} = \Pi$, see also [15], Lemma 4.4, for 2-dimensional Laguerre planes. In the 2-dimensional case such a Laguerre plane is ovoidal, that is, embeddable into 3-dimensional projective space as the geometry of plane sections of a cone with base an oval (minus the vertex of the cone). There are however examples of 2-dimensional Laguerre planes in which $\mathcal{E}$ consists of precisely two parallel classes, see [15]. This then yields 3-dimensional generalized quadrangles that are not translation generalized quadrangles and admit two concurrent axes of symmetries.

4 Generalized Quadrangles with two Opposite Projective Axes of Symmetry

Here, we let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized quadrangle containing two opposite projective axes of symmetry L_1, L_2. Let G_1 and G_2 be the corresponding symmetry groups, and put $G = \langle G_1, G_2 \rangle$, the group generated by G_1 and G_2. We first prove a general transitivity result, and then apply it to the topological case. We start with some lemmas.

Lemma 4.1 *Every line of Γ not contained in $\{L_1, L_2\}^\perp$ intersects some member of $\{L_1, L_2\}^\perp$.*

Proof. Let L be an arbitrary line of Γ. Since L does not belong to $\{L_1, L_2\}^\perp$ it is opposite L_1 or L_2 (or both). Without loss of generality, we may assume that L is opposite L_1. If L belongs to $\{L_1, L_2\}^{\perp\perp}$, then the assertion is trivial. If L does not belong to $\{L_1, L_2\}^{\perp\perp}$, then, since L_1 is a projective line, the traces $\{L_1, L\}^\perp$ and $\{L_1, L_2\}^\perp$ have a unique line M in common. Now L meets M and M belongs to $\{L_1, L_2\}^\perp$. □

Lemma 4.2 *Let Ω be the subgeometry of Γ consisting of the points of Γ not incident with a line of $\{L_1, L_2\}^\perp$ and with line set $\mathcal{L} \setminus (\{L_1, L_2\}^\perp \cup \{L_1, L_2\}^{\perp\perp})$. Then Ω is a connected geometry (in the sense that the incidence graph of Ω is a connected graph).*

Proof. Let a, b be two distinct points of Ω, and let a' and b' be the unique points incident with L_1 and collinear with a and b, respectively. If $a' \neq b'$, then the unique point c collinear with a and incident with bb' belongs to Ω and we have the chain $a\mathtt{I}ac\mathtt{I}c\mathtt{I}bb'\mathtt{I}b$ connecting a with b. If $a' = b'$, then we choose a point d' on L_1 and a point d of Ω collinear with d' in Γ. The foregoing implies that a is connected with d and d is connected with b in Ω. Hence a is connected with b in Ω. $\square$

Remark 4.3 More generally, one shows, with exactly the same arguments, that, if a generalized quadrangle Γ_1 contains a full subquadrangle Γ_2 (i.e., every point of Γ_1 incident with a line of Γ_2 belongs to Γ_2), then the *complement* of Γ_2 in Γ_1 is a connected geometry.

We denote by Δ the thin subquadrangle with line set $\{L_1, L_2\}^\perp \cup \{L_1, L_2\}^{\perp\perp}$. Its point set comprises precisely all point rows of lines in $\{L_1, L_2\}^\perp$.

Lemma 4.4 *The smallest subquadrangle of Γ containing all points of Δ and at least one point not in Δ is Γ itself.*

Proof. Let Γ' be a subquadrangle containing Δ and some point a not in Δ. By Lemma 4.1, all lines of Γ incident with a belong to Γ'. This implies, by [32], Proposition 1.8.1, that the subquadrangle Γ' is ideal in Γ, that is, that every pencil in Γ' coincides with the corresponding pencil in Γ. Since also all points on L_1 belong to Γ', the subquadrangle Γ' is full in Γ by the fual statement of Proposition 1.8.1 in [32]. Hence Γ' coincides with Γ by [32], Proposition 1.8.2. $\square$

We can now show:

Theorem 4.5 (*i*) *The group G acts sharply transitively on the point set of Ω.*

(*ii*) *If Z is the (normal) subgroup of G fixing Δ pointwise, then $Z \leq Z(G)$.*

(*iii*) *G/Z acts faithfully and 2-transitively on the span $\{L_1, L_2\}^{\perp\perp}$, or, equivalently, on the point row determined by any line in $\{L_1, L_2\}^\perp$. Also, the subgroups G_1Z/Z and G_2Z/Z are root groups of a Moufang set with little projective group G/Z acting on the above mentioned sets.*

(iv) If $G_1Z/Z \leq [G_{L_1}/Z, G_1Z/Z]$, then G is perfect.

Proof. (i): First we remark that, due to the transitivity properties of the symmetry groups G_1 and G_2, every line of $\{L_1, L_2\}^{\perp\perp}$ is an axis of symmetry, and all corresponding symmetry groups are conjugate in G to G_1 and to G_2. In particular, G_1 is conjugate to G_2.

Now let a, b be two distinct points of Ω. By Lemma 4.2, we may assume that a and b are collinear in Ω, and hence in Γ. By Lemma 4.1 ab is incident with a unique point y of Δ. Let L be the unique line of $\{L_1, L_2\}^{\perp\perp}$ incident with y. Then L is an axis of symmetry and so there is a symmetry g about L mapping a onto b. Clearly $g \in G$ and so G acts transitively on the point set of Ω.

Now suppose some $g \in G$ fixes some point a of Ω. Since both G_1 and G_2 fix all lines of $\{L_1, L_2\}^{\perp}$, g also fixes all elements of $\{L_1, L_2\}^{\perp}$. Now let $M \in \{L_1, L_2\}^{\perp\perp}$, and let a' be the unique point incident with M and collinear with a. Then a' is incident with a unique member of $\{L_1, L_2\}^{\perp}$, and since this member is fixed under g, also a', and hence M is fixed under g. We conclude that g fixes Δ pointwise, and also a, so it fixes the smallest subquadrangle containing Δ and a pointwise. By Lemma 4.4, g is the identity. This proves (i).

(ii): If $h \in Z$ and $g \in G_1$, then clearly $[g, h]$ belongs to G_1, but also fixes L_2; hence $[h, G_1] = \{\mathrm{id}\}$. Similarly, $[h, G_2] = \{\mathrm{id}\}$. Hence $Z \leq Z(G)$.

(iii): Let X denote either the set $\{L_1, L_2\}^{\perp\perp}$, or the set of points incident with some arbitrary but fixed line M of $\{L_1, L_2\}^{\perp}$. It is clear that G_i, and hence also G_iZ/Z, fixes a unique element x_i of X and acts sharply transitively on $X \setminus \{x_i\}$, $i = 1, 2$. Hence G acts 2-transitively on X. Since Z is precisely the kernel of that action, G/Z still acts 2-transitively on X, but also faithfully. Moreover, since G_i consists of all collineations fixing all lines concurrent with L_i, it is normal in G_{L_i}, $i = 1, 2$, and G_1 is conjugate to G_2. So, in view of section 2.4 the assertion follows.

(iv): For $g \in G_{L_1}$ and $g_1 \in G_1$ arbitrary, the commutator $[g, g_1]$ belongs to G_1. Hence if $g_1'Z$, with $g_1' \in G_1$, is the product of commutators of the form $[g, g_1]Z$, say

$$g_1'Z = \prod_i [g^{(i)}, g_1^{(i)}]Z,$$

then the product $\prod_i [g^{(i)}, g_1^{(i)}]$ belongs to both $g_1'Z$ and G_1, which is clearly g_1'. Hence if $G_1Z/Z \leq [G_{L_1}/Z, G_1Z/Z]$, then $G_1 \leq [G_{L_1}, G_1]$ and by conjugation, $G_2 \leq [G_{L_2}, G_2]$, implying $G_1, G_2 \leq [G, G]$, i.e., $\langle G_1, G_2 \rangle = G \leq [G, G]$ and so G is perfect. □

We now treat a rather special case.

Theorem 4.6 *Assume that* $\mathbb{K}$ *is a field with* $\operatorname{char}\mathbb{K} \neq 2$ *such that a perfect central extension of* $\mathsf{PSL}_2(\mathbb{K})$ *with center of order 2 and in which every involution is central, is isomorphic to* $\mathsf{SL}_2(\mathbb{K})$.

If $G/Z \cong \mathsf{PSL}_2(\mathbb{K})$ *with its natural action on* $\{L_1, L_2\}^{\perp\perp}$, *and if some point* x *of* Ω *is antiregular in* Γ, *then* Γ *is isomorphic to the dual of* $\mathsf{W}(\mathbb{K})$.

Proof. We denote the action of G on $\mathcal{P}$ by $p.g$ for $g \in G$ and $p \in \mathcal{P}$. We start by showing that $|Z| = 2$. If $h \in Z$, $h \neq \mathrm{id}$, then $x.h \neq x$ (as otherwise h would fix a subquadrangle containing Δ and x, contradicting Lemma 4.4). Since Z is the subgroup of G fixing Δ pointwise, Z fixes every line of $\{L_1, L_2\}^{\perp}$ and $\{L_1, L_2\}^{\perp\perp}$. Hence, if x' is collinear with x and belongs to the point set of Δ, then x' is also collinear with $x.h$. Hence, if x'_1, x'_2, x'_3 are three points of Δ collinear with x, then $\{x'_1, x'_2, x'_3\}^{\perp}$ contains $x.Z$. By antiregularity, $|Z| = |x.Z| \leq 2$.

But, again by the antiregularity of x, there exists a unique point $y \in \{x'_1, x'_2, x'_3\}^{\perp}$, different from x. By transitivity of G, there exists $g \in G$ such that $x.g = y$. Since the three respective lines of $\{L_1, L_2\}^{\perp}$ incident with x'_1, x'_2, x'_3 are fixed under g, also the points x'_1, x'_2, x'_3 are fixed, and so are the unique respective lines of $\{L_1, L_2\}^{\perp\perp}$ incident with x'_1, x'_2, x'_3. Since $\mathsf{PSL}_2(\mathbb{K})$ acts Zassenhaus transitively (i.e., 2-transitively, but the stabilizer of three distinct elements is trivial), g fixes $\{L_1, L_2\}^{\perp\perp}$ elementwise, and so $g \in Z$. Hence Z is non-trivial and is isomorphic to the group of two elements.

We now claim that G/Z satisfies the condition stated in (iv) of Theorem 4.5. Let $a, t \in \mathbb{K}^{\times}$ and consider the following commutator.

$$\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & (a^2-1)t \\ 0 & 1 \end{pmatrix},$$

which implies the condition, if we can choose $a \notin \{-1, 0, 1\}$, i.e., if $|\mathbb{K}| > 3$. But if $|\mathbb{K}| = 3$, then Γ has order $(3, 3)$ and the result follows from the uniqueness of quadrangles with these orders, see [13]. Hence G is a perfect central extension of $\mathsf{PSL}_2(\mathbb{K})$ with center of order 2.

Now we claim that every involution in G is central in G (hence belongs to Z). Let $g \in G$ be an involution and suppose, by way of contradiction, that g does not belong to Z. Then, by Theorem 4.5(ii), g induces a non-trivial permutation $\mathfrak{g}$ on $\{L_1, L_2\}^{\perp\perp}$. Let M be an arbitrary line of $\{L_1, L_2\}^{\perp}$. Then we know that $M.g = M$ and that M contains at least one point y with $y.g \neq y$ (otherwise g fixes all elements of $\{L_1, L_2\}^{\perp\perp}$). Let N be any line incident with y, but not contained in Δ. Then $N.g$ is not concurrent with N, and so we can choose a line K meeting both N and $N.g$, with $K \neq M$. The line K meets Δ in a unique point u by Lemma 4.1. Denote by L the unique line of Δ incident with u and meeting M. Since L is a projective line, there is a unique thin full subquadrangle containing L and N, and since it contains K and M, it also contains $N.g$. Hence it is determined by N and $N.g$, and consequently it is fixed under g (as $(N.g).g = N$). It follows that g fixes L. Hence $\mathfrak{g}$ fixes at least one element, and it is easy to see that, in $\mathsf{PSL}_2(K)$, any element fixing precisely one element of $\{L_1, L_2\}^{\perp\perp}$ has order p, when $\operatorname{char}\mathbb{K} = p$, or infinity, when $\operatorname{char}\mathbb{K} = 0$. Since we assume $\operatorname{char}\mathbb{K} \neq 2$, this implies that $\mathfrak{g}$, and hence also g, fixes a second line L' of $\{L_1, L_2\}^{\perp\perp}$, with $L' \neq L$.

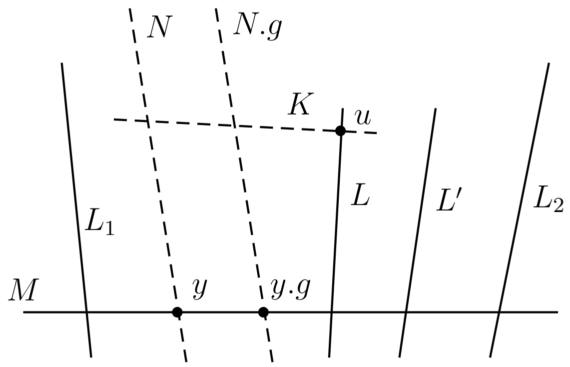

Figure 2:

Now suppose that g fixes some line J of Ω that, viewed as a line of Γ, meets L in some point v. Let w be any point on J distinct from v, then by sharp transitivity of G on the point set of Ω, g coincides with the symmetry about L mapping w to $w.g$, contradicting $\operatorname{char}\mathbb{K} \neq 2$ again. In fact, the same argument shows that no line outside Δ can be fixed under g.

Now let J be a line concurrent with both L and M, but distinct from both of them, and, likewise, let J' be a line concurrent with both L' and M, but distinct from both of them. As before, there is a unique full thin subquadrangle Δ^* containing J and J', and Δ^* contains a unique line L^* of $\{L_1, L_2\}^{\perp\perp}$, with $L \neq L^* \neq L'$. Now let A be an arbitrary line distinct from M meeting both $J.g$ and $J'.g$. Clearly, both A and J are mutually opposite, and also opposite L^*. Hence this (dual) triad has at least one (dual) centre (since L^* is a projective line), which we denote by C. The line C meets A in a point a. Hence, this point a belongs to the unique full thin subquadrangles determined by J and J', and by $J.g$ and $J'.g$, respectively. Since both of these subquadrangles contain M, both of them also contain the line B through a meeting M. Clearly, these subquadrangles only have M and B is common, and they are mapped to each other under g. Consequently, since $M.g = M$, we must have $B.g = B$. Since we showed above that g cannot fix a line outside Δ, the line B must belong to Δ. Since B meets L^*, this implies $B = L^*$. But this contradicts our assumption that $g \notin Z$. Our claim is proved.

Therefore, by the assumption on $\mathbb{K}$ and perfect central extensions of $\mathsf{PSL}_2(\mathbb{K})$ we conclude that $G \cong \mathsf{SL}_2(\mathbb{K})$.

We now describe Γ in terms of $\mathsf{SL}_2(\mathbb{K})$. To this end we note that the root groups of $\mathsf{PSL}_2(\mathbb{K})$ lift uniquely to isomorphic subgroups in $\mathsf{SL}_2(\mathbb{K})$. Hence the groups G_1 and G_2 and their conjugates are uniquely determined. We also call them root groups of $\mathsf{SL}_2(\mathbb{K})$.

Let x again be as before. By the sharp transitivity of $\mathsf{SL}_2(\mathbb{K})$, we can identify in a unique way every point y of Ω with the element $g \in \mathsf{SL}_2(\mathbb{K})$ such that $x.g = y$. The elements g, h are, as points of Γ, collinear if and only if $x.g, x.h$ are collinear if and only if $x.gh^{-1}, x$ are collinear if and only if gh^{-1} belongs to a root group of $\mathsf{SL}_2(\mathbb{K})$. Since Γ does not contain triangles, this is enough to uniquely reconstruct Ω. Using a result of Harm Pralle [16], we can uniquely complete Ω to a generalized quadrangle. Hence Γ is uniquely determined by $\mathsf{SL}_2(\mathbb{K})$, and since the dual of $\mathsf{W}(\mathbb{K})$ satisfies the assumptions, we conclude that Γ is isomorphic to the dual of $\mathsf{W}(\mathbb{K})$.

We can also avoid Pralle's result and proceed by direct construction. Let us briefly sketch this approach. Since in the previous paragraph, we already constructed the points and lines of Ω, it suffices to reconstruct the points of Δ, the collinearity in Δ, and the collinearity between points of Δ and points of Ω. The points of Δ are identified with the right cosets of the normalizers of the root groups. Note that, since the root groups form a complete conjugacy class, this set of right cosets of their normalizers is identical with the set of left cosets. A coset $N(R)g$ of the normalizer of the root group R is collinear with the point of Ω determined by the element h if and only if $h \in N(R)g$. Furthermore two cosets $N(R)g$ and $N(S)h$ of the normalizers of the respective root groups R, S are collinear as points of Δ, and hence of Γ, if and only if either $R = S$ (they are right cosets of the normalizer of the same root group; the joining line corresponds to a member of $\{L_1, L_2\}^\perp$) or $g^{-1}N(R)g = h^{-1}N(S)h$ (they are left coset of the normailizer of the same root group; the joining line corresponds with a member of $\{L_1, L_2\}^{\perp\perp}$). This uniquely determines Γ and hence, once again, Γ is unique and must be dual to $\mathsf{W}(\mathbb{K})$. $\square$

Remark 4.7 The argument in the previous proof that established the line B inside the intersection of two full thin subquadrangles of Γ in fact shows that the line M of Γ is a regular projective line. Another way of organizing the proof would be to first use this argument to conclude that M is projective, and then to use the fact that an involution in the perp-plane in M must fix all points on some line of that perp-plane (since it cannot be a Baer-involution by the fact that it only fixes L, L' in Δ). Since this approach requires some more definitions and results, as the reader can see, we chose to argue directly in Γ instead of in the perp-plane in M.

Remark 4.8 We do not know of any perfect central 2-fold extension of $\mathsf{PSL}_2(\mathbb{K})$, $\mathbb{K}$ a field with $\operatorname{char}\mathbb{K} \neq 2$, in which every involution is central other than $\mathsf{SL}_2(\mathbb{K})$; compare [21] for a discussion for related universal central extensions of a perfect group and [4], Satz 25.7, for the statement without the assumption on involutions for finite fields $\mathbb{K}$. In fact, it is conjectured that this is a characterisation of $\mathsf{SL}_2(\mathbb{K})$. In this case the assumptions made on perfect central 2-fold extensions of $\mathsf{PSL}_2(\mathbb{K})$ in Theorem 4.6 become redundant. Also in the topological setting of the following theorem where G is a Lie group the only such extensions are $\mathsf{SL}_2(\mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

We now apply this to the topological case.

Theorem 4.9 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a compact connected generalized quadrangle with topological parameters (p, p), $p \in \mathbb{N}$. Suppose Γ has two opposite axes of symmetry, say L_1, L_2. Then $p \in \{1, 2\}$ and Γ is isomorphic to the dual of $\mathsf{W}(\mathbb{R})$ (for $p = 1$) or to the dual of $\mathsf{W}(\mathbb{C})$ (for $p = 2$).*

Proof. First we note that, as in the previous section, since Γ has equal topological parameters, the axes L_1, L_2 of symmetry are projective lines. Furthermore, equal topological parameters and [8] imply that $p \in \{1, 2, 4\}$.

The group G/Z acts 2-transitively and faithfully on the points of Δ by (iii) of Theorem 4.5 and thus is a Lie group by [17], 96.15. Hence G also is a Lie group because Z has order 2 as seen in the proof of Theorem 4.6 and so G is locally isomorphic to G/Z. Moreover, G/Z has dimension at most $3p$, by (i) of Theorem 4.5, and the set X of (iii) of the proof of Theorem 4.5 is a topological sphere of dimension p. Hence G/Z is isomorphic to the little projective group of a Moufang set acting on a sphere with topological dimension $p \in \{1, 2, 4\}$ and thus has dimension at most $3p \in \{3, 6, 12\}$. From the list of groups in Lemma 2.1 and their dimensions we see that the only possibilities are $G/Z \cong \mathsf{PSL}_2(\mathbb{R})$ and $G/Z \cong \mathsf{PSL}_2(\mathbb{C})$. Hence $p \in \{1, 2\}$ and Γ is antiregular.

The connected Lie groups that are 2-fold covering groups of $\mathsf{PSL}_2(\mathbb{R})$ or $\mathsf{PSL}_2(\mathbb{C})$ are well known. compare [30], Ch IV. The unimodular group $\mathsf{SL}_2(\mathbb{C})$ is simply connected and thus

G must be isomorphic to $\mathsf{SL}_2(\mathbb{C})$ in case $G/Z \cong \mathsf{PSL}_2(\mathbb{C})$. A 2-fold covering of $\mathsf{PSL}_2(\mathbb{R})$ is obtained from the simply connected covering $\widetilde{\mathsf{PSL}_2}(\mathbb{R})$ of $\mathsf{PSL}_2(\mathbb{R})$ by factoring out a subgroup of index 2 in the centre of $\widetilde{\mathsf{PSL}_2}(\mathbb{R})$. Since the centre of $\widetilde{\mathsf{PSL}_2}(\mathbb{R})$ is isomorphic to $\mathbb{Z}$, there is, up to isomorphism, only one 2-fold covering of $\mathsf{PSL}_2(\mathbb{R})$, namely $\mathsf{SL}_2(\mathbb{R})$. The result now follows from Theorem 4.6. □

Note that for $p \in \{1, 2\}$ the points of Ω induce the structure of a $2p$-dimensional Minkowski plane Δ^* in Δ, see [18, 14] for the definition and basic properties of Minkowski planes. Indeed, the centre Z of G has order 2 as seen in the first part of the proof of Theorem 4.6. The non-identity collineation τ in Z fixes precisely the points of Δ and thus satisfies that assumptions made in Theorems 5.29 and 5.36 of [20] so that in case $p = 2$

$$\mathcal{M}(\Gamma, \tau) = (X, \{X \cap q^\perp \ : \ q \in P \setminus X\}, \mathcal{X})$$

and in case $p = 1$

$$\mathcal{M}(\Gamma, \tau, \varphi) = (X, \{X \cap q^\perp : q \in P \setminus X\} \cup \{\varphi^{-1}(X \cap q^\perp) : q \in P \setminus X\}, \mathcal{X}),$$

with incidence given by inclusion, is a 4- or 2-dimensional Minkowski plane, respectively, where X is the space of fixed points of τ and $\mathcal{X}$ is the space of fixed lines of τ and where in the latter case $\varphi : \Delta \to \Delta$ is an isomorphism such that $|T^\perp| + |\varphi(T)^\perp| = 2$ for any triad T in Δ. Note that by (2.15) of [20] a compact antiregular generalized quadrangle with topological parameters $(2, 2)$ is centric and one with topological parameters $(1, 1)$ is non-centric and that in the latter case there exists such an isomorphism $\varphi : \Delta \to \Delta$ as above; see [20], Corollary 5.32.

Furthermore, the group G fixes all lines of one parallel class in Δ^* so that G is a group in one of the kernels of the Minkowski plane. Since G has dimension $3p$ by (i) of Theorem 4.5, we see that one kernel of Δ^* has maximal dimension $3p$.

If $p = 2$, a kernel of dimension 6 implies that Δ^* must be the classical Minkowski plane over the complex numbers; see [23], Proposition 3.2. (This is the geometry of non-trivial plane sections of a ruled quadric in 3-dimensional projective space over $\mathbb{C}$.) But then Γ is classical too. If $p = 1$, we know by Satz 5.9 of [18] (see also [14], Theorem 44.10) that Δ^* consists of two halves of a classical Minkowski plane over $\mathbb{R}$. However, by (5.32) of [20], each generalized quadrangle corresponding to these halves, in particular Γ, is classical.

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