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Quasilocal Energy and Conservation Laws in General Relativity

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Abstract

In this work we investigate current research on quasilocal energy and conservation laws in general relativity. We explore the derivations and motivations for the Brown and York quasilocal energy and the Epp invariant quasilocal energy. We obtain expressions for the quasilocal energy of the radially inhomogeneous Lemaître-Tolman geometry via both the Brown and York and the Epp definitions. We then make a perturbative comparison between the energy predicted by Newtonian cosmology and the quasilocal energy of a Friedmann-Lemaître-Robertson-Walker universe transformed into locally inertial Fermi normal coordinates. It is found that by transforming to Fermi normal coordinates the magnitude of the difference in energy between these cosmological models is reduced.

Recent developments on the utility of a rigid quasilocal frame (RQF) in quasilocal conservation laws are investigated. We apply the RQF construction to a Lemaître-Tolman universe and prove the existence of such a frame at the center of spherical symmetry. We obtain an explicit function in terms of the components of the metric that allow a congruence of observers to remain expansion- and shear-free for all time.

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1 Introduction

1.1 Energy in General Relativity

The concept of energy in the theory of general relativity is not well understood and has evaded a universally accepted definition since the theory was proposed almost a century ago. While elegant descriptions exist for the energy of electromagnetic and mechanical systems a concrete notion of gravitational energy remains absent. The issue is not one of the existence of gravitational energy, with numerous examples of gravitationally bound systems within our own solar system existing in reduced energy states. Instead it is the type of definition that we seek. Gravitational energy is not localisable, making the search for a point-like definition futile. This has led to the quasilocal description of energy, a substantial step towards an accepted notion of energy in general relativity.

The non-localisability of gravitational energy arises from Einstein's strong equivalence principle. The strong equivalence principle states that anywhere in any gravitational field it is always possible to find a local inertial frame of reference such that in a sufficiently small region of spacetime all non-gravitational laws of nature take on the same form as in the absence of gravity. Because of this an observer in such a frame of reference could not carry out any physical measurement that would distinguish this frame from one in the absence of a gravitational field, hence concluding there to be zero gravitational energy. As a result any frame invariant definition of gravitational energy at a point must be trivially zero, since there exists at least one frame in which it is indeed zero in an arbitrarily small neighbourhood. As Misner, Thorne and Wheeler state, anybody who searches for such a point-like definition is "looking for the right answer to the wrong question" [10]. With this understood the search shifts towards a definition of energy valid for *extended* but *finite* spacetime domains, that is, at the *quasilocal* level.

Quasilocal definitions are based on the general relativistic understanding that mass-energy and angular momentum distort spacetime. These deviations from a flat spacetime, manifest as curvature of spacetime itself, can be measured to provide us with an indirect evaluation of the mass-energy and angular momentum within a region. For example, during a 1919 solar eclipse Arthur Eddington verified Einstein's general relativity by observing the gravitational bending of light around the Sun [22]. This observation was the first to show that mass, and in turn energy, can curve the surrounding spacetime. Likewise, in 1918 Lense and Thirring used general relativity to predict that a massive rotating body can cause an orbiting object to precess by its influence on the properties of the surrounding spacetime [23][24]. As a result we can understand that by comparing the properties of a closed surface in a curved spacetime to that in a flat spacetime, we can attempt to quantify the energy and angular momentum contained within the surface. In this way quasilocal measurements avoid a point-like definition in favour of measurements on surfaces.

The observed accelerated expansion of the universe reveals a significant motivation for investigating quasilocal energy. After the success of Einstein's general relativity at the solar system scale the theory has become the framework for cosmological models in order to understand the universe at large. The current standard model assumes that the universe is homogeneous and isotropic using the Friedmann-Lemaître-Robertson-Walker (FLRW) geometry. In 1998 observations of the luminosity distances of distant supernovae gave evidence to support the claim that the universe is expanding at an accelerated rate [19]. If the geometry of the universe is homogenous and isotropic then these observations suggest that there must be some mysterious "dark energy" driving the expansion. To account for this cosmic acceleration the matter content of the standard model has been adjusted to include dark energy in order to overcome the universal gravitational attractiveness of matter which would lead to deceleration. The most recent PLANCK collaboration data on the cosmic microwave background best fits a universe with 68% dark energy in the form of a cosmological constant, Λ, 26.8% non-baryonic cold dark matter (CDM) and 4.9% ordinary matter [7]. These parameters give the ΛCDM standard cosmological model.

Inhomogeneous cosmological models have been suggested as an alternative to dark energy for explaining the apparently accelerated expansion of the universe. This may be possible since the assumption of homogeneity is an over simplification that neglects the effects of local inhomogeneity. Indeed, although the universe started out very smooth we now observe a high level of structure even beyond the scale of galaxy clusters. The universe contains voids, typically of diameter $30/h$ Mpc, surrounded and threaded by filaments of galaxies such that statistical homogeneity is only achieved at scales above $100/h$ Mpc. Here h is related to the Hubble constant, H_0 by $H_0 = 100h$ km s^{-1} Mpc^{-1}. In the standard model, departures from homogeneity are modeled using Newtonian N-body simulations in a background given by a uniformly expanding box. The expansion of this background box is adjusted to fit the ΛCDM model. Although these simulated universes have many features that match observations, in using the gravitational physics of a rigid Newtonian spacetime and putting in the expansion from the outside, such approaches could miss essential features of general relativity. By the first principles of general relativity spacetime is non-rigid and the expansion rate and growth of structure should be locally coupled and nonuniform. Fundamental issues remain as to what extent the N-body simulations reflect the possibilities of inhomogeneous models available in the full theory of general relativity.

In this report we will explore quasilocal energy concepts in the context of a inhomogeneous geometry. The most simple inhomogeneous geometry is the Lemaître-Tolman geometry which is an exact solution of Einstein's field equations discovered independently by Lemaître in 1933 [13] and Tolman in 1934 [14]. This solution is spherically symmetric but allows for radial inhomogeneity. As a simple toy model it describes an expanding or collapsing universe of pressureless dust and provides us with a setting to investigate the effects inhomogeneity may have on observations and expansion itself. It is expected that the quasilocal energy of a spacetime domain in an expanding universe will include contributions from the kinetic energy of the expansion of space. The kinetic energy of expansion is poorly understood or quantified beyond a Newtonian cosmological approximation in which the universe is modelled as a spatially homogenous sphere of dust. In order to understand the effect of inhomogeneity on the expansion of space, it would be enlightening to consider the relative differences in the quasilocal energy of various cosmological models. Therefore gaining an agreed upon definition of quasilocal energy and applying this to simple inhomogeneous geometry will be a useful first step towards understanding these concepts more broadly.

The study of energy and angular momentum at the quasilocal level has given rise to numerous definitions over the past three decades. Amongst various proposals that of Brown and York in 1993 [2] was notable. They constructed a quasilocal energy based on a *total* (matter plus gravitational) stress-energy-momentum tensor. This tensor is intended to be a generalization of the stress-energy-momentum tensor found in the study of fluid mechanics. Derived from this was a total energy density that when integrated over a closed surface gave a notion of the total (matter plus gravitational) energy within the volume bounded by the surface. The essence of the Brown and York definition is the relationship between the curvature of a closed two surface and the total energy contained within it. The Brown and York definition was extended by Epp in 2000 [3]. The Epp definition of quasilocal energy is derived from an analogy with the invariant mass in special relativity, and as a result turns out to be invariant under radial boosts of the boundary of the spacetime domain. These are just two rather similar examples from a myriad of different constructions; yet despite the diverse range an agreed upon definition for quasilocal energy remains to be found.

As with any physical quantity it is important to understand how quasilocal energy changes with respect to time and the mechanisms through which change occurs. In the 2012 work of Epp, Mann and McGrath [8] the Brown and York definition of quasilocal energy is considered in the context of a conservation law. In general relativity a conservation law typically starts with the identity

$$\nabla_a \left(T^{ab}_{\text{mat}} \Psi_b\right) = \left(\nabla_a T^{ab}_{\text{mat}}\right) \Psi_b + T^{ab}_{\text{mat}} \nabla_{(a} \Psi_{b)} \tag{1.1}$$

where ∇_a is the covariant derivative operator, T^{ab}_{mat} is the stress-energy-momentum tensor (standard fluid mechanics form) and Ψ_b is a vector field. Typically, $\nabla_a T^{ab}_{\text{mat}} = 0$ by the definition of T^{ab}_{mat}. Also, if a spacetime contains a particular symmetry then there may exist a vector field such that $\nabla_{(a} \Psi_{b)} = 0$. In this case Ψ_b is called a *Killing* vector field, named after Wilhelm Killing. A simple example is an isolated object in a spacetime with a timelike Killing vector, that is, a time symmetry. If we want to consider the energy of this object we let $\Psi_b = u_b$, where u^b is a 4-velocity. In this case both terms on the right of (1.1) are zero, giving the conserved quantity $T^{ab}_{\text{mat}} u_b$ which may be identified as a conserved mass-energy of the isolated object.

In general such conservations law are effectively blind to gravitational physics, for T^{ab}_{mat} may be zero even in a region through which gravitational energy is flowing. In [8] Epp, Mann and McGrath use a quasilocal approach to extend (1.1) to a generic spacetime that does not always have a Killing vector field and replace the stress-energy-momentum tensor, T^{ab}_{mat}, with the total (matter plus gravitational) stress-energy-momentum tensor introduced earlier. McGrath, Epp and Mann motivate the need for a general relativistic quasilocal conservation law through a simple example [18]. In most situations we can account for a change in a physical quantity in a volume by considering fluxes through the surface enclosing it. By considering a relativistic accelerating observer in an electromagnetic field McGrath, Epp and Mann show that a “bulk term” (volume integral) is required to explain the transfer of energy into an accelerating box. This is because the standard Poynting vector approach, involving surface fluxes, accounts for only *half* the rate of change of electromagnetic energy contained inside the box. Similarly, a bulk term arises when considering a conservation law in a generic spacetime that does not admit any Killing vector, that is, it does not have any symmetries. McGrath, Epp and Mann [18] introduce a quasilocal construction to transform this bulk term into a surface flux, to remain in keeping with the classical understanding of a conservation law.

The field of quasilocal energy and conservation laws in general relativity is extensive and diverse, making a complete discussion beyond the scope of this introduction. For more information on work in the area of quasilocal energy the reader is directed to the review article [1] which discusses a wider range of definitions and open questions that arise in the study of quasilocal energy.

1.2 Outline of Research

The Lemaître-Tolman model is an ideal candidate for initial investigations into the quasilocal energy of inhomogeneous cosmological models. In section 4.1 we present a calculation of the Brown and York and Epp quasilocal energies for a Lemaître-Tolman universe. This original calculation is the first such application of these quasilocal definitions to the Lemaître-Tolman model we are aware of and is verified by comparison in a limiting case to the results of Afshar [12], and in turn, the Newtonian cosmology.

The Epp definition of quasilocal energy is applied to a small spherical region of a FLRW universe in Fermi normal coordinates. This original work is intended to extend the perturbative comparison made by Afshar [12] in comparing quasilocal energy to the energy predicted by a Newtonian approximation. We find that transforming a spatially flat FLRW universe to Fermi normal coordinates gives a quasilocal energy that agrees with the Newtonian up to order r^3 and offers a reduced error than that found by Afshar [12] in higher order terms. We also calculate the energy for a $k = \pm 1$ (open or closed) FLRW universe in Fermi normal coordinates. The interpretation of these results from a physical perspective are presented in chapter 5.

Finally, we explore the utility of rigid quasilocal frames for the calculation of quasilocal conservation laws in general relativity. Rigid quasilocal frames offer a natural setting for the calculation of a conservation law, where changes in quasilocal quantities can be accounted for purely by surface fluxes rather than changes in shape and size of the spacetime region. These frames are introduced in detail in section 3.1. The most significant result we have obtained is the proof of the existence of such a rigid quasilocal frame at the center of spherical symmetry in a Lemaître-Tolman universe, presented in section 4.2. This is followed by a discussion on the utility of rigid quasilocal frames for cosmological applications in chapter 5.

2 Quasilocal Energy

In this chapter we will introduce the geometric formalisms used throughout this report and motivate and derive the quasilocal energy definitions given by Brown and York [2] and Epp [3]. This is followed by a brief introduction to Newtonian cosmology and a discussion on the physical interpretation of the quasilocal energy in a FLRW universe.

2.1 Geometric Formalism

Consider a spacetime, $M = \Sigma \otimes \mathbb{R}$, where Σ represents 3-space and the real line interval, $\mathbb{R}$, represents time. We denote the metric on M by g_{ab}. Quasilocal energy is defined as the energy contained within a domain on a space-like slice of the 4-dimensional manifold, M. In both the Brown and York and the Epp definitions the quasilocal energy of a finite spatial domain on such a slice is characterised by a surface integral on the boundary of that domain. Although the boundary on the space-like slice need only be a closed compact 2-surface we will use a spherical surface denoted by Ω. In such a way the quasilocal energy takes the form of a 2-dimensional integral over Ω.

Since Ω is a time-like 2-surface we must have a time-like and space-like unit normal denoted by $\frac{1}{c}u_a$ and n_a respectively. For each of these unit vectors we can calculate an extrinsic curvature defined as

$$l_{ab} \equiv \mathcal{P}_a{}^c \mathcal{P}_b{}^d \nabla_{(c} u_{d)}$$
$$k_{ab} \equiv \mathcal{P}_a{}^c \mathcal{P}_b{}^d \nabla_{(c} n_{d)}$$

where $\mathcal{P}_b{}^a = \delta_b{}^a + \frac{1}{c^2} u^a u_b - n^a n_b$ is the surface projection operator on the 2-surface and ∇_a is the covariant derivative operator on M. The extrinsic curvature is a quantity defined for surfaces that are embedded in a higher dimensional spacetime and is used to characterise the curvature of the surface. The trace of the extrinsic curvatures, $k \equiv \sigma^{ab} k_{ab}$ and $l \equiv \sigma^{ab} l_{ab}$ are vital in the definitions of quasilocal energy that we investigate in this report. Here σ_{ab} is the induced metric on Ω and is found to be $\sigma_{ab} = g_{ab} - n_a n_b + \frac{1}{c^2} u_a u_b$ by applying the projection operator to the 4-metric g_{ab}.

The study of conservation laws require us to extend these geometric ideas over a finite time interval. In order to achieve this we consider the history of the boundary, Ω, and call this a worldtube denoted by $\mathcal{B}$. The worldtube forms a 3-dimensional spacetime which is topologically the product of a closed 2-surface and the real line interval. The worldtube, $\mathcal{B}$, has an induced metric $\gamma_{ab} = g_{ab} - n_a n_b$ where n_a is the space-like outward directed unit normal vector on Ω and hence also a unit normal vector on $\mathcal{B}$.

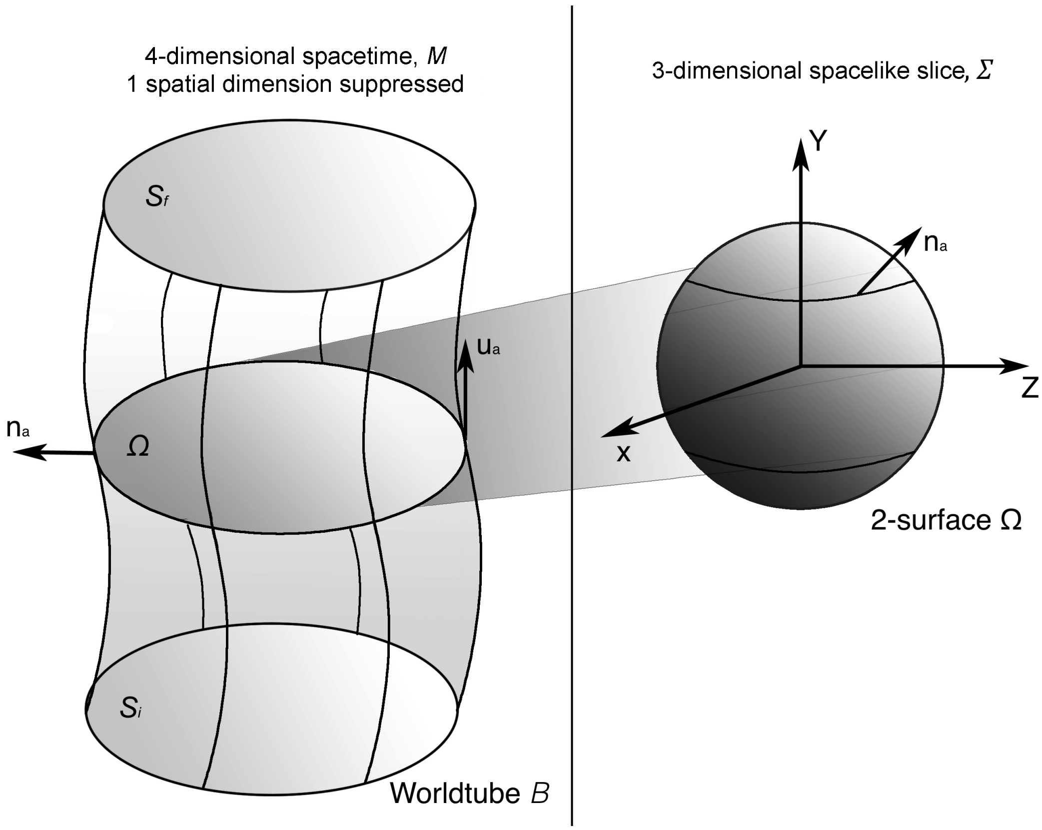

Figure 2.1: The worldtube $\mathcal{B}$ with initial and final space-like hypersurfaces $\mathcal{S}_i$ and $\mathcal{S}_f$. The right image is the projection of a spacelike slice into the three spatial dimensions. One spatial and one time dimension is suppressed on the left and right respectively for this representation.

The worldtube forms an open ended tube in 4-dimensional spacetime through which we can calculate fluxes in order to understand changes in quasilocal energy. However, to quantify the change in quasilocal energy over a finite time interval we must add space-like "end caps" to the worldtube so that it forms a closed three surface. We denote these additional space-like surfaces as $\mathcal{S}_i$ and $\mathcal{S}_f$, each has a time-like future-directed unit normal vector $\frac{1}{c}u^a_{\mathcal{S}}$ that is tangent to $\mathcal{B}$. Figure 2.1 gives a visual representation of the worldtube. Table 2.1 gives a summary of the quantities we will use in this report, including those introduced in this section.

Now consider the worldtube, $\mathcal{B}$, as a congruence of time-like worldlines. The unit vector u^a then represents the 4-velocity of the observers in this congruence. The worldtube can be understood as the history of a 2-sphere's worth of observers bounding a finite spatial volume. The time development of this congruence is characterised by the tensor field commonly denoted by $\theta_{ab} = \sigma_a{}^c \sigma_b{}^d \nabla_c u_d$. We can interpret θ_{ab} by considering the simple example of a set of observers that form a triangle. If the set of observers maintains its shape but increases in size, then this *expansion* is given by the trace, $\theta = \sigma^{ab}\theta_{ab}$. If the shape of this set of observers also changes, then we can quantify this as a *shear*, given by

Table 2.1: Summary of the mathematical quantities we will use in this report.

	Metric	Covariant Derivative	Unit Normal	Intrinsic Curvature	Extrinsic Curvature	Conjugate Momentum
M	g_{ab}	∇_a		$\mathcal{R}_{abcd}$		
Σ	h_{ij}	D_a	u_a	R_{abcd}	K_{ab}	P^{ab}
$\mathcal{B}$	γ_{ab}	$\mathcal{D}_a$	n_a		Θ_{ab}	π^{ab}
Ω	σ_{ab}		n_a, u_a		k_{ab}, l_{ab}	

the trace free part, $\theta_{<ab>} = \theta_{(ab)} - \frac{1}{2}\theta\sigma_{ab}$. Finally, any rotation of the shape is given by the *vorticity* which is described by the antisymmetric part, $\theta_{[ab]}$. The shear and expansion of a congruence of observers will be important when we consider rigid quasilocal frames in section 3.1.

In this report we will use indices in the start of the Latin alphabet for quantities on M, such as $a, b \in \{0, 1, 2, 3\}$, Greek indices for quantities on $\mathcal{B}$, such as $\mu, \nu \in \{0, 2, 3\}$ and indices in the middle of the Latin alphabet for quantities on Ω, such as $i, j \in \{2, 3\}$. We will also let a prime and an over-dot denote a partial derivative with respect to r and t respectively.

2.2 Quasilocal Energy Definitions

2.2.1 Brown and York

The Brown and York definition of quasilocal energy was proposed in 1993 [2]. This definition is derived from a Hamilton-Jacobi analysis of the action functional. In order for the reader to understand the physical motivation for this definition we present an outline of the derivation given by Brown and York in [2].

The Brown and York definition of quasilocal energy is based on integrating a quantity over a closed surface which bounds a finite region of spacetime. The interpretation of this integral is a quasilocal energy contained in the bounded region. The surface quantity Brown and York use is the projection of a total (matter plus gravitational) stress-energy-momentum tensor onto the observers' 4-velocity (where "observers" are situated on the boundary). The total stress-energy-momentum tensor is found by considering the boundary term of the action functional

$$S^1 = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,\mathcal{R} + \frac{1}{\kappa}\int_{\mathcal{S}_i}^{\mathcal{S}_f} d^3x\sqrt{h}\,K - \frac{1}{\kappa}\int_{\mathcal{B}} d^3x\sqrt{-\gamma}\,\Theta + S^m \tag{2.1}$$

where S^m is the matter action, $d^n x$ are the appropriate volume or surface elements and the other quantities are as given in Table 2.1. Here we use the index free symbol corresponding to an extrinsic curvature to denote the trace and $g = \det(g_{ab})$ (and similarly for the induced metrics). We take the variation of this action with respect to the metric and matter fields to obtain (see appendix A.1)

$$\delta S^1 = \frac{1}{2\kappa}\int_M d^4x\,\sqrt{-g}\,G_{ab}\delta g^{ab} + \int_{S_i}^{S_f} d^3x\sqrt{h}\,P^{ab}\delta h_{ab} + \int_\Omega d^3x\,\pi^{\mu\nu}\,\delta\gamma_{\mu\nu}$$
$$+\,(\text{boundary terms coming form the matter action})$$

where G_{ab} is the Einstein tensor and $\pi^{\mu\nu}$ and P^{ab} are the canonical momentum on the surfaces $\mathcal{B}$ and Σ respectively (see Table 2.1). If following the general case presented by Brown and York we would subtract a functional, S_0, of γ_{ij} such that $S = S^1 - S^0$. However we will set $S^0 = 0$ for the purposes of this derivation as this is still a valid result [2]. By restricting the variation of S to variations among classical solutions the terms giving the equations of motion vanish and we are left with

$$\delta S_{\text{cl}} = \int_{t'}^{t''} d^3x\; P^{ab}\,\delta h_{ab} + \int_{{}^3B} d^3x\;\pi^{\mu\nu}\,\delta\gamma_{\mu\nu} + \begin{pmatrix}\text{terms involving variations}\\ \text{in the matter fields}\end{pmatrix}. \tag{2.2}$$

At this point we consider an analogous Hamilton-Jacobi equation for the energy, which is in the form of a surface stress-energy-momentum tensor defined as, $T^{ab}_{\mathcal{B}} \equiv \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ab}}$ [2]. Since we have $\frac{\delta S_{\text{cl}}}{\delta\gamma_{ab}} = \pi^{ab}$ from (2.2) we find $T^{ab}_{\mathcal{B}} = \frac{2}{\sqrt{-\gamma}}\pi^{ab}$. Then from the definition $\pi^{ab} \equiv -\frac{1}{2\kappa}\sqrt{-\gamma}\left(\Theta\gamma^{ab} - \Theta^{ab}\right)$ [2] we find

$$T^{ab}_{\mathcal{B}} = -\frac{1}{\kappa}\left(\Theta\gamma^{ab} - \Theta^{ab}\right) \tag{2.3}$$

where $\kappa = \frac{8\pi G}{c^4}$. This is the total (matter plus gravitational) stress-energy-momentum tensor which includes contributions from both the matter and gravitational fields.

The total stress-energy-momentum tensor (2.3) can be decomposed into a surface energy density ϵ by projection onto the observers' 4-velocity u^a [2]

$$\epsilon \equiv \frac{1}{c^2}u^a u^b T^{\mathcal{B}}_{ab} \tag{2.4}$$

where ϵ is in units of energy per unit area. The surface energy density is the total (matter plus gravitational) energy density measured by the observers. From this interpretation of ϵ Brown and York arrived at [2]

$$E_{\text{abs}} = \int_\Omega d^2x\sqrt{|\sigma|}\epsilon \tag{2.5}$$

as the absolute quasilocal energy contained within the spatial region bounded by Ω.

Now that we have a definition of quasilocal energy, (2.6), we need to consider what the zero point of this energy is, that is, we need to find the vacuum contribution. A vacuum subtraction energy is required to obtain a sensible definition of energy for use when comparing measurements, and also when dealing with infinite volumes. This subtracted energy is the reference energy. Typically, the reference energy is calculated by an isometric embedding of the two-surface, Ω, into a flat Minkowski spacetime and calculating the corresponding quasilocal energy. The details of this method are included in section 4.1.1 and an alternative choice of the reference spacetime is discussed in section 2.4.

Epp, Mann and McGrath [8] show that the surface energy density is proportional to the trace of the extrinsic curvature, k, of Ω. More precisely, $\epsilon = -\frac{k}{\kappa}$ [1]. Therefore the Brown and York definition of quasilocal energy can also be expressed as

$$E_{\mathrm{BY}} = -\frac{1}{8\pi G}\int_{\Omega} d^2x\sqrt{|\sigma|}k - E_{\mathrm{ref}} \tag{2.6}$$

in units with $c = 1$ and where E_{ref} is a suitable reference energy.

The relationship between the surface energy density and the trace of the extrinsic curvature gives us an intuitive understanding of quasilocal energy. Consider a spherical 2-surface Ω. If we project each point in an area element of this surface one unit distance radially outward, the fractional expansion of this area element is given by k. In flat Minkowski $\mathbb{R}^3$ space we would have $k = 2/r$ for a spherical surface [8]. Now, if we have some matter or energy inside Ω by Einstein's field equations we know that the surrounding spacetime will be curved, so that its volume is greater than we would expect by measuring the surface area and using Euclidean geometry. In fact, we would have $k < 2/r$. The result is that the mass-energy decreases k, makes E_{BY} less negative and makes the referenced energy positive. Therefore (2.6) with a suitable reference energy is an appropriate definition of total (matter plus gravitational) energy.

2.2.2 Epp

The work of Epp in 2000 [3] extends the Brown and York definition of quasilocal energy to an Invariant Quasilocal Energy (IQE). Epp provided a definition of energy, which unlike the Brown and York energy, is invariant under local boosts of the set of observers on the bounding 2-surface Ω. This concept was introduced by an analogy to the invariant mass in the special relativistic formula $E^2 - \vec{p}^2 = m^2$.

Epp's definition of the IQE involves the traces of both extrinsic curvatures of the two-surface, Ω. We have already discussed the physical significance of the trace of k_{ab} in section 2.2.1 as a measure of energy so now we must consider the interpretation of $l \equiv \sigma^{ab}l_{ab}$. Since l_{ab} is related to the future directed time-like unit vector it is understandable that l measures the expansion of Ω in the time-direction defined by the observers' 4-velocity u^a. As such l can be interpreted as a radial momentum surface density [17]. Epp uses this interpretation of l as a momentum density and that of k as an energy density to motivate the analogy

$$E^2 - \vec{p}^2 \rightarrow \frac{1}{(8\pi)^2}\left(k^2 - l^2\right). \tag{2.7}$$

[1] Epp, Mann and McGrath [8] introduce a sign difference in their definition of $T^{ab}_{\mathcal{B}}$ and thus obtain $\epsilon = -\frac{k}{\kappa}$ while Afshar [12] and Brown and York [2] have $\epsilon = \frac{k}{\kappa}$. This difference is reconciled by a sign ambiguity in the definition of k_{ab} and therefore to be consistent with our definition of k_{ab} we choose the former convention.

Since l is interpreted as a radial momentum density the question arises, does the above analogy account for angular momentum, since $\vec{p}$ is the total momentum of the system? It turns out that this expression implicitly does include contributions from angular momentum as is demonstrated by Epp in [3]. Through this analogy Epp defines the IQE as [3]

$$E_{\text{Epp}} = -\frac{1}{8\pi G}\int_{\Omega} d^2x\sqrt{k^2 - l^2} - E_{\text{ref}} \tag{2.8}$$

where the reference energy is again calculated by an isometric embedding of Ω in a Minkowski spacetime.

The most straightforward method for calculating the IQE is to first find the mean curvature vector, H^c, since the norm of this gives us the required quantity, $\sqrt{H^c H_c} = \sqrt{k^2 - l^2}$. We will use the (corrected) definition of H^c given by Afshar [12]

$$H^c = \sum_{s=1}^{2} K^c{}_{ab}\xi_s{}^a\xi_s{}^b \quad \text{where} \quad K^c{}_{ab} = n^c\nabla_a n_b - u^c\nabla_a u_b \tag{2.9}$$

and $\xi_1{}^a$, $\xi_2{}^a$ are unit vectors that span the tangent space of Ω (note that this definition of H^c is a factor of 2 greater than that given in [3] for the mean curvature vector, h^c, where $\sqrt{h^c h_c} = \frac{1}{2}\sqrt{k^2 - l^2}$).

2.3 Energy in Newtonian Cosmology

With a new physical definition it is important that in appropriate limits we are able to reduce the result down to well established physics. In general relativity we may show that a new result is equivalent to Newtonian physics for a suitable region of space in a weak gravitational field. Since quasilocal energy can be calculated for a cosmological model, comparison to Newtonian cosmology for small regions is a sensible method to validate a new definition of energy.

Newtonian cosmology is based on the simplifying assumption that the universe consists of an expanding sphere of radius r filled with a homogeneous fluid of density $\rho(t)$. As a result it neglects the fact that in general relativity the stress-energy-momentum tensor is also sourced by pressures and anisotropic stresses. However, it can still be used as an approximation in the limiting case of small r. By calculating the gravitational potential energy of mass elements within the fluid, and evaluating the kinetic energy of the fluid, it is possible to arrive at the energy [12]

$$E_{\text{classical}} = \frac{4\pi}{3}\rho_0 a_{N0}^3 a_N(t)^{-3} r^3 - \frac{2\pi}{5}\rho_0 a_{N0}^3 a_N(t)^{-5} r^5 k \tag{2.10}$$

for a spherical region of radius r centered on $r = 0$, where $a_N(t)$ is the scale factor and $k_N \equiv \frac{8\pi G}{3}\rho_0 a_{N0}^2 - \dot{a}_{N0}^2$ is a constant. The scale factor is such that at any time the radius of the sphere is $r(t) = r_0\frac{a_N(t)}{a_{N0}}$ where ρ_0 is the initial fluid density and a_{N0} is the initial scale factor at $t = 0$.

In the work of Afshar [12] the Epp quasilocal energy in a FLRW universe has already been compared to the Newtonian cosmology. Therefore, when we calculate the quasilocal energy in a Lemaître-Tolman universe we need only perform the intermediate step to show that this energy reduces to the FLRW energy for an appropriate limiting case. We will also extend the comparison made by Afshar [12] using a Fermi normal coordinate system to reduce the difference between the quasilocal energy and the energy in (2.10).

2.4 A New Choice of Reference Energy

The choice of the reference spacetime made when calculating quasilocal energy is completely arbitrary. While the Minkowski spacetime appears to give the most physical result that agrees with Newtonian cosmology to leading order [12], we can make other choices for various reasons. In particular, consider a flat ($k = 0$) FLRW reference spacetime to compare the relative energies of an open or closed universe.

We find the Epp quasilocal energy for a FLRW universe with a spatially flat FLRW reference spacetime to be

$$E_{\mathrm{FLRW}} = \frac{ra(t)}{G}\left(\sqrt{1 - r^2\dot{a}^2(t)} - \sqrt{1 - kr^2 - r^2\dot{a}^2(t)}\right). \tag{2.11}$$

It is immediately apparent that a closed $k = 1$ universe will have a positive quasilocal energy for all $r > 0$. On the other hand, a $k = -1$ open universe will have negative quasilocal energy for all $r > 0$.

This result can be understood by considering the matter content of a sphere of fixed radius r. If the sphere is analogous to a closed universe, then the ratio of matter content to vacuum energy is such that eventually the sphere will collapse due to gravitational attraction. On the other hand, if the sphere is analogous to an open universe we expect vacuum energy to dominate over matter. Since matter and vacuum contribute positive and negative energy respectively we would expect the "closed" spherical universe to have a more positive energy, and an open universe to have negative energy, relative to the critical, $k = 0$ case.

Although we can interpret (2.11) in terms of vacuum and matter energies this appears to contradict our understanding of binding energy. For example, we typically consider a bound particle to have negative energy equal in magnitude to the amount of energy required to free the particle. Similarly, we would expect a closed, bounded, universe to have negative energy with respect to the flat $k = 0$ case. However, the choice of sign in the definition of quasilocal energy is simply a matter of convention. The sign is chosen to give agreement with the positive mass theorem of general relativity which states that the total energy of an asymptotically flat universe must be non-negative [21]. Therefore, with an appropriate choice of sign convention the quasilocal energy would not contradict our understanding of the binding energy involved in bounded systems.

3 Quasilocal Conservation Laws

3.1 Rigid Quasilocal Frames

In classical Newtonian mechanics the assumption of rigid body motion is key in understanding the motion of a system of particles. In the Newtonian case internal forces are neglected and only external forces considered. In this way a system can be simplified to an object having six degrees of freedom, three translational and three rotational, with arbitrary time dependence. However, in 1910 Herglotz [5] and Noether [6] found that the relativistic equivalent, a 3-parameter family of time-like worldlines satisfying Born's rigidity conditions [4], does not have these six degrees of freedom, instead having only three. It was not until recent work by Epp, Mann and McGrath [8] that the concept of a Rigid Quasilocal Frame (RQF) was suggested as a resolution to this apparent limitation to the study of rigid motion in special and general relativity.

The RQF in flat spacetime has been demonstrated to admit 6 *conformal* Killing vectors[1] corresponding to the six required degrees of freedom for rigid body motion [9]. This is achieved through the quasilocal nature of a RQF, as a 2-dimensional set of points consisting of the boundary of a finite spatial volume, instead of the Newtonian definition of a three-dimensional set of points in a volume. More precisely, a RQF is defined as a congruence of time-like worldlines forming a worldtube as described in section 2.1 with the additional constraints that this congruence be expansion and shear free.

In order to construct a rigid quasilocal frame we refer to the explanation of the tensor field θ_{ab} given in section 2.1. The requirement that the congruence be expansion and shear free gives us a set of rigidity conditions. That is, we require that $\theta = \theta_{<ab>} = 0$ and hence $\theta_{(ab)} = 0$ which gives three differential constraints. These constraints are equivalent to the three conditions $\partial\sigma_{ij}/\partial t = 0$ [8].

The definition of a RQF implies that each observer on the bounding two surface is permanently at rest with respect to their nearest neighbours, so it is indeed a *rigid* frame. Epp, Mann and McGrath [8] introduce a general form for the induced metric on $\mathcal{B}$ with the use of a coordinate system $x^\mu = (t, x^i)$ and by setting $u^\mu = N^{-1}\delta^\mu_t$, where N is the

[1] A conformal Killing vector field is a vector field X on a manifold (M, g) such that $\mathcal{L}_X g = \phi g$ for some smooth ϕ on M, where $\mathcal{L}_X$ is the Lie derivative [16]. If ϕ is zero then X is a Killing vector field.

lapse function such that $u^a u_a = -c^2$. This general form of the induced metric, $\gamma_{\mu\nu}$, is

$$\gamma_{\mu\nu} = \begin{pmatrix} -c^2N^2 & Nu_j \\ Nu_i & \sigma_{ij} - \frac{1}{c^2}u_i u_j \end{pmatrix} \tag{3.1}$$

where σ_{ij} and the shift covector u_i are the spatial (x^i) coordinate components of σ_{ab} and u_a respectively. In a generic spacetime, by comparing the induced metric on the surface of the RQF with the general form, we can demand rigidity by setting σ_{ij} equal to the metric of a 2-sphere.

The RQF construction provides us with an appropriate setting in which to investigate conservations laws. This is because it isolates change due to expansion and change of shape of the boundary from fluxes of energy, momentum or angular momentum across the boundary. An additional powerful property of RQFs is their adaptability to curved space, as we will investigate in the following section.

3.2 Fermi Normal Coordinates

In order to study RQFs in small regions of curved spacetime it is appropriate to consider the concept of *Fermi normal coordinates*. Misner, Thorne and Wheeler [10] define Fermi normal coordinates to be the "local coordinates of an observer's proper reference frame", for an arbitrary accelerated rotating observer. This coordinate system provides us with a locally inertial reference frame in which to construct a RQF.

Let $X^a = (cT, X^I)$, where $I = 1, 2, 3$ denote the Fermi normal coordinates in the neighbourhood of a time-like worldline $\mathcal{C}$ with arbitrary acceleration in a generic spacetime. The components of the metric in these coordinates are then [11]

$$\begin{aligned} g_{00} &= -\left(1 + \frac{1}{c^2}A_K X^K\right)^2 + \frac{1}{c^2}R^2 W_K W_L P^{KL} - {}^F\mathring{R}_{0K0L}X^K X^L + \mathcal{O}(R^3) \\ g_{0J} &= \frac{1}{c}\epsilon_{JKL}W^K X^L - \frac{2}{3}{}^F\mathring{R}_{0KJL}X^K X^L + \mathcal{O}(R^3) \\ g_{IJ} &= \delta_{IJ} - \frac{1}{3}{}^F\mathring{R}_{IKJL}X^K X^L + \mathcal{O}(R^3) \end{aligned} \tag{3.2}$$

where $R^2 = \delta_{IJ}X^I X^J$, $A_K(T)$ is the proper acceleration along the worldline $\mathcal{C}$, $W_K(T)$ is the proper rate of rotation of the spatial axes along $\mathcal{C}$, $P^{KL} = \delta^{KL} - X^K X^L/R^2$ projects vectors perpendicular to the radial direction and ${}^F\mathring{R}_{abcd}(T)$ are the Fermi normal coordinate components of the Riemann curvature tensor evaluated on $\mathcal{C}$. The time T is proper time along $\mathcal{C}$ and an overset circle denotes that a quantity is evaluated on $\mathcal{C}$ unless the quantity is defined to be on $\mathcal{C}$ already, such as for A_K and W_K [8].

To transform a generic spacetime into Fermi normal coordinates we must calculate the non-zero components of the Riemann curvature tensor, ${}^F R_{ABCD}$. If R_{abcd} denotes the original Riemann tensor, then we can find the transformed tensor by ${}^F R_{ABCD} = R_{abcd}e^a_A e^b_B e^c_C e^d_D$. Here the observer tetrad, (e^a_0, e^a_I), is such that $e^a_0 = \frac{u^a}{c} = \frac{1}{Nc}\delta^a_0$ and

$e^I_a e^J_b \eta_{IJ} = g_{ab}$ [10].

We now introduce a second coordinate system to represent the 2-parameter family of observers in the neighbourhood of the worldline $\mathcal{C}$. This 2-sphere of observers gives the time-like worldtube, $\mathcal{B}$, surrounding $\mathcal{C}$. This set of coordinates is given by $x^\alpha = (\tau, \rho, x^i)$. Following Epp, Mann and McGrath [8] we let $x^i = (\Theta, \Phi)$ where Θ and Φ are the standard spherical coordinates, and we introduce the coordinate transformation

$$\begin{aligned} T(\tau,\rho,\Theta,\Phi) &= \tau, \\ X^I(\tau,\rho,\Theta,\Phi) &= \rho\rho^I(\Theta,\Phi) + \rho^3 f^I(\tau,\Theta,\Phi) + \mathcal{O}\big(\rho^4\big) \end{aligned} \tag{3.3}$$

where $\rho^K(\Theta,\Phi) = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$ are the standard direction cosines of a radial unit vector in spherical coordinates in Euclidean three-space [8]. The function $f^I(\tau,\Theta,\Phi)$ gives us the freedom to adjust the observer's worldlines (defined by constant values of ρ, Θ and Φ) such that the RQF rigidity conditions are satisfied. In particular, we demand that the observer's 2-metric, σ_{ij} be equal to $\rho^2\mathbb{S}_{ij}$, where $\mathbb{S}_{ij}$ is the metric of a 2-sphere. Figure 3.1 depicts the coordinate transformations required to set up a RQF at an arbitrary radial position around a worldline $\mathcal{C}$. The choice of $T = \tau$ above is justified since the rigidity conditions are invariant under a time reparametrisation [8]. The induced metric for the observers on a sphere of fixed ρ is compared to the general form given by (3.1) to obtain the relevant physical quantities.

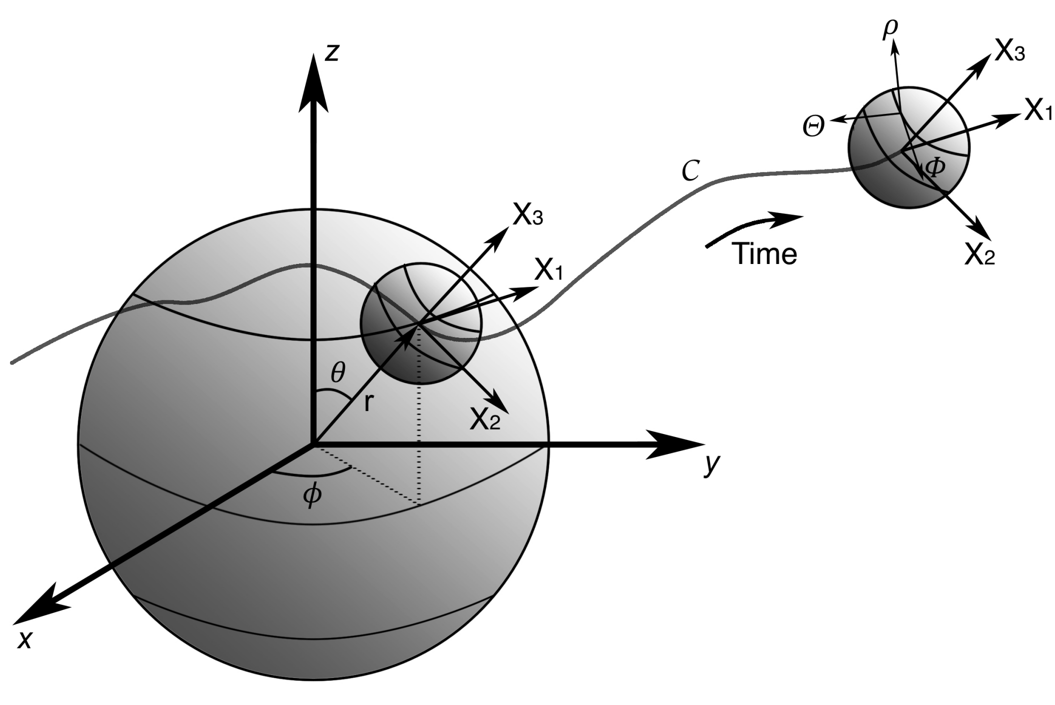

Figure 3.1: The transformation from a spherical coordinate system (r,θ,ϕ) into Fermi normal coordinates (X_1,X_2,X_3) and the transformation from Fermi normal coordinates into observer adapted coordinates (ρ,Θ,Φ) that label the worldlines of observers around the worldline $\mathcal{C}$.

The function $f^I(\tau, \Theta, \Phi)$ introduced in (B.13) can be decomposed as

$$f^I(\tau, \Theta, \Phi) = F(\tau, \Theta, \Phi)\rho^I(\Theta, \Phi) + f^i(\tau, \Theta, \Phi)\mathbb{B}_i^I(\Theta, \Phi) \tag{3.4}$$

where F gives radial, or normal variations of the observer's worldlines and the f^i give angular, or tangential, variations to the observer's worldlines. Also, the boost generators, $\mathbb{B}_i^I$, are functions that have the property that $\mathbb{B}_i^I \partial_i$ are conformal Killing vectors on the unit round sphere in Euclidean space [8].

3.3 Conservation Laws

In general relativity a conservation law is typically constructed by considering the quantity in (1.1). However there are three reasons [8] why this construction is not sufficient for producing an energy conservation law. Firstly, a generic spacetime does not always admit any Killing vectors, making the use of the identity (1.1) limited. Secondly if a suitable Killing vector does exist and still does not equal the observer's four velocity then the resulting conserved quantity is not in general equal to the energy. Finally, the identity (1.1) is limited by the fact that T^{ab}_{mat} does not include gravitational energy. For example, if $T^{ab}_{\text{mat}} = 0$ then we have a trivial result even if gravitational energy does exist, making this conservation law effectively blind to gravitational physics. The concept of RQFs and the total stess-energy-momentum tensor introduced in section 2.2.1 resolves all three of these issues.

We are now familiar with the definition of quasilocal energy using the Brown and York total (matter plus gravitational) stress-energy-momentum tensor. Since this tensor is defined on the worldtube boundary, $\mathcal{B}$, we expect to find an analog of (1.1) on this spacetime. In such a way if ψ^a is an arbitrary vector field tangent to $\mathcal{B}$ then we can write [8]

$$D_a \left(T^{ab}_{\mathcal{B}} \psi_b \right) = \left(D_a T^{ab}_{\mathcal{B}} \right) \psi_b + T^{ab}_{\mathcal{B}} D_{(a} \psi_{b)} \tag{3.5}$$

where D_a is the covariant derivative on $\mathcal{B}$ and $T^{ab}_{\mathcal{B}}$ is the total matter plus gravitational stress-energy-momentum tensor from (2.3). Integrating (3.5) over a section of the world-tube bounded by initial and final space-like slices, $\mathcal{S}_i$ and $\mathcal{S}_f$, gives

$$\frac{1}{c} \int_{\mathcal{S}_f - \mathcal{S}_i} d\mathcal{S} \; T^{ab}_{\mathcal{B}} \psi_b \tilde{u}_a = - \int_{\mathcal{B}} \left[\left(D_a T^{ab}_{\mathcal{B}} \right) \psi_b + T^{ab}_{\mathcal{B}} D_{(a} \psi_{b)} \right] \; d\mathcal{B}. \tag{3.6}$$

where $\tilde{u}_a$ is the future-directed unit normal to the spatial end caps, $\mathcal{S}_{i,f}$. Now if we choose $c\psi^a = u^a$, the 4-velocity of the RQF observers, this gives us an energy conservation equation. In this case the left-hand side of (3.6) becomes $\frac{1}{c^2} \tilde{u}_a u_b T^{ab}_{\mathcal{B}}$. If the observers are at rest with respect to $\mathcal{S}_{i,f}$ then $\tilde{u}^a = u^a$ and this reduces to $\frac{1}{c^2} u_a u_b T^{ab}_{\mathcal{B}} = \epsilon$, giving the energy density that when integrated over $\mathcal{S}_{i,f}$ would give us the change in quasilocal energy between these surfaces. However, since the observers are not in general at rest with respect to the endcaps, Epp, Mann and McGrath [8] introduce a boost transformation $\tilde{u}^a = \alpha(u^a - \beta^a)$ where β^a is a shift vector and α is an inverse lapse function that corresponds to the γ-factor of the associated Lorentz transformation. This results in the more general expression for the left-hand side of (3.6) [8]

$$\int_{\mathcal{S}_f-\mathcal{S}_i} d\mathcal{S}\ \alpha\left(\epsilon+\beta_a\mathcal{P}^a\right) \tag{3.7}$$

since $T^{ab}_{\mathcal{B}}u_b=\epsilon\frac{u^a}{c}+c\mathcal{P}^a$ where $\mathcal{P}^a$ is the surface momentum density, $\mathcal{P}_a=-\frac{1}{c^2}\sigma_a{}^b u^c T^{\mathcal{B}}_{bc},$.

Next we apply the Gauss-Codazzi identity and the Einstein equation to the first term on the right of (3.6). These relations are $D_aT^{ab}_{\mathcal{B}}=\frac{2}{\sqrt{-\gamma}}D_a\pi^{ab}=-\frac{1}{\kappa}n_aG^{ab}$ and $G_{ab}=\kappa T_{ab}$ [18]. This gives $\left(D_aT^{ab}_{\mathcal{B}}\right)\psi_b=-T^{ab}n_a\psi_b$ by the definition of $T^{ab}_{\mathcal{B}}$, (2.3). This is an important result, because unlike the case of the matter stress-energy-momentum tensor where $\nabla_aT^{ab}_{\mathrm{mat}}=0$, the divergence of $T^{ab}_{\mathcal{B}}$ on the surface of the worldtube is not zero. This then demands the existence of external sources in the form of matter fluxes passing through $\mathcal{B}$ that are interacting with the RQF system. These fluxes are associated with the motion of the RQF system that are manifest as accelerations and precision rates of inertial gyroscopes [8]. The precession of inertial gyroscopes is understood by considering a torque free gyroscope being transported around a circle and thus undergoing acceleration. In Newtonian physics the gyroscope would maintain its direction despite the acceleration, yet a relativistic calculations reveals that the gyroscope would precess, known as Thomas precession.

Consider the last term in the conservation law (3.6). While a generic spacetime does not admit any Killing vectors, we have already stated in section 3.1 that the two-surface bounding a spatial volume does indeed admit six conformal Killing vectors. If we take ψ^a as one of these conformal Killing vectors and let $c\psi_a=u^a$ we find the second term on the right-hand side of (3.6) to be

$$\frac{1}{c}T^{ab}_{\mathcal{B}}D_au_b=\frac{1}{c}\alpha_a\mathcal{P}^a+\mathcal{S}^{(ab)}\theta_{(ab)}=\frac{1}{c}\alpha_a\mathcal{P}^a \tag{3.8}$$

where $\alpha^a=\sigma^a_b a^b$ is the projection of the observers' 4-acceleration, $a^a\equiv u^b\nabla_b u^a$, tangent to $\mathcal{B}$, S^{ab} is a spatial stress on the RQF and we have used the RQF condition $\theta_{(ab)}=0$ for the second equality. Epp, Mann and McGrath claim that $\alpha_a\mathcal{P}^a$ is a "simple, exact expression for the outward-directed geometrical flux of gravitational energy across the boundary of a RQF" [8]. With these results the conservation law for the case $\psi^a=u^a/c$ becomes

$$\int_{\mathcal{S}_f-\mathcal{S}_i} d\mathcal{S}\ \alpha\left(\epsilon+\beta_a\mathcal{P}^a\right)=\frac{1}{c}\int_{\Delta\mathcal{B}}d\mathcal{B}\left(n^au^bT^{\mathrm{mat}}_{ab}-\alpha_a\mathcal{P}^a\right) \tag{3.9}$$

where we can consider the first term on the right to be a matter flux and the second a geometrical, or gravitational, flux across the boundary $\mathcal{B}$. The right-hand side of (3.9) is simply the difference in the quasilocal energy of the spatial end caps. We will make use of general expressions for the evaluation of the individual terms in (3.9) for a generic spacetime found by Epp, Mann and McGrath [8].

4 Results

4.1 Quasilocal Energy of a Lemaître-Tolman Universe

4.1.1 Brown and York

In this section we present the main results for a calculation of the Brown and York quasilocal energy in Lemaître-Tolman Universe. The Lemaître-Tolman geometry is described in comoving coordinates (t, r, θ, ϕ) by the metric

$$ds^2 = -dt^2 + \frac{A'^2(r,t)}{1-\kappa(r)}dr^2 + A^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.1}$$

in units with $c = 1$ where $A(r,t)$ and $k(r)$ are arbitrary functions. We calculate the quasilocal energy contained within a sphere of comoving radius r bounded by the 2-surface Ω. The normal to this 2-surface with respect to the Lemaître-Tolman metric is given by

$$n^a = \left(0 \;\; , \;\; \frac{\sqrt{1-\kappa(r)}}{A'(r,t)} \;\; , \;\; 0 \;\; , \;\; 0 \right) \tag{4.2}$$

Using (4.2) and the metric (4.1) we find the extrinsic curvature to be

$$k_{ab} = \mathrm{diag}\Big(0 \;\; , \;\; A(r,t)\sqrt{1-\kappa(r)} \;\; , \;\; A(r,t)\sqrt{1-\kappa(r)}\sin^2(\theta)\Big) \tag{4.3}$$

and hence

$$k = g^{ab}k_{ab} = \frac{2}{A(r,t)}\sqrt{1-\kappa(r)}. \tag{4.4}$$

From the Brown and York definition of quasilocal energy, (2.6), we calculate an (absolute) unreferenced energy of

$$E_{\mathrm{abs}} = -\frac{1}{8\pi G}\int_\Omega d\theta\; d\phi\; A^2(r,t)\sin^2(\theta)\, k = -\frac{1}{G}A(r,t)\sqrt{1-\kappa(r)}. \tag{4.5}$$

To calculate the reference energy we follow the method of Afshar [12]. The reference spacetime is chosen to be Minkowski with the understanding that this would have zero energy. This reference spacetime is denoted by $\bar{M}$ with spherical coordinates $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$ and the metric

$$d\bar{s}^2 = -d\bar{t}^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2\sin^2\bar{\theta}d\bar{\phi}^2.$$

For an isometric embedding $\psi : \Omega \to \bar{M}$ we use the ansatz of Afshar [12]

$$\psi(\theta, \phi) = (\bar{t}(t, r), \bar{r}(t, r), \theta, \phi)$$

where the choice of the angular components is justified by the fact that the spacetime is spherically symmetric. To demand isometry we find must set $\bar{r}(t, r) = A(r, t)$ and hence the reference energy is

$$E_{\text{ref}} = -\frac{1}{G}A(r, t) \tag{4.6}$$

using the Brown and York definition (2.6). The referenced quasilocal energy is therefore

$$E_{\text{BY}} = E_{\text{abs}} - E_{\text{ref}} = \frac{1}{G}A(r, t)\left(1 - \sqrt{1 - \kappa(r)}\right). \tag{4.7}$$

4.1.2 Epp

The calculation of the Epp definition of quasilocal energy requires a different approach to the Brown and York case. We present an outline here and the detailed calculations in the appendix (B.1). First we identify the unit vectors that span the tangent and normal space of the bounding surface which is again a sphere of radius r. The unit normals are $u^a = (1, 0, 0, 0)$ and n^a as given in (4.2). The two unit vectors that span the space tangent to Ω are

$$\xi_1^a = \left(\ 0\ ,\ 0\ ,\ \frac{1}{A(r, t)}\ ,\ 0\ \right) \text{ and } \xi_2^a = \left(\ 0\ ,\ 0\ ,\ 0\ ,\ \frac{1}{A(r, t)\sin(\theta)}\ \right). \tag{4.8}$$

From the above unit vectors and the metric (4.1) we use (2.9) to obtain the mean curvature vector

$$H^a = \left(\frac{-2\dot{A}(r, t)}{A(r, t)}, \frac{2(1 - \kappa(r))}{A(r, t)A'(r, t)},\ 0\ ,\ 0\ \right).$$

After calculating $\sqrt{H^a H_a}$ we find the unreferenced Epp quasilocal energy from (2.8) to be

$$E_{\text{abs}} = -\frac{1}{G}A(r, t)\sqrt{1 - \kappa(r) - \dot{A}^2(r, t)}. \tag{4.9}$$

The reference energy is again found using an isometric embedding of Ω into a Minkowski spacetime giving a referenced energy of

$$E_{\text{Epp}} = E_{\text{abs}} - E_{\text{ref}} = \frac{1}{G}A(r, t)\left(1 - \sqrt{1 - \kappa(r) - \dot{A}^2(r, t)}\right). \tag{4.10}$$

If we set $A(r, t) = ra(t)$ and $\kappa(r) = kr^2$ then we obtain the FLRW metric from the Lemaître-Tolman metric and the quasilocal energy given here reduces to that found for a FLRW universe by Afshar [12]. This verifies the validity of our calculations since the result found by Afshar gives the Newtonian limit in a limiting case and we are able to reduce our result to that of Afshar also in a limiting case.

4.2 Existence of a Rigid Quasilocal Frame in a Lemaître-Tolman Universe

In order to consider a quasilocal conservation law for a Lemaître-Tolman universe we first need to prove the existence of a RQF in this spacetime. Additional details of these calculations are provided in the appendix B.2.

We will construct a rigid quasilocal frame in a neighborhood of the worldline $\mathcal{C}$, where $\mathcal{C}$ is positioned on the origin ($r = 0$) for all time.

The first step involves transforming the Lemaître-Tolman metric into the locally inertial Fermi normal coordinates, (cT, X_1, X_2, X_3) using (3.2). To achieve this we calculate the non-zero components of the Riemann tensor and carry out the coordinate transformation using the tetrad given in [15]. This tetrad sets the X_1 axis along the polar direction, the X_2 axis along the azimuthal direction and the X_3 axis along the radial direction (as depicted in Figure 3.1). We then use the components of the Riemann tensor in Fermi normal coordinates to define the functions

$$\begin{aligned}
K_1(r,t) &\equiv {}^F R_{0101} = {}^F R_{0202} = -\frac{\ddot{A}(r,t)}{A} \\
K_2(r,t) &\equiv {}^F R_{0303} = -\frac{\ddot{A}'(r,t)}{A'(r,t)} \\
K_3(r,t) &\equiv {}^F R_{3131} = {}^F R_{3232} = \frac{1}{2}\frac{1}{A(r,t)A'(r,t)}\frac{\partial}{\partial r}\left(\dot{A}(r,t)^2 + \kappa(r)\right) \\
K_4(r,t) &\equiv {}^F R_{1212} = \frac{1}{A(r,t)^2}\left(\dot{A}(r,t)^2 + \kappa(r)\right)
\end{aligned} \tag{4.11}$$

which in the following we will denote using the notation $K_i = K_i(r,t)$ for $i = 1, \ldots, 4$.

To construct the RQF in a neighborhood of the worldline $\mathcal{C}$ we transform the metric using

$$\begin{aligned}
T(\tau, \rho, \Theta, \Phi) &= \tau \\
X^I(t, \rho, \Theta, \Phi) &= \rho\hat{\rho}^I(\Theta, \Phi) + F(t)\rho^3\hat{\rho}^I(\Theta, \Phi)
\end{aligned} \tag{4.12}$$

where we choose $F = F(\tau) = F(T)$ on the understanding that this universe is spherically symmetric at $r = 0$ and ρ, the radius of the RQF, is fixed. This new coordinate system, $(\tau, \rho, \Theta, \Phi)$, is the observer adapted coordinates. For a given ρ, fixed values of Θ and Φ correspond to a worldline in the congruence of observers on the worldtube boundary. To find the induced metric on the worldtube, $\mathcal{B}$, we use the relation $\gamma_{ab} = g_{ab} - n_a n_b$. Here n_a is the space-like unit normal to $\mathcal{B}$ (in the radial direction) calculated from the metric in observer adapted coordinates.

The induced metric on $\mathcal{B}$ is found to be

$$\gamma_{\mu\nu} = \begin{pmatrix} -1+\rho^6\dot{F}^2 - WV & 0 & 0 \\ 0 & \frac{\rho^2\left(1+\rho^2 F\right)^2}{3}\left(3-\mathring{K}_3 W\right) & 0 \\ 0 & 0 & \frac{\rho^2\sin^2\Theta\left(1+\rho^2 F\right)^2}{3}\left(3-WU\right) \end{pmatrix} \tag{4.13}$$

where $U = \left(\mathring{K}_4 \sin^2\Theta + \mathring{K}_3\cos^2\Theta\right)$, $V = \left(\mathring{K}_1\sin^2\Theta + \mathring{K}_2\cos^2\Theta\right)$ and $W = (\rho^2 + 2\rho^4 F + \rho^6 F^2)$. We can compare (4.13) with the general form of the induced metric (3.1) to find the lapse, $N = -1 + \rho^6\dot{F}^2 - WV$ and shift $u_i = 0$. Since the shift is zero we can form the rigidity conditions by equating the two lower diagonal elements of $\gamma_{\mu\nu}$ with $\rho^2\mathbb{S}_{ij}$. This gives us the two equations

$$\begin{aligned} 1 &= \frac{(1+\rho^2F)^2}{3}\left(3-\mathring{K}_3(\rho^2+2\rho^4F+\rho^6F^2)\right) \\ 1 &= \frac{(1+\rho^2F)^2}{3}\left(3-(\mathring{K}_3\cos^2\Theta+\mathring{K}_4\sin^2\Theta)\left(\rho^2+2\rho^4F+\rho^6F^2\right)\right). \end{aligned} \tag{4.14}$$

To solve these two algebraic equations we first recall the use of the overset circle to denote that a quantity is evaluated on the world line $\mathcal{C}$. So in this case we evaluate K_3 and K_4 at $r = 0$. To reduce (4.14) to one algebraic equation we use the result of Mashhoon [15] which gives

$$\mathring{K}_3 = \mathcal{W}(T) + \mathcal{O}(r) \qquad \text{and} \qquad \mathring{K}_4 = \mathcal{W}(T) + \mathcal{O}(r)$$

where the form of the function $\mathcal{W}(T)$ and a full justification of this result is given in [15]. Therefore at $r = 0$, $K_3 = K_4$ and we find the solution for $F(T)$ to be

$$F(T) = \frac{-2\mathring{K}_3\rho^4 \pm \sqrt{6\rho^3\mathring{K}_3 \pm 2\rho^6\mathring{K}_3\sqrt{-12\mathring{K}_3\rho^2+9}}}{2\rho^6\mathring{K}_3}. \tag{4.15}$$

where there is no correlation between the $\pm$ signs, so we have a set of four non-trivial solutions in total. Since there exists a non-trivial transformation defined by $F(T)$ we have proved the existence of an RQF centered on a worldline at the $r = 0$ position of a Lemaître-Tolman universe.

Alternatively, if we were to construct this RQF about $r \neq 0$ we would no longer have $K_3 = K_4$ in general and the only solution would be

$$F = -\frac{1}{\rho^2}, \tag{4.16}$$

which is a trivial result since the transformation of the spatial coordinates in (4.12) becomes $X^I = 0$. This is expected since the assumption that $F = F(t)$ would no longer hold if we shift the RQF away from the center of spherical symmetry, indeed, only at $r = 0$ can we assume the transformation to the RQF to be spherically symmetric.

Now that we have found that a rigid quasilocal frame exists in a Lemaître-Tolman spacetime, we calculate the individual terms in the conservation law introduced in section 3.1. This involves finding the change in quasilocal energy in the RQF due to the fluxes through the worldtube boundary. We find the energy density to be

$$\epsilon = -\frac{2}{\kappa\rho} + \frac{\rho}{\kappa}\left(K_4 \sin^2\Theta + K_3\cos^2\Theta\right). \tag{4.17}$$

When we integrate the energy density over the sphere we find that the change in quasilocal energy is

$$\Delta\mathrm{QE} = \left[\frac{8\pi\rho^3}{3\kappa}\left(K_4 + 2K_3\right)\right]_{T_i}^{T_f}, \tag{4.18}$$

and since we have comoving coordinates $\beta_a = 0$ and thus (4.18) gives the left hand side of the conservation law (3.9). We find that the only non-zero terms on the right-hand side of the conservation law correspond to the matter fluxes through $\mathcal{B}$. Therefore we can see that the change in quasilocal energy contained within the RQF constructed here is entirely associated with the "motion" of the RQF system [8].

4.3 Comparison to Newtonian Cosmology with Fermi Normal Coordinates

Fermi normal coordinates provide a frame of reference that is locally inertial and as such "closer" to a Newtonian limit. Because of this nature of Fermi normal coordinates we are interested in comparing the quasilocal energy calculated in these coordinates with the Newtonian cosmology. We will transform the FLRW metric into Fermi normal coordinates and then calculate the corresponding quasilocal energy. Since the transformation to Fermi normal coordinates is only exact up to quadratic order in r, we do not expect the coordinate invariance of the quasilocal energy to hold, hence offering a closer comparison to Newtonian cosmology.

After transforming the FLRW metric in coordinates (t, ρ, θ, ϕ),

$$ds^2 = -dt^2 + \frac{a(t)^2}{(1+\frac{k}{4}\rho^2)^2}\left(d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{4.19}$$

to Fermi normal coordinates using the transformation (3.2) and then to spherical polar coordinates (τ, r, Θ, Φ) we obtain the metric

$$g_{ab} = \begin{pmatrix} -1 + r^2\frac{\ddot{a}(t)}{a(t)} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{r^2}{3}\left(3 - r^2\left(\frac{\dot{a}^2(t)+k}{a^2(t)}\right)\right) & 0 \\ 0 & 0 & 0 & \frac{r^2\sin^2\Theta}{3}\left(3 - r^2\left(\frac{\dot{a}^2(t)+k}{a^2(t)}\right)\right) \end{pmatrix}. \tag{4.20}$$

The quasilocal energy is then calculated with respect to (4.20), following the same method as performed for the Lemaître-Tolman case (see B.3). We find the unreferenced Epp definition of quasilocal energy to be

$$E_{\text{abs}} = -\frac{r}{3G}\sqrt{\frac{1}{a^4(t)}\left(3a^2(t) - 2r^2\dot{a}^2(t) - 2r^2k\right)^2 - \frac{r^6\dot{a}^2(t)}{a(t)}\frac{\left(\ddot{a}(t)a(t) - \dot{a}^2(t) - k\right)}{\left(1 - \frac{r^2\ddot{a}(t)}{a(t)}\right)}} \tag{4.21}$$

and the reference energy to be

$$E_{\text{ref}} = -\frac{r}{\sqrt{3}G}\sqrt{3 - r^2\frac{\left(\dot{a}^2(t) + k\right)}{a^2(t)}}. \tag{4.22}$$

To compare the referenced energy to Newtonian cosmology we perform a Taylor expansion of the above energy and reference energy about $r = 0$. Before carrying out this expansion, we substitute for $\dot{a}$ using the Friedmann equation for a matter dominated universe

$$\frac{\dot{a}^2(t)}{a^2(t)} + \frac{k}{a^2(t)} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a(t)}\right)^3 \tag{4.23}$$

where ρ_0 and a_0 are the matter density and scale factor at $t = 0$ respectively. Therefore we can set $\dot{a}(t) = \sqrt{\frac{8\pi G}{3a}\rho_0 a_0^3 - k}$. Performing the Taylor expansion gives

$$E = E_{\text{abs}} - E_{\text{ref}} = \frac{4\pi}{3}\frac{r^3\rho_0 a_0^3}{a^3(t)} - \frac{8\pi^2\rho_0^2}{81}\frac{Gr^5a_0^6}{a^6(t)} + \mathcal{O}(r^7). \tag{4.24}$$

Now for the spatially flat $k = 0$ case we find the difference between (4.24) and the Newtonian energy (2.10) to be

$$|E - E_{\text{Newtonian}}| = \frac{8\pi^2}{81}\frac{G\rho_0^2 a_0^6 r^5}{a^6(t)} \tag{4.25}$$

since $a_N(t) = a(t)$ and $k_N = k$ when $k = 0$ [12]. This difference is a factor of 9 smaller than that found by Afshar [12] in the original FLRW coordinates. This shows that in Fermi normal coordinates the Epp quasilocal energy is slightly closer to the Newtonian result, by almost one order of magnitude in the r^5 term.

In addition to the preceding result we have also noted that in at least two cases the quasilocal energy calculated in Fermi normal coordinates is independent of the representation of the original spatial coordinates. That is, the quasilocal energy of two different representations of the FLRW metric is the same *without* the need to transform the coordinates in the final result. We prove this observations holds for a general scaling of the spatial coordinates, and show it holds for two different representations of the FLRW metric in appendix B.3.

5 Discussion

The most important result in this report has been the calculation of the Brown and York and the Epp quasilocal energies for a Lemaître-Tolman universe. This original calculation has confirmed the observation made by Afshar [12] regarding the ambiguity of the Brown and York definition in non-stationary spacetimes. For example, we see that the Brown and York quasilocal energy in the Lemaître-Tolman case (4.7) gives a trivial result if $\kappa(r) = 0$ for any choice of $A(r,t)$. Since a $\kappa(r) = 0$ universe would not contain zero energy in general, this confirms the statement in [12] that the Epp definition is the preferable choice for defining quasilocal energy.

The comparison between quasilocal energy of a flat FLRW universe and Newtonian cosmology in the appendix of [12] has been extended in this report. We have carried out this comparison by a perturbative expansion of the quasilocal energy evaluated in the Fermi normal coordinate frame. The aim was to obtain a reduced error with respect to the Newtonian cosmology. This was achieved, with a reduction in the error by approximately one order of magnitude. The difference in quasilocal energy in the Fermi normal coordinates appears to affect terms above leading order in r by a factor of $-1/9$. The result is that the quasilocal energy in a Fermi normal frame is, for at least small r, less than in the original FLRW coordinates. This is intuitively understood, since Fermi normal coordinates, being adapted to a local inertial frame, will have "less curvature". Therefore the quasilocal energy will be less by our understanding of the relationship between the curvature of a region of spacetime and the total (matter plus gravitational) energy in that region. We also extended [12] by calculating the FLRW quasilocal energy in the $k = \pm 1$ case using Fermi normal coordinates. This calculation showed no change in the form of the energy once the value of $\dot{a}(t)$ had been evaluated using the Friedmann equation with the appropriate choice of k.

We have found that calculating the quasilocal energy in Fermi normal coordinates gives a result in terms of proper distance. In addition, the calculation of the quasilocal energy in Fermi normal coordinates is independent of a scaling of the spatial coordinates, as we prove in (B.3). This could have been expected since Fermi normal coordinates are intended to give a locally inertial frame in terms of proper distances [20]. For this reason the locally flat Fermi normal coordinates appear to be a natural testing ground for quasilocal energy definitions, as the Newtonian cosmology is also given in terms of a proper distance.

Our next important result was the proof of the existence of a rigid quasilocal frame in a Lemaître-Tolman geometry. We found that it is possible to construct a non-trivial congruence of rigid observers in a neighbourhood of the center of spherical symmetry for such a universe. This construction depends on the existence of the function $F(T)$ that "perturbs" the observers' worldlines in order to maintain rigidity. We found an explicit non-trivial expression for this function, $F(T)$. This is indeed the most simple case for a Lemaître-Tolman universe. Moving away from the center of symmetry increases the complexity of the situation. The starting point for these more general calculations are presented in the appendix (B.2).

Although the rigid quasilocal construction is vital for understanding conservation laws, limitations for cosmological applications have become apparent in this investigation. The foremost issue arises through the use of Fermi normal coordinates which give only an approximate transformation exact up to quadratic order. Indeed, as we demonstrated the quasilocal energy of a FLRW universe in Fermi normal coordinates differs from the original coordinate system at order r^5. In cosmological applications we are interested in going beyond the perturbative approximation, to scales that involve curvature that would be neglected by the locally inertial Fermi normal coordinates. On the other hand, rigid quasilocal frames may still be useful in understanding expansion in a small region of spacetime. For example, the function $F(T)$ found in our results is intended to "undo" the expansion and shear of the bounding two-surface, and as such it may contain information that we could associate with the expansion of space. We have not explored the implications of this idea yet, but it could be a focus for future investigations.

An example of interest for general cosmological applications is to consider the quasilocal energy on a large expanding sphere outside a bound structure such as a cluster of galaxies, in comparison to a similar sphere in a void region of vastly smaller density. We are hence interested in the existence of rigid frames without the need for the locally flat Fermi normal coordinates, since these rigid frames simplify the calculations involved, particularly for conservation laws. One possible method is to consider radial boosts of observers such that they maintain their position on a spherical shell of constant radius. However, this would only be suitable for a spherically symmetric situation in which we can ignore changes in shape due to shear. Investigation of these ideas, along with comparisons between the quasilocal energies of various cosmological geometries would be an appropriate future extension of this research.

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A Technical Background

A.1 Brown and York Quasilocal Energy

This section is supplementary to the sketch derivation of the Brown and York quasilocal energy presented in section 2.2.1. It is intended to offer the reader details of the mathematical steps which are omitted in the Brown and York paper [2].

In particular consider the variation of the first two terms of the action (2.1). First we take the variation of the trace of the extrinsic curvature, $K = h^{ab}K_{ab}$,

$$
\begin{aligned}
\delta(K) &= \delta(h^{ab}K_{ab}) = K_{ab}\delta h^{ab} + h^{ab}\delta K_{ab} \\
&= K_{ab}\delta h^{ab} + h^{ab}\delta\left(h_a{}^c\nabla_c u_b\right) \\
&= K_{ab}\delta h^{ab} + h^{ab}\delta\left(h_a{}^c\left(\partial_c u_b - \Gamma^d_{cb}u_d\right)\right) \\
&= K_{ab}\delta h^{ab} + h^{ab}\left(\left(\delta h_a{}^c\right)\nabla_c u_b + h_a{}^c\partial_c\delta u_b - h_a{}^c\Gamma^d_{cb}\delta u_d - u_d h_a{}^c\delta\Gamma^d_{cb}\right) \\
&= K_{ab}\delta h^{ab} + h^{ab}\left(h_a{}^c\nabla_c\delta u_b - h_a{}^c\left(\delta\Gamma^d_{cb}\right)u_d\right) \\
&= K_{ab}\delta h^{ab} + h^{ab}h_a{}^c\nabla_c\delta u_b - h^{ab}h_a{}^c u_d\left(\frac{g^{de}}{2}\left(\nabla_c\delta g_{be} + \nabla_b\delta g_{ce} - \nabla_e\delta g_{cb}\right)\right) \\
&= K_{ab}\delta h^{ab} + h^{ab}\nabla_a\delta u_b - \frac{h^{bc}}{2}g^{de}\left(\nabla_c u_d\delta g_{be} - \delta g_{be}\nabla_c u_d\right) - \frac{h^{bc}}{2}u_d g^{de}\left(\nabla_b\delta g_{ce} - \nabla_e\delta g_{cb}\right)
\end{aligned}
$$

but since the connection is metric-compatible we have that $h^{bc}g^{de}\nabla_c u_d = h^{bc}\nabla_c u^e = K^{be}$ so the above expression becomes

$$
\begin{aligned}
\delta(K) &= K_{ab}\delta h^{ab} + h^{bc}\nabla_c\delta u_b + \frac{1}{2}\delta g_{be}K^{be} - \frac{1}{2}h^{bc}\nabla_c u^e\delta g_{be} - \frac{h^{bc}}{2}u_d g^{de}\left(\nabla_b\delta g_{ce} - \nabla_e\delta g_{cb}\right) \\
&= -\frac{1}{2}K^{ab}\delta h_{ab} + h^{bc}\nabla_c\left(\delta u_b - \frac{1}{2}u^e\delta g_{be}\right) - \frac{h^{bc}}{2}u^e\left(\nabla_b\delta g_{ce} - \nabla_e\delta g_{cb}\right).
\end{aligned}
$$

We have used here the relations $\delta h_{ab}K^{ab} = \left(\delta g_{ab} + \frac{u_a}{c^2}\delta u_b + \frac{u_b}{c^2}\delta u_a\right)K^{ab} = \delta g_{ab}K^{ab}$ and $u_a K^{ab} = u_b K^{ab} = 0$.

Now, we can take the variation of the action to be

$$
\begin{aligned}
\delta(S) =& \frac{1}{2\kappa}\int_M d^4x\left(\sqrt{-g}(\delta R_{ab})g^{ab} + \sqrt{-g}R_{ab}\delta g^{ab} + R\left(\delta\sqrt{-g}\right)\right) + \frac{1}{\kappa}\int_{S_i}^{S_f} d^3x\left(K\delta\sqrt{h} + \sqrt{h}\delta K\right) \\
=& \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\left(R_{ab} - \frac{1}{2}g_{ab}\right)\delta g^{ab} + \frac{1}{2\kappa}\int_M d^4x g^{ab}\sqrt{-g}\delta R_{ab}
\end{aligned}
$$

$$+\frac{1}{\kappa}\int_{S_i}^{S_f} d^3x\sqrt{h}\left(\frac{1}{2}h^{ab}\left(\delta h_{ab}\right)K+\delta K\right).$$

Next we make use of the relation $g^{ab}\delta R_{ab} = \nabla^a v_a$ where $v_a = \nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right)$ (see Wald [16] (E.1.15)), so the second integral in the preceding expression becomes

$$\begin{aligned}\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\nabla^a v_a &= \frac{1}{2\kappa}\int_{S_i}^{S_f} d^3x\sqrt{h}n^a v_a\\ &= \frac{1}{2\kappa}\int_{S_i}^{S_f} d^3x\sqrt{h}n^a h^{bc}\left(\nabla_c\left(\delta g_{ab}\right)-\nabla_a\left(\delta g_{bc}\right)\right).\end{aligned}$$

Combining the above result with that found for δK the total variation is then

$$\begin{aligned}\delta S =&\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}G_{ab}\delta g^{ab} + \frac{1}{2\kappa}\int_{S_i}^{S_f} d^3x\sqrt{h}\left(h^{ab}K - K^{ab}\right)\delta h_{ab}\\ &+\frac{1}{\kappa}\int_{S_i}^{S_f} d^3x\sqrt{h}\left(h^{bc}\nabla_c\left(\delta u_b - \frac{1}{2}u^e\delta g_{be}\right)\right)\\ &+\frac{1}{2\kappa}\int_{S_i}^{S_f} d^3x\sqrt{h}\left(h^{bc}u^a\nabla_c\delta g_{ab} - u^a h^{bc}\nabla_a\delta g_{bc} - h^{bc}u^a\nabla_b\delta g_{ca} + h^{bc}u^a\nabla_a\delta g_{cb}\right)\\ =&\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}G_{ab}\delta g^{ab} + \int_{S_i}^{S_f} d^3x\sqrt{h}P^{ab}\delta h_{ab}.\end{aligned}$$

By a similar calculation, if we consider the boundary of the boundary then

$$\delta S = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}G_{ab}\delta g^{ab} + \int_{S_i}^{S_f} d^3x\sqrt{h}P^{ab}\delta h_{ab} + \int_\Omega d^3x\pi^{ij}\delta\gamma_{ij}$$

which gives the required result.

A.2 Quasilocal Conservation Laws

The following is a derivation of (3.8) that was presented in [8] without proof.

$$\begin{aligned}\frac{1}{c}T_{\mathcal{B}}^{ab}D_a u_b &= \frac{1}{c}T_{\mathcal{B}}^{ab}\gamma_a{}^c\nabla_c u_b\\ &= \frac{1}{c}T_{\mathcal{B}}^{ab}\left(\sigma_b{}^c - \frac{1}{c^2}u_a u^c\right)\nabla_c u_b\\ &= \frac{1}{c}T^{\mathcal{B}}_{ab}\left(-\frac{1}{c^2}\right)a^b u^a + \frac{1}{c}T_{\mathcal{B}}^{ab}\sigma_a{}^c\sigma_b{}^k\nabla_c u_k\\ &= \frac{1}{c}T^{\mathcal{B}}_{lk}\left(-\frac{1}{c^2}\right)a^b\left(\delta^l{}_b + \frac{1}{c^2}u_b u^l - n_b n^l\right)u^k + \frac{1}{c}\sigma_c{}^l T_{\mathcal{B}}^{cd}\sigma_d{}^k\nabla_l u_k\\ &= \frac{1}{c}\left(-\frac{1}{c^2}\right)T^{\mathcal{B}}_{lk}a^b\sigma^l{}_b u^k + \frac{1}{c}\sigma^a{}_c\sigma^b{}_d T_{\mathcal{B}}^{cd}\sigma_a{}^l\sigma_b{}^k\nabla_l u_k\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{c}\left(-\frac{1}{c^2}\right) T^{\mathcal{B}}_{lk} a^b \sigma_{ab} \sigma^{al} u^k + \frac{1}{c}\sigma^{ac}\sigma_{bd} T^{\mathcal{B}}_{cd} \sigma_a{}^l \sigma_b{}^k \nabla_l u_k \\
&= \frac{1}{c}\alpha_a \mathcal{P}^a + \mathcal{S}^{(ab)}\theta_{(ab)}
\end{aligned}
$$

where the fourth line follows from the fact that $u_a a^a = 0$ and $n_b u^a \nabla_a u^b = 0$ from Wald (3.3.6) [16].

The following is a outline of the results of Epp, Mann and McGrath in [8] that we use to calculate the terms in the quasilocal conservation law (3.9).

Firstly, in a general spacetime the energy density is [8]

$$
\epsilon = -\frac{2}{\kappa\rho} - \frac{\rho}{\kappa}\left[\left(\frac{3}{c^2}W_I W_J + \delta^{KL}\, {}^F\!\mathring{R}_{IKJL}\right)\rho^I\rho^J - \frac{1}{2}\delta^{IJ}\, {}^F\!\mathring{R}_{IJKL}\right]. \tag{A.1}
$$

Similarly the momentum density is [8]

$$
\mathcal{P}_i = \frac{1}{c\kappa}\rho W_I \mathbb{R}^I_i + \rho^2\left[-\frac{1}{\kappa c}\left(c\mathring{\mathcal{B}}_{IJ} + \frac{2}{c^3}W_I A_J\right)\mathbb{R}^I_i \rho^J - \frac{1}{2}\mathring{T}^{\text{mat}}_{0I}\mathbb{B}^I_i\right] + \mathcal{O}\left(\rho^3\right) \tag{A.2}
$$

where $\mathcal{B}_{IJ}$ is the magnetic part of the Weyl tensor and $\mathbb{R}^I_i = -\mathbb{E}^j_i \mathbb{B}^I_j$ are the rotation generator counterparts to $\mathbb{B}^I_i$ (where $\mathbb{E}^j_i$ is the volume form associated with $\mathbb{S}_{ij}$).

Turning now to the right-hand side of the conservation law we must consider the matter and geometrical fluxes. Since $d\mathcal{B}/c = \rho^2 d\mathbb{S}\, dt\, N$ and N is the lapse function we need to find the fluxes times the lapse function, Epp, Mann and McGrath give the general expression for these terms [8]

$$
\begin{aligned}
N\left(n^a u^b T^{\text{mat}}_{ab}\right) &= \underbrace{-\rho^I \mathring{S}_I}_{\star} + \rho\left(\frac{1}{3}\frac{\partial \mathring{T}^{\text{mat}}_{00}}{\partial t} + \frac{1}{3c^2}\mathring{S}_I A^I + \Psi_{mat}\right) + \mathcal{O}(\rho^2), \\
N\left(-\alpha\cdot\mathcal{P}\right) &= \underbrace{-\frac{c^2}{8\pi G}\epsilon_{IJK} r^I A^J W^K}_{\star} + r\left(-\frac{1}{3c^2}\mathring{S}_I A^I - \frac{1}{3}\frac{c^2}{8\pi G}\frac{\partial W^2}{\partial t} + \Psi_{geo}\right) + \mathcal{O}(\rho^2),
\end{aligned}
$$

where $\mathring{S}_I := -c\mathring{T}_{0I}$ is the matter energy flux in the X^I direction and Ψ_{mat} and Ψ_{geo} are flux terms corresponding to $l = 2$ spherical harmonics, and hence give zero contribution to the energy change when integrated over a sphere. Similarly, the terms with a star beneath correspond to $l = 1$ spherical harmonics and integrate to zero over angles. Therefore, after adding these terms and integrating over a sphere the quasilocal conservation law becomes

$$
\int_{\mathcal{S}_f - \mathcal{S}_i} d\mathcal{S}\ \alpha\left(\epsilon + \beta_a \mathcal{P}^a\right) = \left[\frac{4\pi r^3 T^{\text{mat}}_{00}}{3} - \frac{r^3 c^2 W^2}{6G}\right]^{t_f}_{t_i} \tag{A.3}
$$

where the first and second terms on the right-hand side are associated with matter and gravitational fluxes respectively.

B Technical Details of the Original Calculations

B.1 Epp Quasilocal Energy for a Lemaître-Tolman Universe

We present a detailed outline of the Epp quasilocal energy in a Lemaître-Tolman universe. The calculation of the Brown and York quasilocal energy, which involves just one extrinsic curvature, follows similar steps.

Firstly, to obtain the mean curvature vector we expand (2.9) as

$$H^c = \left(n^c \nabla_a n_b - u^c \nabla_a u_b\right) {\xi_1}^a {\xi_1}^b + \left(n^c \nabla_a n_b - u^c \nabla_a u_b\right) {\xi_2}^a {\xi_2}^b \tag{B.1}$$

so the 0^{th} component is

$$\begin{aligned} H^0 &= \left(n^0 \nabla_a n_b - u^0 \nabla_a u_b\right) \left({\xi_1}^a {\xi_1}^b + {\xi_2}^a {\xi_2}^b\right) \\ &= -\left(\partial_a u_b - {\Gamma^d}_{ab} u_d\right) \left({\xi_1}^a {\xi_1}^b + {\xi_2}^a {\xi_2}^b\right) \\ &= -\left(\partial_2 u_2 - {\Gamma^0}_{22} u_0\right) {\xi_1}^2 {\xi_1}^2 - \left(\partial_3 u_3 - {\Gamma^0}_{33} u_0\right) {\xi_2}^3 {\xi_2}^3 \\ &= \left({\Gamma^0}_{22} {\xi_1}^2 {\xi_1}^2 + {\Gamma^0}_{33} {\xi_2}^3 {\xi_2}^3\right) \end{aligned}$$

However, ${\Gamma^0}_{22} = A(r,t)\dot{A}(r,t)$ and ${\Gamma^0}_{33} = A(r,t)\dot{A}(r,t)\sin^2\theta$, and thus

$$H^0 = -\frac{A\dot{A}}{A^2} - \frac{A\dot{A}\sin^2\theta}{A^2\sin^2\theta} = -\frac{2\dot{A}}{A}.$$

Similarly, we calculate H^1

$$\begin{aligned} H^1 &= \left(n^1 \nabla_a n_b - u^1 \nabla_a u_b\right) \left({\xi_1}^a {\xi_1}^b + {\xi_2}^a {\xi_2}^b\right) \\ &= \left(\partial_a n_b n^1 - n^1 {\Gamma^d}_{ab} n_b\right) \left({\xi_1}^a {\xi_1}^b + {\xi_2}^a {\xi_2}^b\right) \\ &= \left(\partial_2 n_2 n^1 - {\Gamma^1}_{22} n_1 n^1\right) {\xi_1}^2 {\xi_1}^2 + \left(\partial_3 n_3 n^1 - {\Gamma^1}_{33} n_1 n^1\right) {\xi_2}^3 {\xi_2}^3 \\ &= -{\Gamma^1}_{22} {\xi_1}^2 {\xi_1}^2 - {\Gamma^1}_{33} {\xi_2}^3 {\xi_2}^3 \end{aligned}$$

But ${\Gamma^1}_{22} = -\frac{(1-\kappa)A}{A'}$ and ${\Gamma^1}_{33} = -\frac{(1-\kappa)A\sin^2\theta}{A'}$, and thus

$$H^1 = \frac{(1-\kappa)A}{A'A^2} + \frac{(1-\kappa)A\sin^2\theta}{A'A^2\sin^2\theta} = 2\frac{(1-\kappa)}{AA'}.$$

The remaining two components of the mean curvature vector are zero because the unit vectors, u^a and n^a are only non-zero in the first and second components respectively, and H^λ is only non-zero if $n^a \neq 0$ or $u^a \neq 0$. Therefore

$$H^c = \left(\frac{-2\dot{A}(r,t)}{A(r,t)}, \frac{2(1-\kappa(r))}{A(r,t)A'(r,t)}, \ 0 \ , \ 0 \right) \tag{B.2}$$

and

$$\sqrt{H^c H_c} = \frac{2\sqrt{1-\kappa(r)-\dot{A}^2(r,t)}}{A(r,t)}. \tag{B.3}$$

The Epp quasilocal energy is then obtained using the definition (2.8). The reference energy is obtained by a similar calculation.

B.2 Existence of an Rigid Quasilocal Frame in a Lemaître-Tolman Universe

This section contains supplementary details for the proof presented in section 4.2. The first step involves transforming the Lemaître-Tolman metric into Fermi normal coordinates, for this we must find the non-zero components of the Riemann tensor with respect to (4.1). These components are found to be

$$\begin{aligned}
R_{0101} &= \frac{A'(r,t)\ddot{A}'(r,t)}{\kappa(r)-1} \\
R_{0202} &= -A(r,t)\ddot{A}(r,t) \\
R_{0303} &= -A(r,t)\sin^2(\theta)\ddot{A}(r,t) \\
R_{1212} &= \frac{A(r,t)A'(r,t)\left(2\dot{A}(r,t)\dot{A}'(r,t)+\kappa'(r)\right)}{2(1-\kappa(r))} \\
R_{1313} &= \frac{\sin^2(\theta)A(r,t)A'(r,t)\left(2\dot{A}(r,t)\dot{A}'(r,t)+\kappa'(r)\right)}{2(1-\kappa(r))} \\
R_{2323} &= A(r,t)^2\sin^2(\theta)\left(\dot{A}(r,t)^2+\kappa(r)\right)
\end{aligned} \tag{B.4}$$

To calculate the non-zero components of the Riemann tensor in Fermi normal coordinates we use the tetrad [15]

$$e_0{}^a = u^a = \frac{1}{c}\delta_0{}^a \tag{B.5}$$

$$e_1{}^a = \left(0, 0, \frac{1}{A(r,t)}, 0\right) \tag{B.6}$$

$$e_2{}^a = \left(0, 0, 0, \frac{1}{A(r,t)\sin(\theta)}\right) \tag{B.7}$$

$$e_3{}^a = \left(0, \frac{\sqrt{1-\kappa(r)}}{A'(r,t)}, 0, 0\right) \tag{B.8}$$

where u^a is the velocity of comoving observers in this universe. Applying this transformation gives the non-zero components of the Riemann tensor in (4.11).

Now using (3.2) the Lemaître-Tolman metric in Fermi normal coordinates (cT, X_1, X_2, X_3) becomes

$$g_{IJ} = \begin{pmatrix} 1 - \frac{1\left(\mathring{K}_4 X_2^2 + \mathring{K}_3 X_3^2\right)}{3} & \frac{1}{3}\mathring{K}_4 X_2 X_1 & \frac{1}{3}\mathring{K}_3 X_1 X_3 \\ \frac{1}{3}\mathring{K}_4 X_2 X_1 & 1 - \frac{1\left(\mathring{K}_3 X_3^2 + \mathring{K}_4 X_1^2\right)}{3} & \frac{1}{3}\mathring{K}_3 X_2 X_3 \\ \frac{1}{3}\mathring{K}_3 X_1 X_3 & \frac{1}{3}\mathring{K}_3 X_2 X_3 & 1 - \frac{\mathring{K}_3\left(X_1^2 + X_2^2\right)}{3} \end{pmatrix}$$

with $g_{00} = -1 - \left(\mathring{K}_1\left(X_1^2 + X_2^2\right) + \mathring{K}_2 X_3^2\right)$ and $g_{0I} = 0$.

Using the transformation to observer adapted coordinates (4.12) we obtain the metric

$$g_{IJ} = \begin{pmatrix} \left(1+3\rho^2 F\right)^2 & 0 & 0 \\ 0 & \frac{\rho^2\left(1+\rho^2 F\right)^2}{3}\left(3 - \mathring{K}_3 G\right) & 0 \\ 0 & 0 & \frac{\rho^2 \sin^2\Theta\left(1+\rho^2 F\right)^2}{3}\left(3 - WU\right) \end{pmatrix}$$

with $g_{00} = -1 + \rho^6 \dot{F}^2 - WV$, $g_{01} = \rho^3\left(1+3\rho^2 F\right)\dot{F}$ and $g_{02} = g_{03} = 0$. From this we can calculate the induced metric, $\gamma_{\mu\nu}$, with the remaining steps presented in section 4.2.

The following is an incomplete extension of section 4.2 intended to generalise the RQF construction to the neighbourhood of a general worldline in a Lemaître-Tolman universe. Since this is an aside and not pertinent to any results presented in this work, we give only a sketch of the main steps.

If we are to construct a RQF about a worldline at $r \neq 0$, then we must allow F to be a function of the angles and time, that is, $F = F(T, \Theta, \Phi)$. Therefore the transformation to observer adapted coordinates becomes

$$\begin{aligned} T &= t \\ X^I &= \rho\rho^I + \rho^3\rho^I F(t, \Theta, \Phi) + \mathcal{O}(\rho^4) \end{aligned}$$

where with this transformation we obtain the following induced metric

$$\sigma_{ab} = \begin{pmatrix} -\frac{r^2}{3}\left(-3r^4\left(\frac{\partial F}{\partial\Theta}\right)^2 + V + K_3 U\right) & 0 \\ 0 & \begin{matrix} -\frac{r^2\sin^2\Theta}{3}\left(-3r^4\left(\frac{\partial F}{\partial\Phi}\right)^2 + V\right. \\ \left. + U\left(K_3\cos^2\Theta + K_4\sin^2\Theta\right)\right) \end{matrix} \end{pmatrix}$$

where $V = -3 - 3r^4F^2 - 6r^2F$ and $U = r^2 + 4r^4F + 6r^6F^2 + 4r^8F^3 + r^{10}F^4$. Therefore the rigidity condition implies that $F(t, \Theta, \Phi)$ satisfies the following set of differential equations

$$3 = 3r^4\left(\frac{\partial F}{\partial\Theta}\right)^2 - V - K_3 U$$

$$3 = 3r^4 \left(\frac{\partial F}{\partial \Phi}\right)^2 - V - U\left(K_3 \cos^2\Theta + K_4 \sin^2\Theta\right)$$

which reduce to the result obtained in 4.2 at the center of spherical symmetry.

B.3 Quasilocal Energy of a FLRW Universe in Fermi Normal Coordinates

In this section we present the details of the calculations leading to the result in section 4.3. Firstly, the non-zero components of the Riemann tensor are calculated with respect to the FLRW metric (4.19) to be

$$R_{0101} = R_{0202} = R_{0303} = -\frac{a\ddot{a}}{(1+\frac{k}{4}\rho^2)}$$
$$R_{1212} = R_{1313} = R_{2323} = \frac{(k+\dot{a}^2)a^2}{(1+\frac{k}{4}\rho^2)^4}$$

Using the appropriate tetrad given by,

$$e_0^a = (-1,0,0,0) \qquad e_1^a = \left(0, \frac{1-\frac{k}{4}\rho^2}{a(t)}, 0, 0\right)$$
$$e_2^a = \left(0,0,\frac{1-\frac{k}{4}\rho^2}{\rho a(t)},0\right) \quad e_3^a = \left(0,0,0,\frac{1-\frac{k}{4}\rho^2}{\rho\sin\theta a(t)}\right)$$

we find

$${}^F\!R_{0101} = {}^F\!R_{0202} = {}^F\!R_{0303} = -\frac{\ddot{a}}{a}$$
$${}^F\!R_{1212} = {}^F\!R_{1313} = {}^F\!R_{2323} = \frac{k+\dot{a}^2}{a^2}.$$

Now, via the transformation (3.2) we obtain the metric in Fermi normal coordinates as

$$g_{AB} =$$
$$\begin{pmatrix} -1+\frac{\ddot{a}}{a}\left(X_1^2+X_2^2+X_3^2\right) & 0 & 0 & 0 \\ 0 & 1-\frac{1}{3}\Lambda\left(X_2^2+X_3^2\right) & \frac{1}{3}\Lambda X_1X_2 & \frac{1}{3}\Lambda X_1X_3 \\ 0 & \frac{1}{3}\Lambda X_1X_2 & 1-\frac{1}{3}\Lambda(X_1^2+X_3^2) & \frac{1}{3}\Lambda X_2X_3 \\ 0 & \frac{1}{3}\Lambda X_1X_3 & \frac{1}{3}\Lambda X_2X_3 & 1-\frac{1}{3}\Lambda\left(X_1^2+X_2^2\right) \end{pmatrix}$$

where $\Lambda = \left(\frac{\dot{a}^2+k}{a^2}\right)$. We then carry out the transformation to spherical polar coordinates (τ, r, θ, ϕ) to obtain the metric

$$ds_F^2 = \left(-1+\frac{r^2\ddot{a}}{a}\right)d\tau^2 + dr^2 + \frac{r^2}{3}\left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{B.9}$$

In order to calculate the quasilocal energy with respect to (B.9) we need the unit vectors that span the normal and tangent space of a spherical two-surface centered on $r = 0$, these are

$$u^a = \left(\frac{1}{\sqrt{\frac{\ddot{a}r^2}{a^2} - 1}} , \ 0 \ , \ 0 \ , \ 0 \right), \qquad n^a = (\ 0 \ , \ 1 \ , \ 0 \ , \ 0 \ )$$

$$\xi_1^a = \left(\ 0 \ , \ 0 \ , \frac{\sqrt{3}}{r\sqrt{3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)}} , \ 0 \right), \quad \xi_2^a = \left(\ 0 \ , \ 0 \ , \ 0 \ , \frac{\sqrt{3}}{r\sqrt{3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)}} \right).$$

The mean curvature vector is calculated from the definition given in (2.9). We begin with the 0$^{\text{th}}$ component

$$\begin{aligned} H^0 &= \left(n^0\nabla_a n_b - u^0\nabla_a u_b\right)\left({\xi_1}^a{\xi_1}^b + {\xi_2}^a{\xi_2}^b\right) \\ &= -\left(u^0\nabla_a u_b\right)\left({\xi_1}^a{\xi_1}^b + {\xi_2}^a\xi_2^b\right) \\ &= -\left(u^0\partial_a u_b - {\Gamma^d}_{ab}u_d u^0\right)\left({\xi_1}^a{\xi_1}^b + {\xi_2}^a{\xi_2}^b\right). \end{aligned}$$

As the vectors ${\xi_1}^a$ and ${\xi_2}^a$ each only have one non-zero component this expression becomes

$$\begin{aligned} H^0 &= -\left(u^0\partial_2 u_2 - {\Gamma^d}_{22}u_d u^0\right){\xi_1}^2{\xi_1}^2 - \left(u^0\partial_3 u_3 - {\Gamma^d}_{33}u_d u^0\right){\xi_2}^3{\xi_2}^3 \\ &= {\Gamma^0}_{22}u_0u^0{\xi_1}^2{\xi_1}^2 + {\Gamma^0}_{33}u_0u^0{\xi_2}^3{\xi_2}^3 \\ &= -{\Gamma^0}_{22}{\xi_1}^2{\xi_1}^2 - {\Gamma^0}_{33}{\xi_2}^3{\xi_2}^3 \end{aligned}$$

since u^a also has only one non-zero component and $u_0u^0 = -1$. Substituting for the Christoffel symbols we obtain

$$\begin{aligned} H^0 &= \frac{1}{3}\frac{r^4\dot{a}}{a^2}\frac{(\ddot{a}a - \dot{a}^2 - k)}{(a - r^2\ddot{a})}\frac{3}{r^2\left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right)} + \frac{1}{3}\frac{r^4\dot{a}}{a^2}\frac{(\ddot{a}a - \dot{a}^2 - k)}{(a - r^2\ddot{a})}\frac{3}{r^2\sin^2\theta\left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right)} \\ &= \frac{2r^2\dot{a}}{a^2}\frac{(\ddot{a}a - \dot{a}^2 - k)}{(a - r^2\ddot{a})\left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right)} \end{aligned}$$

Similarly,

$$\begin{aligned} H^1 &= \left(n^1\nabla_a n_b - u^1\nabla_a u_b\right)\left({\xi_1}^a{\xi_1}^b + {\xi_2}^a{\xi_2}^b\right) \\ &= \left(n^1\partial_a n_b - {\Gamma^d}_{ab}n_d n^1\right)\left({\xi_1}^a{\xi_1}^b + {\xi_2}^a{\xi_2}^b\right) \\ &= \left(\partial_2 n_2 n^1 - {\Gamma^d}_{22}n_d n^1\right){\xi_1}^2{\xi_1}^2 + \left(\partial_3 n_3 n^1 - {\Gamma^d}_{33}n_d n^1\right){\xi_2}^3{\xi_2}^3 \\ &= -{\Gamma^d}_{22}n_d n^1{\xi_1}^2{\xi_1}^2 - {\Gamma^d}_{33}n_d n^1{\xi_2}^3{\xi_2}^3 \\ &= -{\Gamma^1}_{22}n_1 n^1{\xi_1}^2{\xi_1}^2 - {\Gamma^1}_{33}n_1 n^1{\xi_2}^3{\xi_2}^3 \\ &= \frac{2}{3ra^2}\left(3a^2 - 2r^2\dot{a}^2 - 2r^2k\right)\frac{3}{\left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right)} \\ &= \frac{2}{ra^2}\frac{(3a^2 - 2r^2\dot{a}^2 - 2r^2k)}{\left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right)} \end{aligned}$$

The remaining two components are both zero. Therefore

$$\begin{aligned} H^c H_c &= g_{00} H^0 H^0 + g_{11} H^1 H^1 \\ &= \frac{4}{\left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right)^2} \left(\frac{1}{r^2 a^4} \left(3a^2 - 2r^2\dot{a}^2 - 2r^2 k\right)^2 - \frac{r^4 \dot{a}^2}{a^6} \frac{\left(\ddot{a}a - \dot{a}^2 - k\right)^2}{\left(1 - \frac{r^2 \ddot{a}}{a}\right)} \right). \end{aligned}$$

With an isometric embedding such that $\bar{r} = r\sqrt{1 - \frac{r^2}{3}\left(\frac{\dot{a}^2+k}{a^2}\right)}$ and using $\sqrt{|\sigma|} = \frac{1}{3} r^2 \left(3 - r^2\left(\frac{\dot{a}^2+k}{a^2}\right)\right) \sin\theta$ we obtain the quasilocal energy (4.21) and reference energy (4.22).

Now, we stated that the quasilocal energy in Fermi normal coordinates is independent of a scaling of the coordinates, to prove this consider the following metric

$$ds^2 = -dt^2 + \frac{s a(t)^2}{\left(1 + \frac{sk}{4}(x^2+y^2+z^2)\right)^2} \left(dx^2 + dy^2 + dz^2\right) \tag{B.10}$$

where $k \in \{-1, 0, 1\}$ and $s \in \mathbb{R}$. Note that (B.10) is simply (4.19) expressed in cartesian coordinates where the spatial coordinates are scaled by a factor of $\sqrt{s}$. Using the Maple tensor package we calculate the following non-zero components of the Riemann tensor with respect to the metric (B.10)

$$\begin{aligned} R_{0101} = R_{0202} = R_{0303} &= -\frac{s a \ddot{a}}{(1 + \frac{sk}{4}(x^2+y^2+z^2))^2} \\ R_{1212} = R_{1313} = R_{2323} &= \frac{s^2 (k + \dot{a}^2) a^2}{(1 + \frac{sk}{4}(x^2+y^2+z^2))^4} \end{aligned}$$

Therefore, using the tetrad: $e_0^a = (-1, 0, 0, 0)$, $e_1^a = \left(0, \frac{1 - \frac{sk}{4} R^2}{\sqrt{s} a(t)}, 0, 0\right)$, $e_2^a = \left(0, 0, \frac{1 - \frac{sk}{4} R^2}{\sqrt{s} a(t)}, 0\right)$ and $e_3^a = \left(0, 0, 0, \frac{1 - \frac{sk}{4} R^2}{\sqrt{s} a(t)}, \right)$ (where $r = x^2 + y^2 + z^2$), we obtain

$$\begin{aligned} {}^F R_{0101} = {}^F R_{0202} = {}^F R_{0303} &= -\frac{\ddot{a}}{a} \\ {}^F R_{1212} = {}^F R_{1313} = {}^F R_{2323} &= \frac{k + \dot{a}^2}{a^2} \end{aligned} \tag{B.11}$$

Since the Fermi normal coordinates depend only on these components of the Riemann tensor we can immediately see that the result will be equal to that obtained above and hence the quasilocal energy calculated in Fermi normal coordinates is invariant under a scaling of the spatial coordinates.

The above property of Fermi normal coordinates can be extended to a less trivial transformation of the spatial coordinates. We find that the evaluation of the quasilocal energy from two different coordinate representations of the FLRW metric returns the same result in Fermi normal coordinates. Consider the FLRW metric in the form

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \right), \tag{B.12}$$

with the transformation between the metrics (B.12) and (4.19) given by

$$\rho = \frac{2}{kr}\left(1+\sqrt{1-kr^2}\right) \;\leftrightarrow\; r = \frac{\rho}{(1+\frac{k}{4}\rho^2)} \tag{B.13}$$

where clearly the time component remains the same. We find that non-zero components of the Riemann tensor with respect to the metric (4.19) are identical to (B.11) once transformed with the tetrad: $e_0^a = (-1,0,0,0)$, $e_1^a = \left(0, \frac{\sqrt{1-kr^2}}{a(t)}, 0, 0\right)$, $e_2^a = \left(0,0,\frac{1}{ra(t)},0\right)$ and $e_3^a = \left(0,0,0,\frac{1}{r\sin\theta\, a(t)}\right)$. Therefore once the metric is transformed into to Fermi normal coordinates and the quasilocal energy calculated it will again be equal to (4.21) with reference energy (4.22).

To see that the above result is not a trivial property of the Epp quasilocal energy consider the effect of calculating the quasilocal energy with respect to the two different metrics without making the transformation to Fermi normal coordinates. For the energy with respect to (B.12) we state the result of Afshar [12]

$$E_1 = \frac{1}{G} r a(t)\left[1-\sqrt{1-kr^2-r^2\dot{a}^2(t)}\right]. \tag{B.14}$$

Now, we have calculated the Epp energy with respect to (4.19) to be

$$E_2 = -\frac{a^2(t)\rho^2}{G\left(1+\frac{k}{4}\rho^2\right)^2}\sqrt{\frac{\left(\frac{k}{4}\rho^2-1\right)^2}{a(t)^2\rho^2} - \frac{\dot{a}(t)^2}{a(t)^2}} + \frac{a(t)\rho}{G\left(1+\frac{k}{4}\rho^2\right)} \tag{B.15}$$

which after the coordinate transformation (B.13) returns the expected result (B.14). Clearly, a scaling of the spatial coordinates would also not give the same quasilocal energy without making the required transformation. This is precisely what we would expect since quasilocal energy should be a coordinate invariant quantity. However, as we have seen in at least two cases the quasilocal energy in Fermi normal coordinates appears to be independent of the representation of the original spatial coordinates. That is, the quasilocal energy is the same *without* the need for an appropriate transformation.