RESEARCH REPORT

SHORT WAVE GROUPS IN DEEP WATER

P. J. Bryant

No. 18

November 1981

Department of Mathematics

University of Canterbury
Christchurch
New Zealand

SHORT WAVE GROUPS IN DEEP WATER

By P.J. Bryant

Mathematics Department, University of Canterbury,

Christchurch, New Zealand.

Waves on the water surface propagate typically in wave groups, each group consisting of a small number of waves. The envelopes of the wave groups change, in general, as the groups propagate. Particular envelope shapes remain constant for certain ranges of the group height to wavelength ratio ε and the group length to wavelength ratio k_0. Envelopes for groups containing a large number of waves ($k_0 \gg 1$) of small amplitude ($\varepsilon \ll 1$) are modelled by the cubic Schrödinger equation. Short periodic groups of permanent envelope exist only for larger values of ε. A numerical method is described for obtaining solutions of the nonlinear water wave equations representing periodic wave groups of permanent envelope without small ε or large k_0 assumptions. The method, which is based on the fast Fourier transform technique, has applications elsewhere in nonlinear wave problems. Examples of short wave groups of permanent envelope are presented.

1. Introduction

The action of wind on the water surface is to generate wave groups whose group lengths are a small multiple of the lengths of the waves in the groups. The envelopes of such groups change, in general, as the groups propagate, whether or not the wind is present. This investigation is concerned with calculating those freely propagating periodic wave groups whose envelopes have permanent shape. It is possible that the envelopes of permanent shape may be the asymptotic forms of the envelopes of developing wave groups.

Zakharov (1968) derived equations for nonlinear periodic waves on deep water in the small amplitude approximation. He showed further that if all wavenumbers lie in a narrow waveband, the envelope of the wave train satisfies the cubic Schrödinger equation. This equation has solutions describing envelopes of permanent shape, which because of the narrow waveband assumption, enclose groups whose length is large compared with the lengths of the waves inside them. Yuen & Lake (1980) reviewed many of the interesting properties that have been found for the wave group solutions of the cubic Schrödinger equation, and for their relation to the long-time evolution of narrow waveband small amplitude wave trains. One such property, described originally by Zakharov & Shabat (1971), is that a small amplitude narrow waveband wave group evolves asymptotically into a wave group of permanent envelope.

In an earlier investigation (Bryant, 1979), wave group solutions were derived from the nonlinear wave equations without making the narrow waveband approximation, but with the small amplitude approximation still retained. The wave group solutions tended towards uniform wave train solutions as the group length to wavelength ratio was decreased, without short wave groups being found. It was clear that the small amplitude approximation was the limiting factor, so a new approach to the nonlinear wave equations has been devised which makes neither the narrow waveband nor the small amplitude approximations.

An important part of the new approach is the use of the fast Fourier transform technique, following the successful application of this technique to nonlinear wave phenomena by Fornberg & Whitham (1978), where more information about it may be found. The calculations were made on the Prime 750 at the University of Canterbury using Fortran 77, with the fast Fourier transform subroutines adapted from the Harwell Subroutine Library.

2. Governing equations

The nondimensional equations for the irrotational motion associated with a wave train propagating in the x-direction on deep water are

$$\phi_{xx} + \phi_{yy} = 0\,, \quad y < \varepsilon\eta\,, \tag{2.1a}$$

$$\phi_x, \phi_y \to 0\,, \quad y \to -\infty\,, \tag{2.1b}$$

$$\eta_t - \phi_y + \varepsilon\eta_x\phi_x = 0\,, \quad y = \varepsilon\eta\,, \tag{2.1c}$$

$$\phi_t + \eta + \tfrac{1}{2}\varepsilon\phi_x^2 + \tfrac{1}{2}\varepsilon\phi_y^2 = 0\,, \quad y = \varepsilon\eta\,. \tag{2.1d}$$

The nondimensional water surface displacement is $\eta(x,t)$, and $\phi(x,y,t)$ is the nondimensional velocity potential. Periodic wave group solutions of permanent envelope are sought for which $2\pi\ell$ is the wavelength of a typical wave in the group, $2\pi L$ is the group length, and a is the maximum group height above the mean horizontal surface $y = 0$. The length scale ℓ is used in the derivation of equations (2.1), and the wave slope parameter is $\varepsilon = a/\ell$. All nondimensional wavenumbers k are integers, corresponding to dimensional wavenumbers k/L, but the ratio $k_0 = L/\ell$ need not be an integer.

Periodic wave group solutions of the cubic Schrödinger equation have the form

$$\eta = \sum_k a_k \cos\left\{\frac{kx}{k_0} - \left(1 + \frac{k - k_0}{2k_0} + \alpha\varepsilon^2\right)t\right\} \tag{2.2a}$$

$$= \left[\sum_k a_k \cos\frac{k - k_0}{k_0}(x - \tfrac{1}{2}t)\right]\cos\{x - (1 + \alpha\varepsilon^2)t\}$$

$$- \left[\sum_k a_k \sin\frac{k - k_0}{k_0}(x - \tfrac{1}{2}t)\right]\sin\{x - (1 + \alpha\varepsilon^2)t\}\,, \tag{2.2b}$$

describing a group propagating with the group velocity $\tfrac{1}{2}$ whose amplitudes a_k are symmetric about k_0. It should be noted that the wave frequency is

$$\omega(k) = \omega(k_0) + (k - k_0)\left(\frac{d\omega}{dk}\right)_{k_0} + \tfrac{1}{2}(k - k_0)^2\left(\frac{d^2\omega}{dk^2}\right)_{k_0} + \ldots$$

$$= 1 + \frac{k - k_0}{2k_0} + 0\left(\frac{k - k_0}{k_0}\right)^2 .$$

The narrow waveband assumption is equivalent to stating that the wavenumbers k satisfy

$$\frac{k - k_0}{k_0} \sim \varepsilon \ll 1 .$$

The frequency correction $\alpha\varepsilon^2$ is such that α tends to 0.25 as k_0 becomes large at small ε. The velocity potential associated with the surface displacement, equation (2.2a), is

$$\phi = \sum_k b_k \exp\frac{ky}{k_0}\sin\left\{\frac{kx}{k_0} - \left(1 + \frac{k - k_0}{2k_0} + \alpha\varepsilon^2\right)t\right\}, \tag{2.2c}$$

where $b_k = a_k(1 + 0(\varepsilon))$.

The generalisation of equations (2.2) that satisfies equations (2.1) with the full nonlinear boundary conditions is

$$\eta = \sum_j\sum_k a_{jk} \cos\left\{\frac{k}{k_0}(x - \tfrac{1}{2}t) - j(\tfrac{1}{2} + \alpha\varepsilon^2)t\right\}, \tag{2.3a}$$

$$\phi = \sum_j\sum_k b_{jk} \exp\frac{ky}{k_0}\sin\left\{\frac{k}{k_0}(x - \tfrac{1}{2}t) - j(\tfrac{1}{2} + \alpha\varepsilon^2)t\right\}. \tag{2.3b}$$

The index j = 0 describes the surface displacement and water motion which is steady relative to the group structure. The index j = 1 refers to the amplitudes contained in equations (2.2) lying in a waveband about k_0. Each higher value of j describes a set of amplitudes in a waveband lying about jk_0, the maximum surface displacement amplitude in each of these wavebands being estimated by the ratio ε^{j-1} of the maximum amplitude in the waveband for j = 1. When equations (2.3a,b) are substituted into equations (2.1c,d), with the function $\exp(k\varepsilon\eta/k_0)$ expanded as a power series, the latter equations may be written respectively

$$F = \sum_m\sum_n F_{mn} \sin\left\{\frac{n}{k_0}(x - \tfrac{1}{2}t) - m(\tfrac{1}{2} + \alpha\varepsilon^2)t\right\} = 0, \tag{2.4a}$$

$$G = \sum_m\sum_n G_{mn} \cos\left\{\frac{n}{k_0}(x - \tfrac{1}{2}t) - m(\tfrac{1}{2} + \alpha\varepsilon^2)t\right\} = 0, \tag{2.4b}$$

where F_{mn}, G_{mn} are functions of all amplitudes a_{jk}, b_{jk} and α.

Equations (2.4a,b) are valid for all x and t, implying that

$$F_{mn} = G_{mn} = 0, \quad \text{all } m,n. \tag{2.5}$$

It is possible in principle to calculate F_{mn}, G_{mn} as functions of a_{jk}, b_{jk}, and α, and then to solve equations (2.5) for a_{jk}, b_{jk}, and α. A likely strategy for the algebraic calculation would be that of successive approximations to F_{mn}, G_{mn}, followed by successive approximations to a_{jk}, b_{jk}, and α. A numerical method has been devised instead which uses the fast Fourier transform technique to calculate F_{mn}, G_{mn}, and Newton's method of solving nonlinear algebraic equations to find a_{jk}, b_{jk}, and α.

3. Numerical method of solution

The surface displacement η and velocity potential ϕ in equations (2.3a,b) are substituted now into the nonlinear boundary conditions (2.1c,d) in order to obtain explicit relations for F,G in equations (2.4a,b). The abbreviations

$$c_{jk} = \cos\{(k/k_0)(x-\tfrac{1}{2}t) - j(\tfrac{1}{2}+\alpha\varepsilon^2)t\},$$

$$s_{jk} = \sin\{(k/k_0)(x-\tfrac{1}{2}t) - j(\tfrac{1}{2}+\alpha\varepsilon^2)t\}$$

are used in the expansions.

$$\begin{aligned} F = {} & \sum_j\sum_k \{k/(2k_0) + j(\tfrac{1}{2}+\alpha\varepsilon^2)\}a_{jk}s_{jk} - \sum_j\sum_k (k/k_0)\exp(k\varepsilon\eta/k_0)b_{jk}s_{jk} \\ & - \varepsilon[\sum_j\sum_k (k/k_0)a_{jk}s_{jk}] \times [\sum_j\sum_k (k/k_0)\exp(k\varepsilon\eta/k_0)b_{jk}c_{jk}] = 0, \end{aligned} \tag{3.1a}$$

$$\begin{aligned} G = {} & -\sum_j\sum_k \{k/2k_0) + j(\tfrac{1}{2}+\alpha\varepsilon^2)\}\exp(k\varepsilon\eta/k_0)b_{jk}c_{jk} + \sum_j\sum_k a_{jk}c_{jk} \\ & + \tfrac{1}{2}\varepsilon[\sum_j\sum_k (k/k_0)\exp(k\varepsilon\eta/k_0)b_{jk}c_{jk}]^2 \\ & + \tfrac{1}{2}\varepsilon[\sum_j\sum_k (k/k_0)\exp(k\varepsilon\eta/k_0)b_{jk}s_{jk}]^2 = 0. \end{aligned} \tag{3.1b}$$

Also, from the definition of ε,

$$H = \sum_j\sum_k a_{jk} - 1 = 0. \tag{3.1c}$$

The partial derivatives needed for Newton's method are now listed.

$$\partial F/\partial a_{jk} = \{k/(2k_0) + j(\tfrac{1}{2}+\alpha\varepsilon^2)\}s_{jk} - \varepsilon c_{jk}\sum_p\sum_q (q/k_0)^2 \exp(q\varepsilon\eta/k_0)b_{pq}s_{pq}$$
$$- \varepsilon(k/k_0)s_{jk}\sum_p\sum_q (q/k_0)\exp(q\varepsilon\eta/k_0)b_{pq}c_{pq}$$
$$- \varepsilon^2 c_{jk}[\sum_p\sum_q (q/k_0)a_{pq}s_{pq}] \times [\sum_p\sum_q (q/k_0)^2\exp(q\varepsilon\eta/k_0)b_{pq}c_{pq}] , \qquad (3.2a)$$

$$\partial F/\partial b_{jk} = - (k/k_0)\exp(k\varepsilon\eta/k_0)s_{jk}$$
$$- \varepsilon(k/k_0)\exp(k\varepsilon\eta/k_0)c_{jk}\sum_p\sum_q (q/k_0)a_{pq}s_{pq} , \qquad (3.2b)$$

$$\partial F/\partial\alpha = \varepsilon^2 \sum_p\sum_q p a_{pq}s_{pq} , \qquad (3.2c)$$

$$\partial G/\partial a_{jk} = - \varepsilon c_{jk}\sum_p\sum_q \{q/(2k_0) + p(\tfrac{1}{2}+\alpha\varepsilon^2)\}(q/k_0)\exp(q\varepsilon\eta/k_0)b_{pq}c_{pq} + c_{jk}$$
$$+ \varepsilon^2 c_{jk}[\sum_p\sum_q (q/k_0)\exp(q\varepsilon\eta/k_0)b_{pq}c_{pq}]$$
$$\times [\sum_p\sum_q (q/k_0)^2\exp(q\varepsilon\eta/k_0)b_{pq}c_{pq}]$$
$$+ \varepsilon^2 c_{jk}[\sum_p\sum_q (q/k_0)\exp(q\varepsilon\eta/k_0)b_{pq}s_{pq}]$$
$$\times [\sum_p\sum_q (q/k_0)^2\exp(q\varepsilon\eta/k_0)b_{pq}s_{pq}] , \qquad (3.2d)$$

$$\partial G/\partial b_{jk} = - \{k/(2k_0) + j(\tfrac{1}{2}+\alpha\varepsilon^2)\}\exp(k\varepsilon\eta/k_0)c_{jk}$$
$$+ \varepsilon(k/k_0)\exp(k\varepsilon\eta/k_0)c_{jk}\sum_p\sum_q (q/k_0)\exp(q\varepsilon\eta/k_0)b_{pq}c_{pq}$$
$$+ \varepsilon(k/k_0)\exp(k\varepsilon\eta/k_0)s_{jk}\sum_p\sum_q (q/k_0)\exp(q\varepsilon\eta/k_0)b_{pq}s_{pq} , \qquad (3.2e)$$

$$\partial G/\partial\alpha = - \varepsilon^2 \sum_p\sum_q \exp(q\varepsilon\eta/k_0)p b_{pq}c_{pq} , \qquad (3.2f)$$

$$\partial H/\partial a_{jk} = 1 , \quad \partial H/\partial b_{jk} = 0 , \quad \partial H/\partial\alpha = 0 . \qquad (3.3)$$

Suppose that it is required to calculate all surface displacement amplitudes a_{jk} having magnitudes greater than some small quantity such as 10^{-3}, and that the total number of such amplitudes is N. The corresponding wavenumber ranges, found by trial and error, are

$$k_1(j) \leqslant k \leqslant k_2(j) , \quad j = 0,1,\ldots,J ,$$

where $k_1(0) = 1$, $k_1(j) \geqslant 1$, and

$$\sum_{j=0}^{J} (k_2(j) - k_1(j) + 1) = N . \qquad (3.4)$$

A given wavenumber k may occur in the ranges defined by more than one value of j. The 2N + 1 unknown quantities to be calculated are a_{jk}, b_{jk} for the values of j and k above, and α. An initial estimate is made of the 2N+1 unknowns and this is substituted into all of equations (3.1,3.2). Next, a double Fourier transform over x and t is made of each of these equations to calculate $F_{mn}, G_{mn}, (\partial F/\partial a_{jk})_{mn}, \ldots, (\partial G/\partial \alpha)_{mn}$ for $k_1(m) \leqslant n \leqslant k_2(m)$, $m = 0,1,\ldots,J$.

Newton's method applied to F as a function of the 2N+1 variables a_{jk}, b_{jk}, and α, is described by

$$\sum_j \sum_k (\partial F/\partial a_{jk})(a_{jk}-a'_{jk}) + \sum_j \sum_k (\partial F/\partial b_{jk})(b_{jk}-b'_{jk}) + (\partial F/\partial \alpha)(\alpha-\alpha') = F, \tag{3.5}$$

where $'$ denotes the new value of each variable, and F and its derivatives are calculated at the old values of the variables. The double Fourier transform of equation (3.5) is

$$\sum_j \sum_k (\partial F/\partial a_{jk})_{mn}(a_{jk}-a'_{jk}) + \sum_j \sum_k (\partial F/\partial b_{jk})_{mn}(b_{jk}-b'_{jk}) + (\partial F/\partial \alpha)_{mn}(\alpha-\alpha') = F_{mn}, \tag{3.6a}$$

for $k_1(m) \leqslant n \leqslant k_2(m)$, $m = 0,1,\ldots,J$. Similarly

$$\sum_j \sum_k (\partial G/\partial a_{jk})_{mn}(a_{jk}-a'_{jk}) + \sum_j \sum_k (\partial G/\partial b_{jk})_{mn}(b_{jk}-b'_{jk}) + (\partial G/\partial \alpha)_{mn}(\alpha-\alpha') = G_{mn}, \tag{3.6b}$$

over the same N values as equation (3.6a), and

$$\sum_j \sum_k (\partial H/\partial a_{jk})(a_{jk}-a'_{jk}) = H. \tag{3.6c}$$

Since all coefficients of the set of 2N+1 linear equations (3.6) are known, the set may be solved to obtain the new values of each of the 2N+1 variables a_{jk}, b_{jk}, and α.

Calculations were begun with the solution of the cubic Schrödinger equation at wave slope $\varepsilon = 0.05$ and group length to wavelength ratio $k_0 = 20$. The wave slope ε was then increased step by step at constant

k_0, and k_0 was decreased step by step at constant ε, with each solution being used as the initial estimate for the next solution. The number of wavenumbers N was also increased step by step for each value of ε and k_0, the criterion being that the first and last amplitude a_{jk} for each waveband should have a magnitude less than 10^{-3}, unless the lower limit $k_1(j)$ was already 1. Newton's method converged well in most examples calculated, the main exception being examples near to uniform wave train solutions where care had to be taken to keep the two types of solution apart. The accuracy of the solutions could be tested by determining F_{mn} and G_{mn} in equations (2.4) not only for the N wavenumbers included in the calculation but for other wavenumbers also.

4. Short wave groups

The parts of the surface displacement and horizontal velocity at the water surface which are steady relative to the group structure are given, from equations (2.3), by

$$\eta_0(x,t) = \sum_{k=1}^{k_2(0)} a_{0k} \cos(k/k_0)(x-\tfrac{1}{2}t) , \tag{4.1a}$$

$$u_0(x,t) = \sum_{k=1}^{k_2(0)} (k/k_0) b_{0k} \cos(k/k_0)(x-\tfrac{1}{2}t) .$$

Both have negative minima below the group maxima, representing a set-down and counter-current below each group maximum. Although cause and effect cannot be separated, the growth and decay of the amplitude of a wave train as it moves through a group maximum is influenced directly by the convergence and divergence regions respectively of the horizontal velocity $u_0(x,t)$. This nonlinear velocity field contributes to the balance between linear dispersion and the Stokes nonlinear wave speed correction which results in the envelope having permanent shape.

Three examples of groups of permanent envelope are presented with $\varepsilon = 0.25$ and $k_0 = 5, 2.5$, and 1.25. The envelope and one position of the wave train is sketched for each example, as well as the horizontal velocity field $u_0(x,t)$. The 2N+1 variables a_{jk}, b_{jk}, and α for each example are tabulated at the end of the section.

The first example, with $\varepsilon = 0.25, k_0 = 5$, is sketched in figure 1. As in all three examples, the height of the envelope above the mean level exceeds the depth of the envelope below the mean level at the centre point (with $\eta_0(x,t)$ included), the ratio of height to depth in this case being 1.28. The horizontal velocity field $u_0(x,t)$, which is confined here to about one wavelength at the centre of the group, contributes to the balance with the linear dispersion which keeps the envelope of the group constant with an effective length of about three wavelengths.

The second example, with $\varepsilon = 0.25, k_0 = 2.5$, is sketched in figure 2, two group lengths being shown. An interesting feature is that, as in the previous example, three wavelengths occur in each group, and the effective part of the horizontal velocity field $u_0(x,t)$ is almost the same as before. Although calculation of this example uses a group length to wavelength ratio $k_0 = 2.5$, the amplitude of the surface harmonic at $k = 3$ far exceeds that at $k = 2$. The ratio k_0, at given ε, determines the wave group, but it may be interpreted as a central wave-number in the dominant waveband only in the linear limit as ε tends to zero.

The third example, with $\varepsilon = 0.25, k_0 = 1.25$, is sketched in figure 3, four group lengths being shown. The central part of the horizontal velocity field $u_0(x,t)$ in each group length is almost identical in shape with that in the previous two examples. This example falls in the transition range between two and one wavelengths in each group length. The evolution of the wave train as it moves through the group is made clearer in figure 4, which is a perspective sketch of the water surface over two group lengths in a frame of reference moving with the group structure, with time increasing over one group period up the figure. Figure 3 is a cross-section of this sketch one quarter of the group period from the initial time. The crest of the wave does not move through the group with a uniform velocity, but is moved forward in phase as it passes through the group minimum. Sketches of a number of examples with $\varepsilon = 0.25$ have shown that the lower limit in k_0 for two wavelengths in each group, both waves moving uniformly through the group structure, is at about $k_0 = 1.35$, and the upper limit in k_0 for one wavelength in each group, the wave moving uniformly through the group structure, is at about $k_0 = 1.15$. The group structure at the smaller values of k_0 consists of a small periodic rise and fall together of the wave crests, and corresponding oscillation of the wave shapes, as the wave train propagates across the water surface.

The method of solution includes a spectral analysis of the water surface displacement and of the complete water velocity field, enabling properties such as momentum or energy integrals to be calculated immediately. It includes also a spectral analysis of the nonlinear water surface boundary conditions, from which it may be deduced which harmonics should be added if higher precision is required. The fast Fourier transform technique, combined either with time-stepping (Fornberg & Whitham, 1978), or with Newton's method as above, offers many possibilities for the solution of nonlinear wave problems.

Table of harmonics

The harmonics for the three examples are listed, stating only those harmonics whose surface displacement amplitudes exceed 10^{-3} in magnitude.

$\varepsilon = 0.25, \quad k_0 = 5, \quad \alpha = 0.207.$

a_{0k}, k=1,...,8	-0.003	-0.004	-0.004	-0.004	-0.003	-0.002
	-0.001	-0.001				
b_{0k}, k=1,...,8	-0.033	-0.024	-0.016	-0.010	-0.006	-0.003
	-0.003	-0.002				
a_{1k}, k=2,...,15	0.003	0.017	0.088	0.219	0.215	0.136
	0.078	0.044	0.025	0.015	0.008	0.005
	0.003	0.002				
b_{1k}, k=2,...,15	0.005	0.024	0.098	0.218	0.195	0.114
	0.061	0.033	0.018	0.060	0.006	0.003
	0.002	0.001				
a_{2k}, k=8,...,21	0.002	0.006	0.013	0.018	0.020	0.018
	0.015	0.011	0.008	0.006	0.004	0.003
	0.002	0.001				
b_{2k}, k=8,...,21	0.0	0.001	0.001	0.00	0.001	0.001
	0.001	0.001	0.0	0.0	0.0	0.0
	0.0	0.0				
a_{3k}, k=15,...,25	0.001	0.002	0.003	0.003	0.003	0.003
	0.003	0.002	0.002	0.002	0.001	
b_{3k}, k=15,...,25	0.0	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0	

$\varepsilon = 0.25, \quad k_0 = 2.5, \quad \alpha = 0.213.$

$a_{0k}, k=1,\ldots,5$	-0.009	-0.007	-0.004	-0.002	-0.001	
$b_{0k}, k=1,\ldots,5$	-0.048	-0.020	-0.008	-0.003	-0.001	
$a_{1k}, k=1,\ldots,8$	0.005	0.163	0.442	0.162	0.053	0.018
	0.006	0.002				
$b_{1k}, k=1,\ldots,8$	0.008	0.181	0.402	0.127	0.037	0.012
	0.004	0.001				
$a_{2k}, k=4,\ldots,11$	0.004	0.020	0.040	0.031	0.018	0.009
	0.004	0.002				
$b_{2k}, k=4,\ldots,11$	0.0	0.001	0.002	0.001	0.0	0.0
	0.0	0.0				
$a_{3k}, k=7,\ldots,14$	0.001	0.004	0.007	0.007	0.005	0.003
	0.002	0.001				
$b_{3k}, k=7,\ldots,14$	0.0	0.0	0.0	0.0	0.0	0.0
	0.0	0.0				
$a_{4k}, k=12,\ldots,15$	0.001	0.002	0.002	0.001		
$b_{4k}, k=12,\ldots,15$	0.0	0.0	0.0	0.0		

$\varepsilon = 0.25, \quad k_0 = 1.25, \quad \alpha = 0.080.$

$a_{0k}, k=1,2$	-0.015	-0.003		
$b_{0k}, k=1,2$	-0.043	-0.005		
$a_{1k}, k=1,\ldots,4$	0.682	0.189	0.015	0.001
$b_{1k}, k=1,\ldots,4$	0.760	0.149	0.011	0.001
$a_{2k}, k=2,\ldots,5$	0.049	0.041	0.012	0.002
$b_{2k}, k=2,\ldots,5$	0.002	0.001	0.0	0.0
$a_{3k}, k=3,\ldots,6$	0.005	0.008	0.005	0.002
$b_{3k}, k=3,\ldots,6$	0.0	0.0	0.0	0.0
$a_{4k}, k=5,6$	0.002	0.001		
$b_{4k}, k=5,6$	0.0	0.0		

References

Bryant, P.J. 1979 Nonlinear wave groups in deep water. *Stud. Appl. Math.* **61**, 1-30.

Fornberg, B., & Whitham, G.B. 1978 A numerical and theoretical study of certain nonlinear wave phenomena. *Phil. Trans. Roy. Soc. Lond. A* **289**, 373-404.

Yuen, Henry C., & Lake, Bruce M. 1980 Instabilities of waves on deep water. *Ann. Rev. Fluid Mech.* **12**, 303-34.

Zakharov, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zh. Prikl. Mekh. Tekh. Fiz.* **9**, 86-94 (*J. Appl. Mech. Tech. Phys.* **2**, 190-4).

Zakharov, V.E., & Šhabat, A.B. 1971 Exact theory of two-dimensional self-focusing and one-dimensional self-modulating waves in nonlinear media. *Zh. Eksp. Teor. Fiz.* **61**, 118-134 (*Sov. Phys. JETP* **34**, 62-69).

MATHEMATICAL RESEARCH REPORTS

University of Canterbury

1. (November 1976) BRYANT, P.J. (Reprinted March 1978)
Permanent wave structures on an open two layer fluid.
2. (December 1976) WEIR, G.J.
Conformal Killing tensors in reducible spaces.
3. (January 1977) WEIR, G.J. & KERR, R.P.
Diverging type-D metrics.
4. (February 1977) KERR, R.P. & WILSON, W.B.
Singularities in the Kerr-Schild metrics.
5. (May 1977) PETERSEN, G.M.
Algebras for matrix limitation.
6. (August 1977) GRAY, A.G.
Topological methods in second order arithmetic.
7. (September 1977) WAYLEN, P.C.
Green functions in the early universe.
8. (November 1977) BEATSON, R.K.
The asymptotic cost of Lagrange interpolatory side conditions in the space C(T).
9. (November 1977) BEATSON, R.K.
The degree of monotone approximation.
10. (November 1977) BREACH, D.R.
On the non-existence of 5-(24,12,6) and 4-(23,11,6) designs.
11. (February 1978) BRYANT, P.J. & LAING, A.K.
Permanent wave structures and resonant triads in a layered fluid.
12. (August 1978) BALL, R.D.
Morse theory with applications and extensions.
13. (September 1978) BRYANT, P.J.
Nonlinear wave groups in deep water.
14. (September 1978) BREACH, D.R.
Some 2-(2n+1,n,n-1) designs with multiple extensions.
15. (October 1980) CHOW, SHUE-SUM
Singular value decomposition and its applications.
16. (January 1981) BREACH, D.R.
Some Remarks on a Family of t-designs.
17. (August 1981) BREACH, D.R.
Star Gazing in Affine Planes.
18. (November 1981) BRYANT, P.J.
Short Wave Groups in Deep Water.

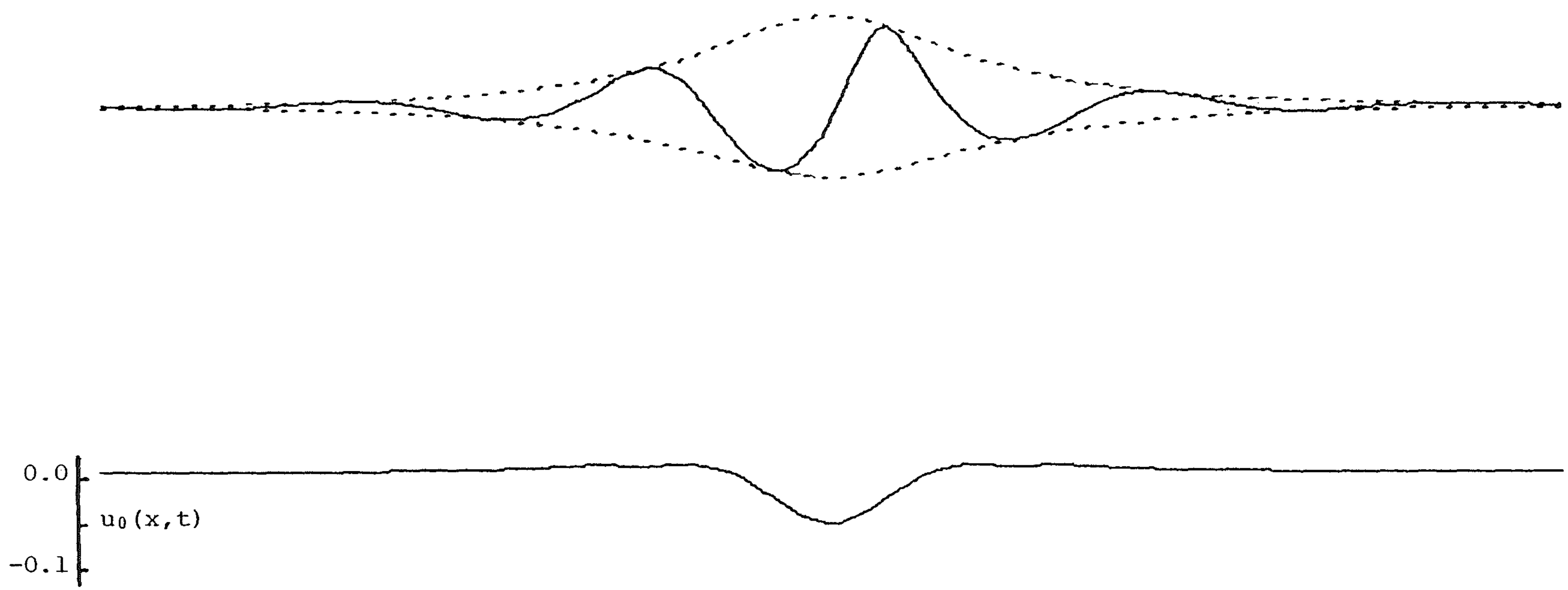

Figure 1. The wave group of permanent envelope for $\varepsilon = 0.25$, $k_0 = 5$, (a) one group length with one position of the water surface and the envelope of permanent shape, horizontal contraction 2.5π, (b) the part of the horizontal velocity field at the water surface which is steady relative to the group structure.

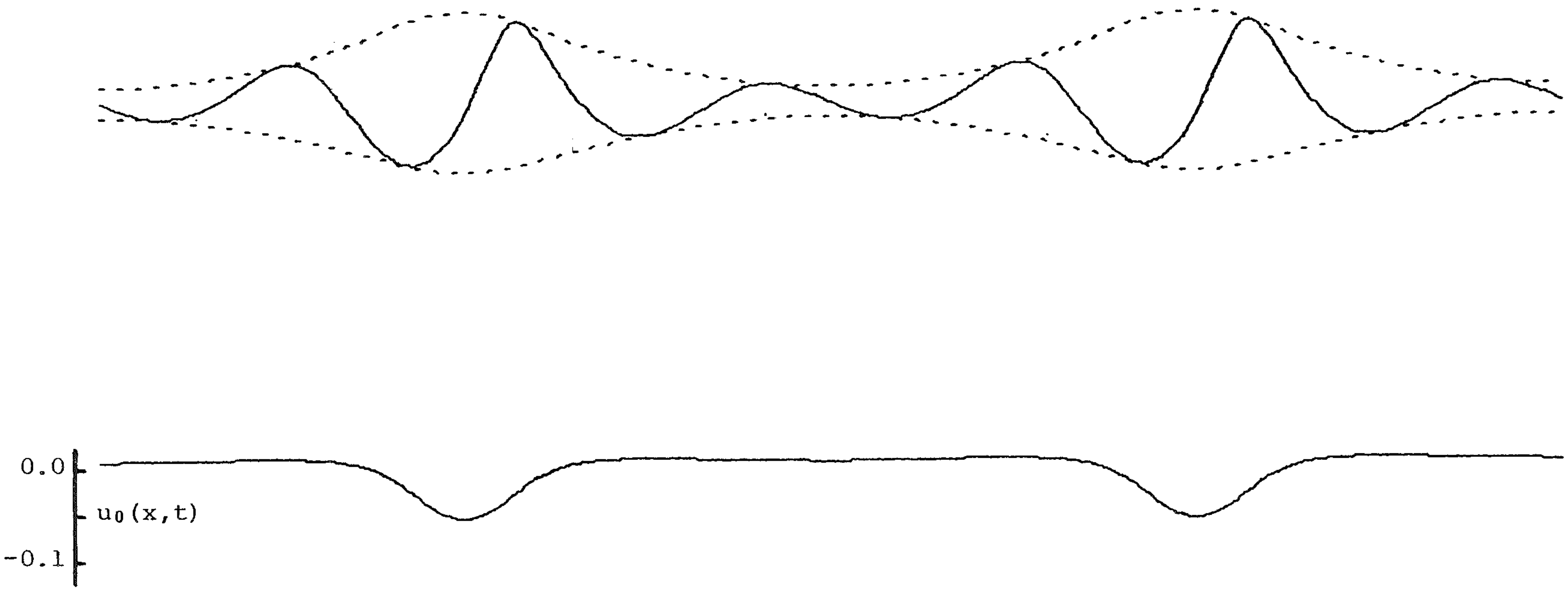

Figure 2. Wave groups of permanent envelope for $\varepsilon = 0.25$, $k_0 = 2.5$, (a) two group lengths with one position of the water surface and the envelope of permanent shape, horizontal contraction 2.5π, (b) the part of the horizontal velocity field at the water surface which is steady relative to the group structure.

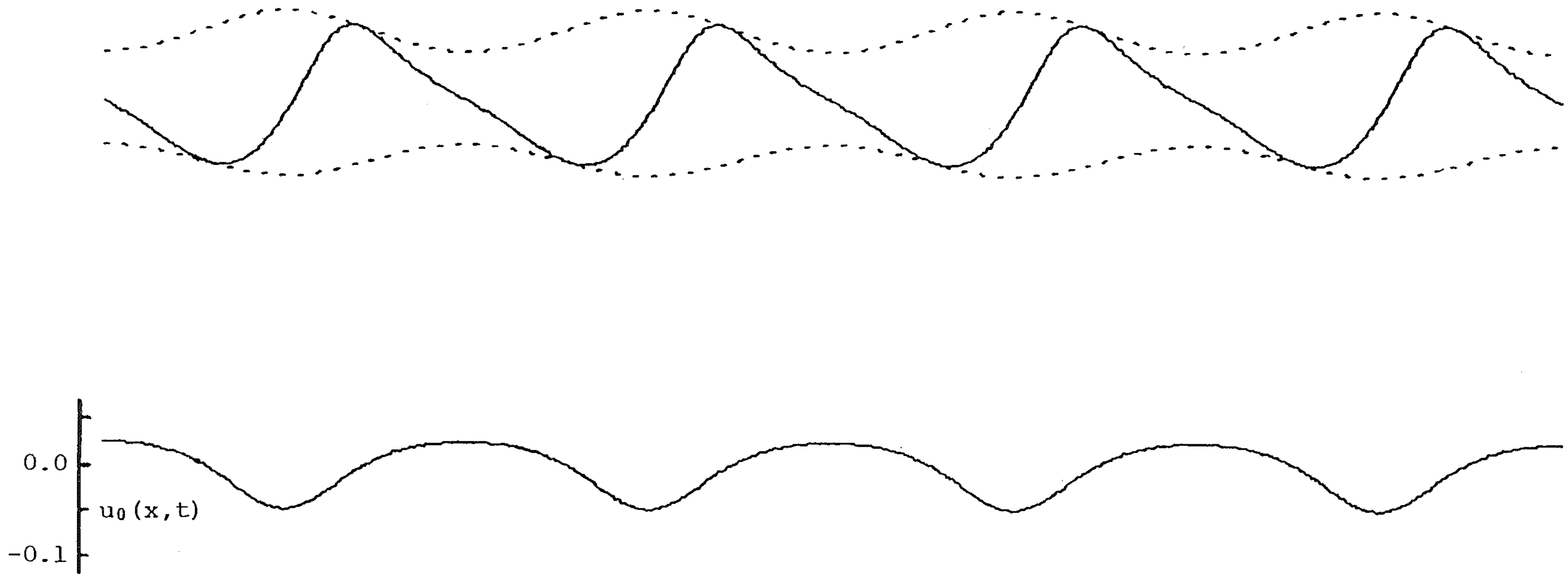

Figure 3. Wave groups of permanent envelope for $\varepsilon = 0.25$, $k_0 = 1.25$, (a) four group lengths with one position of the water surface and the envelope of permanent shape, horizontal contraction 2.5π, (b) the part of the horizontal velocity field at the water surface which is steady relative to the group structure.

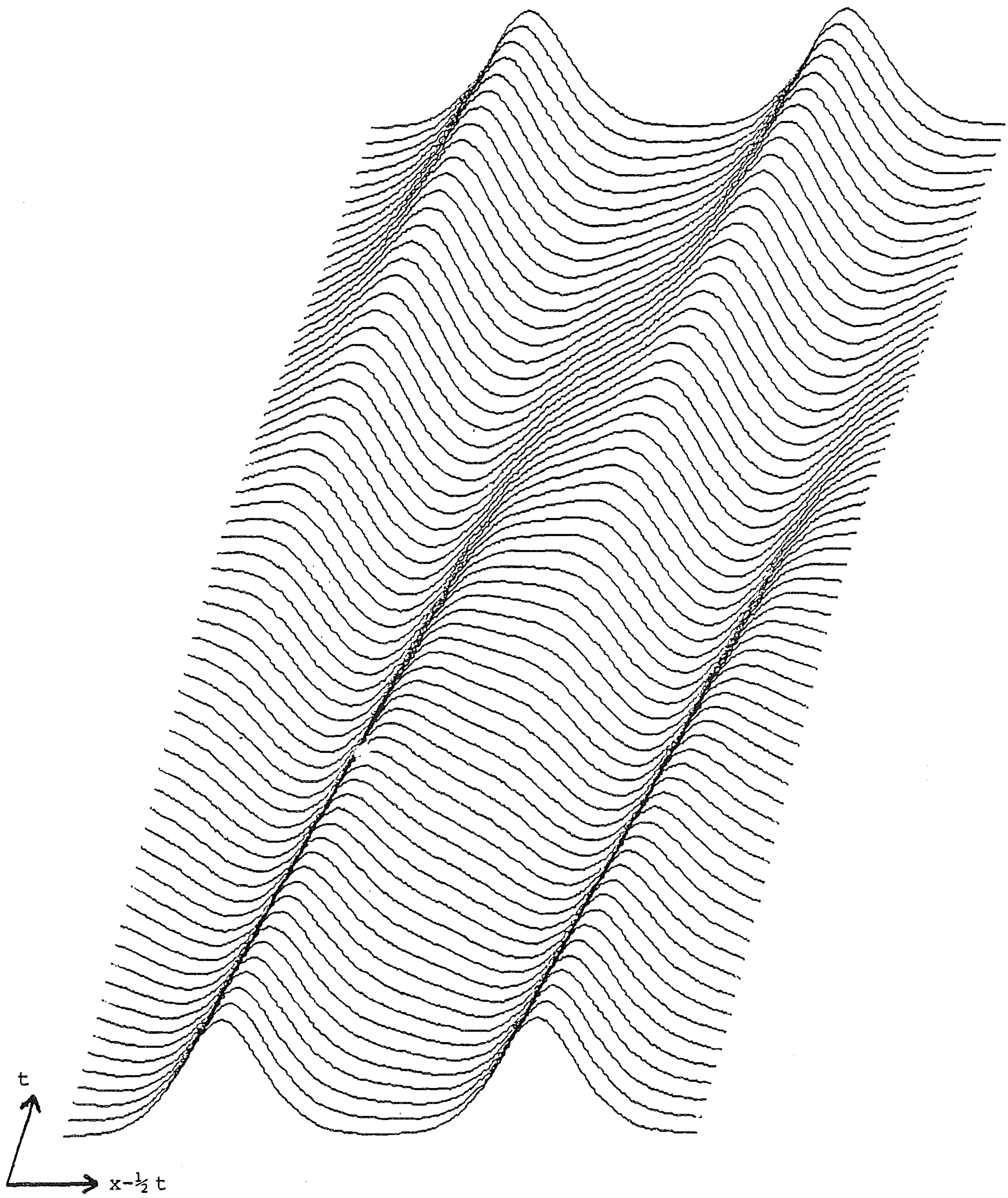

Figure 4. Perspective view of the evolution in time of the water surface in two group lengths at $\varepsilon = 0.25$, $k_0 = 1.25$, in a frame of reference moving with the group structure, horizontal contraction 2.5π.