A STUDY OF THE TRANSPORTATION OF GRAVEL BY TURBULENT WATER FLOWS.


by

STEPHEN M. THOMPSON, B.E. (Hons.)

1963
# CONTENTS:

<table>
<thead>
<tr>
<th>Section No.</th>
<th>Title</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Synopsis</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>Acknowledgements</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>Introduction</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>Field Observations</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>Turbulent flow independent of viscous effects</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>The mixing length theory for turbulent diffusion</td>
<td>17</td>
</tr>
<tr>
<td>7</td>
<td>The parameters of the mixing length theory which must be determined empirically</td>
<td>20</td>
</tr>
<tr>
<td>8</td>
<td>The effect of suspended sediment on the turbulent structure of the fluid</td>
<td>22</td>
</tr>
<tr>
<td>9</td>
<td>Introduction to an experimental measurement of flow</td>
<td>26</td>
</tr>
<tr>
<td>10</td>
<td>The pitot tube apparatus</td>
<td>29</td>
</tr>
<tr>
<td>11</td>
<td>Reduction of each velocity profile to a shear parameter and a roughness parameter</td>
<td>34</td>
</tr>
<tr>
<td>12</td>
<td>The sources of error in velocity measurement</td>
<td>37</td>
</tr>
<tr>
<td>13</td>
<td>The probable error of the velocity measurement due to pitot tube procedure</td>
<td>36</td>
</tr>
<tr>
<td>14</td>
<td>Error in the velocity profile due to imprecise measurement of depth</td>
<td>43</td>
</tr>
<tr>
<td>15</td>
<td>A comparison of estimated errors with actual deviations from the logarithmic formula</td>
<td>45</td>
</tr>
<tr>
<td>16</td>
<td>Reasons for preferring the logarithmic formula for the velocity profile</td>
<td>46</td>
</tr>
<tr>
<td>17</td>
<td>Measuring the equivalent roughness size</td>
<td>49</td>
</tr>
<tr>
<td>18</td>
<td>Secondary flows</td>
<td>53</td>
</tr>
</tbody>
</table>
Boundary shear and bed motion 55
What is Einstein's approach? 57
Kalinske's approach 59
The exposure of particles of different size in the bed surface 60
Einstein's description of the flow for the purpose of calculating bed load transport 61
Introduction to the bed load transport experiments 63
Details of the bed load transport experimental method and apparatus 66
The rate of entrainment 72
Length of travel 77
The program of the bed load experiments 79
The non uniformity of the flow and the vibrations of the flume 81
The experimental results compared with Einstein's bed load formula 84
The entrainment site plate 90
The glass balls 93
The importance of adjacent bed configuration 95
The variation with time of the entrainment rate at a location 100
Direct measurement of the hydraulic force on a particle in a stream bed 105

Introduction to a new explanation for the entrainment of gravel by water 110
Details of a new explanation for the entrainment of gravel by water 112
A model for the velocity fluctuations which explains the impulsive nature of the forces which act on a rough boundary in shearing turbulent flow 116
<table>
<thead>
<tr>
<th>Section No.</th>
<th>Title</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>Comment on some of the well known formulas for the discharge of bed load sediment</td>
<td>123</td>
</tr>
<tr>
<td>40</td>
<td>Summary of Conclusions</td>
<td>128</td>
</tr>
<tr>
<td>41</td>
<td>Bibliography</td>
<td>132</td>
</tr>
<tr>
<td>42</td>
<td>Nomenclature</td>
<td>135</td>
</tr>
<tr>
<td>43</td>
<td>Appendix 1. Tabulated velocity measurements</td>
<td>140</td>
</tr>
<tr>
<td>44</td>
<td>Appendix 2. Einstein's contribution to the theory of bed load transport</td>
<td>148</td>
</tr>
<tr>
<td>45</td>
<td>Appendix 3. Tabulated transport measurements</td>
<td>157</td>
</tr>
<tr>
<td>Figure No.</td>
<td>Title</td>
<td>Page No.</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------------------------------------------------------</td>
<td>----------</td>
</tr>
<tr>
<td>1</td>
<td>General diagram of a velocity profile</td>
<td>33</td>
</tr>
<tr>
<td>2</td>
<td>Velocity profiles measured over large gravel 25-27.2.63</td>
<td>36</td>
</tr>
<tr>
<td>3</td>
<td>Velocity profile pattern for the width of the channel 18.2.63</td>
<td>54</td>
</tr>
<tr>
<td>4</td>
<td>Plot of proportion of sample versus delay exceeded 7.1.63</td>
<td>75</td>
</tr>
<tr>
<td>5</td>
<td>Entrainment rate measurements for a constant discharge 19.1.63</td>
<td>82</td>
</tr>
<tr>
<td>6</td>
<td>First comparison of measurements with Einstein’s formula</td>
<td>86</td>
</tr>
<tr>
<td>7</td>
<td>Second comparison of measurements with Einstein’s formula</td>
<td>89</td>
</tr>
<tr>
<td>8</td>
<td>Entrainment rate, site plates similar, locations vary 15.1.63</td>
<td>92</td>
</tr>
<tr>
<td>9</td>
<td>Entrainment rate, glass balls similar, locations vary 12.1.63</td>
<td>94</td>
</tr>
<tr>
<td>10</td>
<td>Entrainment rate, anomalous behaviour at location E 8.1.63</td>
<td>96</td>
</tr>
<tr>
<td>11</td>
<td>Entrainment rate, obstruction at location E 20.1.63</td>
<td>98</td>
</tr>
<tr>
<td>12</td>
<td>Entrainment rate, obstructions at location E 23.1.63</td>
<td>99</td>
</tr>
<tr>
<td>13</td>
<td>Entrainment rate, 120 measurements at Location E 23.1.63</td>
<td>102</td>
</tr>
<tr>
<td>14</td>
<td>Transducer apparatus for measuring the force on a particle</td>
<td>106</td>
</tr>
<tr>
<td>15</td>
<td>Magnification spectra of the transducer apparatus</td>
<td>107</td>
</tr>
<tr>
<td>16</td>
<td>Transport diagram as plotted by Brown</td>
<td>124</td>
</tr>
<tr>
<td>17</td>
<td>Transport diagram as plotted by Einstein</td>
<td>149</td>
</tr>
<tr>
<td>18</td>
<td>Entrainment rate, measurements made during 1962</td>
<td>158</td>
</tr>
<tr>
<td>19</td>
<td>Entrainment rate, nine largest samples measured</td>
<td>159</td>
</tr>
<tr>
<td>Photo No.</td>
<td>Title</td>
<td>Page No.</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------------------------------------------------------</td>
<td>----------</td>
</tr>
<tr>
<td>1</td>
<td>Waian River at Kaimai</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>About 50 gpm on a fine sand beach</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>Waikariri River, Canterbury</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>55 gpm on fine sand</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>41 gpm on fine sand</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>Outlet Lake Forsyth, Canterbury</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>Outlet Lake Forsyth, Canterbury</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>Recirculating Flume</td>
<td>27</td>
</tr>
<tr>
<td>9</td>
<td>Pitot and pitotstatic tubes in position</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>Pitot Tube mount and manometer tubes</td>
<td>30</td>
</tr>
<tr>
<td>11</td>
<td>Pitot Tube as for reading one (lowest)</td>
<td>32</td>
</tr>
<tr>
<td>12</td>
<td>Pitot Tube as for sixth reading</td>
<td>32</td>
</tr>
<tr>
<td>13</td>
<td>Pitot Tube as for tenth reading (highest)</td>
<td>32</td>
</tr>
<tr>
<td>14</td>
<td>Perspex sighting window</td>
<td>65</td>
</tr>
<tr>
<td>15</td>
<td>Entrainment Locations on $\frac{1}{8}$ to 3/16&quot; gravel</td>
<td>66</td>
</tr>
<tr>
<td>16</td>
<td>Entrainment Locations on 3/16&quot; to $\frac{3}{32}$ gravel</td>
<td>66</td>
</tr>
<tr>
<td>17</td>
<td>Site plate</td>
<td>67</td>
</tr>
<tr>
<td>18</td>
<td>Three different gravels used</td>
<td>67</td>
</tr>
<tr>
<td>19</td>
<td>Choosing the more spherical glass balls</td>
<td>69</td>
</tr>
</tbody>
</table>
I. SYNOPSIS

Some aspects of the motion of gravel particles (> 2 mm) when entrained by water is considered. The flow is turbulent and the longitudinal slope, flat as in natural rivers.

First a few field observations are described and it is shown how these observations conflict with existing theories for sediment transport.

Secondly, a method for measuring the characteristics of turbulent water shearing at a rough boundary is described. The method involves measuring the velocity profile with a pitot tube and then finding the formula of best fit which is in the form:

$$u = x \frac{y}{\varepsilon_0} \frac{y}{\varepsilon_0}$$

Measurements were made and the technique is shown to be sufficiently precise. The results are analysed using a digital computer and are presented.

Thirdly, a large number of entrainments were observed. Each entrainment was of a single glass ball from a specially prepared site on the rough bed. A careful analysis of the measurements made have enabled a number of final conclusions to be drawn which relate to the mechanism of entrainment.

Finally, a new mechanism of entrainment is postulated. The turbulence is considered as a large number of distinct rotating eddies. At the centre of each eddy is a small low pressure region and when it acts on a particle the eddy imparts a strong force to the particle. The effect is limited by the short time for which this force acts on the particle. Einstein has previously related entrainment to fluid pressure fluctuations, but the writer believes that the lift forces are ten times as strong as those postulated by Einstein.
2. ACKNOWLEDGEMENTS:

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Senior Lecturer in Civil Engineering
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3. INTRODUCTION

Interest in this subject was first excited by the variety of channel patterns found in natural rivers, particularly in mountain regions where the writer has spent many holidays. On reading up some of the existing theories for transport of gravel, concepts were found which though apparently accepted, conflicted with the writer's notions.

The interactions between a turbulent flow of water and a rough boundary of stones is treated in a most unsatisfactory way in the engineering literature. Previous attempts to interpret experimental results for turbulent flow have always involved a rather ill defined but all explaining "turbulence factor". Discrepancies between predicted motion of bed material and the observed motion vary widely and turbulence factors ranging from two to six are invoked to account for these discrepancies.

The writer has attempted to substitute for the ill defined "turbulence factor", an "equivalent roughness size" which is proportional to the scale of the turbulence. A technique is developed for measuring this effective roughness size directly from velocity measurements near the boundary. When tried the method was only partially successful but the scatter of the results appears to be due to the shortcomings of the particular flows used. Contrary to a widely held view, the discussion of errors shows that the method though laborious, could precisely define the flow and so prove useful in research. That is, statistical methods seem to be able to overcome the difficulty of measuring a velocity profile near a rough boundary.
Direct observation of stones in a stream bed when on the point of motion is an obvious way to determine the worth of various conflicting notions about the forces which act there. The writer's preconceptions were based on undirected and presumably some unconscious observation. It was essential that a technique be developed for measuring some property of the motion of the stones independently of any personal bias or assumption. However, techniques for measuring fundamental observable quantities relating to the process of bed load transport are few.

The third part of this research was the development of such a technique which measures the "entrainment rate". The measurements only relate to a single place on the bed, called the entrainment site, and it was found that a number of firm conclusions could be established about the amount of variation between different sites on the bed. Thousands of entrainments were observed to obtain the necessary data, and without the benefit of this experience the reader may find difficulty in accepting some statements made here.

An arbitrary model for the turbulent fluctuations is introduced and shown to satisfactorily explain two well established results. These results are:

1. The ratio of the shear velocity to the velocity gradient equals four tenths of the distance from the "boundary" of the flow.
   \[
   \frac{u_v}{\frac{du}{dy}} = 0.4 \gamma
   \]

2. At the threshold of sediment motion Shields entrainment function equals about .06
   \[
   \frac{\rho u_v^2}{(\sigma - \rho)gD} = 0.06
   \]

This model for the turbulent velocity fluctuations is then shown to explain some features of the entrainment of stones which may be observed, but which cannot be satisfactorily explained by the conventional model for the turbulent velocity fluctuations. These fluctuations are restricted in such a way that the useful results from the turbulent theory of random fluctuations are still valid, but in addition the entrainment of gravel is also explained.
The behaviour of natural rivers often appears to contradict the existing theories for sediment motion. For instance single stones have been seen by the writer to break the surface in flooded rivers. In a glass sided flume I, Hill has photographed 2" gravel being transported by 6" of water on a steep slope. Single stones suddenly rise from the bed to the surface apparently propelled by forces originating in the water. These observations contradict the picture of bed load as the rolling of large particles along the bed.

Photograph 1 was taken half a mile upstream of the Ferry Bridge across the Waikau at Hamer. Note that this braided river is degrading. Note also that the island has resisted the bank erosion attacks of the river. Thus it seems that braiding is not a result of an excessive tendency to attack the banks as some geomorphologists assert.

Compare photographs 2 and 3. One shows a flow of about 50 gpm over a fine sand beach. The other shows a flow of about 5000 cusecs across the Canterbury Plains in the Waikariri River. The similarity of the channel patterns suggest that the process of braiding might be studied in small scale models.

A series of experiments lasting a fortnight were conducted on a large deposit of clean uniform 0.3 mm sand at Waitoku Beach. A portable low head centrifugal pump was used which was capable of a maximum discharge of 110 gallons per minute. The water was made to flow over various slopes and to form its own natural channel. Different discharges were observed to result in radically different channel patterns even on the same slope. A consistent reproducible set of patterns was found in the range of slopes .01 to .06 and discharges 8 gpm to 50 gpm.

* I. Hill Ph.D student at the University of Canterbury 1962
Photo 1

Waiau River at Hanmer
Photo 2. About 50 gpm on a fine sand beach.

Photo 3. Waimakariri River, Canterbury.
The meandering flow in photograph 5 occurred when the straight flow in photograph 4 of 55 gpm was reduced to 41 gpm. Antidunes were a dominant feature, especially in photograph 4. The outlet of Lake Forsyth in photographs 6 and 7 shows similar features to those shown in photograph 4, but on a much larger scale. The scale factor for both the width and the sediment size was about 200. (i.e. 1 ft : 300 yds, .2 m : 3") Note particularly that the antidune waves are proportionately much less pronounced at the large scale.

For the flatter slopes (i.e. .01) on the .2 mm sand it was found that dunes formed. These dunes were flat topped with large holes between them.

The channel patterns were not comparable with any seen by the writer in streams and rivers owing to the dominance of the bed configuration at the small scale.

On the steeper slopes (i.e. .05) firm sand became loose when water flowed over it. A one foot steel rule would penetrate one inch under its own weight when the water depth was only a 3\text{\textfrac{1}{4}}. The channel divided and wandered about and formed a braided pattern similar to that in photograph 2. The islands changed in position rapidly, and an island two feet by three inches might be formed and completely obliterated in ten minutes. The sand in the centre of these islands was however much
Photo 4. 55 gpm on fine sand.

Photo 5. 41 gpm on fine sand.
Photo 6. Outlet Lake Forsyth, Canterbury.

Photo 7. Outlet Lake Forsyth, Canterbury.
finer than in the adjacent channels. The motion of the sand in these conditions could well be described as the down slope movement of a fluidized grain dispersion. These tentative observations suggest the importance of Du Boys' and Bagnold's theories.

The small depths of water and the fine sand through which it flowed resulted in laminar sublayer effects being important in the small scale experiments. The seepage loss or gain along the channel was unmeasured. No deposits of sand of any other size could be found within eighty miles of Christchurch. It was therefore decided not to persist with these experiments because:

1. they were unsuitable for testing any existing theory owing to the small range in variables possible;

2. the process of gravel motion in streams is free from laminar sublayer effects which occurred in the small channels.

For particles larger than 2 mm diameter viscous fluid drag is unimportant (Rubey), the behaviour of a dispersion of grains in fluid is dynamically similar to that of larger gravel (Bagnold) and ripples and dunes do not occur (Shields). Thus any further model experiments on channel patterns which are intended for study of rivers in gravel should be restricted to granular material at least as large as 2 mm. Since seepage through such coarse material will be large it seems that such a model study would have to be conducted in a flume in which seepage normal to the bed surface could be eliminated.

* P. du Boys Ann. d. Ponts et Chauss.; (5) 18 (1879) p. 149
*** W. W. Rubey American Journ. of Science V25 (1933) p. 335
5. TURBULENT FLOW INDEPENDENT OF VISCOS EFFECTS

By restricting this study to flow over gravels coarser than 2 mm diameter it is possible to neglect the viscosity of the water. The freedom from viscous effects may be established by considering three standard boundary layer and sediment transport results.

Open channel flow over gravel will always be turbulent. The thickness of the laminar sublayer next to the gravel for slow flows will be:

\[ \delta = \frac{11.6 \cdot \sqrt{y}}{u_x} \]

But sediment motion will only occur when the flow is so fast that:

\[ \frac{\rho u_x^2}{\sigma - \rho} g d > 0.06 \quad \text{Shields} \]

Thus the smallest value of \( u_x \) which need be considered in the study of the motion of gravel in water is:

\[ u_x \text{ min} = \sqrt{1.65 \times 62.4 \times \frac{2}{305} \times 0.06} \]

\[ = 0.2 \, \text{ft/sec} \]

The largest laminar sublayer thickness encountered will therefore be:

\[ \delta_{max} = \frac{11.6 \times 0.00001 \times 305}{2} \]

\[ = 0.18 \, \text{mm} \]

But this small thickness of laminar sublayer could not exist on a gravel bed because the roughness projections would be at least five times as high as the theoretical sublayer thickness.
6. THE MIXING LENGTH THEORY FOR TURBULENT DIFFUSION

Equations for \( \text{1} \) the logarithmic velocity profile and \( \text{2} \) the suspended sediment concentration profile may be developed using the mixing length theory. This approach is often criticised for being over simplified but is worthy of study as it explains the physical significance of the parameters in these equations:

Consider the downward rate of transfer of

\[
\begin{array}{|c|c|}
\hline
\text{1} & \text{2} \\
\hline
\text{momentum} & \text{sediment} \\
\hline
T = \rho \ell \frac{V}{Z} \frac{dU}{dy} & wC = \ell \frac{V}{Z} \frac{dC}{dy} \\
\hline
\end{array}
\]

(1)

\( \frac{V}{Z} \) is the volume of fluid interchanged vertically across a unit area in a second. \( \ell \) is the average vertical distance travelled by the interchanged volumes.

Prandtl suggested that

\[
\frac{V}{Z} \propto \ell \frac{dU}{dy}
\]

(2)

This made the shear proportional to the square of the velocity gradient as had been observed.

The relationship is dimensionally homogeneous.

Keulegan suggested that

\[
\ell \propto \sqrt{y} \left( 1 - \frac{y}{h} \right)
\]

(3)

This conforms to the condition that the vertical transfer distance is zero at the bed and the surface. The relation also conforms to Prandtl's hypothesis that the mixing length is only dependent on the geometry of the flow and so the relation is dimensionally homogeneous. Substitute

these two expressions in (1) & (2).

$T \propto \rho l^2 \left( \frac{d\epsilon}{dy} \right)^2$

$T \propto \rho y^2 \left( 1 - \frac{y}{h} \right) \frac{d\epsilon}{dy}$

$w \propto \epsilon \left( \frac{d\epsilon}{dy} \right)^2$

$w \propto y^2 \left( 1 - \frac{y}{h} \right) \frac{d\epsilon}{dy}$

Introduce a proportionality const. $K^2$.

From simple strics

Substitute $u_\infty$ for $\frac{T_0}{\rho}$

Adopt this expression for $\frac{d\epsilon}{dy}$

Rearrange

Integrate

<table>
<thead>
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<th>$u$</th>
<th>$\frac{d\epsilon}{dy}$</th>
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<tr>
<td>$u_\infty \frac{dy}{y}$</td>
<td>$\frac{dy}{y}$</td>
</tr>
<tr>
<td>$u_\infty \log \frac{y}{y_0}$</td>
<td>$\frac{dy}{y}$</td>
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$y_0$ is the depth where $u = 0$.

$C$ is the depth where $C = C_b$.

The writer pictures the turbulent fluctuations as the effect of an aggregate of distinct rotating eddies. The eddies are stacked up like the stones in a strata of gravel. In the same way that an aggregate of gravel is sometimes represented by a regular arrangement of uniformly sized solid spheres, the turbulence will be considered to be a regular arrangement of uniformly sized rotating spheres of fluid.

The turbulent diffusion near any point will depend on the rotation rate of the eddies "$\omega$" radians per sec., the diameter of the eddies "$a$" and the spacing of the eddy centres "$S_p$" near the point.

Introducing the mixing length concept in terms of these three factors:

<table>
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<th>$u$</th>
<th>$\frac{d\epsilon}{dy}$</th>
</tr>
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<tr>
<td>$u_\infty \frac{dy}{y}$</td>
<td>$\frac{dy}{y}$</td>
</tr>
<tr>
<td>$u_\infty \log \frac{y}{y_0}$</td>
<td>$\frac{dy}{y}$</td>
</tr>
</tbody>
</table>

$y_0$ is the depth where $u = 0$. $C$ is the depth where $C = C_b$.
(1) the rate of interchange of fluid by the turbulence will be

\[ \frac{|V|}{2} = \frac{n a^3 \omega}{6 \beta r^2} \]

and

(2) the mean distance of transfer of the interchanged fluid

\[ l = \text{the distance between the centroid of two halves of a sphere} \]

\[ = \frac{3}{8} a \]

The Boussinesq diffusion coefficient of the turbulence is the product \( \frac{|V|}{2} l \). From the point of view of both the velocity profile and the sediment concentration profile, this coefficient completely defines the turbulence. The Prandtl and Karman hypotheses described above define the way that this property varies from place to place in the fluid. The writer's hypothesis shows how this property is related to the structure of the turbulence at a single place.

These hypotheses are not a very firm base on which to build a sediment transport theory and so it would be remarkable if the theory developed in this thesis were correct in detail. However the details have been carefully knit into a description of all the known turbulent fluid processes near a rough boundary, and in particular the physical magnitude of every detail has been estimated. The writer does not claim that these details are correct, but rather that they could be correct.
7. THE PARAMETERS IN THE MIXING LENGTH THEORY EQUATIONS WHICH MUST BE DETERMINED EMPIRICALLY

The similarity basis for the derivation of the mixing length equations suggest that the proportionality constant $K$ should be a universal constant. A large body of experimental data indicates that the value of $K$ (which is known as Karman's constant) is about 0.4.

\[ K = 0.4 \]

Keulegcn who first presented in English the Mixing Length equation for the velocity profile, used the two classic sets of data of Nikuradse, and Bazin, to illustrate its usefulness. To obtain various different rough boundaries Nikuradse glued sand grains to the walls of pipes in close packed formation. The grains were of uniform size and their diameter has been adopted as the "roughness size" designated $k$. Keulegen showed that the distance of the origin of $y$ below the level of zero flow velocity is a thirtieth of the "roughness size"

\[ y_o = \frac{k}{30} \]

Keulegen also showed how the equivalent sand "roughness size" can be estimated for any irregular boundary from a knowledge of the effect of the boundary on the flow over it. The fluid shear stress at the boundary must be so large that no laminar sublayer is formed. The limiting condition was calculated as \( \frac{h u^*}{v} \leq 55 \) and provided \( \frac{h u^*}{v} \).

exceeds this value the effect of the boundary on the flow is completely determined by its $k$ value. Thus if $k$ is determined for one flow then this value is applicable to all flows over the same configuration.

Einstein and El Samm measured the velocity profiles for various
flows of water over a close packed arrangement of $1\frac{1}{4}$" diameter hemispheres. They found that the origin of the velocity distribution was $1/5 \times 1\frac{1}{4}$" below the plane of the top of the hemispheres, and that
\[ y_o = \frac{1}{30} \times 1\frac{1}{2}'' \]
as Keulegan had found.

Similar measurements were made for flows over naturally graded
river gravel, whose $D_{50} = 1\frac{1}{2}''$ In this case the origin of
the velocity distribution was found to be $1/5 \times 2\frac{1}{2}''$ below the level of a
board placed on the rough bed. Consistent with this result
\[ y_o = \frac{1}{30} \times 2\frac{1}{2}'' \]
gave a good fit for the velocity profile.

They assumed that $D_{67}/D_{50} = 5/3$ for natural gravel which
means that $D_{67} = 2\frac{1}{2}''$, and it follows that:

(1) $k = D_{67} = 1.7 D_{50}$ for close packed naturally graded
gravel.

(2) The origin of the velocity distribution is $1/5 \times k$ below the top of
the roughness projections.

According to R. W. Fookes, a better fit is obtained for loosely
packed gravel such as is found in the field when $k = 3D_{50}$.

8. THE EFFECT OF SUSPENDED SEDIMENT ON THE TURBULENT STRUCTURE IN THE FLUID

The expression "turbulent structure" is used here rather loosely to mean the size, speed and spacing of the eddies which make up the turbulence. The writer believes that these eddies play an important part in the entrainment of gravel particles on stream beds. Two important questions will be considered.

(1) Is the mixing length theory useful in describing the established motion of coarse sand and gravel?

(2) Does the presence of suspended sediment alter the turbulent structure and so affect the motion of coarser particles along the bed?

An independent confirmation of the value of the parameters in the mixing length theory is available in the equation for the sediment concentration profile:

\[ \frac{C}{C_b} = \left( \frac{h-y}{h-b} \right) \frac{b}{y} \]

This is usually written:

\[ \frac{C}{C_b} = \left( \frac{h-y}{h-b} \right) \frac{b}{y} \]

For particles smaller than 0.05 mm the exponent \( z \) becomes so small that the distribution of sediment is almost uniform. For particles larger than 2 mm the exponent becomes so large that the actual amount of sediment discharged is very small. Thus this equation for suspended sediment is only useful when calculating the discharge of sand sizes.
Brush has summarised the present state of knowledge of the process which this equation purports to describe. For small particles (within the Stokes range), or relatively flat beds, the above equation appears to be adequately confirmed both in the laboratory and in the field. For larger sizes, or greater concentrations and irregular beds, the above equation is usually correct in form but the exponent of the measured concentration profile $z$ is different from the predicted value and is usually too small.

If suspended sediment causes a significant alteration in the behaviour of the fluid then it will be a factor determining the threshold of the bed load motion of gravel. Thus the circumstances under which the suspended sediment profile exponent $z$ is different from its predicted value should be examined carefully.

The first two equations, labelled (1), in the presentation of the mixing length theory may be written as follows:

$$
\begin{align*}
\lambda &= \frac{\nu}{2} \rho \left( \frac{dU}{dy} \right) \\
\nu C &= \frac{\lambda}{2} \frac{dC}{dy} \\
&= \frac{E_m \rho}{\rho} \frac{dU}{dy} = \frac{E_s}{\rho} \frac{dC}{dy}
\end{align*}
$$

$E_m$ and $E_s$ are commonly called the Boussinesq Diffusion Coefficients, and the larger their value then the more intense is the diffusion process. Their ratio is defined as $\beta$.

$$
\beta = \frac{E_s}{E_m}
$$

The strict application of the mixing length theory to suspended sediment transport assumes that $\beta = 1$.

*L. M. Brush Journ. of Geophysical Res. V67 N4 (1962)*
Drush has demonstrated recently that for 0.20 mm diameter glass balls $\beta$ does equal unity. He has also demonstrated that for 0.32 mm and 0.55 mm diameter glass balls at low concentration that $\beta$ is 0.50 and .15 respectively. This result suggests that particles larger than .2 mm are less efficiently diffused than the mixing length theory predicts.

Measured concentration profiles of these larger sizes in open channel flows actually indicate a more efficient diffusion process than the mixing length theory predicts for such channels. (For instance Colby and Hembree have shown that the exponent $z \propto w^{1.7}$ on the basis of their field gaugings in sand bedded rivers, not $z \propto w^{1.0}$ as the mixing length theory suggests.) It seems that different sizes of particles respond differently to the turbulent motion of the fluid.

Drush's experiment was with an axisymmetric submerged jet. This is a complex turbulent flow which does not admit simple estimates of eddy size and intensity and so a detailed analysis of the discrepancy between his findings and the findings of the many workers with open channel flows (e.g. Colby and Hembree) is not possible. The results do suggest that there is an additional lifting mechanism for the larger grains in open channel flows. Turbulent interchange is being progressively replaced by this mechanism in the transport process as the size of particle increases.

Thus the first question is answered in the negative. The mixing length theory ceases to explain satisfactorily the transport of particles of medium sand size and larger, (i.e. greater than 0.2 mm diameter.) Because the mixing length theory does not describe the motion of the larger particles, it is not possible to determine from measurements of the particle concentration whether the actual turbulent structure of the fluid has been affected by these larger particles.

It does seem likely that the turbulent structure will be affected when the concentration of solid particles is appreciable. Bagnold arbitrarily estimates that the shear due to the residual fluid turbulence in his coarse shear flows is reduced by a quarter when the volume concentration of sediment is one per cent and by three-quarters when the concentration is thirty per cent. Thus it can be expected that the action of the turbulent fluid on the stationary bed grains will start to change at the threshold of motion of coarse sand and gravel.

There is a large range of circumstances (i.e. grain size and concentration) for which the mixing length theory satisfactorily describes the concentration profile. For these circumstances it is unlikely that the amount of suspended sediment affects the motion of gravel at all.

When the concentration of fine sediment is very high however, the measured, profiles of sediment concentration of all sizes have values of $x$ which are smaller than the predicted value $\frac{W}{4U^*}$. That is the turbulent interchange is more intense in the fluid grain mixture than in a similar flow of clear fluid. It is also known that turbid water flows faster than clear water when the depth and slope are the same. The turbulent structure must be affected by the presence of high concentrations of fine sediment. As the motion of gravel only occurs during floods in natural rivers when the water is turbid, the amount of fines suspended will sometimes be large enough to affect the transport of gravel and so the answer to the second question depends on the sediment concentration in any particular case.

Any study of stream transport involves two phases,
(1) the fluid phase and (2) the solid phase. When sediment
motion only takes place near the bed then most of the fluid phase is
uncontaminated by the solid phase and its motion may be studied as a
separate topic. The next ten sections are about this topic - the fluid
flow. Can the fluid flow be measured with sufficient accuracy to predict
from the measurements the effect the fluid has on loose grains at the bed?

The flows to be considered are so fast that when the fluid flows
around the roughness irregularities at the boundary it becomes unstable
and eddies are formed. These eddies are formed because the fluid must
remain continuous about the irregularities and yet cannot do so without
becoming unstable. The size of the eddies will be comparable with the
size of the irregularities in which they form. Ignoring the possible
significance of the shape of the turbulence (presumably a secondary effect)
it follows that the boundary can be characterised by one number
which is proportional to this size, the size of the irregularities and
therefore of the eddies.

Before considering experiments in which motion of solids is studied
a series of measurements of the flow are described. The available methods
were limited by certain details of the experimental flume and a rather
novel way for estimating the equivalent roughness size \( k \) is developed.
The flume used for the experiments was of the recirculating type and did
not include a discharge measuring device in the circuit. Thus it was
necessary to measure the flow by means of a pitot tube. To calculate
equivalent roughness by standard results it is further necessary to
integrate the velocity profile to obtain the discharge. When running the
flume at a sufficiently high discharge and slope to cause entrainment at
the bed the water surface was irregular and unsuitable for accurate
measurement of slope or depth. See photograph 8.
It was therefore more satisfactory to describe the flow directly by means of its velocity profile if this could be measured with sufficient precision. By measuring the profile near to the rough bed in the central half of the channel it was not necessary to consider side friction effects. Secondary flows were found to have an effect near the bed and this is considered later. "Surface Friction" effect was avoided by taking the profile over the bottom two thirds of the depth. Integration to obtain discharge was unnecessary.

Steady flow was essential because measurement procedures were slow. Uniform flow though desirable was not essential because the flow was measured at a single section. Thus non-uniform flows could be measured and included in the data.
The pitot tube was made from a hypodermic needle, O.D. .06" and I.D. .03". The small size minimized distortion of the flow when measuring near the boundary. Also the small size reduced the displacement of the effective centre due to the velocity gradient to such a small value that the depth values were unaffected. However the hypodermic needle would allow only a very slow flow of water through its small tube and the manometers were slow to reach equilibrium after a change in setting.

To reduce the delay between setting and reading the manometers were made as small as the meniscus would allow (i.e. \( \frac{2}{3} \)" I.D.), and the tubes were rapidly adjusted to a near estimate of the final reading each time the pitot was set at a new depth. The pitot static tube was a .125" O.D. brass tube with a hole drilled right through. This hole was positioned one inch above the pitot tube. See photo. 7.

The pitot tube was attached to a vernier screw which could be set to the nearest .001 ft. The manometers were read with a vernier point gauge which read to the nearest .01 inch. See photo. 10.

At each setting of the pitot tube, ten piezometric head values were read and averaged. This took four minutes. Twenty readings were taken for the static head and their average value used for calculating the velocity head values. A traverse consisted of settings at ten different depths and took fifty minutes to measure. The average velocity head at each depth was converted to velocity in feet per second. The result of each traverse was thus expressed as a ten by two array of numbers, ten depths each with its measured velocity.

The bottom setting was obtained by lowering the pitot till it touched the bed, and so stopped vibrating, then lifting it just clear again. A brace was then inserted which stopped the vibration.
The next five settings were at .002 ft. intervals above the bottom reading.

Four more settings were made above this again at roughly equal increments of velocity. Thus considering the bottom reading as depth .010 ft., the array of ten depths might be

.010  .012  .014  .016  .018  .020  .040  .060  .140  .240 ft.

Lower Readings

Upper Readings

See figure 1 and photos 11, 12, 13.
Pitot tube as for reading one (lowest).

Pitot tube as for sixth reading.

Pitot tube as for tenth reading (highest).
Photos 11, 12, 13.
Figure 1

General diagram of a velocity profile in the flume.
II. THE REDUCTION OF EACH VELOCITY PROFILE TO A SHEAR PARAMETER AND A ROUGHNESS PARAMETER

Reduction of data was performed numerically. A 1620 I.B.M. electronic digital computer was used and calculations carried to eight significant figures. Thus there were no graphical averaging procedures and only small errors due to rounding off.

The logarithmic velocity profile was chosen as the most suitable curve to fit the measurements. By using this curve the array of six pairs of numbers defining the profile was reduced to two parameters which define the curve of best fit. A number \( y_a \) (for \( u = 0 \) in \( u = 2.5 u_a \log \frac{y}{y_0} \)) has been correlated by past workers directly with the roughness size. Thus for a large proportion of the results this number should be the same because the roughness was constant. The other number, the slope

\[
u_a = \frac{d u}{\log y/y_0} \]

has been correlated with the square root of the boundary shear. For constant roughness condition this number becomes the only flow parameter. (See Fig. 1.)

In order to increase the precision of the lower readings so that they are more comparable with the upper readings, it was decided to average the bottom six readings. This was achieved by choosing an arbitrary origin for \( y \) and calculating a regression line of \( u \) on \( \log y \) for this array of six pairs. The values of \( u \) on this line corresponding to the extreme values of \( y \) in the range were then calculated.

These two values of \( u \) were then combined with the other four upper readings to make a new array of six pairs. This second array was at roughly equal increments of \( u \) and \( \log y \) and of similar precision.

A whole family of regression lines of \( u \) on \( \log y \) were then calculated, each with a different origin of \( y \). The residual was calculated for each line, that is the root mean square deviation of the
readings from the line. The origin of best fit is the origin for which the regression line has the least residual. The expected value of the origin is the same for all flows but the values calculated in this way showed considerable scatter. The expected origin was estimated as the mean of all the calculated values.

In order to obtain comparable values for the slope parameter for the different flows regression lines were calculated with the origin at its (estimated) expected value. See figure 2 which is a typical set of six profiles passing through the same origin $y_0$. 
Velocity profiles measured over large gravel Feb. 1963
Each point on the velocity profile was defined by a value of depth \( y \), which was set at a predetermined figure, and a value of velocity head which was measured. The velocity \( u \) was calculated for each depth then a regression line of \( u \) on \( \log y \) was fitted to the profile. Both the value of \( u \) and of \( \log y \) are subject to error but seeing that the regression line was calculated by minimizing the mean square deviation from the line in the \( u \) direction, the errors of \( y \) will be considered in terms of the corresponding error that is produced in \( u \).

The depth was set on a \( \frac{1}{1000} \) th foot vernier attached to the pitot tube. The range of the reading error would therefore be \( \frac{1}{1000} \) th foot. However, the rod supporting the pitot tube was easily bent and the range of setting error is considered to be \( \frac{2}{1000} \) th foot. The corresponding range of error in \( u \) depends on the distance from the bed. Small errors in \( y \) can cause relatively large corresponding errors in \( u \) near the bed where the velocity gradient is steep. This important aspect of the errors will be discussed in detail below under a separate heading.

The velocity head was read by a point gauge attached to a \( \frac{1}{100} \) inch vernier. The point gauge was lowered successively into the static head manometer then the total head manometer and the difference in level became the velocity head. The range of reading error is therefore \( \frac{2}{100} \) th inch. However, the water level in both manometers fluctuated up and down in a random fashion with peaks every minute or so. The range of the fluctuations was \( \frac{2}{100} \) th inch or more depending on the total head being measured. The reading error was unimportant compared with these fluctuations. The fluctuations will be examined below and shown to be unimportant compared with the errors in \( y \).
The pitot and pitostatic tubes were connected by two independent plastic hoses to two open manometers which were mounted side by side. See photograph 10. A pointed rod was lowered till it touched the water surface in first one then the other of the open manometer tubes. This rod was attached to a vertical vernier by which the difference in water level in the two open tubes could be read to the nearest 1/100th inch. This difference in level is the velocity head.

In both manometers the water level fluctuated up and down. The fluctuations were random with long periods, comparable to a minute. A single reading of the velocity head could be in error as a result of these fluctuations in the water level and so repeated readings were taken and averaged.

A typical record of a velocity profile measurement follows:

(February 15)

<table>
<thead>
<tr>
<th>Time</th>
<th>Depth y ft.</th>
<th>Readings 1/100 inch units</th>
<th>Mean Reading</th>
<th>Velocity Head H</th>
<th>Range RH</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.15 am</td>
<td>Static head</td>
<td>-1 0 0 0 1 0 1 0 0 0</td>
<td>0.1</td>
<td>57.2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>57 57 57 57 57 57 56 56 55</td>
<td>57.0</td>
<td>64.0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>.012</td>
<td>63 64 64 64 64 63 63 64</td>
<td>63.8</td>
<td>71.6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.014</td>
<td>71 71 71 72 72 72 72 72 72</td>
<td>77.4</td>
<td>77.6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.016</td>
<td>78 78 78 77 77 77 77 77 77</td>
<td>83.6</td>
<td>83.6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.018</td>
<td>64 64 64 64 64 63 63 63 64</td>
<td>83.4</td>
<td>80.0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>.020</td>
<td>91 90 90 90 90 90 90 90 90</td>
<td>89.8</td>
<td>90</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Static head</td>
<td>-1 1 0 0 1 1 0 0 0 0</td>
<td>-0.5</td>
<td>136.5</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.040</td>
<td>135 37 37 37 37 37 37 37 37</td>
<td>136.7</td>
<td>136.7</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>.060</td>
<td>168 69 68 67 67 67 67 67 67</td>
<td>169.4</td>
<td>169.4</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>.240</td>
<td>251 52 51 51 50 50 50 50 50</td>
<td>250.7</td>
<td>250.7</td>
<td>3</td>
</tr>
<tr>
<td>11.00 am</td>
<td></td>
<td>322 22 22 22 22 22 22 22 22</td>
<td>322.1</td>
<td>322.4</td>
<td>4</td>
</tr>
</tbody>
</table>
The velocity head in the table is obtained by subtracting the static head from the piezometric head. The last column in the table is the difference between the largest and smallest of the sample of ten readings in each row. It is seen that this range of the fluctuations is larger than the reading error and so the reading error may be disregarded.

In order to estimate the probable error of the velocity profile it is necessary to consider the fluctuations in water level. Each "sample" of ten readings took about three minutes and so would include at least one complete fluctuation of water level. It may therefore be considered a fair sample from a "population" which represents the water level in the manometer at different times.

The following three important properties of the Gauss Normal Error Law will be used.

(1) The expected range of a sample of ten is 3.08 times the population standard deviation. *

(2) The mean of a sample of ten has a standard deviation which is \(1/\sqrt{10}\) times the population standard deviation. **

(3) The probable difference between a variate and its mean is .6745 times its standard deviation. ***

* C. E. Weatherburn, "A first course in Mathematical Statistics"
  Camb. Univ. Press 2nd Ed. 1949 § 91

** Ibid § 22

*** Ibid § 21
As a starting point the "expected value" for the range of the sample of ten must be found from the data. It was found to increase with the velocity head \( H \) according to the following relationship.

Range in piezometric head readings = \( .03\sqrt{H} \)

Range in static head readings = \( .02 \)

\[
\therefore \text{Range in velocity head readings} \quad R_h = \sqrt{(0.03 \sqrt{H})^2 + (0.02)^2}
\]

\[
= 0.03\sqrt{H} + \frac{1}{2} \quad \text{inches.}
\]

The way in which this relationship was arrived at is shown in the following table where the actual ranges of 267 samples of ten are tabulated and averaged. The above equation for \( R_h \) is simply an algebraic expression approximating the average values in this table.
All the samples were ten readings. Unit is $\frac{1}{100}$

<table>
<thead>
<tr>
<th>Date</th>
<th>Range of Static Head Fluctuation</th>
<th>$R_n$</th>
<th>Range of Pileometric Head Fluctuation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20-49 50-99 100-199 200-299 300-399 400-499</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Feb 11</td>
<td>1,2</td>
<td>2,2,3,2,2,2,1</td>
<td>4,2,5</td>
</tr>
<tr>
<td>Feb 12</td>
<td>3,2,3,2</td>
<td>4,3,3,5,3,11</td>
<td>3,2,11,10</td>
</tr>
<tr>
<td></td>
<td>3,1</td>
<td>2,2</td>
<td>3,3,2,5,8</td>
</tr>
<tr>
<td>Feb 14</td>
<td>2,4</td>
<td>5,2,4,2,2,2</td>
<td>2,2</td>
</tr>
<tr>
<td></td>
<td>3,1</td>
<td>1,2</td>
<td>5,4,1,3</td>
</tr>
<tr>
<td></td>
<td>2,2,2,2</td>
<td>3</td>
<td>5,2,3,2,2,2,3</td>
</tr>
<tr>
<td></td>
<td>3,1</td>
<td>2</td>
<td>5,2,2,2,2,2,3</td>
</tr>
<tr>
<td></td>
<td>2,3</td>
<td>The profile tabulated</td>
<td>7,4,5,2,2,2,2,14</td>
</tr>
<tr>
<td></td>
<td>0,3</td>
<td>in full on page 38</td>
<td>4,2,3,2,1,3</td>
</tr>
<tr>
<td>Feb 16</td>
<td>2,1</td>
<td>2,1,1,1,2,2,2,4,5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1,2</td>
<td>2,1,3,1,2,4,4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1,2</td>
<td>2,2,2,2,2,2,2,2</td>
<td>3,5</td>
</tr>
<tr>
<td></td>
<td>1,1</td>
<td>0,2,1,2,2,2,6,4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1,2</td>
<td>2,2,2,2,2,2,2,2,2</td>
<td>3,5</td>
</tr>
<tr>
<td></td>
<td>2,3</td>
<td>3,2,2,1,2</td>
<td>1,5,2</td>
</tr>
<tr>
<td></td>
<td>1,2</td>
<td>1,3,3,2,1,2,3,5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>2,2</td>
<td>2,2,2,4,2,4,2,3</td>
<td>3,4</td>
</tr>
<tr>
<td>Feb 19</td>
<td>2,1</td>
<td>2,4,2,3,3,5</td>
<td>6,1</td>
</tr>
<tr>
<td></td>
<td>2,2</td>
<td>1,1,1,1,2,3,2,3</td>
<td>4,5</td>
</tr>
<tr>
<td></td>
<td>3,3</td>
<td>2,3,1,2,2,2,5</td>
<td>3,5</td>
</tr>
<tr>
<td>Total</td>
<td>61</td>
<td>21</td>
<td>225</td>
</tr>
<tr>
<td>Number</td>
<td>42</td>
<td>12</td>
<td>95</td>
</tr>
<tr>
<td>Average Range</td>
<td>1.93</td>
<td>1.75</td>
<td>2.57</td>
</tr>
<tr>
<td>Mean Head H</td>
<td>32</td>
<td>71</td>
<td>141</td>
</tr>
<tr>
<td>Estimate of expected range $\frac{0.03}{H}$</td>
<td>1.70</td>
<td>2.53</td>
<td>3.56</td>
</tr>
<tr>
<td>$\left(\frac{0.03}{H} - \frac{\text{Average Range}}{\text{Range}}\right)$</td>
<td>-.05</td>
<td>.16</td>
<td>-.16</td>
</tr>
</tbody>
</table>
Now from the statistical theory quoted above:

\[
R = 3.08 \times (\text{population s. d.}) = 3.08 \times \sqrt{10} \times (\text{sample mean s. d.}) = 3.08 \times \sqrt{10} \times \left( \frac{\text{probable difference between the}}{.0745} \right) \times (\text{sample mean and the population mean}) = 14.5 E_H
\]

\(E_H\) is called the probable error in the velocity head. Rearranging this equation and substituting for \(R\), the expression found above,

\[
E_H = \frac{R}{14.5} = .002 \sqrt{H + \frac{1}{12}} \text{ inches}
\]

The corresponding probable error in velocity is \(E_v\).

\[
E_v = \sqrt{\frac{2g(H + E_H)}{12}} - \sqrt{\frac{2g H}{12}} \text{ ft/sec.}
\]

\[
= \sqrt{\frac{2g H}{12}} \left[ \left(1 + \frac{E_H}{H} \right)^{\frac{1}{2}} - 1 \right]
\]

\[
= \sqrt{\frac{2g H}{12}} \left[ \frac{E_H}{H} \right] \quad \text{because} \quad \frac{E_H}{H} \ll 1
\]

\[
= .0023 \sqrt{1 + \frac{1}{2H}} \text{ ft/second.}
\]

Actual values of \(H\) were always between 1/4 and 4\(\text{"}\) and so the probable error in the velocity was between

.0025 ft/sec and .0032 ft/sec.

The conclusion is that the probable error (expected deviation) from the true velocity profile caused by pitot tube imprecision is \(\geq .003\) ft/sec. Therefore the maximum deviation which can be attributed to this cause is

\[
\frac{3.0}{.0745} \times .0032 = \geq .015 \text{ ft/sec.}
\]
ERROR IN THE VELOCITY PROFILE DUE TO IMPRECISE MEASUREMENT OF DEPTH

The pitot tube was attached to a vernier gauge which read in .001 ft units. The sources of error in measuring depth were three.

(1) Reading error  + .0005
(2) Bending of pitot tube support (estimated)  + .0005
(3) Displacement of the effective centre of the pitot opening  + .0000

TOTAL  + .001 ft

The probable error will be half this value, + .0005 ft

The third of these errors is caused by the velocity gradient and so will be a maximum near the bed. The magnitude of this error has been estimated using Preston's reported results. The error is small because of the small diameter of the pitot tube but it is mentioned because it is a systematic error which would drastically affect the optimisation calculation if it were large.

The velocity error caused by an error in depth will be largest near the bed where the velocity gradient is steepest. The following velocity profile is typical of the measured profiles and will be used for calculating errors.

\[ u = 2 \log \frac{y}{y_0} \]

The probable error in velocity, corresponding to a probable error of ± .0005 in the depth \( y \) depends on the value of \( y \) as follows:

When \( y = .016 \text{ ft} \), probable error in \( u \) = \( 2 \log \frac{.017}{.016} = .0526 \text{ ft/sec} \)

When \( y = .026 \)  "  "  "  \( u = 2 \log \frac{.027}{.026} = .0324 \text{ ft/sec} \)

When \( y = .046 \)  "  "  "  \( u = 2 \log \frac{.047}{.046} = .0186 \text{ ft/sec} \)

When \( y = .086 \)  "  "  "  \( u = 2 \log \frac{.087}{.086} = .0096 \text{ ft/sec} \)

When \( y = .146 \)  "  "  "  \( u = 2 \log \frac{.147}{.146} = .0060 \text{ ft/sec} \)

When \( y = .246 \)  "  "  "  \( u = 2 \log \frac{.247}{.246} = .0036 \text{ ft/sec} \)

Root mean square = \( .027 \text{ ft/sec} \)

I.e. ± \( .014 \text{ ft/sec} \)

The probable root mean square deviation of the six measured points from the curve of best fit is ± .014 ft/sec.

This value of probable error is as large as the possible error from other sources. Therefore future workers should concentrate on more precise measurement of depth.
15. A Comparison of Estimated Errors with Actual Deviations from the Logarithmic Formula

It is obvious from the previous paragraph that the errors on the two readings nearest to the bed are so large as to dwarf any other errors. The optimisation calculation (performed by the electronic computer) is particularly sensitive to small errors in these same two readings, and tends to fit the straight line closely to them, imprecise as they are. This close fit to these two points is at the expense of less accurately fitting of the other four more precisely measured points.

The expected mean deviation of the six points from the line is the sum of the two errors estimated above.

(i) Probable error due to the imprecise measurement of piezometric head
\[ + .003 \text{ ft/sec} \]

(ii) Probable error due to the imprecise setting of depth
\[ + .014 \text{ ft/sec} \]

TOTAL \[ + .017 \text{ ft/sec} \]

The actual mean deviation of the measured points from the logarithmic formula which best fits the profile was calculated for each profile. 61 profiles were measured each consisting of six measured "points" and resulting from averaging 50 separate piezometric total head readings. The actual mean deviation of these \( 61 \times 50 = 5490 \) measurements was
\[ + .024 \text{ ft/sec} \]

No measurements made have been discarded from this sample. The actual value for the mean root mean square deviation is satisfactorily comparable with the expected value.
16. REASONS FOR PREFERING THE LOGARITHMIC FORMULA FOR THE VELOCITY PROFILE

Fitting a measured profile of \( u \) on \( y \) to a logarithmic formula in the form \( u = A U_* \log \frac{y}{y_0} \) amounts to determining values for the parameters \( U_* \) and \( y_0 \).

It was found when calculating the value of the parameters which gave the logarithmic formula of "best fit" to the measurements, that there was a wide range of values with approximately equal "goodness of fit". That is a four fold range in \( y_0 \) and a corresponding 40% range in \( U_* \). Thus a velocity profile measurement defines a whole family of values of \( U_* \) and \( y_0 \) which will describe it equally well.

Previous workers have correlated \( U_* \) with the square root of the fluid shear stress at the boundary and \( y_0 \) with the height of roughness projections. When the object of flow measurement is to determine the effective roughness of the boundary then the value of \( U_* \) is determined directly from a measurement of the shear stress, and then the value of roughness size \( k = \frac{30y_0}{U_*} \) is calculated using the log formula of "best fit" to the velocity profile. However the insensitivity of the logarithmic formula to large variation in the value of \( y_0 \) (as has been noted above) means that \( y_0 \) can only be roughly determined by this calculation.

Other formulas are no better than the logarithmic formula in this respect. From the standpoint of the purely empirical treatment of data the log formula for the mean flow velocity derived by Keulegan is much the same as the Manning formula.
\[ u = 5.75 \, U_* \log_{10} \frac{12.27 \, R}{k} \text{ Keulegan} \]

\[ \frac{1}{a} = 5.75 \sqrt{R \, S \, g} \, 1.4 \left( \frac{R}{k} \right)^{1/6} \]

to within 5% for \( 5 < \frac{R}{k} < 500 \)

\[ = \frac{1.49}{(0.032 \, k^{1/6})} \, R^{3/5} \, S^{1/2} \text{ Manning} \]

\[ \begin{align*}
L \text{ Manning units in feet units.}
\end{align*} \]

In engineering design there is no advantage in adopting the Keulegan formula when the same result can be obtained by using the much better documented Manning formula. The advantage of this previous experience with the Manning formula is particularly evident when the effect of the fluid boundary depends on obstructions such as bed ripples or vegetation and when complicating factors such as suspended fines are important in determining the discharge.

On the other hand research into the transport of gravel by streams is greatly facilitated if a logarithmic formula is adopted because in this case the roughness parameter \( y_0 \) can (presumably?) be predicted in terms of the grading and closeness of packing of the gravel. When the value of \( y_0 \) is known, two velocity measurements at known spacing in a vertical enable the whole velocity profile at that vertical to be accurately defined by a logarithmic formula. That is two velocity measurements and a bed material grading curve completely define the flow.
The most important advantage of the logarithmic equation in research is that it is consistent with the Mixing Length Theory which explains the form of the equation. Admittedly the explanation is oversimplified but it enables certain properties of the turbulent structure to be inferred from velocity measurements. The writer has attempted to relate the inferred turbulent structure to the observed entrainment process, and the log formula, being dimensionally homogeneous, is a useful starting point for these speculations.

Compare these features with the formulas of the type,

\[ u \propto y^k \]

These power laws include complicated dimensional parameters and no explanation is available for their form. Though adequate for engineering design these formulas are clearly inferior when used in research.
Inferring the effective fluid shear stress and corresponding roughness at a boundary from velocity measurements is an established technique for smooth boundaries. The Stanton tube is a pitot tube placed against the boundary which measures the average effect of the velocity in a known region. The measurement is extremely sensitive to the distance of the tube from the boundary, and the size of the tube. These two sources of difficulty are overcome by accurately machining the tube and placing it against the boundary.

For rough boundaries this technique cannot be used because the measurement depends too much on the way in which the pitot tube happens to rest among the roughness projections.

Inference of effective roughness size from resistance formulas has already been mentioned. Resistance formulas relate the discharge to the sum of the pressure and gravity gradients in the flow and thereby obtain a parameter which indicates the average resistance of the whole boundary. In sediment transport the most important shear stress is the value at the centre of the bed of the channel. The relationship between this shear stress and the average resistance of the whole section depends on the shape of the section which is usually different in model and prototype.

In the laboratory it is difficult to measure the water surface slope of flows which are fast enough to entrain gravel particles. The particular flume available for the writer's experiments did not have a discharge measuring device. Thus the resistance formula method for estimating the equivalent roughness size was of no use in the experiments.

A third method for estimating the equivalent roughness has been developed by the writer. The method uses a large number of velocity
profile measurements. Forty four separate profiles were measured over the
same roughness and the log formula of best fit calculated for each.

\[ u = (5.75 \, u_\ast) \, \log_{10} \frac{y}{y_0} \]

The values of \( y_0 \) were found to range from \( .001 \) feet to \( .004 \) feet and the
mean value of \( y_0 \) was close to \( .002 \) feet.

\[ y_0 = .002 \, f. \]

This value of \( y_0 \) gave a
log formula of reasonably good fit for all the profiles:

Mean R.M.S. deviation for 44 profiles with:

(1) optimum \( y_0 \) values:

(11) \( y_0 = .002 \) feet

\[ \begin{array}{ccc}
(1) & \text{optimum } y_0 \text{ values} & .024 \, \text{ft/sec} \\
(11) & y_0 = .002 \, \text{foot} & .030 \, \text{ft/sec} \\
\end{array} \]

These values represent a scatter of about 2% in the velocities measured.

The agreement between the observed scatter, i.e. \( .03 \) ft/sec and
the estimated scatter \( .02 \) ft/sec is reasonable. The method seems
promising even though it is rather laborious and no experiments were
carried out on two other roughness configurations.

<table>
<thead>
<tr>
<th>Roughness of</th>
<th>Rounded Greysanche Gravel</th>
<th>Number of Velocity Profiles Measured</th>
<th>Mean ( y_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Passing B.S. Sieve Retained B.S. Sieve Geom. Mean D feet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/16&quot;</td>
<td>3/16&quot;</td>
<td>.014</td>
<td>11</td>
</tr>
<tr>
<td>1/8&quot;</td>
<td>3/16&quot;</td>
<td>.020</td>
<td>44</td>
</tr>
<tr>
<td>1/16&quot;</td>
<td>1/8&quot;</td>
<td>.029</td>
<td>6</td>
</tr>
</tbody>
</table>
The measurements with the smallest gravel were unsatisfactory because an undular hydraulic jump formed at the measuring section. This was the result of the particular combination of slope and discharge used. There was no time in which alternative flows could be tried.

In the case of medium sized gravel, it was found that
\[ y_o = 0.002 \, \text{ft} = \frac{D}{10} \] (Writer)

This should be comparable with Keulegan's formula (which was derived by a resistance formula approach) for a roughness of closely packed grains.
\[ y_o = k/30 \] (Keulegan)

Fookes has suggested that field measurements indicate that \( k = 3D \)

therefore
\[ y_o = \frac{D}{10} \] (Fookes)

This last result is the same as was found in this research.

The voids between the grains in loosely packed gravel are larger than in closely packed gravel as the diagram above illustrates. Keulegan's "\( K \)" is the grain size in Nikuradse's roughness experiments and Fookes's "\( D \)" is the mean grain size of gravel in natural rivers. The threefold difference in the equivalent roughness size is probably a result of the different degree of packing. The writer believes this to be a demonstration that it is the dimensions of the voids not the projections that determine the resistance of a gravel roughness.

For the largest gravel $y_0 = \frac{D}{6}$. This gravel was sprinkled on the already rough bed and then sprayed with varnish. It is probable that the voids were even larger than they would have been in a loosely packed arrangement.

The six velocity profiles measured over this large gravel were the last to be measured. The experimental technique benefited from the previous experience, the most notable improvement being the freedom from any downstream control. This was achieved by adding an additional pump to the recirculating system. The thirty six points which comprise the six profiles are plotted on figure 2. The magnitude and the random nature of the scatter of the points is well illustrated. This type of scatter is typical of the majority of the profiles measured.

The point marked with a question mark appears to be subjected to a gross error, possibly a mistake in reading. The numerical procedures did not eliminate gross errors, and there are presumably a few other similar mistakes.
The roughness configuration appeared to be uniform as may be seen from the photos. However, different entrainment rates were measured at different locations and led to suspicions that there were significant variations in the roughness on the bed. To investigate this possibility a set of ten velocity profiles were measured across a section. The results have been plotted in figure 3.

The isovels show a symmetrical pattern of the type usually attributed to secondary flows and the effect of the sides. Straight parallel equally spaced isovels have also been drawn to show the assumed flow pattern which is used when comparing entrainment measurements.

\[ y_0 = 0.02 \text{ and } u_0 = \frac{1.971}{5.75} \text{ in this particular case.} \]

The \( y_0 \) values which give the velocity profiles of best fit were found to vary in a consistent pattern also. Lower values of \( y_0 \) were found to give the best fit at the quarter points, and relatively high values at the centre.

There were also very high values of \( y_0 \) for the velocity profiles of best fit near the sides of the channel. These high values were associated with low goodness of fit. The explanation is that the friction of the sides of the flume reduces the velocity near the surface by a greater proportion than at lower levels. The resulting velocity profile is not logarithmic and high \( y_0 \) values arise when this non logarithmic profile is forced to fit a logarithmic formula.
$y = \text{distance from bed in ft.}$

Water surface

$\downarrow$

<table>
<thead>
<tr>
<th>$y$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
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<td>1.6</td>
<td>1.7</td>
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<td>1.9</td>
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<td>1.7</td>
<td>1.9</td>
<td>2.0</td>
<td>2.1</td>
<td>2.2</td>
<td>2.3</td>
<td>2.4</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9</td>
<td>1.4</td>
<td>1.7</td>
<td>1.9</td>
<td>2.0</td>
<td>2.1</td>
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<td>2.3</td>
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<tr>
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<td>1.7</td>
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<td>2.1</td>
<td>2.2</td>
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<td>2.4</td>
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<tr>
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<tr>
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<td>2.2</td>
<td>2.3</td>
<td>2.4</td>
<td>2.5</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Value of $y$ at which $u=0$ for best fit:

$y_0 = 0.0881 \quad 0.0210 \quad 0.0163 \quad 0.0200 \quad 0.0242 \quad 0.0272 \quad 0.0231 \quad 0.0165 \quad 0.00506$

Date: 10-2-63

Decimal indicates position of measured mean velocity ft/sec.

Heavy lines are isovels.

Dashed lines are assumed isovels.

Velocity profile pattern for the width of the channel.
Consider a motor car on a steep hill with its brakes on. There is a shear force between the car and the road equal to the product weight times slope. This shear can increase by either change in weight or slope. If the road is greasy and the applied shear force is increased then the car will slide if the shear exceeds the resistance of the tires. Alternatively if the brakes are poorly adjusted then the brakes will fail and the car will roll. A third type of failure could occur if for instance a shackle pin has fallen out. Thus the value of the shear force is of no importance until it exceeds the resistance of some part of the support. Notice also that immediately after failure the dynamic resistance can be somewhat less than the applied force and acceleration will occur.

Consider now a stream bed of coarse alluvium. As the shear applied by the fluid increases the bed will finally fail. In this case as with the motor car at least three different modes of failure are possible.

1. Particles which project above the general level of the boundary may be dislodged and start to roll and slide along. The force on the particle which dislodges it, is the result of it projecting into the flow. Thus the shear on each particle at threshold is calculated as the shear stress times a packing fraction. The packing fraction is the number of projections per unit area. This approach has been developed by C. M. White.*

2. Small particles may be lifted out of the bed and skim along a short distance. The force which lifts the particle originates in the flow. Thus it ultimately depends on the shear in that the shear is one of the factors which determines the flow. This approach has been developed by H. A. Einstein.** The experiments described below verified the details of this type of failure.

3. The top layer of the bed, which may be many grain diameters deep, may flow as a dispersion of grains in fluid. This occurs on very steep scree slopes. It has also been reported to occur in the Waikariri River by T. Nevin. R. A. Bagnold has described experiments which measured the shear resistance of a shearing dispersion of large grains at high concentration. This approach seems to be directly applicable to the calculation of discharge rates of sediment. However the initiation of a shearing dispersion in rivers with gravel beds and relatively small slopes is still unexplained and unmeasured.

The dynamic shear resistance on a grain after failure in the first case will be similar to the static resistance at failure. However this is certainly not true in the second case and is unlikely to be true in the third case. Thus as with the motor car, the most important value of the shear, the threshold value, depends on the type of failure considered. Formulas for transport rate based on the difference (actual shear less threshold shear) are only applicable to the first type of motion. Note that this type of motion is limited to a layer one grain thick.

Accurate and detailed description of the motion of the first particles to move when the flow is being increased are important. Many flume investigators have neglected to report this important description and field observations are entirely lacking.


Einstein says he developed his theory from observation of flume experiments with uniform one inch gravel. The stationary bed and moving bed load consisted of the same kind of grains and there was a steady intensive exchange of particles between the two.

In Einstein's own words:

"(a) Bed load movement is to be considered as the motion of bed particles in quick steps with comparatively long intermediate periods of rest. Thus bed load movement is a slow downstream motion of a certain top layer of the bed.

(b) The average step of a certain particle seems always to be the same even if the hydraulic conditions or the composition of the bed changes.

(c) Different rates of transportation are produced by a change in the average time between two steps."

Consider a section of unit width

Every particle passing this section is on its way to complete a step of constant length \( L \). This length is physically represented by the probable step which a particle will make when it is entrained, that is the average step. All particles which are entrained from a bed area of length \( L \) upstream of a section will together form the transport past the section in a unit of time.

The volume of sediment which passes a section in a second is obtained by multiplying together the number of particles in the surface of the area \((L \times 1)\) by the rate at which particles in the surface entrain and by the volume of one particle.

Thus

\[
\frac{q \text{\, immersed}}{(\sigma - \rho) g} = \left(\frac{i_b L}{A_1 D^2}\right) \times \left(\frac{P_3}{A_2 D^3}\right)
\]

\[
= \frac{A_2}{A_1} i_b D \times L \times \frac{P_3}{A_2 D^3}
\]

\( q \text{\, immersed} \) is the immersed weight of transported particles per second per unit width.

\( (\sigma - \rho) g \) is the immersed specific weight of the sediment.

\( i_b \) is the proportion of particles of size \( D \) in the bed material.

\( L \) is the average step.

\( A_1 D^2 \) is the bed surface area per exposed particle

\( A_2 D^3 \) is the volume per particle

\( P_3 \) is the rate of entrainment of exposed particles.

The three factors \( i_b \), \( L \) and \( P_3 \) will be discussed below under separate headings.
At this stage in the discussion it is instructive to compare Kalinske's approach in developing a similar equation. 

\( \eta \) is introduced as the proportion of the bed area taking the fluid shear, its numerical value depending on the closeness of packing of the grains. Thus the number of grains per unit area taking the shear is 

\[ \frac{\eta}{A_1 D^2} \]. Each of these grains is assumed to move when the instantaneous fluid velocity above it \( u \) exceeds a critical value \( u_c \). The grain velocity equals the amount by which the fluid velocity exceeds \( u_c \). Thus the mean grain velocity is the mean value of this excess.

\[ \overline{u_g} = \frac{u - u_c}{\log \rho} \]

The value rate of bed load movement is obtained by multiplying together the number of particles moving per unit area by their mean velocity and by the volume of one particle.

Thus 

\[ \frac{q_{im}}{(\sigma - \rho) \gamma} = \frac{\eta}{A_1 D^2} \times \overline{u_g} \times \frac{\pi}{6} D^3 \]

This equation has been made to fit sediment transport observations as well as Einstein's equation does. However the writer rejects this equation for two reasons.

For low rates of transport near the threshold of bed movement Kalinske's equation implies the relatively slow motion of a number of particles but the observed motion is rather quick movements of a very few particles.

For high rates of transport when a layer of the bed more than one grain deep is moving the value of \( \eta \) is greater than one. Thus the "packing fraction" notion of \( \eta \) becomes meaningless. Also the "slip velocity" \( u_c \) will presumably be different from the threshold value under these conditions.

22. The Exposure of Particles of a Certain Size in the Bed Surface

Consider the factor \( \frac{A_2}{A_1} \cdot i_b \) in Einstein's equation (1)_A. It is an easily evaluated property of the grains which make up the bed. Even if the bed consist of a mixture of grains of different sizes then the expression can be evaluated for any particular size using a grading curve of the bed material.

The expression \( \frac{A_2}{A_1} \cdot i_b \cdot D \) may be described as the volume of sediment of size \( D \) in a unit area of bed surface which is exposed in such a way that it could be entrained by a fast flow. For example if the grading curve of a bed material sample shows that one fifth of the bed material by volume is in the size range \( \frac{1}{5} \) to \( \frac{3}{4} \) then the expression is approximated as follows:

\[
\frac{A_2}{A_1} \cdot i_b \cdot D = \text{number of particles in size range exposed} \times \text{volume per particle}
\]

\[
= 1/5 \times \frac{\pi r^2}{2} \cdot \frac{4}{\pi} \cdot \frac{1}{64}
\]

\[
= .002 \text{ cu ft per sq. ft.}
\]

In estimating the transport of bed load over a natural stream bed the transport of each size range present is calculated separately and the results for all the ranges added together to give the total transport rate. A superior feature of Einstein's Approach is that in this way it allows the moving material to have a different grading curve from the bed material.
In his original presentation Einstein described the flow by the depth slope product, sometimes called the bed shear stress

\[ T = \rho g \cdot h \cdot s \]

He developed a procedure which reduced this shear for narrow channels to allow for the bank resistance. He also describes an elaborate procedure to obtain an estimate of the mean shear in natural rivers by averaging both the slope and the depth over a long reach.

In the "Modified Einstein Procedure" adapted by the U. S. Geological Survey all these refinements have been scrapped. The flow is described by the mean velocity, mean depth and a roughness size. The velocity is measured at a single section with a current meter. The roughness size is taken as the sieve size which passes 65% of a bed material sample. This modification considerably reduced the field work necessary without affecting the accuracy of the final estimate of the transport when the method was applied to sandy bed rivers in the mid west United States.

The logarithmic velocity distribution with Keulegan's constants is adopted and so the mean velocity is given as follows.

\[ \bar{u} = 5.75 \cdot u_{\infty} \cdot \log \frac{2\gamma}{k} \]

\[ u = 5.75 \cdot u_{\bar{u}} \cdot \log \frac{2\gamma\bar{R}}{k} \]

In his original presentation Einstein included a factor which increased the roughness size when dunes formed on the bed. This factor has been dropped in the "Modified Procedure". There is another factor which allows for the effect of a laminar sublayer. However for gravel particles 2 mm and larger this factor is always unity.

The gauging data are reduced to one flow parameter the shear velocity \( u_* \) thus

\[
u_* = \frac{u}{5.75 \log_{10} \frac{12.75k}{y}}\]

This parameter has been correlated with the depth slope product by many workers.
24. INTRODUCTION TO THE BED LOAD TRANSPORT EXPERIMENTS

These experiments were an attempt to make a quantitative test of the details of Einstein's approach to the transport of gravel. Only a few special conditions were covered and so the results do not provide any useful engineering data. However, two new procedures of measurement have been developed and the results have furnished strong evidence supporting some speculations about the process of entrainment.

The interaction of the moving particles with each other appears to be an important feature of the transport of gravel. These interactions are unimportant at the ill-defined upper limit of the moving particles because the particles are sparse and their behavior must be largely governed by the turbulent fluid around them. If the lifting force which counteracts the weight of the particles is the drag of upward flow filaments then the particles would be described as suspended sediment. It is doubtful if such a suspension mechanism exists at all for gravel (i.e., 2 mm diameter and larger), but if it does exist then it can only play a minor role in the whole process of transport. This last point has been confirmed by some movie films of flowing ½ inch gravel in a glass-sided flume. These photographs show the particles to have smooth trajectories between encounters with the bed, not a waltzing motion such as "suspended" particles would have.

Instead of "suspension" Einstein describes a mechanism in which the fluid plucks particles out of the stream bed. The formula he has derived fits observations of low rates of transport when the particles move independently of each other. This same mechanism could also explain the lifting of the sparse uppermost particles of a "live" gravel riverbed when interactions between most of the moving particles below are frequent, but in this case Einstein's formula for the total transport is no longer applicable. The experiments were devised to investigate the details of this
plucking mechanism.

The motion of glass balls over a gravel bed was observed in a laboratory flume. Only one ball was free to move so that there could be no effects from other moving grains. The gravel which formed the bed of the flume was stuck down with varnish and was intended to cause a turbulent structure in the flow the same as occurs in rivers.
Photo 15  Entrainment locations on 1/8" - 3/16" gravel.

Photo 16  Entrainment locations on 3/16" - 1/4" gravel.
Photo 17

Site plate.

Large $\frac{3}{8}$" - $\frac{1}{4}$"

Medium $\frac{1}{4}$" - $\frac{3}{16}$"

Small $\frac{3}{16}$" - $\frac{1}{6}$"
The aim was to achieve model flow conditions dynamically similar to those in natural gravel bedded rivers. The depth, roughness size and slope were such that no laminar sublayer would occur and the particles so large that viscous drag was negligible compared with form drag. The additional effects of suspended fine sediment and of shear failure of the gravel bed were not considered.

The procedure was to place a glass ball on the bed with long tweezers and measure with a stop watch the delay from the time it was free to move until it moved. The distance the ball travelled before coming to rest was also measured in some cases. A run usually consisted of about a hundred such trials with the flow of water remaining steady throughout.

The laboratory flume was rectangular in section, fifty feet long by eighteen inches wide. The depth was between four and five inches and the bed slope constant at .008. The measurements were made within five inches of the centerline of the flume to avoid side effects. Only the last ten feet of the flume was used so that there was a forty foot reach upstream for a boundary layer to develop. The velocity traverse measurements were conducted in conjunction with the transport experiments, and the results of these measurements used to describe the flow. See photo B.

The glass balls were about 3/16 inches diameter, nearly spherical and of the same density as gravel. The balls were chosen from a large collection of assorted glass ballotini. They were measured with a micrometer gauge and chosen so that the difference between the maximum and minimum diameters was less than .010". The mean diameters were in the range .175" to .193". See photo 19.
Photo 19 Choosing the more spherical glass balls.
The gravel bed was made from rounded greywacke river gravel which passed \( \frac{1}{16} \text{"} \) and was retained on \( 3/16 \text{"} \) B.S. Sieves. The gravel layer was \( \frac{1}{16} \text{"} \) thick and stuck to marine plywood with varnish. The plywood was bolted to the steel bottom of the flume. The sides were painted steel plate.

Undulations in the general bed level were not noticeable. The final slope and plane-ness of the bed were checked after assembly by filling the flume with still water and measuring the depth. The depths at each side and on the centerline were measured every two feet along the flume with a steel rule. The results are tabulated below.

<table>
<thead>
<tr>
<th>Distance from Entry</th>
<th>2&quot;</th>
<th>4&quot;</th>
<th>6&quot;</th>
<th>8&quot;</th>
<th>10&quot;</th>
<th>12&quot;</th>
<th>14&quot;</th>
<th>16&quot;</th>
<th>18&quot;</th>
<th>20&quot;</th>
<th>22&quot;</th>
<th>24&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHS</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>19</td>
<td>21</td>
<td>22</td>
<td>24</td>
<td>25</td>
<td>27</td>
</tr>
<tr>
<td>Q</td>
<td>10</td>
<td>12</td>
<td>13</td>
<td>16</td>
<td>16</td>
<td>17</td>
<td>19</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>25</td>
<td>26</td>
</tr>
<tr>
<td>NIS</td>
<td>11</td>
<td>13</td>
<td>13</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>19</td>
<td>21</td>
<td>22</td>
<td>24</td>
<td>26</td>
<td>28</td>
</tr>
<tr>
<td>Mean</td>
<td>10.3</td>
<td>12.3</td>
<td>13.3</td>
<td>15.3</td>
<td>15.3</td>
<td>17.7</td>
<td>19.0</td>
<td>21.0</td>
<td>22.0</td>
<td>23.7</td>
<td>25.3</td>
<td>27.0</td>
</tr>
<tr>
<td>Expected</td>
<td>10.0</td>
<td>11.6</td>
<td>13.1</td>
<td>14.6</td>
<td>15.6</td>
<td>17.7</td>
<td>19.2</td>
<td>20.6</td>
<td>22.3</td>
<td>23.6</td>
<td>25.4</td>
<td>26.9</td>
</tr>
<tr>
<td>Difference</td>
<td>.3</td>
<td>.7</td>
<td>.2</td>
<td>.7</td>
<td>.1</td>
<td>0</td>
<td>.2</td>
<td>.2</td>
<td>.3</td>
<td>.1</td>
<td>.1</td>
<td>.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Distance from Entry</th>
<th>25&quot;</th>
<th>26&quot;</th>
<th>30&quot;</th>
<th>32&quot;</th>
<th>34&quot;</th>
<th>36&quot;</th>
<th>38&quot;</th>
<th>40&quot;</th>
<th>42&quot;</th>
<th>44&quot;</th>
<th>46&quot;</th>
<th>48&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHS</td>
<td>28</td>
<td>30</td>
<td>31</td>
<td>33</td>
<td>34</td>
<td>35</td>
<td>36</td>
<td>40</td>
<td>40</td>
<td>42</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>Q</td>
<td>28</td>
<td>29</td>
<td>31</td>
<td>34</td>
<td>34</td>
<td>36</td>
<td>38</td>
<td>40</td>
<td>41</td>
<td>42</td>
<td>44</td>
<td>45</td>
</tr>
<tr>
<td>NIS</td>
<td>25</td>
<td>30</td>
<td>31</td>
<td>33</td>
<td>35</td>
<td>35</td>
<td>36</td>
<td>39</td>
<td>41</td>
<td>42</td>
<td>43</td>
<td>46</td>
</tr>
<tr>
<td>Mean</td>
<td>27.3</td>
<td>29.7</td>
<td>31.0</td>
<td>33.3</td>
<td>34.3</td>
<td>35.7</td>
<td>38.0</td>
<td>39.7</td>
<td>40.7</td>
<td>42.0</td>
<td>44.0</td>
<td>45.3</td>
</tr>
<tr>
<td>Expected</td>
<td>28.4</td>
<td>30.0</td>
<td>31.5</td>
<td>33.1</td>
<td>34.6</td>
<td>36.1</td>
<td>37.7</td>
<td>39.2</td>
<td>40.7</td>
<td>42.3</td>
<td>43.8</td>
<td>45.3</td>
</tr>
<tr>
<td>Difference</td>
<td>1.1</td>
<td>.3</td>
<td>.5</td>
<td>.2</td>
<td>.3</td>
<td>.4</td>
<td>.3</td>
<td>.5</td>
<td>0</td>
<td>.3</td>
<td>.2</td>
<td>0</td>
</tr>
</tbody>
</table>

The individual depth measurements could vary up to \( \frac{1}{8} \text{"} \) at a single section depending how the end of the steel rule rested among the stones.
Nevertheless, the mean of the three measurements at each section was found to be close to the value for a true plane at a longitudinal slope of .008. At only three sections was the deviation of the mean level from the expected value more than 1/16" and was never more than 1/8".

The steel recirculating tilting flume used was built by R.F. Hince in 1957 for use in research. He describes it in detail in his thesis submitted to the University of Canterbury in 1958.

The straightness and planeness of the sides and bed was considered excellent before the experiments began but there were still some indications that the flow was unsymmetrical at the measuring section. The variation of the flow across the measuring section has already been described.
Glass balls were placed on the bed and the time until they were entrained was measured. The results of a run were a set of numbers representing the observed delay times. For example a typical set of actual measurements are:

<table>
<thead>
<tr>
<th>Delay exceeded in seconds (January 7)</th>
<th>30</th>
<th>16</th>
<th>16</th>
<th>3</th>
<th>4</th>
<th>8</th>
<th>8</th>
<th>19</th>
<th>9</th>
<th>4</th>
<th>21</th>
<th>4</th>
<th>29</th>
<th>43</th>
<th>7</th>
<th>15</th>
<th>7</th>
<th>3</th>
<th>8</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>3</td>
<td>2</td>
<td>26</td>
<td>31</td>
<td>38</td>
<td>6</td>
<td>11</td>
<td>35</td>
<td>4</td>
<td>4</td>
<td>9</td>
<td>19</td>
<td>37</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>60+</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>21</td>
<td>17</td>
<td>7</td>
<td>8</td>
<td>3</td>
<td>36</td>
<td>6</td>
<td>1</td>
<td>12</td>
<td>9</td>
<td>2</td>
<td>1</td>
<td>16</td>
<td>5</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>30</td>
<td>8</td>
<td>2</td>
<td>15</td>
<td>26</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The glass balls when placed on the bed were intermittently subjected to jolts which at times were large enough to cause entrainment. The numbers give an indication of the average frequency of the large jolts and of their random distribution with time. A more elegant way to express these results will now be explained.

If \( P_1 \) is the probability of an entrainment (either one or more than one entrainment) in one second and is independent of the time elapsed without entrainment, then \((1 - P_1)^x P_1\) is the probability of a delay of between \( x \) and \( x + 1 \) seconds. (Note that \( 0 < P_1 < 1 \)).

Thus the probability of a delay exceeding \( x \) seconds where \( P_1 = 1 - P_1 \) is

\[
F_x = q x P_1 (q + q^2 + q^3 + \ldots) = \frac{q x P_1}{(1 - q)} = (1 - P_1)^x
\]

Similarly if \( P_{\frac{1}{k}} \) is the probability of an entrainment in \( \frac{1}{k} \) seconds then

\[
F_x = (1 - P_{\frac{1}{k}})^x
\]
Therefore
\[ 1 - R \approx (1 - \frac{P_3}{n})^n \]

Whence
\[ P_3 < \frac{n P_n}{n} \quad \text{if} \quad n > 1 \]

The rate of entrainment \( P_3 \) is defined as the value of \( n P_n \)

\[ \lim_{n \to \infty} (1 - \frac{P_3}{n})^n = \left( \lim_{n \to \infty} (1 - \frac{P_3}{n}) \right)^{P_3} = e^{-P_3} \quad \text{(2)} \]

\( P_3 \) is analogous to the force of interest used in compound interest theory. It is an absolute measure of rate which scales directly with the unit of time used and so \( P_3 \) may be called the entrainment rate.

To estimate \( P_3 \) the results are tabulated in order of size, and the proportion \( \frac{n_x}{n} \) exceeding a delay \( x \) is found. For example the results already quoted may be tabulated as follows:

<table>
<thead>
<tr>
<th>Delay secs</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. in sample</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>No. exceeding ( x )</td>
<td>72</td>
<td>66</td>
<td>60</td>
<td>55</td>
<td>48</td>
<td>43</td>
<td>41</td>
<td>36</td>
<td>30</td>
<td>25</td>
<td>25</td>
<td>24</td>
<td>23</td>
<td>23</td>
</tr>
<tr>
<td>Proportion exceeding ( \frac{n_x}{n} )</td>
<td>1.00</td>
<td>.93</td>
<td>.85</td>
<td>.77</td>
<td>.65</td>
<td>.57</td>
<td>.51</td>
<td>.42</td>
<td>.35</td>
<td>.34</td>
<td>.32</td>
<td>.32</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The expected value of \( \frac{n_x}{n} \), the last row in the table, is the ratio of the probability of a delay exceeding \( x \) seconds and the probability of a
delay exceeding one second. This is because all delays less than one second have been discarded in order to ensure that the manner of placing the ball does not affect the results. Thus the expected value of

\[ \pi_x = \frac{F_x}{F_1} = e^{-3x}/e^{-3} = e^{-3}(e^{-1}) \]

and so

\[ \log_e \pi_x = -3(x-1) \]

(iii)

A plot of \( \log \pi_x \) versus \( x \) see Figure A enables two conclusions to be reached.

(i) If a straight line results then the assumption that the probability of entrainment is independent of time is confirmed.

(ii) The slope of the straight line of best fit is an estimate of \( \pi_x \).

To calculate \( \pi_x \) the value of \( x \) corresponding to \( \pi_x = .5 \) is substituted in the equation

\[ \pi_x = -\log_e \frac{.5}{x-1} = \frac{.693}{x-1} \]

For example from Figure A the required value of \( x \) is 8.3 seconds and so

\[ \pi_x = \frac{.693}{8.3-1.0} = .095 \]  entrainments per second.

Thus the sample of 71 observations may be completely described by this one number.

The objection might be raised that if the entrainment at a site is a regular event with a fixed period then a random distribution of measurements would still result because of the arbitrary entry into the sequence when the glass ball is placed on the bed. In answer it is pointed out that such a situation would mean that \( \pi_x \) would vary linearly with \( x \) and the graph of \( \log \pi_x \) versus \( x \) would be concave toward small \( x \) values. Also very large delays would not occur.
Date 7.1.63   Sample 71 trials   \( P_3 = 0.095 \) per sec.

Plot of proportion of sample versus delay exceeded.
The actual graphs of \( \log T_x \) versus \( x \) tend to be concave toward the large \( x \) values. This could indicate that the rate of entrainment at site \( \Omega \) fluctuates about its mean with fluctuations of long period. The same curvature results when one sample is drawn from two or more locations on the bed where the entrainment rate is different.
Once entrained the glass balls travelled by bouncing along the bed. For example a ball might bounce first after one inch then come to rest two inches farther on in a slight depression. A rest of less than a second would be followed by eight inches of rapid skimming motion, apparently rolling before stopping suddenly in another depression. This second rest would be longer and the ball would then be retrieved by the tweezers four seconds after it came to rest.

Each encounter with the bed generally causes a change in the direction of movement. Encounters in which the ball's direction of movement when viewed from above is unchanged would go undetected and so it is difficult to estimate the length of each free jump. At the fastest flows the balls averaged above two short rests and were entrained three times in all before resting long enough to be retrieved. On the other hand at the slow flows when entrainments occurred one a quarter hour, the first rest was invariably long enough to enable the ball to be retrieved.

The length of travel seemed to depend on the number of suitable resting places in the path of the ball. The balls bounced off roughness projections until, by chance, they landed in a depression in such a way that they were stopped dead. If the depression was shallow then the ball was entrained immediately and if the depression was deep then the ball remained. Experiments with different roughness sizes confirmed this description. These experiments were as follows.

<table>
<thead>
<tr>
<th>Roughness</th>
<th>Length of travel of 3/16&quot; glass ball</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small stones 1/8&quot; to 3/16&quot;</td>
<td>greater than 10 feet</td>
</tr>
<tr>
<td>Medium stones 3/16&quot; to 3/8&quot;</td>
<td>of the order of 1.5 feet</td>
</tr>
<tr>
<td>Large stones 3/8&quot; to 1/2&quot;</td>
<td>of the order of 1.0 feet</td>
</tr>
</tbody>
</table>
The distance that the moving particle travels between encounters is determined by the spacing of the roughness projections. The proportion of encounters in which the particle will be brought to rest will be some function of the ratio roughness height to particle diameter. For a roughness of gravel both the spacing and the height of the projections are proportional to the "equivalent roughness size" $k$. The average length of travel may therefore be expressed as the following product:

$$ L = k \times \text{function} \left( \frac{k}{D} \right) $$

In nearly all the experiments described below the roughness size $k$ and the particle size $D$ remained the same. The expression for length of travel was consequently simplified as follows.

Since $\frac{k}{D}$ is constant

Then $L = \left( \text{"constant} \ 1^\text{st} \right) \times k$

and $L = \left( \text{"constant} \ 2^\text{nd} \right) \times D$.

This last expression is the same as the one assumed by Einstein in developing his bed load formula. Therefore it seems worthwhile measuring the travel distances of the glass balls and attempting to verify Einstein's formula. For a particular ratio $k/D$ Einstein's assumption that $L \propto D$ at the threshold of motion is reasonable, and for only this special case will verification be attempted.
The object of the bed load experiments was to measure the two variables:

1. Entrainment rate $P_s$
2. Average length of travel $L$

The variables are the basis of Einstein's approach to the problem of bed load transport. It should be possible to relate their product to the Shield entrainment function in the same way as Einstein's "Bed Load Formula".

In the course of the experiments the rather disappointing discovery was made that under steady flow conditions the entrainment rate will be substantially different at different locations on the bed, and on different occasions at the same location. The program of experiments was therefore altered and extended so that the factors which determine this variation might be isolated.

A summary of the results of all the bed load experiments is tabulated in Appendix 3. Details of the measurements are explained under the following six headings:

1. The nonuniformity of the flow and the vibrations of the flume.
2. The experimental results are compared with Einstein's "Bed Load Formula".
3. The entrainment "site plate".
4. The glass balls.
5. The importance of adjacent bed configuration

6. The variation with time of the entrainment rate at a location.
It was noticed that the slope of the water surface in the measuring reach was less than the slope of the bed. The roughness of the water surface and the bed meant that measurement of the amount of this divergence was very difficult and inaccurate. The importance of this nonuniformity as a factor affecting the entrainment rate was clearly demonstrated however.

The pump was run at a constant discharge throughout an experiment in which the level of the water in the flume's downstream tank was raised by stages. The flow in the measuring reach must have been under downstream control because the depth at the measuring section also increased by stages. The increase in water level in the downstream tank was achieved by increasing the total volume of water in the recirculating system.

A small change in the water surface slope at the measuring section significantly affected the entrainment rate. This was demonstrated by a series of measurements of the entrainment rate at different water depths, and the results are tabulated immediately below. Note that the discharge was constant throughout.

<table>
<thead>
<tr>
<th>Depth at the measuring section in feet</th>
<th>Entrainment rate per second</th>
</tr>
</thead>
<tbody>
<tr>
<td>.437</td>
<td>.30</td>
</tr>
<tr>
<td>.510</td>
<td>.036</td>
</tr>
<tr>
<td>.573</td>
<td>.0046</td>
</tr>
<tr>
<td>.583</td>
<td>.0008</td>
</tr>
</tbody>
</table>

The entrainment measurements are plotted in figure 5.
Date 19.1.63.
Smooth curves are for the assumed randomly distributed populations, i.e., infinite samples. Heavier lines are the plots of the measurements, i.e., actual small samples.

The measurements show how a small variation in the downstream control drastically affects the entrainment rate at the measuring section.

Entrainment rate measurements for a constant discharge in the flume...
It was intended that the velocity traverse technique would give a satisfactory measure of the shear stress at a section even in nonuniform flows. However, the rather crude experiment described here shows that the entrainment rate is highly sensitive to variation in the flow pattern along the streamlines. Therefore it seems likely that "nonuniformity" is a major source of error in the attempt to correlate the entrainment rate measurements and the flow measurements.

On the other hand the six velocity profiles over the larger gravel which are plotted in figure 2 were not subject to this source of error because there was no downstream control of the flow. In fact these results were much more consistent than the more numerous results over the smaller gravel.

It was also noticed that the flume at the measuring section was vibrating up and down with a frequency of 6 cycles per second and an amplitude of about .003". The flume was fifty feet long and only supported at three points as a continuous two span beam, and it would require extensive alterations to reduce the vibrations. Therefore the vibrations were tolerated.
The experimental results are compared with Einstein’s bed load formula.

Two comparisons are made. The first is without any modifications by the writer which would improve the goodness of fit. The flow function is then arbitrarily altered so that the writer’s data fit Einstein’s “Bed Load Formula”. This enables the scatter of the writer’s measurements to be compared with that of previous workers.

1. The flow function \( \psi \) was calculated as follows. Keulegan’s result for uniform flow is adopted, that the slope of the logarithmic velocity profile is

\[
\chi = 5.75 u_\star = \frac{u}{\log_\sigma \frac{y}{y_0}}
\]

Colby’s form of Einstein’s flow function is adopted

\[
\psi = 4 \frac{(\sigma - \rho) g D}{\rho u_\star^2}
\]

Substituting the values

\[
\frac{\sigma - \rho}{\rho} = 1.65
\]

\[
D = \frac{y}{16} = \frac{1}{64} \frac{y}{\chi^2}
\]

\[
u_\star = \frac{\chi}{5.75}
\]

then

\[
\psi = 4 \times 1.65 \times 5.75^2 = \frac{1}{\chi^2}
\]

The bed load function \( \phi \) is calculated as follows.

The function is

\[
\phi = \frac{Z_{im}}{(\sigma - \rho) g D W}
\]

The equation expressing "Einstein's Approach" is (from section 20)

\[ \frac{2L_{in}}{(\sigma - \rho)g} = \frac{A_2}{A_1} D L \frac{P_2}{P_1} \]

Therefore

\[ \phi = \frac{A_2}{A_1} \frac{L}{P_2} \]

Substituting the values:

\[ A_1 = \frac{\pi}{4} \]
\[ A_2 = \frac{\pi}{6} \]
\[ w = 0.816 \sqrt{\frac{\sigma - \rho}{\rho}} g D \]

Rubey's formula:

\[ = 0.816 \sqrt{1.65 \times 32.2 \times \frac{1}{64}} \]
\[ = 0.744 \text{ ft/sec} \]

then

\[ \phi = \frac{\pi}{6} \cdot \frac{4}{\pi} \cdot \frac{L P_2}{0.744} \]
\[ = 0.9 L P_2 \quad (2) \]

Both \( \psi \) and \( \phi \) are calculated and tabulated in Appendix 3. The values of \( \psi \) and \( \phi \) for each run define a point which should lie on a line representing Einstein's Bed Load Formula. The twenty four points are plotted on figure 6 and are seen to lie near the lower limit of transport for sediment mixtures. This limit was determined by Einstein on the basis of the measurements of the U.S. Waterways.

2. In the second comparison the ratio of the shear velocity to the slope of the logarithmic velocity profile is taken as half of Keulegan's value.

Keulegan \( \times = 5.75 u_k \) for uniform flow.

Writer \( \times = 12.25 u_k \) for steep flow under downstream control

First comparison of measurements with Einstein's Formula.
Thus

$$\psi = \frac{50}{x^2}$$  \hspace{1cm} (3)

This modification makes the measurements plot in the same region of the $\phi - \psi$ plane as the Gilbert and Moyer Peter data. The modification might be regarded as an arbitrary allowance for the non uniformity of the flow.

An expression for the entrainment rate $P_3$ was derived from Einstein's formula. This expression enables an independent check to be made on Einstein's formula, a check which does not involve the expression for length of travel. 

$$L = \frac{\lambda D}{1-p} \hspace{1cm} \text{See page 153.}$$

Now the bed load formula may be written,

$$\phi = \frac{p}{A(1-p)} \hspace{1cm} \text{See Appendix 2.}$$

The bed load function has been shown to be

$$\phi = qL \frac{P_3}{1-p} = q \frac{\lambda D}{1-p} P_3$$

Therefore

$$q \frac{\lambda D}{1-p} P_3 = \frac{P}{A(1-p)}$$

and

$$P_3 = \frac{P}{q \lambda D A}$$

Note that the factor $(1-p)$ has been eliminated.

Einstein suggests that $\lambda = 100$ (though the average travel measured at the small values of $P_3$ indicate a value of about $\lambda = 20$). Einstein's curve which fits the Gilbert and Moyer Peter data has $A = 39$, in the experiments $D = \frac{1}{64}$ and so substituting these two values,

$$P_3 = \frac{64 P}{q \times 100 \times 39} = \frac{P}{55} \int_{2.75}^{\infty} e^{-t^2} dt$$

** Moyer-Peter, Faure and Muller: Schweiz. Bauzeitung, Band 105 1935
This equation for the entrainment rate has been inferred entirely from Einstein's publications without modification. The line corresponding to the equation is dotted on figure 7.

For each run in Table One values of $\psi$ and $\psi_z$ were calculated. Forty one points are thereby defined on the $\psi - \psi_z$ plane and are plotted on figure 7. The points are seen to scatter about the line about as much as the Gilbert and Meyer Peter data. See Figure 17.

It has been demonstrated that by making an arbitrary adjustment to allow for the non-uniformity of the flow it is possible to make the measurements fit Einstein's formula. It is not considered worthwhile developing this comparison with Einstein's formula any further because of the necessity for arbitrary assumptions. Also the physical description of entrainment implied in Einstein's publications differs in some important details from the writer's findings. These differences will be described below.
See figure 17 for comparison of the scatter of these points to the points from the data of Gilbert, and Meyer-Peter which Einstein plotted.

Second comparison of measurements with Einstein's Formula.
The tendency for a particle to entrain from a stream bed depends on how it is located in the bed. These experiments involved replacing a glass ball in the same place many times and such a place is called an entrainment site.

Six identical "site plates" were made. They consisted of a brass plate on which five 11/32" diameter steel balls rested in five holes. The 3/16" diameter glass ball was placed on a pin between the steel balls so that the top of all balls was flush. The site extended for 3/4" on three sides and 5/8" upstream of the glass ball. Sufficient stones were cleaned off the bottom of the flume so that the site plate could rest on the bottom. When in position the tops of the balls were at the general level of the tops of the roughness projections on the adjacent bed, that is about 1/64" below a foot rule laid flat on the bed.

The site plates were placed at six "locations". One location was used in conjunction with another apparatus and so only five locations were used for measuring entrainment. The locations were designated A, B, C, D, E as indicated in the photograph 16.

An experiment demonstrating that the site plates are identical was as follows. Five delay times were measured for each of the five locations in turn. Then the site plates were changed to new locations and the process repeated for every combination of site plate and location. The results (January 15) follow.
When five samples of 25 trials are selected, one sample for each site plate, the entrainment rate for all samples as shown on figure 8 is much the same. Thus the site plates may be considered to be identical.

However when five samples of 25 trials are selected, one sample for each location, the entrainment rates obtained differ widely. Thus the entrainment rate varies at different locations on the bed.
Diagram showing locations.

Figure 8

Proportion of sample $T_{10}$

Continuous line - 25 trials made on one "site plate", 5 at each "location". Note the similarity of the five samples of 25 which indicates that the "site plates" are identical.

Dashed line - 25 trials made at a "location", 5 with each "site plate". Note the significant variation (four fold) between the five locations.

Mean entrainment rate, $P_3 = 0.154$ per sec. Date 15-1-63.

Entrainment rate with site plates similar, locations vary.
The tendency of a particle to entrain from a stream bed also depends on the size and shape of that particle. The particles used in the experiments were carefully selected from a large collection of assorted glass ballotini which varied in both size and shape. See photograph 19.

An experiment demonstrating that the differences in size of the various balls used was unimportant was as follows. Two sizes of glass balls were chosen for comparison. (1) mean diameter .179"

(11) mean diameter .195"

The entrainment rate for each size of ball on each of the five locations was measured. The flow was steady throughout. The results (January 12) are plotted in Figure 9.

It is seen in figure 9 that there is again a four fold variation in the entrainment rate between the locations. There is only a small difference however between the two values of the rate for each size at the same location. It may be concluded that the difference between the entrainment characteristics of the various glass balls used is small in comparison with the errors of measurement. (The anomalous result for location E is explained below.)
Date 12-1-63
Mean $p_s = 0.015$ per sec.
Total no. of trials 267

Location on bed | A | B | C | D | E |
---|---|---|---|---|---|
No. trials with large balls | 24 | 38 | 22 | 28 | 30 |
No. trials with small balls | 22 | 32 | 20 | 30 | 21 |

Entrainment rate, glass balls similar, locations vary.
The entrainment rate for the five locations was found to include about a four fold range of values on all occasions. The site plates which extend about the entrainment site at least $\frac{1}{8}$ in all directions were shown to be identical. Therefore the different entrainment rates cannot be explained as a result of different amounts of exposure to the flow. See figure 8 above.

The experiment already described with the two sizes of balls (January 12) indicates that the rate of entrainment is reasonably defined by 25 trials. Thus the four fold range cannot be explained as random variation, (except possibly for location E). See figure 9 above.

Investigation of the anomalous behaviour at location E has led to a more satisfactory explanation of the observed variation at all locations. On January 8 it was found that the entrainment rate at E was four times the average rate of all the locations. On the same occasion if an 11/32" steel ball was placed in a larger than usual depression 1" upstream of the site, then the rate was the same as the average of the other locations. The following day at a slower flow the effect was confirmed. This time placing the obstruction in the depression reduced the entrainment rate from six times the average, to the average value. See figure 10.

Further anomalous behaviour at location E has already been reported, that is on January 12 when there was a three fold difference between two measurements. Also on January 23 six separate measurements were made at E and five were similar, the odd one being half the average value of the other five. See figures 9 and 13.

A special experiment was devised to see whether the entrainment rate at other locations could be altered by placing steel balls on the adjacent
Date 8.1.63  Each sample 20 trials  Mean $P_3 = 0.09$ per sec

$E_1$ - depression in bed upstream.

$E_2$ - depression filled.

Entrainment rate, anomalous behaviour at location $E$. 

Figure 10
bed. Also the extent of the area of bed which affects the entrainment at a point might be measured this way.

The entrainment rate at D was measured three times. (January 20). The first time with the bed in normal condition ("before") then with a pattern of 11/32" steel balls 1" upstream ("obstructed") and then with the balls removed again ("after"). The results are plotted in figure 11 and show that the obstructions increased the rate. (For location E the obstruction decreased the rate.)

A similar experiment at location C (January 21) resulted in the obstructions decreasing the rate. At location B (January 22) the obstructions did not alter the rate at all.

A different upstream obstruction was found which increased the rate at location E on January 23. Downstream obstructions were also tried on this occasion. See figure 12. The first downstream obstruction was so close that it increased the height to which the glass ball had to rise to be entrained. The other downstream obstruction did not obstruct the trajectory of the particle and so did not affect the entrainment rate at all.

The important conclusion to be drawn from these diverse observations is that the entrainment rate at a site can be altered in two ways. A change in the roughness configuration up to at least an inch upstream and three quarters of an inch to the side can affect the turbulent structure of the flow over the site. Alternatively an increase in the size of the obstruction against which the free glass ball is temporarily lodged decreases the entrainment rate because it increases the height the ball must be lifted.
Figure 11

Date 20·1·63 Location B Each of the 3 samples was 20 trials.

Entrainment rate, obstructions at location B.
Entrainment rate, obstructions at location E.
The explanation for the variation of entrainment rate between locations at the end of the previous section is inadequate to explain the variation of the rate at a particular location. It would be expected that some of the locations would be consistently relatively faster or slower than the other locations depending on their respective adjacent bed configurations. However when the five locations are listed in order of entrainment rate, then this order is different for different dates.

For instance:

<table>
<thead>
<tr>
<th>Date in January</th>
<th>Location</th>
<th>Relatively fast Entrainment</th>
<th>Relatively slow Entrainment</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>E&lt;sub&gt;1&lt;/sub&gt;</td>
<td>D, A</td>
<td>E&lt;sub&gt;2&lt;/sub&gt;, C, B</td>
</tr>
<tr>
<td>12</td>
<td>B</td>
<td>E&lt;sub&gt;1&lt;/sub&gt;, D, A</td>
<td>E&lt;sub&gt;2&lt;/sub&gt;, C</td>
</tr>
<tr>
<td>15</td>
<td>-</td>
<td>D, B, E</td>
<td>A, C</td>
</tr>
<tr>
<td>17</td>
<td>-</td>
<td>D, A, E</td>
<td>C, B</td>
</tr>
<tr>
<td>18</td>
<td>-</td>
<td>D, B, E</td>
<td>C, A</td>
</tr>
<tr>
<td>21</td>
<td>B</td>
<td>D, C&lt;sub&gt;1&lt;/sub&gt;, A</td>
<td>E, C&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td>23</td>
<td>A</td>
<td>D, B, E&lt;sub&gt;1&lt;/sub&gt;</td>
<td>C, E&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
</tbody>
</table>

Subscripts indicate two different measurements at the same location during a run.

Location D is seen to be consistently fast and location C consistently slow. The other locations A, B and E were inconsistent being both faster and slower than the average on different occasions.

To investigate these inconsistencies a hundred and twenty consecutive trials (i.e. delay measurements) were made at Location E. The trials averaged a minute each so that the complete set took two hours to measure.
The delay times in the chronological order of measurement follow.

(February 23) Consecutive readings at Location E

<table>
<thead>
<tr>
<th>No.</th>
<th>19</th>
<th>16</th>
<th>5</th>
<th>2</th>
<th>12</th>
<th>12</th>
<th>3</th>
<th>36</th>
<th>1</th>
<th>10</th>
<th>7</th>
<th>1</th>
<th>36</th>
<th>1</th>
<th>4</th>
<th>39</th>
<th>1</th>
<th>45</th>
<th>53</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>2</td>
<td>9</td>
<td>11</td>
<td>40</td>
<td>7</td>
<td>1</td>
<td>40</td>
<td>7</td>
<td>21</td>
<td>22</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>27</td>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>No.</td>
<td>3</td>
<td>10</td>
<td>22</td>
<td>32</td>
<td>1</td>
<td>1</td>
<td>25</td>
<td>34</td>
<td>17</td>
<td>16</td>
<td>5</td>
<td>44</td>
<td>19</td>
<td>17</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>27</td>
</tr>
<tr>
<td>No.</td>
<td>4</td>
<td>3</td>
<td>19</td>
<td>1</td>
<td>3</td>
<td>16</td>
<td>1</td>
<td>138</td>
<td>12</td>
<td>7</td>
<td>5</td>
<td>1</td>
<td>7</td>
<td>3</td>
<td>11</td>
<td>4</td>
<td>20</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>No.</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>6</td>
<td>7</td>
<td>21</td>
<td>10</td>
<td>2</td>
<td>6</td>
<td>17</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>12</td>
<td>5</td>
<td>1</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>No.</td>
<td>6</td>
<td>10</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td>28</td>
<td>10</td>
<td>17</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>56</td>
<td>1</td>
<td>28</td>
<td>16</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Unit = one second. The numbers 1 to 6 refer to consecutive samples of 20.

The above table shows how the long and short delays occur in random succession. However, in order to test whether the rate of entrainment was truly constant over this two hour period, the set of a hundred and twenty is treated as six independent samples of twenty. Each sample thus indicates the average rate of entrainment in a twenty minute period.

The six samples are plotted in figure 13 and the values of entrainment rate are in order of size: No. 3 = .06, No. 2 = .09, No. 1 = .10, No. 6 = .10, No. 4 = .11, No. 5 = .11 per second. It appears that the third sample of twenty represents an entrainment rate at E considerably less than any of the other samples which are all similar to each other. That is, for the third twenty minutes the entrainment rate was different from its value for the rest of the two hour period.

On the same figure the measurements for the other four locations on the bed are shown as dashed lines. The four fold range of variation of the entrainment rate between locations on this occasion was similar to the range noticed on the six other occasions mentioned at the beginning of this section.

The explanation proposed to explain these variations with time at
Figure 13

Date 23.1.63
All samples were 20 trials
P3 values shown are per sec.

Entrainment rate, 120 measurements at location E.
a location is consistent with the previous explanation of the variation between locations. Certain features on the bed act as eddy generators and are constantly initiating eddies which are convected away by the flow. There may be more than one eddy generator near a site and the eddies from any particular generator will vary in size. Some eddy generators are unreliable and may cease to operate after having initiated a succession of eddies for the previous hour or so.

Location E is notable in that one particular feature on the bed which functioned as a strong eddy generator was found. A rather larger than usual depression in the bed (see photo 16 which shows the position of the depression relative to the entrainment site) was found to cause very fast entrainment. If this depression was filled then the entrainment rate was reduced as has been reported already in the previous section, 33.

A strong and stable eddy generator must exist upstream of location D and so the entrainment rate at D is consistently faster than the average for the particular flow. This eddy generator was not found. The locations A, B and E must have strong eddy generators but which are comparatively unstable and only operate for some of the time. Thus their different behaviour on different occasions is explained.

Location C was the only location near the centre line of the flow, and was the only location at which the entrainment rate was consistently slower than the average. The velocity measurements already described indicate that secondary flows affect the central region of the flow reducing the velocity. This would be likely to be related to the consistently slow entrainment rate at C.

The reduced velocities at the centre line of the flow were shown to be a stable feature throughout a six hour run. This steadiness is a characteristic of true secondary flows of the kind being considered here. The entrainment rate at the locations A, B and E was different at different times and so their variation could not be related to secondary flows which are essentially steady.
This finding suggests that it is the size of the depressions that determines the effect of a particular roughness configuration. Note that this explanation is consistent with the previous explanation for the difference between the effective roughness size of close packed and loose packed gravel of the same size, \( y = D/30 \) and \( y = D/10 \) respectively. This finding also seems to explain the errors that have been encountered when direct measurements are made of the boundary shear stress. These errors are because the clearance gaps between the shear plate and the rest of the boundary have a drastic effect upon the turbulent structure.

The above explanation of "eddy generators" of varying reliability can satisfactorily explain the entrainment observations made. The extent of these observations is shown in Appendix 3.
An apparatus was made to measure small forces electronically and a 3/16" dia. ball was attached. The ball was the same size and had the same position relative to the flow as the glass balls which were entrained. A 1.2 mm dia. bronze rod passed through a small hole in the bottom of the flume and connected the ball to the electronic device which was enclosed in a glass chamber. The air pressure in this chamber just balanced the hydrostatic pressure so that no water or air flowed through the small hole. See figure 14.

The response of the device was found to be dominated by resonant vibrations. The "magnification factor" spectra was measured by attaching the device to a controlled exciting force. Two different spectra corresponding to two different springs in the device are shown on figure 15. The spectra of the random exciting force included components in the resonant frequency range where the magnification factor is large and indeterminate. This meant that forces comparable with the weight of a particle i.e. 100 dynes, gave a response of the order 1000 dynes. A typical response trace from the electronic device is attached to figure 15.

When Einstein and El Sammi measured the fluctuating force of the fluid at a point on a rough boundary they reported a dominant "period of about 1/20th second". The roughness was made of 1/4" diameter balls. It is pointed out that if eddies at a spacing of 1/4" pass over a point at 2.5 ft per second then the recurring period will be 1/20th second.

In the writer's experiment the roughness was 1/8" gravel and the velocity near the bed was 2.0 ft per second, and so the recurring period of eddies would be 1/96th second. The random nature of the roughness configuration resulted in an equivalent roughness of 1/8" which corresponds to a recurring period of 1/24th second. These two recurring periods
Figure 14

Transducer Apparatus for Measuring Force

Full Size
Figure 15

Oscillations were with a constant peak acceleration of \( \pm 1.09 \text{ g} \).

**MAGNIFICATION SPECTRA OF THE TRANSUDER**
bracket the recurring period of the resonant frequency of the device. It is therefore not surprising to find that the device was excited at its resonant frequency thereby making its response meaningless.

The limitations of the type of apparatus used may be calculated. The magnitude of the force to be measured is of the same order as the weight of a 3/16" d.c. glass ball, that is 100 dynes. Full scale deflection on the most sensitive displacement transducers is ±3 microns. Thus to obtain a useful measuring range the stiffest spring that can be used has a constant of 100 dynes per micron. The mass of the moving parts of the transducer plus attached ball cannot be reduced to less than 2 gms. The maximum attainable resonant frequency is therefore,

\[
\nu = \frac{1}{2\pi} \sqrt{\frac{\text{Spring Constant}}{\text{mass}}} = \frac{1}{2\pi} \sqrt{\frac{100 \times 10}{2}} = 100 \text{ c/s}
\]

The core of the eddies is smaller than the spacing, and the lift pressure in the core can be very large. It is the short time which this force acts (i.e. about \( \frac{1}{200} \) sec) which limits its lifting impact. The response of a spring loaded device to a "true impact" will always be a resonant vibration. Therefore a force measuring device must have a resonant frequency much greater than the 100 c/s which is apparently the maximum attainable.
Two other ways to attack this problem of measuring directly the impact on a stationary particle may be feasible.

1. Interpret the response of a heavily damped force transducer.

2. Photograph with a high speed movie camera the initial motion of the entrained particle. This second approach is being pursued (1963) by I. Hill at the University of Canterbury.
INTRODUCTION TO A NEW EXPLANATION FOR THE ENTRAINMENT OF GRAVEL BY WATER

It was shown in section 5 that the velocity distribution of a given discharge of water over a stationary gravel bed is independent of the viscosity and depends only on the size of the gravel. The laminar sub-layer which would exist at the same discharge over a smooth boundary is disrupted by the large gravel particles and the fluid shear stress is reacted at the bed by a non-uniform pressure distribution on the roughness projections, i.e. form drag.

This shear stress is transferred through the fluid by a turbulent shear mechanism which is largely governed by the behaviour of the larger eddies. These larger eddies which at the boundary are caused by the roughness elements do not themselves dissipate energy by viscous action, but transfer their energy by inertial motions to smaller eddies. It is the smallest eddies which dissipate the energy of the flow by viscous action to heat. The larger eddies determine the shape of the velocity profile but are themselves determined entirely by the geometry of the boundary of the flow.

The classic explanation for the motion of large particles along a stream bed is based on the concept of each roughness projection contributing a portion of the total shear stress at the bed. Each particle which projects above the bed is supposed to roll to a more stable position when the instantaneous drag from the flow on it is sufficient to dislodge it. According to this theory, relatively small particles between the larger particles would not be entrained. If the force on each particle depends on the amount by which it projects into the flow then the larger particles will share all the shear stress. Therefore the motion that takes place will be by the larger sizes only. Any theory that implies this result is obviously unsatisfactory.
To overcome this objection to the classic approach Einstein has suggested that the lifting forces exerted by the fluid flow on the bed are randomly distributed with time at any point. All areas on the bed are considered equally likely entrainment sites regardless of whether they project above the general level or not. Entrainment occurs when the fluid "happens" to apply a sufficiently large lift force over a sufficiently small particle in the bed.

The writer has investigated Einstein's approach in the experiments described above. The fluid is shown to be quite capable of entraining a particle which does not project into the flow at all, and therefore Einstein's approach is adopted as the most suitable way to attack the theoretical question of when entrainment of gravel begins.
A good picture of the entrainment mechanism is obtained if the turbulent fluctuations of the fluid at the bed are considered to be a set of distinct rotating circular eddies. This model of the flow is developed because it can satisfactorily explain all the experimental observations. The speculations made about the shape, size and behaviour of the eddies are based on a number of standard results and formulas and are confirmed by the circumstantial evidence of the experiments.

Consider as an idealized form for an eddy, a vortex filament with a core of rotating fluid.

The filament is convected along the bed and when its axis is normal to the bed a lifting force is imparted to those particles it passes over.
When the lifting force is greater than the particle's immersed weight, the particle starts to move up out of the bed. The speed at which the eddy is convected along the bed is comparable with the flow speed and so the eddy can only act on each particle for a short time. The net effect is therefore an impulse and the momentum gained by the particle is equal to this impulse,

\[
(\text{Force}) \times (\text{time force acts}) = (\text{mass}) \times (\text{initial velocity})
\]

therefore,

\[
(\text{initial velocity}) = \frac{(\text{force}) \times (\text{time force acts})}{(\text{mass})}
\]

Entrainment occurs when this initial velocity produced by the impact of the eddy is sufficient to project the particle above the obstruction against which it is lodged.

The force which the eddy exerts on the particle is the product of the suction pressure of the eddy core and the area over which it acts. The suction or lift pressure of a vortex is,

\[
\text{Press.} \, 1 - \text{Press.} \, 2 = \frac{1}{4} \rho \omega^2 a^2 \sqrt{\frac{\pi}{4}}
\]

see diagram on previous page.

The area over which this pressure acts depends on whether the eddy is larger or smaller than the particle, that is whether \( a \geq D \). Therefore two cases must be distinguished:

1. \( \text{force} = \frac{1}{2} \rho \omega^2 a^2 \frac{\pi}{4} D^2 \) when \( a > D \)

2. \( \text{force} = \frac{1}{2} \rho \omega^2 a^2 \frac{\pi}{4} a^2 \) when \( a < D \).

The time the force acts on the particle depends on the speed at which the eddy is convected past. This speed will be the same as the mean flow speed at the top of the roughness projections. This speed may be calculated using the logarithmic velocity profile with Keulegan's
empirical constants. Thus at a distance above the top of the roughness projections \( \left( \frac{k}{c} \right) \) equal to the distance of the theoretical bed below the top of the roughness \( \left( \frac{k}{c} \right) \) the velocity is:

\[
U_i = 5.75 U_{*k} \log_{10} \frac{30 \times \frac{1}{3} k}{k}
\]

\[
= 5.75 U_{*k}
\]

Note that this formula for \( U_i \) is not very sensitive to the arbitrary level chosen for substitution i.e. \( \frac{1}{3} k \). The time the eddy can act on the particle may now be calculated and again two cases must be distinguished:

(i) time force acts = \( \frac{a}{5.75 U_{*k}} \) when \( a > D \)

(ii) time force acts = \( \frac{D}{5.75 U_{*k}} \) when \( a < D \)

The actual mass of the particle is effectively increased by the "virtual mass" for potential flow around it. The rapid initiation of the motion would not allow any boundary layer to develop and so the initial flow would be essentially potential flow. Therefore the effective density is:

\[
\sigma_f = \text{solid density} + \frac{1}{2} \text{ fluid density}
\]

\[
= (2.65 + .50) 1.94 \text{ slugs/cu ft}
\]

The effective mass of the particle is therefore:

\[
\text{mass} = \sigma_f \frac{\pi}{6} D^3
\]

The initial velocity of the particle when jolted by the eddy may now be calculated.

(i) \( a > D \)

\[
U_i = \frac{\text{force} \times \text{time force acts}}{\text{mass}}
\]

\[
= \left( \frac{1}{2} \rho \omega^2 a^2 \frac{\pi}{6} D^2 \right) \left( \frac{a}{5.75 U_{*k}} \right)
\]

\[
= \left( \sigma_f \frac{\pi}{6} D^3 \right)
\]

A simple device will be introduced now so that the two cases \( a \gtrsim D \) can be expressed in one equation.

\[
\nu_i = \frac{3}{8 \times 5.75} \frac{\rho \omega^2 a^3}{\sigma_i u_m D} \times \frac{a}{D}
\]

The significance of the dashed brackets will be obvious if the last three equations are compared.

This equation expresses the new approach to the entrainment of gravel by water. Both forms of the equation are dimensionally homogeneous and they only differ by the factor \( \frac{a}{D} \). Before this new approach can be usefully applied, expressions must be found for \( \nu_i, \omega \) and \( \alpha \) in terms of measurable flow, fluid and sediment properties.
A MODEL FOR THE VELOCITY FLUCTUATIONS WHICH EXPLAINS THE IMPULSIVE NATURE OF THE FORCES WHICH ACT ON A ROUGH BOUNDARY IN SHEARING TURBULENT FLOW

The random fluctuations in velocity which are imposed on the mean flow velocity in turbulent flow are considered to be the result of an aggregate of distinct rotating spheres in the fluid. These rotating spheres, or eddies are convected along at the mean flow speed and the randomness of the velocity fluctuations is a result of the random orientation of the axes of rotation. The spheres are all of diameter "d" and are close-packed so that the spacing of their centres is also "d". The angular velocity of the spheres is "ω" radians per second.

The consequence of assuming this simple model for the turbulent fluctuations will now be considered.

The mixing length theory considers the shear in a fluid as the product of three factors:

1. The rate of interchange of volume across unit area ...... "|V|/2"
2. The mean distance of transfer of this volume ............ "L"
3. The gradient of momentum ..................................... "du/Δy"

Therefore

\[ \tau = \frac{|V|}{2} \times L \times \frac{du}{Δy} \]

At some point near a rough boundary sol which the shear is "\( \tau = \eta \)", the mean eddy size will be "a". Introduce the length \( y = y_1 \) as the distance of this point from the theoretical bed. Consideration of the turbulence model shows that:
\[
\frac{\sqrt{V}}{2} = \frac{\pi a^3}{6} \times \frac{6}{2\pi} \times \frac{1}{\alpha^2} = \frac{\omega a}{\sqrt{2}}
\]

and
\[
\ell = 2 \int_0^{\frac{\pi}{2}} \frac{\pi}{4} (\cos^2 \theta)^2 \sin \theta \frac{\partial \alpha}{\partial \theta} d\theta = \frac{3a}{8^3}
\]

\[
\therefore \quad C_o = \frac{\omega a}{12} \times \frac{3a^3}{8} \times (\frac{\partial \alpha}{\partial \theta}), \quad = \rho u_x^2
\]

\[
\therefore \quad \left( \frac{du}{dy} \right)_1 = \frac{32u_x^2}{15a^2}
\]

(The rate of mass interchange across a unit of area is the product of half the volume of an eddy by the number of half revolutions per second and the number of eddies per unit of area. The mean distance of transfer, is the distance that the centroid of a hemisphere moves when it rotates through half a revolution about the centre of the spherical surface.)

According to Karmann, the velocity gradient in a shearing turbulent fluid is proportional to the distance from the boundary. He first introduced the empirical statement

\[
\frac{du}{dy} = \frac{U_x}{4y} \quad \text{where } a \text{ is Karmann's Constant}
\]

\[
\therefore \quad \left( \frac{du}{dy} \right)_1 = \frac{U_x}{4y_1} = \frac{32u_x^2}{15a^2}
\]

whence

\[
\omega a^2 = 12.671U_x
\]

Note that this result is solely a consequence of Karmann's Law in conjunction with the turbulent "model".

The writer's explanation for entrainment which is based on Einstein's approach is now introduced. Consider Newton's Second Law of Motion in the form:

\[
\text{impulse} = \text{change in momentum},
\]

i.e. \( \text{pressure} \times \text{area} \times \text{time} = \text{mass} \times \text{initial velocity} \).

Substitute in this law the relevant expressions derived from considering the eddy structure.
Assume the eddies are small with \( a < D \)

\[
\frac{1}{4} \rho \omega^2 a^2 \times \frac{\pi}{4} a^2 \times \frac{D}{u_i} = \sigma_i \frac{\pi}{6} D^3 \times u_i
\]

Note that it is necessary to introduce the speed at which the eddies are convected along the bed i.e. \( u_1 \). This speed will be the mean flow speed at the level of the eddy centres i.e. \( y = y_1 \) and so from Keulegan's logarithmic velocity profile in which \( u^* \) is the (arbitrary) equivalent roughness size,

\[
u_1 = 5.75 \log \frac{30y_1}{k}
\]

Substitute this value and rearrange to obtain the initial velocity of the entrained particle

\[
u_i = \frac{3}{8 \times 5.75} \frac{\sigma_i}{\sigma_1} \left( \frac{\omega a_0^2}{u_* D} \right) \log \frac{30y_1}{k} u_*
\]

\[
u_i = 10.7 \frac{\sigma_i}{\sigma_1 D^2} \log \frac{30y_1}{k} u_* \quad \text{from (1)}
\]

Note that this result is a consequence of Newton's Law (assuming an impulsive action) and the Keulegan velocity profile.

The particle is now moving up out of the rough bed assisted by buoyancy but retarded by gravity and the (potential) flow of fluid around it. Since force = mass \( \times \) acceleration

\[
\frac{\pi}{6} (\sigma - \rho) g D^3 = \sigma_i \frac{\pi}{6} D^3 \times \nu \frac{d\nu}{dy}
\]

Integrating w.r.t. \( y \)

\[
(\sigma - \rho) g y = \sigma_i \frac{\nu^2}{2}
\]

The particle must be projected to the level \( y = y_1 \) to be entrained. Therefore the initial velocity necessary to cause entrainment against the constant (?) resistance to the particles upward motion is

\[
u_i \text{ min} = \sqrt{2g y_1 \frac{\sigma - \rho}{\sigma_i}} \quad \text{(3)}
\]
For entrainment to occur

\[ U_i > U_{i \min} \]

Substituting from (2) and (3)

\[ \frac{10.7 \rho y_i^2 U_k}{\alpha_i D^2 \log \frac{30 y_i}{h}} > \sqrt{2g y_i \alpha_i - \frac{\rho}{\alpha_i}} \]

Square both sides and rearrange

\[ \left( \frac{\rho}{\sigma - \rho} \right) y_i D^2 > \frac{10.75}{\log \frac{30 y_i}{h}} \left( \frac{\sigma_i}{\rho} \right) \left( \frac{D}{y_i} \right)^3 \]

Put \( D = y_i = \frac{k}{3} \) and \( \alpha_i = (2.65 + .50) \rho \)

\[ \left( \frac{\rho}{\sigma - \rho} \right) y_i D^2 > 0.06 \]

This result is a statement of Shields' condition for the threshold of entrainment.

\( D = \frac{k}{3} \) is Fokkes suggestion which has been mentioned above. Putting \( y_i = \frac{k}{3} \) is consistent with the experimental finding of Einstein and El Samni for flow over a regular roughness of large spheres. They found that the value of velocity which best characterised the flow past the roughness was at a distance \( y = .35k \) from the theoretical bed. The expression for \( \alpha_i \) is obtained by adding to the density of gravel half the density of water as an allowance for the resistance of the potential flow around the particle. This allowance is a result of classical hydrodynamics, see Milne - Thomson.*


N.B. (1) The factor \( \frac{\sigma_i}{\rho} \) on the R.H.S. implies that Shields result will not be true with different fluid-grain combinations.

(2) The factor \( \left( \frac{D}{y_i} \right)^3 \) on the R.H.S. may be written as the product \( \left( \frac{k}{3} \right)^3 \left( \frac{D}{y_i} \right)^3 \) The first part of this product is a constant because of the way in which \( k \) is arbitrarily defined.
The second factor depends on how the grains pack together to form the roughness, and the high value of the index shows that the threshold is very sensitive to this closeness of packing. For naturally deposited gravels it is found that \( k = 3D \) and this value has been assumed here.

Refer again to the equation derived from Karman's Law in this section

\[
\omega a^2 = 12.8 u_x y, \quad \text{same as (1)}
\]

\[
= \frac{12.8}{3} u_x k
\]

The lift pressure of an eddy is (see section 37.)

\[
\frac{1}{k} \rho \omega^2 a^2 = \frac{1}{k} \rho \left( \frac{12.8}{3} \frac{u_x k}{a} \right)^2
\]

\[
= 4.54 \left( \frac{k}{a} \right)^2 \rho u_x^2
\]

Einstein and El Samad measured directly the fluctuating pressure on a rough bed averaged over an area comparable with \( k^2 \). They found that the mean lift pressure was

\[
C_L = \frac{1}{k} \rho u^2 = 0.178 \frac{1}{k} \rho \left( 6.75 u_x \log \frac{30}{k} \frac{h}{k} \right)^2
\]

\[
= 3.08 \rho u_x^2
\]

The fluctuations were found to be "normally" distributed with a standard deviation .364 times the mean.

Without specifically saying so Einstein assumed a lift constant \( C_L^* \) of twice this value and then states that entrainment occurs when the instantaneous lift on a particle is just larger than its immersed weight. The max. lift pressure that occurs during the fluctuations at a point on the bed will be about three standard deviations larger than the mean lift. Therefore according to Einstein entrainment occurs when

\[
2 \cdot 3.08 \rho u_x^2 \left( 1 + 3 \times 3.64 \right) \frac{F}{D^2} \frac{D}{k} - \left( \sigma - \rho \right) g \frac{F}{k} \frac{D^3}{k^3}
\]

\[
\text{i.e. } \frac{\rho u_x^2}{\left( \sigma - \rho \right) g \frac{F}{D}} \geq .06
\]

Apart from his arbitrarily doubling of the lift Coefficient \( C_L^* \), Einstein appears to have predicted the threshold of entrainment from indirect measurements.
Evaluating Einstein's expression for the maximum lift pressure and the writer's expression for the lift pressure in an eddy it is possible to determine the scale of the turbulence at the bed.

\[
2 \times 3.08 \times (1 + 3 \times 364) \rho u_k^2 = 454 \left( \frac{b}{a} \right)^2 \rho u_k^2
\]

implies that

\[a = \frac{b}{1.7}\text{ Einstein.}\]

It is seen that Einstein's argument requires large eddies whose diameter is about half the equivalent roughness size. That is if the turbulence is of the type proposed here.

The writer believes that in fact the eddies are much smaller. As a consequence of this, the lift pressures exerted on the bed are much larger but only act for a short time over any one particle. The eddies at the bed are caused by separation on the roughness projections which project a distance \(\frac{k}{5}\) above the level where the mean flow velocity is zero. (Einstein and El Samii). It is therefore reasonable to assume that the eddy diameters will also be \(\frac{k}{5}\)

\[a = \frac{b}{5.0}\text{ Writer.}\]

whence the lift pressure in each eddy is

\[4.54 \times 5^2 \rho u_k^2 = 113 \rho u_k^2.\]

That is, the lift pressure is a hundred times as strong as the bed shear stress.
This lift pressure is very large and if it acted continuously on a single grain then the grain would be accelerated at about 10 g upwards. At the threshold of entrainment however this pressure only acts long enough to project the particle above the obstruction against which it is lodged. Thus the impulsive nature of the effect of this pressure is demonstrated. The essential difference between Einstein's "lift forces" and the writer's "lift forces" is that the latter act impulsively. Their time of action is very important in determining the effect produced, but this time of action will not be related to the particle settling velocity as Einstein proposes.

The time for which the lift forces act at a point on the bed will be the quotient of the scale of the turbulence near the bed divided by the flow velocity near the bed.
When presenting experimental data it is customary to express the measurements in the form of dimensionless functions and to plot the functions one against the other on a graph. The success or failure of the experiments is then judged by the degree of scatter of the plotted points, and the most highly successful research is when the points plot on a line. There is only a small amount of data reported in the extensive literature on sediment transport which refers to the transport of coarse material. This data has been plotted many times using different dimensionless functions but the most widely accepted parameters today seem to be:

(1) the sediment discharge function
\[ \phi = \frac{2i_m}{\sqrt{\frac{gD^2}{\rho}}} \]
and

(2) the fluid flow function
\[ \frac{1}{\psi} = \frac{\rho a_x^2}{(\sigma - \rho) g D} \]

Different conclusions may be supported using the same functions and the same data. In particular the scatter of the data may be hidden by plotting the points on a very steep or very flat slope across the page. The fairest way to plot data once the functions have been chosen would appear to be to select the scales so that the points lie in a line at roughly 45° across the page. Figure 16 is a plot of \( \phi \) versus \( \frac{1}{\psi} \) with the scales selected in this way. The data is from the now "classic" experiments which are listed on the figure. It is seen that the data scatters over a band and obviously a number of different curves could fit this limited range equally well.

When the flow is very fast a stage is reached when a flowing dispersion of grains forms a complete layer between the turbulent fluid and the stationary bed.
Einstein's Gauss-law parameter $p$:

$$ p = \int_{2\nu} e^{-x^2} dx $$

$\phi = 50/\phi_0$.

Experimental points plotted are for gravel particles in the range of sizes 0.8 mm to 28.6 mm. They are from the data published by Gilbert and Meyer-Peter. This is the same data as Einstein used.

Transport diagram as plotted by Brown.
For coarse sediment in the absence of bed configuration the slowest flow for this to occur appears to be somewhere in the range of flows represented by the following values of \( \frac{1}{\psi} \):

\[ 1 < \frac{1}{\psi} < 4.4 \]

For faster flows established bed load motion is said to occur.

Bagnold has attempted to explain quantitatively how for established bed load motion the amount of sediment discharged depends on the flow rate. Most of the experimental data he used was complicated by bed configuration. Nevertheless the result of his investigation, as far as established flow of coarse sediment is concerned, may be fairly represented by the equation,

\[ \phi = 4 \left( \frac{1}{\psi} \right)^{3.2} \]

Du Boys developed an equation for established bed load transport without an experimental basis, which for the case when flow rate is very much less than unity may be represented thus,

\[ \phi \propto \left( \frac{1}{\psi} \right)^{2} \]

Brown simply fitted the same experimental data as Einstein collected with a power law of best fit which was,

\[ \phi = 40 \left( \frac{1}{\psi} \right)^{3} \]

All three lines have been plotted on figure 1G along with the experimental data mentioned. None of the lines is particularly convincing when plotted with this data for coarse sediment.


For slow flows near the threshold of sediment motion it is customary to introduce into bed load formulas some factor involving the difference between \( \frac{1}{U} \) and a constant. Einstein has published the most convincing treatment of this factor (see appendix 2) in which he developed the function \( \Phi \) defined as the proportion of bed area at any instant where the fluid is actively entraining. This function has also been plotted on figure 16. The product of Einstein's function \( \Phi \) and the other three formulas is seen to improve their fit to the data as is shown by the continuous black lines on figure 16. These three curves may be expressed thus:

\[
\Phi = 4 \left( \frac{1}{U} \right)^{\frac{3}{2}} P \\
\Phi = 12 \left( \frac{1}{U} \right)^{2} P \\
\Phi = 40 \left( \frac{1}{U} \right)^{3} P
\]

These curves may be expressed thus:

\[
\Phi = 4 \left( \frac{1}{U} \right)^{\frac{3}{2}} P \quad \text{Bagnold - Einstein} \\
\Phi = 12 \left( \frac{1}{U} \right)^{2} P \quad \text{du Boys - Einstein} \\
\Phi = 40 \left( \frac{1}{U} \right)^{3} P \quad \text{Brown - Einstein}
\]

How does the entrainment process which is described in the preceding section fit into the context of the bed load formulas described in this section? The answer to this question follows and should show how the academic considerations of the bulk of this thesis may lead to some practical guides for engineers engaged on sediment problems. It is also intended to demonstrate the limited relevance of the mathematical statistics used by some previous investigators, e.g., Einstein and Kalinske.

Entrainment is still assumed to be caused by eddies which are convected over the bed by the fluid. Up to this stage those eddies have been assumed to be all the same size, and the condition that each and every eddy is capable of causing entrainment has been expressed thus:

\[
\frac{\rho k^2}{(\sigma + \rho) \rho D} \left( \frac{3 \rho}{\sigma} \right) \left( \frac{k}{3 D} \right)^3 \geq 0.06
\]

which employing the notation \( \frac{1}{U} \) is

\[
\frac{1}{U} \left( \frac{3 \rho}{\sigma} \right) \left( \frac{k}{3 D} \right)^3 \geq 0.06
\]

Now define a new function \( \frac{1}{\psi_i} \), called the eddy strength, 
\[
\frac{1}{\psi_i} = \frac{1}{\psi} \left( \frac{3 \sigma}{\alpha} \right) \left( \frac{k}{3 \beta} \right)^3 
\]
This modified flow function is a measure of the effectiveness of the eddies in the flow so described, for causing entrainment. It will be used from now on as a measure of "eddy strength" and each single eddy will be considered to have a \( \frac{1}{\psi_i} \) value. The assumption that all eddies are the same size will be dispensed with and the eddies in any particular flow will be considered to have a whole range of \( \frac{1}{\psi_i} \) values, that is eddies of many different sizes will be assumed to exist in the turbulence.

The statistical theory of turbulence assumes that the velocity fluctuations are distributed according to the Gauss Normal Error Law. Here for the case of turbulent shear flow it is assumed that the strength of the eddies convected past the boundary are distributed according to this same bell shaped distribution.

\[
f = \frac{1/\psi_{50}}{\sqrt{2\pi}} \times e^{-\frac{\psi_{50}^2}{2 \psi_m^2}} \]

It follows that the proportion of eddies whose strength is greater than a threshold strength \( \frac{1}{\psi_0} \) may be calculated thus:
\[
P = \int_{\frac{1}{\psi_0}}^{\infty} f \, d\frac{1}{\psi_i} \\
= \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\psi_0}-\frac{1}{\psi_m}}^{\infty} e^{-\frac{t^2}{2}} \, dt.
\]
\( t \) is only a variable of the integration.

(Einstein stated:

1. explicitly that \( \frac{1}{\psi_{50}} = 0.364 \frac{1}{\psi_m} \) and

2. implicitly that \( \frac{1}{\psi_0} = 0.073 \)

and so

\[ P = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \left(2\psi_m - 2.75\right) e^{-\frac{t^2}{2}} dt. \]

The function \( P \) is simply the proportion of the eddies which are capable of entraining a predetermined size of sediment. A check of figure 16 will show that the value of \( P \) (as defined by Einstein) differs appreciably from unity only for values of \( \frac{1}{\psi} \left(\frac{1}{\psi_m}\right) \leq 0.1 \).

Note also that for still smaller values of \( \frac{1}{\psi} \) the curve of \( P \) rapidly becomes so steep that it dominates any expression of which it is a factor.

\( P \) only affects the formulas in which it is a factor for values of \( \frac{1}{\psi} \) in the range,

\[ 0.03 < \frac{1}{\psi} < 1. \]

For smaller \( \frac{1}{\psi} \) it is zero and for larger \( \frac{1}{\psi} \) it is unity. Thus \( \frac{1}{\psi} \) is very reasonably described as a "threshold factor".

The modification of the usual flow function \( \frac{1}{\psi} \) by the factors \( \rho/\sigma \) and \( (k/D)^3 \) could not be avoided when the entrainment theory was developed in section 38.

The second of these factors, \( (k/D)^3 \), has a high index and so is sensitive to small changes in the ratio \( k/D \). This sensitiveness could possibly explain a large part of the scatter of transport data, particularly where the sediment is not all the same size. Quantitative confirmation of the significance of this roughness factor \( (k/D)^3 \) could lead to the ability to predict the degree of scour protection afforded by the coarser material in a graded gravel.
Established bed load motion of coarse sediment is difficult to observe in the field, and there has been very little laboratory investigation of this subject. As a consequence modern bed load formulas are basically no improvement on the drag theory first proposed by DuBoys in 1879. In the meantime the writer has fitted the data on figure 16 with a line which may be expressed as follows

\[ \phi = 50 \left( \frac{1}{\psi} \right)^{2.9} \times \left( \frac{1}{\sqrt{2\pi}} \right) \int_{2\psi - 3}^{\infty} e^{-t^2} \, dt \]

Established motion factor, Threshold factor, (Einstein's \( \psi \)).

It is emphasised that the numerical parameters in this formula have been chosen simply to fit the limited range of scattered data on figure 16. The formula has been plotted in red. The significant point to note is the way that the bed load transport function \( \phi \) has been expressed as the product of an "established motion factor" \( \left( \frac{1}{\psi} \right)^{2.9} \) and a "threshold factor" \( \psi \).

Improvement on this last formula for bed load transport of coarse sediment will follow:

1. a proper understanding of the shearing mechanism within a flowing dispersion of grains, and
2. an understanding of how the fluid transmits lift as well as drag to the uppermost layer of grains. An important conclusion from this investigation is relevant to this last problem, in that the turbulent fluid forces on the grains have been found to be impulsive in their action. This conclusion appears to be new and could be a step toward the proper understanding of the motion of coarse sediment.
The report of each observation has been followed by the relevant conclusion in the same section. These conclusions will now be listed.

1. Study of channel patterns on a small scale in fine sand is complicated by viscous sublayer effects. These effects are absent in natural streams in coarse material and so it is difficult to relate the small scale observations to the large scale natural phenomena.

2. Previous investigators have found medium and coarse sand to be more efficiently diffused than the turbulent diffusion theory predicts. This result suggests that there must be another lifting mechanism for the coarser sediments.

3. It is possible to measure the velocity profile near a rough boundary of gravel to a precision of ± .02 ft/sec even for the very steep velocity gradient characteristic of turbulent flow. This profile is related to an arbitrary origin for the distance from the boundary.

4. By choosing the best origin for the distance from the bed it is possible to fit this velocity profile by a formula of the type \( u = x \log y/y_0 \) so that the mean square deviation of measured points from the line is the same as its expected value.

5. A roughness configuration seems to be characterized by a single average \( y_0 \) value for all flow velocities. Unfortunately values of \( y_0 \) for individual velocity profiles were found to scatter over a wide range but this scatter has been attributed to the noticeable non-uniformity of the flow.
6. Near to the threshold of motion, the repeated entrainment of the same particle from the same site on the bed is a random event.

7. Even though the entrainments occur at irregular intervals of time it is still possible to define and measure an "entrainment rate" which is a local property of the fluid flow.

8. The entrainment rate varies from location to location on the bed even for identical "site" conditions and flow conditions.

9. The entrainment rate varies from time to time at a single location even though the bed configuration and the flow conditions remain unaltered.

10. It is the depressions in the upstream roughness configuration which have the most important effect on the entrainment rate at a point, not the projections.

11. The fluid shear stress at the bed is transmitted to the roughness configuration by a series of large forces which only act for a very short time at any one point. Because of the impulsive nature of the fluid attack on bed grains, direct measurement of the force on a single grain can only be accomplished by a device capable of measuring impulses.

The conclusions so far have been the direct result of specific observations. Most of these observations were made during flume experiments when measurements were made of the fluid velocity – depth profile and the particulate entrainment rate. The measurements have been carefully interpreted to avoid bias or assumption but the results are consequently very limited in their scope and so only qualitative conclusions can be drawn. The entrainment rate measurements are noteworthy for the many different sources of variation found and as a consequence of this, the data is insufficient to establish any result with even a low statistical confidence level.
Einstein and Kalinske both used a random time function in developing their respective bed load transport formulas, the function being used to represent turbulent fluctuations of pressure and velocity. They adopted the Gaussian Normal Error Law for the relative frequency distribution and thereby were able to specify the distribution of the turbulent fluctuations with the mean value and the standard deviation. However neither of these investigators considered the frequency spectra of their random function and yet the dominant frequency of the presumably unimodal frequency spectra would surely be as significant a variable as the standard deviation.

The writer has introduced this frequency variable into the analysis but has avoided mathematical statements in preference for an admittedly arbitrary and hypothetical description of the physical details. HunterHouse would call it a "phenomenological approach". This descriptive approach is more suitable in this case where none of the assumptions necessary for a more mathematical treatment have been experimentally confirmed.

The most significant outcome of this descriptive treatment is that the equivalent roughness size of a gravel boundary and the actual size of the gravel are shown to be separate variables, both playing an important part in the process of entrainment. The equivalent roughness size has been described as a measure of the scale of the turbulence at the boundary, that is the size of the eddies. The intuitive picture of the turbulent water swirling among the roughness projections has been formulated into a theory which enables the various parts of the process to be understood in quantitative terms. The large number of observations in the flume experiment and a few additional observations made elsewhere have reinforced the writer's belief that the fluid forces can be very large, exerting local suction pressures of the order of a hundred times the boundary shear stress. But, the effect of these pressures on free particles is limited by the short duration of their action.
<table>
<thead>
<tr>
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**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
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<td>16</td>
<td>Eddy diameter</td>
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<td>A</td>
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<td>Einstein's universal constant associated with ( \phi ).</td>
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<td>A particles ratio of projected area to ( D^2 ) and volume to ( D^3 ).</td>
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<td>An arbitrary value for the depth ( y ).</td>
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<td>Local volumetric concentration of solid sediment in the fluid.</td>
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<td>( E_v )</td>
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<td>Probable error in velocity corresponding to ( E_h ).</td>
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<td>Statistical relative frequency.</td>
</tr>
<tr>
<td>( F_{x,y} F_1 )</td>
<td></td>
<td>72</td>
<td>Probability of a delay exceeding ( x ) or one seconds.</td>
</tr>
<tr>
<td>g</td>
<td>ft/s²</td>
<td>8</td>
<td>Acceleration due to gravity.</td>
</tr>
<tr>
<td>h</td>
<td>ft</td>
<td>18</td>
<td>Total depth, value of ( y ) at the free water surface.</td>
</tr>
<tr>
<td>H</td>
<td></td>
<td>1/100&quot;, 36</td>
<td>Velocity head (600 ( u^2/g )).</td>
</tr>
</tbody>
</table>

### Nomenclature Continued

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Dimension</th>
<th>Page first used</th>
<th>Meaning</th>
</tr>
</thead>
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<tr>
<td>$l_b$</td>
<td>ft</td>
<td>58</td>
<td>Proportion by weight of particles of a given size in the bed material.</td>
</tr>
<tr>
<td>$k$</td>
<td>ft</td>
<td>20</td>
<td>Equivalent roughness, grain size of equivalent closely packed uniform sand.</td>
</tr>
<tr>
<td>$K$</td>
<td>ft</td>
<td>18</td>
<td>Karman's turbulence constant</td>
</tr>
<tr>
<td>$\ell$</td>
<td>ft</td>
<td>17</td>
<td>Prandtl's mixing length of the turbulence.</td>
</tr>
<tr>
<td>$L$</td>
<td>ft</td>
<td>57</td>
<td>Einstein's average length of travel of a particle per entrainment.</td>
</tr>
<tr>
<td>$n$</td>
<td>per sec 72</td>
<td></td>
<td>Number of small units of time in a second.</td>
</tr>
<tr>
<td>$p$</td>
<td>per sec 72</td>
<td></td>
<td>Einstein's proportion of bed area where the flow is actively entraining. This is interpreted by the writer as the proportion of eddies stronger than $\frac{1}{\ell}$ i.e. capable of entraining a particle.</td>
</tr>
<tr>
<td>$p_a$</td>
<td>per sec 58</td>
<td></td>
<td>Entrainment rate.</td>
</tr>
<tr>
<td>$p_{1}, p_{2}$</td>
<td>per sec 72</td>
<td></td>
<td>Probability of entrainment in 1 and $\frac{1}{n}$ secs respectively.</td>
</tr>
<tr>
<td>$q_1$</td>
<td>per sec 72</td>
<td></td>
<td>Probability of no entrainment in a sec. = $1 - R$</td>
</tr>
<tr>
<td>$q$</td>
<td>cuft/s</td>
<td></td>
<td>Volume of sediment transported past unit section per sec.</td>
</tr>
<tr>
<td>$q_{im}$</td>
<td>lbm/s 58</td>
<td></td>
<td>Immersed weight of sediment transported past unit section per sec.</td>
</tr>
<tr>
<td>$R_H$</td>
<td>1/100m</td>
<td>36</td>
<td>Range of a sample of ten measurements of $H$.</td>
</tr>
<tr>
<td>$R$</td>
<td>ft</td>
<td>47</td>
<td>Hydraulic radius, area of section divided by wetted perimeter.</td>
</tr>
<tr>
<td>$S$</td>
<td></td>
<td>47</td>
<td>Slope of hydraulic grade line.</td>
</tr>
<tr>
<td>$S_p$</td>
<td>ft</td>
<td>18</td>
<td>Average spacing between eddy centres.</td>
</tr>
<tr>
<td>Symbol</td>
<td>Dimension</td>
<td>Page first used</td>
<td>Meaning</td>
</tr>
<tr>
<td>--------</td>
<td>-----------</td>
<td>----------------</td>
<td>---------</td>
</tr>
<tr>
<td>t</td>
<td>ft/s</td>
<td>126</td>
<td>Variable of integration.</td>
</tr>
<tr>
<td>u</td>
<td>ft/s</td>
<td>5</td>
<td>Mean local velocity in direction of mean streamlines.</td>
</tr>
<tr>
<td>(\ddot{u})</td>
<td>ft/s</td>
<td>47</td>
<td>Mean velocity in a vertical from the bed to the water surface.</td>
</tr>
<tr>
<td>(u_{ws})</td>
<td>ft/s</td>
<td>5</td>
<td>Shear velocity, square root of boundary shear stress divided by fluid density.</td>
</tr>
<tr>
<td>(u_{g})</td>
<td>ft/s</td>
<td>59</td>
<td>Mean velocity of transported grains (Kalinske).</td>
</tr>
<tr>
<td>(u_{c})</td>
<td>ft/s</td>
<td>59</td>
<td>Critical value of fluid velocity when particle motion starts (Kalinske).</td>
</tr>
<tr>
<td>(u_1)</td>
<td>ft/s</td>
<td>114</td>
<td>Mean velocity of eddies past the bed.</td>
</tr>
<tr>
<td>(\frac{\partial u}{\partial y})</td>
<td>per sec</td>
<td>117</td>
<td>Effective mean velocity gradient near a rough bed.</td>
</tr>
<tr>
<td>v</td>
<td>ft/s</td>
<td>118</td>
<td>Vertical component of the velocity of a moving particle.</td>
</tr>
<tr>
<td>(v_1)</td>
<td>ft/s</td>
<td>114</td>
<td>Value of v immediately after the impact of an eddy.</td>
</tr>
<tr>
<td>(\frac{V}{2})</td>
<td>ft/s</td>
<td>17</td>
<td>Prandtl’s volumetric turbulent exchange rate.</td>
</tr>
<tr>
<td>w</td>
<td>ft/s</td>
<td>17</td>
<td>Terminal settling velocity of a sediment particle in the fluid.</td>
</tr>
<tr>
<td>x</td>
<td>sec</td>
<td>22</td>
<td>Period of time in seconds.</td>
</tr>
<tr>
<td>X</td>
<td>ft/s</td>
<td>5</td>
<td>Slope of semilogarithmic velocity profile.</td>
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<tr>
<td>y</td>
<td>ft</td>
<td>5</td>
<td>Depth, distance upward from an arbitrary origin in the bed.</td>
</tr>
<tr>
<td>(y_o)</td>
<td>ft</td>
<td>5</td>
<td>Constant of integration, the distance of the origin of y below the level where the velocity u is theoretically zero.</td>
</tr>
</tbody>
</table>
### NOMENCLATURE Continued

<table>
<thead>
<tr>
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<th>Page first used</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_d )</td>
<td>ft</td>
<td>116</td>
<td>Value of the depth ( y ) at the centre of the eddies next to the bed.</td>
</tr>
<tr>
<td>( z )</td>
<td></td>
<td>22</td>
<td>Exponent of sediment concentration profile ((v/\nu_n))</td>
</tr>
</tbody>
</table>

### GREEK LETTERS USED

- \( \beta \) | 23 | Ratio \( \varepsilon_s/\varepsilon_m \) |
- \( \varepsilon \) | ft | 16 | Laminar sublayer thickness. |
- \( \varepsilon_m/\varepsilon_s \) | 23 | Boussinesq turbulent diffusion coefficients for momentum and sediment. |
- \( \eta \) | 59 | Proportion of bed area taking shear, Kalinske's packing fraction. |
- \( \lambda \) | 87 | Travel between attempts at deposition measured in particle diameters. |
- \( \nu \) | ft\(^2\)/s | 16 | Kinematic viscosity of the fluid. |
- \( \Pi \) | | 73 | Proportion of a sample of delay measurements exceeding \( x \) seconds. |
- \( \rho \) | Slugs/ft\(^3\) | 6 | Mass density of the fluid. |
- \( \sigma \) | | 8 | Mass density of the solid material |
- \( \sigma_r \) | | 114 | Virtual mass density of a particle \((\sigma + \frac{1}{2} \rho)\) |
- \( \tau \) | lbs/sq ft | 17 | Shear stress at any plane in the fluid. |
- \( \tau_o \) | | 116 | Fluid shear stress at the bed. |
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Dimensions</th>
<th>Page first used</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td></td>
<td>54</td>
<td>Einstein's bedload sediment transport function</td>
</tr>
<tr>
<td>$\psi$</td>
<td></td>
<td>84</td>
<td>Einstein's fluid flow function</td>
</tr>
<tr>
<td>$\overline{\psi}$</td>
<td></td>
<td>126</td>
<td>Measure of the strength of a single eddy.</td>
</tr>
<tr>
<td>$\overline{\psi_m}$</td>
<td>$\overline{\psi_p}$</td>
<td>126</td>
<td>For a given flow these are the mean, standard deviation, and minimum entrainment strength respectively of the population of eddy strengths present.</td>
</tr>
</tbody>
</table>
A large number of velocity profiles were measured between 26th September 1962 and 27th February 1963. Parameters describing curves of best fit to the profiles were worked out using an electronic computer, I.B.M. 1620. An apparently satisfactory measuring technique was adopted at the end of November and all earlier profiles have been disregarded. The parameters of the last 61 profiles are tabulated at the end of this appendix.

The computer calculation was checked in one instance using a desk calculator. The "Fortran" source program used to obtain the tabulated data was as follows.

```
DIMENSION K(10), H(10), B(10), A(10), AA(6)
READ 1, A1, AX2, N
PRINT 1, A1, AX2, N
1 FORMAT (F10.2, F10.4, I5)
AX2L = LOG (AX2) / 2.30259
9 READ 2, K(1), K(2), K(3), K(4), K(5), K(6), K(7), K(8), K(9), K(10),
      L3, L4, L5, L6
2 FORMAT (15, 15, 15, 15, 15, 15, 15, 15, 15, 15)
      AM = 0
      AS = 0
      DO401 = 1, 6
      X = I - 1
      A(I) = LOG (A1 + X*2) / 2.30259
      AM = AM + A(I)

40 AM = AM*62
      DO31 = 1, 10
      H(I) = K(I)

3 H(I) = CR (.00536667#H(I) )
      BM = 0.
      ADR = 0.
      DO41 = 1, 6
      BM = BM + B(I)

4 AB = AM*A(I) - N(I)
      BB = (6. * AB - BM)/N(I)
      B(5) = BB/6. - BB#(AM/6 - A(I))
      B(6) = BB/6. + BB#(AM - A(I)) / 6.
      BM = 0.
```
DO 441=5,10
44 IM=IM+B(I)
   N = N + 1
   LA=10
   L2=20
   N=M/V*.006+.5
   DO30 I=5,10
90 K(I)=I*1000.+5
   PRINT 6, L1, L2, L3, LA, L5, L6, N
   PUNCH 6, L1, L2, L3, LA, L5, L6, N
   PRINT 6, K(5), K(6), K(7), K(8), K(9), K(10), M
6 FORMAT(L5, L5, L5, L5, L5, L5, L5)
   IF (SENSE SWITCH 4) 92,9
92 IF (SENSE SWITCH 2) 50,51
50 N2 = 1
   GO TO 52
51 N2 = 2
   GO TO 52
52 IF (SENSE SWITCH 1) 53,54
53 N1 = 1
   GO TO 56
54 N1 = 2
56 A(I)=1000.0
   A(1)=10
   A(2)=20
   A(3)=30
   A(4)=40
   A(5)=50
   A(6)=60
   DO 7 I=1,6
      A(I)=A(I)+4
      J=I+4
7 B(I)=B(J)
   DO5 J=1,4,2
      AM = 0.
      AN = 0.
      AB = 0.
      XJ = J-1.
      DO26 I=1,6
         AA(I) = LOG(A(I) +.1**XJ)/2.30259
         AM = AM+AA(I)
         AN = AN+AA(I)**2
      26 AB = AM+AA(I)**2
         BB=6.**AB-A**B)//5.**AS,-A**2
         AX1L=(AN-AB)/BB/6.
      S=0.
500 S=S+6*(B(I) - BB**A(A(I)-AX1L))**2
   IF (N1-2) 27,10,10

27 IF (S-S1)29,6,8
29 S1 = S
   AX1= AM/6.+.0005
   BM=BB+.0005
   AX11= AX11
   IF (N2-2) 10,5,5
5 CONTINUE
8 AX1 = IFP ((2.39259 * AX1))
61 = SCR (61 /6.)
   A1=A(I)+.1*XJ-.2
   PRINT 94,SI,AX1,AM, BBJ, A1
   PUNCH 94,SL,AX1,AMJ, BB1, A1
94 FORMAT (F8.5,F7.4,F7.3,F7.3,F6.3)
N1=2
   IF (N2-2)10,21,21
10 IF (AX1-AX2L)5,15,15
21 IF (SELECT SWITCH 3)9,22
23 PAUSE
GOTO9
15 S2=100
   DO 16M=-1,61
   AM=0
   XM = M
16 DO23L = 1,6
   AA(I)=LOG (A(I)+.1*XJ-.44+.01*XJ) / 2.30259
23 S=SAA+AA(I)
   BM=BM/(AM-6. *AX2L)
   S=0.
   DO24L=1,6
24 S=S+(B(I)-BB*(AA(I)-AX2L) )*.02
   IF (S-S2) 17, 16, 16
17 S2=S
   A2 = AM/6. + .0:05
16 BB2=BM+.0005
18 A2=A(I)+.1*XJ-.42+.01*XJ
   S2 = SCR (61 /6.)
   PRINT 94,S2,AX2, AX22, BB2, A2
   PUNCH 94,S2,AX2, AX2, BB2, A2
   IF (N1-2) 20,21,21
20 N2 = 2
GO TO 5
END
<table>
<thead>
<tr>
<th>Number and Date</th>
<th>Measured Velocity u Ft/sec</th>
<th>Regression line parameters</th>
<th>For optimum value of y</th>
<th>For y = .002 ft</th>
<th>Entrain rate per sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dec</td>
<td>Arbitrary values of depth y ft</td>
<td>u = 0</td>
<td>u = u</td>
<td>x</td>
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<td>1 1</td>
<td>1.275 1.541 1.847 2.178 2.434 2.699</td>
<td>.00107</td>
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<td>1.14</td>
<td>.0057</td>
</tr>
<tr>
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<td>.020</td>
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<td>.0119</td>
</tr>
<tr>
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<td>.0150</td>
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<td>.0136</td>
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### MEDIUM STONE ROUGHNESS 3/16" - 1/4" Continued

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<tr>
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<td>1.628 2.171 2.559 2.994 3.375 3.765</td>
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<tr>
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<td>15</td>
<td>1.648 2.128 2.591 2.957 3.335 3.730</td>
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<tr>
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<td>17</td>
<td>1.451 1.626 2.053 2.456 2.779 3.162</td>
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<tr>
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<td>18</td>
<td>1.329 1.545 1.913 2.298 2.576 2.908</td>
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<td>1.745 2.052 2.550 3.002 3.352 3.800</td>
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<tr>
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</tr>
<tr>
<td>-----</td>
<td>----</td>
<td>------</td>
</tr>
<tr>
<td>41</td>
<td>16</td>
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<td>19</td>
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<td>43</td>
<td>19</td>
<td>1.082</td>
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<tr>
<td>44</td>
<td>19</td>
<td>1.094</td>
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</tbody>
</table>

| 44  | 0.06614 | 44 | 1.0012 | 44 | 1.3361 |

Mean \( y_0 \): .001966
Mean \( 0.2256 \% \): Mean \( 0.3244 \% \):

| roughness parameter | goodness of fit | goodness of fit |

<table>
<thead>
<tr>
<th>SUBJECT TO SIDE EFFECT</th>
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</thead>
<tbody>
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</tr>
<tr>
<td>14</td>
</tr>
<tr>
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45
<table>
<thead>
<tr>
<th>No.</th>
<th>Date</th>
<th>Velocity u ft/sec</th>
<th>Depth y ft</th>
<th>Regression line parameters</th>
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<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>For optimum value of y</td>
</tr>
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<td>x₀ ft</td>
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Not worthwhile working out totals and means

Undular hydraulic jump formed at the measuring reach
<table>
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<tr>
<th>No. Date</th>
<th>Velocity u ft/sec</th>
<th>For optimum value of y</th>
<th>For y = .005 ft</th>
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<tr>
<td></td>
<td>Feb</td>
<td>y ft x ft/s</td>
<td>e ft/s</td>
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<td>.0297 2.09  .0199</td>
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Mean $\bar{y} = 0.00524$ The mean = 0.0252 ft/s The mean = 0.0315 ft/s

roughness parameter goodness of fit goodness of fit
Only two of Einstein’s many publications in English have been directly concerned with bed load transport, that is the motion of sediment near to the bed. In the first of these Einstein developed a functional relationship on the basis of a similarity argument. This relationship was first published in 1942 as follows

\[
\frac{g}{\kappa D} = \text{function of } \left(\frac{(\sigma - \rho g D)}{\eta_c}\right)
\]

The argument is that a dimensionless function involving only properties of the sediment and the discharge of sediment,

\[
\phi = \frac{g}{\kappa D}
\]

will equal a function of a function which includes only properties of the sediment and the fluid shear stress at the bed,

\[
\psi = \frac{(\sigma - \rho g D)}{\eta_c}
\]

The particular form of the relationship was justified by a plot of experimental flume data as \(\phi\) versus \(\psi\), see figure 17. The scatter of the plotted points from a line was considered to be small thereby establishing the physical significance of these two dimensionless functions. Neither the similarity argument nor the plot of experimental data is particularly convincing.

\[\text{H. A. Einstein} \quad \text{Trans. A.S.C.E. V107 1942 p561}\]

\[\text{H. A. Einstein} \quad \text{U.S. Dept. of Agric. Technical Bulletin No.1026 1950}\]
FIGURE 17.

EXPERIMENTAL POINTS

- Uniform Gravels
  D = 0.8mm to 28.6mm

- Graded Gravel
  D = 3.7mm

CURVES

A. Einstein 1942
Trans. A.S.C.E.
Lower Line for Mixture

B. Einstein 1942
Trans. A.S.C.E.
Uniform Sediment

C. Cotting and Hemree 1955
U.S. Geol. Surv. W.S.P. 1357
Modified Levin Procedure

D. Einstein 1950
Original Einstein Procedure

E. Einstein 1942
TR. A.S.C.E.

F. Thompson (Written)
Deviation of calculated values from curve D.
The significant contribution of Einstein is not this bed load function and formula, but his approach to bed load motion. In 1942 he outlined this approach in purely qualitative terms and this outline has already been quoted in section 20 above. In 1950 Einstein considered this approach again this time in greater detail. He attempted to develop it into a quantitative approach which would be consistent with the empirical formula published in 1942. This quantitative treatment of the Einstein approach is very poorly presented in the original Agriculture Department Bulletin and so an outline of the treatment will now be presented.

The particles which form the top layer of the bed are acted on by the fluid which flows over them. The fluid pressure on the top of each particle is less than the pore pressure in the bed which acts on the bottom of them. This difference in pressure is the sum of
1. the hydrostatic difference in head \( \rho g \frac{D^2}{2} \)
2. the velocity head of the flowing water \( \frac{u^2}{2g} \) where \( u \) is the velocity of the filament of fluid over the top of the particles.

Einstein assumes that a particle entrains when the net uplift of the fluid pressure exceeds the particle's weight. That is when:
\[
\frac{1}{2} \rho u^2 + \frac{\pi}{4} D^2 + \rho g \frac{D^3}{8} \geq \rho g \frac{D^3}{2}
\]
and rearranging, this condition becomes
\[
\frac{1}{2} \rho \frac{u^2}{g} \geq \frac{1}{3} (\sigma - \rho) g D
\]
The velocity \( u \) is still undefined except for the statement that "u is the velocity at the top of the (entraining) particle." The flow is always turbulent and so the value of \( u \) fluctuates with time at any point. Einstein and El-Sawi measured the pressure (not the velocity) fluctuations and showed that they varied according to a Gauss Normal Error Law where \( \frac{\frac{\sum u^2}{n}}{\text{mean}} \) was 2.75 times the standard deviation of the pressure fluctuations.

Entrainment can occur when this fluctuating lift pressure, $\frac{1}{2} \rho u^2$ exceeds the threshold value, $2/3 (\sigma - \rho) g D$ which is just sufficient to support the particle. The proportion of time that entrainment can occur at a single location is called $P$ and may be calculated if the distribution of the pressure fluctuations is known.

A typical fluctuating pressure trace.

$$P = \text{proportion of time that } \frac{1}{2} \rho u^2 \geq \frac{2}{3} (\sigma - \rho) g D$$

$$= \int_{\sqrt{2} \rho u^2 / 2.75}^{\infty} \left( \text{relative frequency of } \frac{1}{2} \rho u^2 \right) d \frac{1}{2} \rho u^2$$

Adopting the Gauss Normal Error Law for the relative frequency distribution:

$$P = \frac{1}{\sqrt{2\pi}} \int_{t_0}^{\infty} e^{-t^2} dt$$

where $t$ is just a variable of integration

but $t_0 = \frac{\sqrt{2} (\sigma - \rho) g D - \frac{2}{3} \rho u^2}{\frac{1}{2} \rho u^2 / 2.75}$ is the value of $t$ at the threshold.

Rearranging $t_0 = \frac{4 \times 2.75}{3} \left( \frac{\sigma - \rho) g D}{\rho u^2} \right) - 2.75$

then introducing $U_t$

$$t_0 = \left( \frac{4 \times 2.75}{3} \left( \frac{U_t}{U} \right)^2 \right) \left( \frac{(\sigma - \rho) g D}{\rho u^2} \right) - 2.75$$

which may be written $t_0 = B U - 2.75$
(1) According to Einstein \( \Phi \) is a universal constant which implies that \( \frac{U_e}{U} \) is a constant ratio. That is the velocity of the fluid at the top of the entraining particles is a constant ratio of the shear velocity.

(2) The function \( \Phi \) is called the flow function and is the reciprocal of the well known Shields entrainment function. It is the same flow function that Einstein proposed in 1942 as has already been mentioned.

(3) The constant 2.75 is the ratio of the mean lift to the standard deviation of the pressure fluctuations as measured directly by Einstein and El Samii.

So far there has been no obvious objection which could be raised about this theoretical treatment. However, two assumptions are now made which do not bear close scrutiny. The first of these concerns the rate of entrainment and the second, the length of travel after entrainment.

Particles are assumed to entrain whenever the fluid lift is sufficiently strong as has been explained. It is then assumed that the rate of entrainment of particles exposed to this lift is proportional to the quotient of the particle settling velocity divided by the particle diameter. Introducing a constant of proportionality of six tenths it follows that

\[
P = \frac{6 \, w \, L}{D} \text{ per second.}
\]

The Evidence for this expression is entirely "dimensional" (?) and is devoid of any physical explanation. The constant of proportionality was not mentioned explicitly by Einstein but this value is necessary if the resultant equation is to be the same as the curve published with Einstein's bulletin.

Once entrained the particle can only deposit itself on that part of the bed area where the fluid is not actively entraining, that is a proportion \( (1 - P) \) of the area. Einstein then states that the distance of
travel between attempts at deposition is, on the average, a hundred particle diameters, regardless of the roughness \( \psi \) or the rate of flow \( \phi \):

\[
L = \frac{100D}{1-p}
\]

Now using Einstein's approach as outlined in section 20 the discharge of sediment per unit width is:

\[
Q = \frac{2}{3} D \rho \frac{L}{P} = \frac{2}{3} D \times \frac{6WP}{D} \times \frac{100D}{1-p}
\]

\[
\frac{Q}{WD} = \frac{40P}{1-p}
\]

which may be written

\[
\phi = \text{function of } \psi
\]

Summarising the treatment just presented here and in section 20:

1. The volumetric sediment discharge per unit width is equated to the product of the number of exposed particles per unit area, the volume of each particle, its rate of entrainment and average travel

\[
Q = \frac{1}{2} D^3 \rho \frac{L}{P}
\]

2. The proportion of bed area where the fluid is actively entraining is calculated assuming that there is a fluctuating lift pressure on the bed exerted by the fluid whose mean value is proportional to the bed shear stress. Thus:

\[
p = \frac{1}{\pi} \int_0^\infty B \psi^{-2.75} e^{-t^2} dt
\]

\( \psi \) is a dimensionless flow function incorporating the bed shear stress and \( B \) is a dimensionless "universal" constant whose value must be determined by measurement. Note that this formula is not sensitive to small changes in the value of the constant which is here put as 2.75.
(3) The dubious assumption is made that 

$$ p^2 \propto \frac{P_0}{D} $$

(4) A second dubious assumption is made that 

$$ l \propto \frac{D}{1-p} $$

Substituting the last two expressions in the first equation and dividing by $WD$ to make each side dimensionless,

$$ \frac{p}{WD} = \frac{1/A}{1-p} $$

where $A$ is a second dimensionless "universal" constant whose value must also be determined empirically. Since

$$ \frac{p}{WD} = \phi $$

and

$$ p = \text{a function of } \psi $$

it follows that $\phi = \text{a function of } \psi$, all the functions being defined above.

The writer has found that by putting

$$ A = 39 $$
$$ B = .202 $$

then this equation accurately fits the $\phi - \psi$ curve published by Einstein in 1950. The reader may wonder why the values quoted by Einstein for his Universal Constants $A$ and $B$ are not used here. The answer is that Einstein's original presentation of the equation for his curve is incorrect and so his values for $A$ and $B$ are not relevant.

Two algebraic mistakes are made in the treatment as published by Einstein in 1950.

(1) On page 34 of the U.S. Dept. of Agric. Tech. Bull. 1026 it is stated that

$$ \frac{p}{1-p} = A\phi $$
and then on page 37 this is restated in the incorrect form

\[ P = \frac{A \phi}{1 - A \phi} \]

(2) On page 36 of the same bulletin a meaningless statement is made that

\[ n \] must be understood on an absolute basis.\"

However, this idea has been persisted with, resulting in an incorrect form for the Causs Normal Error Law expression for \( P \). Unfortunately these two errors made the equation numbered fifty seven on page 37 read as follows,

\[ P = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{B \mu - \frac{1}{2} \sigma} e^{-t^2} dt = \frac{A \phi}{1 - A \phi} \]

when it should have been

\[ P = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{B \mu - \frac{1}{2} \sigma} e^{-t^2} dt = \frac{A \phi}{1 + A \phi} \]

or more simply by reverting the limits of integration

\[ P = \frac{1}{\sqrt{2\pi}} \int_{B \mu - \frac{1}{2} \sigma}^{\infty} e^{-t^2} dt = \frac{A \phi}{1 + A \phi} \]

The corrected form of the \( \phi - \psi \) equation is plotted on figure 17 with the values for the constants \( A, B \) and \( \mu_0 \) as have been given above. Except for the short section labelled F on the figure 17 the calculated curve coincides with the curve published by Einstein in 1950 which is labelled D. The other curves on this figure are all the different forms for Einstein's \( \phi - \psi \) relationship which have been published. The plotted points are the same as those used by Einstein in determining the values for his empirical constants. However, the points for sediment finer than course sand have been omitted because they are not relevant to this study.
When Einstein published his equation for the $\phi - \psi$ curve in 1930 he included it with a procedure for computing the total transport of sediment in rivers. This procedure was taken up by the staff of the U.S. Geological Survey who tested it and modified it for their own use in stream gauging in the United States. The Einstein $\phi - \psi$ curve remained an important feature of their modified procedure.

The modifications introduced were as follows.

1. A measurement of mean velocity was substituted for the depth slope product. It was then necessary to calculate the shear velocity using Keulegan's formula

$$U_s = \frac{U}{0.75 \log D_{0.5}}$$

2. In order to obtain a value of the flow function which was comparable with Einstein's value the flow function was arbitrarily multiplied by four tenths.

$$\psi = \frac{A (\sigma - \rho) a D}{\rho U_s^2}$$

3. The "universal" constant $A$ was also arbitrarily halved. Thus rearranging the equation for Einstein's $\phi - \psi$ curve,

$$\phi = \frac{\int_{-\infty}^{\infty} 2\psi \cdot f e^{-\psi^2} d\psi}{20 \left( \sqrt{2\pi} - \int_{-\infty}^{\infty} e^{-\psi^2} d\psi \right)}$$

As before the discharge of sediment per unit width is calculated from $\phi$ thus:

$$q = wD\phi$$

*Colby and Nembree  U.S.G.S. Water Supply Paper No. 1357


and

* D. R. Colby and others  U.S.G.S. Water Supply Paper No. 1593

The first bed load experiments were in December and the results seemed to be very good. See figures 18 and 19. An extensive program was embarked on in January to relate the shear of the flow to the entrainment rate. The following table is a summary of all the measurements made.

The letters A, B, C, D, E, T in the table refer to the various locations on the bed. These locations are shown in photograph 16.

All the samples which included more than 100 trials and one sample of 68 trials are plotted in figure 19. The samples were the largest measured and so they should be the most reliable. The figure 19 shows the range of values of shear and entrainment rate covered by the experiments.
Entrainment rate measurements made during 1962.
FIGURE 19

Date | 15  | 23  | 21  | 10  | 8   | 9   | 12  | 17  | 18
---|-----|-----|-----|-----|-----|-----|-----|-----|-----
No. of Trials | 125 | 100 | 120 | 141 | 95  | 120 | 267 | 140 | 70
P3 per sec. | .15  | .085 | .026 | .015% | .00075

Entrainment rate, nine largest samples measured.
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<th>Shear</th>
<th>Location</th>
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<th>Travel</th>
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<th>$\gamma$</th>
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*Note: The table represents data collected from various locations and conditions, including the number of trials and measurements for travel and other relevant metrics.*