On the development of non-classical mathematics

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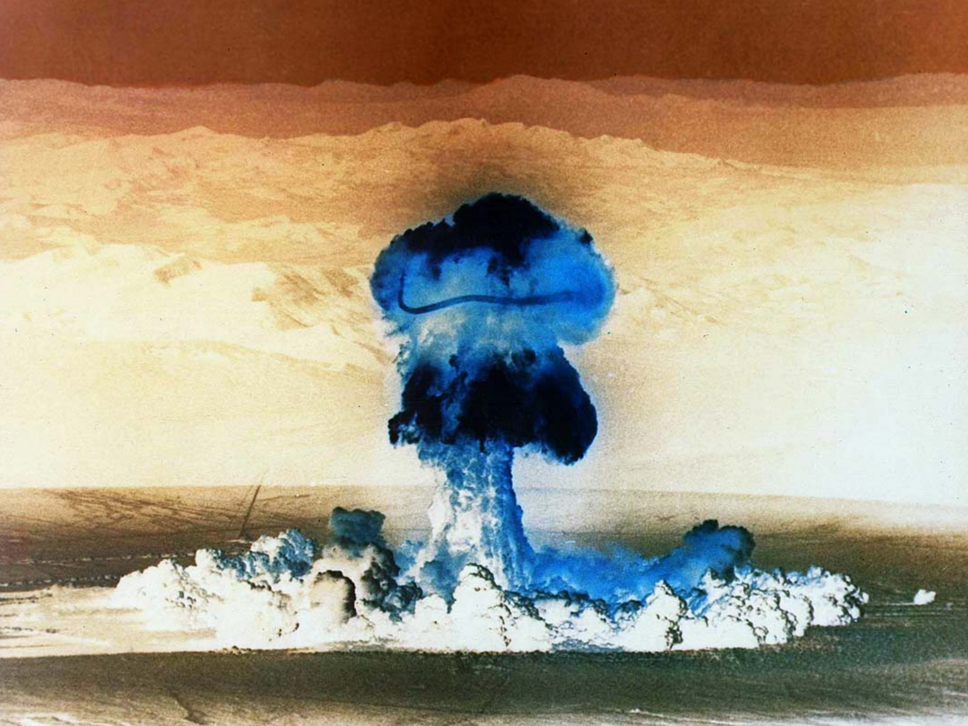
An apology and a contention.

WOLF IN SHEEPS CLOTHING.

There lies a traitor in our midst...

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ P \wedge \neg P\end{array}}{?}$$



"Indeed, even at this stage, I predict a time when there will be mathematical investigations of calculi containing contradictions, and people will actually be proud of having emancipated themselves from consistency."

— Ludwig Wittgenstein (1930)

Paraconsistent logic

Definition

A logic is *paraconsistent* if it does not validate explosion,

$$A, \neg A \vdash B$$

for arbitrary A and B.

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There are many paraconsistent logics.

Paradoxes of material implication

1. $(p \wedge \neg p) \rightarrow q$ (paradox of entailment)
2. $p \rightarrow (q \rightarrow p)$ (weakening)
3. $\neg p \rightarrow (p \rightarrow q)$ (explosion)
4. $p \rightarrow (q \vee \neg q)$
5. $(p \rightarrow q) \vee (q \rightarrow r)$
6. $\neg(p \rightarrow q) \rightarrow (p \wedge \neg q)$

Relevance?

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(Belnap’s criterion: the antecedent and consequent must share a (propositional) variable.)

A question of discharge policy.

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B} \to \mathsf{I}$$

	Multiple discharge	Single discharge
Vacuous discharge	“classical”	affine
No vacuous discharge	relevant	linear

An observation

Later mathematicians will regard set theory as a disease from which one has recovered.

— Henri Poincaré (1854-1912)

Paraconsistent Mathematics

Advantages / Disadvantages

On the locatedness of sets

Definition

A set S of real numbers is *order located* if, given $x \in \mathbf{R}$, either

$$\forall_{y \in S}(y \leqslant x) \qquad \text{or} \qquad \exists_{y \in S}(x < y).$$

The significance of $17,387,594,889$?

The Least Upper Bound Principle.

The least-upper-bound principle is constructively not provable.

The *constructive* least-upper-bound priciple is.

The Least Upper Bound Principle.

The least-upper-bound principle is constructively not provable.

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The moral: to constructively prove something similar to a classical theorem, we often need to either strengthen the hypotheses or weaken the conclusion.

The paraconsistent story.

The Sorites paradox (if time allows).

The paraconsistent story.

The Sorites paradox (if time allows).

> "Das is nicht Mathematik. Das ist Theologie."
>
> — Gordan, concerning Hilbert's basis theorem.

Vague (inconsistent?) sequences

The phenomenon of strictly increasing bounded sequences which do not converge (of course, for different reasons) is well-known in constructive mathematics: *Specker* sequences.

Brouwerian examples

The role of Brouwerian examples (in constructive mathematics) is to demonstrate that some statement is constructively not provable by showing it implies some nonconstructive principle.

We may adopt the same strategy in other non-classical mathematics.

Example.

Suppose that $p \vee q$ and $\neg q$ are true statements. Form the set

$$S_p := \{x \in \mathbf{R} : x = 0 \vee (x = 1 \wedge p)\}$$

Doing mathematics paraconsistently.

Mathematical theorems are typically of the form

$$\forall x \in A(x \in B).$$

What is the “implication” here?

The idea is to validate

$$\text{If } x \in A \vdash x \in B, \text{ then } \vdash \forall_{x \in A} x \in B.$$

Stronger than the material conditional (they obey modus ponens); but weaker than relevant implication, since they weaken and obey de Morgan exchanges.

But using $\vdash$ to define these means that they *do not contrapose*.

So $\forall_{x \in A} x \in B$ does not prove $\forall_{x \notin B} x \notin A$.

Example: Heine-Borel for $[0, 1]$.

Definitions.

Definitional issues arise.

Definition

f is continuous at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

What do we mean by "if"? And "whenever"?

$$f \text{ continuous } \overset{?}{\Leftrightarrow} \forall_{\varepsilon>0} \exists_{\delta>0} \left[|x - x_0| < \delta \overset{?}{\Rightarrow} |f(x) - f(x_0)| < \varepsilon \right].$$

$\rightarrow$ seems too strong. $\vdash$ seems too weak.


BBC

Can constructive and (say) paraconsistent mathematics be compared?

Thank you for listening!

References

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