Analytical and numerical bedrock reconstruction in glacier flows from free surface elevation data

Mathieu Sellier *a), Alelign Gessese* & Christian Heining**

*Department of Mechanical Engineering, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand

**Applied Mechanics and Fluid Dynamics, University of Bayreuth, Universitätsstr., 95440 Bayreuth, Germany

Summary This paper presents a simple analytical and numerical approach to reconstruct the bedrock in glacier flows from given free surface data. The approach relies on the Shallow Ice Approximation in one space dimension to describe the glacier flow dynamics. Remarkably, we show that this complex, non-linear partial differential equation can be integrated once to yield and explicit relationship between the ice free surface elevation and the ice depth and therefore the bedrock. The applicability of the proposed approach is broadened by proposing a transient numerical scheme capable of reconstructing the ice depth and underlying bedrock topography.

DESCRIPTION OF THE DIRECT PROBLEM

Because the ice-mass is such a good indicator of climatic changes, it has been under intense scrutiny in the recent past. One of the major challenges glaciologists face when it comes to understanding ice flows is that while information at the glacier surface is easily accessible, basal quantities are notoriously difficult to access and assess [1]. Current techniques to indirectly infer the bedrock topography rely on airborne radar measurements, a difficult and costly operation. This paper explores the possibility of inferring the bedrock topography from the knowledge of the free surface elevation.

Consider a one-dimensional glacier flow as illustrated in Figure 1. The y-axis is aligned with gravity, pointing upwards, and the x-axis, perpendicular to it, points in the flow direction. The glacier free surface elevation is denoted by S(x,t), the glacier thickness by H(x,t), and the bedrock by Z(x). Clearly, these three quantities are related through S=Z+B.

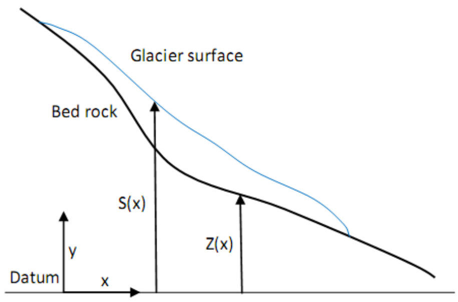

Figure 1: Problem description and notations

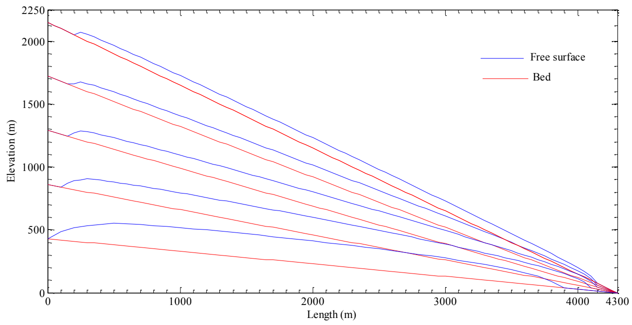

Figure 2: Glacier evolution on a flat bed of different slope

The starting point of the analysis is the Shallow Ice Approximation (SIA) which is known to describe well the evolution of shallow glaciers providing the ice may be assumed incompressible with a constant density, inertia may be neglected, and the ratio of the characteristic length scale in the cross-flow direction to the one of the flow direction is small [1]. Under these conditions, the glacier depth satisfies the following non-linear partial differential equation (PDE)

$$\frac{\partial H}{\partial t} = a + \frac{2(\rho g)^3}{5}\frac{\partial}{\partial x}\left(D\frac{\partial S}{\partial x}\right) \tag{1}$$

where $D = H^3\left|\frac{\partial S}{\partial x}\right|^2 (AH^2)$ in the absence of basal sliding. In eq. (1), a(x,t) is the ablation/accumulation coefficient on the surface of the glacier, ρ the density of the ice, g the acceleration of gravity, and A is Glen's flow parameter, considered constant here. This second order pde has been extensively studied in its direct form which consists in inferring the glacier thickness for a given bedrock form and known accumulation/ablation coefficient, the traditional forward problem. An illustrative solution of eq. (1) for a benchmark problem inspired from Le Meur et al [2] is shown in Figure 2. Solutions are obtained using a centred Finite Difference spatial discretization with an explicit time integrator subject to a zero ice thickness boundary condition at both ends of the computational domain. The positivity of the solution is explicitly enforced throughout the solution procedure. For this benchmark test, the flat bedrock is described by $Z(x) = Z_0 - \alpha x$ where α is the bedrock slope, Z_0 is bedrock topographic elevation at $x = 0$, $a(x) = a_0(0.01x - 2)$ if $x \le 300$ and $a(x) = a_0(1.158 - 5.263 \times 10^{-4}x)$ if $x > 300$. Tests are carried out for values of α ranging from 0.1 and 0.5 with interval of 0.1 and $a_0 = 0.5$. The resolution of the computational mesh is $\Delta x = 50\text{m}$ and $\Delta t = 0.1\,\text{year}$. As shown in Figure 2, the glacier evolution increases with the bed slope meaning that the steeper the slope the thinner the glacier as expected. These results are in a good agreement with the

a) Corresponding author. Email: mathieu.sellier@canterbury.ac.nz.

results published in [2].Thus the algorithm is capable of modelling glacier evolution from a known bedrock elevation and given flow parameters in the case of constant slope bedrock topography.

INVERSE PROBLEM SOLUTION

The inverse problem consists in inferring the bedrock topographic elevation from the known glacier free surface elevation and a known ablation accumulation distribution over the entire glacier. The classical approach for dealing with such inverse problems relies on PDE constrained optimization whereby, an objective expressing the mismatch between the computed free surface elevation and the actual one is first defined and the bedrock form is optimized in order to minimize this objective function. The approach we propose here is a direct approach in the sense that it only requires the solution of a PDE. Indeed, the SIA still holds and may be rewritten as

$$\frac{\partial H}{\partial t} = a + \frac{2(\rho g)^3}{5} \frac{\partial}{\partial x}(TM) \tag{2}$$

where $T = H^3(AH^2)$. The function $M = \frac{\partial s}{\partial x}\left|\frac{\partial s}{\partial x}\right|^2$ is known for a given glacier surface S(x). For a glacier in a steady state, eq. (2) is an ordinary differential equation which admits the following exact solution

$$H = \left[C_0 - \int_0^x a dx \left(\frac{2A(\rho g)^3}{5}\right)^{-1}\right]^{1/5} \tag{3}$$

where $C_0 = \frac{2A(\rho g)^3}{5} H_0^5 M_0^2 \int_0^{x_0} a dx$ is the constant of integration which requires H and its first derivative to be known at location x_0. Thus, unlike most inverse problems, an analytical solution to the bedrock reconstruction problem exists. This solution was tested for a benchmark problem where the bedrock has a bump and is described by $Z(x) = Z_0 - 0.5x + 100\exp\left(-\frac{(x-2000)^2}{200^2}\right)$. The ablation/accumulation coefficient is as given above with $a_0 = 5$. The glacier free surface was first inferred by solving the direct problem until a steady state is reached and then using the free surface elevation as an input to the inverse problem. The reconstruction results can be seen in Figure 3.

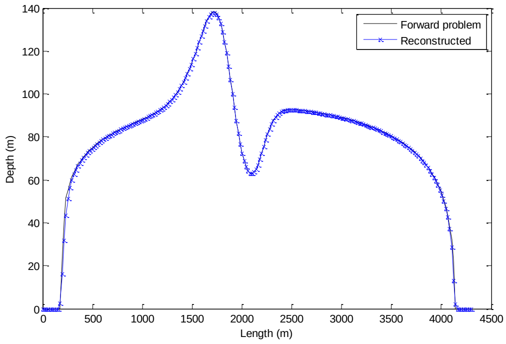

Figure 3: Actual and reconstructed glacier depth (steady analytical approach)

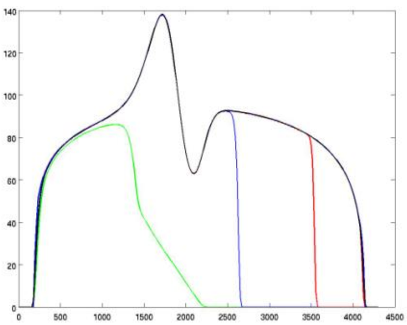

Figure 4: Actual and reconstructed bedrock (unsteady numerical approach)

Figure 3 clearly confirms the validity of this simple analytical approach since the naked eye cannot distinguish the actual and reconstructed glacier depth and bedrock topography. This should, of course, come as no surprise since both the forward and inverse problems fulfill the same differential equation but could provide an attractive alternative to the currently used methods to reconstruct the bedrock in glacier flows. The proposed analytical approach suffers from limitations: it is restricted to one-dimensional glacier flows; it requires the knowledge of an integration constant; it is inherently restricted to steady situations. To circumvent these limitations, it is possible to solve eq. (2) directly using an implicit time integrator since explicit ones are unstable for this case. The discrete form of eq. (2) is

$$\frac{H_i^{n+1} - H_i^n}{\Delta t} = \frac{(TM)_{i+1/2}^{n+1} - (TM)_{i-1/2}^{n+1}}{\Delta x} \tag{4}$$

The resulting system of algebraic equations is solved exactly using Newton's method which usually converges within 10 iterations. The convergence of the reconstructed ice depth towards the actual one can be seen in Figure 4. As time increases, the reconstructed ice thickness slowly tends towards the actual one and ultimately both are undistinguishable. Future work will investigate the sensitivity of the reconstruction algorithm to noisy input data and use actual glacier data available in the literature.

References

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