On-brane data for braneworld stars

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Stellar structure in braneworlds is markedly different from that in ordinary general relativity. As an indispensable first step towards a more general analysis, we completely solve the "on brane" 4-dimensional Gauss and Codazzi equations for an arbitrary static spherically symmetric star in a Randall–Sundrum type II braneworld. We then indicate how this on-brane boundary data should be propagated into the bulk in order to determine the full 5-dimensional spacetime geometry. Finally, we demonstrate how this procedure can be generalized to solid objects such as planets.

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I. INTRODUCTION

One physically important problem in the braneworld scenario is the development of a full understanding of stellar structure and black holes [1, 2, 3, 4, 5]. What is already known is that stellar structure in braneworlds is rather different from that in ordinary general relativity. The key point is that if one is confined to making physical measurements on the brane, then the restricted "on-brane" version of the Einstein equations does not form a complete system for specifying the brane geometry [6, 7, 8]. The problem lies in the fact that the bulk 5-dimensional Weyl tensor feeds into the "on-brane" equations and so connects the brane to the bulk.

On the other hand, one may solve the "on-brane" version of the Einstein equations to obtain a consistent class of boundary data that satisfies the 4-dimensional Gauss and Codazzi equations [6], analogously to the initial data in the standard $(3+1)$ decomposition of globally hyperbolic manifolds in general relativity. Although the relevant partial differential equations are now elliptic rather than hyperbolic, one can use the boundary data in a completely analogous fashion — as input into the 5-dimensional bulk Einstein equations in order to "propagate" the 4-dimensional geometry off the brane and into the bulk [1, 5].

In this Letter we present an algorithm for completely solving the 4-dimensional Gauss and Codazzi equations for a static spherically symmetric star on the brane. This provides the most general boundary data, suitable for then determining the bulk geometry by an appropriate relaxation method [1].

Specifically, we will use Gaussian normal coordinates adapted to a timelike 4-dimensional hypersurface (spacelike normal) in a 5-dimensional geometry,

$$\mathrm{d}s_5{}^2 = \mathrm{d}\eta^2 + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b \tag{1}$$

and assume a type II Randall–Sundrum braneworld [9], in which the bulk metric is a 5-dimensional Einstein space, (*i.e.*, with Ricci tensor proportional to the metric), but the 4-dimensional Lorentzian signature metric g_{ab} is as yet undetermined.

If we impose $\mathbb{Z}_2$ symmetry on this spacetime, and tune the 5-dimensional bulk cosmological constant and brane tension appropriately, then the junction conditions together with the projected 5-dimensional Einstein equations (the Gauss equations) reduce to the induced but incomplete on-brane "Einstein equations" in the form [2, 6]

$$\begin{aligned} G_{ab} \;=\; & 8\pi\, T_{ab} - \Lambda_4 g_{ab} - \mathcal{E}_{ab} \\ & -\frac{\kappa^2}{4}\left[(T^2)_{ab} - \tfrac{1}{3}T\,T_{ab} - \tfrac{1}{2}g_{ab}\left(T\cdot T - \tfrac{1}{3}T^2\right)\right]. \end{aligned} \tag{2}$$

where κ^2 is a constant inversely proportional to the brane tension, Λ_4 is an on-brane cosmological term, the nonlocal term $\mathcal{E}_{ab}$ is simply the projection of the 5-dimensional Weyl tensor onto the brane, and in addition to the usual stress-energy term there is also a nonlinear term quadratic in stress-energy. In addition to (2) one finds that the extrinsic curvature

$$K_{ab} = \frac{1}{2}\frac{\partial g_{ab}}{\partial \eta}, \tag{3}$$

is related to the on-brane fields via

$$K_{ab} = -\frac{\kappa}{2}\,\left[T_{ab} - \tfrac{1}{3}g_{ab}T\right] - \frac{8\pi}{\kappa}\,g_{ab}. \tag{4}$$

The Codazzi equation

$$\nabla_b[K^{ab} - K g^{ab}] = 0, \tag{5}$$

is then equivalent to 4-dimensional stress-energy conservation for the on-brane matter: $\nabla^a T_{ab} = 0$. Eqs. (2)

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and (4) complete the specification of the boundary data by effectively supplying the "on brane" metric and its normal derivative.

The only truly general thing we know about the nonlocal Weyl tensor projection term $\mathcal{E}_{ab}$ is that it is traceless, $\mathcal{E}^a{}_a = 0$. Together with (2) this implies that

$$R = -8\pi\, T + 4\Lambda_4 - \tfrac{1}{4}\kappa^2 \left[T \cdot T - \tfrac{1}{3}T^2\right]. \tag{6}$$

In the vacuum region outside a star this reduces to

$$R = 4\Lambda_4, \tag{7}$$

whereas inside the star it takes the form

$$R = [\text{nonlinear-source}]. \tag{8}$$

If for instance we are dealing with a perfect fluid then

$$R = 8\pi(\rho - 3p) + 4\Lambda_4 + \tfrac{1}{4}\kappa^2 \left[(\rho^2 - 3p^2) + \tfrac{1}{3}(\rho - 3p)^2\right]. \tag{9}$$

If we are restricted to making physical measurements on the brane, then (6) is the only really general thing we can say. Eqs. (2), (6) are *much* weaker than the 4-dimensional Einstein equations and so the solution space will be much more general. Of course, we must also use the extrinsic curvature constraint (4) to complete the boundary data, and then ultimately probe the bulk geometry off the brane, and this will indirectly provide further restrictions.

Our aim here is to solve Eqs. (2) and (4) in full generality for arbitrary static spherically symmetric solutions on the brane, to provide appropriate data to "propagate" into the bulk [1] (via Eqs. (32) below), to thereby test the consistency of candidates for realistic stellar models.

II. VACUUM [$\Lambda_4 = 0$]

In this section we will consider the case when $\Lambda_4 = 0$. For a braneworld star, in the vacuum region outside the surface we are interested in solving $R = 0$. In the usual way we can locally adopt on-brane coordinates such that

$$\mathrm{d}s^2 = -\exp[-2\phi(r)]\,\mathrm{d}t^2 + \frac{dr^2}{B(r)} + r^2 \mathrm{d}\Omega^2. \tag{10}$$

If we now impose the condition $R = 0$ we have one differential constraint connecting two unknowns — therefore there will be a nondenumerable infinity of solutions parameterized by some arbitrary function of r. Various specific cases have already been discussed in the literature, but no attempt has been made at extracting the general solution.

A. Special cases

Known special case solutions include:
— "Reissner-Nordström-like" [3]

$$B(r) = \exp[-2\phi(r)] = 1 - \frac{2M}{r} + \frac{Z}{r^2}. \tag{11}$$

Note that the parameter Z is *not* an electric charge, but should be thought of as a tidal distortion parameter.
— "Spatial Schwarzschild" [4]

$$B(r) = 1 - \frac{2M(1+\epsilon)}{r}; \tag{12}$$

$$\exp[-2\phi(r)] = \left(\frac{\epsilon + \sqrt{1 - \frac{2M(1+\epsilon)}{r}}}{1+\epsilon}\right)^2. \tag{13}$$

(This geometry is also discussed in a rather different context in ref. [10].)
— "Temporal Schwarzschild" [3, 4]

$$B(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{a}{r(1 - \frac{3}{2}\frac{M}{r})}\right); \tag{14}$$

$$\exp[-2\phi(r)] = 1 - \frac{2M}{r}. \tag{15}$$

However these are all very specific special cases; and are in no way general.

B. General vacuum solution

Let us first write the metric in the form

$$\mathrm{d}s^2 = -\exp\left[-2\int_r^\infty g(\bar{r})\,\mathrm{d}\bar{r}\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2\,\mathrm{d}\Omega^2. \tag{16}$$

The function $g(r)$ is interpreted as the locally-measured acceleration due to gravity; it is positive for a inward acceleration.

Now calculate the Ricci scalar R

$$R = \frac{(2r + r^2 g)B' + (2r^2g^2 + 2r^2g' + 4rg + 2)B - 2}{r^2}. \tag{17}$$

For the vacuum case we set this equal to zero, which yields

$$(2r + r^2 g)B' + (2r^2g^2 + 2r^2g' + 4rg + 2)B - 2 = 0. \tag{18}$$

If we treat $g(r)$ as input and view this as a differential equation for $B(r)$, it is a first-order linear ODE, and hence explicitly solvable. The integrating factor is

$$F(r; r_0) = \exp\left\{\int_{r_0}^r \frac{1 + 2\bar{r}g(\bar{r}) + \bar{r}^2 g(\bar{r})^2 + \bar{r}^2 g'(\bar{r})}{\bar{r}(1 + \bar{r}g(\bar{r})/2)}\,\mathrm{d}\bar{r}\right\}, \tag{19}$$

where r_0 is any convenient reference point, and the general solution is

$$B(r) = \left\{\int_{r_0}^r \frac{F(\bar{r}; r_0)}{\bar{r}\,(1 + \bar{r}g(\bar{r})/2)}\,\mathrm{d}\bar{r} + B(r_0)\right\} F(r; r_0)^{-1}. \tag{20}$$

Whereas r_0 is an arbitrary gauge parameter, the constant $B(r_0)$ is related to physical parameters such as the mass and post-Newtonian corrections. In the case of the "temporal Schwarzschild" solution (14), (15), for example, $B(r_0) = 2(a + r_0) - 3M$.

Alternatively one may use $B(r)$ as input to generate a first-order ODE quadratic in $g(r)$ — a Riccati equation. However, this has no comparable general solution. The algorithm above is a modification of Lake's construction for generating spherically symmetric perfect fluid spacetimes [11]. We suspect that an isotropic coordinate version, based on [12], may also be viable.

This general solution can be somewhat simplified by using integration by parts on the integrating factor to remove the derivatives of $g(r)$. We find

$$F(r;r_0) = \frac{(1+rg(r)/2)^2}{(1+r_0+g(r_0)/2)^2} \times \exp\left\{\int_{r_0}^{r} \frac{1+\bar{r}g(\bar{r})+\bar{r}^2 g(\bar{r})^2}{\bar{r}(1+\bar{r}g(\bar{r})/2)}\,\mathrm{d}\bar{r}\right\}, \tag{21}$$

$$= \frac{r\left[1+\frac{1}{2}rg(r)\right]^2}{r_0\left[1+\frac{1}{2}r_0 g(r_0)\right]^2}\,F_2(r;r_0), \tag{22}$$

where we have defined

$$F_2(r;r_0) \equiv \exp\left\{\int_{r_0}^{r} \frac{g(\bar{r})[1+2\bar{r}g(\bar{r})]}{2(1+\frac{1}{2}\bar{r}g(\bar{r}))}\,\mathrm{d}\bar{r}\right\}. \tag{23}$$

Eq. (20) then becomes

$$B(r) = \left\{\int_{r_0}^{r} (1+\bar{r}g(\bar{r})/2)\,F_2(\bar{r};r_0)\,\mathrm{d}\bar{r} + r_0\,\left[1+\tfrac{1}{2}r_0 g(r_0)\right]^2\,B(r_0)\right\} \times\left\{r\,\left[1+\tfrac{1}{2}rg(r)\right]^2\,F_2(r;r_0)\right\}^{-1}. \tag{24}$$

This is now the explicit general (static spherically symmetric) solution of the equation $R = 0$ using the function $g(r)$ and the constant $B(r_0)$ as arbitrary input. The construction is completely algorithmic.

There is a potential subtlety if the Schwarzschild coordinate r does not increase monotonically with respect to the outward radial proper distance, ℓ_r. For ordinary stars in general relativity the monotonicity of r with respect to ℓ_r is guaranteed by the null energy condition. In braneworld stars there is no particular reason to believe in the monotonicity of $r(\ell_r)$, but our construction will still hold piecewise on monotonic intervals.

III. STELLAR INTERIOR

We now want to solve $R = S(r)$, with $S(r)$ a specified source. We can now consider arbitrary values of Λ_4 without additional complications by including a possibly non-zero Λ_4 in $S(r)$. Proceeding exactly as above we find

$$B(r) = \left\{\int_{r_0}^{r} \frac{1+\bar{r}^2 S(\bar{r})/2}{\bar{r}\,[1+\bar{r}g(\bar{r})/2]} F(\bar{r};r_0)\,\mathrm{d}\bar{r} + B(r_0)\right\} \times F(r;r_0)^{-1}, \tag{25}$$

with the same integrating factor $F(r;r_0)$ of Eq. (21). Equivalently

$$B(r) = \left\{\int_{r_0}^{r} \left(1+\tfrac{1}{2}\bar{r}^2 S(\bar{r})\right)\left(1+\tfrac{1}{2}\bar{r}g(\bar{r})\right)F_2(\bar{r})\,\mathrm{d}\bar{r} + r_0\,\left[1+\tfrac{1}{2}r_0 g(r_0)\right]^2\,B(r_0)\right\} \times\left\{r\,\left[1+\tfrac{1}{2}rg(r)\right]^2\,F(r;r_0)\right\}^{-1}. \tag{26}$$

Once the source $S(r)$ is specified, this is fully general. In addition one must specify an arbitrary function $g(r)$, and a single arbitrary constant $B(r_0)$, and so algorithmically determine the metric on the brane.

Of course the source $S(r)$ is somewhat restricted in that it is an algebraic function of the on-brane stress-energy tensor, which is itself restricted by 4-dimensional energy-momentum conservation (equivalent to the Codazzi equation (5) as mentioned above). For a static perfect fluid with spherical symmetry the stress-energy has the form

$$T^{ab} = \begin{bmatrix} \rho(r)\;\exp\left[+2\int_r^\infty g(\bar{r})\,\mathrm{d}\bar{r}\right] & 0 & 0 & 0 \\ 0 & p(r)\;B(r) & 0 & 0 \\ 0 & 0 & p(r)/r^2 & 0 \\ 0 & 0 & 0 & p(r)/(r^2\sin^2\theta) \end{bmatrix}. \tag{27}$$

The equation of energy–momentum conservation gives

$$\frac{dp}{dr} = -g(r)\;[\rho+p], \tag{28}$$

which, being a linear first-order ODE, has the exact closed-form solution

$$p(r) = \exp\left[-\int_{r_0}^{r} g(\bar{r})\ \mathrm{d}\bar{r}\right] \times \left\{p(r_0) - \int_{r_0}^{r} g(\bar{r})\ \rho(\bar{r})\ \exp\left[+\int_{r_0}^{\bar{r}} g(\tilde{r})\ \mathrm{d}\tilde{r}\right]\ \mathrm{d}\bar{r}\right\}. \tag{29}$$

The lesson now is that to find all possible conserved tensors T^{ab} one is free to specify the function $\rho(r)$ and the number $p(r_0)$ arbitrarily, and thereby calculate $p(r)$, which now yields the full tensor T^{ab}. From $T^{ab}(r)$ we now calculate $S(r)$, thereby fixing the intrinsic geometry on the brane. By eq. (4) this also automatically generates the most general possible candidate for the extrinsic curvature K^{ab} compatible with the assumed symmetries. It must be emphasised that only *some* of these braneworld geometries are physically meaningful, because one now needs to extrapolate them off the brane to see if the "graviton" is still bound [13].

IV. EXTRAPOLATING OFF THE BRANE

The full algorithm is:

- Step 1: Pick an arbitrary density distribution $\rho(r)$ of matter inside the star; an arbitrary function $g(r)$; and a single number $p(r_0)$. Calculate $p(r)$ and so evaluate the stress-energy tensor T^{ab} and the source term

$$S(r) = -8\pi T + 4\Lambda_4 - \tfrac{1}{4}\kappa^2 \left[T \cdot T - \tfrac{1}{3}T^2\right] \tag{30}$$

- Step 2: Armed with this source term $S(r)$ and the previously chosen $g(r)$, pick one additional number $B(r_0)$ in order to calculate the function $B(r)$ and thereby generate a candidate metric g_{ab} for the on-brane physics.

  By calculating the 4-dimensional Einstein tensor for this candidate metric, one can rearrange Eq. (2) to calculate $\mathcal{E}_{ab} = {}^{(5)}C_{\eta a \eta b}$ and so the find the projection of the five-dimensional Weyl tensor on the brane.

- Step 3: Using this on-brane metric, and the on-brane stress-energy calculated in Step 1, use Eq (4) to calculate the extrinsic curvature K_{ab}.

- Step 4: Evolve the metric off the brane. Sufficiently near the brane one can certainly use normal coordinates and so the standard result

$$^{(5)}R_{\eta a \eta b} = \frac{\partial K_{ab}}{\partial \eta} + K_{am}\ K^m{}_b \tag{31}$$

  applies. Rearrange this to yield:

$$\frac{\partial^2 g_{ab}}{\partial \eta^2} = -\frac{1}{2}\ g^{a'b'}\ \ \frac{\partial g_{aa'}}{\partial \eta}\ \frac{\partial g_{b'b}}{\partial \eta} + 2\ {}^{(5)}R_{\eta a \eta b}. \tag{32}$$

  Here $^{(5)}R_{\eta a \eta b}$ is calculable in terms of the projection of the five-dimensional Weyl tensor and the bulk cosmological constant.

This now provides a well-determined set of equations for extending the on-brane metric into the bulk. Whether or not the resulting braneworld model is actually viable depends on whether or not the bulk geometry is "well behaved" — in particular, is the graviton bound or unbound [13].

We must make the caveat that the Gaussian normal coordinates system (1) is likely to break down at some stage as one moves away from the brane. This is not really a fundamental objection but more of a technical issue. While it is easiest to set up the "on-brane" boundary conditions using Gaussian normal coordinates, as a practical matter when it comes to numerically solving for the bulk geometry one should be prepared to dynamically adjust the coordinate system.

V. SOLID PLANETS

If we are dealing with a situation of spherical symmetry that does not correspond to a perfect fluid, such as a solid planet, the radial and transverse pressures would be different and we would have

$$\begin{aligned} R \ &= \ 8\pi(\rho - p_r - 2p_t) + 4\Lambda_4 \\ &\quad + \tfrac{1}{4}\kappa^2 \left[(\rho^2 - p_r^2 - 2p_t^2) + \tfrac{1}{3}(\rho - p_r - 2p_t)^2\right]. \end{aligned} \tag{33}$$

The other major change arises in the on-brane stress-energy tensor, which becomes

$$T^{ab} = \begin{bmatrix} \rho(r)\ \exp\left[+2\int_r^\infty g(\bar{r})\ \mathrm{d}\bar{r}\right] & 0 & 0 & 0 \\ 0 & p_r(r)\ B(r) & 0 & 0 \\ 0 & 0 & p_t(r)/r^2 & 0 \\ 0 & 0 & 0 & p_t(r)/(r^2 \sin^2\theta) \end{bmatrix}. \tag{34}$$

Then covariant conservation gives the slightly more complicated equation

$$\frac{dp_r}{dr} = \frac{2}{r}(p_t - p_r) - g(r)\ [\rho + p_r]. \tag{35}$$

This is still a linear first-order ODE, and has the exact closed-form solution

$$p_r(r) = r^{-2} \exp\left[-\int_{r_0}^{r} g(\bar{r})\ \mathrm{d}\bar{r}\right] \times \left\{ \int_{r_0}^{r} \exp\left[+\int_{r_0}^{\bar{r}} g(\tilde{r})\ \mathrm{d}\tilde{r}\right] \left(2\,\bar{r} p_t(\bar{r}) - g(\bar{r})\ \rho(\bar{r})\ \bar{r}^2\right)\ \mathrm{d}\bar{r} + r_0^2\ p_r(r_0) \right\}. \tag{36}$$

The lesson now is that to find all possible conserved tensors T^{ab} one is free to specify the functions $\rho(r)$, $p_t(r)$, $g(r)$, and the number $p_r(r_0)$ arbitrarily, and thereby calculate $p_r(r)$ which now yields the full stress-energy tensor T^{ab}. This is now used to calculate $S(r)$. Once $B(r_0)$ is specified the on-brane intrinsic geometry (encoded in $B(r)$) is calculable. *Ipso facto*, this procedure also generates the most general possible candidate for the extrinsic curvature K^{ab} compatible with the assumed symmetries. Thus spherically symmetric solid objects such as planets are not much more difficult to deal with than are fluid objects such as stars.

VI. CONCLUSION

What we have done in this article is to find the most general set of "on-brane data" suitable for characterizing an arbitrary static spherically symmetric braneworld star. This on-brane data is characterized by two arbitrary functions $g(r)$, and $\rho(r)$, and two arbitrary constants $B(r_0)$, and $p(r_0)$. This is an enormous dataset, much more general than the various special cases discussed in [3, 4, 5]. After suitable notational modifications, this boundary data can be used as input into Wiseman's relaxation algorithm for determining the bulk geometry [1]. Although Wiseman's algorithm [1] is general, his numerical examples were restricted to special choices of boundary data. Hopefully, by characterizing the most general set of boundary data, we are providing the ingredients for a more systematic treatment of the numerical problem in future work.

Ultimately, it will be the bulk geometry that determines whether or not a specific set of on-brane data leads to a physically meaningful star [13]. We believe the technique we have developed here could be quite powerful because it is completely algorithmic. Ideally, we would hope that broad statements about the class of possible stellar structure functions $S(r)$ could made. For example, by a suitable characterization of possible pathological singularities in the bulk geometry, it might be possible to rule out particular classes of structure functions.

We conclude that braneworld stars are potentially *much* more complicated than standard general relativistic stars, and emphasise that the coupling to the bulk will play an important role in restricting the possible functions $g(r)$ which enter the Tolman-Oppenheimer-Volkoff-like equation (28).

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Note added: After this work was completed a paper by Bronnikov and Kim appeared [14] which also derives the pure vacuum solution (21)–(24), but in the context of braneworld wormholes. That analysis does not consider the presence of matter, nor does it consider the extrinsic curvature.

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