CLOSURE ALGEBRAS

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ABSTRACT

This thesis investigates the properties and behaviour of closure algebras. Closure algebras generalize the concepts of topological closure, algebraic closure and logical consequence. A closure algebra consists of a unary function $\mathbf{C}$, defined on the power set $\mathbf{P}(X)$ of a given set X, and satisfying three axioms : $A \subseteq \mathbf{C}(A)$, $\mathbf{C}(\mathbf{C}(A)) = \mathbf{C}(A)$, $A \subseteq B \Rightarrow \mathbf{C}(A) \subseteq \mathbf{C}(B)$, for each $A, B \in \mathbf{P}(X)$. These structures are considered along with their dual spaces; topological spaces constructed by generalizing the methods of M.H. Stone in his work on the representations of boolean algebras. A representation theorem for T_1 spaces is obtained.

The notions of subalgebra, homomorphism and congruence are defined for closure algebras, so that the definitions generalize standard usage, and enable analogues of some of the major theorems of Universal Algebra to be proved. Using these definitions and the definition of a closure product, it becomes possible to obtain some detailed results about the structure of the dual spaces.

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INTRODUCTION

The concept of a closure algebra arose more or less independently in a number of areas of mathematics, as a device used to describe some of the structural properties of various systems.

The first publication using closure operations seems to have been Moore [1] in 1910, who used them in the field of Analysis. Birkhoff pioneered the use of closure operations in Lattice Theory and General Algebra in Birkhoff [2]. Independently, Tarski used a "consequence operator", satisfying the closure axioms, to describe the concept of logical consequence, in the course of a lecture to the Polish Mathematical Society in 1928. The lecture was later developed as two published papers, the translations of which appear as articles III, IV in Tarski [1]. In articles V, VII of [1], which were published in 1930, 1936 respectively, Tarski expanded considerably on his initial use of closure operations and established them firmly as a useful tool of Metamathematics. This approach to deductive systems was summarized and continued in a long article by Suszko, Suszko [1]. In [1] Kuratowski demonstrated that a suitable closure operation could be used to characterize the structure of an arbitrary topological space, and his methods have been used extensively since.

The first studies of closure operations, considered as mathematical systems in their own right, were Ore [1], [2]. This work was continued, and the link between closure operations and Galois connections was investigated in Ore [3] and Everett [1].

In the same year, 1944, McKinsey and Tarski published an intensive inquiry into the algebraic structure of a topological space, McKinsey and Tarski [1]. Continuing this line of research McKinsey [1] and McKinsey and Tarski [1], [2] laid the foundations for the use of special closure operations to provide algebraic semantics for some systems of modal logic. This work was extended and developed in Lemmon [1], [2].

A paper containing a particularly important result was Schmidt [1] which appeared in 1952. In this paper Schmidt provided a characterization, in purely set-theoretic terms of the class of finitary closure algebras. An analogous result for the class of cocompact closure algebras was proved in Brown [1] and later appeared in Brown and Suszko [1]. This result has been particularly useful for the work in this thesis. The monograph , Brown and Suszko [1], combines closure algebras and Category Theory to produce some interesting metamathematical results.

The point of view of this thesis is that closure algebras are deserving of more study as mathematical systems than has been accorded them in the past. In chapter I the basic definitions and the required results are laid down; some new results are obtained on concepts related to cocompactness. Chapter II contains an analysis of the behaviour of the maximal consistent sets of a closure algebra. Chapter III is concerned with a special type of dual space. The dual spaces of closure algebras have been investigated in the papers by Ore and Everett. However my definition is different from theirs, and has the advantage of always yielding a topological space as the dual and not just a second closure algebra. Indeed this dual space is always T_1, and in chapter IV I show that every T_1 space may be represented as the dual space on a closure algebra. In chapter V some of the standard notions of Universal Algebra are modified so as to apply to closure algebras, and with these definitions it is proved that analogues of many basic algebraic results hold. Chapter VI is devoted to two methods of constructing new closure algebras from given ones, and to comparing the structures of the new dual spaces with the structures of the dual spaces on the given closure algebras.

CHAPTER I : CLOSURE OPERATIONS

This chapter is a summary of the basic facts about closure operations and closure systems. Sections 1.1 to 1.5 give the initial definitions, and sections 1.6 to 1.8 establish the connections between closure operations and closure systems.

1.1 Definition. Let X be a fixed set, and $\mathbf{C}$ a function on $\mathbf{P}(X)$, the power set of X. $\mathbf{C}$ is said to be a closure operation on X whenever $\mathbf{C}$ is increasing, idempotent and order preserving ; i.e. $\mathbf{C} : \mathbf{P}(X) \to \mathbf{P}(X)$ and :

I $\quad A \subseteq \mathbf{C}(A)$,

II $\quad \mathbf{C}(\mathbf{C}(A)) = \mathbf{C}(A)$,

III $\quad A \subseteq B \Rightarrow \mathbf{C}(A) \subseteq \mathbf{C}(B)$, $\quad A, B \in \mathbf{P}(X)$.[1]

1.2 Definition. If $\mathbf{C}$ is a closure operation on X, then the images, under $\mathbf{C}$, of members of $\mathbf{P}(X)$ are called closed sets.

1.3 Theorem. If $\mathbf{C}$ is a closure operation on X then for every $A \in \mathbf{P}(X)$, A is closed iff $A = \mathbf{C}(A)$.

[1] A slight gain in generality can be made by defining the function $\mathbf{C}$ on any ordered set, rather than on the power set of some given set. In this case the set inclusion symbol of Definition 1.1 is replaced by the non-strict form of the ordering symbol, and the definition proceeds as above. For my purposes the given definition is sufficiently general.

Proof. If A is closed then

$$A = \mathbf{C}(B), \text{ for some } B \in \mathbf{P}(X), \text{ by 1.2;}$$

so that $\mathbf{C}(A) = \mathbf{C}(\mathbf{C}(B))$.

However by 1.1 (II), $\mathbf{C}(\mathbf{C}(B)) = \mathbf{C}(B)$ and hence

$$A = \mathbf{C}(B) = \mathbf{C}(A).$$

Conversely, if $A = \mathbf{C}(A)$ then A is closed by 1.2.

Because of 1.1 (I) we have as a corollory that A is closed iff $A \supseteq \mathbf{C}(A)$. Since $X \supseteq \mathbf{C}(X)$, we will always have that X is closed.

1.4 Properties of closure operations

In addition to the properties listed in 1.1, $\mathbf{C}$ may satisfy one or more of the following :

IV $\mathbf{C}(A \cup B) = \mathbf{C}(A) \cup \mathbf{C}(B)$ - additivity

V $\mathbf{C}(\{x\})$[1] $= \{x\}$ - closure of singletons

VI $x \neq y \Rightarrow \mathbf{C}(x) \neq \mathbf{C}(y)$ - determination of points by closures.

VII $\mathbf{C}(\phi) = \phi$ - closure of null set.

1.5 Definition. If $\mathbf{C}$ is a closure operation on X, then the pair $(X, \mathbf{C})$ will be referred to as a closure algebra.

1.6 Definition. If X is a fixed set and $Y \subseteq \mathbf{P}(X)$ is closed under arbitrary intersections then Y is called a closure system.

[1] From now on I write $\mathbf{C}(x)$ for $\mathbf{C}(\{x\})$.

In theorem 1.7 it is shown that the family of closed subsets of a closure algebra is a closure system; and in the following theorem that any closure system generates a closure algebra by associating with each set $A \in \mathbf{P}(X)$ the smallest member of the closure system that contains A. Both these results are due to E.H. Moore [1].

<u>1.7 Theorem.</u> If $(X, \mathbf{C})$ is a closure algebra then

$$Y = \{\mathbf{C}(A) : A \in \mathbf{P}(X)\}$$

is a closure system.

Proof.

Choose $\alpha \subseteq Y$. For each $B \in \alpha$ we have

$$\bigcap_{A\in\alpha} A \subseteq B$$

$$\Rightarrow \quad \mathbf{C}(\bigcap_{A\in\alpha} A) \subseteq \mathbf{C}(B), \text{ by 1.1 (III)}$$

$$\Rightarrow \quad \mathbf{C}(\bigcap_{A\in\alpha} A) \subseteq B. \quad \text{by 1.3 and since } B \in Y.$$

Since this holds for each $B \in \alpha$ we have

$$\mathbf{C}(\bigcap_{A\in\alpha} A) \subseteq \bigcap_{B\in\alpha} B = \bigcap_{A\in\alpha} A ;$$

because $\bigcap_{A\in\alpha} A \in \mathbf{P}(X)$, the definition of Y yields that

$$\bigcap_{A\in\alpha} A \in Y, \text{ so that } Y \text{ is a closure system.}$$

1.8 Theorem. Let $Y \subseteq \mathbf{P}(X)$ be a closure system. Define $\mathbf{C} : \mathbf{P}(X) \to \mathbf{P}(X)$ by

$$\mathbf{C}(A) = \bigcap_{\substack{B \in Y \\ B \supseteq A}} B, \text{ for each } A \in \mathbf{P}(X),$$

then $(X, \mathbf{C})$ is a closure algebra.

Proof. (i) Since each $B \supseteq A$, $\mathbf{C}(A) = \bigcap_{\substack{B \in Y \\ B \supseteq A}} B \supseteq A$.

(ii) Since Y is closed with respect to arbitrary intersections,

$$\mathbf{C}(A) = \bigcap_{\substack{B \in Y \\ B \supseteq A}} B \in Y$$

$$\Rightarrow \quad \mathbf{C}(A) \in \{D \in Y : D \supseteq \mathbf{C}(A)\}.$$

$$\Rightarrow \quad \mathbf{C}(A) \supseteq \bigcap_{\substack{D \in Y \\ D \supseteq \mathbf{C}(A)}} D = \mathbf{C}(\mathbf{C}(A)).$$

By part (i), $\mathbf{C}(A) \subseteq \mathbf{C}(\mathbf{C}(A))$ so that

$$\mathbf{C}(A) = \mathbf{C}(\mathbf{C}(A)).$$

(iii) Assume $A \subseteq B$, $A, B \in \mathbf{P}(X)$, then

$$\{D \in Y : D \supseteq A\} \supseteq \{D \in Y : D \supseteq B\}$$

$$\Rightarrow \quad \bigcap_{\substack{D \in Y \\ D \supseteq A}} D \subseteq \bigcap_{\substack{D \in Y \\ D \supseteq B}} D$$

$$\Rightarrow \quad \mathbf{C}(A) \subseteq \mathbf{C}(B).$$

This establishes the three properties of definition 1.1 and shows $\mathbf{C}$ to be a closure operation.

The next two theorems show that we have stability with respect to the constructions of theorems 1.7 and 1.8.

<u>1.9 Theorem.</u> Let $(X, \mathbf{C})$ be a closure algebra and let Y be the closure system formed as in 1.7; let $\mathbf{C}_1$ be the closure operation formed from Y as in 1.8, then $\mathbf{C} = \mathbf{C}_1$.

Proof. Let $A \in \mathbf{P}(X)$, then $\mathbf{C}(A) \in Y$ and $A \subseteq \mathbf{C}(A)$. Hence $\mathbf{C}(A) \in \{B : B \in Y, B \supseteq A\}$, yielding that

$$\mathbf{C}(A) \supseteq \bigcap_{\substack{B \in Y \\ B \supseteq A}} B = \mathbf{C}_1(A).$$

On the other hand, if $B \in Y$ and $B \supseteq A$ then $B = \mathbf{C}(B)$, by the definition of Y, and 1.3, and $\mathbf{C}(B) \supseteq \mathbf{C}(A)$ by definition 1.1 (III), consequently $B \supseteq \mathbf{C}(A)$. This holds for each B with $B \in Y$ and $B \supseteq A$, so that :

$$\mathbf{C}_1(A) = \bigcap_{\substack{B \in Y \\ B \supseteq A}} B \supseteq \mathbf{C}(A).$$

This shows that $\mathbf{C}_1(A) = \mathbf{C}(A)$, $A \in \mathbf{P}(X)$, demonstrating that $\mathbf{C}_1 = \mathbf{C}$.

<u>1.10 Theorem.</u> Let $Y \subseteq \mathbf{P}(X)$ be a closure system and let $\mathbf{C}$ be the closure operation constructed from Y as in 1.8. Let $Z = \{\mathbf{C}(A) : A \in \mathbf{P}(X)\}$ as in 1.7, then $Y = Z$.

Proof. If $z \in Z$ then $z = \mathbf{C}(A)$, for some $A \in \mathbf{P}(X)$, and by the definition of $\mathbf{C}$, $z = \bigcap_{\substack{B \in Y \\ B \supseteq A}} B$.

Since Y is closed under arbitrary intersections, $z \in Y$, so that $Z \subseteq Y$.

On the other hand, if $y \in Y$, then

$$y = \bigcap_{\substack{B \in Y \\ B \supseteq y}} B = \mathbf{C}(y).$$

Now $y \in \mathbf{P}(X)$ so that $\mathbf{C}(y) \in Z$; i.e. $y \in Z$. Hence $Y \subseteq Z$ and we may conclude $Y = Z$.

The importance of the last four theorems is that they show that it does not make much difference whether we talk about closure algebras or closure systems; since each closure algebra carries with it its unique closure system, and vice versa. Given a closure algebra (system) I will refer to the closure system (algebra) arising from the construction of theorem 1.7 (1.8) as the associated closure system (algebra).

We turn now to some of the most familiar examples of closure algebras along with their associated closure systems.

1.11 Examples

(i) Let X be the carrier of any universal algebra. Define $\mathbf{C}$ on $\mathbf{P}(X)$ by taking $\mathbf{C}(A)$ to be the subalgebra generated by A. Then $(X, \mathbf{C})$ is a closure algebra, and the associated closure system is the lattice of subalgebras.

(ii) Let X be a lattice (or more particularly, a Boolean algebra). Define $\mathbf{C}$ by taking $\mathbf{C}(A)$ to be the smallest filter containing A, for each $A \in \mathbf{P}(X)$. $(X,\mathbf{C})$ is a closure algebra, and the associated closure system is the family of all filters of X.

(iii) Let X be a topological space. Define $\mathbf{C}$ to be the Kuratowski closure operator that characterizes the topology. $\mathbf{C}$ is a closure operation that also satisfies IV and VII of 1.4, and in the case of a T_1 space, satisfies V also. The associated closure system is the family of topologically closed sets.

(iv) Let X be the set of formulas of a logic. Define $\mathbf{C}$ by taking $\mathbf{C}(A)$ to be the set of all logical consequences of A, i.e. augment A by the axioms of the logic and make $\mathbf{C}(A)$ the set of all formulas that are derivable from this augmented set using the given rules of inference. The associated closure system is the set of all theories of the logic.

(v) Let X,Y be two sets and $f : \mathbf{P}(X) \to \mathbf{P}(Y)$, $g : \mathbf{P}(Y) \to \mathbf{P}(X)$ be two mappings such that

$$A \subseteq B \Rightarrow f(B) \subseteq f(A)$$

$$C \subseteq D \Rightarrow g(D) \subseteq g(C)$$

$$A \subseteq gof(A),\ C \subseteq gof(C) \qquad A,\ B \in \mathbf{P}(X), C, D \in \mathbf{P}(Y).$$

The pair of mappings f, g constitute what is called a Galois connection, and the operators gof, fog are closure operations on X,Y respectively.

<u>1.12 Definition</u>. If $\mathbf{C}_1$, $\mathbf{C}_2$ are two closure operations on a set X then $\mathbf{C}_2$ is said to contain $\mathbf{C}_1$ ($\mathbf{C}_1 \subseteq \mathbf{C}_2$) iff $\mathbf{C}_1(A) \subseteq \mathbf{C}_2(A)$, for each $A \in \mathbf{P}(X)$.

The next theorem is sometimes useful in establishing the partial ordering [1] of definition 1.12.

<u>1.13 Theorem (O. Ore [2])</u>. If $\mathbf{C}_1$, $\mathbf{C}_2$ are two closure operations on a set X then $\mathbf{C}_1 \subseteq \mathbf{C}_2$ iff

$$\mathbf{C}_2(A) = \mathbf{C}_1(\mathbf{C}_2(A)), \text{ for each } A \in \mathbf{P}(X)$$

Proof. If $\mathbf{C}_1 \subseteq \mathbf{C}_2$ then $\mathbf{C}_1(\mathbf{C}_2(A)) \subseteq \mathbf{C}_2(\mathbf{C}_2(A))$ for each $A \in \mathbf{P}(X)$, by 1.12, so that $\mathbf{C}_1(\mathbf{C}_2(A)) \subseteq \mathbf{C}_2(A)$ by 1.1 II. Furthermore, $\mathbf{C}_1(\mathbf{C}_2(A)) \supseteq \mathbf{C}_2(A)$ since $\mathbf{C}_1$ is a closure operation and satisfies 1.1 I.
Hence $\mathbf{C}_1(\mathbf{C}_2(A)) = \mathbf{C}_2(A)$ for each $A \in \mathbf{P}(X)$.

Conversely, if $\mathbf{C}_1(\mathbf{C}_2(A)) = \mathbf{C}_2(A)$ for each $A \in \mathbf{P}(X)$ then $\mathbf{C}_1 \subseteq \mathbf{C}_2$ because for each $A \in \mathbf{P}(X)$:

$$\begin{aligned} & A \subseteq \mathbf{C}_2(A) && \text{- by 1.1 I} \\ \Rightarrow\ & \mathbf{C}_1(A) \subseteq \mathbf{C}_1(\mathbf{C}_2(A)) && \text{- by 1.1 III} \\ \Rightarrow\ & \mathbf{C}_1(A) \subseteq \mathbf{C}_2(A) && \text{- by assumption} \end{aligned}$$

In many of the remaining major theorems in this chapter, results are obtained by applying the axiom of choice,

[1] G. Birkhoff [1] has shown that the family of all closure operations on a set forms a complete lattice.

in the form of Zorn's lemma, to various subsets of $\mathbf{P}(X)$, for X the carrier of some closure algebra $(X,\mathbf{C})$. Definitions 1.14 to 1.16 set up the technical apparatus for the easy application of Zorn's lemma.

1.14 Definition. A family of sets in which every finite subfamily has an upper bound[1] is said to be directed (upwards).

1.15 Definition. A family of sets is said to be inductive iff every directed subfamily has its least upper bound in the family.

1.16 Theorem (Zorn's lemma). Every non-empty inductive family of sets has a maximal element.

In some of the following results it turns out to be more convenient to deal with chains of sets; i.e. families of sets linearly ordered by inclusion, than with directed families. For our purposes the two concepts are equivalent, since a family of sets is inductive iff every chain in the family has a least upper bound in the family. A proof of this standard result is given in P. Cohn [1] (pp.33-34). In proofs depending on inductive families of sets, either chains or directed subfamilies will be used

[1] The ordering throughout these sections is by set inclusion.

without comment.

We turn now to one of the most important classes of closure algebras, those that are finitary. To assist with the expression of this idea, I introduce some notation for the finite subsets of a set. If A is a set I write A_f for an arbitrary, finite subset of A; I take $f \in F_A$ to index the family of finite subsets of A (dropping the subscript in "F_A" when there is no ambiguity).

1.17 Definition. A closure operation $\mathbf{C}$, on a set X is said to be finitary iff for each $A \in \mathbf{P}(X)$, whenever x is in $\mathbf{C}(A)$ then x is in the closure of some finite subset of A. A closure algebra $(X, \mathbf{C})$ is said to be finitary whenever $\mathbf{C}$ is a finitary closure operation. In the above notation, $(X, \mathbf{C})$ is finitary iff

$$\mathbf{C}(A) = \bigcup_{f \in F_A} \mathbf{C}(A_f), \text{ for each } A \in \mathbf{P}(X).$$

In the examples given in 1.11; (i), (ii), (iv) are all examples of finitary closure algebras. Indeed because of the importance of this condition for the algebras of example (i), the name "finitary" is often replaced by "algebraic".

1.18 Theorem (Schmidt [1]). A closure algebra $(X, \mathbf{C})$ is finitary iff the associated closure system is inductive.

Proof. Let $(X, \mathbf{C})$ be finitary and suppose that $\{\alpha_t\}_{t \in T}$ is a chain of closed sets with $A = \bigcup_{t \in T} \alpha_t$; and that A_f is any finite subset of A, so that $A_f \subseteq \bigcup_{t \in T} \alpha_t$.

Now for each $x \in A_f$, there is some $t_x \in T$ such that $x \in \alpha_{t_x}$. Letting $T' = \{t_x : x \in A_f\} \subseteq T$, we have that $A_f \subseteq \bigcup_{t \in T'} \alpha_t$.

Since A_f is finite, T' is finite; hence $\bigcup_{t \in T'} \alpha_t$ is αt_0 for some $t_0 \in T$, since $\{\alpha_t\}_{t \in T}$ is a chain.

This yields that $A_f \subseteq \alpha t_0$.

$$\Rightarrow \quad \mathbf{C}(A_f) \subseteq \mathbf{C}(\alpha_{t_0}) \quad \text{by 1.1 III}$$

$$\Rightarrow \quad \mathbf{C}(A_f) \subseteq \alpha t_0 \quad \text{, since } \alpha_{t_0} \text{ is closed.}$$

Now, $\mathbf{C}(A) = \bigcup_{f \in F_A} \mathbf{C}(A_f)$, since $(X, \mathbf{C})$ is finitary

$\subseteq \bigcup_{t \in T} \alpha_t$, since by the foregoing, every $\mathbf{C}(A_f)$ is contained in some α_{t_0}, $t_0 \in T$. But $\bigcup_{t \in T} \alpha_t = A$, so that $\mathbf{C}(A) \subseteq A$ and hence A is closed. This shows the closure system to be inductive.

Conversely, assume the closure system to be inductive and take $A \in \mathbf{P}(X)$. Put $\Gamma = \{\mathbf{C}(A_f) : f \in F_A\}$. Now if $\mathbf{C}(A_{f_1}), \ldots, \mathbf{C}(A_{f_n}) \in \Gamma$ then $\bigcup_{i=1}^{n} A_{f_i} \subseteq A$; furthermore, since $A_{f_1}, \ldots, A_{f_n}$ are finite, $\bigcup_{i=1}^{n} A_{f_i}$ is finite.

Hence $\mathbf{C}(\bigcup_{i=1}^{n} A_{f_i}) \in \Gamma$, and is an upper bound for $\mathbf{C}(A_{f_1}),\ldots,\mathbf{C}(A_{f_n})$; since if $A_{f_j} \subseteq \bigcup_{i=1}^{n} A_{f_i}$ then $\mathbf{C}(A_{f_j}) \subseteq \mathbf{C}(\bigcup_{i=1}^{n} A_{f_i})$. That is to say Γ is a directed subfamily of an inductive family, and hence has a least upper bound, $\bigcup_{f \in F_A} \mathbf{C}(A_f)$, in the closure system.

Since $\mathbf{C}(A)$ is an upper bound for Γ, we have

$$\bigcup_{f \in F_A} \mathbf{C}(A_f) \subseteq \mathbf{C}(A).$$

On the other hand : $A_f \subseteq \mathbf{C}(A_f)$, $f \in F_A$, by 1.1 I.

$$\Rightarrow \quad A = \bigcup_{f \in F_A} A_f \subseteq \bigcup_{f \in F_A} \mathbf{C}(A_f)$$

$$\Rightarrow \quad \mathbf{C}(A) \subseteq \mathbf{C}(\bigcup_{f \in F_A} \mathbf{C}(A_f)) = \bigcup_{f \in F_A} \mathbf{C}(A_f)$$

since by the above $\bigcup_{f \in F_A} \mathbf{C}(A_f)$ is closed.

Hence $\mathbf{C}(A) = \bigcup_{f \in F_A} \mathbf{C}(A_f)$.

Since A was arbitrary this establishes that $(X, \mathbf{C})$ is finitary.

A second class of closure algebras that can be characterized in terms of the inductiveness of a family of subsets of the individual carriers, has been given by Brown in [1] ; the next sections develop the concepts necessary to formulate Brown's result. The definition of consistency is due to Tarski [1], and the term cocompact was suggested by R.A. Bull.

<u>1.19 Definition.</u> A set $A \in \mathbf{P}(X)$ is said to be consistent with respect to a closure operation $\mathbf{C}$, defined on X, iff $\mathbf{C}(A) \neq X$; otherwise A is said to be inconsistent with respect to $\mathbf{C}$. Where the closure operation $\mathbf{C}$ is understood, the words "with respect to $\mathbf{C}$" will be omitted.

<u>1.20 Definition.</u> A closure algebra $(X, \mathbf{C})$ is said to be cocompact iff every inconsistent set has a finite, inconsistent subset.

<u>1.21 Theorem (Brown [1]).</u> A closure algebra $(X, \mathbf{C})$ is cocompact iff the family of all its consistent sets is inductive.

Proof. Assume that $(X, \mathbf{C})$ is cocompact, and that $\{\alpha_t\}_{t\in T}$ is a chain of consistent subsets of X. It has to be shown that $\bigcup_{t\in T} \alpha_t$ is consistent. Assuming otherwise, if $\bigcup_{t\in T}\alpha_t$ is inconsistent, then there exists a set Y, $Y \subseteq \bigcup_{t\in T}\alpha_t$, Y finite and Y inconsistent, by the cocompactness of $(X,\mathbf{C})$.

Considering Y; for each $y \in Y$ there is some $t_y \in T$ such that $y \in \alpha_{t_y}$.

Hence $Y \subseteq \bigcup_{t\in T'} \alpha_t$ where $T' = \{t_y : y \in Y\}$.

Since Y is finite, T' is finite; and because $\{\alpha_t\}_{t\in T}$ is a chain, there is some $t_0 \in T$ such that $Y = \bigcup_{t\in T'} \alpha_t \subseteq \alpha_{t_0}$. But then $X = \mathbf{C}(Y) \subseteq \mathbf{C}(\alpha t_0)$, contradicting the consistency of α_{t_0} . Hence $\bigcup_{t\in T} \alpha_t$ is consistent, and the family of consistent sets is inductive.

Conversely, suppose that $(X, \mathbf{C})$ is a closure algebra with an inductive family of consistent sets. Consider any $A \subseteq X$; it will be shown that if $\mathbf{C}(A_f) \neq X$, for each $f \in F_A$ then $\mathbf{C}(A) \neq X$.

Let $\Gamma = \{\mathbf{C}(A_f) : f \in F_A\}$. Then as in the proof of 1.18, Γ is a directed subfamily of an inductive family, and so $\bigcup_{f\in F_A} \mathbf{C}(A_f)$ is consistent.

Now $A \subseteq \bigcup_{f\in F_A} A_f \subseteq \bigcup_{f\in F_A} \mathbf{C}(A_f)$, since each $A_f \subseteq \mathbf{C}(A_f)$, by 1.1 III.

So that $\mathbf{C}(A) \subseteq \mathbf{C}(\bigcup_{f\in F_A} \mathbf{C}(A_f)) \neq X$, since $\bigcup_{f\in F_A} \mathbf{C}(A_f)$ is consistent.

Hence $(X,\mathbf{C})$ is cocompact.

The next few sections connect the notion of cocompactness with two other properties.

<u>1.22 Definition</u>. Let $(X, \mathbf{C})$ be a closure algebra. If every consistent set can be extended to a maximal consistent set then $(X, \mathbf{C})$ is said to have the Lindenbaum property.

<u>1.23 Definition.</u> Let $(X, \mathbf{C})$ be a closure algebra. If for each $A \in \mathbf{P}(X)$, A can be extended to a maximal consistent set whenever every finite subset of A can be extended to a maximal consistent set, then $(X, \mathbf{C})$ is said to have the finite extension property (f.e.p.)

<u>1.24 Theorem.</u> If $(X, \mathbf{C})$ is a closure algebra then $(X,\mathbf{C})$ has the f.e.p. iff the family of all subsets of maximal consistent sets is inductive.

Proof. Assume that $(X, \mathbf{C})$ has the f.e.p. and let $\mathcal{H}$ be the family of all subsets of maximal consistent sets, with $\{\alpha_t\}_{t\in T}$ a chain in $\mathcal{H}$. Consider $\bigcup_{t\in T}\alpha_t$; if this is not a member of $\mathcal{H}$ then $\bigcup_{t\in T}\alpha_t$ cannot be extended to a maximal consistent set. By the f.e.p. $\bigcup_{t\in T}\alpha_t$ has a finite subset Y that cannot be extended to a maximal consistent set.

For each $y \in Y$, there is some $t_y \in T$ such that $y \in \alpha_{t_y}$. Put $T' = \{t_y : y \in Y\}$. Since Y is finite, T' is finite so that

$$Y \subseteq \bigcup_{t\in T'} \alpha_t = \alpha_{t_0} \text{ for some } t_0 \in T, \text{ since } \{\alpha_t\}_{t\in T}$$

is a chain. But then α_{t_0} cannot be extended to a maximal consistent set, contradicting $\alpha_{t_0} \in \mathcal{H}$. Hence $\bigcup_{t\in T}\alpha_t \in \mathcal{H}$, and $\mathcal{H}$ is inductive.

Conversely, assume $\mathcal{H}$ is inductive and choose $A \in \mathbf{P}(X)$ with every finite subset of A contained in a maximal consistent set. Let $f \in F$ index the finite subsets of A. Then $\{A_f\}_{f\in F} \subseteq \mathcal{H}$, and $\{A_f\}_{f\in F}$ is a directed subfamily since it is closed under finite unions. Since $\mathcal{H}$ is inductive we have that $\bigcup_{f\in F} A_f \in \mathcal{H}$, which is to say that $A = \bigcup_{f\in F} A_f$ can be extended to a maximal consistent set. Hence $(X,\mathbf{C})$ has the f.e.p.

One of the principal interests in cocompact closure algebras derives from the following theorem, the earliest version of which Tarski ([1], p.98) ascribes to Lindenbaum.

<u>1.25 Theorem (Lindenbaum)</u>. If $(X,\mathbf{C})$ is a cocompact closure algebra then $(X,\mathbf{C})$ has the Lindenbaum property.

Proof. Let $(X,\mathbf{C})$ be cocompact and $A \in \mathbf{P}(X)$ be consistent. Let $\mathcal{H}$ denote the family of all consistent members of $\mathbf{P}(X)$ that contain A. Since $A \in \mathcal{H}$, $\mathcal{H} \neq \phi$. Let $\{\alpha_t\}_{t\in T}$ be any chain in $\mathcal{H}$, then $\bigcup_{t\in T} \alpha_t$ is consistent, since by cocompactness and theorem 1.21 the family of consistent sets is inductive. Furthermore, since $A \subseteq \alpha_t$ for each $t \in T$, $A \subseteq \bigcup_{t\in T} \alpha_t$ so that $\bigcup_{t\in T} \alpha_t \in \mathcal{H}$ and $\mathcal{H}$ is also inductive.

Since $\mathcal{H}$ is a non-empty, inductive family of sets, we have by Zorn's lemma (1.16) that $\mathcal{H}$ has a maximal element Δ. Δ is maximal consistent in $(X,\mathbf{C})$ because,

$\xi \in \mathbf{P}(X)$ and ξ is consistent and $\Delta \subseteq \xi$

$\Rightarrow$ ξ is consistent and $A \subseteq \xi$ and $\Delta \subseteq \xi$

$\Rightarrow$ $\xi \in \mathcal{H}$ and $\Delta \subseteq \xi$

$\Rightarrow$ $\Delta = \xi$, since Δ is maximal in $\mathcal{H}$. Hence Δ is maximal consistent and $A \subseteq \Delta$ so that $(X,\mathbf{C})$ has the Lindenbaum property.

<u>1.26 Theorem</u>. If (X, $\mathbf{C}$) is a closure algebra, then (X,$\mathbf{C}$) is cocompact iff (X, $\mathbf{C}$) has the Lindenbaum property and the f.e.p.

Proof. If (X,$\mathbf{C}$) is cocompact then (X, $\mathbf{C}$) has the Lidenbaum property by 1.25 . To prove the f.e.p., assume that $A \in \mathbf{P}(X)$ is such that every finite subset of A is extendable to a maximal consistent set. Then each finite subset of A is consistent. Since (X, $\mathbf{C}$) is cocompact A is consistent and by the Lindenbaum property can be extended to a maximal consistent set.

Conversely, assume that (X, $\mathbf{C}$) has the Lindenbaum property and the f.e.p.. Choose $A \in \mathbf{P}(X)$ to be inconsistent; then A cannot be extended to a maximal consistent set, and so by the f.e.p. there is a finite subset of A that cannot be extended to a maximal consistent set. By the Lindenbaum property this finite subset of A is inconsistent, so that (X, $\mathbf{C}$) is cocompact.

The next three results will be used later on in the thesis.

<u>1.27 Lemma</u>. If (X, $\mathbf{C}$) is a closure algebra then every maximal consistent member of $\mathbf{P}(X)$ is a closed member of $\mathbf{P}(X)$.
Proof. Let Δ be a maximal consistent, then

$\mathbf{C}(\Delta) \neq X$; hence by 1.1 II $\mathbf{C}(\mathbf{C}(\Delta)) \neq X$, so that $\mathbf{C}(\Delta)$ is also consistent. But $\Delta \subseteq \mathbf{C}(\Delta)$, so by the maximality of Δ we must have $\Delta = \mathbf{C}(\Delta)$, i.e. Δ is a closed set.

<u>1.28 Lemma</u>. If $(X, \mathbf{C})$ is a closure algebra and $\{\alpha_t\}_{t\in T}$ is a chain of subsets of X, then $\{\mathbf{C}(\alpha_t)\}_{t\in T}$ is also a chain of subsets of X.

Proof. If $\mathbf{C}(\alpha_{t_1}) \nsubseteq \mathbf{C}(\alpha_{t_2})$ then $\alpha_{t_1} \nsubseteq \alpha_{t_2}$, by 1.1 III, and hence $\alpha t_2 \subseteq \alpha_{t_1}$, since $\{\alpha_t\}_{t\in T}$ is a chain. But now $\mathbf{C}(\alpha_{t_2}) \subseteq \mathbf{C}(\alpha_{t_1})$. Hence $\{\mathbf{C}(\alpha_t)\}_{t\in T}$ is a chain.

<u>1.29 Theorem (Tarski [1][1])</u>. If $(X, \mathbf{C})$ is a finitary closure algebra and $\{\alpha_t\}_{t\in T}$ is a chain of subsets of X, then $\bigcup_{t\in T} \mathbf{C}(\alpha_t) = \mathbf{C}(\bigcup_{t\in T} \alpha_t)$.

Proof. Since $(X, \mathbf{C})$ is finitary we have by 1.18 that the associated closure system is inductive. Now since $\{\alpha_t\}_{t\in T}$ is a chain we have by 1.28 that $\{\mathbf{C}(\alpha_t)\}_{t\in T}$ is a chain of closed sets. By the inductiveness of the closure system,

$\bigcup_{t\in T} \mathbf{C}(\alpha_t)$ is closed, and :

$$\alpha_t \subseteq \mathbf{C}(\alpha_t) \text{ for each } t \quad T, \text{ by 1.1 I}$$

$$\Rightarrow \bigcup_{t\in T} \alpha_t \subseteq \bigcup_{t\in T} \mathbf{C}(\alpha_t).$$

$$\Rightarrow \mathbf{C}(\bigcup_{t\in T} \alpha_t) \subseteq \mathbf{C}(\bigcup_{t\in T} \mathbf{C}(\alpha_t)), \text{ by 1.1 III.}$$

$$\Rightarrow \mathbf{C}(\bigcup_{t\in T} \alpha_t) \subseteq \bigcup_{t\in T} \mathbf{C}(\alpha_t), \text{ since } \bigcup_{t\in T} \mathbf{C}(\alpha_t) \text{ is closed.}$$

- yielding one containment. For the other containment :

[1] "Fundamental Concepts of the Methodology of the Deductive Sciences"; originally appeared (in German) in Monatshefte für Mathematik und Physik vol.37(1930)pp361-404.

$$\alpha_t \subseteq \bigcup_{t\in T} \alpha_t, \text{ for each } t \in T.$$

$$\Rightarrow \quad \mathbf{C}(\alpha_t) \subseteq \mathbf{C}(\bigcup_{t\in T}\alpha_t), \text{ for each } t \in T.$$

$$\Rightarrow \quad \bigcup_{t\in T}\mathbf{C}(\alpha_t) \subseteq \mathbf{C}(\bigcup_{t\in T}\alpha_t).$$

Hence $\bigcup_{t\in T} \mathbf{C}(\alpha_t) = \mathbf{C}(\bigcup_{t\in T}\alpha_t)$.

We now move on to define finitely generated closure algebras, and then to establish the relationships that hold between finitely generated, finitary and cocompact closure algebras.

<u>1.30 Definition</u>. A closure algebra $(X, \mathbf{C})$ is said to be finitely generated iff $\mathbf{C}(X_f) = X$ for some finite $X_f \in \mathbf{P}(X)$.

<u>1.31 Theorem</u>. If $(X, \mathbf{C})$ is a cocompact closure algebra then $(X, \mathbf{C})$ is finitely generated.

Proof. Since $\mathbf{C}(X) = X$ we have that X is inconsistent. By cocompactness, there is some $X_f \subseteq X$, X_f finite with $\mathbf{C}(X_f) = X$. Hence $(X, \mathbf{C})$ is finitely generated with generating set X_f.

<u>1.32 Theorem</u>. If $(X, \mathbf{C})$ is a finitary, finitely generated closure algebra, then $(X, \mathbf{C})$ is cocompact.

Proof. Let $\{\alpha_t\}_{t\in T}$ be any chain of consistent sets. By 1.28 $\{\mathbf{C}(\alpha_t)\}_{t\in T}$ is also a chain of sets, and for each $t \in T$

$\mathbf{C}(\alpha_t)$ is consistent, since

$$\mathbf{C}(\mathbf{C}(\alpha_t)) = \mathbf{C}(\alpha_t) \neq X.$$

Hence $\{\mathbf{C}(\alpha_t)\}_{t\in T}$ is a chain of closed consistent sets. If we assume that $\{\alpha_t\}_{t\in T}$ does not have a consistent least upper bound, then $\mathbf{C}(\bigcup_{t\in T}\alpha_t) = X$, and $\bigcup_{t\in T}\mathbf{C}(\alpha_t) = X$, by 1.29.

Since X is finitely generated there is some $X_f \in \mathbf{P}(X)$, such that $\mathbf{C}(X_f) = X$ and X_f is finite. Now we must have $X_f \subseteq \bigcup_{t\in T}\mathbf{C}(\alpha_t)$, so that for each $x \in X_f$ we may choose $t_x \in T$ such that $x \in \mathbf{C}(\alpha_{t_x})$.

Put $T' = \{t_x : x \in X_f\}$; since X_f is finite, T' is finite and $X_f \subseteq \bigcup_{t\in T'}\mathbf{C}(\alpha_t) \subseteq \mathbf{C}(\alpha_{t_0})$ for some $t_0 \in T$, since T' is finite and $\{\mathbf{C}(\alpha_t)\}_{t\in T}$ is a chain.

But now
$$\begin{aligned} X &= \mathbf{C}(X_f) \\ &\subseteq \mathbf{C}(\bigcup_{t\in T'}\mathbf{C}(\alpha_t) \\ &\subseteq \mathbf{C}(\mathbf{C}(\alpha_{t_0}), \text{ for some } t_0 \in T \\ &= \mathbf{C}(\alpha_{t_0}) \text{ by 1.1 II.} \end{aligned}$$

Contradicting the consistency of α_{t_0}. Hence $\{\alpha_t\}_{t\in T}$ does have a consistent least upper bound, the family of consistent sets is inductive, and $(X, \mathbf{C})$ is cocompact by 1.21.

I do not know the origin of theorem 1.32. It has occurrences in various forms in the literature, and was suggested to me by a proposition set as a problem in Cohn [1],p.86.

CHAPTER II : MAXIMAL CONSISTENT SETS

In the latter part of chapter I, two important points about cocompact closure algebras emerged. Firstly, since every finitary, finitely-generated closure algebra is cocompact, we have that the closure algebra corresponding to any universal algebra (in the sense of example 1.11(i)) is cocompact whenever the universal algebra is finitely generated. On the side of mathematical logic there are a number of standard results which show that the closure algebras corresponding with logics (in the sense of example 1.11 (iv)) will be in many cases cocompact. These two points guarantee that the class of cocompact closure algebras is large enough to be of interest.

Secondly, in a cocompact closure algebra we may extend any consistent set to a maximal consistent set; this assures us that there is a reasonably rich class of maximal consistent sets in any non-trivial, cocompact closure algebra.

Chapter II commences by examining some of the behaviour of maximal consistent sets in a cocompact closure algebra, and then goes on to define a second closure operation in terms of the intersections of maximal consistent sets, in 2.3 and 2.4. The structure of the resulting closure algebra is compared with the structure of the original closure algebra in 2.5 to 2.8.

In the latter half of the chapter a notion of relative consistency (motivated from logic) is set up and investigated.

It is shown in 2.13 to be closely linked with the closure operation that was defined in the first half of the chapter, whenever this closure operation is defined on a co-compact or finitary closure algebra. In 2.14 a notion of absolute consistency is defined, and in 2.15 this definition is used to generalize a well known result of group theory.

<u>2.1 Definition</u>. Let $(X, \mathbf{C})$ be a closure algebra and let $\mathbb{M}_X$ denote the family of all maximal consistent subsets of X. Define a function $\mathbf{S} : \mathbf{P}(X) \to \mathbf{P}(\mathbb{M}_X)$ by :

$$\mathbf{S}(A) = \{\Delta \in \mathbb{M}_X : A \subseteq \Delta\}, \text{ for each } A \in \mathbf{P}(X).$$

<u>2.2 Theorem</u>. Let $(X, \mathbf{C})$ be a closure algebra, $\mathbf{S}$ as in 2.1, and let A, B, $\in \mathbf{P}(X)$, then :

(i) $A \subseteq B \Rightarrow \mathbf{S}(B) \subseteq \mathbf{S}(A)$

(ii) $\mathbf{S}(\mathbf{C}(A)) = \mathbf{S}(A)$.

Proof. (i) Let $A \subseteq B$; if $\Delta \in \mathbf{S}(B)$ then $B \subseteq \Delta$, and hence $A \subseteq \Delta$ so that $\Delta \in \mathbf{S}(A)$. We have shown that $\mathbf{S}(B) \subseteq \mathbf{S}(A)$.
(ii) Since $A \subseteq \mathbf{C}(A)$ by 1.1 II , we have that $\mathbf{S}(\mathbf{C}(A)) \subseteq \mathbf{S}(A)$ by the first part of this theorem. For the other containment:

$$\begin{aligned} \Delta \in \mathbf{S}(A) &\Rightarrow A \subseteq \Delta \\ &\Rightarrow \mathbf{C}(A) \subseteq \mathbf{C}(\Delta), \text{ by 1.1 III} \\ &\Rightarrow \mathbf{C}(A) \subseteq \Delta, \text{ since } \Delta \text{ is closed by 1.27.} \end{aligned}$$

Hence $\mathbf{S}(A) \subseteq \mathbf{S}(\mathbf{C}(A))$, yielding that

$\mathbf{S}(A) = \mathbf{S}(\mathbf{C}(A))$.

2.3 Definition. Let $(X, \mathbf{C})$ be a closure algebra.

Define $[\![\ \ ]\!] : \mathbf{P}(X) \to \mathbf{P}(X)$ by ;

$$[\![A]\!] = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta \quad , \text{ for each } A \in \mathbf{P}(X).$$

2.4 Theorem. Let $(X, \mathbf{C})$ be a closure algebra with $[\![\ \ ]\!]$ defined as in 2.3; then $(X, [\![\ \ ]\!])$ is a closure algebra.

Proof. Let $A, B \in \mathbf{P}(X)$, then :

(i) For every $\Delta \in \mathbf{S}(A)$, $A \subseteq \Delta$ by the definition of $\mathbf{S}(A)$ hence $A \subseteq \bigcap_{\Delta \in \mathbf{S}(A)} \Delta = [\![A]\!]$.

(ii) For each $\Delta \in \mathbb{M}_X$, if $\Delta \supseteq A$ then $\Delta \in \mathbf{S}(A)$ and $\Delta \supseteq \bigcap_{\Delta \in \mathbf{S}(A)} \Delta = [\![A]\!]$. On the other hand, if $\Delta \supseteq [\![A]\!]$ then $\Delta \supseteq A$ by part (i) so that;

$$\Delta \supseteq A \text{ iff } \Delta \supseteq [\![A]\!] \text{ , for each } \Delta \in \mathbb{M}_X$$

$$\Rightarrow \quad \Delta \in \mathbf{S}(A) \text{ iff } \Delta \in \mathbf{S}([\![A]\!]) \text{ for each } \Delta \in \mathbb{M}_X$$

$$\Rightarrow \quad \bigcap_{\Delta \in \mathbf{S}(A)} \Delta = \bigcap_{\Delta \in \mathbf{S}([\![A]\!])} \Delta$$

$$\Rightarrow \quad [\![A]\!] = [\![[\![A]\!]]\!] \text{ .}$$

(iii)

$A \subseteq B \quad \Rightarrow \mathbf{S}(A) \supseteq \mathbf{S}(B)$, by 2.2 (i)

$$\Rightarrow \bigcap_{\Delta \in \mathbf{S}(A)} \Delta \subseteq \bigcap_{\Delta \in \mathbf{S}(B)} \Delta$$

$$\Rightarrow [\![A]\!] \subseteq [\![B]\!] \text{ .}$$

This shows that $[\![\ \ ]\!]$ satisfies the defining conditions for a closure operator on X; i.e. $(X, [\![\ \ ]\!])$ is a closure algebra.

2.5 Theorem. Let $(X, \mathbf{C})$ be a closure algebra, then $\mathbf{C} \subseteq [\![\ ]\!]$.

Proof. Let $A \in \mathbf{P}(X)$. By 1.27, each $\Delta \in \mathbf{S}(A)$ is a closed set and hence is a member of the closure system associated with $(X, \mathbf{C})$. It follows that $\bigcap_{\Delta \in \mathbf{S}(A)} \Delta = [\![A]\!]$ is also a member of the same closure system and is thus a closed set. Hence $\mathbf{C}([\![A]\!]) = [\![A]\!]$ by 1.3, and $\mathbf{C} \subseteq [\![A]\!]$, by 1.13.

2.6 Theorem. Let $(X, \mathbf{C})$ be a cocompact closure algebra, then for each $A \in \mathbf{P}(X)$, A is consistent with respect to $\mathbf{C}$ iff A is consistent with respect to $[\![\ ]\!]$.

Proof. If A is consistent with respect to $[\![\ ]\!]$, then $[\![A]\!] \neq X$. Hence $\mathbf{C}(A) \neq X$, since $\mathbf{C}(A) \subseteq [\![A]\!]$, by 2.5, and A is consistent with respect to $\mathbf{C}$. Conversely, if A is consistent with respect to $\mathbf{C}$ then A can be extended to a maximal consistent set by the cocompactness of $(X, \mathbf{C})$ and theorem 1.25. Hence $\mathbf{S}(A) \neq \phi$, which gives us that $[\![A]\!] = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta \neq X$[1] so that A is consistent with respect to $[\![\ ]\!]$.

[1] The convention is adopted throughout the thesis that the intersection of any family of proper subsets of a set X is equal to X iff the family is empty.

Now that the closure operation $[\![\ ]\!]$ has been defined, the question arises as to whether, given a suitable, initial closure algebra $(X, \mathbf{C})$, it might be possible to define a third closure operation on X by treating the sets that are maximal consistent with respect to $[\![\ ]\!]$ in the same way as the sets that are maximal consistent with respect to $\mathbf{C}$ were treated to define $[\![\ ]\!]$. Might it be possible to define a whole series of closure operations in this manner ? By showing that the family of maximal consistent sets remain unchanged when any closure operation defined under 2.3 is considered, theorem 2.7 shows that no new closure operations will arise, whenever $(X,\mathbf{C})$ is cocompact.

<u>2.7 Theorem</u>. Let $(X, \mathbf{C})$ be a cocompact closure algebra, then for each $A \in \mathbf{P}(X)$, A is maximal consistent with respect to $\mathbf{C}$ iff A is maximal consistent with respect to $[\![\ ]\!]$.

Proof. If A is maximal consistent with respect to $\mathbf{C}$, then A is consistent with respect to $[\![\ ]\!]$, by 2.6. If $x \notin A$ then $\mathbf{C}(A \cup \{x\}) = X$, because A is maximal consistent with respect to $\mathbf{C}$. Hence $[\![A \cup \{x\}]\!] = X$ by 2.5, and so A is maximal consistent with respect to $[\![\ ]\!]$.

On the other hand if A is maximal consistent with respect to $[\![\ ]\!]$, then $[\![A]\!] \neq X$ and $[\![A \cup \{x\}]\!] = X$, for each $x \notin A$. By 2.6 $\mathbf{C}(A) \neq X$, yielding the required consistency. Furthermore if $\mathbf{C}(A \cup \{x\}) \neq X$, for some $x \notin A$, then $A \cup \{x\}$ is consistent with respect to $[\![\ ]\!]$, by 2.6, contradicting $[\![A \cup \{x\}]\!] = X$. Hence $\mathbf{C}(A \cup \{x\}) = X$, and A is maximal consistent with respect to $\mathbf{C}$.

<u>Theorem 2.8</u> If $(X, \mathbf{C})$ is a cocompact closure algebra then $(X, [\![\ ]\!])$ is a cocompact closure algebra.

Proof. Let $\{\alpha_t\}_{t\in T} \subseteq \mathbf{P}(X)$ be a chain of subsets each of which is consistent with respect to $[\![\ ]\!]$. Then by 2.6 $\{\alpha_t\}_{t\in T}$ is a chain of sets each of which is consistent with respect to $\mathbf{C}$. By the cocompactness of $(X, \mathbf{C})$ and theorem 1.21, $\{\alpha_t\}_{t\in T}$ has a least upper bound that is consistent with respect to $\mathbf{C}$. By theorem 2.6 this least upper bound is consistent with respect to $[\![\ ]\!]$. This holds for each such chain $\{\alpha_t\}_{t\in T}$, which establishes that the family of sets that are consistent with respect to $[\![\ ]\!]$ is inductive, and hence that $(X, [\![\ ]\!])$ is cocompact by 1.21.

The definition of the operator $[\![\ ]\!]$ was motivated by considering the behaviour of consistent and complete sets of formulas in logics, and from the behaviour of ultrafilters in Boolean Algebra.

The intersection of a family of maximal consistent sets has also been investigated in the theory of nilpotent groups. In this case the closure algebra is as in 1.11 (i) and the maximal consistent sets are the maximal proper subgroups. The Frattini subgroup is obtained by taking the intersection of all the maximal proper subgroups and it may be shown (Cf. Kuros [1], para. 62) that finite nilpotent groups can be characterized in terms of the Frattini subgroup. In the next few sections we look at the concepts of relative and absolute consistency, both of which can be formulated in closure theoretic terms, and use them to provide an alternative description of the closure operation $[\![\ ]\!]$. As theorem 2.15 I obtain a generalization of a group-theoretic result.

<u>2.9 Definition</u>. Let $(X, \mathbf{C})$ be a closure algebra, $x \in X$, and $A \in \mathbf{P}(X)$. We say that x is consistent relative to A iff the following condition holds :

$$\text{if } \mathbf{C}(\overset{A}{\cancel{X}}) \neq X \text{ then } \mathbf{C}(A \cup \{x\}) \neq X.$$

We denote by A' the set of all members of X that are consistent relative to A.

<u>2.10 Theorem</u>. If $(X, \mathbf{C})$ is a closure algebra then

$$\mathbf{C}(A) \subseteq A' \text{ for each } A \in \mathbf{P}(X).$$

Proof. For each $A \in \mathbf{P}(X)$, $x \in X$:

$$\begin{aligned} x \in \mathbf{C}(A) &\Rightarrow \mathbf{C}(A) \cup \{x\} = \mathbf{C}(A) \\ &\Rightarrow \mathbf{C}(\mathbf{C}(A) \cup \{x\}) = \mathbf{C}(\mathbf{C}(A)) \\ &\Rightarrow \mathbf{C}(\mathbf{C}(A) \cup \{x\}) = \mathbf{C}(A), \text{ by 1.1 II} \end{aligned}$$

Hence :

$$\begin{aligned} \mathbf{C}(A) &\subseteq \mathbf{C}(A \cup \{x\}), \text{by 1.1 III} \\ &\subseteq \mathbf{C}(\mathbf{C}(A) \cup \{x\}), \text{ by 1.1 III,} \\ &\qquad \text{since } A \subseteq \mathbf{C}(A) \\ &= \mathbf{C}(A), \end{aligned}$$

yielding that $\mathbf{C}(A) = \mathbf{C}(A \cup \{x\})$ for each $x \in \mathbf{C}(A)$, so that if $\mathbf{C}(A) \neq X$, then $\mathbf{C}(A \cup \{x\}) \neq X$. By definition 2.9, $x \in A'$ for each $x \in \mathbf{C}(A)$. Hence $\mathbf{C}(A) \subseteq A'$ for each $A \in \mathbf{P}(X)$.

<u>2.11 Theorem.</u> If $(X, \mathbf{C})$ is a closure algebra, and $A \in \mathbf{P}(X)$ then A' is consistent iff A' is maximal consistent.

Proof. If A' is maximal consistent then A' is consistent. For the converse, suppose that A' is consistent and that $A' \subset B$[1], for some consistent $B \in \mathbf{P}(X)$.

Let $x \in B \backslash A'$. Since $x \notin A'$ we have that

$$\mathbf{C}(A) \neq X \text{ and } \mathbf{C}(A \cup \{x\}) = X, \text{ by } 2.9.$$

Since $x \in B$ and $A \subseteq A' \subset B$ we have $A \cup \{x\} \subseteq B$, and so $\mathbf{C}(A \cup \{x\}) \subseteq \mathbf{C}(B)$, by 1.1 III.

Thus $\mathbf{C}(B) = X$, contradicting the consistency of B. Hence there is no consistent set B that strictly contains A', and so A' is maximal consistent.

<u>2.12 Theorem.</u> If $(X, \mathbf{C})$ is a closure algebra and $A \in \mathbf{P}(X)$, then $A = A'$ iff either $A = X$ or A is maximal consistent.

Proof. A' is consistent iff A' is maximal consistent, by 2.11.

$\Rightarrow$ A' is inconsistent or A' is maximal consistent. Hence if $A = A'$ then

either A is inconsistent or A' is maximal consistent

[1] Throughout the thesis, '$\subset$' will be used to denote strict inclusion.

$\Rightarrow$ either $\mathbf{C}(A) = X$ or A' is maximal consistent

$\Rightarrow$ either $A' = X$ or A' is maximal consistent,

since $\mathbf{C}(A) \subseteq A'$

$\Rightarrow$ either $A = X$ or A is maximal consistent,

since $A = A'$.

Conversely, if $A = X$ then :

$$\begin{aligned} X &= A \\ &\subseteq \mathbf{C}(A), \text{by 1.1 I}, \\ &\subseteq A', \quad \text{by 2.10}, \\ &\subseteq X, \end{aligned}$$

so that $A = A'$. Also if A is maximal consistent and $x \notin A$ then :

$$\mathbf{C}(A) \neq X \text{ and } \mathbf{C}(A \cup \{x\}) = X$$

$$\Rightarrow \quad x \notin A' \quad \text{by 2.9}$$

Hence $A' \subseteq A$; but, as above, $A \subseteq \mathbf{C}(A) \subseteq A'$, so that in both cases $A = A'$.

<u>2.13 Theorem</u>. Let $(X, \mathbf{C})$ be a cocompact or a finitary closure algebra, then,

$$[\![A]\!] = \bigcap_{B \supseteq A} B', \text{ for each } A \in \mathbf{P}(X).$$

Proof. In both cases we have for each $x \in X$:

$$x \notin [\![A]\!] \Rightarrow x \notin \bigcap_{\Delta \in \mathbf{S}(A)} \Delta \quad , \text{ by } 2.3$$

$$\Rightarrow x \notin \Delta, \text{ for some } \Delta \in \mathbf{S}(A)$$

$$\Rightarrow x \notin \Delta, \text{ for some } \Delta \in M_X, \ \Delta \supseteq A.$$

$$\Rightarrow x \notin \Delta', \ \Delta \supseteq A,$$

since $\Delta = \Delta'$ for each maximal consistent Δ, by 2.12

$$\Rightarrow x \notin \bigcap_{B \supseteq A} B'.$$

Therefore $\bigcap_{B \supseteq A} B' \subseteq [\![A]\!]$, for each $A \in \mathbf{P}(X)$.

For the opposite inclusion I treat the cocompact and finitary cases separately.

If $(X, \mathbf{C})$ is finitary then choose any $A \in \mathbf{P}(X)$ and suppose that $x \notin \bigcap_{B \supseteq A} B'$. Now there is some $B \supseteq A$, with $x \notin B'$. Hence there is some $B \supseteq A$ with $\mathbf{C}(B) \neq X$ and $\mathbf{C}(B \cup \{x\}) = X$. Put $\Gamma = \{D \in \mathbf{P}(X) : D \supseteq B, x \notin D, \mathbf{C}(D) = D\}$. Since $x \notin B' \supseteq \mathbf{C}(B)$, by 2.10, we have that $\mathbf{C}(B) \in \Gamma$ so that $\Gamma \neq \phi$.

Furthermore, if $\{\alpha_t\}_{t \in T}$ is a chain in Γ, then $\bigcup_{t \in T} \alpha_t$ is also in Γ. For $\bigcup_{t \in T} \alpha_t$ is closed by the assumption that $(X, \mathbf{C})$ is finitary and 1.18; and $x \notin \alpha_t$, for each $t \in T$, by the construction of Γ, so that $x \notin \bigcup_{t \in T} \alpha_t$. This shows that Γ is inductive and non-empty, so that Γ has a maximal element Δ, by 1.16 (Zorn's lemma). Now since Δ is closed and $x \notin \Delta$, we have that Δ is consistent. Furthermore, if $\Delta \subset E$, then $\Delta \subset \mathbf{C}(E)$, so that $\mathbf{C}(E) \notin \Gamma$, otherwise the maximality of Δ in Γ would be contradicted. Since $\mathbf{C}(E) \notin \Gamma$ but $\mathbf{C}(E) \supseteq B$, we must have that $x \in \mathbf{C}(E)$. Hence $X = \mathbf{C}(B \cup \{x\}) \subseteq \mathbf{C}(E)$, and E is inconsistent. This shows that Δ is maximal consistent in $(X, \mathbf{C})$ and $x \notin \Delta$. Now $\Delta \supseteq B \supseteq A$ so $\Delta \in \mathbf{S}(A)$ and hence $x \notin \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$. This yields the inclusion $[\![A]\!] \subseteq \bigcap_{B \supseteq A} B'$ for the case when $(X, \mathbf{C})$ is finitary.

If $(X, \mathbf{C})$ is cocompact then for each $x \in X$:

$$x \notin \bigcap_{B \supseteq A} B'$$

$\Rightarrow$ $B \supseteq A$ and $x \notin B'$, for some $B \in \mathbf{P}(X)$

$\Rightarrow$ $B \supseteq A$ and $\mathbf{C}(B) \neq X$ and $\mathbf{C}(B \cup \{x\}) = X$, by definition of B'

$\Rightarrow$ $B \supseteq A$ and $\Delta \supseteq B$ and $\mathbf{C}(B \cup \{x\}) = X$ for some $\Delta \in \mathbf{M}_X$, since B is consistent and may be extended to a maximal consistent set Δ by the cocompactness of $(X, \mathbf{C})$ and 1.25.

We now have that $\Delta \supseteq A$ and that if $x \in \Delta$ then $\mathbf{C}(\Delta) \supseteq \mathbf{C}(B \cup \{x\}) = X$, for some $\Delta \in \mathbf{M}_X$. Since this would contradict the consistency of Δ , we must have :

$\Delta \supseteq A$ and $x \notin \Delta$, for some $\Delta \in \mathbf{M}_X$

$\Rightarrow$ $\Delta \in \mathbf{S}(A)$ and $x \notin \Delta$

$\Rightarrow$ $x \notin \bigcap_{\Delta \in \mathbf{S}(A)} \Delta = [\![A]\!]$.

I have shown that $x \notin \bigcap_{B \supseteq A} B'$ implies $x \notin [\![A]\!]$ for each $x \in X$, so that :

$$[\![A]\!] \subseteq \bigcap_{B \supseteq A} B', \text{ for each } A \in \mathbf{P}(X).$$

The theorem tells us that $[\![A]\!]$ is precisely the set of all members of X that are consistent relative to every superset of A, whenever (X, **C**) is finitary or cocompact.

2.14 Definition. Let (X, **C**) be a closure algebra. For each $x \in X$, x is said to be absolutely consistent iff x is consistent relative to every subset of X; i.e. x is absolutely consistent iff

$$x \in A', \text{ for each } A \in \mathbf{P}(X)$$

2.15 Theorem. If (X, **C**) is a closure algebra that is either cocompact or finitary then the set of absolutely consistent elements is given by $[\![\phi]\!]$.

Proof. x is absolutely consistent

$\iff x \in B'$, for each $B \in \mathbf{P}(X)$

$\iff x \in B'$, for each $B \in \mathbf{P}(X)$ with $B \supseteq \phi$,

$\iff x \in \bigcap_{B \supseteq \phi} B'$

$\iff x \in [\![\phi]\!]$

In group theory the set of absolutely consistent elements is generally referred to as the set of non-generators of a group. The Frattini subgroup may be defined as the intersection of all the maximal proper subgroups. The proof that the two are equivalent is attributed in Kuros [1] to B.H. Neumann and H. Zassenhaus.

CHAPTER III THE DUAL SPACE

Some of the best results on Boolean algebras were proved by M.H. Stone in [1], [2]. In these papers, Stone constructed a topological space on the set of all ultrafilters of a Boolean algebra. This space turns out to be a compact, Hausdorff space with a basis of clopen sets. Furthermore the Boolean algebra can be recaptured from the topological space, since under a very natural mapping the Boolean algebra is isomorphic to the family of all clopen subsets of the space.

In [1], Bloom and Brown extend the methods of Stone so as to provide a topology on the family of all maximal consistent subsets of any closure algebra, $(X, \mathbf{C})$ having the following properties :

there is a unary function $(-a)$ and a binary function $(a \vee b)$ defined on X and satisfying :

(i) $a \in \mathbf{C}(A) \iff \mathbf{C}(A \cup \{-a\}) = X$, for each $a \in X$, $A \in \mathbf{P}(X)$;

(ii) $\mathbf{C}(A \cup \{a\}) \cap \mathbf{C}(A \cup \{b\}) = \mathbf{C}(A \cup \{a \vee b\})$, for each $a, b \in X$, $A \in \mathbf{P}(X)$. For Stone, and for Bloom and Brown, the topology on $\mathbf{M}X$ is constructed by taking as a basis the family $\{\mathbf{S}(x) : x \in X\}$[1]. Under the restrictions of the operations defined on X this basis is equivalent to

[1] $\mathbf{S}$ is defined as in 2.1; and from now on I write $\mathbf{S}(x)$ for $\mathbf{S}(\{x\})$.

$\{\mathbf{S}(A_f) : A_f \subseteq X, A_f \text{ finite}\}$, and under these restrictions Bloom and Brown are able to show that the topological space generalizing the Stone space is still a compact, Hausdorff space with a basis of clopen sets.

However, the only condition that is essential to this method of constructing a topology on $\mathbb{M}_X$ is that property of $\mathbf{S}$ expressed in 3.1. Using this I topologize $\mathbb{M}_X$ by taking $\{\mathbf{S}(A_f) : A_f \subseteq X, A_f \text{ finite}\}$ as a basis. The set $\{\mathbf{S}(x) : x \in X\}$ now appears only as a subbasis. The properties of this topology differ significantly from those of the two topologies outlined above. Firstly the topological structure is independent of any algebraic structure on the carrier of the underlying closure algebra, and depends only on the behaviour of the closure operation itself. This topology can be defined for any closure algebra, and will be non-trivial whenever there are some maximal consistent sets. Secondly there is no correspondence between elements of the closure algebra and clopen subsets of the topological space. The principal difficulty here is the impossibility of either identifying or distinguishing between sets of the form $\mathbf{S}(x)$ and sets of the form $\mathbf{S}(A_f)$. This disadvantage is not too great since, as there are no operations defined on X, there is no reason to look for structural connections between X and any subset of $\mathbf{P}(\mathbb{M}_X)$, unless these connections can be defined in purely closure-theoretic terms. The topology given in this chapter generalizes both of the above examples, and the gain in generality makes it possible to see what properties depend only on the closure algebra.

3.1 Theorem. Let $(X, \mathbf{C})$ be a closure algebra, $\mathbb{M}_X$ the family of maximal consistent subsets of X, and $\mathbf{S}$ the function defined in 2.1.

If $k \in K$ indexes a subfamily of $\mathbf{P}(X)$ then :

$$\bigcap_{k\in K} \mathbf{S}(A_k) = \mathbf{S}(\bigcup_{k\in K} A_k)$$

Proof. We have $\bigcap_{k\in K} \mathbf{S}(A_k) = \mathbf{S}(\bigcup_{k\in K} A_k)$ because :

$$\begin{aligned}\Delta \in \bigcap_{k\in K} \mathbf{S}(A_k) &\Longleftrightarrow \Delta \in \mathbf{S}(A_k), \text{ for each } k \in K\\ &\Longleftrightarrow A_k \subseteq \Delta, \text{ for each } k \in K,\\ &\quad\ \text{by definition 2.1}\\ &\Longleftrightarrow \bigcup_{k\in K} A_k \subseteq \Delta\\ &\Longleftrightarrow \Delta \in \mathbf{S}(\bigcup_{k\in K} A_k).\end{aligned}$$

I now use the images under $\mathbf{S}$ of the finite subsets of X to define a topology on $\mathbb{M}_X$.

3.2 Theorem. Let $(X, \mathbf{C})$ be a closure algebra and $\mathbb{M}_X$ the family of maximal consistent sets. Define $\beta = \{\mathbf{S}(A_f) : A_f \in \mathbf{P}(X), A_f \text{ finite}\}$, then β is the base for a topology τ on M_X and the set $\sigma = \{\mathbf{S}(x) : x \in X\}$ is a subbase for this topology.

Proof. I will show that ϕ, $M_X \in \tau$ and that τ is closed under arbitrary unions and pairwise intersections.

Set $\tau = \{\bigcup_{\mathbf{S}(A_f)\in\beta'} \mathbf{S}(A_f) : \beta' \subseteq \beta\}$.

(i) For each $\Delta \in M_X$ we may choose a finite subset of Δ, say Δ_f, so that :

$$\Delta \in \mathbf{S}(\Delta_f) \text{ and } \mathbf{S}(\Delta_f) \in \beta.$$

Therefore $\Delta \in \bigcup_{\mathbf{S}(\Delta_f)\in\beta} \mathbf{S}(\Delta_f)$, for each $\Delta \in M_X$,

and hence $M_X \subseteq \bigcup_{\mathbf{S}(\Delta_f)\in\beta} \mathbf{S}(\Delta_f)$. On the other hand,

$$\mathbf{S}(\Delta_f) \subseteq M_X \text{ , for each } \Delta_f \in \mathbf{P}(X)$$

so that $\bigcup_{\mathbf{S}(\Delta_f)\in\beta} \mathbf{S}(\Delta_f) \subseteq M_X$, yielding that

$$M_X = \bigcup_{\mathbf{S}(\Delta_f)\in\beta} \mathbf{S}(\Delta_f) \in \tau.$$

(ii) Taking the empty subset of β, we have that

$$\phi = \bigcup_{\mathbf{S}(A_f)\in\phi} \mathbf{S}(A_f) \text{ , so that } \phi \in \tau.$$

(iii) Let $k \in K$ index $\{0_k\}_{k\in K} \subseteq \tau$.

Then for each $k \in K$, $0_k = \bigcup_{\mathbf{S}(A_f)\in\beta_k} \mathbf{S}(A_f)$, for some $\beta_k \subseteq \beta$,

since each $0_k \in \tau$.

Now

$$\bigcup_{k\in K} 0_k = \bigcup_{k\in K}\left(\bigcup_{S(A_f)\in\beta_k} S(A_f)\right)$$

$$= \bigcup_{S(A_f)\in\bigcup_{k\in K}\beta_k}\left(S(A_f)\right)$$

Since $\beta_k \subseteq \beta$ for each $k \in K$, we have that $\bigcup_{k\in K}\beta_k \subseteq \beta$,

so that $\bigcup_{k\in K} 0_k \in \tau$.

(iv) If $0_1, 0_2 \in \tau$ then

$0_1 = \bigcup_{S(A_{f_1})\in\beta_1} S(Af_1)$ and $0_2 = \bigcup_{S(A_{f_2})\in\beta_2} S(A_{f_2})$, for

some $\beta_1, \beta_2 \subseteq \beta$.

Consequently $0_1 \cap 0_2 = \left(\bigcup_{S(A_{f_1})\in\beta_1} S(A_{f_1})\right)\cap\left(\bigcup_{S(A_{f_2})\in\beta_2} S(A_{f_2})\right)$

$$= \bigcup_{S(A_{f_1})\in\beta_1}\left(\bigcup_{S(A_{f_2})\in\beta_2}(S(A_{f_1})\cap S(A_{f_2}))\right)$$

Now by 3.1, $S(A_{f_1}) \cap S(A_{f_2}) = S(A_{f_1} \cup A_{f_2})$ for each $A_{f_1} \in \beta_1$, $A_{f_2} \in \beta_2$, and since A_{f_1}, A_{f_2} are finite, $A_{f_1} \cup A_{f_2}$ is finite. Therefore $S(A_{f_1} \cup A_{f_2}) \in \beta$, for each $A_{f_1} \in \beta_1$, $A_{f_2} \in \beta_2$, and so $S(A_{f_1}) \cap S(A_{f_2}) \in \tau$, for each $A_{f_1} \in \beta_1$, $A_{f_2} \in \beta_2$. By two applications of part (iii) of this theorem we have that :

$$\bigcup_{S(A_{f_1})\in\beta_1} \left[\bigcup_{S(A_{f_2})\in\beta_2} (S(A_{f_1}) \cap S(A_{f_2})) \right] \in \tau \text{ and so}$$

$O_1 \cap O_2 \in \tau$.

By (i)..., (iv) τ is a topology for M_X and β is a base for τ. I now show that $\sigma = \{S(x) : x \in X\}$ is a subbase for τ. Choose any $S(A_f) \in \beta$, then

$$S(A_f) = S(\bigcup_{x\in A_f} \{x\})$$

$$= \bigcap_{x\in A_f} S(x)$$ - by 3.1, so that any member of the base may be expressed as the intersection of a finite number of members of σ, and so σ is a subbase for τ.

If (X, C) is any closure algebra then the pair (M_X, τ) will denote the above topological space and will be called the dual space of the closure algebra (X, C). The structure of this dual space is of considerable interest as it reflects some of the properties of the closure algebra. Recollect from theorem 1.24 that a closure algebra is cocompact iff it has the Lindenbaum property and the f.e.p. In the next two theorems I show the equivalence of these properties in the closure algebra with set theoretic properties of the dual space.

3.4 Theorem. Let (X, C) be a closure algebra (M_X, τ) the dual space, and β the base for the topology τ. (X, C) has the finite extension property (f.e.p.) iff every subfamily of β with the finite intersection property (f.i.p.) has a non-empty intersection.

Proof. Assume that $(X, \mathbf{C})$ has the f.e.p. and that $\Gamma = \{\mathbf{S}(A_f)\}_{f \in F}$ is a subfamily of β such that $\bigcap_{f \in F} \mathbf{S}(A_f) = \phi$. I will show that Γ does not have the f.i.p. and the required implication will follow by contraposition.

If $\bigcap_{f \in F} \mathbf{S}(A_f) = \phi$ then $\mathbf{S}[\bigcup_{f \in F} A_f] = \phi$, by 3.1, hence there is no $\Delta \in \mathbb{M}_X$ such that $\bigcup_{f \in F} A_f \subseteq \Delta$.

Since $(X, \mathbf{C})$ has the f.e.p. we have that $\bigcup_{f \in F} A_f$ has a finite subset G such that G is contained in no maximal consistent set. Since $G \subseteq \bigcup_{f \in F} A_f$, we may choose, for each $x \in G$ a set A_{f_x}, $f_x \in F$, such that $x \in A_{f_x}$. Then $G \subseteq \bigcup_{x \in G} A_{f_x}$, and so $\bigcup_{x \in G} A_{f_x}$ is contained in no maximal consistent set; i.e. $\mathbf{S}(\bigcup_{x \in G} A_{f_x}) = \phi$. It follows that $\bigcap_{x \in G} \mathbf{S}(A_{f_x}) = \phi$, by 3.1, and since G is finite, it follows that Γ does not have the f.i.p.

Conversely, assume that each subfamily of β with the f.i.p. has a non empty intersection, and assume that $A \in \mathbf{P}(X)$ is contained in no maximal consistent set so that $\mathbf{S}(A) = \phi$. If we let $f \in F$ index the finite subsets of A, then :

$$A = \bigcup_{f \in F} A_f$$

$$\Rightarrow \quad \mathbf{S}(\bigcup_{f \in F} A_f) = \mathbf{S}(A) = \phi$$

$$\Rightarrow \quad \bigcap_{f \in F} \mathbf{S}(A_f) = \phi, \text{ by } 3.1$$

$$\Rightarrow \bigcap_{f \in F'} \mathbf{S}(A_f) = \phi, \text{ where } F' \subseteq F \text{ is finite;}$$

since $\{\mathbf{S}(A_f)\}_{f \in F} \subseteq \beta$

$$\Rightarrow \mathbf{S}\left(\bigcup_{f \in F'} A_f\right) = \phi, \text{ by 3.1.}$$

But $\bigcup_{f \in F'} A_f$ is the finite union of finite sets and is hence a finite subset of A. This shows that A has a finite subset that cannot be extended to any maximal consistent set, so that (X, $\mathbf{C}$) has the f.e.p.

<u>3.5 Theorem</u>. Let (X, $\mathbf{C}$) be a closure algebra and ($\mathcal{M}_X$, τ) its dual space. (X, $\mathbf{C}$) has the Lindenbaum property iff the image of every consistent set under $\mathbf{S}$ is non-empty.

Proof. If (X, $\mathbf{C}$) has the Lindenbaum property then each consistent set A can be extended to a maximal consistent set Δ, so that $\Delta \in \mathbf{S}(A) \neq \phi$.

On the other hand if the image under $\mathbf{S}$ of each consistent set is non-empty, and $A \in \mathbf{P}(X)$ is consistent then :

$$\mathbf{S}(A) \neq \phi.$$

$$\Rightarrow \Delta \in \mathbf{S}(A), \text{ for some } \Delta \in \mathcal{M}_X$$

$$\Rightarrow A \subseteq \Delta, \quad \text{for some } \Delta \in \mathcal{M}_{X_1}$$

which is to say that A can be extended to a maximal consistent set, and that (X, $\mathbf{C}$) has the Lindenbaum property.

The next three results examine the topological status of the images, under $\mathbf{S}$, of the subsets of X. I first show that the topological closure of any image under $\mathbf{S}$ has a definite form. In the second theorem I show that, providing the closure algebra is cocompact, every image of a set under $\mathbf{S}$ is closed. This result is used to prove the third theorem, that the dual space of a cocompact closure algebra has a basis of clopen sets. This generalizes the result in the Stone representation theorem that every Boolean space has a basis of clopen sets.

<u>3.6 Theorem</u>. If $(X, \mathbf{C})$ is a closure algebra and $(\mathbb{M}_X, \tau)$ the dual space, we have for each $A \in \mathbf{P}(X)$:

$\overline{\mathbf{S}(A)} = \mathbb{M}_X \setminus \bigcup \{\mathbf{S}(B_f) : \mathbf{S}(B_f) \cap \mathbf{S}(A) = \phi,\ B_f \text{ finite}\}$, where $\overline{\mathbf{S}(A)}$ denotes the topological closure of $\mathbf{S}(A)$.

Proof. Let $\Gamma = \{\mathbf{S}(B_f) : \mathbf{S}(B_f) \cap \mathbf{S}(A) = \phi,\ B_f \text{ finite}\}$, and let $k \in K$ index Γ. Putting $T = \bigcup\limits_{k \in K} \mathbf{S}(B_{f_k})$, I will show that $\overline{\mathbf{S}(A)} \subseteq \mathbb{M}_X \setminus T$. Since T is the union of members of the base, T is open and $\mathbb{M}_X \setminus T$ closed, so that it is sufficient to show that $\mathbf{S}(A) \subseteq \mathbb{M}_X \setminus T$. If $\Delta \in \mathbf{S}(A)$ then $A \subseteq \Delta$, so that $B_{f_k} \nsubseteq \Delta$ for each $k \in K$. Otherwise $B_{f_k} \cup A \subseteq \Delta$ would imply that

$$\begin{aligned}\Delta \in \mathbf{S}(B_{f_k} \cup A) &= \mathbf{S}(B_{f_k}) \cap \mathbf{S}(A) \\ &= \phi \text{ , contradicting } A \subseteq \Delta \text{ .}\end{aligned}$$

I assume $A \neq \phi$ since the result holds trivially in this case.

Since $B_{f_k} \not\subseteq \Delta$ for each $k \in K$,

$$\Delta \notin \mathbf{S}(B_{f_k}) \text{ for each } k \in K$$

$$\Rightarrow \Delta \notin \bigcup_{k\in K} \mathbf{S}(B_{f_k})$$

$$\Rightarrow \Delta \in M_X \setminus \bigcup_{k\in K} \mathbf{S}(B_{f_k}) = M_X \setminus T.$$

Also if Q is a topologically closed subset of M_X, and $\mathbf{S}(A) \subseteq Q$, then $M_X \setminus T \subseteq Q$ because :

Q closed $\Rightarrow Q = M_X \setminus \bigcup_{j\in J} \mathbf{S}(B_{f_j})$, where $j \in J$ indexes a suitable subfamily of the base

$$\Rightarrow Q \cap (\bigcup_{j\in J} \mathbf{S}(B_{f_j}) = \phi$$

$$\Rightarrow \bigcup_{j\in J} (Q \cap \mathbf{S}(B_{f_j}) = \phi.$$

$$\Rightarrow Q \cap \mathbf{S}(B_{f_j}) = \phi, \text{ for each } j \in J$$

$$\Rightarrow \mathbf{S}(A) \cap \mathbf{S}(B_{f_j}) = \phi, \text{ for each } j \in J,$$

since $\mathbf{S}(A) \subseteq Q$

$$\Rightarrow \mathbf{S}(B_{f_j}) \in \Gamma, \text{ for each } j \in J$$

$$\Rightarrow \bigcup_{j\in J} \mathbf{S}(B_{f_j}) \subseteq \bigcup_{k\in K} \mathbf{S}(B_{f_k})$$

$$\Rightarrow M_X \setminus \bigcup_{j\in J} \mathbf{S}(B_{f_j}) \supseteq M_X \setminus \bigcup_{k\in K} \mathbf{S}(B_{f_k})$$

$$\Rightarrow Q \supseteq M_X \setminus T.$$

From these two facts it follows that

$$\begin{aligned} \overline{\mathbf{S}(A)} &= M_X \setminus T \\ &= M_X \setminus \bigcup \{\mathbf{S}(B_f) \;:\; \mathbf{S}(B_f) \cap \mathbf{S}(A) = \phi, B_f \text{ finite}\}. \end{aligned}$$

3.7 Theorem. Let $(X, \mathbf{C})$ be a cocompact closure algebra and $(\mathbb{M}_X, \tau)$ its dual space, then $\mathbf{S}(A)$ is topologically closed for each $A \in \mathbf{P}(X)$.

Proof. Defining Γ, T as in the last theorem, we have that $\mathbf{S}(A) \subseteq \overline{\mathbf{S}(A)} = \mathbb{M}_X \setminus T$. To show that $\mathbf{S}(A)$ is closed I need only show that $\mathbb{M}_X \setminus T \subseteq \mathbf{S}(A)$. I will show that if $\Delta \notin \mathbf{S}(A)$ then $\Delta \notin \mathbb{M}_X \setminus T$. First note that :

$$\Delta \notin \mathbf{S}(A)$$

$$\Rightarrow \quad A \nsubseteq \Delta$$

$\Rightarrow \quad A \cup \Delta$ is inconsistent, since Δ is a maximal consistent set

$\Rightarrow \quad A \cup \Delta$ has a finite, inconsistent subset, by the cocompactness of $(X, \mathbf{C})$. Let this finite inconsistent subset be A_f. We have that $\mathbf{S}(\Delta \cap A_f) \in \Gamma$, because $\Delta \cap A_f$ is finite, and :

$$\begin{aligned} \mathbf{S}(A) \cap \mathbf{S}(\Delta \cap A_f) &= \mathbf{S}[A \cup (\Delta \cap A_f)] \quad - \text{ by } 3.1 \\ &= \mathbf{S}[(A \cup \Delta) \cap (A \cup A_f)] \\ &= \mathbf{S}(A \cup A_f) \end{aligned}$$

because $A_f \subseteq A \cup \Delta$ and hence $A \cup A_f \subseteq A \cup \Delta$, yielding that $(A \cup \Delta) \cap (A \cup A_f) = A \cup A_f$.

Furthermore, $A_f \subseteq A \cup A_f$, so that $\mathbf{S}(A_f) \supseteq \mathbf{S}(A \cup A_f)$ by 2.1. Because A_f is inconsistent, $\mathbf{S}(A_f) = \phi$; hence $\mathbf{S}(A) \cap \mathbf{S}(\Delta \cap A_f) = \phi$, and $\mathbf{S}(\Delta \cap A_f) \in \Gamma$. We now have that $\mathbf{S}(\Delta \cap A_f) \subseteq \bigcup_{k \in K} \mathbf{S}(A_{f_k}) = T$.

But $\Delta \cap A_f \subseteq \Delta \Rightarrow \Delta \in \mathbf{S}(\Delta \cap A_f)$

$\Rightarrow \Delta \in T$

$\Rightarrow \Delta \notin \mathbb{M}_X \backslash T.$

Hence $\mathbf{S}(A) = \mathbb{M}_X \backslash T$, and must be closed.

3.8 Theorem. If $(X, \mathbf{C})$ is a cocompact closure algebra and $(\mathbb{M}_X, \tau)$ the dual space, then τ has a basis of clopen sets.

Proof. Taking the usual base β of images of finite subsets of X under $\mathbf{S}$ we have that each member of β is open by definition, and that each member of β is closed by theorem 3.7.

In chapter II the closure operation $[\![\ ]\!]$ was investigated, and there was some attention paid to when $[\![\ ]\!] = \mathbf{C}$ for a closure algebra $(X, \mathbf{C})$. This condition also has topological significance in that it modifies the behaviour of the mapping $\mathbf{S}$ and hence also the degree to which the topology on $\mathbb{M}_X$ is linked to the closure structure on $(X, \mathbf{C})$. Tarski first proved for the propositional calculus that every theory was the intersection of those complete theories that contained it, and so this condition is labelled as follows :

3.9 Definition. If $(X, \mathbf{C})$ is a closure algebra then $(X, \mathbf{C})$ is said to satisfy the Tarski condition iff

$[\![A]\!] = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta = \mathbf{C}(A)$, for each $A \in \mathbf{P}(X)$.

<u>3.10 Theorem</u>. Let $(X, \mathbf{C})$ be a closure algebra, $(\mathcal{M}_X, \tau)$ the dual space, then $(X, \mathbf{C})$ satisfies the Tarski condition iff $\mathbf{C}(D) = \mathbf{C}(E) \Longleftrightarrow \mathbf{S}(D) = \mathbf{S}(E)$ for each $D, E \in \mathbf{P}(X)$.

Proof. I have to show that :

$$\mathbf{C}(A) = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta \quad \text{for each } A \in \mathbf{P}(X) \text{ iff}$$

$$\mathbf{C}(D) = \mathbf{C}(E) \Longleftrightarrow \mathbf{S}(D) = \mathbf{S}(E), \text{ for each } D, E \in \mathbf{P}(X).$$

Assume that $\mathbf{C}(A) = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$, for each $A \in \mathbf{P}(X)$, then :

(i) If $\mathbf{C}(D) = \mathbf{C}(E)$ then $\mathbf{S}(D) = \mathbf{S}(E)$ because, for each $\Delta \in \mathcal{M}_X$:

$$\begin{aligned}
\Delta \in \mathbf{S}(D) &\Rightarrow D \subseteq \Delta \\
&\Rightarrow \mathbf{C}(D) \subseteq \mathbf{C}(\Delta) && \text{- by 1.1 III} \\
&\Rightarrow \mathbf{C}(D) \subseteq \Delta, && \text{by 1.27} \\
&\Rightarrow \mathbf{C}(E) \subseteq \Delta \\
&\Rightarrow \Delta \in \mathbf{S}(\mathbf{C}(E)) \\
&\Rightarrow \Delta \in \mathbf{S}(E), \text{ by 1.27.}
\end{aligned}$$

Similarly, $\mathbf{S}(E) \subseteq \mathbf{S}(D)$ so that

$$\mathbf{C}(D) = \mathbf{C}(E) \Rightarrow \mathbf{S}(D) = \mathbf{S}(E).$$

(ii) If $\mathbf{S}(D) = \mathbf{S}(E)$, then $\bigcap_{\Delta \in \mathbf{S}(D)} \Delta = \bigcap_{\Delta \in \mathbf{S}(E)} \Delta$ and so

$\mathbf{C}(D) = \mathbf{C}(E)$ by the hypothesis. By (i), (ii) we have that $\mathbf{C}(D) = \mathbf{C}(E) \Longleftrightarrow \mathbf{S}(D) = \mathbf{S}(E)$, for each $D, E \in \mathbf{P}(X)$.

Conversely, assuming that $\mathbf{C}(D) = \mathbf{C}(E) \Longleftrightarrow \mathbf{S}(D) = \mathbf{S}(E)$, for each $D, E \in \mathbf{P}(X)$. I have to show that $\mathbf{C}(A) = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$, for each $A \in \mathbf{P}(X)$.

By 1.27, each $\Delta \in \mathbf{S}(A)$ is closed, so that $\bigcap_{\Delta \in \mathbf{S}(A)} \Delta$ is closed by 1.7, and I may write $\bigcap_{\Delta \in \mathbf{S}(A)} \Delta = \mathbf{C}(B)$, for some $B \in \mathbf{P}(X)$. Now

$$\mathbf{C}(A) \subseteq [\![A]\!] = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta = \mathbf{C}(B), \text{ by 2.5, which yields}$$

$$\mathbf{S}(\mathbf{C}(B)) \subseteq \mathbf{S}(\mathbf{C}(A)), \text{ by 2.1}$$

$$\Rightarrow \mathbf{S}(B) \subseteq \mathbf{S}(A), \text{ by 2.1 .}$$

Furthermore, $\mathbf{S}(A) \subseteq \mathbf{S}(B)$ because , for each $\Delta \in \mathbb{M}_X$:

$$\Delta \in \mathbf{S}(A) \Rightarrow \Delta \supseteq \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$$

$$\Rightarrow \Delta \supseteq \mathbf{C}(B) \supseteq B$$

$$\Rightarrow \Delta \in \mathbf{S}(B).$$

So that $\mathbf{S}(A) = \mathbf{S}(B)$ and therefore :

$$\mathbf{C}(A) = \mathbf{C}(B) = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta, \text{ by the assumption.}$$

Corollary. Since by 2.1 $\mathbf{S}(A) = \mathbf{S}(\mathbf{C}(A))$, for each $A \in \mathbf{P}(X)$, we have that $\mathbf{S}(D) = \mathbf{S}(E) \Longleftrightarrow \mathbf{S}(\mathbf{C}(D)) = \mathbf{S}(\mathbf{C}(E))$, so that the statement of theorem 3.10 is equivalent to : $(X, \mathbf{C})$ satisfies the Tarski condition iff the restriction of $\mathbf{S}$ to the closed subsets of X is a one-to-one function.

3.11 Theorem. Let $(X, \mathbf{C})$ be a closure algebra and $(\mathbb{M}_X, \tau)$ its dual space then $(X, \mathbf{C})$ satisfies the Tarski condition iff there are sufficiently many maximal consistent sets so as to distinguish points and closed sets in the closure algebra; i.e. $\mathbf{C}(A) = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$ for each $A \in \mathbf{P}(X)$ iff for each

$x \in X$, $B \in \mathbf{P}(X)$,

$$x \notin \mathbf{C}(B) \Rightarrow \text{there is some } \Delta \in M_X \text{ such that}$$

$\mathbf{C}(B) \subseteq \Delta$, $x \notin \Delta$.

Proof. Assuming that $\mathbf{C}(A) = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$ for each $A \in \mathbf{P}(X)$,

then for each $x \in X$, $x \notin \mathbf{C}(B)$

$$\Rightarrow x \notin \bigcap_{\Delta \in \mathbf{S}(B)} \Delta, \text{ by assumption.}$$

$$\Rightarrow x \notin \Delta, \text{ for some } \Delta \in \mathbf{S}(B)$$

$$\Rightarrow \text{there exists } \Delta \in M_X \text{ such that}$$

$$\mathbf{C}(B) \subseteq \Delta \text{ and } x \notin \Delta.$$

Conversely, assuming that for each $x \in X$, $B \in \mathbf{P}(X)$, $x \notin \mathbf{C}(B) \Rightarrow$ there is some $\Delta \in M_X$ such that $\mathbf{C}(B) \subseteq \Delta$, $x \notin \Delta$, then : $\mathbf{C}(A) \supseteq \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$ for each $A \in \mathbf{P}(X)$, because,

$$x \notin \mathbf{C}(A) \Rightarrow x \notin \Delta \text{ , for some } \Delta \in \mathbf{S}(A), \text{ by assumption}$$

$$\Rightarrow x \notin \bigcap_{\Delta \in \mathbf{S}(A)} \Delta \quad .$$

Furthermore, $\mathbf{C}(A) \subseteq \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$, by theorem 2.5, so that $\mathbf{C}(A) = \bigcap_{\Delta \in \mathbf{S}(A)} \Delta$, for each $A \in \mathbf{P}(X)$.

So far the proofs relating properties in the closure algebra to properties in the dual space have moved from the former to the latter. Under the restriction of the Tarski condition it is possible to prove some results in the opposite direction. I first give a definition from General Topology.

<u>3.12 Definition</u>. A topological space is compact iff every family of closed sets with the finite intersection property (f.i.p.) has a non-empty intersection.

<u>3.13 Theorem</u>. Let $(X, \mathbf{C})$ be a closure algebra, and let the dual space $(\mathbb{M}_X, \tau)$ be topologically compact and such that $\mathbf{S}(A)$ is closed in $\mathbb{M}_X$, for each $A \in \mathbf{P}(X)$. If $(X, \mathbf{C})$ satisfies the Tarski condition then $(X, \mathbf{C})$ is finitary and cocompact.

Proof. To prove that $(X, \mathbf{C})$ is finitary, I must show that $\mathbf{C}(A) = \bigcup_{f \in F} \mathbf{C}(A_f)$, for each $A \in \mathbf{P}(X)$, where $f \in F$ indexes the family of finite subsets of A. We have that $\bigcup_{f \in F} \mathbf{C}(A_f) \subseteq \mathbf{C}(A)$ because each $A_f \subseteq A$ means that $\mathbf{C}(A_f) \subseteq \mathbf{C}(A)$ by 1.1 III. To show that $\mathbf{C}(A) \subseteq \bigcup_{f \in F} \mathbf{C}(A_f)$, I shall suppose that $x \notin \bigcup_{f \in F} \mathbf{C}(A_f)$ and prove that $x \notin \mathbf{C}(A)$.

Letting $\Gamma = \{\mathbf{S}(A_f) \setminus \mathbf{S}(x) \; : \; f \in F\}$, I shall show that Γ is a family of non-empty sets with the f.i.p.. This will lead to an application of compactness to show that $\bigcap_{f \in F} \mathbf{S}(A_f) \setminus \mathbf{S}(x) \neq \phi$.

Firstly each $\mathbf{S}(A_f) \setminus \mathbf{S}(x) \in \Gamma$ is non-empty because :

$$x \notin \bigcup_{f \in F} \mathbf{C}(A_f) \Rightarrow x \notin \mathbf{C}(A_f), \text{ for each } f \in F$$

$$\Rightarrow \text{for each } f \in F, \text{ there is some}$$

$\Delta \in \mathbb{M}_X$ such that $\mathbf{C}(A_f) \subseteq \Delta$ and $x \notin \Delta$, by 3.11, since $(X, \mathbf{C})$ satisfies the Tarski condition

$$\Rightarrow \Delta \in \mathbf{S}(\mathbf{C}(A_f)) \text{ and } \Delta \notin \mathbf{S}(x), \text{ for some } \Delta \in \mathbb{M}_X$$

$$\Rightarrow \Delta \in \mathbf{S}(A_f) \text{ and } \Delta \notin \mathbf{S}(x), \text{ for some } \Delta \in \mathbb{M}_X, \text{by } 2.1$$

$$\Rightarrow \Delta \in \mathbf{S}(A_f)\backslash\mathbf{S}(x), \quad \text{for some } \Delta \in \mathbb{M}_X$$

$$\Rightarrow \mathbf{S}(A_f)\backslash\mathbf{S}(x) \neq \phi, \quad \text{for each } f \in F.$$

Furthermore, each $\mathbf{S}(A_f)\backslash\mathbf{S}(x) \in \Gamma$ is closed in $(\mathbb{M}_X, \tau)$ because for each $f \in F$: $\mathbf{S}(A_f)$ is closed, by hypothesis, and $\mathbf{S}(x)$ is open since it is a member of the base. Hence $\mathbf{S}(A_f)$ is closed and $\mathbb{M}_X\backslash\mathbf{S}(x)$ is closed so that

$$\mathbf{S}(A_f)\backslash\mathbf{S}(x) = \mathbf{S}(A_f) \cap \mathbb{M}_X\backslash\mathbf{S}(x) \text{ is closed.}$$

To prove that Γ has the f.i.p. , note that for each $F' \subseteq F$:

$$\bigcap_{f\in F'} (\mathbf{S}(A_f)\backslash\mathbf{S}(x)) = \bigcap_{f\in F'} (\mathbf{S}(A_f) \cap \mathbb{M}_X\backslash\mathbf{S}(x))$$

$$= (\bigcap_{f\in F'} \mathbf{S}(A_f)) \cap \mathbb{M}_X\backslash\mathbf{S}(x)$$

$$= \mathbf{S}(\bigcup_{f\in F'} A_f) \cap \mathbb{M}_X\backslash\mathbf{S}(x), \text{ by } 3.1$$

$$= \mathbf{S}(\bigcup_{f\in F'} A_f)\backslash\mathbf{S}(x).$$

Now when F' is finite, $\bigcup_{f\in F'} A_f$ is a finite subset of A, so that

$$\bigcap_{f\in F'} (\mathbf{S}(A_f)\backslash\mathbf{S}(x)) = \mathbf{S}(\bigcup_{f\in F'} A_f)\backslash\mathbf{S}(x) \in \Gamma,$$

and is therefore non-empty. Hence Γ has the f.i.p.

Therefore $\mathbf{S}(A)\backslash\mathbf{S}(x) = \mathbf{S}(\bigcup_{f\in F} A_f)\backslash\mathbf{S}(x)$

$$= (\bigcap_{f\in F}\mathbf{S}(A_f))\backslash\mathbf{S}(x)$$

$$= \bigcap_{f\in F}(\mathbf{S}(A_f)\backslash\mathbf{S}(x)) \neq \phi, \text{ since } \Gamma$$

has the f.i.p. and $(\mathbb{M}_X,\tau)$ is compact. We now have that $x \notin \mathbf{C}(A)$ as required, because :

$$\mathbf{S}(A)\backslash\mathbf{S}(x) \neq \phi.$$

$\Rightarrow$ for some $\Delta \in \mathbb{M}_X$, $\Delta \in \mathbf{S}(A)$, $\Delta \notin \mathbf{S}(x)$

$\Rightarrow$ for some $\Delta \in \mathbb{M}_X$, $A \subseteq \Delta$, $x \notin \Delta$

$\Rightarrow$ for some $\Delta \in \mathbb{M}_X$, $\mathbf{C}(A) \subseteq \Delta$, $x \notin \Delta$.

$\Rightarrow$ $x \notin \mathbf{C}(A)$, so that :

$\mathbf{C}(A) \subseteq \bigcup_{f\in F}\mathbf{C}(A_f)$, and as outlined above, $(X, \mathbf{C})$ is finitary.

To show that $(X, \mathbf{C})$ is cocompact I show that X has a finite inconsistent subset; since $(X, \mathbf{C})$ is finitary, cocompactness will follow by 1.32

Consider the base $\beta = \{\mathbf{S}(A_f) : A_f \subseteq X, A_f \text{ finite}\}$. By assumption each member of β is closed, and we may assume each member to be non-empty, since if A_{f_0} is such that $\phi = \mathbf{S}(A_{f_0}) \in \beta$ then

$$\mathbf{C}(A_{f_0}) = \bigcap_{\Delta\in\mathbf{S}(A_{f_0})}\Delta, \text{ by the Tarski condition}$$

$$= \bigcap_{\Delta\in\phi}\Delta = X, \text{ so that } A_{f_0} \text{ is a finite in-}$$

consistent set, and the result is proved. Now

$$\bigcap_{S(A_f)\in\beta} S(A_f) = S(\bigcup_{S(A_f)\in\beta} A_f) \quad \text{- by 3.1}$$

$$= S(X)$$

$$= \phi, \text{ since no maximal consistent set may contain } X.$$

Therefore, by the compactness of $(\mathbb{M}_X, \tau)$, there is some finite subfamily of β with an empty intersection i.e. there is some $A_{f_1}, \ldots, A_{f_n}$ such that

$$\bigcap_{i=1}^{n} S(A_{f_i}) = \phi.$$

$$\Rightarrow \quad S(\bigcup_{i=1}^{n} A_{f_i}) = \phi, \text{ by 3.1.}$$

Now $\bigcup_{i=1}^{n} A_{f_i}$ is the finite union of finite sets, and is hence finite; and by the Tarski condition :

$$C(\bigcup_{i=1}^{n} A_{f_i}) = \bigcap_{\Delta\in S(\bigcup_{i=1}^{n} A_{f_i})} \Delta$$

$$= \bigcap_{\Delta\in\phi} \Delta$$

$= X$, yielding that $\bigcup_{i=1}^{n} A_{f_i}$ is a finite inconsistent set. Hence $(X, \mathbf{C})$ is cocompact.

<u>3.14 Theorem</u>. Let $(X, \mathbf{C})$ be a closure algebra, and $(\mathbb{M}_X, \tau)$ its dual space be as in 3.13. Then a set A is clopen

in $(\mathbb{M}_X,\tau)$ iff $A = \bigcup_{i=1}^{n} \mathbf{S}(A_{f_i})$ where $\mathbf{S}(A_{f_i}) \in \beta$, for $1 \leqslant i \leqslant n$.

Proof. Since each $\mathbf{S}(A_f) \in \beta$ is clopen by theorems 3.7 and 3.8, we have that finite unions of members of β are clopen.

On the other hand, if A is clopen in $(\mathbb{M}_X,\tau)$: then since A is open, $A = \bigcup_{f\in F} \mathbf{S}(A_f)$ where $f \in F$ indexes a suitable family of the base. If for any $f_0 \in F$, $A\setminus \mathbf{S}(A_{f_0}) = \phi$ then $A = \mathbf{S}(A_{f_0})$ and the theorem is proved. Hence we may assume that $A\setminus \mathbf{S}(A_f) \neq \phi$ for each $f \in F$.

Furthermore, since A is closed and $\mathbb{M}_X\setminus \mathbf{S}(A_f)$ is closed, for each $f \in F$, $A\setminus \mathbf{S}(A_f)$ is closed for each $f \in F$. Finally, since

$$\bigcap_{f\in F} A\setminus \mathbf{S}(A_f) = A\setminus \bigcup_{f\in F} \mathbf{S}(A_f)$$

$$= A\setminus A$$

$$= \phi$$

, we have that $\{A\setminus \mathbf{S}(A_f)\}_{f\in F}$ is a family of non-empty, closed sets, and has an empty intersection. Since $(\mathbb{M}_X,\tau)$ is compact, $\{A\setminus \mathbf{S}(A_f)\}_{f\in F}$ cannot have the f.i.p., i.e. there is some $f_1,\ldots,f_n \in F$ such that :

$$\bigcap_{i=1}^{n} (A\setminus \mathbf{S}(A_{f_i})) = \phi$$

$$\Rightarrow A\setminus \bigcup_{i=1}^{n} \mathbf{S}(A_{f_i}) = \phi$$

$$\Rightarrow A = \bigcup_{i=1}^{n} \mathbf{S}(A_{f_i}),$$ proving the theorem.

CHAPTER IV THE REPRESENTATION OF T_1- SPACES

This chapter uses results from the three previous chapters to establish some of the properties of the dual space (M_X, τ), of a closure algebra $(X, \mathbf{C})$. I first show that any dual space is T_1, and then that in the case of a cocompact closure algebra the dual space is Hausdorff, indeed regular.

The major result of the chapter is 4.5 which shows that any T_1-space can be represented as the dual space of a closure algebra. In view of theorem 4.1, T_1- spaces can be characterized as follows : a topological space is T_1 iff it is homeomorphic to the dual space of a closure algebra.

As a corollary to the chapter I obtain the result that the dual space of a cocompact closure algebra is metrizable and separable. This result however does not seem to be reversible.

<u>4.1 Theorem</u>. If $(X, \mathbf{C})$ is a closure algebra then the dual space (M_X, τ) is a T_1-space, i.e. singleton sets are closed.

Proof. Let $\Delta \in M_X$, $\Gamma = \{A_f : A_f \text{ is finite}, A_f \not\subseteq \Delta\}$, and let $f \in F$ index Γ. I shall prove that $\{\Delta\} = M_X \setminus \bigcup_{f \in F} \mathbf{S}(A_f)$, by showing that $\Delta \in M_X \setminus \bigcup_{f \in F} \mathbf{S}(A_f)$, and that if $\Delta' \in M_X \setminus \bigcup_{f \in F} \mathbf{S}(A_f)$

then $\Delta' = \Delta$. It will follow that

$$\{\Delta\} = \mathbb{M}_X \setminus \bigcup_{f\in F} \mathbf{S}(A_f), \text{ and is closed.}$$

Since $A_f \nsubseteq \Delta$ for each $f \in F$, we have that

$$\Delta \notin \mathbf{S}(A_f), \text{ for each } f \in F.$$

$$\Rightarrow \quad \Delta \notin \bigcup_{f\in F} \mathbf{S}(A_f)$$

$$\Rightarrow \quad \Delta \in \mathbb{M}_X \setminus \bigcup_{f\in F} \mathbf{S}(A_f).$$

Furthermore, if $\Delta' \in \mathbb{M}_X \setminus \bigcup_{f\in F} \mathbf{S}(A_f)$, choose an $x \notin \Delta$, so that $\{x\} \nsubseteq \Delta$. Now we must have $\{x\} = A_{f_0}$, for some $f_0 \in F$, by the construction of Γ.

Since $\Delta' \in \mathbb{M}_X \setminus \bigcup_{f\in F} \mathbf{S}(A_f)$ we have that

$$\Delta' \notin \mathbf{S}(A_f), \text{ for each } f \in F, \text{ in particular}$$

$$\Delta' \notin \mathbf{S}(A_{f_0})$$

$$\Rightarrow A_{f_0} \nsubseteq \Delta'$$

$$\Rightarrow \{x\} \nsubseteq \Delta'$$

$$\Rightarrow x \notin \Delta'.$$

This holds for each $x \notin \Delta$, hence $\Delta' \subseteq \Delta$. But Δ' is a maximal consistent subset of X so that $\Delta' = \Delta$. Therefore $\{\Delta\} = \mathbb{M}_X \setminus \bigcup_{f\in F} \mathbf{S}(A_f)$. Finally, for each $f \in F$, $\mathbf{S}(A_f)$ is a member of the base β of $(\mathbb{M}_X, \tau)$. This means that $\bigcup_{f\in F} \mathbf{S}(A_f)$ is open and that $\mathbb{M}_X \setminus \bigcup_{f\in F} \mathbf{S}(A_f)$ is closed.

Since Δ was an arbitrary member of $\mathbb{M}_X$, we have that each singleton subset of $\mathbb{M}_X$ is closed, and that $\mathbb{M}_X$ is a T_1-space.

<u>4.2 Theorem</u>. Let $(X, \mathbf{C})$ be a closure algebra and (M_X, τ) its dual space. If $(X, \mathbf{C})$ is cocompact then (M_X, τ) is Hausdorff; i.e. distinct points have disjoint neighbourhoods.

Proof. Let $\Delta_1, \Delta_2 \in M_X$ and assume that for any pair of sets $N_1, N_2 \subseteq M_X$ that are neighbourhoods of Δ_1, Δ_2 respectively, we have that $N_1 \cap N_2 \neq \phi$. I shall show that in this case $\Delta_1 = \Delta_2$, and the result will follow by contraposition.

Consider the set $\Delta_1 \cup \Delta_2 \subseteq X$, and let A_f be any finite subset of $\Delta_1 \cup \Delta_2$.

If $A_f \subseteq \Delta_1$, then A_f is consistent, since $\mathbf{C}(A_f) \subseteq \mathbf{C}(\Delta_1) = \Delta_1 \neq X$, by 1.27.

Similarly, if $A_f \subseteq \Delta_2$, then A_f is consistent.

Furthermore, if $A_f = A_{f_1} \cup A_{f_2}$, with $A_{f_1} \subseteq \Delta_1$, $A_{f_2} \subseteq \Delta_2$, then $\Delta_1 \in \mathbf{S}(A_{f_1})$, $\Delta_2 \in \mathbf{S}(A_{f_2})$. Since A_{f_1}, A_{f_2} are finite, $\mathbf{S}(A_{f_1})$, $\mathbf{S}(A_{f_2})$ are members of the base and are thus neighbourhoods of Δ_1, Δ_2 respectively. Hence

$$\mathbf{S}(A_{f_1}) \cap \mathbf{S}(A_{f_2}) \neq \phi \quad \text{by the assumption that}$$

Δ_1, Δ_2 have no pair of disjoint neighbourhoods.

$$\Rightarrow \quad \mathbf{S}(A_{f_1} \cup A_{f_2}) \neq \phi \text{ , by 3.1}$$

$$\Rightarrow \quad \mathbf{S}(A_f) \qquad \neq \phi, \text{ since } A_f = A_{f_1} \cup A_{f_2}.$$

In this case A_f is contained in some maximal consistent set, and is itself consistent.

I have shown that all finite subsets of $\Delta_1 \cup \Delta_2$ are consistent, and by the cocompactness of $(X, \mathbf{C})$ it follows that $\Delta_1 \cup \Delta_2$ is consistent. Finally, $\Delta_1 \subseteq \Delta_1 \cup \Delta_2$ and $\Delta_2 \subseteq \Delta_1 \cup \Delta_2$; however, Δ_1, Δ_2 are maximal consistent so that :

$$\Delta_1 = \Delta_1 \cup \Delta_2, \quad \Delta_2 = \Delta_1 \cup \Delta_2 \text{ , and hence}$$
$$\Delta_1 = \Delta_2 \ .$$

The result follows by contraposition.

We now introduce the concept of regularity so that the topological separation properties of $(\mathbb{M}_X, \tau)$ can be investigated a little further.

<u>4.3 Definition.</u> (Dugundji [1]). A Hausdorff space is regular iff for each point y in the space and for each closed set A not containing y there are disjoint neighbourhoods. That is, there is a neighbourhood u of y and an open set $0 \supseteq A$ such that $u \cap 0 = \phi$.

<u>4.4 Theorem.</u> If $(X, \mathbf{C})$ is a cocompact closure algebra, then the dual space $(\mathbb{M}_X, \tau)$ is regular.

Proof. Let T be any closed set in $\mathbb{M}_X$, $\Delta \notin T$. For a suitable indexing $f \in F$ of some subset of the base :

$$\mathbb{M}_X \setminus T = \bigcup_{f \in F} \mathbf{S}(A_f), \text{ since } \mathbb{M}_X \setminus T \text{ is open}$$

$$\Rightarrow \quad \Delta \in \bigcup_{f \in F} \mathbf{S}(A_f), \text{ since } \Delta \notin T$$

$$\Rightarrow \quad \Delta \in \mathbf{S}(A_{f_0}), \text{ for some } f_0 \in F, \text{ which is to say}$$

that $\mathbf{S}(A_{f_0})$ is an open neighbourhood of Δ. I put

$0 = \mathbb{M}_X \setminus \mathbf{S}(A_{f_0})$. By the cocompactness of $(X, \mathbf{C})$ and theorem 3.7, $\mathbf{S}(A_{f_0})$ is closed so that 0 is open, and by construction $0 \cap \mathbf{S}(A_{f_0}) = \phi$. It only remains to be shown that $0 \supseteq T$, and it will follow that $(\mathbb{M}_X, \tau)$ is regular. We have :

$$\mathbb{M}_X \setminus T = \bigcup_{f \in F} \mathbf{S}(A_f)$$

$$\Rightarrow T = \mathbb{M}_X \setminus \bigcup_{f \in F} \mathbf{S}(A_f)$$

$$\Rightarrow T \subseteq \mathbb{M}_X \setminus \mathbf{S}(A_{f_0}), \text{ since } f_0 \in F.$$

$$\Rightarrow T \subseteq 0, \text{ by the construction of } 0.$$

In 4.1 it was shown that the dual space of a closure algebra is always a T_1-space. The possibility of so representing every T_1-space arises; the next theorem gives the appropriate result.

<u>4.5 Theorem</u>. Every T_1 space is homeomorphic to the dual space of a closure algebra.

Proof. Let Y be any T_1 space, and denote the family of all open subsets of Y, by X. I define a closure operation $\mathbf{C}$, on $\mathbf{P}(X)$ as follows :

$$\mathbf{C}(A) = \{u \in X : \bigcap_{\alpha \in A} \alpha \subseteq u\}, \text{ for each } A \in \mathbf{P}(X).$$

Firstly I show that $\mathbf{C}$ is a closure operation.

(i) $\quad u \in A \Rightarrow u \supseteq \bigcap_{\alpha\in A} \alpha$

$\Rightarrow u \in \mathbf{C}(A)$, so that

$A \subseteq \mathbf{C}(A)$, for each $A \in \mathbf{P}(X)$.

(ii) $\quad$ If $u \in \mathbf{C}(A)$, then $\bigcap_{\alpha\in A} \alpha \subseteq u$, so that

$$\bigcap_{u\in \mathbf{C}(A)} u \supseteq \bigcap_{\alpha\in A} \alpha$$

$$\Rightarrow \quad \{u \in X: \bigcap_{\alpha\in \mathbf{C}(A)} \alpha \subseteq u\} \subseteq \{u \in X: \bigcap_{\alpha\in A} \alpha \subseteq u\}.$$

$$\Rightarrow \quad \mathbf{C}(\mathbf{C}(A)) \subseteq \mathbf{C}(A).$$

Coupled with (i), this gives $\mathbf{C}(\mathbf{C}(A)) = \mathbf{C}(A)$, for each $A \in \mathbf{P}(X)$.

(iii) $\quad A \subseteq B \Rightarrow \bigcap_{\alpha\in A} \alpha \supseteq \bigcap_{\alpha\in B} \alpha.$

$$\Rightarrow \{u \in X : \bigcap_{\alpha\in A} \alpha \subseteq u\} \subseteq \{u \in X: \bigcap_{\alpha\in B} \alpha \subseteq u\}$$

$$\Rightarrow \mathbf{C}(A) \subseteq \mathbf{C}(B), \text{ for each } A, B \in \mathbf{P}(X).$$

(i), (ii), (iii) yield that $\mathbf{C}$ is a closure operation.

We turn now to a description of the family of maximal consistent sets, M_X. For each $y \in Y$, I define $\&_y = \{u \in X : y \in u\}$, and now prove the following lemma about $\&_y$:

$$Y\backslash\{x\} \in \&_y \quad \text{iff } x \neq y.$$

This holds since : $x = y \Rightarrow y \notin Y\backslash\{x\}$.

$\Rightarrow Y\backslash\{x\} \notin \&_y$, by construction of $\&_y$,

and : $x \neq y \Rightarrow y \in Y\backslash\{x\}$, while because Y is T_1, $\{x\}$ is closed and $Y\backslash\{x\}$ is open, yielding that $Y\backslash\{x\} \in X$. Hence $Y\backslash\{x\} \in \&_y$ in this case.

I now show that $\mathbb{M}_X = \{\&_y : y \in Y\}$.

Let $y \in Y$, then $\&_y$ is closed since :

$$u \in \mathbf{C}(\&_y) \Rightarrow u \in X,\ u \supseteq \bigcap_{\alpha \in \&_y} \alpha$$

$$\Rightarrow u \supseteq \{y\}, \text{ since } y \in \alpha$$

for each $\alpha \in \&_y$, by the construction of $\&_y$.

$$\Rightarrow y \in u$$

$$\Rightarrow u \in \&_y. \quad \text{Hence } \&_y \text{ is closed by 1.3.}$$

Using this we have that $\&_y$ is consistent, because $Y\backslash\{y\} \in X$ since $\{y\}$ closed and $Y\backslash\{y\}$ open and, $Y\backslash\{y\} \notin \&_y$ by the lemma. Hence

$$\mathbf{C}(\&_y) = \&_y \neq X, \text{ and } \&_y \text{ is consistent.}$$

Furthermore, $\&_y$ is maximal consistent, since it can be shown that $\mathbf{C}(\&_y \cup \{v\}) = X$, for each v with $v \in X\backslash\&_y$ as follows.

If $v \in X$ and $v \notin \&_y$ then $y \notin v$, and $y \notin \bigcap_{\alpha \in \&_y} \alpha \cap v$.

Furthermore, if $x \neq y$, $x \in Y$ then $Y\backslash\{x\} \in \&_y$, by the lemma. Hence there is some $\alpha_0 \in \&_y$ such that $x \notin \alpha_0$ and so $x \notin \bigcap_{\alpha \in \&_y} \alpha$, for each $x \in Y$, with $x \neq y$. This shows that

$$\bigcap_{\alpha \in \mathcal{E}_y} \alpha \cap v = \phi$$

$$\Rightarrow \bigcap_{\alpha \in \mathcal{E}_y \cup \{v\}} \alpha = \phi.$$

$$\Rightarrow \{u \in X : \bigcap_{\alpha \in \mathcal{E}_y \cup \{v\}} \alpha \subseteq u\} = X$$

$$\Rightarrow \mathbf{C}(\mathcal{E}_y \cup \{v\}) = X.$$

Hence $\mathcal{E}_y$ is maximal consistent, and

$$\{\mathcal{E}_y : y \in Y\} \subseteq \mathbb{M}_X.$$

For the opposite inclusion, I show that if $\Delta \in \mathbb{M}_X$ then $\Delta = \mathcal{E}_y$, for some $y \in Y$. We have that

$$\bigcap_{\alpha \in \Delta} \alpha \neq \phi \quad \text{because}$$

$$\bigcap_{\alpha \in \Delta} \alpha = \phi$$

$$\Rightarrow \quad u \supseteq \bigcap_{\alpha \in \Delta} \alpha, \text{ for each } u \in X$$

$$\Rightarrow \quad X = \{u \in X : \bigcap_{\alpha \in \Delta} \alpha \subseteq u\}$$

$$\Rightarrow \quad \mathbf{C}(\Delta) = X$$

contrary to Δ being consistent.

In view of this take $y \in \bigcap_{\alpha \in \Delta} \alpha$. We have that

$$\Delta \subseteq \mathcal{E}_y \quad \text{because :}$$

$$u \in \Delta$$

$$\Rightarrow \quad y \in \bigcap_{\alpha \in \Delta} \alpha \subseteq u$$

$$\Rightarrow \quad u \in \mathcal{E}_y$$

But $\&_y$ is consistent and Δ is maximal consistent, so $\Delta = \&_y$. We now have that $\mathbb{M}_X = \{\&_y : y \in Y\}$.

I now prove that Y and $(\mathbb{M}_X, \tau)$ are homeomorphic. Consider the mapping $\theta : Y \to \mathbb{M}_X$ defined by

$$\theta(y) = \&_y, \text{ for each } y \in Y.$$

θ is clearly a function, and since $\mathbb{M}_X = \{\&_y : y \in Y\}$, θ maps onto $\mathbb{M}_X$. θ is one to one because, for each $x,y \in Y$:

$$\begin{aligned} & x \neq y \\ \Rightarrow\quad & \{x\} \neq \{y\} \\ \Rightarrow\quad & Y\backslash\{x\} \neq Y\backslash\{y\}. \end{aligned}$$

Now $Y\backslash\{x\} \notin \&_x$ and $Y\backslash\{y\} \notin \&_y$, by the lemma, while $Y\backslash\{x\} \in \&_y$ and $Y\backslash\{y\} \in \&_x$, by the lemma, and since $x \neq y$. Hence $\&_x \neq \&_y$ and so

$$\theta(x) \neq \theta(y), \text{ for each } x,y \in Y.$$

I now show that θ^{-1} is a continuous function. Suppose $u \subseteq Y$ and that u is open in Y, then

$$\begin{aligned} \theta[u] &= \{\theta(x) : x \in u\} \\ &= \{\&_x : x \in u\} \\ &= \{\&_x : u \in \&_x\} \\ &= \{\Delta \in \mathbb{M}_X : u \in \Delta\}, \text{ as shown above} \\ &= \mathbf{S}(u). \end{aligned}$$

Now $\{u\}$ is a singleton subset of X, and hence $\mathbf{S}(u)$ is in the base for $(\mathbb{M}_X, \tau)$, yielding that $\theta[u]$ is open. This shows that θ^{-1} is continuous.

To show the continuity of θ, I need the following result : for each $y \in Y$, A_f any finite subset of X;

$$A_f \subseteq \&_y \iff \bigcap_{\alpha \in A_f} \alpha \in \&_y.$$

This holds because : $A_f \subseteq \&_y \Rightarrow \alpha \in \&_y$, for each $\alpha \in A_f$.

$\Rightarrow y \in \alpha$ and $\alpha \in X$, for each $\alpha \in A_f$.

$\Rightarrow y \in \bigcap_{\alpha \in A_f} \alpha$ and $\bigcap_{\alpha \in A_f} \alpha \in X$, since A_f is finite.

$\Rightarrow \bigcap_{\alpha \in A_f} \alpha \in \&_y$.

Conversely, if $\bigcap_{\alpha \in A_f} \alpha \in \&_y$, then for each $\alpha \in A_f$, $y \in \bigcap_{\alpha \in A_f} \alpha \subseteq \alpha$, so that $\alpha \in \&_y$ for each $\alpha \in A_f$.

$\Rightarrow A_f \subseteq \&_y$.

To see that θ is continuous, assume that $0 \subseteq \mathbb{M}_X$ is open. Then $0 = \bigcup_{f \in F} \mathbf{S}(A_f)$, where $\{\mathbf{S}(A_f)\}_{f \in F} \subseteq \beta$, the base for $(\mathbb{M}_X, \tau)$. With this,

$$\begin{aligned}
\theta^{-1}[0] &= \theta^{-1}[\bigcup_{f \in F} \mathbf{S}(A_f)] \\
&= \bigcup_{f \in F} (\theta^{-1}[\mathbf{S}(A_f)] \text{ , since } \theta \text{ is a function} \\
&= \bigcup_{f \in F} (\theta^{-1}[\{\Delta \in \mathbb{M}_X : A_f \subseteq \Delta\}]) \\
&= \bigcup_{f \in F} (\theta^{-1}[\{\&_y : y \in Y, A_f \subseteq \&_y\}]) \text{, since } \mathbb{M}_X = \{\&_y : y \in Y\} \\
&= \bigcup_{f \in F} (\theta^{-1}[\{\&_y : y \in Y, \bigcap_{\alpha \in A_f} \alpha \in \&_y\}]) \text{,}
\end{aligned}$$

by the above result.

$= \bigcup_{f\in F} (\theta^{-1}[\{\&_y : y \in Y,\ y \in \bigcap_{\alpha\in A_f} \alpha\}])$, by the definition of $\&_y$.

$= \bigcup_{f\in F} (\{y \in Y : y \in \bigcap_{\alpha\in A_f} \alpha\})$.

$= \bigcup_{f\in F} (\bigcap_{\alpha\in A_f} \alpha)$.

Now each A_f is a finite set of open subsets of Y, and hence $\bigcap_{\alpha\in A_f} \alpha$ is an open subset of Y, and $\bigcup_{f\in F} (\bigcap_{\alpha\in A_f} \alpha)$ is an open subset of Y. This proves that θ is continuous and that under θ, $\mathbf{M}_X$, Y are homeomorphic.

The final result for this chapter will show that the dual space of a cocompact closure algebra with a countable carrier is metrizable and separable. The following result from General Topology, due to Urysohn is required : a regular T_1-space whose topology has a countable base is metrizable and separable. A reference is Kelley[1], p.125.

4.6 Theorem. Let (X, **C**) be a cocompact closure algebra with a countable carrier, then $(\mathbf{M}_X,\tau)$ is metrizable and separable.

Proof. By 4.1 $(\mathbf{M}_X,\tau)$ is a T_1-space and by the cocompactness of (X, **C**) and theorem 4.4, $(\mathbf{M}_X,\tau)$ is regular. Due to the above theorem of Urysohn, it only remains to be shown that the base β is countable for the result to follow. Consider the following subsets of β :

$$\beta_1 = \{\mathbf{S}(A_f) : A_f \text{ has one member}\}$$

$$\beta_2 = \{\mathbf{S}(A_f) : A_f \text{ has two members}\}$$

$$\vdots$$

$$\beta_n = \{\mathbf{S}(A_f) : A_f \text{ has n members}\},$$

$$\vdots$$

Then for each $n < \omega$, β_n is countable, since the carrier X is countable, and only has countably many subsets of finite cardinality n. Furthermore, $\beta = \bigcup_{n<\omega} \beta_n$ is countable, establishing the theorem.

CHAPTER V. SUBALGEBRAS, HOMOMORPHISMS AND CONGRUENCES

Chapter V is wholly concerned with the algebraic aspects of a closure algebra. I start out by defining subalgebras, homomorphisms and congruences for closure algebras, and then prove some of the basic results needed in the mechanics of succeeding theorems. The definitions are quite natural and are shown to generalize the usual notions of subalgebra, homomorphism and congruence from Universal Algebra. Analogues of the homomorphism theorem and the first and second isomorphism theorems are proved. The formulation of these theorems and some of the strategies for their proofs follow the versions in Grätzer [1], as does the proof of the analogue of the Zassenhaus lemma. The proof that the join of two closure congruences is a closure congruence, is also based on ideas and definitions from Grätzer [1] ; particular references are given. Finally the version of the Jordan-Holder-Schreier theorem is modelled almost exactly on the treatment given by Gould in Gould [1].

<u>5.1 Definition</u>. Let $(X,\mathbf{C})$, $(Y,\mathbf{C}_1)$ be closure algebras. We say that $(Y, \mathbf{C}_1)$ is a subalgebra of $(X, \mathbf{C})$ iff Y is a subset of X that is closed with respect to $\mathbf{C}$, and $\mathbf{C}_1(A) = \mathbf{C}(A)$, for each $A \in \mathbf{P}(Y)$.

5.1 Definition. Let $(X,\mathbf{C})$, $(Y,\mathbf{C}_1)$ be closure algebras, and $\theta : X \to Y$ a function such that :

$$\theta[\mathbf{C}(A)] = \mathbf{C}_1(\theta[A]), \text{ for each } A \in \mathbf{P}(X), \text{ then}$$

θ is called a (closure) homomorphism. If θ is also one-to-one and onto then θ is called a (closure) isomorphism, and $(X,\mathbf{C})$ $(Y,\mathbf{C}_1)$ are said to be isomorphic. In this case we will write : $(X, \mathbf{C}) \cong (Y, \mathbf{C}_1)$.

5.3 Definition. Let $(X, \mathbf{C})$ be a closure algebra, and let R be an equivalence relation on X. We define the induced relation $\equiv$ on $\mathbf{P}(X)$ by :

$A \equiv B(R)$ iff for each $a \in A$ there is some $b \in B$ such that aRb, and for each $b \in B$ there is some $a \in A$ such that aRb.

We have that this induced relation is an equivalence relation because :

(i) $\equiv$ is reflexive, since if $A \in \mathbf{P}(X)$ and $a \in A$ then aRa, since R is reflexive, so that $A \equiv A(R)$.

(ii) $\equiv$ is symmetric, from the obvious symmetry in the definition of $\equiv$.

(iii) $\equiv$ is transitive, since if $A,B,C \in \mathbf{P}(X)$, and $A \equiv B(R)$, $B \equiv C(R)$, and $a \in A$; there is some $b \in B$ such that aRb, and some $c \in C$ such that bRc. Hence aRc, since R is transitive. Similarly, if $c \in C$ then there is some $a \in A$ such that aRc; yielding $A \equiv C(R)$.

For each $A \in \mathbf{P}(X)$ we denote by $[A]_R$ that subset of X that is precisely those elements which are related to some member of A; i.e.,

$[A]_R = \{x \in X : xRy$, for some $y \in A\}$. It is common in the literative that $[\ ]_R$ is a closure operation, specifically : $A \subseteq [A]_R = [[A]_R]_R$ and $A \subseteq B \Rightarrow [A]_R \subseteq [B]_R$, for each $A,B \in \mathbf{P}(X)$. These facts will be used in proofs without further comment.

It will often be convenient to work with sets of the form $[A]_R$ rather than with arbitrary subsets of X, and for this reason I now prove the following lemma.

<u>5.4 Lemma</u>. $A \equiv B(R)$ iff $[A]_R = [B]_R$.

Proof. Suppose that $A \equiv B(R)$, then $[A]_R = [B]_R$, because :

$x \in [A]_R$

$\Rightarrow$ xRy, for some $y \in A$

$\Rightarrow$ xRy and yRz, for some $y \in A$, $z \in B$, since $A \equiv B(R)$

$\Rightarrow$ xRz, for some $z \in B$, since R is transitive.

$\Rightarrow$ $x \in [B]_R$

Similarly, if $x \in [B]_R$, then $x \in [A]_R$, so that $[A]_R = [B]_R$.

Conversely, if $[A]_R = [B]_R$, then $A \equiv B(R)$, because :

$x \in A$

$\Rightarrow$ $x \in [A]_R$

$\Rightarrow$ $x \in [B]_R$

$\Rightarrow$ xRy, for some $y \in B$.

Similarly, if $y \in B$ then xRy, for some $x \in A$, and hence $A \equiv B(R)$.

5.5 Definition. If $(X, \mathbf{C})$ is a closure algebra, and R is an equivalence relation on X, then R is said to be a (closure) congruence of $(X, \mathbf{C})$ whenever :

$$A \equiv B(R) \Rightarrow \mathbf{C}(A) \equiv \mathbf{C}(B)(R), \text{ for each } A,B \in \mathbf{P}(X) \text{ where}$$

$\equiv$ is the equivalence relation induced by R on $\mathbf{P}(X)$.

For an equivalence relation R, as above, the following definition of a closure congruence was suggested by R.A. Bull : $\mathbf{C}([A]_R) \subseteq [\mathbf{C}(A)]_R$, for each $A \in \mathbf{P}(X)$. It was only somewhat later that the two definitions were seen to be equivalent. Since it is often easier to work with this second definition, the equivalence is proved in the next theorem.

5.6 Theorem. If $(X, \mathbf{C})$ is a closure algebra, and R is an equivalence relation on X, then R is a congruence of $(X, \mathbf{C})$ iff $\mathbf{C}([A]_R) \subseteq [\mathbf{C}(A)]_R$, for each $A \in \mathbf{P}(X)$.

Proof. Assume that R is a congruence of $(X, \mathbf{C})$, then R is an equivalence relation, so that :

$$[\mathbf{C}(A)]_R = [[\mathbf{C}(A)]_R]_R, \text{ for each } A \in \mathbf{P}(X)$$

$$\Rightarrow \mathbf{C}(A) \equiv [\mathbf{C}(A)]_R(R), \text{ for each } A \in \mathbf{P}(X) \text{ by lemma 5.4}$$

$$\Rightarrow \mathbf{C}(\mathbf{C}(A)) \equiv \mathbf{C}([\mathbf{C}(A)]_R)(R), \text{ since } R \text{ is a congurence}$$

$$\Rightarrow \mathbf{C}(A) \equiv \mathbf{C}([\mathbf{C}(A)]_R)(R), \text{ by 1.1 II.}$$

$$\Rightarrow [\mathbf{C}(A)]_R = [\mathbf{C}([\mathbf{C}(A)]_R)]_R, \text{ for each } A \in \mathbf{P}(X), \text{ by 5.4.}$$

Furthermore, $A \subseteq \mathbf{C}(A)$, by 1.1

$$\Rightarrow [A]_R \subseteq [\mathbf{C}(A)]_R$$

so that $\mathbf{C}([A]_R) \subseteq \mathbf{C}([\mathbf{C}(A)]_R)$, by 1.1

$$\subseteq [\mathbf{C}([\mathbf{C}(A)]_R)]_R$$

$$= [\mathbf{C}(A)]_R, \text{ from above.}$$

This shows that $\mathbf{C}([A]_R) \subseteq [\mathbf{C}(A)]_R$, for each $A \in \mathbf{P}(X)$.

Conversely, assume that $\mathbf{C}([A]_R) \subseteq [\mathbf{C}(A)]_R$, for each $A \in \mathbf{P}(X)$, and that $A \equiv B(R)$. Then $\mathbf{C}(A) \equiv \mathbf{C}(B)(R)$ because

$$\mathbf{C}(A) \subseteq \mathbf{C}([A]_R), \text{ since } A \subseteq [A]_R$$

$$= \mathbf{C}([B]_R), \text{ since } A \equiv B(R), \text{ and so } [A]_R = [B]_R, \text{ by 5.4.}$$

$$\subseteq [\mathbf{C}(B)]_R, \text{ by hypothesis.}$$

Hence $[\mathbf{C}(A)]_R \subseteq [[\mathbf{C}(B)_R]_R$, since $[\ \ ]_R$ is a closure operation

$$= [\mathbf{C}(B)]_R, \text{ for the same reason.}$$

Similarly, it may be shown that $[\mathbf{C}(B)]_R \subseteq [\mathbf{C}(A)]_R$, so that

$$[\mathbf{C}(A)]_R = [\mathbf{C}(B)]_R ,$$

Hence $\mathbf{C}(A) \equiv \mathbf{C}(B)(R)$, by theorem 5.4.

5.7 Theorem. Let $(X, \mathbf{C})$ be a closure algebra, and R a congruence of $(X, \mathbf{C})$. Let $[x] = [\{x\}]_R$ and $X/R = \{[x] : x \in X\}$. Define a function $\mathbf{C}_R : \mathbf{P}(X/R) \to \mathbf{P}(X/R)$ by :

$$\mathbf{C}_R(A) = \{[x] : x \in \mathbf{C}(\bigcup_{[y] \in A} [y])\} \text{ , for each } A \in \mathbf{P}(X/R)$$

then $(X/R, \mathbf{C}_R)$ is a closure algebra (called the factor algebra of X by R).

Proof. To see that $(X/R, \mathbf{C}_R)$ is a closure algebra, note that $\mathbf{C}_R$ is clearly a function, and that it is also a closure operation on X/R, since if $A,B \in \mathbf{P}(X)$ then:

(i) $A \subseteq \mathbf{C}_R(A)$ because;

$$
\begin{aligned}
[x] \in A &\Rightarrow x \in [x] \text{ and } [x] \in A \\
&\Rightarrow x \in \bigcup_{[y]\in A} [y] \\
&\Rightarrow x \in \mathbf{C}\Big(\bigcup_{[y]\in A} [y]\Big) \\
&\Rightarrow [x] \in \mathbf{C}_R(A), \text{ by the definition of } \mathbf{C}_R.
\end{aligned}
$$

(ii) Note that :

$$
\begin{aligned}
y \in \bigcup_{[x]\in \mathbf{C}_R(A)} [x] &\iff y \in [x], \text{ for some } [x] \in \mathbf{C}_R(A), \\
&\iff [y] \in \mathbf{C}_R(A), \\
&\quad\ \text{since } y \in [x] \iff [y] = [x] \\
&\iff y \in \mathbf{C}\Big(\bigcup_{[x]\in A} [x]\Big), \text{ by the definition of } \mathbf{C}_R.
\end{aligned}
$$

On account of this we have that

$$\bigcup_{[x]\in \mathbf{C}_R(A)} [x] = \mathbf{C}\Big(\bigcup_{[x]\in A} [x]\Big), \text{ and so}$$

$$\mathbf{C}\Big(\bigcup_{[x]\in \mathbf{C}_R(A)} [x]\Big) = \mathbf{C}\Big(\mathbf{C}\Big(\bigcup_{[x]\in A} [x]\Big)\Big) = \mathbf{C}\Big(\bigcup_{[x]\in A} [x]\Big), \text{ by 1.1}$$

Hence

$$
\begin{aligned}
\mathbf{C}_R(\mathbf{C}_R(A)) &= \Big\{[y] : y \in \mathbf{C}\Big(\bigcup_{[x]\in \mathbf{C}_R(A)} [x]\Big)\Big\} \\
&= \Big\{[y] : y \in \mathbf{C}\Big(\bigcup_{[x]\in A} [x]\Big)\Big\} \\
&= \mathbf{C}_R(A).
\end{aligned}
$$

(iii) $A \subseteq B \Rightarrow \bigcup_{[x]\in A} [x] \subseteq \bigcup_{[x]\in B} [x]$

$$\Rightarrow \mathbf{C}\Big(\bigcup_{[x]\in A} [x]\Big) \subseteq \mathbf{C}\Big(\bigcup_{[x]\in B} [x]\Big)$$

$$\Rightarrow \{[y] : y \in \mathbf{C}\Big(\bigcup_{[x]\in A} [x]\Big)\} \subseteq \{[y] : y \in \mathbf{C}\Big(\bigcup_{[x]\in B} [x]\Big)\}$$

$$\Rightarrow \mathbf{C}_R(A) \subseteq \mathbf{C}_R(B).$$

By (i), (ii), (iii) $(X/R, \mathbf{C}_R)$ is a closure algebra.

In the next few sections of this chapter I will build up the various relationships between (closure) homomorphisms and (closure) congruences. In order to motivate the results, as well as to show that the initial definitions are not without some interest, I will now demonstrate that the concept of a closure homomorphism generalizes the concept of a homomorphism from Universal Algebra, and that the concept of a closure congruence generalizes the concept of a congruence that is used in Universal Algebra. It is clear that my definition of a subalgebra includes the ordinary sense of the term whenever one is considering a closure algebra of the type in example 1.11 (i).

To enable me to do this, I now borrow two standard results from Universal Algebra. Consider two universal algebras, U,V, with carriers X,Y and associated closure algebras $(X, \mathbf{C}_1)$, $(Y, \mathbf{C}_2)$ as in 1.11 (i). If R is a congruence of U, and $\theta : X \to Y$ a homomorphism of U onto V, and $A \in \mathbf{P}(X)$, $B \in \mathbf{P}(Y)$ then :

(i) A is a subalgebra of U implies that $\theta[A]$ is a subalgebra of V, and B a subalgebra of V implies that

$$\theta^{-1}[B] = \{x \in A : \theta(x) \in B\} \text{ is a subalgebra of U.}$$

A reference is Grätzer [1] pp.36,37.

(ii) If A is a subalgebra of U, then $[A]_R$ is a subalgebra of U.

Using these two results we have the following two theorems.

<u>5.8 Theorem</u>. If U,V are two universal algebras with carriers X,Y, and associated closure algebras $(X, \mathbf{C}_1)$ $(Y, \mathbf{C}_2)$ respectively, and θ is a homomorphism of U onto V, then θ is a closure homomorphism of $(X,\mathbf{C}_1)$ onto $(Y,\mathbf{C}_2)$.

Proof. We have that θ is a function of X onto Y, so it only remains to be shown that

$$\theta[\mathbf{C}_1(A)] = \mathbf{C}_2(\theta[x]), \text{ for each } A \in \mathbf{P}(X).$$

Let $A \in \mathbf{P}(X)$, then $A \subseteq \mathbf{C}_1(A)$

$$\Rightarrow \theta[A] \subseteq \theta[\mathbf{C}_1(A)]$$

$$\Rightarrow \mathbf{C}_2(\theta[A]) \subseteq \mathbf{C}_2(\theta[\mathbf{C}_1(A)]$$

$$= \theta[\mathbf{C}_1(A)], \text{ since } \theta[\mathbf{C}_1(A)] \text{ is a subalgebra}$$

of V, by result (i), and hence is closed in $(Y, \mathbf{C}_2)$. This establishes that $\mathbf{C}_2(\theta[A]) \subseteq \theta[\mathbf{C}_1(A)]$; the opposite inclusion holds because, for each $A \in \mathbf{P}(X)$:

$$x \in A \Rightarrow \theta(x) \in \theta[A]$$

$$\Rightarrow \theta(x) \in \mathbf{C}_2(\theta[A])$$

$$\Rightarrow x \in \theta^{-1}[\mathbf{C}_2(\theta[A])].$$

Hence $\quad A \subseteq \theta^{-1}[\mathbf{C}_2(\theta[A])]$.

$$\Rightarrow \mathbf{C}_1(A) \subseteq \mathbf{C}_1(\theta^{-1}[\mathbf{C}_2(\theta[A])])$$
$$= \theta^{-1}[\mathbf{C}_2(\theta[A])], \text{ since } \theta^{-1}[\mathbf{C}_2(\theta[A])]$$

is a subalgebra of U by result (i), and so is closed in $(X, \mathbf{C}_1)$.

$$\Rightarrow \theta[\mathbf{C}_1(A)] \subseteq \theta[\theta^{-1}[\mathbf{C}_2(\theta[A])]] \quad ,$$
$$= \mathbf{C}_2(\theta[A]), \text{ since } \theta \text{ is onto.}$$

Hence $\theta[\mathbf{C}_1(A)] = \mathbf{C}_2(\theta[A])$, and θ is a closure homomorphism.

5.9 Theorem. If U is a universal algebra with associated closure system $(X, \mathbf{C})$, and R is a congruence of U, then R is a closure congruence of $(X,\mathbf{C})$.

Proof. Since R is an equivalence relation on X, it only remains to be shown that $\mathbf{C}([A]_R) \subseteq [\mathbf{C}(A)]_R$, and R will be a closure congruence by 5.6. Firstly, since $\mathbf{C}(A)$ is a subalgebra of U, we have by result (ii) that $[\mathbf{C}(A)]_R$ is a subalgebra of U, and is hence closed in $(X,\mathbf{C})$. Also

$$A \subseteq \mathbf{C}(A), \text{ for each } A \in \mathbf{P}(X)$$
$$\Rightarrow [A]_R \subset [\mathbf{C}(A)]_R$$
$$\Rightarrow \mathbf{C}([A]_R) \subseteq \mathbf{C}([\mathbf{C}(A)]_R)$$
$$= [\mathbf{C}(A)]_R \text{ , since } [\mathbf{C}(A)]_R \text{ is closed in } (X,\mathbf{C}).$$

Hence R is a closure congruence.

In general, of course, we will have that the concepts of closure homomorphism and closure congruence are wider than their analogues in Universal Algebra, since any function between two universal algebras that preserved the lattice of subalgebras would be a closure homomorphism, and would generate a closure congruence by the next theorem.

5.10 Theorem. If θ is a homomorphism of a closure algebra $(X, \mathbf{C}_1)$ onto a closure algebra $(Y, \mathbf{C}_2)$, then the relation R, defined by $xRy \Longleftrightarrow \theta(x) = \theta(y)$, is a closure congruence that satisfies :

$A \equiv B(R)$ iff $\theta[A] = \theta[B]$, for each $A,B \in \mathbf{P}(X)$.

Proof. Since θ is a function, we have by a standard result that R is an equivalence relation on X. I will first show that $A \equiv B(R)$ iff $\theta[A] = \theta[B]$, for each $A,B \in \mathbf{P}(X)$; and it will quickly follow that R is a congruence. Let $A,B \in \mathbf{P}(X)$ and assume that $A \equiv B(R)$, then :

$$\begin{aligned}
& x \in \theta[A] \\
\Rightarrow\ & x = \theta(a), \text{ for some } a \in A \\
\Rightarrow\ & x = \theta(a) \text{ and } aRb, \text{ for some } a \in A,\ b \in B, \\
& \text{since } A \equiv B(R) \\
\Rightarrow\ & x = \theta(a) = \theta(b), \text{ for some } b \in B \\
\Rightarrow\ & x \in \theta[B].
\end{aligned}$$

Similarly, if $y \in \theta[B]$ we may show that $y \in \theta[A]$ so that $\theta[A] = \theta[B]$.

Conversely, assume that $\theta[A] = \theta[B]$, then :

$$\begin{aligned}
& a \in A \\
\Rightarrow\ & \theta(a) \in \theta[A] \\
\Rightarrow\ & \theta(a) \in \theta[B] \\
\Rightarrow\ & \theta(a) = \theta(b), \text{ for some } b \in B
\end{aligned}$$

$\Rightarrow$ aRb, for some $b \in B$.

Similarly, given $b \in B$ we may find $a \in A$ such that aRb. This yields that $A \equiv B(R)$.

Now R is a congruence because :

$$\begin{aligned} A \equiv B(R) &\Rightarrow \theta[A] = \theta[B] \text{ , by above} \\ &\Rightarrow \mathbf{C}_2(\theta[A]) = \mathbf{C}_2(\theta[B]) \\ &\Rightarrow \theta[\mathbf{C}_1(A)] = \theta[\mathbf{C}_1(B)] \text{ , since } \theta \\ &\quad \text{is a homomorphism.} \\ &\Rightarrow \mathbf{C}_1(A) \equiv \mathbf{C}_1(B)(R)\text{, by above.} \end{aligned}$$

We will refer to this congruence R as the congruence generated by θ.

The definitions of closure homomorphism and closure congruence have been chosen as one way of generalizing the notions of homomorphism and congruence used in Universal Algebra. They are the definitions one would expect to get if the closure structure of a universal algebra was being studied in itself, while the algebraic structure (in terms of the elements and operations of the universal algebra) that gave rise to the closure algebra, was being ignored.

The remainder of this chapter indicates the suitability of the definitions presented so far by showing that the behaviour of a closure algebra with respect to homomorphic images, factor algebras and subalgebras is analogous to the behaviour of a universal algebra with respect to the usual definitions of these constructions. The next theorem is the closure-theoretic version of the homomorphism theorem of universal algebra, and shows that homomorphisms and factor algebras are linked in the appropriate way.

5.11 Theorem. Let $(X, \mathbf{C}_1)$, $(Y, \mathbf{C}_2)$ be two closure algebras, and let θ be a homomorphism from $(X, \mathbf{C}_1)$ onto $(Y, \mathbf{C}_2)$. Let $(X/R, \mathbf{C}_R)$ be the factor algebra, where R is the congruence generated by θ, then :

$$(X/R, \mathbf{C}_R) \cong (Y, \mathbf{C}_2).$$

Proof. Define a mapping $\xi : X/R \to Y$ by $\xi([x]) = \theta(x)$, for each $x \in X$.

(i) Since $[x] = [y] \iff xRy$

$\iff \theta(x) = \theta(y)$, we have that ξ is a one-to-one function.

(ii) If $y \in Y$, then there is some $x \in X$ such that $y = \theta(x)$, since θ is an onto function. Hence there is some $[x] \in X/R$ such that $\xi([x]) = \theta(x) = y$, and we have that ξ is onto Y.

(iii) Let $A \subseteq X/R$, then for the condition on $\mathbf{C}_2$ and $\mathbf{C}_R$:

$\xi[\mathbf{C}_R(A)] = \xi[\{[x] : x \in \mathbf{C}_1(\bigcup_{[z]\in A}[z])\}]$, by the definition of $\mathbf{C}_R$

$= \{\theta(x) : x \in \mathbf{C}_1(\bigcup_{[z]\in A}[z])\}$, by the definition of ξ

$= \theta[\mathbf{C}_1(\bigcup_{[z]\in A}[z])]$

$= \mathbf{C}_2(\theta[\bigcup_{[z]\in A}[z]])$, since θ is a homomorphism

$= \mathbf{C}_2(\{\theta(x) : x \in \bigcup_{[z]\in A}[z]\})$

$= \mathbf{C}_2(\{\theta(x) : [x] \in A\})$, since $x \in [y]$ iff $[x] = [y]$

$= \mathbf{C}_2(\xi[\{[x] : [x] \in A\}]$

$= \mathbf{C}_2(\xi[A])$.

This shows that $(X/R, \mathbf{C}_R) \cong (Y, \mathbf{C}_2)$.

In the next section I prove three lemmata that will be used in later results.

<u>5.12 Lemmata</u>. (a) If $\theta : X \to Y$ is a homomorphism of $(X, \mathbf{C}_1)$ onto $(Y, \mathbf{C}_2)$, then θ is an onto mapping of the closed sets of $(X, \mathbf{C}_1)$ to the closed sets of $(Y, \mathbf{C}_2)$.

Proof. Since $\theta[\mathbf{C}_1(A)] = \mathbf{C}_2(\theta[A])$, for each $A \in \mathbf{P}(X)$ we have that θ maps each closed subset of $(X, \mathbf{C}_1)$ to a closed subset of $(Y, \mathbf{C}_2)$. Moreover, if $B = \{x \in X : \theta(x) \in \mathbf{C}_2(D)\}$, for some $D \in \mathbf{P}(Y)$, then :

$x \in \mathbf{C}_1(B) \Rightarrow \theta(x) \in \theta[\mathbf{C}_1(B)]$

$\Rightarrow \theta(x) \in \mathbf{C}_2(\theta[B])$, since θ is a homomorphism

$\Rightarrow \theta(x) \in \mathbf{C}_2(\mathbf{C}_2(D))$, from the definition of B and since θ is onto

$\Rightarrow \theta(x) \in \mathbf{C}_2(D)$

$\Rightarrow x \in B$.

Hence B is a closed set and $\theta[B] = \mathbf{C}_2(D)$, showing that θ maps a closed subset of X to each closed subset of Y.

(b) If θ is a homomorphism of $(X, \mathbf{C}_1)$ into $(Y, \mathbf{C}_2)$ then $(\theta[X], \mathbf{C}_2)$ is a subalgebra of $(Y, \mathbf{C}_2)$.

Proof.

$$\begin{aligned} \theta[X] &= \theta[\mathbf{C}_1(X)] \\ &= \mathbf{C}_2(\theta[X]), \end{aligned}$$

and from definition 5.1, $(\mathbf{C}_2(\theta[X]), \mathbf{C}_2)$ is a subalgebra of $(Y, \mathbf{C}_2)$.

(c) If $(Y,\mathbf{C})$ is a subalgebra of a closure algebra $(X,\mathbf{C})$ then $([Y]_R, \mathbf{C})$ is a subalgebra of $(X,\mathbf{C})$, for every congruence R of $(X,\mathbf{C})$.

Proof. By definition 5.1, I need only show that $[Y]_R$ is closed in $(X,\mathbf{C})$. Since R is a congruence :

$$\begin{aligned} \mathbf{C}([Y]_R) &\subseteq [\mathbf{C}(Y)]_R, \text{ by theorem 5.6} \\ &= [Y]_R, \end{aligned}$$

since $(Y,\mathbf{C})$ is a subalgebra of $(X,\mathbf{C})$ and Y is closed in $(X,\mathbf{C})$, so that $\mathbf{C}(Y) = Y$. This shows that $[Y]_R$ is closed, and establishes the lemma.

The next result, along with 5.10 and 5.11, gives the principal relationships between closure homomorphisms and closure congruences. Again it is an analogue of a standard result for universal algebras, the factor theorem.

<u>5.13 Theorem</u>. Let $(X,\mathbf{C})$ be a closure algebra, and R a congruence of $(X,\mathbf{C})$ that gives rise to a factor algebra $(X/R, \mathbf{C}_R)$. Then $\theta : X \to X/R$, defined by $\theta(x) = [x]$, is a closure homomorphism from $(X,\mathbf{C})$ onto $(X/R, \mathbf{C}_R)$.

Proof. θ is clearly an onto function. It is also a homomorphism because $\theta[\mathbf{C}(A)] = \mathbf{C}_R(\theta[A])$. Firstly $\mathbf{C}_R(\theta[A]) \subseteq \theta[\mathbf{C}(A)]$, since :

$$[x] \in \mathbf{C}_R(\theta[A])$$

$$\Rightarrow \quad x \in \mathbf{C}\Big(\bigcup_{[z]\in\theta[A]} [z]\Big), \text{ by the definition of } \mathbf{C}_R$$

$$\Rightarrow \quad x \in \mathbf{C}\Big(\bigcup_{z\in A} [z]\Big)$$

$$\Rightarrow \quad x \in \mathbf{C}([A]_R)$$

$$\Rightarrow \quad x \in [\mathbf{C}(A)]_R \text{ , since } R \text{ is a congruence}$$

$$\Rightarrow \quad xRy, \text{ for some } y \in \mathbf{C}(A)$$

$$\Rightarrow \quad [x] = [y] \text{ , for some } y \in \mathbf{C}(A)$$

$$\Rightarrow \quad [x] \in \{[y] : y \in \mathbf{C}(A)\}$$

$$\Rightarrow \quad [x] \in \theta[\mathbf{C}(A)].$$

Secondly, $\theta[\mathbf{C}(A)] \subseteq \mathbf{C}_R(\theta[A])$ since :

$$[x] \in \theta[\mathbf{C}(A)]$$

$$\Rightarrow \quad [x] = [y] \text{ , for some } y \in \mathbf{C}(A).$$

Now $y \in \mathbf{C}(A) \Rightarrow y \in \mathbf{C}([A]_R)$, since $A \subseteq [A]_R$ yields that $\mathbf{C}(A) \subseteq \mathbf{C}([A]_R)$, by 1.1.

$$\Rightarrow \quad y \in \mathbf{C}\Big(\bigcup_{z\in A} [z]\Big)$$

$$\Rightarrow \quad y \in \mathbf{C}\Big(\bigcup_{[z]\in\theta[A]} [z]\Big)$$

$$\Rightarrow \quad [y] \in \mathbf{C}_R(\theta[A]), \text{ by the definition of } \mathbf{C}_R$$

$$\Rightarrow \quad [x] \in \mathbf{C}_R(\theta[A]), \text{ since } [x] = [y].$$

Hence $\theta[\mathbf{C}(A)] = \mathbf{C}_R(\theta[A])$, and θ is a homomorphism.

In the next sections the analogues for closure algebras, of the first and second isomorphism theorems of Universal Algebra are proved. The particular formulation that I have chosen follows Grätzer [1], pp. 58-62.

<u>5.14 Theorem</u>. Let $(X,\mathbf{C})$ be a closure algebra, $(Y,\mathbf{C})$ a subalgebra of $(X,\mathbf{C})$ and R a congruence of $(X,\mathbf{C})$. If Q is the restriction of R to Y and S is the restriction of R to $[Y]_R$, then :

$$([Y]_R/S,\ \mathbf{C}_S) \cong (Y/Q,\ \mathbf{C}_Q)\ .$$

Proof. Let $\theta : X \to X/R$ be the natural mapping $\theta(x) = [x]_R$, and let θ_1 be the restriction of θ to Y. Then θ_1 is a homomorphism of Y into X/R, by 5.13. We now compare the sets $\theta[Y] = \{[x]_R : x \in Y\}$ and $[Y]_R/S$. We have $[Y]_R/S \subseteq \theta[Y]$ because

$$
\begin{aligned}
& [x]_R \in [Y]_R/S \\
\Rightarrow\ & [x]_R = [z]_S, \text{ for some } z \in [Y]_R \\
\Rightarrow\ & [x]_R = [z]_R, \text{ for some } z \in [Y]_R, \text{ since } S \\
& \qquad\qquad \text{is the restriction of } R \text{ to} \\
& \qquad\qquad [Y]_R. \\
\Rightarrow\ & xRz, \text{ for some } z \in [Y]_R \\
\Rightarrow\ & xRz \text{ and } zRy, \text{ for some } y \in Y \\
\Rightarrow\ & xRy, \text{ for some } y \in R, \text{ by the transitivity of } R. \\
\Rightarrow\ & [x]_R = [y]_R, \text{ for some } y \in Y \\
\Rightarrow\ & [x]_R \in \{[y]_R : y \in Y\} = \theta[Y].
\end{aligned}
$$

Also $\theta[Y] \subseteq [Y]_R/S$ because :

$$
\begin{aligned}
& [x]_R \in \theta[Y] \\
\Rightarrow\ & [x]_R = [y]_R, \text{ for some } y \in Y \\
\Rightarrow\ & [x]_R = [y]_R, \text{ for some } y \in [Y]_R \\
\Rightarrow\ & [x]_R = [y]_R \text{ and } [y]_S \in [Y]_R/S, \text{ for some} \\
& \qquad\qquad y \in [Y]_R
\end{aligned}
$$

$$\Rightarrow \quad [x]_R = [y]_R \in [Y]_R/S, \text{ since } S \text{ is the restriction of } R \text{ to } [Y]_R.$$

Hence $\theta[Y] = [Y]_R/S$, yielding that $\theta_1 : Y \to [Y]_R/S$ is an onto homomorphism , and by 5.10, generates a congruence R_1 on Y. Furthermore, if $x,y \in Y$ then

$$\begin{aligned} xR_1 y &\iff \theta_1(x) = \theta_1(y) \\ &\iff \theta(x) = \theta(y), \text{ since } x,y \in Y \\ &\iff xRy, \text{ for } x,y \in Y \\ &\iff xQy, \text{ since } x,y \in Y \text{ and } Q \text{ is the restriction of } R \text{ to } Y; \end{aligned}$$

so that $(Y/Q, \mathbf{C}_Q)$ is the factor algebra generated by θ_1.
We now have :

$$\begin{aligned} (Y/Q, \mathbf{C}_Q) &= (Y/R_1, \mathbf{C}_{R_1}) \\ &\cong ([Y]_R/S, \mathbf{C}_S), \text{ by 5.11.} \end{aligned}$$

For the special case when $[Y]_R = X$, $S = R$, and we have :

$$(X/R, \mathbf{C}_R) \cong (Y/Q, \mathbf{C}_Q).$$

To prepare the way for 5.17, the closure-theoretic version of the second isomorphism theorem, the next definition gives a way of describing the congruence relations of a factor algebra. The reference is Grätzer [1] pp 59f. The following notation will be used :
If S,R are two congruences on a closure algebra $(X,\mathbf{C})$, $S \leqslant R$ will indicate that S is contained in R as an equivalence relation, i.e. $S \leqslant R$ iff $xRy \Rightarrow xSy$, for each $x,y \in X$.

<u>5.15 Definition</u>. Let R,S, with $R \leqslant S$ be two congruences on $(X,\mathbf{C})$. We define S/R on X/R by :

$$[x]_R S/R [y]_R \text{ iff } xSy.$$

To show that S/R is well defined, suppose that $[x_1]_R = [x_2]_R$ and $[y_1]_R = [y_2]_R$, then :

$$\begin{aligned}
[x_1]_R S/R [y_1]_R &\Rightarrow x_1Sy_1 \text{ and } x_1Rx_2 \text{ and } y_1Ry_2 \\
&\Rightarrow x_1Sy_1 \text{ and } x_1Sx_2 \text{ and } y_1Sy_2\text{, since } R \leqslant S \\
&\Rightarrow x_2Sx_1 \text{ and } x_1Sy_1 \text{ and } y_1Sy_2\text{, by the symmetry of } S \\
&\Rightarrow x_2Sy_2\text{, by the transitivity of } S \\
&\Rightarrow [x_2]_R S/R [y_2]_R.
\end{aligned}$$

<u>5.16 Theorem</u>. Given the notation and conditions of 5.15, S/R is a congruence of $(X/R, \mathbf{C}_R)$.

Proof. That S/R is an equivalence relation follows immediately from the fact that S is an equivalence relation. Now let $A,B \in \mathbf{P}(X/R)$, and put

$$P = \bigcup_{[a]_R \in A} [a]_R \text{ and } Q = \bigcup_{[b]_R \in B} [b]_R \text{ , so that } P,Q \in \mathbf{P}(X).$$

Assuming that $A \equiv B(S/R)$, I first show that $P \equiv Q(S)$ and then that $\mathbf{C}_R(A) \equiv \mathbf{C}_R(B)(S/R)$. If $a \in P$, then $[a]_R \in A$, and there exists $[b]_R \in B$ such that $[a]_R S/R[b]_R$ since $A \equiv B(S/R)$. Hence there exists $b \in Q$ such that aSb. Similarly, if $b \in Q$ then there exists $a \in P$ such that aSb. It follows that $P \equiv Q(S)$, and since S is a congruence we have that $\mathbf{C}(P) \equiv \mathbf{C}(Q)(S)$.

Recollect that $\mathbf{C}_R(A) = \{[x]_R : x \in \mathbf{C}(P)\}$ and that $\mathbf{C}_R(B) = \{[x]_R : x \in \mathbf{C}(Q)\}$. Now,

$$\begin{aligned} & [x]_R \in \mathbf{C}_R(A) \\ \Rightarrow\ & x \in \mathbf{C}(P) \\ \Rightarrow\ & \text{there is some } y \in \mathbf{C}(Q) \text{ such that } xSy, \\ & \text{since } \mathbf{C}(P) \equiv \mathbf{C}(Q)(S) \\ \Rightarrow\ & \text{there is some } [y]_R \in \mathbf{C}_R(B) \text{ such that} \\ & [x]_R S/R [y]_R. \end{aligned}$$

Similarly, if $[y] \in \mathbf{C}_R(B)$ then there is some $[x]_R \in \mathbf{C}_R(A)$ such that $[x]_R S/R[y]_R$, yielding that

$$\mathbf{C}_R(A) \equiv \mathbf{C}_R(B)(S/R).$$

Thus if $A \equiv B(S/R)$ then $\mathbf{C}(A) \equiv \mathbf{C}(B)(S/R)$, so that S/R is a congruence.

5.17 Theorem. Let $(X, \mathbf{C})$ be a closure algebra, and let R,S be congruences of $(X, \mathbf{C})$ with $R \leqslant S$). Then S/R is a congruence of $(X/R, \mathbf{C}_R)$ by 5.16, and

$$(X/S,\ \mathbf{C}_S) \cong (X/R \big/ S/R, \mathbf{C}_{S/R}).$$

Proof. Let $\theta : X/R \to X/S$ be defined by taking $\theta([x]_R) = [x]_S$, for each $x \in X$. Clearly θ is onto X/S, and it is also a function because :

$$\begin{aligned} & [x]_R = [y]_R \\ \Rightarrow\ & xRy \\ \Rightarrow\ & xSy, \text{ since } R \leqslant S \\ \Rightarrow\ & [x]_S = [y]_S \\ \Rightarrow\ & \theta([x]_R) = \theta([y]_R). \end{aligned}$$

To show that θ is a homomorphism, I take any $A \in \mathbf{P}(X/R)$, and first show that $\mathbf{C}(\bigcup_{[y]_R \in A} [y]_R) \equiv \mathbf{C}(\bigcup_{[y]_S \in \theta[A]} [y]_S)\ (S)$. $z \in \bigcup_{[y]_R \in A} [y]_R$.

$\Rightarrow z \in [y]_R$, for some y with $[y]_R \in A$.

$\Rightarrow z \in [y]_R$, for some y with $[y]_S \in \theta[A]$.

$\Rightarrow z \in [y]_S$, for some y with $[y]_S \in \theta[A]$, since $R \leqslant S$

$\Rightarrow zSy$, for some $y \in \bigcup_{[y]_S \in \theta[A]} [y]_S$.

Also $z \in \bigcup_{[y]_S \in \theta[A]} [y]_S$

$\Rightarrow z \in [y]_S$, for some y with $[y]_S \in \theta[A]$

$\Rightarrow z \in [y]_S$, for some y with $[y]_R \in A$.

$\Rightarrow ySz$, for some $y \in \bigcup_{[y]_R \in A} [y]_R$.

Hence $\bigcup_{[y]_R \in A} [y]_R \equiv \bigcup_{[y]_S \in \theta[A]} [y]_S \quad (S)$.

and so $\mathbf{C}(\bigcup_{[y]_R \in A} [y]_R) \equiv \mathbf{C}(\bigcup_{[y]_S \in \theta[A]} [y]_S)\ (S)$, since S is a congruence.

I now show that $\theta[\mathbf{C}_R(A)] = \mathbf{C}_S(\theta[A])$. Firstly $\theta[\mathbf{C}_R(A)] \subseteq \mathbf{C}_S(\theta[A])$ because :

$[x]_S \in \theta[\mathbf{C}_R(A)]$

$\Rightarrow \quad [x]_R \in \mathbf{C}_R(A)$

$\Rightarrow \quad x \in \mathbf{C}\left(\bigcup_{[y]_R \in A} [y]_R\right)$, from the definition of $\mathbf{C}_R$

$\Rightarrow \quad xSw$, for some $w \in \mathbf{C}\left(\bigcup_{[y]_S \in \theta[A]} [y]_S\right)$,

since $\mathbf{C}\left(\bigcup_{[y]_R \in A} [y]_R\right) \equiv \mathbf{C}\left(\bigcup_{[y]_S \in \theta[A]} [y]_S\right)$ (S)

$\Rightarrow \quad [x]_S = [w]_S$ and $[w]_S \in \mathbf{C}_S(\theta[A])$, from the definition of $\mathbf{C}_S$

$\Rightarrow \quad [x]_S \in \mathbf{C}_S(\theta[A])$.

On the other hand $\mathbf{C}_S(\theta[A]) \subseteq \theta[\mathbf{C}_R(A)]$ because :

$[x]_S \in \mathbf{C}_S(\theta[A])$

$\Rightarrow \quad x \in \mathbf{C}\left(\bigcup_{[y]_S \in \theta[A]} [y]_S\right)$

$\Rightarrow \quad xSw$, for some $w \in \mathbf{C}\left(\bigcup_{[y]_R \in A} [y]_R\right)$, since

$\mathbf{C}\left(\bigcup_{[y]_R \in A} [y]_R\right) \equiv \mathbf{C}\left(\bigcup_{[y]_S \in \theta[A]} [y]_S\right)$ (S)

$\Rightarrow \quad [x]_S = [w]_S$ and $[w]_R \in \mathbf{C}_R(A)$, from the definition of $\mathbf{C}_R$

$\Rightarrow \quad [x]_S = [w]_S \in \theta[\mathbf{C}_R(A)]$.

Hence θ is a homomorphism of $(X/R, \mathbf{C}_R)$ onto $(X/S, \mathbf{C}_S)$.

Furthermore,

$\theta([x]_R) = \theta([y]_R)$

$\Longleftrightarrow \quad [x]_S = [y]_S$

$\Longleftrightarrow \quad xSy$

$\iff \quad [x]_R S/R [y]_R$, by the definition of S/R,

so that the congruence induced by θ on X/R is precisely the congruence S/R. By theorem 5.11 applied to $\theta: X/R \to X/S$,

$$(X/S, \mathbf{C}_S) \cong (X/R \big/ S/R, \mathbf{C}_{S/R}).$$

Sections 5.18 to 5.21 give the necessary definitions and results to show that the Zassenhaus lemma and the Jordan-Holder-Schreier theorem of Universal Algebra can be set up in closure algebras, providing that the appropriate algebraic constructions are replaced by their closure-theoretic analogues. The particular formulation of everything prior to, and including the Zassenhaus lemma has been taken from Grätzer [1], p.74, Exercises 61-67. The version of the Jordan-Holder-Schreier theorem is almost a copy of Gould [1].

The first task is to show that in the case of a finitary closure algebra there is an easily described join for any pair of congruences. The method of describing the join of two equivalence relations is essentially due to Grätzer [1], pp.18,19.

5.18 Definition. Let $(X, \mathbf{C})$ be a finitary closure algebra with R, S congruences of $(X, \mathbf{C})$. If $\phi \neq A \in \mathbf{P}(X)$, define the following chain of sets :

$$A_0 = A,\ A_1 = [A_0]_R,\ A_2 = [A_1]_S,\ A_3 = [A_2]_R,\ \ldots,$$

and define $R \vee S$ on X^2 by $x(R\vee S)y$ iff there is some $n < \omega$ such that $x \in y_n$ $(= \{y\}_n)$.

<u>5.19 Theorem</u>. Given the definitions of 5.18, RvS is a congruence, and is the join of R and S.

Proof. I first show that RvS is an equivalence relation:

(i) For each $x \in X$, $x \in \{x\} = x_1$, so that $x(RvS)x$ by the definition of RvS, and RvS is reflexive.

(ii) Assume $x(RvS)y$, then $x \in y_n$, for some $n < \omega$

$\Rightarrow$ $x \in [---[[[\{y\}]_R]_S]_R\ldots]_Q$, for a finite sequence where either $Q = R$ or $Q = S$.

$\Rightarrow$ there is a sequence $x_1,\ldots,x_{n-1} \in X$ such that

$x_1Ry,\ x_2Sx_1,\ x_3Rx_2,\ \ldots,\ xQx_{n-1}$

$\Rightarrow$ $x_{n-1}Qx,\ldots,x_2Rx_3,\ x_1Sx_2,\ yRx_1$, by the symmetry of R and of S.

$\Rightarrow$ either $x_{n-1}Rx,\ \ldots,\ x_1Sx_2,\ yRx_1$, if $Q = R$,

or $xRx,\ x_{n-1}Sx,\ldots,x_1Sx_2,\ yRx_1$, if $Q = S$.

$\Rightarrow$ either $y \in x_n$ or $y \in x_{n+1}$

$\Rightarrow$ $y(RvS)x$, so that RvS is symmetric.

(iii) Assume $x(RvS)y$, $y(RvS)z$, then

$x \in y_m$, $y \in z_n$, for some $m,n < \omega$.

$\Rightarrow$ $x \in y_m$ and either $[\{y\}]_R \subseteq [z_n]_R$

or $[\{y\}]_S \subseteq [z_n]_S$

$\Rightarrow$ $x \in y_m$ and $y_1 \subseteq z_{n+1}$;

by a finite number of repetitions of this process we have :

$$x \in y_m \text{ and } y_m \subseteq z_{n+m}$$

$$\Rightarrow \quad x \in z_{n+m}$$

$$\Rightarrow \quad x(R\vee S)z, \text{ so that } R\vee S \text{ is transitive.}$$

We have that $R\vee S$ is an equivalence relation; to show that it is a congruence of $(X,\mathbf{C})$, I show that it satisfies theorem 5.6.

Let $\mathbf{C}(A) \in \mathbf{P}(X)$, then $\mathbf{C}(A)_n$ is closed for each $n < \omega$, by induction, for :

$\mathbf{C}(A)_1 = [\mathbf{C}(A)]_R$, is closed by 5.12(c);
and if $\mathbf{C}(A)_k$ is closed for $k < \omega$ then

$$\mathbf{C}(A)_{k+1} = [\mathbf{C}(A)_k]_R \text{ or } \mathbf{C}(A)_{k+1} = [\mathbf{C}(A)_k]_S,$$

and in either case $\mathbf{C}(A)_{k+1}$ is closed by 5.12(c). By induction $\mathbf{C}(A)_n$ is closed for each $n < \omega$. Now since $\{\mathbf{C}(A)_n\}_{n<\omega}$ is a chain of closed subsets of X, and $(X,\mathbf{C})$ is finitary, then $\bigcup_{n<\omega}\mathbf{C}(A)_n$ is closed by theorem 1.18.

Furthermore, $[\mathbf{C}(A)]_{R\vee S} = \bigcup_{n<\omega}\mathbf{C}(A)_n$ because :

$$\begin{aligned} x \in [\mathbf{C}(A)]_{R\vee S} &\Longleftrightarrow x(R\vee S)y, \text{ for some } y \in \mathbf{C}(A) \\ &\Longleftrightarrow x \in y_n, \text{ for some } n < \omega,\ y \in \mathbf{C}(A) \\ &\Longleftrightarrow x \in \mathbf{C}(A)_n, \text{ for some } n < \omega \\ &\Longleftrightarrow x \in \bigcup_{n<\omega}\mathbf{C}(A)_n. \end{aligned}$$

We may conclude that $[\mathbf{C}(A)]_{RvS}$ is closed in $(X,\mathbf{C})$.

It is a congruence because for each $A \in \mathbf{P}(X)$:

$$A \subseteq \mathbf{C}(A)$$

$$\Rightarrow \quad [A]_{RvS} \subseteq [\mathbf{C}(A)]_{RvS}$$

$$\Rightarrow \mathbf{C}([A]_{RvS}) \subseteq \mathbf{C}([\mathbf{C}(A)]_{RvS})$$

$$= \quad [\mathbf{C}(A)]_{RvS} \text{ , since } [\mathbf{C}(A)]_{RvS} \text{ is closed;}$$

and RvS satisfies the condition of theorem 5.6.

The following argument shows that RvS is the join of R and S.

Firstly, $R \leqslant RvS$, since $xRy \Rightarrow x \in [y]_R$

$\Rightarrow x \in y_1$

$\Rightarrow x(RvS)y.$

Secondly, $S \leqslant RvS$, since $xSy \Rightarrow yRy$ and xSy

$\Rightarrow x \in [[y]_R]_S$

$\Rightarrow x \in y_2$

$\Rightarrow x(RvS)y.$

Finally, if $R \leqslant P$, $S \leqslant P$, for some equivalence relation P, then $RvS \leqslant P$ since :

$x(RvS)y \Rightarrow x \in y_n$, for some $n < \omega$

$\Rightarrow x \in [\ldots[[\{y\}]_R]_S\ldots]_Q$,

where $Q = R$ or $Q = S$

$\Rightarrow x \in [\ldots[[\{y\}]_P]_P\ldots]_P$, where

each occurrence of R or S has been replaced by P, since $R \leqslant P$, $S \leqslant P$.

$\Rightarrow x \in [y]_P$, since P is transitive

$\Rightarrow xPy.$

The next few sections are preliminary to the closure algebraic version of the Zassenhaus lemma.

5.20 Definition. Let $(X,\mathbf{C})$ be a closure algebra, and $(Y,\mathbf{C})$ a subalgebra of $(X,\mathbf{C})$. Let R be a congruence of $(X,\mathbf{C})$ with $R|_Y$ the restriction of R to Y, and S a congruence of $(Y,\mathbf{C})$ with $R|_Y \leqslant S$. Define a relation $R(S)$ on $[Y]_R$ by taking $y_1 R(S) y_2$ iff there is some $x_1, x_2 \in Y$ such that $y_1 R x_1$, $x_1 S x_2$, $x_2 R y_2$, for each $y_1, y_2 \in [Y]_R$.

5.21 Theorem. If the conditions of 5.20 hold then $R(S)$ is a congruence of $([Y]_R,\mathbf{C})$, with $[A]_{R(S)} = [[A]_S]_R$, for each $A \in \mathbf{P}(Y)$.

Proof. I first prove that $[A]_{R(S)} = [[A]_S]_R$.

If $A \in \mathbf{P}(Y)$ and $x \in [[A]_S]_R$, then there exists some $y \in [A]_S$ such that xRy. Note that $y \in Y$, since $A \subseteq Y$ and S is only defined on Y means that $[A]_S \subseteq Y$. Now $y \in [A]_S$ implies that there exists $z \in A \subseteq Y$ such that ySz. Hence we have $y, z \in Y$ such that :

$$xRy,\ ySz,\ zRz,$$

yielding that $xR(S)z$, by the definition of $R(S)$. Since $z \in A$ it follows that $[[A]_S]_R \subseteq [A]_{R(S)}$.

For the opposite inclusion :

$$\begin{aligned}
& x \in [A]_{R(S)} \\
\Rightarrow\quad & \text{there is some } y \in A \text{ such that } xR(S)y \\
\Rightarrow\quad & \text{there is some } y \in A \text{ and } w,z \in Y \text{ such that} \\
& xRw,\ wSz,\ zRy \\
\Rightarrow\quad & xRw,\ wSz,\ zSy, \text{ since } y,z \in Y \text{ and } R|_Y \leqslant S.
\end{aligned}$$

$\Rightarrow$ xRw, wSy, by the transitivity of S.

$\Rightarrow$ xRw and $w \in [A]_S$, since $y \in A$.

$\Rightarrow$ $x \in [[A]_S]_R$.

I now prove that R(S) is a congruence of $(Y,\mathbf{C})$, and then extend this result to R(S) being a congruence of $([Y]_R,\mathbf{C})$. Let $A \in \mathbf{P}(Y)$, then :

$$\begin{aligned}
\mathbf{C}([A]_{R(S)}) &= \mathbf{C}([[A]_S]_R), \text{ from above} \\
&\subseteq [\mathbf{C}([A]_S)]_R, \text{ by 5.6, since R is a congruence} \\
&\subseteq [[\mathbf{C}(A)]_S]_R, \text{ by 5.6, since S is a congruence} \\
&= [\mathbf{C}(A)]_{R(S)}, \text{ from above.}
\end{aligned}$$

Thus R(S) is a congruence of $(Y,\mathbf{C})$ by 5.6.

Now let $A,B \in \mathbf{P}([Y]_R)$; I will show that

$$A \equiv B(R(S)) \Rightarrow \mathbf{C}(A) \equiv \mathbf{C}(B)(R(S)).$$

Assume $A \equiv B(R(S))$, and for each $x \in A \subseteq [Y]_R$, take some $y_x \in Y$ such that xRy_x, and let $A' = \{y_x : x \in A\} \subseteq Y$. Since y_xRx for $x \in A$, for each $y_x \in A'$, we have that $A \equiv A'(R)$, and $A' \subseteq Y$. Similarly, there is some $B' \subseteq Y$ such that $B \equiv B'(R)$, and :

$x \in A'$

$\Rightarrow$ xRy for some $y \in A$, since $A \equiv A'(R)$

$\Rightarrow$ xRy, yRw_1, w_1Sw_2, w_2Rz for some $z \in B$ and $w_1, w_2 \in Y$, since $A \equiv B(R(S))$.

$\Rightarrow$ xRw_1, w_1Sw_2, w_2Rz, transitivity of R

$\Rightarrow$ xRw_1, $w_1\dot{S}w_2$, w_2Rz, zRz', for some $z' \in B'$, since $B \equiv B'(R)$

$\Rightarrow$ $xR(S)z'$, for some $z' \in B$, since $w_1, w_2 \in Y$.

Similarly, if $z \in B'$ then there is some $x \in A'$ such that $xR(S)z$. Hence $A' \equiv B'(R(S))$, for $A', B' \in \mathbf{P}(Y)$. Now $R(S)$ is a congruence on Y, so that $\mathbf{C}(A') \equiv \mathbf{C}(B')(R(S))$. Furthermore, R is a congruence on X, so that $\mathbf{C}(A) \equiv \mathbf{C}(A')(R)$ and $\mathbf{C}(B) \equiv \mathbf{C}(B')(R)$. By an argument similar to the above, $\mathbf{C}(A) \equiv \mathbf{C}(B)(R(S))$. Hence $R(S)$ is a congruence of $([Y]_R, \mathbf{C})$.

<u>5.22 Theorem</u>. Let $(X, \mathbf{C})$ be a closure algebra, $(Y, \mathbf{C})$ a subalgebra of $(X, \mathbf{C})$, and R, S be as in definition 5.20, then

$$([Y]_R/R(S), \mathbf{C}_{R(S)}) \cong (Y/S, \mathbf{C}_S)$$

Proof. I first show, as a lemma, that if $[y]_{R(S)} \in [Y]_R/R(S))$, then we may take y to be in Y. This is so because if $z \in [y]_{R(S)}$, for $y \in [Y]_R$, then

$zR(S)y$ and $y \in [Y]_R$.

$\Rightarrow$ zRw_1, $w_1\dot{S}w_2$, w_2Ry and yRy_1, for some $w_1, w_2, y_1 \in Y$.

$\Rightarrow$ zRw_1, $w_1\dot{S}w_2$, w_2Ry_1, for w_1, w_2, $y_1 \in Y$, since R is transitive

$\Rightarrow$ $zR(S)y_1$

$\Rightarrow$ $yR(S)y_1$, since $R(S)$ is transitive and $zR(S)y$

$\Rightarrow$ $[y]_{R(S)} = [y_1]_{R(S)}$.

Throughout the proof we will take $y \in Y$ whenever $[y]_{R(S)} \in [Y]_R/R(S)$.

Now define $\theta : [Y]_R/R(S) \to Y/S$ by $\theta([y]_{R(S)}) = [y]_S$, for each $[y]_{R(S)} \in [Y]_R$, i.e. for each $y \in Y$. Clearly θ maps $[Y]R/_{R(S)}$ onto Y/S, and θ is a function since :

$$[y_1]_{R(S)} = [y_2]_{R(S)} \text{ , with } y_1, y_2 \in Y$$

$\Rightarrow$ $y_1 R(S) y_2$, with $y_1, y_2 \in Y$

$\Rightarrow$ $y_1 R w_1$, $w_1 S w_2$, $w_2 R y_2$, for some $y_1, y_2, w_1, w_2 \in Y$

$\Rightarrow$ $y_1 S w_1$, $w_1 S w_2$, $w_2 S y_2$, since $R|_Y \leqslant S$

$\Rightarrow$ $y_1 S y_2$, by the transitivity of S.

$\Rightarrow$ $[y_1]_S = [y_2]_S$

$\Rightarrow$ $\theta([y_1]_{R(S)}) = \theta([y_2]_{R(S)})$.

Also θ is one-to-one, since :

$$\theta([y_1]_{R(S)}) = \theta([y_2]_{R(S)}) \text{, with } y_1, y_2 \in Y$$

$\Rightarrow$ $[y_1]_S = [y_2]_S$, with $y_1, y_2 \in Y$.

$\Rightarrow$ $y_1 S y_2$, with $y_1, y_2 \in Y$.

$\Rightarrow$ $y_1 R y_1$, $y_1 S y_2$, $y_2 R y_2$, with $y_1, y_2 \in Y$, since R is reflexive.

$\Rightarrow$ $y_1 R(S) y_2$

$\Rightarrow$ $[y_1]_{R(S)} = [y_2]_{R(S)}$.

To show that θ is a homomorphism, I first show that:

$$\left[\bigcup_{[y]_{R(S)} \in A} [y]_{R(S)}\right]_{R(S)} = \left[\bigcup_{[y]_S \in \theta[A]} [y]_S\right]_{R(S)} \text{, for}$$

each $A \subseteq [Y]_R/R(S)$. For one inclusion :

$$x \in \left[\bigcup_{[y]_{R(S)} \in A} [y]_{R(S)}\right]_{R(S)}$$

$\Rightarrow$ $xR(S)z, z \in [y]_{R(S)}$ for some $y \in Y$ with $[y]_{R(S)} \in A$

$\Rightarrow$ $xR(S)y$, for some $y \in Y$ with $[y]_{R(S)} \in A$

$\Rightarrow$ $xR(S)y$, for some $y \in Y$ with $[y]_S \in \theta[A]$, since θ is an onto function.

$\Rightarrow$ $x \in \left[\bigcup_{[y]_S \in \theta[A]} [y]_S\right]_{R(S)}$.

For the opposite inclusion :

$$x \in \left[\bigcup_{[y]_S \in \theta[A]} [y]_S\right]_{R(S)}$$

$\Rightarrow$ $xR(S)z$, $z \in [y]_S$, for some $y \in Y$ with $[y]_S \in \theta[A]$.

$\Rightarrow$ $xR(S)z$, zSy, for some $y \in Y$ with $[y]_S \in \theta[A]$ and $z \in Y$, since S is only defined on Y.

$\Rightarrow$ $xR(S)z$, zRz, zSy, yRy, with $[y]_{R(S)} \in A$ and $y, z \in Y$, since R is reflexive and θ is a one-to-one and onto function

$\Rightarrow$ $xR(S)z$, $zR(S)y$ for some y with $[y]_{R(S)} \in A$.

$\Rightarrow$ $x \in \left[\bigcup_{[y]_{R(S)} \in A} [y]_{R(S)}\right]_{R(S)}$.

Since $\left[\bigcup_{[y]_{R(S)} \in A} [y]_{R(S)}\right]_{R(S)} = \left[\bigcup_{[y]_S \in \theta[A]} [y]_S\right]_{R(S)}$, we have that $\left[\mathbf{C}\left(\bigcup_{[y]_{R(S)} \in A} [y]_{R(S)}\right)\right]_{R(S)} = \left[\mathbf{C}\left(\bigcup_{[y]_S \in \theta[A]} [y]_S\right)\right]_{R(S)}$,

because $R(S)$ is a congruence of $([Y]_R, \mathbf{C})$, and by 5.4.

Using this :

$$\theta[\mathbf{C}_{R(S)}(A)] = \theta[\{[x]_{R(S)} : x \in \mathbf{C}\left(\bigcup_{[y]_{R(S)} \in A} [y]_{R(S)}\right)\}]$$

$$= \theta[\{[x]_{R(S)} : x \in \mathbf{C}\left(\bigcup_{[y]_S \in \theta[A]} [y]_S\right)\}],$$

from above.

$$= \{[x]_S : x \in \mathbf{C}\left(\bigcup_{[y]_S \in \theta[A]} [y]_S\right)\}$$

$$= \mathbf{C}_S(\theta[A]); \text{ for each } A \subseteq [Y]_R/R(S)$$

Hence $([Y]_R/R(S), \mathbf{C}_{R(S)}) \cong (Y/S, \mathbf{C}_S)$.

We proceed with the Zassenhaus lemma.

<u>5.23 Lemma</u>. Let $(Y,\mathbf{C})$, $(X,\mathbf{C})$ be two subalgebras of a finitary closure algebra $(X,\mathbf{C})$ with $Y \cap Z \neq \phi$. Let R and S be congruences on $(Y,\mathbf{C})$, $(Z,\mathbf{C})$ respectively, and let $R|_{Y \cap Z}$, $S|_{Y \cap Z}$ be the restrictions of R and S to $Y \cap Z$. Put $Q = R|_{Y \cap Z} \vee S|_{Y \cap Z}$, then :

$$([Y \cap Z]_R/R(Q), \mathbf{C}_{R(Q)}) \cong ([Y \cap Z]_S/S(Q), \mathbf{C}_{S(Q)})$$

Proof. $(Y \cap Z, \mathbf{C})$ is a subalgebra of $(X,\mathbf{C})$, and Q is a congruence of $(Y \cap Z, \mathbf{C})$ by 5.19 with $(X,\mathbf{C})$ finitary. By the definition of Q, $R|_{Y \cap Z} \leqslant Q$ and $S|_{Y \cap Z} \leqslant Q$, so that by 5.21, $R(Q)$ and $S(Q)$ are congruences on $[Y \cap Z]_R$ and $[Y \cap Z]_S$, respectively. By two applications of theorem 5.22, we have that :

$([Y \cap Z]_R/R(Q), \mathbf{C}_{R(Q)}) \cong (Y \cap Z/Q, \mathbf{C}_Q)$ and

$([Y \cap Z]_R/S(Q), \mathbf{C}_{S(Q)}) \cong (Y \cap Z/Q, \mathbf{C}_Q)$, so that

$([Y \cap Z]_R/R(Q), \mathbf{C}_{R(Q)}) \cong ([Y \cap Z]_R/S(Q), \mathbf{C}_{S(Q)})$.

We now set up the necessary details to use the Zassenhaus lemma to prove an analogue of the Jordan-Holder-Schreier theorem. The treatment follows Gould [1].

<u>5.24 Definition.</u> Let $(X,\mathbf{C})$ be a closure algebra, and let $\{(Y_i,\mathbf{C})\}_{i=0}^{n}$ be a sequence of subalgebras. If $X = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_n$, and for each i, $0 \leq i \leq n$, there is a congruence R_i of $(Y_i,\mathbf{C})$ with $Y_i = [Y_n]_{Ri-1}$, and with Rn the identity congruence, then $\{(Y_i,\mathbf{C})\}_{i=0}^{n}$ is called a normal series.

<u>5.25 Definition.</u> If $(X,\mathbf{C})$ is a closure algebra, and $\{(Y_i,\mathbf{C})\}_{i=0}^{n}$, $\{(Z_j,\mathbf{C})\}_{j=0}^{m}$ are normal series such that each Z_j is a Y_i, then $\{(Y_i,\mathbf{C})\}_{i=0}^{n}$ is called a refinement of $\{(Z_j,\mathbf{C})\}_{j=0}^{m}$.

<u>5.26 Definition.</u> If $(X,\mathbf{C})$ is a closure algebra, and $\{(Y_i,\mathbf{C})\}_{i=0}^{n}$ and $\{(Z_j,\mathbf{C})\}_{j=0}^{m}$ are normal series with congruences $\{R_i\}_{i=0}^{n}$ and $\{S_j\}_{j=0}^{n}$ respectively, then the two series are said to be isomorphic iff $n = m$, $Z_m = Y_n$, and the R_i, S_j can be chosen in such a way that :

$(Y_i/R_i, \mathbf{C}) \cong (Z_{\pi(i)}/S_{\pi(i)}, \mathbf{C})$, where π is a permutation of $i = 0,\dots,n$.

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5.27 Definition. If $(X,\mathbf{C})$ is a closure algebra, and R,S are congruences of $(X,\mathbf{C})$, then R,S are said to be weakly associable over Y, for $(Y,\mathbf{C})$ a subalgebra of $(X,\mathbf{C})$, whenever $[[Y]_R]_S = [Y]_{RvS} = [[Y]_S]_R$.

5.28 Theorem. Suppose that $(X,\mathbf{C})$ is a finitary closure algebra, and that $\{(Y_i,\mathbf{C})\}_{i=0}^{n}$, $\{(Z_j,\mathbf{C})\}_{j=0}^{m}$ are normal series with congruences $\{R_i\}_{i=0}^{n}$, $\{S_j\}_{j=0}^{m}$, respectively. If these congruences can be chosen in such a way that $R_i|_{Y_i \cap Z_j}$ and $S_j|_{Y_i \cap Z_j}$ are weakly associable over $Y_n = Z_m$, for each i, j, then the normal series have isomorphic refinements.

Proof. Define $Y_{ij} = [Y_i \cap Z_j]_{R_i}$, $Z_{ij} = [Y_i \cap Z_j]_{S_j}$; by 5.12(c) these are subalgebras of $(X,\mathbf{C})$.

Set $Q_{ij} = R_i|_{Y_i \cap Z_j} \vee S_j|_{Y_i \cap Z_j}$, for each i,j. By 5.20, each Q_{ij} is a congruence, and by 5.21 we may define the congruence $R_{ij} = R_i(Q_{ij})$; $S_{ij} = S_j(Q_{ij})$, for each i,j. The next three results about the subalgebras Y_{ij}, Z_{ij} will enable us to show that these form normal series. The proofs are only given for the Y_{ij} , since the Z_{ij} are similar.

(i) $$\begin{aligned} Y_{i0} &= [Y_i \cap Z_0]_{R_i} \\ &= [Y_i \cap X]_{R_i} \\ &= [Y_i]_{R_i} \\ &= Y_i, \text{ since } R_i \text{ is only defined on } Y_i. \end{aligned}$$

(ii) $$Y_{im} = [Y_i \cap Z_m]_{R_i}$$
$$= [Y_i \cap Y_n]_{R_i}, \text{ since } Y_n = Z_m$$
$$= [Y_n]_{R_i}, \text{ since } Y_n \subseteq Y_i, \text{ for each } i, \; 1 \leqslant i \leqslant n$$
$$= Y_{i+1}, \text{ since } (Y_i, \mathbf{C})_{i=0}^{n} \text{ is a normal series.}$$

(iii) $$Y_{ij} = [Y_i \cap Z_j]_{R_i}$$
$$\supseteq [Y_i \cap Z_{j+1}]_{R_i}, \text{ since } Z_j \supseteq Z_{j+1}$$
$$= Y_{ij+1}.$$

From (i), (ii), (iii), we have that :

$$X = Y_{00} \supseteq Y_{01} \supseteq \dots \supseteq Y_{0m} = Y_1 \supseteq \dots \supseteq Y_n;$$

similarly,

$$X = Z_{00} \supseteq Z_{10} \supseteq \dots \supseteq Z_{n0} = Z_1 \supseteq \dots \supseteq Z_m = Y_n$$

We now prove that the first of these is a normal series; the proof for the second is similar. Firstly note that because of (i) and (ii) above, we have that any algebra in the series may be expressed as Y_{ij} in such a way that the succeeding algebra may be expressed as Y_{ij+1}. It will thus be sufficient to prove that $[Y_n]_{R_{ij}} = Y_{ij+1}$, for each j, $0 \leqslant j \leqslant m$. This holds because :

$$[Y_n]_{R_{ij}} = [Y_n]_{R_i(Q_{ij})}, \text{ by the definition of } R_{ij}$$
$$= [[Y_n]_{Q_{ij}}]_{R_i}, \text{ by 5.21, since } Y_n \subseteq Y_i \cap Y_j.$$
$$= \left[[Y_n]_{R_i}|_{Y_i \cap Z_j} \vee S_j|_{Y_i \cap Z_j}\right]_{R_i}, \text{ by the definition of } Q_{ij}.$$

$= \left[\left[[Y_n]S_j|_{Y_i \cap Z_j}\right]R_i|_{Y_i \cap Z_j}\right]_{R_i}$ by weak associability.

$= [[Y_n]S_j|_{Y_i \cap Z_j}]_{R_i}$, since $R_i|_{Y_i \cap Z_j} \leqslant R_i$, and

$$[Y_n]S_j|_{Y_i \cap Z_j} \subseteq Y_i \cap Z_j.$$

$= \left[[Z_m]S_j|_{Y_i \cap Z_j}\right]_{R_i}$, since $Y_n = Z_m$.

$= [[Z_m]_{S_j} \cap (Y_i \cap Z_j)]_{R_i}$, since $S_j|_{Y_i \cap Z_j}$ is the restriction of S_j to $Y_i \cap Z_j$, and $Z_m \subseteq Y_i \cap Z_j$.

$= [Z_{j+1} \cap (Y_i \cap Z_j)]_{R_i}$, by the properties of S_j.

$= [Y_i \cap Z_{j+1}]$, since $Z_{j+1} \subseteq Z_j$.

$= Y_{ij+1}$.

Hence the congruences $\{R_{ij}\}_{i=0,j=0}^{n\ ,m}$ are such as to make $\{(Y_{ij},\mathbf{C})\}_{i=0,j=0}^{n\ ,m}$ a normal series. Similarly, $\{(Z_{ij},\mathbf{C})\}_{i=0,j=0}^{n\ ,m}$ is a normal series.

By (i), we have that each $Y_i = Y_{i0}$ (and similarly, each $Z_j = Z_{0j}$) so that both these series are refinements of their respective original series. Finally, the use of the Zassenhaus lemma (5.33) proves the appropriate isomorphism of the subalgebras in the series i.e.

$$(Y_{ij}/R_{ij}, \mathbf{C}_{R_{ij}}) \cong (Z_{ij}/S_{ij}, \mathbf{C}_{S_{ij}}), \text{ for each } i,j.$$

This follows because $(X,\mathbf{C})$ is finitary, and :

$$Y_{ij}/R_{ij} = [Y_i \cap Z_j]_{R_i}/R_i(Q_{ij}),$$

$$Z_{ij}/S_{ij} = [Y_i \cap Z_j]_{S_j} \Big/ S_j(Q_{ij}) \ ,$$
$$\text{where } Q_{ij} = R_i|_{Y_i \cap Z_j} \vee S_j|_{Y_i \cap Z_j} \ ;$$

satisfying all the conditions of 5.23.

Hence the two original series have isomorphic refinements.

CHAPTER VI PRODUCTS AND HOMOMORPHIC IMAGES

One of the most common ways of creating new structures from a given set of structures is to define operations on the cartesian product of the carriers of the given structures. In the first part of this chapter we look at one way in which this may be done for closure algebras. In order to provide the link between the structure on the product and the structure on the component algebras we may apply a number of different restrictions. Two of the more obvious are the restrictions that either the inverse projections or the projections preserve closed sets. If we require that the inverse projections preserve closed sets then there are two closure operations which can be readily defined on the product: the largest and the smallest (where the ordering is by definition 1.12) closure operations that satisfy the restriction. The largest such closure operation is the standard, product closure operation of General Topology; and the smallest is also an operator of interest to topologists and sometimes referred to as the "box topology". Since both of these closure operations have been studied in depth I pass over them. The restriction that the projections preserve closed sets does not seem to be strong enough in itself to yield worthwhile results. However, we shall strengthen this to the requirement that the projection maps be homomorphisms from the closure product to the component algebras, and look at the largest closure operation satisfying this requirement. With this some strong connections can be shown to hold between a product closure algebra and the component algebras, and between the dual space of a product closure algebra and the dual spaces of the component algebras. Furthermore this closure operation on the product is capable of being given an extremely simple definition,(6.2). The first half of chapter VI investigates this closure algebra.

The product algebra of a family of universal algebras also gives rise to a closure algebra in the manner of example 1.11(i), and has closed sets that are preserved under all the projections. However, all attempts to characterize this most interesting closure algebra, in closure-theoretic terms have yielded nothing. Possibly it is only adequately described in terms of the underlying algebraic operations on the component algebras.

The latter half of the chapter investigates the homomorphic images of a closure algebra, in particular when they are finitely generated, finitary or cocompact, and concludes with a major theorem on the relationship between the dual space of a closure algebra and the dual space of any of its homomorphic images.

6.1 Definition. Let $\{(X_i, \mathbf{C}_i)\}_{i\in I}$ be an arbitrary family of closure algebras, and take $\prod_{i\in I} X_i$ to be the cartesian product of $\{X_i\}_{i\in I}$. For each $i \in I$ define :

$\theta_i : \prod_{j\in I} X_j \to X_i$ by $\theta_i(x)$ = the ith coordinate of x.

θ_i will be referred to as the ith projection.

<u>6.2 Definition</u>. Let $\{(X_i, \mathbf{C}_i)\}$ be a family of closure algebras. We define an operation on $\mathbf{P}(\prod_{i\in I} X_i)$ by

$\mathbf{C}(A) = \prod_{i\in I} \mathbf{C}_i(\theta_i(A_i))$, for each $A \in \mathbf{P}(\prod_{i\in I} X_i)$.

<u>6.3 Theorem</u>. Using the notations of 6.1, 6.2, and putting $X = \prod_{i\in I} X_i$, we have that $(X, \mathbf{C})$ is a closure algebra.

Proof. For any $A, B \in \mathbf{P}(X)$, we have :

(i) $\quad x \in A \Rightarrow \theta_i(x) \in \theta_i[A]$, for each $i \in I$

$\Rightarrow \theta_i(x) \in \mathbf{C}_i(\theta_i[A])$, since $\mathbf{C}_i$ is a closure operation, for each $i \in I$.

$$\Rightarrow x = \prod_{i\in I} \theta_i(x) \in \prod_{i\in I} \mathbf{C}_i(\theta_i[A])$$

$$= \mathbf{C}(A).$$

Hence $A \subseteq \mathbf{C}(A)$.

(ii) $\quad$ Since $\mathbf{C}(A) = \prod_{i\in I} \mathbf{C}_i(\theta_i[A])$, we have that

$$\theta_i[\mathbf{C}(A)] = \mathbf{C}_i(\theta_i[A]), \text{ for each } i \in I$$

so that
$$\mathbf{C}(\mathbf{C}(A)) = \prod_{i\in I} \mathbf{C}_i(\theta_i[\mathbf{C}(A)]$$

$$= \prod_{i\in I} \mathbf{C}_i(\mathbf{C}_i(\theta[A])$$

$$= \prod_{i\in I} \mathbf{C}_i(\theta[A]), \text{ since } \mathbf{C}_i \text{ is a}$$

closure operator for each $i \in I$,

$$= \mathbf{C}(A).$$

(iii) Assume that $A \subseteq B$, then,

$$\theta_i[A] \subseteq \theta_i[B] \quad \text{, for each } i \in I$$

$$\Rightarrow \mathbf{C}_i(\theta_i[A]) \subseteq \mathbf{C}_i(\theta_i[B]) \text{, for each } i \in I$$

$$\Rightarrow \prod_{i\in I} \mathbf{C}_i(\theta_i[A]) \subseteq \prod_{i\in I} \mathbf{C}_i(\theta_i[B])$$

$$\Rightarrow \mathbf{C}(A) \subseteq \mathbf{C}(B).$$

The next theorem shows that, with this definition of a product closure algebra, the finite product of finitary closure algebras is finitary. Indeed the condition is necessary and sufficient.

<u>6.4 Theorem</u>. Let $\{(X_i, \mathbf{C}_i)\}_{i\in I}$ be a finite family of closure algebras, then the product closure algebra $(X, \mathbf{C})$, where $X = \prod_{i\in I} X_i$, is finitary iff $(X_i, \mathbf{C}_i)$ is finitary for each $i \in I$.

Proof. Assume that $(X_i, \mathbf{C}_i)$ is finitary, for each $i \in I$, and let $A \subseteq X$ with $x \in \mathbf{C}(A)$. Firstly,

$\theta_i(x) \in \theta_i[\mathbf{C}(A)] = \mathbf{C}_i(\theta_i[A])$, for each $i \in I$; and each $(X_i, \mathbf{C}_i)$ is finitary, so that $\theta_i(x) \in \mathbf{C}_i(A_i)$, where A_i is some finite subset of $\theta_i[A]$, for each $i \in I$. Fix $i \in I$; for each $\alpha \in A_i$ choose an $x_\alpha \in A$ such that $\theta_i(x_\alpha)=\alpha$. This is always possible by the definition of the projections. Construct the subset $u_i = \{x_\alpha : \alpha \in A_i\}$ of A, noting that u_i is finite, and that $A_i = \theta_i[u_i]$. Construct such a set u_i for each $i \in I$, and consider $\bigcup_{i\in I} u_i$.

(i) Since u_i is finite for each $i \in I$, and the index set I is finite by assumption, we have that $\bigcup_{i \in I} u_i$ is finite.

(ii) Since $u_i \subseteq A$ for each $i \in I$, $\bigcup_{i \in I} u_i \subseteq A$.

(iii) $\prod_{j \in I} \mathbf{C}_j(A_j) \subseteq \mathbf{C}(\bigcup_{i \in I} u_i)$, because :

$$u_j \subseteq \bigcup_{i \in I} u_i, \text{ for each } j \in I$$

$$\Rightarrow A_j = \theta_j(u_j) \subseteq \theta_j[\bigcup_{i \in I} u_i], \text{ for each } j \in I$$

$$\Rightarrow \mathbf{C}_j(A_j) \subseteq \mathbf{C}_j(\theta_j[\bigcup_{i \in I} u_i]), \text{ for each } j \in I.$$

$$\Rightarrow \prod_{j \in I} \mathbf{C}_j(A_j) \subseteq \prod_{j \in I} \mathbf{C}_j(\theta_j[\bigcup_{i \in I} u_i]$$

$$\Rightarrow \prod_{j \in I} \mathbf{C}_j(A_j) \subseteq \mathbf{C}(\bigcup_{i \in I} u_i) .$$

Now $(X, \mathbf{C})$ is finitary, because for arbitrary $x \in X$, $A \in \mathbf{P}(X)$,

$$x \in \mathbf{C}(A) \Rightarrow \theta_j(x) \in \mathbf{C}_j(A_j), \text{ for each } j \in I.$$

$$\Rightarrow x \in \prod_{j \in I} \mathbf{C}_j(A_j)$$

$$\Rightarrow x \in \mathbf{C}(\bigcup_{i \in I} u_i), \text{ by (iii) above,}$$

and by (i), (ii), above $\bigcup_{i \in I} u_i$ is a finite subset of A.

Conversely, assume $(X, \mathbf{C})$ finitary and fix $i \in I$. Choose $A_i \subseteq X_i$, $x_i \in \mathbf{C}_i(A_i)$ and consider the product $\prod_{j \in I} Q_j$, where $Q_i = A_i$

and $Q_j = X_j$ for $j \neq i$.

We have $\theta_i[\mathbf{C}(\prod_{j\in I} Q_j)] = \theta_i[\prod_{k\in I} \mathbf{C}_k(\theta_k[\prod_{j\in I} Q_j])]$,

from the definition of $\mathbf{C}$

$$= \theta_i[\prod_{k\in I} \mathbf{C}_k(Q_k)]$$

$$= \mathbf{C}_i(Q_i)$$

$$= \mathbf{C}_i(A_i).$$

Therefore $x_i \in \theta_i[\mathbf{C}(\prod_{j\in I} Q_j)]$,and we may choose $x \in \mathbf{C}(\prod_{j\in I} Q_j)$ such that $\theta_i(x) = x_i$. Since $(X,\mathbf{C})$ is finitary there exists a finite subset, A_f of $\prod_{j\in I} Q_j$ such that $x \in \mathbf{C}(A_f)$. But then

$$x_i = \theta_i(x) \in \theta_i[\mathbf{C}(A_f)]$$

$$= \theta_i[\prod_{j\in I} \mathbf{C}_j(\theta_j[A_f])].$$

$$= \mathbf{C}_i(\theta_i[A_f]); \text{ and}$$

$$\theta_i[A_f] \subseteq \theta_i[\prod_{j\in I} Q_j] = A_i, \text{ since } A_f \subseteq \prod_{j\in I} Q_j.$$

Furthermore, since A_f is finite and θ_i is a function, $\theta_i[A_f]$ is finite. I have thus constructed a finite subset, $\theta_i[A_f]$ of A_i such that $x_i \in \mathbf{C}_i(\theta_i[A_f])$.

Hence $(X_i, \mathbf{C}_i)$ is finitary for any $i \in I$.

6.5 Theorem. If $\{(X_i,\mathbf{C}_i)\}_{i\in I}$ is a finite family of closure algebras, then the product closure algebra, $(X,\mathbf{C})$ is finitely generated iff $(X_i,\mathbf{C}_i)$ is finitely generated for each $i \in I$.

Proof. Assume $(X_i, \mathbf{C}_i)$ is finitely generated for each $i \in I$. Then for each $i \in I$ there exists a finite subset, A_i of X_i such that $\mathbf{C}_i(A_i) = X_i$. By the definition of the product closure operation,

$$\begin{aligned} \mathbf{C}(\prod_{i\in I} A_i) &= \prod_{j\in I} \mathbf{C}_j(\theta_j[\prod_{i\in I} A_i]) \\ &= \prod_{j\in I} \mathbf{C}_j(A_j) \\ &= \prod_{j\in I} X_j \\ &= X. \end{aligned}$$

Since each A_i is finite and I is finite, we have that $\prod_{i\in I} A_i$ is a finite generating set for X.

Conversely, given that $(X,\mathbf{C})$ is finitely generated, we have there exists an $X_f \subseteq X$, X_f finite with

$$\begin{aligned} &\mathbf{C}(X_f) = X \\ \Rightarrow &\prod_{i\in I} \mathbf{C}_i(\theta_i[X_f]) = \prod_{i\in I} X_i \end{aligned}$$

$\Rightarrow \mathbf{C}_i(\theta_i[X_f]) = X_i$, for each $i \in I$, by taking projections of both sides. Since X_f is finite, and each θ_i is a function, each $\theta_i[X_f]$ is finite, and hence each $(X_i, \mathbf{C}_i)$ is finitely generated.

6.6 Theorem. Let $\{(X_i,\mathbf{C}_i)\}$ be a finite family of closure algebras. The product $(X,\mathbf{C})$ is cocompact iff $(X_i,\mathbf{C}_i)$ is cocompact for each $i \in I$.

Proof. Assume that $(X_i,\mathbf{C}_i)$ is cocompact for each $i \in I$, and that $A \subseteq X = \prod_{i\in I} X_i$ is inconsistent with respect to $\mathbf{C}$, so that

$$\mathbf{C}(A) = X$$

$$\Rightarrow \prod_{i\in I} \mathbf{C}_i(\theta_i[A]) = \prod_{i\in I} X_i$$

$$\Rightarrow \mathbf{C}_i(\theta_i[A]) = X_i,$$ for each $i \in I$, by taking projections of both sides.

Now for each $i \in I$, $(X_i, \mathbf{C}_i)$ is cocompact, so there is an $\alpha_i \subseteq \theta_i[A]$ such that $\mathbf{C}_i(\alpha_i) = X_i$, with each α_i finite. Since $\alpha_i \subseteq \theta_i[A]$ and α_i is finite, we may find a set $u_i \subseteq A$ such that u_i is finite and $\theta_i[u_i] = \alpha_i$. Choosing such a u_i for each $i \in I$, and considering $\bigcup_{i\in I} u_i$, we have that :

(i) $\bigcup_{i\in I} u_i \subseteq A$.

(ii) $\bigcup_{i\in I} u_i$ is finite, since each u_i is finite and I is finite.

(iii) $\theta_j[\bigcup_{i\in I} u_i] \supseteq \theta_j[u_j] = \alpha_j$, for each $j \in I$, since $\bigcup_{i\in I} u_i \supseteq u_j$;

so that
$$\begin{aligned}\mathbf{C}(\bigcup_{i\in I} u_i) &= \prod_{j\in I} \mathbf{C}_j(\theta_j[\bigcup_{i\in I} u_i]) \\ &\supseteq \prod_{j\in I} \mathbf{C}_j(\alpha_j) \\ &= \prod_{j\in I} X_j \\ &= X.\end{aligned}$$

By (i), (ii), (iii) we have that $\bigcup_{i\in I} u_i$ is a finite, inconsistent subset of A. This holds for every inconsistent subset of A, and hence $(X, \mathbf{C})$ is cocompact.

Conversely, assume that $(X,\mathbf{C})$ is cocompact; fix $i \in I$ and assume that $A_i \subseteq X_i$ is inconsistent with respect to $\mathbf{C}_i$. Consider the set $\prod_{j\in I} Q_j$,

where $Q_i = A_i$

and $Q_j = X_j$, for $j \neq i$.

We have that
$$\mathbf{C}\left(\prod_{j\in I} Q_j\right) = \prod_{k\in I} \mathbf{C}_k\left(\theta_k\left[\prod_{j\in I} Q_j\right]\right)$$
$$= \prod_{k\in I} \mathbf{C}_k(Q_k)$$
$$= \prod_{k\in I} X_k, \text{ since } Q_k = X_k,$$
for each $k \neq i$, and $\mathbf{C}_i(Q_i) = \mathbf{C}_i(A_i) = X_i$. Thus $\prod_{j\in I} Q_j$ is inconsistent, and, by the cocompactness of $(X,\mathbf{C})$, has a finite inconsistent subset, A_f.

We have $X = \mathbf{C}(A_f) = \prod_{j\in I} \mathbf{C}_j(\theta_j[A_f])$

$\Rightarrow$ $X_i = \mathbf{C}_i(\theta_i[A_f])$, taking the ith projection of each side.

$\Rightarrow$ $\theta_i[A_f]$ is inconsistent with respect to $\mathbf{C}_i$.

We also have that $\theta_i[A_f]$ is finite, since A_f is finite and θ_i is a function. Finally $\theta_i[A_f] \subseteq A_i$, since $A_f \subseteq \prod_{j\in I} Q_j$, and $Q_i = A_i$. Hence $(X_i, \mathbf{C}_i)$ is cocompact for any $i \in I$.

Having established that the product of a finite family of cocompact closure algebras is cocompact, we have that the dual space of such a product will be non-trivial (since every consistent set may be extended to a maximal consistent set by 1.25) and perhaps of some interest.

The next theorem sets out the relationships between the dual spaces of a finite family of cocompact closure algebras and the dual space of their product.

6.7 Definition. If Y and Z are topological spaces and $f : Y \to Z$ is a function establishing that Y is homeomorphic to $f[Y]$, a subspace of Z, then f is called a topological embedding of Y into Z.

6.8 Theorem. Let $\{(X_i, \mathbf{C}_i)\}_{i\in I}$ be a finite family of cocompact closure algebras, and let $(X, \mathbf{C})$ be the product of the family. For each $i \in I$, let $(\mathbf{M}_{X_i}, \tau_i)$ be the dual space of $(X_i, \mathbf{C}_i)$, let $\mathbf{S}_i : \mathbf{P}(X_i) \to \mathbf{P}(\mathbf{M}_{X_i})$ be the mapping of definition 2.1, and let $(\mathbf{M}_X, \tau)$ be the dual space of $(X, \mathbf{C})$. Then $(\mathbf{M}_{X_i}, \tau_i)$ can be topologically embedded in $(\mathbf{M}_X, \tau)$, for each $i \in I$, and if $\{\xi_i\}_{i\in I}$ is the family of embedding functions then $\mathbf{M}_X = \bigcup_{i\in I} \xi_i[\mathbf{M}_{X_i}]$, and $\xi_i[\mathbf{M}_{X_i}] \cap \xi_j[\mathbf{M}_{X_j}] = \phi$ for each pair i, j with $i \neq j$.

Proof. I first obtain a description of $\mathbf{M}_X$ in terms of the members of the sets $\mathbf{M}_{X_i}$; indeed $\mathbf{M}_X$ is precisely the family of all sets of the form $\prod_{i\in I} Q_i$, where $Q_j = \Delta_j$, for some unique $j \in I$, $\Delta_j \in \mathbf{M}_{X_j}$, and $Q_i = X_i$, for each $i \neq j$.

I first show that any set of the above form is maximal consistent in $(X, \mathbf{C})$. We have :

$$\mathbf{C}(\prod_{i\in I} Q_i) = \prod_{i\in I} \mathbf{C}_i(Q_i)$$

$$= \prod_{i\in I} Q_i, \text{ since } \mathbf{C}_j(\Delta_j) = \Delta_j, \text{ and}$$

$$\mathbf{C}_i(X_i) = X_i \text{ ,for each } i \neq j.$$

$$\neq X, \text{ since } \Delta_j \neq X_j;$$

so that $\prod_{i\in I} Q_i$ is consistent. Furthermore, a set of the form $\prod_{i\in I} Q_i$ is maximal consistent; for let

$\prod_{i\in I} Q_i \subset A \subseteq X,$

then there is some $x \in A$ with $x \notin \prod_{i\in I} Q_i$, and so $\theta_i(x) \notin \theta_i$, for some $i \in I$.

But for each $i \neq j$, $\theta_i(x) \in X_i = Q_i$, so we must have $\theta_j(x) \notin \Delta_j = Q_j$. Now consider $\prod_{i\in I} Q_i \cup \{x\}$.

We have

$$\mathbf{C}_j(\theta_j[\prod_{i\in I} Q_i \cup \{x\}]$$

$$= \mathbf{C}_j(\Delta_j \cup \{\theta_j(x)\}$$

$$= X_j \text{ since } \Delta_j \subset \Delta_j \cup \{\theta_j(x)\} \text{ and } \Delta_j \text{ is maximal consistent.}$$

For each $k \neq j$,

$$\mathbf{C}_k(\theta_k[\prod_{i\in I} Q_i \cup \{x\}]$$

$$= \mathbf{C}_k(X_k \cup \{\theta_k(x)\})$$

$$= \mathbf{C}_k(X_k)$$

$$= X_k.$$

Combining these : $\mathbf{C}(A) \supseteq \mathbf{C}[\prod_{i\in I} Q_i \cup \{x\}]$

$$= \prod_{k\in I} \mathbf{C}_k(\theta_k[\prod_{i\in I} Q_i \cup \{x\}])$$

$$= \prod_{k\in I} X_k, \text{ from above}$$

$$= X, \text{ so that } A \text{ is inconsistent,}$$

and $\prod_{i\in I} Q_i$ is maximal consistent.

On the other hand, every maximal consistent subset of X is of this form, because if $\Delta \in \mathbf{M}_X$ then

$$\Delta = \mathbf{C}(\Delta) = \prod_{i\in I} \mathbf{C}_i(\theta_i[\Delta]),$$

and for some $j \in I$, $\theta_j[\Delta]$ must be consistent with respect to $\mathbf{C}_j$, otherwise Δ is inconsistent with respect to $\mathbf{C}$. We may extend $\theta_j[\Delta]$ to $\Delta_j \in \mathbf{M}_{X_j}$, by 1.25, yielding that

$$\Delta \subseteq \prod_{i\in I} Q_i, \text{ where } Q_j = \Delta_j \in X_j$$
$$\text{and } Q_i = X_i, \text{ for } i \neq j.$$

But $\prod_{i\in I} Q_i$ is consistent by the above result, and Δ is maximal consistent so that $\Delta = \prod_{i\in I} Q_i$. I have now shown that $\mathbf{M}_X$ is precisely the family of all sets of the form $\prod_{i\in I} Q_i$, where $Q_j = \Delta_j \in \mathbf{M}_{X_j}$, and $Q_i = X_i$ for $i \neq j$.

I now construct a family of functions $\{\xi_j\}_{j\in I}$ with $\xi_j : \mathbf{M}_{X_j} \to \mathbf{M}_X$, defined by

$$\xi_j(\Delta_j) = \prod_{i\in I} Q_i, \text{ where } Q_j = \Delta_j$$
$$\text{and } Q_i = X_i \text{ for } i \neq j, \text{for each } i \in I.$$

From the above description of $\mathfrak{M}_X$, each ξ_j is a function from $\mathfrak{M}_{X_j}$ into $\mathfrak{M}_X$. Furthermore, since for each $j \in I$, $\theta_j[\xi_j(\Delta_j)] = \Delta_j$, from the definition of ξ_j, we have that :

$$\begin{aligned} & \Delta_{j_1} \neq \Delta_{j_2} \\ \Rightarrow \quad & \theta_j[\xi_j(\Delta_{j_1})] \neq \theta_j[\xi_j(\Delta_{j_2})], \\ \Rightarrow \quad & \xi_j(\Delta_{j_1}) \neq \xi_j(\Delta_{j_2}), \text{ since } \theta_j \text{ is a function,} \end{aligned}$$

so that each ξ_j is a one-to-one function.

To prepare the way for the discussion of the continuity of ξ_j and ξ_j^{-1}, for each $j \in I$, I now deduce two properties of the mappings $\{\xi_i\}_{i\in I}$.

(i) $\bigcup_{i\in I} \xi_i[\mathbf{S}_i(A_i)] = \mathbf{S}(\prod_{i\in I} A_i)$, where $A_i \subseteq X_i$, for each $i \in I$.

For one inclusion :

$$\begin{aligned} \Delta \in \bigcup_{i\in I} \xi_i[\mathbf{S}_i(A_i)] \quad & \Rightarrow \quad \Delta \in \xi_i[\mathbf{S}_i(A_i)], \text{ for some } i\in I \\ & \Rightarrow \quad \Delta = \xi_i(\Delta_i), \text{for some } \Delta_i \in \mathbf{S}_i(A_i), \\ & \qquad\qquad \text{for some } i \in I \\ & \Rightarrow \quad \Delta \in \mathfrak{M}_X,\ \Delta_i \supseteq A_i, \text{ for some } \\ & \qquad\qquad i \in I, \end{aligned}$$

from the above description of $\mathfrak{M}_X$, and the definition of $\mathbf{S}_i$,

$$\Rightarrow \quad \Delta \in \mathfrak{M}_X,\ \Delta = \prod_{j\in I} Q_j \supseteq \prod_{j\in I} A_j,$$

since $Q_i = \Delta_i \supseteq A_i$ and $Q_j = X_j$ for $j \neq i$.

$$\Rightarrow \quad \Delta \in \mathbf{S}(\prod_{j\in I} A_j) = \mathbf{S}(\prod_{i\in I} A_i).$$

For the opposite inclusion :

$$\Delta \in \mathbf{S}(\prod_{i\in I} A_i) \Rightarrow \Delta \in \mathbf{M}_X,\ \Delta \supseteq \prod_{i\in I} A_i$$

$$\Rightarrow \Delta = \prod_{i\in I} Q_i \supseteq \prod_{i\in I} A_i, \text{ from the}$$

above description of $\mathbf{M}_X$.

$\Rightarrow Q_j = \Delta_j \supseteq A_j$, by taking the jth projection of both sides, for a suitable $j \in I$.

$$\Rightarrow \Delta_j \in \mathbf{S}_j(A_j) \quad \text{,for some } j \in I$$

$$\Rightarrow \xi_j(\Delta_j) \in \xi_j[\mathbf{S}_j(A_j)] \text{ for some } j \in I.$$

$$\Rightarrow \Delta \in \xi_j[\mathbf{S}_j(A_j)] \text{ , for some } j \in J \text{ ,}$$

since $\xi_j(\Delta_j) = \Delta$.

$$\Rightarrow \Delta \in \bigcup_{i\in I} \xi_i[\mathbf{S}_i(A_i)].$$

(ii) If $\{A_j\}_{j\in I}$ is a family of sets with $A_j \subseteq X_j$, for each $j \in I$, then for fixed $i \in I$,

$$\xi_i[\mathbf{S}_i(A_i)] = \bigcup_{j\in I}\Big(\xi_j[\mathbf{S}_j(A_j)] \cap \xi_i[\mathbf{M}_{X_i}]\Big).$$

This holds because : $\bigcup_{j\in I}\Big(\xi_j[\mathbf{S}_j(A_j)] \cap \xi_i[\mathbf{M}_{X_i}]\Big)$

$$= \bigcup_{j\in I}\Big(\xi_j[\mathbf{S}_j(A_j)] \cap \xi_i[\mathbf{S}_i(\phi)]\Big)$$

$$= \bigcup_{j\in I} (\xi_j[\mathbf{S}_j(A_j)]) \cap \xi_i[\mathbf{S}_i(\phi)], \text{ since}$$

the independence of i,j permits us to apply the distributive law for sets. Now if $i \neq j$, then

$$\xi_j[\mathbf{S}_j(A_j)] \cap \xi_i[\mathbf{S}_i(\phi)] = \phi, \text{ since}$$

$$\Delta \in \xi_j[\mathbf{S}_j(A_j)] \cap \xi_i[\mathbf{S}_i(\phi)]$$

$\Rightarrow \Delta \in \xi_j[\mathbf{S}_j(A_j)]$ and $\Delta \in \xi_i[\mathbf{S}_i(\phi)]$

$\Rightarrow \theta_j[\Delta]$ is maximal consistent in $(X_j, \mathbf{C}_j)$

and $\theta_i[\Delta]$ is maximal consistent in $(X_i, \mathbf{C}_i)$,

which is impossible for $i \neq j$, using the description of the members of $\mathbf{M}_X$. On the other hand, if $i = j$, then :

$$\xi_i[\mathbf{S}_i(A_i)] \cap \xi_i[\mathbf{S}_i(\phi)] = \xi_i[\mathbf{S}_i(A_i) \cap \mathbf{S}_i(\phi)] \ ,$$

since ξ_i is a one-to-one function

$$= \xi_i[\mathbf{S}_i(A_i \cup \phi)], \text{ by } 3.1$$

$$= \xi_i[\mathbf{S}_i(A_i)] \ ,$$

completing the result.

For each $i \in I$, ξ_i^{-1} is continuous. Fix $i \in I$, and let O be an open subset of $\mathbf{M}_{X_i}$, then $O = \bigcup_{f \in F} \mathbf{S}_i(A_{i_f})$, where each A_{i_f} is a finite subset of X_i. Retaining the same index set F, construct a class of families, $\{\{A_{j_f}\}_{f \in F} : j \in I, j \neq i\}$ where each A_{j_f} is a finite subset of X_j. We have :

$$\xi_i[O] = \xi_i[\bigcup_{f \in F} \mathbf{S}_i(A_{i_f})]$$

$$= \bigcup_{f \in F} \xi_i[\mathbf{S}_i(A_{i_f})]$$

$$= \bigcup_{f \in F} \left(\bigcup_{j \in I} \xi_j[\mathbf{S}_j(A_{j_f})] \cap \xi_i[\mathbf{M}_{X_i}] \right)$$

by property (ii)

$$= \bigcup_{f \in F} \left(\bigcup_{j \in I} \xi_j[\mathbf{S}_j(A_{j_f})] \right) \cap \xi_i[\mathbf{M}_{X_i}] .$$

$$= \bigcup_{f\in F}[\mathbf{S}(\prod_{j\in I} A_{j_f})] \cap \xi_i[\mathbf{M}_{X_i}], \text{ by property (i).}$$

Now each A_{j_f} is a finite subset of X_j, and I is finite so that $\prod_{j\in I} A_{j_f}$ is finite, yielding that $\mathbf{S}(\prod_{j\in I} A_{j_f})$ is a member of the base of $(\mathbf{M}_X, \tau)$ for each $f \in F$. This means that $\xi_i[O]$ is the intersection of $\xi_i[\mathbf{M}_{X_i}]$ with an open subset of $(\mathbf{M}_X, \tau)$ and hence is open with respect to the relative topology of $\xi_i[\mathbf{M}_{X_i}]$ as a subspace of $(\mathbf{M}_X, \tau)$.

For the continuity of each ξ_i, fix $i \in I$, and assume that u is open in $\xi_i[\mathbf{M}_{X_i}]$ with respect to the relative topology then :

$$u = \xi_i[\mathbf{M}_{X_i}] \cap O, \text{ where O is open in } (\mathbf{M}_X, \tau)$$

$$\Rightarrow u = \xi_i[\mathbf{M}_{X_i}] \cap \bigcup_{f\in F} \mathbf{S}(A_f), \text{ where } f \in F \text{ is indexing}$$

a subfamily of the base of $(\mathbf{M}_X, \tau)$

$$\Rightarrow u = \bigcup_{f\in F}(\xi_i[\mathbf{M}_{X_i}] \cap \mathbf{S}(A_f)), \text{ since } \xi_i[\mathbf{M}_{X_i}]$$

is independent of the index f.

Now for each $f \in F$,

$$\begin{aligned} &\mathbf{S}(A_f) \\ &= \mathbf{S}(\prod_{j\in I} \theta_j(A_f)) \\ &= \bigcup_{j\in I} \xi_j[\mathbf{S}_j(\theta_j[A_f])], \text{ by property (i).} \end{aligned}$$

Therefore $$u = \bigcup_{f\in F}\left\{\bigcup_{j\in I}(\xi_j[\mathbf{S}_j(\theta_j[A_f])]) \cap \xi_i[\mathbf{M}_{X_i}]\right\}$$

$$= \bigcup_{f\in F} (\xi_i[\mathbf{S}_i(\theta_i[A_f])]),$$ by property (ii).

Hence

$$\xi_i^{-1}[u] = \xi_i^{-1}[\bigcup_{f\in F} (\xi_i[\mathbf{S}_i(\theta_i[A_f])])]$$

$$= \bigcup_{f\in F} (\xi_i^{-1}[\xi_i[\mathbf{S}_i(\theta_i[A_f])]]),$$ since ξ is a function

$$= \bigcup_{f\in F} (\mathbf{S}_i(\theta_i[A_f])),$$ since ξ is a one-to-one function.

Now for each $f \in F$, A_f is finite, and so $\theta_i[A_f]$ is finite, since θ_i is a function. Hence each $\mathbf{S}_i(\theta_i[A_f])$ is a member of the base of $(\mathbf{M}_{X_i}, \tau_i)$ and so $\xi_i^{-1}[u] = \bigcup_{f\in F} \mathbf{S}_i(\theta_i[A_f])$ is open in $(\mathbf{M}_{X_i}, \tau_i)$.

I have now shown that each ξ_i is a homeomorphism embedding $(\mathbf{M}_{X_i}, \tau_i)$ in $(\mathbf{M}_X, \tau)$. It only remains to show that $\mathbf{M}_X = \bigcup_{i\in I} \xi_i[\mathbf{M}_{X_i}]$, and that if $i \neq j$, then $\xi[\mathbf{M}_{X_i}] \cap \xi_j[\mathbf{M}_{X_j}] = \phi$.

From the definitions of the ξ_i, we immediately have that $\bigcup_{i\in I} \xi_i[\mathbf{M}_{X_i}] \subseteq \mathbf{M}_X$. On the other hand, if $\Delta \in \mathbf{M}_X$ then we have shown that $\Delta = \prod_{i\in I} Q_i$, where for some unique $j \in I$, $Q_j = \Delta_j \in \mathbf{M}_{X_j}$ and $Q_i = X_i$, for each $i \neq j$.

From the definition of ξ_j, $\Delta = \prod_{i\in I} Q_i = \xi_j(\Delta_j)$, $\Delta_j \in \mathbf{M}_{X_j}$

$$\Rightarrow \Delta \in \xi_j[\mathbf{M}_{X_j}].$$

$$\Rightarrow \Delta \in \bigcup_{i\in I} \xi_i[\mathbf{M}_{X_i}].$$

Hence $\mathbf{M}_X = \bigcup_{i\in I} \xi_i[\mathbf{M}_{X_i}]$.

For the second result, choose any $i,j \in I$ with $i \neq j$, then $\xi_i[\mathbf{M}_{X_i}] \cap \xi_j[\mathbf{M}_{X_j}] = \phi$ because :

$$\Delta \in \xi_i[\mathbf{M}_{X_i}] \cap \xi_j[\mathbf{M}_{X_j}]$$

$$\Rightarrow \quad \Delta \in \xi_i[\mathbf{M}_{X_i}] \text{ and } \Delta \in \xi_j[\mathbf{M}_{X_j}].$$

$$\Rightarrow \quad \Delta = \prod_{k \in I} Q_k, \quad \text{where } Q_i = \Delta_i \in \mathbf{M}_{X_i}$$
$$\text{and} \quad Q_k = X_k, \text{ for } k \neq i,$$

and $$\Delta = \prod_{k \in I} R_k, \quad \text{where } R_j = \Delta_j \in \mathbf{M}_{X_j}$$
$$\text{and} \quad R_k = X_k, \text{ for } k \neq j.$$

$$\Rightarrow \quad Q_i = R_i \text{ and } Q_i = \Delta_i \text{ and } R_i = X_i$$

$$\Rightarrow \quad \Delta_i = X_i, \text{ contradicting the consistency of } \Delta_i \text{ with respect to } \mathbf{C}_i.$$

The remainder of chapter VI is devoted to studying the properties of homomorphic images of closure algebras, and the properties of the dual spaces of such images.
I start by showing that the properties of being finitary and finitely generated are invariant under homomorphisms.

6.9 Theorem. Let $(X,\mathbf{C})$ be a closure algebra and $(Y,\mathbf{C}_1)$ a homomorphic image of $(X,\mathbf{C})$, under a homomorphism θ. If $(X,\mathbf{C})$ is finitely generated, then so is $(Y,\mathbf{C}_1)$.

Proof. Assume $(X,\mathbf{C})$ is finitely generated, and let X_f be a finite generating set for X i.e. $\mathbf{C}(X_f) = X$. Consider $\theta[X_f]$; since X_f is finite and θ is a function, $\theta[X_f]$ is finite.

Furthermore,

$$\mathbf{C}_1(\theta[X_f]) = \theta[\mathbf{C}(X_f)], \text{ since } \theta \text{ is a homomorphism}$$
$$= \theta[X]$$
$$= Y,$$

so that $(Y, \mathbf{C}_1)$ is finitely generated, with generating set $\theta[X_f]$.

<u>6.10 Theorem</u>. If $(X,\mathbf{C})$ is a finitary closure algebra, and $(Y, \mathbf{C}_1)$ is a homomorphic image of $(X,\mathbf{C})$, then $(Y,\mathbf{C}_1)$ is finitary.

Proof. Take θ to be the homomorphism, and let $\{\alpha_t\}_{t\in T}$ be a chain of closed subsets of Y. If $t_1, t_2 \in T$, then :

$$\theta^{-1}[\alpha_{t_1}] \nsubseteq \theta^{-1}[\alpha_{t_2}]$$
$$\Rightarrow \quad x \in \theta^{-1}[\alpha_{t_1}],\ x \notin \theta^{-1}[\alpha_{t_2}], \text{ for some } x \in X.$$
$$\Rightarrow \quad \theta(x) \in \alpha_{t_1},\ \theta(x) \notin \alpha_{t_2}$$
$$\Rightarrow \quad \alpha_{t_1} \nsubseteq \alpha_{t_2}.$$
$$\Rightarrow \quad \alpha_{t_2} \subseteq \alpha_{t_1}, \text{ since } \{\alpha_t\}_{t\in T} \text{ is a chain.}$$
$$\Rightarrow \quad \theta^{-1}[\alpha_{t_2}] \subseteq \theta^{-1}[\alpha_{t_1}],$$

so that $\{\theta^{-1}[\alpha_t]\}_{t\in T}$ is a chain of subsets of X. Moreover each $\theta^{-1}[\alpha_t]$ is closed in X, since α_t is closed in Y, and by 5.12 (a). Because $(X,\mathbf{C})$ is finitary , $\bigcup_{t\in T}\theta^{-1}[\alpha_t]$ is closed in $(X,\mathbf{C})$, by 1.18, and so $\theta[\bigcup_{t\in T}\theta^{-1}[\alpha_t]]$ is closed in $(Y, \mathbf{C}_1)$ by 5.12(a).

But :

$$\theta[\bigcup_{t\in T}\theta^{-1}[\alpha_t]] = \bigcup_{t\in T}\theta[\theta^{-1}[\alpha_t]] \quad , \text{ since } \theta \text{ a function}$$
$$= \bigcup_{t\in T}\alpha_t, \text{ since } \theta \text{ is onto.}$$

Which is to say that $\{\alpha_t\}_{t\in T}$ has a closed least upper bound, and hence the family of closed subsets of $(Y, \mathbf{C}_1)$ is inductive. By 1.18, $(Y, \mathbf{C}_1)$ is finitary.

<u>6.11 Theorem</u>. If $(X, \mathbf{C})$ is a finitary, cocompact closure algebra, then every homomorphic image of $(X, \mathbf{C})$ is finitary and cocompact.

Proof. If $(X,\mathbf{C})$ is cocompact and finitary, then $(X,\mathbf{C})$ is finitely generated and finitary, by 1.31. Hence by 6.9 and 6.10, every homomorphic image of $(X,\mathbf{C})$ is finitely generated and finitary. But then every homomorphic image of $(X,\mathbf{C})$ is cocompact and finitary, by 1.32.

In preparation to comparing the dual space of a closure algebra with the dual space of one of its homomorphic images, I now prove two theorems on the behaviour of consistent and maximal consistent sets under a homomorphism.

<u>6.12 Theorem</u>. If $(Y, \mathbf{C}_1)$ is the homomorphic image of a closure algebra $(X,\mathbf{C})$ under a homomorphism θ, and B is a consistent subset of Y, then $\theta^{-1}[B]$ is a consistent subset of X.

Proof. Assume that B is consistent in Y, and that $\theta^{-1}[B]$ is inconsistent in X. Then $\mathbf{C}(\theta^{-1}[B]) = X$, and

$B = \theta[\theta^{-1}[B]]$, since θ is onto,

$$\begin{aligned} \text{so that } \mathbf{C}_1(B) &= \mathbf{C}_1(\theta[\theta^{-1}[B]]) \\ &= \theta[\mathbf{C}(\theta^{-1}[B])] \\ &= \theta[X] \\ &= Y, \end{aligned}$$

contradicting the consistency of B in Y. Hence $\theta^{-1}[B]$ is consistent in X.

6.13 Theorem. Let $(Y, \mathbf{C}_1)$ be the homomorphic image of a closure algebra $(X, \mathbf{C})$ under θ. If $\Delta \in \mathbf{M}_X$ then either $\theta[\Delta] = Y$ or $\theta[\Delta] \in \mathbf{M}_Y$.

Proof. Assume that $\Delta \in \mathbf{M}_X$, then by 1.27 Δ is closed in X, and by 5.12(a) $\theta[\Delta]$ is closed in Y. Now if $\theta[\Delta] \neq Y$, then $\mathbf{C}(\theta[\Delta]) = \theta[\Delta] \neq Y$, so that $\theta[\Delta]$ is consistent with respect to $\mathbf{C}_1$. Suppose that $\theta[\Delta]$ is not maximal consistent. Then there exists a set B, consistent in Y, such that $\theta[\Delta] \subset B$, and $\Delta \subseteq \theta^{-1}[\theta[\Delta]] \subset \theta^{-1}[B]$, since θ is a function. But B is consistent in Y, so that $\theta^{-1}[B]$ is consistent in X, by 6.12. Thus the maximal consistency of Δ has been contradicted, and we must have that $\theta[\Delta] \in \mathbf{M}_Y$ whenever $\theta[\Delta] \neq Y$.

The next theorem and its three corollaries establish the relationship between the dual space of a closure algebra and the dual space of one of its homomorphic images. Starting with a specified subset of each dual space, I construct topologies for both subsets and prove that they are the relative topologies in each case. Then these subspaces of the dual space are shown to be related, with that for the given algebra being the continuous, one-to-one image of that for the image algebra. The corollaries set out the conditions for the two dual spaces to be homeomorphic.

6.14 Theorem. Let $(X, \mathbf{C}_1)$ be a closure algebra and $(Y, \mathbf{C}_2)$ its image under a homomorphism θ. Let the dual spaces be $(\mathbf{M}_X, \tau_1)$, $(\mathbf{M}_Y, \tau_2)$, respectively, and define $P \subseteq \mathbf{M}_X$, $Q \subseteq \mathbf{M}_Y$ as follows :

$$P = \{\Delta \in \mathbf{M}_X : \theta[\Delta] = \Gamma \text{ , for some } \Gamma \in \mathbf{M}_Y\}$$
$$Q = \{\Gamma \in \mathbf{M}_Y : \Gamma = \theta[\Delta] \text{ , for some } \Delta \in \mathbf{M}_X\}$$

Let $\mathbf{S}_1|_P$, $\mathbf{S}_2|_Q$ be the restrictions of the mappings $\mathbf{S}_1$, $\mathbf{S}_2$, defined as in 2.1, to P and Q respectively, and let

$$\beta_1 = \{\mathbf{S}_1|_P(A_f) : A_f \text{ is finite, } A_f \in \mathbf{P}(X)\}$$
$$\beta_2 = \{\mathbf{S}_2|_Q(B_f) : B_f \text{ is finite, } B_f \in \mathbf{P}(Y)\}$$

Then β_1 is the base for a topology τ_1' on P, and β_2 is the base for a topology τ_2' on Q. Furthermore, (P, τ_1') is a subspace of $(\mathbf{M}_X, \tau_1)$, and (Q, τ_2') is a subspace of $(\mathbf{M}_Y, \tau_2)$, and (P, τ_1') is the continuous, one-to-one image of (Q, τ_2').

Proof. First note that from the definition of P and Q, $P = \phi \iff Q = \phi$, and in this case $\beta_1 = \beta_2 = \phi$,and the theorem holds trivially. I assume that $P \neq \phi$, $Q \neq \phi$, and show that the mappings $\mathbf{S}_1|_P$, $\mathbf{S}_2|_Q$ satisfy theorem 3.1. Since this is the only assumption made for theorem 3.2, which proves that sets of the form β_1, β_2 constitute bases for the topologies on the associated dual spaces, it will follow that (P, τ_1'), (Q, τ_2') are topological spaces with bases β_1, β_2. Consider the set P and the mapping $\mathbf{S}_1|_P$ and let $\{A_k\}_{k\in K} \subseteq \mathbf{P}(P)$, then :

$$\begin{aligned}
\Delta \in \bigcap_{k\in K} \mathbf{S}_1|_P(A_k) &\iff \Delta \in \mathbf{S}_1|_P(A_k), \text{ for each } k \in K \\
&\iff A_k \subseteq \Delta,\ \Delta \in \mathbf{M}_X,\ \theta[\Delta] = \Gamma, \text{ for} \\
&\qquad \text{some } \Gamma \in \mathbf{M}_Y, \text{for each } k \in K. \\
&\iff \bigcup_{k\in K} A_k \subseteq \Delta,\ \Delta \in \mathbf{M}_X,\ \theta[\Delta] = \Gamma, \\
&\qquad \text{for some } \Gamma \in \mathbf{M}_Y \\
&\iff \Delta \in \mathbf{S}_1|_P(\bigcup_{k\in K} A_k).
\end{aligned}$$

Hence $\bigcap_{k\in K} \mathbf{S}_1|_P(A_k) = \mathbf{S}_1|_P(\bigcup_{k\in K} A_k)$, showing that $\mathbf{S}_1|_P$ satisfies theorem 3.1. Applying theorem 3.2 shows that (P, τ_1') is a topological space, where τ_1' is the set of arbitrary unions of members of β_1. Similarly, (Q, τ_2) is a topological space with base β_2.

To prove that (P, τ_1') is a subspace of $(\mathbf{M}_X, \tau_1)$, I show that τ_1' is precisely the family obtained by taking the intersection of each member of τ_1 with P, and is hence the relative topology. Let $O \in \tau_1'$, then $O = \bigcup_{f\in F} \mathbf{S}_1|_P(A_f)$, where $f \in F$ indexes a subset of β_1, and for each $f \in F$,

$$\begin{aligned}
\mathbf{S}_1|_P(A_f) &= \{\Delta \in \mathbf{M}_X : A_f \subseteq \Delta,\ \theta[\Delta] \in \mathbf{M}_Y\} \\
&= \{\Delta \in \mathbf{M}_X : A_f \subseteq \Delta\} \cap \{\Delta \in \mathbf{M}_X : \\
&\qquad\qquad \theta[\Delta] \in \mathbf{M}_Y\} \\
&= \mathbf{S}_1(A_f) \cap P.
\end{aligned}$$

$$\text{Hence } O = \bigcup_{f\in F} S_1|_P(A_f) = \bigcup_{f\in F}(S_1(A_f) \cap P)$$
$$= \bigcup_{f\in F} S_1(A_f) \cap P,$$

so that each member of τ_1' is the intersection of a member of τ_1 with P. On the other hand, each member of τ_1 has the form $\bigcup_{f\in F} S_1(A_f)$, for some indexing $f \in F$ of the base of (M_X, τ_1), so that same equation shows that the intersection of any member of τ_1 with P yields a member of τ_1'. This proves that (P, τ_1') is a subspace of (M_X, τ_1) and it may be proved similarly that (Q, τ_2') is a subspace of (M_Y, τ_2).

I now show that (P, τ_1') is the continuous, one-to-one image of (Q, τ_2'). Define a mapping

$\xi : P \to Q$ by $\xi(\Delta) = \theta[\Delta]$, for each $\Delta \in P$. Now if $\Delta \in P$ then $\theta[\Delta] \in Q$, from the definitions of P,Q and by theorem 6.13, so that ξ maps from all of P into Q. Furthermore ξ is onto, by the definitions of P and Q. ξ is a function, because if $\Delta_1 = \Delta_2$ then $\theta[\Delta_1] = \theta[\Delta_2]$, since θ is a function, and then $\xi(\Delta_1) = \xi(\Delta_2)$. Furthermore, ξ is 1-1, because if $\Delta_1, \Delta_2 \in P$ then

$$\begin{aligned}
& \xi(\Delta_1) = \xi(\Delta_2) \\
\Rightarrow\ & \theta[\Delta_1] = \theta[\Delta_2] \\
\Rightarrow\ & \theta[\Delta_1] \cup \theta[\Delta_2] = \theta[\Delta_1] \\
\Rightarrow\ & \theta[\Delta_1 \cup \Delta_2] = \theta[\Delta_1], \text{ since } \theta \text{ is a function.}
\end{aligned}$$

Hence $\theta[\Delta_1 \cup \Delta_2]$ is consistent, since $\theta[\Delta_1] \in Q \subset M_Y$, and $\theta^{-1}[\theta[\Delta_1 \cup \Delta_2]]$ is consistent by theorem 6.12.

Therefore $\Delta_1 \cup \Delta_2$ is consistent, since $\Delta_1 \cup \Delta_2 \subseteq \theta^{-1}[\theta[\Delta_1 \cup \Delta_2]]$ But $\Delta_1 \subseteq \Delta_1 \cup \Delta_2$ and Δ_1 is maximal consistent, hence $\Delta_1 = \Delta_1 \cup \Delta_2$, and similarly $\Delta_2 = \Delta_1 \cup \Delta_2$, so that $\Delta_1 = \Delta_2$.

Since ξ is a one-to-one and onto function, ξ^{-1} is a one-to-one and onto function, and it only remains to be shown that ξ^{-1} is continuous; i.e. $(\xi^{-1})^{-1} = \xi$ takes open sets to open sets. I first show that ξ maps members of β_1 to members of β_2. Suppose that $\mathbf{S}_1|_P(A_f) \in \beta_1$, the base for (P, τ_1'), then $\xi[\mathbf{S}_1|_P(A_f)] \subseteq \mathbf{S}_2|_Q(\theta[A_f])$ because :

$$\begin{aligned}
& \Gamma \in \xi[\mathbf{S}_1|_P(A_f)] \\
\Rightarrow\quad & \Gamma = \xi(\Delta), \text{ for some } \Delta \in P \text{ with } A_f \subseteq \Delta \\
\Rightarrow\quad & \Gamma = \theta[\Delta], \text{ for some } \Delta \in P \text{ with } \theta[A_f] \subseteq \theta[\Delta] \\
\Rightarrow\quad & \theta[A_f] \subseteq \Gamma \text{ and } \Gamma \in Q \\
\Rightarrow\quad & \Gamma \in \mathbf{S}_2(\theta[A_f]) \text{ and } \Gamma \in Q \\
\Rightarrow\quad & \Gamma \in \mathbf{S}_2|_Q(\theta[A_f]).
\end{aligned}$$

On the other hand $\mathbf{S}_2|_Q(\theta[A_f]) \subseteq \xi[\mathbf{S}_2|_P(A_f)]$, because :

$$\begin{aligned}
& \Gamma \in \mathbf{S}_2|_Q(\theta[A_f]) \\
\Rightarrow\quad & \Gamma \in \mathbf{S}_2(\theta[A_f]) \text{ and } \Gamma \in Q \\
\Rightarrow\quad & \Gamma \supseteq \theta[A_f] \text{ and } \Gamma \in Q \\
\Rightarrow\quad & \Gamma \supseteq \theta[A_f] \text{ and } \Gamma = \theta[\Delta] = \xi(\Delta), \text{ for some} \\
& \qquad \Delta \in P, \text{ by the definition of } P \text{ and } Q. \\
\Rightarrow\quad & \theta[\Delta] \supseteq \theta[A_f],\ \Gamma = \xi(\Delta), \text{ for some } \Delta \in P. \\
\Rightarrow\quad & \theta^{-1}[\theta[\Delta]] \supseteq \theta^{-1}[\theta[A_f]] \supseteq A_f, \quad \Gamma = \xi(\Delta) \text{ for} \\
& \qquad \text{some } \Delta \in P \\
\Rightarrow\quad & \Delta \supseteq A_f,\ \Gamma = \xi(\Delta), \text{ for some } \Delta \in P, \text{ since}
\end{aligned}$$

$\Delta \in P$, $\theta[\Delta] \in Q$ means that $\theta^{-1}[\theta[\Delta]]$ is consistent in X by 6.12. But $\theta^{-1}[\theta[\Delta]] \supseteq \Delta$, so that $\theta^{-1}[\theta[\Delta]] = \Delta$ by the maximal consistency of Δ.

$$\Rightarrow \quad \Delta \in \mathbf{S}_1(A_f) \text{ and } \Gamma = \xi(\Delta), \text{ for some } \Delta \in P$$

$$\Rightarrow \quad \Delta \in \mathbf{S}_1|_P(A_f) \text{ and } \Gamma = \xi(\Delta)$$

$$\Rightarrow \quad \Gamma \in \xi[\mathbf{S}_1|_P(A_f)] \ .$$

Hence $\xi[\mathbf{S}_1|_P(A_f)] = \mathbf{S}_2|_Q(\theta[A_f])$. Further $A_f \subseteq X$ and A_f is finite, so that $\theta[A_f]$ is a finite subset of Y. Thus $\xi[\mathbf{S}_1|_P(A_f)] = \mathbf{S}_2|_Q(\theta[A_f]) \in \beta_2$.

Now if $O \in \tau_1'$ then $O = \bigcup_{f \in F} \mathbf{S}_1|_P(A_f)$, so that

$$\begin{aligned} \xi[O] &= \xi[\bigcup_{f\in F} \mathbf{S}_1|_P(A_f)] \\ &= \bigcup_{f\in F}(\xi[\mathbf{S}_1|_P(A_f)] \text{ , since } \xi \text{ is a function} \\ &= \bigcup_{f\in F}(\mathbf{S}_2|_Q(\theta[A_f])) \in \tau_2' \ . \end{aligned}$$

Hence (P, τ_1') is the continuous, one-to-one image of (Q, τ_2').

<u>6.15 Corollary</u>. Given the notation of theorem 6.14, if θ preserves consistent sets then $(\mathbf{M}_X, \tau_1)$ is the continuous, one-to-one image of $(\mathbf{M}_Y, \tau_2)$.

Proof. If θ preserves consistent sets then each member of $\mathbf{M}_X$ is mapped to a consistent set, and by 6.13 to a maximal consistent set; hence $P = \mathbf{M}_X$ by the definition of P. I will now show that $Q = \mathbf{M}_Y$. If $\Gamma \in \mathbf{M}_Y$ then Γ is consistent, and so $\theta^{-1}[\Gamma]$ is consistent, by 6.12. If $\theta^{-1}[\Gamma]$ is not maximal consistent, then there is some $A \in \mathbf{P}(X)$ such that A is consistent and $A \supset \theta^{-1}[\Gamma]$

$$\Rightarrow \theta[A] \supseteq \theta[\theta^{-1}[\Gamma]], \text{ and there is some } x \in A \text{ with } x \notin \theta^{-1}[\Gamma]$$

$$\Rightarrow \theta[A] \supseteq \Gamma, \text{ and there is some } x \in A \text{ with } \theta(x) \notin \Gamma \text{ , since } \theta \text{ is a function}$$

$$\Rightarrow \theta[A] \supset \Gamma \text{ .}$$

Furthermore A consistent implies $\theta[A]$ consistent, since θ preserves consistent sets. This contradicts the maximal consistency of Γ, so we may take $\theta^{-1}[\Gamma]$ to be maximal consistent in X i.e. $\Delta = \theta^{-1}[\Gamma] \in \mathbf{M}_X$

$$\Rightarrow \theta[\Delta] = \theta[\theta^{-1}[\Gamma]] = \Gamma, \ \Delta \in \mathbf{M}_X, \text{ since } \theta \text{ is an onto function.}$$

$$\Rightarrow \Gamma \in Q, \text{ by the definition of } Q.$$

We now have that $P = \mathbf{M}_X$, $Q = \mathbf{M}_Y$, and by theorem 6.14, that $\mathbf{M}_X$ is the continuous, one-to-one image of $\mathbf{M}_Y$.

6.16 Corollary. Given the notation of theorem 6.14, if θ preserves infinite sets then P is homeomorphic to Q.

Proof. From theorem 6.14, it only remains to be shown that ξ^{-1} maps numbers of τ_2' to members of τ_1' . I first show that ξ^{-1} maps members of β_2 to members of β_1. If $\mathbf{S}_2|_Q(B_f) \in \beta_2$ then $\xi^{-1}[\mathbf{S}_2|_Q(B_f)] \subseteq \mathbf{S}_1|_P(\theta^{-1}[B_f])$, because:

$$\Delta \in \xi^{-1}[\mathbf{S}_2|_Q(B_f)]$$

$$\Rightarrow \xi(\Delta) \in \mathbf{S}_2|_Q(B_f)$$

$\Rightarrow \quad \xi(\Delta) \in Q, \ \xi(\Delta) \supseteq B_f$

$\Rightarrow \quad \Delta \in P, \ \Delta \supseteq \xi^{-1}(\xi(\Delta)) \supseteq \theta^{-1}[B_f]$, since ξ is a one-to-one function

$\Rightarrow \quad \Delta \in \mathbf{S}_1|_P(\theta^{-1}[B_f])$.

Also $\mathbf{S}_1|_P(\theta^{-1}[B_f]) \subseteq \xi^{-1}[\mathbf{S}_2|_Q(B_f)]$, because :

$\Delta \in \mathbf{S}_1|_P(\theta^{-1}[B_f])$

$\Rightarrow \quad \Delta \in P, \ \Delta \supseteq \theta^{-1}[B_f]$

$\Rightarrow \quad \xi(\Delta) = \theta[\Delta] \in Q$ and $\xi(\Delta) = \theta[\Delta] \supseteq \theta[\theta^{-1}[B_f]] = B_f$.

$\Rightarrow \quad \xi(\Delta) \in \mathbf{S}_2|_Q(B_f)$

$\Rightarrow \quad \Delta \in \xi^{-1}[\mathbf{S}_2|_Q(B_f)]$.

Hence $\xi^{-1}[\mathbf{S}_2|_Q(B_f)] = \mathbf{S}_1|_P(\theta^{-1}[B_f])$.

Now if $O \in \tau_2'$ then $O = \bigcup_{f\in F} \mathbf{S}_2|_Q(B_f)$, and

$$\begin{aligned}\xi^{-1}[O] &= \xi^{-1}[\bigcup_{f\in F}\mathbf{S}_2|_Q(B_f)] \\ &= \bigcup_{f\in F}\xi^{-1}[\mathbf{S}_2|_Q(B_f)] \\ &= \bigcup_{f\in F}(\mathbf{S}_1|_P(\theta^{-1}[B_f]), \text{ from the above.}\end{aligned}$$

But since θ preserves infinite sets, θ^{-1} preserves finite sets, and each $\mathbf{S}_1|_P(\theta^{-1}[B_f])$ will be in β_1, the base for τ_1'. Hence $\xi^{-1}[O] \in \tau_1'$.

<u>6.17 Corollary</u>. Given the notation of theorem 6.14, if θ preserves consistent sets and infinite sets, then (M_X, τ_1), (M_Y, τ_2) are homeomorphic.

Proof. By combining theorem 6.14 and corollaries 6.15, 6.16.

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