TWO TOPICS IN THE THEORY OF OPTIMAL
TRAJECTORY ANALYSIS

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Two Topics in the Theory of Optimal Trajectory Analysis

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CONTENTS

PART I - Review of Basic Theory.
1. Introduction.
2. The Mayer Problem.

PART II - Trajectory Optimization for a Rocket with a Generalized Thrust Characteristic.
1. Introduction.
2. General Theory.
3. The Constant Field.
4. The One Dimensional Case.
5. Sounding Rocket.
6. The Constant Power Case.
7. Further Considerations.

PART III - Optimal Trajectories for a Solar Sail Vehicle.
1. Introduction.
2. Force on a Reflecting Surface.
3. Flight in a Uniform Field.
4. Inverse Square Field. The Equations of Motion.
5. Boundary Conditions.
7. The Weierstrass Condition.
10. General Conic Solution.
11. The Three Dimensional Problem.
12. Some Numerical Results.

PART I

REVIEW OF BASIC THEORY
1. Introduction

The work presented in this thesis is concerned with certain problems in the domain of optimal trajectory analysis. The general problem in this field is to determine the 'best' way to use a certain travel vehicle to carry out some particular mission. Thus our vehicle may be a rocket which is designed to operate with constant exhaust velocity or at constant power. Our mission may be to effect a transfer from the orbit of the Earth to the orbit of some destination planet within the solar system. And our criterion, for judging which manner of utilizing our travel vehicle is 'best' may be the requirement that the fuel consumption or the transit time be minimized.

The work which follows consists basically of three parts. In Part I the well known problem in the Calculus of Variations known as the Mayer Problem is outlined, since this is the basic mathematical tool which will be used in Parts II and III. In Part II an investigation is made of the thrust programmes obtained when the thrust characteristic of a rocket is of a more general nature than the constant exhaust velocity thrust characteristic which, for instance, is assumed throughout in [1]. It is shown how well known results for the case of a rocket operating at constant exhaust velocity may be obtained as limits of the more general results obtained in that part. Also, the case of optimization for a rocket operating at constant power is considered as a special case and is solved completely for
the one dimensional case. In Part III, the travel vehicle considered is the so-called 'solar sail'. The thrust for this vehicle is derived from the radiation pressure of the solar radiation reflected from the sail surface. For a vehicle of this type, the question of minimizing fuel consumption does not arise once the vehicle has been assembled in space, so that the payoff criterion of most interest will be minimization of transit time. A more detailed conspectus of the problems considered and the results obtained is presented in the separate introductions to Parts II and III.

2. The Mayer Problem

A basic tool used in tackling problems of trajectory optimization is the solution to the Mayer Problem in the Calculus of Variations. Following section 1.3 of [1], we outline first a formulation of the Mayer problem which is particularly convenient for treating optimal trajectory problems and which is of sufficient generality to be applicable to the most common types of trajectory optimization problem encountered. This formulation, however, will not be sufficiently general to treat all the problems investigated in Part III of the present work. Following section II of [2], we will therefore outline
also another formulation of the Mayer problem which will be sufficiently general to meet the requirements of section 10 of Part III.

The first formulation of the Mayer problem proceeds as follows. We are given n first order differential equations

\[ \dot{x}_i = f_i(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m, t) \]  

(1)

to be satisfied by the n functions \( x_i(t) \) \((i=1,2,\ldots,n)\), known as the state variables, and m functions \( \alpha_j(t) \) \((j=1,2,\ldots,m)\), known as the control variables. Dots denote derivatives with respect to an independent variable \( t \). These equations are to be valid for \( t_0 < t < t_1 \) and the \( \alpha_j \) are defined throughout this interval as continuous functions apart from a finite number of finite discontinuities. The initial values of the \( x_i \) at \( t = t_0 \) are specified by the equations

\[ x_i = x_{i0} \]  

(2)

The functions \( \alpha_j(t) \) and \( x_i(t) \) are required to satisfy certain constraining equations

\[ g_k(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m, t) = 0, \]  

(3)

where \( k = 1,2,\ldots,p < m \) and the \( g_k \) are continuous and possess continuous partial derivatives of sufficiently high order in all their arguments. The values of certain of the state variables are also to be specified at \( t = t_1 \). Thus, for \( t = t_1 \),

\[ x_\ell = x_{\ell1}, \quad \ell = 1,2,\ldots,q \]  

(4)
and $q \leq n$. The control functions are arbitrary within the limits imposed by equations (3), (4). The existence of control functions satisfying the constraints (3), (4) will be assumed.

Let $x_{q+1}, x_{q+2}, \ldots, x_n$ be the values of the $x_i$ at $t = t_1$ not fixed by the constraints (4). Then our problem is to find control functions $\alpha_j$ determining the $x_i$ so that the constraints (3), (4) are satisfied and also so that a given functional

$$J(x_{q+1}, x_{q+2}, \ldots, x_n, t_1)$$

(5)

is minimized, where $t_1$ may or may not be prespecified. $J$ is supposed to be continuous in all its variables and to possess continuous partial derivatives of sufficiently high order.

We introduce certain auxiliary functions $\lambda_i(t), \mu_k(t)$ known as Lagrange multipliers and a quantity $F$ known as the Lagrange expression defined by the equation

$$F = -\lambda_if_i + \mu_ks_k$$

(6)

where we have adopted the usual summation convention; indices repeated in a product imply summation over the range of the index. We are now able to state first necessary conditions for a minimum of $J$.

At points where the $\alpha_j$ are continuous, the Lagrange multipliers $\lambda_i, \mu_k$ must satisfy the differential equations

$$\dot{\lambda}_i = \frac{\partial F}{\partial x_i} \quad i = 1, \ldots, n$$

(7)
and the equations
\[ 0 = \frac{\partial F}{\partial \alpha_j} \quad j = 1, \ldots, m \tag{8} \]

For the so-called normal problem we also have boundary conditions of the form
\[ \lambda_{s_1} = -\frac{\partial J}{\partial x_{s_1}}, \quad s = q+1, \ldots, n \tag{9} \]
\[ \lambda_{i_1} \dot{x}_{i_1} = \frac{\partial J}{\partial t_1}. \tag{10} \]

Equations (1), (3), (7), (8) provide a set of \(2n+m+p\) equations to be satisfied by the same number of functions \(x_i(t), \alpha_j(t), \lambda_i(t), \mu_k(t)\). Equations (2), (4), (9), (10) provide \(2n+1\) end conditions which serve to determine the \(n\) constants \(\lambda_{i_0}\), the \(n\) constants of integration associated with the differential equations (1) and the end point \(t_1\).

At a corner on an optimal trajectory, where some or all of the \(\alpha_j\) are discontinuous, we may apply the Weierstrass-Erdmann Corner Conditions. These state that at a corner, the Lagrange multipliers \(\lambda_i\) and the expression \(\lambda_i \dot{x}_i\) must all be continuous.

A first integral of equations (1), (7) exists when the functions \(f_i, g_k\) are not explicitly dependent upon \(t\). It is given by the equation
\[ \lambda_i \dot{x}_i = \text{constant}. \tag{11} \]

A third necessary condition, which is useful for excluding solutions of the first necessary conditions which do not, in fact, minimize \(J\), is the Weierstrass condition. This states
that for a minimum of $J$, the inequality

$$\lambda_1 \dot{x}_1 \geq \lambda_1 \dot{x}_1$$  \hspace{1cm} (12)

must be satisfied at every point on the minimal trajectory. In this inequality, the quantities $\lambda_1, x_1$ refer to the minimal trajectory, while the quantities $\dot{x}_1$ are any set of values of the $\dot{x}_1$ obtainable from equations (1) by substitution of the minimal values of the $x_1$ and any set of values $A_j$ of the $a_j$ consistent with the constraints (3).

The second formulation of the Mayer problem which we give differs from the first mainly in treating more general boundary conditions. Also, an explicit distinction between state variables and control variables is not made. We consider $n$ unknown functions $x_1(t)$ ($i = 1, 2, \ldots, n$) of an independent variable $t$, which are to be defined for values of $t$ extending over the range $t_0 < t < t_1$. These functions have to satisfy the $m$ first order differential equations

$$\phi_j(t, t_0, t_1, \kappa_k, x_1, \dot{x}_1) = 0 \hspace{1cm} (j = 1, 2, \ldots, m < n), \hspace{1cm} (13)$$

where the $\kappa_k$ ($k = 1, 2, \ldots, q$) are $q$ parameters whose values are also unknown. The values taken by the function $x_1(t)$ at the end points $t = t_0$, $t = t_1$ will be denoted by $x_{10}, x_{11}$, respectively. These values are to satisfy the $p$ equations

$$\psi_\xi(t_0, t_1, x_{10}, x_{11}) = 0 \hspace{1cm} (\xi = 1, 2, \ldots, p < 2n+2). \hspace{1cm} (14)$$

$J$ is a given function of $t_0, t_1, x_{10}, x_{11}$, i.e.,
\[ J = J(t_0, t_1, x_{i0}, x_{i1}) \]  

(15)

It is required to choose the functions \( x_i \), subject to the constraints \((13)\) and \((14)\), the end points \( t_0, t_1 \), and the values of the \( \kappa_k \), so that \( J \) is minimized.

Let \( \lambda_1, \lambda_2, \ldots, \lambda_m \) denote Lagrange multipliers depending upon \( t \) and to be determined subsequently. Let \( F \) be the function defined by the equation

\[ F = \lambda_j \phi_j. \]  

(16)

Let \( \nu_1, \nu_2, \ldots, \nu_p \) be constant multipliers, also to be determined later and define \( H \) by the equation,

\[ H = J + \nu_k \phi_k. \]  

(17)

Then, the functions \( x_i \) which minimize \( J \) necessarily satisfy the second order differential equations

\[ \frac{\partial F}{\partial x_i} \cdot \frac{d}{dt} \left( \frac{\partial F}{\partial x_i} \right) = 0 \quad (i = 1, 2, \ldots, n), \]  

(18)

at all points where the \( \dot{x}_i \) are continuous.

Also, at the end points \( t = t_0, t = t_1 \), it is necessary that

\[ \frac{\partial H}{\partial t_0} + \frac{\partial H}{\partial x_{i0}} \dot{x}_{i0} + \int_{t_0}^{t_1} \frac{\partial F}{\partial t_0} \, dt = 0, \]  

(19)

\[ \frac{\partial H}{\partial t_1} + \frac{\partial H}{\partial x_{i1}} \dot{x}_{i1} + \int_{t_0}^{t_1} \frac{\partial F}{\partial t_1} \, dt = 0, \]  

(20)

\[ \frac{\partial H}{\partial x_{i0}} \left( \frac{\partial F}{\partial x_{i0}} \right) = 0, \]  

(21)

\[ \frac{\partial H}{\partial x_{i1}} \left( \frac{\partial F}{\partial x_{i1}} \right) = 0. \]  

(22)
Together with the \( m \) constraints (13), the \( n \) Euler equations, (equations (18)), which are of the second order, determine the \( m + n \) functions \( x_1, \lambda_1 \), apart from \( 2(m+n) \) constants of integration. The \( (2n+q+2) \) equations (19)-(23), together with the \( p \) constraints (14) and the \( 2m \) equations obtained by setting \( t = t_0, t = t \), in equations (13), determine the \( p \) constants \( \nu_1 \), the end points \( t_0, t \), the parameters \( \kappa_k \) and the \( 2(m+n) \) constants of integration.

The Weierstrass–Erdmann corner conditions state that the following expressions shall be continuous at a discontinuity in \( \dot{x}_1 \):

\[
\frac{\partial F}{\partial \dot{x}_1}, \quad F - \dot{x}_1 \frac{\partial F}{\partial \dot{x}_1}.
\]  

Lastly, writing

\[
K = F - \dot{x}_1 \frac{\partial F}{\partial \dot{x}_1},
\]

it may be shown that, if the \( \phi_j \) are not explicitly dependent upon \( t \), we have a first integral given by

\[
K = \text{constant}.
\]

Connected with the above results there are, of course, certain analytical conditions which must be satisfied by the functions \( \phi_j \) etc. in order that the results quoted can be confidently asserted to be applicable to any particular case.
These matters are considered in detail by Bliss [3].

REFERENCES


PART II

TRAJECTORY OPTIMIZATION FOR A ROCKET WITH A
GENERALIZED THRUST CHARACTERISTIC
1. Introduction

The general theory of optimization of rocket trajectories has been developed in some detail for a rocket having thrust characteristic

\[ f = \frac{cm}{m}, \quad (1.1) \]

where \( f \) is the acceleration of the rocket due to the motor thrust, \( c \) is the exhaust velocity (assumed constant), \( m \) is the mass of the rocket and \( m \) is the rate of expenditure of propellant defined by

\[ \dot{m} = -m. \quad (1.2) \]

The theory shows that three different types of arc may occur in an optimal transfer problem; maximum thrust (M.T.) arcs, intermediate thrust (I.T.) arcs and null thrust (N.T.) arcs. At a corner between arcs of different types, certain necessary conditions must hold but the determination of the positions of corners and the sequence of arc types is a matter of difficulty. An approach to this problem is developed later in this Part, where it is shown that if the thrust characteristic is taken to be

\[ f = kmv/m, \quad (1.3) \]

where \( k \) and \( v \) are constants and \( v \neq 1 \), then the optimal trajectory consists either of a single I.T. arc (if \( f \) is unbounded) and there are no corners, or it consists of N.T. and M.T. arcs and I.T. arcs are definitely excluded. For the problem of minimization of propellant expenditure, it is found
that the optimal trajectory consists of a single I.T. arc when \( v \) is in the range \( 0 < v < 1 \). The theory is applied to the case of the uniform gravitational field and approximate acceleration programmes are obtained for values of \( v \) slightly less than unity. An exact solution is also obtained for all values of \( v \) in the range \( 0 < v < 1 \) for the case of a sounding rocket programmed to obtain maximum height. A feature of the general analysis is that an inequality constraint is imposed on the acceleration, \( f \), rather than on the propellant expenditure rate \( m \). This simplifies the analysis at the cost of adopting a less realistic model.

In a uniform gravitational field, the time sequence of arc types for the special thrust characteristic (1.1) is already known and the methods of this Part merely confirm these results. For more general gravitational fields, however, where the order of the arcs is not known, the sequence of arcs and positions of corners for the standard thrust characteristic (1.1) may be indicated by letting \( v \to 1 \) in the general solution to the problem for the more general thrust characteristic (1.3).

2. General Theory

2.1. Fundamental Equations

Taking \( f \) as defined by equation (1.3) as our thrust characteristic and noting that \( f \) can always be written in the form (1.1), where \( c \) is the exhaust velocity, we see that the
exhaust velocity depends on the rate of expenditure of fuel according to the equation

\[ c = \text{km} \nu^{-1}. \]  

(2.1)

In particular, if \( \nu = 1 \) the exhaust velocity is constant, while if \( \nu = \frac{1}{2} \) we may write

\[ mc^2 = k^2, \]  

(2.2)

showing that in this case the power is constant. Optimization with respect to flight at constant power has been considered by Grodzovsky, Ivanov and Tokarev [1,2].

In the general case, the equations of motion may be written

\[ \dot{\ell}_1 = f \varepsilon_1 + g_1, \]  

(2.3)

\[ \dot{\xi}_1 = v_1, \]  

(2.4)

together with equation (1.2). The constraints are represented by equation (1.3) and the equation

\[ \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 - 1 = 0. \]  

(2.5)

Here, \( v_1 \) are the velocity components of the rocket in an inertial frame, \( \varepsilon_1 \) are the direction cosines of the thrust, \( g_1 \) are the components of gravitational acceleration and \( \xi_1 \) are the position vector components of the rocket.

We shall suppose that \( f \) is bounded by the constraints

\[ 0 \leq f \leq \overline{f}. \]  

(2.6)
It is convenient to replace these inequalities by the equation

\[ f(\vec{r} - \vec{r}) - \alpha^2 = 0 \]  

(2.7)

Thus we have state variables \( v_1, x_1, \vec{M} \) and control variables \( \epsilon_1, m, \vec{F}, \alpha \). The constants \( k, v, \vec{F} \) are given and \( g_1 \) is, in general, a function of position and time, i.e.

\[ g_1 = g_1(x_j, t) \]  

(2.8)

### 2.2. Characteristic Equations

The Lagrange expression, as defined in [3], is

\[ F = - \lambda_1 (f \dot{\epsilon}_1 + \mu_1) - \lambda_{1+3} v_1 + \lambda_7 m + \mu_1 (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 - 1) + \]

\[ + \mu_2 (f - \frac{kmv}{M}) + \mu_3 (f (\vec{r} - \vec{r}) - \alpha) \]  

(2.9)

This leads to the Euler characteristic equations

\[ \dot{\lambda}_1 = - \lambda_{1+3} \]  

(2.10)

\[ \dot{\lambda}_{1+3} = - \lambda_1 \frac{\partial g_1}{\partial x_1} \]  

(2.11)

\[ \dot{\lambda}_7 = \mu_2 \frac{kmv}{M^2} \]  

(2.12)

\[ 0 = \theta f \lambda_1 + 2 \mu_1 \epsilon_1 \]  

(2.13)

\[ 0 = \lambda_7 - \mu_2 \frac{kmv}{M} \]  

(2.14)

\[ 0 = -\lambda_1 \epsilon_1 + \mu_2 + \mu_3 (\vec{F} - \vec{F}) \]  

(2.15)

\[ 0 = -2 \mu_3 \alpha \]  

(2.16)
It follows from equation (2.13) that the thrust must always be aligned with a vector having components \( \lambda_1 \) which, following [3], we shall refer to as the primer \( p \). This satisfies the equation

\[
\ddot{\lambda}_1 = \lambda_1 \frac{\partial g_4}{\partial x_1} \tag{2.17}
\]
as can be seen by eliminating \( \lambda_{1+3} \) between equations (2.10) and (2.11).

Equation (2.16) requires that \( \alpha \) or \( \mu_3 \) must vanish. If \( \alpha \) vanishes, the arc under consideration must be one of null thrust or maximum thrust. If \( \mu_3 \) vanishes, the arc may be one of intermediate thrust and equation (2.15) will define \( \mu_2 \).

Eliminating \( \mu_2 \) between (2.12) and (2.14) we have

\[
\dot{\lambda}_7 = \frac{m}{v_M} \lambda_7 \tag{2.18}
\]

But, from equation (1.2), this may be written

\[
\frac{\dot{\lambda}_7}{\lambda_7} = -\frac{\dot{\hat{m}}}{v_M} \tag{2.19}
\]

and, upon integration, this yields

\[
\lambda_7 = A e^{-(1/v)}, \tag{2.20}
\]

where \( A \) is a constant.

In the special case where the problem is to transfer the rocket between two terminals with minimum expenditure of propellant, we can choose \( J \) as the quantity to be minimized, where
\[ J = \frac{v}{1-v} M_1^{1-1/v} \]  

\[ (2.21) \]

and \( M = M_1 \) at the final time \( t = t_1 \). This is equivalent to minimizing \( M_1 \) for either \( v < 1 \) or \( v > 1 \). Using equation (1.46) in [3], we find that this leads to the end condition

\[ \lambda_\gamma = M_1^{1-1/v} \text{ at } t = t_1. \]  

\[ (2.22) \]

Comparing equations (2.20) and (2.22) we see that

\[ A = 1. \]  

\[ (2.23) \]

Also, since all \( \lambda_\gamma \) are continuous on the optimal trajectory by the corner condition, we see from equations (2.20) and (2.23) that the equation

\[ \lambda_\gamma = M_1^{1-1/v} \]  

\[ (2.24) \]

is true over the whole trajectory.

If \( v = 1 \), equation (2.21) cannot apply and we minimize instead

\[ J = \log \frac{M_0}{M_1}. \]  

\[ (2.25) \]

This also yields \( A = 1 \). Hence we can assume \( A = 1 \) under all circumstances.

2.3. The Weierstrass Condition

The Weierstrass necessary condition for the problem requires that
\[ \lambda^*_i \delta_i - \lambda m \geq \lambda^*_i \delta_i^* - \lambda m^* \]  

where the starred quantities are arbitrary values of the control variables satisfying the constraints (1.3) and (2.7), and the corresponding unstarred quantities are the optimum values.

But from equation (2.20)

\[ \lambda m = A(m^v/k)^1/v \]

which from equation (1.3) gives

\[ \lambda m = A(f/k)^1/v. \]  

We will examine particularly the case of minimization of propellant expenditure for which, by equation (2.23), we may take \( A = 1 \). Substituting from equation (2.27) into condition (2.26) we have

\[ \lambda^*_i \delta_i - k^{-(1/v)f^1/v} \geq \lambda^*_i \delta_i^* - k^{-(1/v)f^*(1/v)}. \]

Since the vectors \( \lambda^*_i \delta_i \) are aligned and \( \lambda^*_i \delta_i^* \) is maximum for variable \( \delta_i^* \) when the vectors \( \lambda^*_i \delta_i^* \) are aligned, we see that this implies

\[ pf - k^{-(1/v)f^1/v} \geq pf^* - k^{-(1/v)f^*(1/v)}, \]

where \( p \) is the magnitude of the primer vector \( \lambda^*_i \).

We now define a function

\[ T(f) = pf - (f/k)^1/v. \]  

This gives
\[ T'(f) = p - \frac{1}{kv} (f/k)^{(1-v)/v}, \quad (2.30) \]
\[ T''(f) = \frac{v-1}{k^2 v^2} (f/k)^{(1-2v)/v}. \quad (2.31) \]

The behaviour of the function \( T(f) \) for different ranges of \( v \) is easily deduced from these equations.

For \( f = 0 \),
\[ T(0) = 0 \]
\[ T'(0) = -\infty, \quad v > 1, \]
\[ = p - 1/k, \quad v = 1, \quad (2.33) \]
\[ = p, \quad 0 < v < 1. \]

By considering the behaviour of \( T''(f) \), we can now sketch the graph of the function \( T(f) \). Figure 1 shows the nature of \( T(f) \) for different ranges of \( v \).

![Graphs of T(f)](image-url)
Now condition (2.28) may be written

\[ T(f) > T(f^*) \]  

(2.34)

so that the optimal value of \( f \) is that which yields a maximum of \( T(f^*) \). We consider separately different ranges of \( \nu \).

a) \( \nu > 1 \)

From Figure 1 we see that

\[ f_x > \overline{f} \Rightarrow \text{N.T. arc} \]

\[ f_x < \overline{f} \Rightarrow \text{M.T. arc} \]

\[ f_x = \overline{f} \Rightarrow \text{either N.T. or M.T. arc} \]

In terms of the primer magnitude \( p \), these results may be written

\[ p > \frac{1}{k} \left( \overline{f}/k \right)^{1-\nu}/\nu \Rightarrow \text{M.T. arc.} \]

\[ p < \frac{1}{k} \left( \overline{f}/k \right)^{1-\nu}/\nu \Rightarrow \text{N.T. arc.} \]

\[ p = \frac{1}{k} \left( \overline{f}/k \right)^{1-\nu}/\nu \Rightarrow \text{M.T. or N.T. arc.} \]

Thus in no case can the acceleration due to the thrust assume any value intermediate between \( f = 0 \) and \( f = \overline{f} \). There are no I.T. arcs.

b) \( \nu = 1 \)

The gradient of the line in the second diagram of Figure 1 is \( p - 1/k \) by equation (2.30). Thus
\[ p > 1/k \Rightarrow M.T. \text{ arc.} \]
\[ p < 1/k \Rightarrow N.T. \text{ arc.} \]
\[ p = 1/k \text{ places no restriction on } f. \]

I.T. arcs are accordingly not excluded in this case.

c) \( 0 < v < 1 \)

From the third diagram in Figure 1 we see that for optimal conditions

\[ f = \min \{ \overline{f}, f_y \}. \tag{2.35} \]

But \( f_y \) is determined by equating \( T'(f) \) to zero. Thus from equation (2.30) we have

\[ f_y = \left( pk^1/v \right)^{1/(1-v)} \tag{2.36} \]

and the optimal acceleration is determined by

\[ f = \min \{ \overline{f}, (pvk^1/v)^{1/(1-v)} \}, \tag{2.37} \]

which yields a unique acceleration programme in terms of \( p \).

If \( \overline{f} = \infty \), then all arcs are I.T. arcs. Clearly, in general, null-thrust arcs are not present in an optimal trajectory (except in the extreme case where \( p \) is identically zero over a finite period of time).

The acceleration given by equation (2.36) may be put in a more convenient form by introducing the variable

\[ s = \frac{v}{1-v}, \tag{2.38} \]

so that
\[ \nu = \frac{s}{1+s}, \quad (2.39) \]

Equation (2.36) may now be written

\[ f = k\eta(kp)^s, \quad (2.40) \]

where

\[ \eta = \left(1 + \frac{1}{s}\right)^{-s}, \quad (2.41) \]

and we have written \( f \) for \( f_Y \). In particular, we note that as \( s \to +\infty, \nu \to 1 - 0 \) and \( \eta \to 1/e \). We can therefore examine the behaviour in the normal case, \( \nu = 1 \), by examining the limiting behaviour of \( I, T \) arcs in the case \( 0 < \nu < 1 \).

Finally in this section, we examine the corner condition and a first integral appropriate to this problem. It is shown in [3] that at a corner connecting arcs of two different types, the quantities \( \lambda_1, \lambda_{1+3}, \) and \( \lambda_7 \) are continuous. From equation (2.10), we see that this implies that \( \mathbb{P}, \mathbb{Q} \) and \( \lambda_7 \) are continuous so that the primer vector is a continuously varying quantity.

The remaining Weierstrass-Erdmann corner condition states that the quantity

\[ F = \lambda_1 \dot{y}_1 + \lambda_{1+3} \dot{x}_1 + \lambda_7 \dot{m} \quad (2.42) \]

is continuous at a corner. Substituting for the derivatives from equations (1.2), (2.3), (2.4) and remembering that \( \lambda_1, \lambda_{1+3} \) are continuous at a corner, we see that

\[ f\lambda_1 \ell_1 - \lambda_7 m = T(f) \quad (2.43) \]

must be continuous at a corner. Thus \( T(f) \) is continuous.
everywhere. In particular, when \( f \) is defined by equation (2.40), we see that \( f \) must be continuous everywhere so that there are no corners on an optimal trajectory. The continuity of \( T(f) \) accordingly yields nothing new.

It remains to consider the form taken by the first integral

\[
F = \text{constant}
\]  

(2.44)

which in [3] is shown to exist when the gravitational field is time invariant. Substituting for the derivatives from equations (1.2), (2.3), (2.4) and using equations (2.10), (2.27) and (2.29), we find that this yields

\[
p \cdot g - \dot{p} \cdot y + T(f) = \text{constant}.
\]  

(2.45)

On an intermediate thrust arc for \( 0 < v < 1 \), the acceleration is given by equation (2.40). Substituting this value of \( f \) into equation (2.45) and using equation (2.29), the first integral becomes

\[
p \cdot g - \dot{p} \cdot y + \frac{n}{s+1} (k')^{s+1} = \text{constant},
\]  

(2.46)

which is the first integral expressed entirely in terms of the magnitude of the primer vector \( p \).

3. The Constant Field

We will now restrict ourselves to the case

\[
0 < v < 1,
\]  

(3.1)
corresponding to values of $s$ in the range

$$0 < s < \infty.$$  \hspace{1cm} (3.2)

For a constant gravitational field, all the field components $g_i$ are constant and equation (2.17) possesses the general solution

$$\lambda_1 = a_1 t + b_1,$$  \hspace{1cm} (3.3)

where $a_1, b_1$ are constants. The primer locus is therefore a straight line and we can choose axes such that the primer vector always lies in a coordinate plane. In general, the magnitude of the primer vector will first decrease and then increase. It can attain, at most, one minimum.

We are interested in the case where the acceleration is given by the equation

$$f = k\eta(kp)^s,$$  \hspace{1cm} (3.4)

assuming that $f$ is not bounded above.

The equations of motion for the general $s$-case are too complex to solve explicitly; however the case $s = 1$ does admit an explicit solution. This corresponds to flight at constant power.

It is of interest to examine the behaviour of $f$ as $s \to \infty$, since this approaches the normal case $v = 1$.

Now $p$ is a function of both $s$ and $t$. In fact
\[ p^2 = \sum_{i=1}^{3} (a_i t + b_i)^2, \quad (3.5) \]

where the dependence of \(a_i, b_i\) upon \(s\) will be decided by the boundary conditions. Suppose that \(a_i \to A_i, b_i \to B_i\) as \(s \to \infty\). Then

\[ p^2 \to P^2 = \sum_{i=1}^{3} (A_i t + B_i)^2 \quad (3.6) \]

as \(s \to \infty\), and, for certain values of \(P\), we can deduce the limiting behaviour of \(f(t)\) from equation (3.4),

(i) where \(P < 1/k\), \(f(t) \to 0\) as \(s \to \infty\), \( (3.7) \)

and

(ii) where \(P > 1/k\), \(f(t) \to \infty\) as \(s \to \infty\).

Clearly, we cannot have an infinite acceleration for a finite time; thus we will assume that for all \(t\)

\[ P \leq 1/k. \quad (3.8) \]

Suppose that the primer vector tends to the limit

\[ P = \Delta t + \vec{B}, \quad (3.9) \]

where \(\Delta \neq 0\). Then the primer locus is a segment of a straight line lying inside a circle whose centre is at the origin and whose radius is \(1/k\). Only if the segment has an end point lying on the circle can a thrust be applied and then only for an infinitesimal time. Thus the only possible limiting behaviour is for the programme to tend to a double impulse, one at each terminal, connected by a null thrust arc. For certain
special boundary conditions, we may have only one impulse or
even no impulses at all.

We must now examine the case where

\[ A = 0. \] (3.10)

We see that

\[ p \to B \] (3.11)

as \( s \to \infty \) and hence \( p \to B \), a constant. If \( B > 1/k \), this would
imply infinite acceleration everywhere. If \( B < 1/k \), the whole
arc would be one of null thrust. Hence, in general

\[ B = 1/k. \] (3.12)

By equation (3.4) the limiting acceleration programme takes the
form

\[ F = ke^{-1}(1)^\infty, \] (3.13)

where \( t \to F \) as \( s \to \infty \).

Now \((1)^\infty\) is an indeterminate form. Its value will depend
upon the manner in which \( p \to 1/k \). Suppose that, in fact, equation
(3.13) yields a definite limit programme \( F(t) \). Then, from
(3.11), we see that the thrust is constant in direction through-
out the manoeuvre. Because of this, there will be an infinite
number of other programmes which will lead to the same values
of \( x_i, v_i \) and the payoff function \( J \). Thus, even if the limit
programme \( F(t) \) does exist, it will not provide a unique optimal
programme for the normal case \( v = 1 \). It will serve, however,
to provide an approximation to the unique optimal programme for
values of \( \nu \) close to 1.

The results obtained above correspond with the known results for the case \( \nu = 1 \) as obtained in [3].

4. The One-Dimensional Case

For the case of rectilinear motion along an \( x \)-axis, the primer possesses but one component given by

\[ \lambda = a't + b'. \quad (4.1) \]

The sign of \( \lambda \) determines the sense of the thrust along \( Ox \). The primer magnitude is now simply

\[ p = |\lambda|. \quad (4.2) \]

Using equation (2.40) the equation of motion (2.3) becomes

\[ \ddot{v} = k^2 \eta (kp)^{s-1} \lambda + g. \quad (4.3) \]

We will consider the sequence of cases where \( s \) is of the form

\[ s = 2n + 1, \quad n = 0, 1, 2, \ldots \quad (4.4) \]

Equation (4.3) becomes

\[ \ddot{v} = k \eta (ka't + kb')^{2n+1} + g. \quad (4.5) \]

Write

\[ a = ka', \quad b = kb', \quad (4.6) \]

so that equation (4.5) becomes
\[ \dot{v} = k\eta (at+b)^{2n+1} + g. \quad (4.7) \]

We will study the optimization problem for a rocket whose motor characteristic is approximately normal, i.e., we shall assume \( n \) to be large.

We take as our general boundary conditions

\[ x = x_0, \quad v = v_0, \quad \text{at} \quad t = t_0, \quad (4.8) \]
\[ x = x_1, \quad v = v_1, \quad \text{at} \quad t = t_1. \]

It will be convenient also to introduce the notation

\[ \bar{x} = x_1 - x_0, \quad \bar{v} = v_1 - v_0, \quad (4.9) \]
\[ \bar{t} = t_1 - t_0. \]

We find, on integrating equation (4.7) twice and inserting the given boundary conditions, that

\[ \bar{v} - \frac{k\eta}{a(2n+2)} \left\{ (at_1+b)^{2n+2} - (at_0+b)^{2n+2} \right\} \quad (4.10) \]

and

\[ \bar{x} = \frac{v_0 \bar{t} - \frac{1}{2}g\bar{t}^2}{a(2n+2)} = \frac{k\eta}{a(2n+2)} x \]
\[ \times \left\{ \frac{(at_1+b)^{2n+3} - (at_0+b)^{2n+3}}{a(2n+3)} - (at_0+b)^{2n+2} \right\} \quad (4.11) \]

These are the equations determining the constants \( a \) and \( b \).

We define the parameter \( \beta \) by the equation

\[ at_1 + b = \beta (at_0 + b), \quad (4.12) \]
from which we will require the results

\[
\frac{b}{a} = \frac{t_1 - \beta t_0}{\beta - 1}, \quad (4.13)
\]

\[
\frac{b}{at_0 + b} = \frac{t_1 - \beta t_0}{t}, \quad (4.14)
\]

\[
\frac{a}{at_0 + b} = \frac{\beta - 1}{t}. \quad (4.15)
\]

Substituting for \(at_1 + b\) from equation (4.12) into equation (4.10), we obtain

\[
\bar{v} - g \bar{t} = \frac{k\eta(\beta^{2n+2}-1)\bar{t}}{(2n+2)(\beta-1)} (at_0 + b)^{2n+1} \quad (4.16)
\]

where we have used equation (4.15).

Similarly, substituting from equation (4.12) into equation (4.11), we have

\[
\bar{x} - \bar{v}_0 \bar{t} - \frac{1}{2}g \bar{t}^2 = \frac{k\eta^2}{(2n+2)(2n+3)} \frac{\beta^{2n+3} - 1 - (2n+3)(\beta-1)}{(\beta-1)^2} (at_0 + b)^{2n+1}. \quad (4.17)
\]

We may now eliminate \((at_0 + b)^{2n+1}\) between equations (4.16), (4.17) to obtain an equation for \(\beta\), namely

\[
\bar{x} - \bar{v}_0 \bar{t} - \frac{1}{2}g \bar{t}^2 = \frac{\bar{v}^2 - g\bar{t}^2 \beta^{2n+3} - (2n+3)\beta + 2n + 2}{2n+3} \frac{\beta^{2n+3} - 1}{(\beta^{2n+3} - 1)(\beta - 1)}. \quad (4.18)
\]

Excluding, for the moment, the cases \(\beta = \pm 1\), we may write equation (4.18) in the form

\[
\beta^{2n+2} [A(\beta - 1) - 1] = B(\beta - 1) - 1, \quad (4.19)
\]

where
\[ A = \left( n + \frac{3}{2} \right)(\gamma + 1) - 1 \]  
(4.20)

\[ B = \left( n + \frac{3}{2} \right)(\gamma - 1) \]  
(4.21)

and

\[ \gamma = \frac{2\bar{\gamma} - (v_0 + v_1)\bar{t}}{(v - g\bar{t})\bar{t}} . \]  
(4.22)

The functions

\[ L(\beta) = \beta^{2n+2}\{A(\beta-1) - 1\} \]  
(4.23)

\[ R(\beta) = B(\beta-1) - 1 \]  
(4.24)

which are the left and right hand members respectively of equation (4.19) may now be sketched in order to locate the roots of equation (4.19). The nature of the roots varies according to the value of the function \( \gamma \) of the boundary conditions. In all cases, however, it is easily shown that \( L(\beta) \) and \( R(\beta) \) are tangential at \( \beta = 1 \).

We will treat two distinct cases to show the two types of limiting behaviour which can occur as \( n \to \infty \). We will then summarize the behaviour for all cases. We first note, however, that \( \gamma \) can be expressed in terms of two functions of the boundary condition which have a clear physical interpretation and which determine in a simple manner the types of roots of equation (4.19) which are encountered. We define two parameters

\[ \Delta_0 = \bar{\gamma} - (v_0 \bar{t} + \frac{1}{2}g\bar{t}^2) \]  
(4.25)

\[ \Delta_1 = \bar{\gamma} - (v_1 \bar{t} - \frac{1}{2}g\bar{t}^2) . \]  
(4.26)
$\Delta_0$ is the difference between the distance which is required to be covered in the set time and the distance which the rocket would travel without motor thrust in the set time starting with the given initial velocity. $\Delta_1$ has a similar interpretation.

We note that

$$\Delta_0 + \Delta_1 = 2x - (v_0 + v_1)t,$$  \hfill (4.27)

$$\Delta_0 - \Delta_1 = (v - g \overline{t})t.$$  \hfill (4.28)

Thus

$$\gamma = \frac{\Delta_0 + \Delta_1}{\Delta_0 - \Delta_1}.$$  \hfill (4.29)

Case I: $\gamma > 1$

This corresponds to two separate classes of boundary conditions, viz.

(i) $\Delta_0 > \Delta_1 > 0$,

(ii) $\Delta_0 < \Delta_1 < 0$.

The graphs of $L(\beta)$, $R(\beta)$ now take the forms sketched in Figure 2.

![Figure 2. $\gamma > 1$ for large $n$]
We see from Figure 2 that there is only one real root for \( \beta \), other than the double root at \( Q(1,-1) \) corresponding to the value \( \beta = 1 \) which we have excluded in deriving equation (3.19). (This case will be considered later; it can only occur under special boundary conditions.) The unique valid root has abscissa \(-1 + \epsilon\), where \( \epsilon \) is small and positive when \( n \) is large.

Assuming \( \beta \) can be expanded thus,

\[
\beta = -1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots ,
\]

substituting into equation (4.19) and equating terms of \( O(n) \) from both members, we obtain

\[
\beta^{2n+2} = e^{-2\alpha_1} = \frac{\Delta_1}{\Delta_0} ,
\]

whence

\[
\alpha_1 = \frac{1}{2} \log \frac{\Delta_1}{\Delta_0} ,
\]

which gives an estimate of \( \epsilon \) to \( O(1/n) \).

Now, from equations (4.31), (4.30), (4.28) and (4.16), we get to \( O(n) \),

\[
k\eta(at_0 + b)^{2n+1} = \frac{\ln \Delta_0}{t^2} .
\]

Thus, from equation (4.7) we have

\[
\dot{v} = k\eta(at_0 + b)^{2n+1} \left( \frac{b}{at_0 + b} + \frac{at}{at_0 + b} \right)^{2n+1} + g ,
\]

which, from equations (4.33), (4.30), (4.14), (4.15), gives

\[
\dot{v} = \frac{\ln \Delta_0}{t^2} \left( 1 \left( 2 - \frac{\alpha_1}{n} \right)^{2n+1} + g ,
\]

(4.34)
where

\[ T = \frac{t - t_0}{t_1 - t_0} \quad (4.35) \]

is a dimensionless time term which varies from \( T = 0 \) at \( t = t_0 \) to \( T = 1 \) at \( t = t_1 \).

Certain properties of the acceleration programme for large \( n \) can be deduced from equation (4.34). Firstly,

\[ \dot{\mathbf{V}} = l_n \Delta_0 / t^2 + g, \quad \text{at} \quad t = t_0, \quad (4.36) \]

and

\[ \dot{\mathbf{V}} = -l_n \Delta_1 / t^2 + g, \quad \text{at} \quad t = t_1. \]

The last result follows from equation (4.31). Also, for \( 0 < T < 1 \), i.e. intermediate values of \( t \), \( \dot{\mathbf{V}} \to g \) as \( n \to \infty \).

Thus, for large \( n \), we tend to a situation where we have impulses at each boundary and a null thrust arc in between. The impulses at \( t = t_0 \) and \( t = t_1 \) are proportional to the quantities \( \Delta_0 \) and \( \Delta_1 \) respectively.

The velocity, obtained by integrating equation (4.34), may be written

\[ \mathbf{v} = v_0 + g(t-t_0) + \frac{\Delta_0}{t} \left\{ 1 - \left( 1 - \left( 2 - \frac{\alpha_1}{n} \right) t \right)^{2n} \right\} \quad (4.37) \]

giving, to zero order,

\[ \mathbf{v} = v_0, \quad \text{at} \quad t = t_0, \]

\[ \mathbf{v} = v_1, \quad \text{at} \quad t = t_1, \quad (4.38) \]

and
\[ v = v_0 + g(t-t_0) + \Delta_0/\tau, \quad t_0 < t < t_1. \quad (4.39) \]

Integrating equation (4.39) with respect to \( t \) from \( t_0 \) to \( t_1 \) and putting \( x = x_0 \) at \( t = t_0 \) we can verify that, with this velocity programme, we attain \( x_1 \) at \( t = t_0 \). It is clear, therefore, that the initial impulse adjusts the velocity such that the rocket subsequently coasts to \( x = x_1 \) in the preassigned time interval \( \tau \), where it receives a second impulse sufficient to give it the preassigned boundary velocity \( v_1 \).

The programme clearly tends to the well-known double impulse programme for the case \( v = 1 \).

Case II: \( 1 > \gamma > 0 \)

This again corresponds to two separate classes of boundary conditions

(i) \( \Delta_0 > \Delta_1, \quad \Delta_1 < 0 < \Delta_1 + \Delta_0, \)

(ii) \( \Delta_0 < \Delta_1, \quad \Delta_1 > 0 > \Delta_1 + \Delta_0, \)

and the graphs of \( L(\beta), R(\beta) \) are as sketched in Figure 3.

![Figure 3.](image)

**Figure 3.** \( 1 > \gamma > 0 \) for large \( n \)
From Figure 3, we see that the only root, other than the excluded double root at $\beta = 1$, may be written

$$\beta = 1 - \epsilon,$$  \hspace{1cm} (4.20)

where $\epsilon$ is small and positive.

Assuming

$$\beta = 1 - \frac{\alpha_1}{n} - \frac{\alpha_2}{n^2} - \cdots$$  \hspace{1cm} (4.21)

and substituting in equation (4.19), we find, working to zero order in $n$, that

$$\left(1 - \frac{\alpha_1}{n}\right)^{2n} = \frac{1 + \alpha_1 (\gamma - 1)}{1 + \alpha_1 (\gamma + 1)}.$$  \hspace{1cm} (4.22)

Now,

$$\left(1 - \frac{\alpha_1}{n}\right)^{2n} \to \exp(-2\alpha_1), \text{ as } n \to \infty.$$  \hspace{1cm} (4.23)

Thus, to zero order, equation (4.22) becomes

$$e^{-2\alpha_1} = \frac{1 + \alpha_1 (\gamma - 1)}{1 + \alpha_1 (\gamma + 1)}.$$  \hspace{1cm} (4.24)

By sketching the graphs of the two members of equation (4.24), we find that there is a zero root and a positive root for $\alpha_1$. Now the value of the root of equation (4.19) lies between two other values of $\beta$ which are easily estimated, namely, the value for which $R(\beta)$ vanishes and the value for which $L(\beta)$ has a point of inflection. From this observation, and working to 0(1/n), we can arrive at the inequalities

$$\frac{1}{1 - \gamma} > \alpha_1 > \frac{\gamma}{1 + \gamma},$$  \hspace{1cm} (4.25)

which show that $\alpha_1$ is bounded below by a number greater than
zero in the range of $\gamma$ considered. We therefore discard the zero root of equation (3.44) and take the positive root.

Using equations (4.41), (4.42), we find that working to zero order in $n$, equation (4.16) gives

$$k\eta = \frac{(\sqrt{-e^T})(1+\alpha_1(1+\gamma))}{\frac{1}{\varepsilon}} \frac{1}{(at_0+b)^{2n+1}}. \quad (4.46)$$

Now, using equations (4.46), (4.44), (4.14), (4.15), we find the acceleration, as given by equation (4.7), may be written

$$\dot{v} = \frac{\Delta_0 - \Delta_1}{\varepsilon^2} \frac{2\alpha_1}{1-e^{-2\alpha_1}} e^{-2\alpha_1 T} + g. \quad (4.47)$$

Thus, for large $n$, the optimal acceleration programme is one of exponential decay, where $\alpha_1$ is given by equation (4.44). As $n \to \infty$, the optimal programme will tend exactly to equation (4.47). However, in the limit, when $v = 1$, the programme (4.47) will only be one of an infinite number of equally optimal programmes satisfying the boundary conditions. As a unique optimal programme for the case $v = 1$, the programme (4.47) is therefore spurious.

In equations (4.5)-(4.7) of Ref. [3] we have conditions for the existence of an I.T. arc for the case $v = 1$ in a uniform field. Using the notation established in this section, these conditions may be written

$$\Delta_0 - \Delta_1 = k\varepsilon T \log \frac{M_0}{M_1}, \quad (4.48)$$
\[ \Delta_0 = k \varepsilon \int_{t_0}^{t_1} \log \frac{M}{M_0} \, dt, \quad (4.49) \]

where \( \varepsilon = \pm 1 \) along the entire trajectory. Equation (4.48) may be written

\[ \Delta_0 - \Delta_1 = k \varepsilon \int_{t_0}^{t_1} \log \frac{M}{M_0} \, dt \quad (4.50) \]

and now, from equations (4.49), (4.50), we have

\[ \Delta_1 = k \varepsilon \int_{t_0}^{t_1} \log \frac{M}{M_0} \, dt. \quad (4.51) \]

The integrands in equations (4.49), (4.51) are always positive and negative respectively. Hence, for an I.T. arc to be possible, the boundary conditions must satisfy the following inequalities:

\begin{align*}
(1) & \text{ if } \varepsilon = +1, \; \Delta_0 > 0, \; \Delta_1 < 0, \\
(ii) & \text{ if } \varepsilon = -1, \; \Delta_0 < 0, \; \Delta_1 > 0. \; (4.52)
\end{align*}

These are precisely the conditions obtained in this section for the existence, for large \( n \), of an I.T. arc of exponential variation. As shown in Figure 4, these conditions correspond to \(|\gamma| < 1\).

Case III: Other values of \( \gamma \)

Cases I and II described above exhibit the two main behavioural types encountered. The excluded cases, \( \beta = 1, \; \beta = -1 \) present no difficulty and may be dealt with by going back to the original acceleration equation, equation (4.7), and integrating.
The results obtained for all the cases have been summarized below. The dependence of $\gamma$ on the boundary conditions is shown in Figure 4.

Figure 4. Dependence of $\gamma$ on boundary conditions

(i) $\gamma = 0$

This corresponds to the case $\beta = 1$ and the boundary conditions $\Delta_0 + \Delta_1 = 0; \Delta_0 \neq \Delta_1$. The acceleration due to the motor thrust is given by

$$f = \frac{\Delta_0 - \Delta_1}{t^2},$$

(i.e. a constant acceleration programme.

(ii) $1 > |\gamma| > 0$

In this case

$$f = \frac{\Delta_0 - \Delta_1}{t^2} \frac{2\alpha_1}{1 - e^{-2\alpha_1}} \ e^{-2\alpha_1 T},$$

where $\alpha_1$ is the non zero root of the equation
\[ e^{-\alpha_i} = \frac{1 + \alpha_i (\gamma - 1)}{1 + \alpha_i (\gamma + 1)}. \quad (4.55) \]

\( \alpha_i \) is positive or negative according as \( \gamma \) is positive or negative. Thus for \( \gamma \) positive, the programme is one of exponential decay, while for \( \gamma \) negative, the programme is one of exponential growth. Moreover, as \( |\gamma| \to 1, |\alpha_i| \to \infty \) so that the programme approaches an impulse at one end followed (or preceded) by a null thrust arc.

(iii) \( |\gamma| = 1 \)

a) \( \gamma = 1, (\Delta_1 = 0, \Delta_0 \neq 0) \),

\[ f = \frac{\ln \Delta_0}{t^2} (1 - (2 - \epsilon)T)^{2n+1}, \quad (4.56) \]

where \( \epsilon \) is given by

\[ (1 - \epsilon)^{2n} = 1/\ln. \quad (4.57) \]

As \( n \to \infty \), the programme tends to an impulse at \( T = 0 \), followed by a null thrust arc and a zero impulse at \( T = 1 \).

b) \( \gamma = -1, (\Delta_0 = 0, \Delta \neq 0) \),

\[ f = \frac{\Delta_1}{t^2} (1 - (2 + \epsilon)T)^{2n+1}, \quad (4.58) \]

where \( \epsilon \) is given by

\[ (1 + \epsilon)^{2n} = \ln. \quad (4.59) \]

As \( n \to \infty \), the programme tends to a zero impulse at \( T = 0 \) followed by a null thrust arc and a non-zero impulse at \( T = 1 \).
(iv) $|\gamma| > 1$

$f$ is given by equation (3.51), where $\varepsilon$ is given by

$$(1 - \varepsilon)^{a_n} = \Delta_1 / \Delta_0.$$  \hspace{1cm} (4.60)

As $n \rightarrow \infty$ the programme tends to a non-zero impulse at $T = 0$ and $T = 1$, connected by a null thrust arc.

(v) $|\gamma| = \infty$

This corresponds to the case $\beta = -1$. $f$ is given by equation (3.51), where $\varepsilon$ is put equal to zero. As $n \rightarrow \infty$ the programme tends to two equal non-zero impulses at $T = 0$ and $T = 1$, connected by a null thrust arc.

(vi) $\gamma = 0/0$

This is the indeterminate case represented by the boundary conditions

$$\Delta_1 = \Delta_0 = 0.$$  \hspace{1cm} (4.61)

In this case conditions are such that the rocket may coast between the boundaries and satisfy both sets of conditions. No expenditure of fuel is necessary.

5. Sounding Rocket

We may apply the previous theory to investigate the problem of programming the thrust of a sounding rocket whose thrust characteristic is given by equation (1.3). We wish to
launch the rocket from rest at the surface of the earth and
programme the thrust optimally with respect to fuel expendi-
ture so that it reaches a given height with zero velocity.
The curvature and rotation of the earth and the variation of
the gravitational field will be neglected. The problem is now
equivalent to the one dimensional case treated in the previous
section except that the final time, $t_f$, is not specified.

We take a vertical $x$-axis and impose a constant gravi-
tational field of magnitude $g$ acting downwards. The boundary
conditions are now, in the notation of the previous section

$$\begin{align*}
x_0 &= 0, \quad v_0 = 0, \quad t_0 = 0, \\
x_1 &= h, \quad v_1 = 0, \quad t_1 = t_f.
\end{align*} \quad (5.1)$$

General theory yields a condition for optimization with
respect to the final time and this is expressed by equation
(1.47) in [3]. This implies that the constant in the first
integral, equation (2.46), is zero so that from conditions (5.1)
we may write

$$\lambda g - \frac{n}{s+1} (k|\lambda|)^{s+1} = 0 \quad \text{at} \quad t = 0 \quad \text{and} \quad t_f, \quad (5.2)$$

where $g$ is positive. Thus, at the boundaries

$$\lambda = 0,$$

or

$$g - \frac{k\lambda}{s+1} (k\lambda)^{s} = 0, \quad \text{if} \quad \lambda \text{ is positive}, \quad (5.3)$$

or
\[ g + \frac{k\eta}{s+1} (-k\lambda)^s = 0, \] if \( \lambda \) is negative.

It is found that the only choice of values which can satisfy the boundary conditions are

\[ g - \frac{k\eta}{s+1} (k\lambda)^s = 0 \quad \text{at} \quad t = 0. \] (5.4)

and

\[ \lambda = 0 \quad \text{at} \quad t = t_1. \] (5.5)

But from equation (4.3) taking \( g \) negative and \( \lambda \) positive, we have

\[ \dot{v} = k\eta (k\lambda)^s - g \] (5.6)

and from equations (4.1), (4.6)

\[ k\lambda = at + b. \] (5.7)

Equations (5.4), (5.5), (5.7) now yield

\[ b^s = \frac{(s+1)g}{k\eta}, \] (5.8)

\[ at_1 + b = 0. \] (5.9)

The acceleration function for the rocket now follows from equations (5.6)-(5.9). We obtain

\[ \dot{v} = (1+s)g(1-T)^s - g, \] (5.10)

where

\[ T = t/t_1. \] (5.11)

The acceleration due to rocket thrust is therefore
\[ f = (1+s)(1-T)^s. \]  \hspace{1cm} (5.12)

This is a monotonically decreasing function of time which vanishes at \( t = t_1 \). The value of \( t_1 \) is obtained by substituting known values in equation (4.11). Working generally in terms of \( s \) rather than the integer variable \( n \) we have, using equations (5.8), (5.9),

\[ t_1 = \sqrt{2\left(1 + \frac{2}{s}\right)h/g}, \]  \hspace{1cm} (5.13)

only the positive root being admissible. The equations (5.12), (5.13) together comprise an exact solution to the problem for general \( s \). Moreover, in the limit, as \( s \to \infty \), the behaviour approaches that already known for the case \( v = 1 \), namely an initial impulse followed by a null thrust arc and a time of arrival given by

\[ t = \sqrt{2h/g}. \]  \hspace{1cm} (5.14)

We can also compute the value of the fuel expenditure in the optimal case. The acceleration due to the rocket and the rate of expenditure of fuel are connected by equation (1.3) which may be written

\[ (f/k)^1/v = m/M^{1/v}. \]  \hspace{1cm} (5.15)

Using equation (1.2) we obtain from equation (5.15)

\[ \int_0^{t_1} (f/k)^1/v \, dt = \int_{M_0}^M \frac{dM}{M^{1/v}}. \]  \hspace{1cm} (5.16)

The right hand side is immediately integrable and substituting for \( f \) from equation (5.12) the left hand side may be integrated
to give
\[
\frac{1}{k} \sqrt{2(1 + \frac{2}{s})} s \left( \frac{g(1+s)}{k} \right)^{1/s} = \frac{s(s+2)}{s+4} \left( M_1^{-1/s} - M_0^{-1/s} \right),
\]
where we have used equation (2.39) to express \( v \) in terms of \( s \).

As \( s \to \infty \), \( (M_1^{-1/s} - M_0^{-1/s}) \to 0 \) and it may be verified that
\[
s(M_1^{-1/s} - M_0^{-1/s}) \to \log \frac{M_0}{M_1} \text{ as } s \to \infty.
\]

From conditions (5.17), (5.18) the fuel expenditure as \( s \to \infty \) is given by
\[
M_1 = M_0 \exp(-\sqrt{2\text{h}g/k}).
\]

6. The Constant Power Case

It was pointed out at the beginning of section 2 that if the power index \( v \) in equation (1.3) is given by the equation
\[
v = \frac{1}{2},
\]
then flight is carried out at constant power. From equations (2.38), (6.1) we find that the value of the parameter \( s \) corresponding to constant power flight is given by
\[
s = 1,
\]
and hence, from equation (2.41), \( \eta \) is given in this case by
\[
\eta = 1/2.
\]
Thus, for constant power flight, equation (4.3), the equation
of motion for the one dimensional problem, takes the particularly simple form

\[ \dot{v} = \frac{1}{2} \xi (at + b) + g, \]  

(6.4)

where we have used equations (4.1), (4.6), (6.2), (6.3).

We will assume, as before, that the boundary conditions are given by equations (4.8), and we will define new variables \( \tau, \xi, \zeta \) by the equations

\[ \tau = \frac{2t - (t_0 + t_1)}{2t_1 - (t_0 + t_1)}, \]  

(6.5)

\[ \xi = \frac{x - x_0}{x_1 - x_0}, \]  

(6.6)

\[ \zeta = \frac{d\xi}{d\tau}. \]  

(6.7)

Equations (6.5)-(6.7), in effect, perform scale changes on the variables \( t, x \) and \( v \) respectively, such that the boundary conditions (4.8) become

\[ \begin{align*} 
\tau_0 &= -1, \quad \tau_1 = +1, \\
\xi_0 &= 0, \quad \xi_1 = +1,
\end{align*} \]  

(6.8)

where the zero subscript denotes initial values and the unit subscript denotes final values. All essentially different boundary conditions may now be specified by the two parameters \( \zeta_0 \) and \( \zeta_1 \), the scaled boundary velocities. The appropriate scaling factor is given by the equation

\[ \zeta = \frac{t}{2X} v, \]  

(6.9)

which is easily deduced from equations (6.5)-(6.7), and where we have used the notation established by equations (4.9).
In terms of the new variables, equation (6.4), the equation of motion, may be written

\[ \frac{d\zeta}{d\tau} = A\tau + B, \]  
\[ (6.10) \]

where

\[ A = \frac{k\ell t^3}{16x}, \]  
\[ (6.11) \]

and

\[ B = \frac{t^2}{16x} \left[k(a(t_0+t_1)+2b)+4g\right], \]  
\[ (6.12) \]

are constants to be determined from the boundary conditions.

Integrating equation (6.10) twice with respect to \( \tau \) and inserting the boundary conditions (6.8) we obtain

\[ \zeta_1 - \zeta_0 = 2B, \]  
\[ (6.13) \]

\[ 2\zeta_0 = 1 - 2B + \frac{3}{2}A, \]  
\[ (6.14) \]

from which we get

\[ A = \frac{3}{2}(\zeta_0 + \zeta_1 - 1), \]  
\[ (6.15) \]

\[ B = \frac{1}{2}(\zeta_1 - \zeta_0). \]  
\[ (6.16) \]

The equation of motion, equation (6.10), may thus be written finally

\[ \frac{d\zeta}{d\tau} = \frac{3}{2}(\zeta_0 + \zeta_1 - 1)\tau + \frac{1}{2}(\zeta_1 - \zeta_0). \]  
\[ (6.17) \]

The constant term \( \frac{1}{2}(\zeta_1 - \zeta_0) \) in the right hand member of equation (6.17), is clearly the 'average' acceleration in the \( \tau \) interval \((-1,1)\) and is directly proportional to the velocity increment demanded by the boundary conditions. There is also a term \( \frac{3}{2}(\zeta_0 + \zeta_1 - 1)\tau \) which has no effect on the velocity increment attained in the interval \((-1,1)\) but which programmes the acceleration in
such a way as to satisfy the boundary conditions placed on the variable $s$.

Equation (6.17) may be written

$$\frac{d\xi}{d\tau} = f + \gamma, \quad (6.18)$$

where $f$ and $\gamma$ are respectively the acceleration due to motor thrust and the acceleration due to gravity, both suitably scaled. Thus

$$\gamma = \frac{F_0}{L} g. \quad (6.19)$$

From equations (6.17), (6.18) we find that the acceleration due to motor thrust is given by

$$f = \frac{3}{2}(\xi_0 + \xi_1 - 1)\tau + \frac{1}{2}(\xi_1 - \xi_0) - \gamma. \quad (6.20)$$

The optimal acceleration programmes for the constant power case ($\nu = \frac{1}{2}$), exhibit features analogous to those observed in the acceleration programmes obtained in section 4 for large $n$, (i.e. $\nu$ slightly less than 1). Two typical programmes of acceleration due to motor thrust, obtained from equation (6.20) are shown in Figure 5.

**Figure 5. Acceleration due to Motor Thrust Constant Power Case**
In programme 1, \( f_0 \) and \( f_1 \) have the same sign and \( f \) increases monotonically with \( \tau \) so that \(|f|\) has a minimum at the first boundary and a maximum at the other. In programme 2, \( f_0 \) and \( f_1 \) have opposite signs and \(|f|\) has relative maxima at both boundaries. In this case the direction of thrust alters during the flight.

The functions of \( \zeta_0 \) and \( \zeta_1 \) occurring in the right hand member of equation (6.20) are easily related to the parameters \( \Delta_0, \Delta_1 \) defined by equations (4.27), (4.28). It is found, using equation (6.9), that

\[
\zeta_0 + \zeta_1 - 1 = - (\Delta_1 + \Delta_0) / 2x, \quad (6.21)
\]

\[
\frac{1}{2} (\zeta_1 - \zeta_0) - \gamma = - (\Delta_1 - \Delta_0) / 4x. \quad (6.22)
\]

If \( F \) is the unscaled acceleration due to motor thrust, then

\[
f = \frac{F}{4x}, \quad (6.23)
\]

and after substituting from equations (6.21)-(6.23) into equation (6.20) we obtain

\[
F = - \left[ 3(\Delta_1 + \Delta_0)\tau + (\Delta_1 - \Delta_0) \right] / t^2, \quad (6.24)
\]

so that the nature of the motor thrust acceleration program is, as before, determined by the position of the point \((\Delta_0, \Delta_1)\) (determined by the boundary conditions), in the \( \Delta_0, \Delta_1 \)-plane. If \( \gamma \) is the parameter defined by equation (4.29) we may easily classify the different programme types in terms of \( \gamma \).
(i) \( \gamma = 0 \).

This corresponds to a constant thrust programme.

(ii) \( 0 < |\gamma| < 1/3 \).

The thrust is always in the same direction and is positive or negative according as \( A_0 \) is greater than or less than \( A_1 \).

(iii) \( |\gamma| > 1/3 \).

The direction of thrust changes during the manoeuvre.

These results are very similar to those obtained at the end of section 4 for the case of large \( n \). However, the critical value of \( \gamma \) for the constant power case is one-third, whereas for the large \( n \) case the critical value of \( \gamma \) was unity.

Lastly, it may be observed that by substituting the value \( s = 1 \) (from equation (6.2)) into the appropriate equations in section 5, the optimal programming of the acceleration due to motor thrust for the Sounding Rocket problem proposed in that section may be deduced for the constant power case. Thus equation (5.12) becomes

\[ r = 2y(1-T), \quad (6.25) \]

and from equation (5.17), the fuel consumption is given by

\[ M_1^0 = M_0^1 = \frac{4}{3gV6/\rho k^2}. \quad (6.26) \]

7. Further Considerations

In section 2.3 we considered the Weierstrass condition for
values of $v$ in the range $v > 0$. For completeness we now consider the behaviour of the Weierstrass function for $v$ in the range $v < 0$.

Substituting $v = 0$ into equation (2.1) we see that

$$mc = k \quad (7.1)$$
a constant. Equation (7.1) in conjunction with equation (1.1) shows that

$$f = \frac{k}{M}. \quad (7.2)$$

Thus the acceleration cannot be varied by varying $c$ when $v = 0$. Clearly, the greatest economy of fuel is attained by maximizing $c$, the exhaust velocity.

If $v < 0$, then the function $T(f)$ has the form shown in Figure 6.

![Figure 6: Graph of $T(f)$ for $v < 0$](image)

$T(f)$ is a monotonically increasing function of $f$ so that to maximize $T(f)$ we choose

$$f = \bar{f} \quad (7.3)$$

where $\bar{f}$ is the upper bound of possible accelerations.
This result is obvious, also, from simpler considerations, for equation (1.3) may be written

\[ m = \left( \frac{k}{\alpha M} \right)^{-1/\nu}. \]

(7.4)

Where \( \nu \) is negative the exponent in the right hand member of equation (7.4) is positive so that as the acceleration \( f \) increases, the fuel expenditure \( m \) decreases. For a hypothetical rocket motor of the type considered, it will clearly be most efficient to run the rocket at maximum thrust when the motor is operative at all.

References


PART III

OPTIMAL TRAJECTORIES FOR A
SOLAR SAIL VEHICLE
1. Introduction

The possibility of using the force exerted by a stream of radiation on a reflecting surface as a means of propulsion within the solar system was investigated by Garwin [1] who concluded that 'solar sailing' offered a practical propulsion system. Tsu [2] considered the equations of motion of a solar sail and by considering cases where the magnitude of the radial acceleration could be neglected, obtained approximate solutions in the form of logarithmic spirals for the path of a sail vehicle, assuming that the angle of incidence of radiation on the sail was constant. He also computed the constant sail angle, as a function of sail strength, minimizing the time of transit between points on a logarithmic spiral trajectory distant \( r_0 \) and \( r \) respectively from the sun. London [3] showed that logarithmic spiral trajectories could, in fact, be obtained from the exact equations with constant sail setting. However, if one wishes to utilize the full potentiality of a solar sail vehicle to carry out some transfer manoeuvre in an optimal manner, one cannot assume that the sail angle will be constant during flight. The problem arises of finding the optimal way of programming the sail angle as a function of time. Kelley [4] used solar sailing as an example to illustrate the 'gradient' method of calculating optimal trajectories. In the present work, the optimal solar sailing problem is treated as a Mayer Problem.

In section 2, the question of the optimal form of the sail itself is considered. It is concluded that, for a given surface
area of sail, a planar sail delivers maximum thrust in any required (possible) direction. Flight in a uniform gravitational and radiation field is considered in section 3. Special solutions, involving constant angle sail settings with the possibility of a corner, are obtained for the problem of minimum time flight. The equations of motion for the inverse square field case are set out in section 4 and are put into a dimensionless form. In section 5 the boundary conditions applicable to departure from and arrival at specified conic orbits are stated. It is shown that a solar sail vehicle can only arrive at a conic orbit 'from outside' that orbit. In section 6, the differential equations to be satisfied by the Lagrange variables for minimum time flight in the inverse square case are obtained. The question of optimal programming of \( \sigma \), the sail strength, is also considered in section 7. An ambiguity of sign for \( \theta \), the sail angle, encountered in section 6 is resolved in section 7, in which the Weierstrass condition is applied. A separate formula is deduced for calculating \( \theta \), when \( \theta \) is small. In section 8 the differential equations for the Lagrange variables are solved analytically from zero order terms for the case of departure from circular orbit after expanding the appropriate functions in powers of \( \sigma \), the sail strength, which is assumed to be small. For cases in which \( \theta \) is approximately constant the boundary conditions are applied in section 9 to the solution found in section 8. An exact solution of the type predicted was computed (Table A.6) and compared with the theoretical results obtained. Zero order solutions for the
case of departure from a general conic orbit are considered in section 9. Some sample families of \( \theta \)-programmes were computed from these solutions. The optimal solar sailing problem in three dimensions is considered in section 11 and zero order solutions for the case of departure from circular orbit are derived. Some numerical results were also computed for the three dimensional case. In section 12, the convergence technique used for obtaining optimal \( \theta \)-programmes from the exact equations is explained. A number of solutions satisfying the boundary conditions for transfer between circular orbits were obtained by this method. Lastly, the computer programmes used to carry out the various calculations are collected in an Appendix, together with tables of values and figures illustrating the results obtained.
2. Force on a Reflecting Surface

Consider the force on a plane, perfectly reflecting, surface due to incident radiation $i$, making an angle $\theta$ with the normal to the surface. The rate of change of momentum of the photons in the beam due to reflection is proportional to $\cos \theta$. Also, a unit area of the surface will be subject to the radiation contained in a beam of cross section $\cos \theta$. Thus, the force $dF$ acting on a given plane element of surface area $dA$ will be given by

$$dF = \sigma \cos^2 \theta \, dA,$$  \hspace{1cm} (2.1)

where $\sigma$ is a measure of the intensity of radiation. In equation (2.1) $\theta$ must satisfy the inequality

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$  \hspace{1cm} (2.2)

There is no meaning given to $F$ for $\theta$ outside this range.

Equation (2.1) applies only to the idealized case of perfect specular reflection. Further consideration of the forces involved in scattering and absorption is given in [5].
In developing the theory which follows, perfect specular reflection is assumed.

The equation (2.1) for the force due to incident radiation is analogous to that assumed by Newton for the resistance encountered by a body moving through the air. Thus, if a stream of particles moving with uniform velocity $V$ are incident at an angle $\theta$ on a plane surface and are reflected in a perfectly elastic manner, we obtain equation (2.1) again for the force developed due to the reflection of the particles, where $\sigma$ is given by

$$\sigma = 2\rho V^2 \quad (2.3)$$

and $\rho$ is the density of the stream.

The well-known problem of Newton in the Calculus of Variations seeks to determine the particular surface of revolution, formed by rotating a curve about an axis parallel to the direction of motion of the incident particles, which minimizes the force on the surface due to the particle stream. The equations (2.1) and (2.3) do not, in practice, give a good characterization of the force on a surface placed in a dense stream of gas, but they may be expected to apply more accurately to the case of present interest, which involves reflection of light quanta. Therefore, the solution to the Newton problem mentioned above may be interpreted as specifying with reasonable accuracy the shape of a space vehicle, with given length and given cross section at one end, which will
minimize the perturbing force due to radiation encountered when
the vehicle is symmetrically placed with respect to the
radiation stream. The details of the solution are given in
sections 9.14, 9.15 of [6].

With regard to the solar sail, however, we are not
interested in minimizing the radiation force; we wish, indeed,
to maximize it. The question therefore arises as to what space
configuration of a solar sail of given surface area will
maximize the thrust developed in a direction making a specified
angle \( \vec{\theta} \) with the direction of radiation.

The most obvious form of a sail giving a thrust in a
direction making an angle \( \vec{\theta} \) with the direction of radiation is
a plane sail whose normal is aligned with the required direction
of thrust. But it is not immediately obvious that this is
necessarily the optimal configuration. Indeed, it is certainly
not the configuration giving the maximum component of force in
the required direction.

Let the normal to a plane sail of unit area make an angle
\( \theta \) with the direction of radiation and let the unit vector \( \hat{e} \)
make an angle \( \vec{\theta} \) with the direction of radiation. Then,
resolving the force on the sail in the direction \( \hat{e} \) we find,
from equation (2.1), that the component of force in this
direction is given by

\[
F(\theta) = \sigma \cos^2 \theta \cos(\vec{\theta} - \theta). \tag{2.4}
\]
Thus, differentiating equation (2.4) with respect to \( \theta \) we have

\[
F'(\theta) = -\sigma \cos \theta \{2 \sin \theta \cos(\bar{\theta} - \theta) - \cos \theta \sin(\bar{\theta} - \theta)\}.
\]  

(2.5)

To find the value of \( \theta \) maximizing \( F(\theta) \), we equate \( F'(\theta) \) to zero and, upon discarding the solution \( \cos \theta = 0 \), which minimizes \( F(\theta) \), we obtain the condition

\[
\tan \beta = 2 \tan \theta
\]

(2.6)

where

\[
\beta = \bar{\theta} - \theta.
\]

(2.7)

This leads to the following solution for \( \theta \) in terms of \( \bar{\theta} \);

\[
\theta = \tan^{-1}\left( -\alpha \sqrt{\alpha^2 + \frac{1}{2}} \right),
\]

(2.8)

where

\[
\alpha = 3/(4 \tan \bar{\theta}).
\]

(2.9)

Equation (2.8) yields two solutions in the range \(-\frac{\pi}{4} < \theta < \frac{\pi}{4}\), one of which corresponds to a minimum of \( F(\theta) \) while the other corresponds to a maximum. The maximizing value is determined by noting that \( \theta \) and \( \bar{\theta} \) have the same sign. A typical solution is shown in Figure 2.2.

![Figure 2.2. Sail Angle, \( \theta \), maximizing Force component in the direction \( \ell \)](image-url)
Because $\theta$ is not identical with $\varphi$, it might be thought that a greater thrust in the direction $\varphi$ would be obtainable by suitably deforming the plane sail so as to take advantage of the greater thrust components along $\varphi$ obtainable at reduced angles. Let us consider the effect of bending a plane, square sail along a line near the middle of the sail, so that the situation is as shown in Figure 2.3. The resultant force vector must still make an angle $\theta$ with the axis Ox so that we have the overriding condition

$$F_x \sin\theta - F_y \cos\theta = 0,$$  \hspace{1cm} (2.10)

where $F_x, F_y$ are the resultant components of force along the axes Ox, Oy respectively. Without loss of generality we may assume that $\theta$ and $\varphi_2$ are positive. From equation (2.1), we see that a unit, square, plane sail, with normal making an angle $\theta$ with the Ox axis, would deliver a thrust

$$F = \sigma \cos^2 \theta.$$  \hspace{1cm} (2.11)

We now proceed to determine the thrust delivered by a comparison sail of the form shown in Figure 2.3 subject to the condition (2.10).

\[ \text{Figure 2.3. A Comparison Sail} \]
Assuming $\delta_1, \delta_2, \delta_r$ are small and expanding the trigonometric functions to second order, we obtain the following expressions for $F_x$ and $F_y$:

$$F_x = \sigma c^3 \{ 1 - \frac{3}{2}t(\delta_1 + \delta_2) + \frac{3}{4}(2t^2 - 1)(\delta_1^2 + \delta_2^2) - 3t\delta_r(\delta_2 - \delta_1) \}, \quad (2.12)$$

$$F_y = \sigma c^2 s \{ 1 + \left( \frac{1}{2t} - t \right)(\delta_1 + \delta_2) + \frac{1}{2}(t^2 - \frac{1}{4})\delta_1^2 + \delta_2^2 + 2\left( \frac{1}{2t} - t \right)\delta_r(\delta_2 - \delta_1) \}, \quad (2.13)$$

where

$$c = \cos \theta, \quad (2.14)$$

$$s = \sin \theta, \quad (2.15)$$

and

$$t = \tan \theta. \quad (2.16)$$

Substituting for $F_x$ and $F_y$ from equations (2.12), (2.13), the condition (2.10) becomes

$$\frac{1}{2}(\delta_1 + \delta_2) + \delta_r(\delta_2 - \delta_1) - t(\delta_1^2 + \delta_2^2) = 0. \quad (2.17)$$

We see from equation (2.17) that $\delta_1 + \delta_2$ must be of the second order of small quantities. We therefore write

$$\delta_1 + \delta_2 = k\delta^2 \quad (2.18)$$

where $k$ is a constant of zero order and

$$\delta_2 = \delta. \quad (2.19)$$

From equations (2.18), (2.19) we have

$$\delta_1 = -\delta + k\delta^2. \quad (2.20)$$
Now, substituting for $\delta_1, \delta_2$ from equations (2.19), (2.20) into equation (2.17), we obtain the following expression for $\delta_r$:

$$
\delta_r = (t - \frac{1}{2}k)\delta. 
$$

(2.21)

Equations (2.19)-(2.21) now express $\delta_1, \delta_2, \delta_r$ in terms of $\delta$ and $k$. Substituting for these quantities into equations (2.12), (2.13) we obtain

$$
F_x = \sigma c^3 (1 - \frac{3}{2} \delta^2 (1 + 2t^2)), 
$$

(2.22)

$$
F_y = \sigma c^2 s (1 - \frac{3}{2} \delta^2 (1 + 2t^2)). 
$$

(2.23)

Thus, the effect of making a bend of the type considered is to reduce both $F_x$ and $F_y$ and hence also the magnitude of the resultant thrust $F$. No advantage, therefore, is to be gained by bending the sail slightly and we will, from now on, consider all our sails to be planar.
3. **Flight in a Uniform Field**

![Diagram](image)

**Fig. 34. Flight in a Uniform Field**

We will consider the two dimensional flight of a plane sail in a gravitational and radiation field, both uniform in magnitude and direction. Let the magnitude of the gravitational force on the sail vehicle be \( g \) and let it be directed at an angle \( \alpha \) to an axis \( Ox \), where \( Oxy \) are rectangular axes taken such that the radiation is directed positively along \( Ox \). The gravitational force components along the \( x \) and \( y \) axes are thus given by

\[
\begin{align*}
    g_x &= g \cos \alpha, \\
    g_y &= g \sin \alpha,
\end{align*}
\]  

(3.1)  

(3.2)

respectively. Let the force on the sail due to radiation at normal incidence be \( \sigma g \). Then, resolving the radiation force according to equation (2.1) we have the following equations of motion:

\[
\begin{align*}
    \dot{x} &= u, \\
    \dot{y} &= v, \\
    \dot{u} &= g_x + \sigma g \cos^2 \theta,
\end{align*}
\]  

(3.3)  

(3.4)  

(3.5)
where \( u \) and \( v \) are the components of the sail's velocity along the \( x \) and \( y \) axes respectively, and \( \theta \) is the angle made by the normal to the plane of the sail with the direction of radiation, as indicated in Fig. 3.1.

The Lagrange expression for the problem is

\[
\mathcal{F} = -\lambda_x u - \lambda_y v - \lambda_u (g_x + \sigma g \cos^3 \theta) - \lambda_v (g_y + \sigma g \cos^2 \theta \sin \theta) \quad (3.7)
\]

and the Euler-Lagrange characteristic equations are found to be

\[
\begin{align*}
\dot{\lambda}_x &= 0, \quad (3.8) \\
\dot{\lambda}_y &= 0, \quad (3.9) \\
\dot{\lambda}_u &= -\lambda_x, \quad (3.10) \\
\dot{\lambda}_v &= -\lambda_y, \quad (3.11) \\
\frac{3 \cos^2 \theta - 2}{3 \sin \theta \cos \theta} &= \frac{\lambda_u}{\lambda_v}. \quad (3.12)
\end{align*}
\]

Equations (3.8)-(3.11) are immediately integrable to give

\[
\begin{align*}
\lambda_x &= a, \quad (3.13) \\
\lambda_y &= b, \quad (3.14) \\
\lambda_u &= c - at, \quad (3.15) \\
\lambda_v &= d - bt, \quad (3.16)
\end{align*}
\]

where \( a, b, c, d \) are constants. Thus from equations (3.12), (3.15) and (3.16), \( \theta \) is determined according to the equations

\[
\frac{3 \cos^2 \theta - 2}{3 \sin \theta \cos \theta} = \frac{c - at}{d - bt}. \quad (3.17)
\]
Making use of the familiar concept of the 'primer locus' we may interpret equation (3.17) pictorially. Thus we suppose \( \lambda_u, \lambda_v \) to be the components of a vector \( \mathbf{u} \), the primer vector, so that \( \mathbf{u} \) varies linearly with respect to time, in accordance with equations (3.15), (3.16). The tip of the primer vector moves along a straight line with constant velocity. This line is called the primer locus and is shown in Fig. 3.2. Let us now resolve the primer vector along axes \( O\lambda_u, \lambda_v \) obtained from the axes \( O\lambda_u, \lambda_v \) by rotation through an angle \( \theta \).

We see that

\[
\lambda_u' = \lambda_u \cos \theta + \lambda_v \sin \theta, \quad (3.18)
\]

\[
\lambda_v' = -\lambda_u \sin \theta + \lambda_v \cos \theta. \quad (3.19)
\]

Then, substituting from equation (3.12) into the quotient of equations (3.18), (3.19) we find that

\[
\frac{\lambda_u'}{\lambda_v'} = \frac{1}{2} \cot \theta. \quad (3.20)
\]

If we define the angle \( \beta \) by the equation

\[\text{Figure 3.2. The Primer Locus}\]
\[ \tan \beta = \frac{\lambda'}{\lambda_u} \]  
\hspace{1cm} (3.21)

then equation (3.20) may be written
\[ \tan \beta = 2 \tan \theta . \]  
\hspace{1cm} (3.22)

In general, equation (3.22) will have two solutions in the interval \([-\pi/2, +\pi/2]\), as indicated in the figure. One of these will be eliminated by the Weierstrass condition.

The Weierstrass condition is considered in detail in section 7 and we may deduce from the work presented there that \(\theta\) is positive or negative according as \(\lambda_v\) is positive or negative. Thus, in Fig. 3.2 it is \(\theta_1\), which is the optimizing angle and the solution \(\theta_2\) must be discarded.

Equation (3.22) is familiar. In fact, it is identical with equation (2.6). The implication is that \(\theta\) is programmed so as to maximize the force component in the direction of the primer vector.

Using the information provided by the Weierstrass condition we may use equation (3.22) to make some general deduction about the \(\theta\)-programme and to consider some special cases.

First we notice that \(\theta\) changes from \(-\pi/2\) to \(+\pi/2\) (or vice versa), when \(\lambda_v\) vanishes, \(\lambda_u\) being negative. This demonstrates the possibility of a corner being present on an optimal trajectory; however, only one such corner may be present. No corner can be present if \(\lambda_u\) is positive when \(\lambda_v\) vanishes, and
the condition for this is, from equations (3.15), (3.16),
\[
\begin{vmatrix}
c & a \\
d & b \\
\end{vmatrix} \times b > 0.
\]

(3.23)

We also note that, with the possible exception of a single point of discontinuity, \( \theta \) is either monotonically increasing or monotonically decreasing. The general form of the \( \theta \)-programme is shown in Fig. 3.3.

The time origin is taken to be the point at which \( \lambda \) vanishes. Other possible general programmes are obtainable from figures 3.3.(i) and 3.3.(ii) by reversing the time axis. As \( t \to \pm \infty \) it will be noticed that \( \theta \) tends to two limiting values denoted by \( \theta_1 \) and \( -\theta_2 \). For a finite manoeuvre the \( \theta \)-programme will consist of some section taken from the general programmes 3(i) or 3(ii) taken between two time limits \( t_b \) and \( t_f \).

While Figure 3.3 shows the most general behaviour of \( \theta \) as a function of time, we must also examine some special cases.

*It will be convenient in this Part to distinguish quantities which are to be calculated at the departure and arrival terminals by subscripts \( b \) and \( f \) respectively.*
The $\theta$-programme is particularly simple if the primer locus passes through the origin. The behaviour in this case is illustrated in Figure 3.4.

![Figure 3.4. Primer Locus passing through the Origin.](image)

If the direction of travel of the point $P$ is as shown by the arrow then, in general, the $\theta$-programme consists initially of a constant negative angle $-\theta_2$. This changes to a constant positive angle $\theta_1$ as $P$ passes through the origin. The actual magnitude of the angles is determined by the gradient of the line $OP$. From equations (3.15), (3.16) we find that the condition that the primer locus pass through the origin is

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 0.$$  \hspace{1cm} (3.24)

It is also possible for the primer locus to degenerate to a point. This occurs when $a$ and $b$ both vanish. In this case $\theta$ is constant for all time.

In order to illustrate the above theory we will consider a particular optimization problem, namely the maximization of the net energy of a sail vehicle in a specified time interval $t_4$. 
Let $V(x, y)$ be the gravitational potential so that

$$g_x = -\frac{\partial V}{\partial x}, \quad (3.25)$$

$$g_y = -\frac{\partial V}{\partial y}. \quad (3.26)$$

Taking a vehicle of unit mass the kinetic energy is given by

$$T = \frac{1}{2}(u^2 + v^2).$$

We may, without loss of generality, take initial values

$$x_0 = 0, \quad (3.27)$$

$$y_0 = 0, \quad (3.28)$$

at an initial time

$$t = 0.$$

The final time will then be $t_f$. Denoting the final values by the subscript $f$ we see that the final net energy is given by

$$E_f = \frac{1}{2}(u_f^2 + v_f^2) + V(x_f, y_f). \quad (3.29)$$

Thus we may define $J$, the payoff function to be minimized, by the equation

$$J = -E_f. \quad (3.30)$$

Since $x_f, y_f, u_f$ and $v_f$ are not prescribed, we have the boundary conditions

$$\lambda_{1i} = -\frac{\partial J}{\partial x_{1i}} = \frac{\partial E_f}{\partial x_{1i}}, \quad (3.31)$$

the subscript $i$ standing for any of the state variables. Equations (3.25), (3.26), (3.29)-(3.31) now give
\begin{align}
\lambda_{x_1} &= -g_x, \\
\lambda_{y_1} &= -g_y, \\
\lambda_{u_1} &= u_1, \\
\lambda_{v_1} &= v_1.
\end{align}

Equations (3.13)-(3.16), (3.32)-(3.35) suffice to determine the constants \(a, b, c, d\). These are
\begin{align}
a &= -g_x, \\
b &= -g_y, \\
c &= u_1 - g_x t_f, \\
d &= v_1 - g_y t_f.
\end{align}

and hence, from equations (3.15), (3.16), the primer vector components are given by
\begin{align}
\lambda_u &= u_1 - g_x (t_f - t), \\
\lambda_v &= v_1 - g_y (t_f - t).
\end{align}

The simplest case to treat will clearly be that for which the gravitational field vanishes. In this case we have
\begin{align}
g_x = g_y = 0
\end{align}

and equations (3.40), (3.41) combine to give
\begin{align}
\mathbf{P} &= \mathbf{v}_f
\end{align}

so that the primer vector is constant with respect to time and coincides with the final velocity. Since there is no gravitational field the potential energy term is absent from
equation (3.29). We now see that under these circumstances the maximization of $R_t$ is equivalent to the maximization of $d_f$ where

$$d_f = \sqrt{u_{f}^2 + v_{f}^2}$$

(3.44)

is the magnitude of the final velocity vector $v_f$. Because of the constancy of the primer vector from equation (3.43) we know that optimal flight is accomplished with constant sail angle. We may therefore select the optimal $\theta$ programme among all possible programmes by selecting the optimal constant flight angle programme among the family of constant flight angle programmes using ordinary calculus techniques.

When $u_b$ is positive the selection of $\theta$ is straightforward as in Figure 3.5.

![Figure 3.5](image)

**Figure 3.5. Zero Gravitational Field Case.**

If $t_f$ is given we draw a contour $\theta$ representing the possible final values of the velocity components obtainable by sailing at any constant angle $\theta$. We then select that angle which maximizes $d_f$, as shown in the figure. The situation is more complicated if $u_b$ is negative. We will investigate the situation analytically.
Assuming, therefore, that the sail angle is constant, equations (3.5), (3.6) may be integrated to give

\[
\begin{align*}
    u_t &= u_b + \sigma t_k \cos^3 \theta, \\
    v_t &= v_b + \sigma t_k \cos^2 \theta \sin \theta.
\end{align*}
\]  

(3.45)  

(3.46)

(Since \( g = 0 \), \( g\sigma \) has been replaced by \( \sigma \).) From equations (3.12), (3.43), we now deduce that

\[
\frac{3 \cos^2 \theta - 2}{3 \sin \theta \cos \theta} = \frac{u_b + \sigma t_k \cos^3 \theta}{v_b + \sigma t_k \cos^2 \theta \sin \theta}.
\]

(3.47)

This equation is equivalent to

\[
\frac{u_b}{v_b} = \frac{3 \cos^2 \theta - 2}{3 \cos \theta \sin \theta} - 2 \frac{\sigma t_k}{v_b} \cos \theta.
\]

(3.48)

Figure 3.6 shows the variation of the function \( F(\theta) \) with respect to \( \theta \), where \( F(\theta) \) denotes the right hand member of equation (3.48).

![Figure 3.6. The Function F(\theta)](image-url)
Different curves are shown corresponding to different values of the parameter $\tau_i$ where

$$\tau_i = - \frac{\sigma t_i}{v_b}.$$  \hspace{1cm} (3.4.9)

Among all the curves $y = F(\theta)$ corresponding to different choices of $\tau_i$ there will be two which will be distinguished by the fact that they have a point of inflexion at which $\frac{dy}{d\theta}$ vanishes. From considerations of symmetry we can assert that these curves will be those corresponding to $\tau_i = \pm \tau_w$ where $\tau_w$ is to be determined. The two points of inflexion will have coordinates $(-\theta_w, c_1)$ and $(+\theta_w, -c_1)$ where $\theta_w$ and $c_1$ are positive constants to be determined.

The angle of flight $\theta$ will be given by the abscissa of a point of intersection of the curves $y = F(\theta)$ and $y = c$ where

$$c = \frac{u_b}{v_b}$$

is a constant. There will normally be either two or four such points of intersection and only one of these points in general will correspond to absolute maximization of $E_k$. It is clear from diagrams such as Figure 3.5 that the value of $\theta$ required will be positive or negative according as $v_b$ is positive or negative, so that from equation (3.4.9) $\theta$ is positive or negative according as $\tau_i$ is negative or positive. We see from Figure 3.6 that this criterion will suffice to determine $\theta$ for all $\tau_i$ if $c_1 > c > -c_1$. We now seek to determine $\theta_w$ and $c_1$ analytically. We have
\[ F(\theta) = \frac{3\cos^2 \theta - 2}{5\cos \theta \sin \theta} + \frac{2\tau \cos \theta}{3} \tag{3.50} \]

from which we obtain
\[ F'(\theta) = \frac{2}{3} \left( \frac{\cos 2\theta - \tau \sin \theta}{\sin^2 2\theta} \right) \tag{3.51} \]

and
\[ F''(\theta) = -\frac{2}{3} \left\{ \frac{2 \cos^2 2\theta - 6\cos 2\theta \tau + 1}{\sin^2 2\theta} + \tau \cos \theta \right\} \tag{3.52} \]

Now \( \theta_w \) will satisfy the conditions
\[ F'(\theta) = F''(\theta) = 0. \tag{3.53} \]

Substituting from equations (3.51), (3.52) into equations (3.53) and expanding in terms of \( \tan \theta \) we obtain the equation
\[ 4\tan^4 \theta - 2\tan^2 \theta - 3 = 0 \tag{3.54} \]

as the equation determining \( \theta_w \). This gives
\[ \tan^2 \theta = \left( 1 \pm \sqrt{4} \right) / 4. \]

The negative sign cannot yield any real \( \theta \) so that \( \theta_w \) is the positive root of the equation
\[ \tan^2 \theta = (1 + \sqrt{13}) / 4, \tag{3.55} \]

so that
\[ \begin{align*}
\tan \theta_w &= \sqrt{1 + \sqrt{13}} / 2, \\
\sin \theta_w &= \sqrt{2 + \sqrt{13}} / \sqrt{3}, \\
\cos \theta_w &= 2 / \sqrt{5 + \sqrt{13}}.
\end{align*} \tag{3.56} \]

Substituting these functions of \( \theta_w \) into equations (3.51), (3.53)
we get $\tau_w$ defined by

$$\tau_w = \frac{1}{12\sqrt[3]{3}} (3 + \sqrt[3]{13})(2 + \sqrt[3]{13})^3. \quad (3.57)$$

Finally, $c_1$ is determined by substituting values for $\theta_w$ into equation (3.50). This gives

$$c_1 = \frac{8}{(\sqrt[3]{13} - 2)\sqrt[3]{1} + \sqrt[3]{13}}. \quad (3.58)$$

We may now discuss the question of how the optimal angle $\theta$ varies with $\tau_f$ for any given initial velocity components. We note first that there is a region of the $(u_b, v_b)$-plane for which the function $\theta(\tau_f)$ is discontinuous. This region is shown shaded in Figure 3.7. In the unshaded region the function $\theta(\tau_f)$ is continuous. In all cases $\theta(\tau_f)$ converges monotonically on zero as $t_f \to \infty$.

![Figure 3.7. Region of $(u_b, v_b)$-plane with Corner](image)

To illustrate the behaviour in the case where $\theta(\tau_f)$ is discontinuous, we will suppose that $v_b$ is negative (so that $\theta$...
is negative) and $c > c_1$ where the line $y = c$, is shown in Figure 3.6. Figure 3.8 shows the points of intersection of the line $y = c$ with the curves $y = F(\theta)$ for different values of $\tau_4$.

![Figure 3.8. Variation of $\theta$ with $\tau_4$.]

Curves are drawn for $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ where $\tau_1 < \tau_2 < \cdots < \tau_5$. For $\tau_1$ there is only one point of intersection, this point corresponding to maximization of $E_1$. At $\tau_2$ however there are two points of intersection; the lesser value of $\theta$ maximizes $E_1$ while the greater value corresponds to a point of inflexion on the function $E_1(\theta)$. At $\tau_4$ it is the greater value of $\theta$ which maximizes $E_1$. For values of $\tau$ between $\tau_2$ and $\tau_4$ there are three points of intersection, the two extreme ones giving relative maxima and the middle one giving a relative minimum. For values of $\tau$ in this range close to $\tau_2$ it is the lesser $\theta$ which maximizes $E_1(\theta)$ absolutely, while for values of $\tau$ close to $\tau_4$ it is the greater $\theta$ which maximizes $E_1(\theta)$ absolutely. At some point $\tau_3$ between $\tau_2$ and $\tau_4$, therefore, there is a
discontinuity in the optimal $\theta$-function, $\theta(\tau_i)$. Lastly, for values of $\tau_i$ greater than $\tau_{i_4}$, say $\tau_{i_5}$, there is only one point of intersection and this defines $\theta(\tau_i)$ unambiguously. Moreover, as $\tau_i \to \infty$, $\theta(\tau_i) \to 0$.

For values of $(u_b, v_b)$ corresponding to points in the unshaded region of Figure 3.7 there is always only one point of intersection of $y = c$ with $y = F(\theta)$ to be considered, (since the sign of $\theta$ is determined by the sign of $v_b$), so that the problems involved in having two relative maxima in the function $E_\xi(\theta)$ do not arise.

The point of discontinuity, when it exists, is easily determinable only in the degenerate case where $v_b = 0$, ($u_b$ must be negative). In this case we easily find that for $t_i < -2u_b/\sigma$ the optimal programme is $\theta = \pi/2$ (zero thrust), while for $t_i > -2u_b/\sigma$ the optimal programme is $\theta = 0$.

The work presented above for the zero gravitational field case suggests that when we investigate the problem of maximizing $E_\xi$ in the case of a non-vanishing gravitational field we may meet the same phenomenon of the existence of more than one programme giving a relative maximum to $E_\xi$. The question as to which of these programmes provides the absolute maximum may only be resolvable by direct calculation.

Now, assuming that $g_x, g_y$ do not both vanish, we will investigate the case in which the primer locus passes through the origin.
The condition that the primer locus be a line through the origin is given by equation (3.24). Using the values of the constants given by equations (3.36)-(3.39) we find that this condition takes the form

$$\mathbf{u}_0 : \mathbf{v} = \mathbf{g}_x : \mathbf{g}_y,$$  \hspace{1cm} (3.59)

In other words, the final velocity vector is aligned (or anti-aligned) with the direction of the gravitational field. Also, because of equations (3.34), (3.35) and because the primer locus is a line through the origin, the primer vector is itself always aligned or anti-aligned with the gravitational field.

Now, if we denote the primer vector by \( \mathbf{p} \) and the final velocity vector by \( \mathbf{v}_f \), we may combine equations (3.40), (3.41) in a vector equation

$$\mathbf{p} = \mathbf{g}(t - t_f) + \mathbf{v}_f,$$  \hspace{1cm} (3.60)

showing that the primer locus is the locus of a point \( \mathbf{p} \) travelling with velocity \( \mathbf{g} \) and, because of equation (3.59), passing through the origin. Also, equations (3.5)-(3.6) combine to give a vector equation

$$\dot{\mathbf{y}} = \mathbf{g} + \sigma \mathbf{g} \cos^2 \theta \mathbf{e}$$  \hspace{1cm} (3.61)

where \( \mathbf{e} \) is a unit vector in the direction of the normal to the sail and \( \theta \) is the angle made by this normal with the Ox axis.

Thus, because of the constancy, (with one discontinuity), of \( \theta \) in equation (3.61), we may write
\[ \dot{\mathbf{y}} = \mathbf{F}_1 \]

before \( P \) passes through the origin, and
\[ \dot{\mathbf{y}} = \mathbf{F}_2 \]

after \( P \) passes through the origin, \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) being constant vectors. The hodograph of \( \mathbf{y} \) is thus quite simple for any choice of initial conditions; in general, it is generated by a point first moving in the direction \( \mathbf{F}_1 \) and then, after \( P \) has passed through the origin, in the direction \( \mathbf{F}_2 \).

Proceeding from an arbitrary starting point \((u_0, v_0)\) at \( t = 0 \) the tip of the velocity vector varies along the hodograph and, from equation (3.59), the end-point, at which an optimal programme will have been realized will be attained when the vectors \( \mathbf{y} \) and \( \mathbf{p} \) are aligned or anti-aligned. Moreover, at this point, we see from equations (3.34), (3.35) that the vectors \( \mathbf{y} \) and \( \mathbf{p} \) will coincide.

An example of the nature of the solutions of the particular type discussed above is shown in Figure 3.9

![Figure 3.9. Hodographs for Special Solution](image_url)
In Figure 3.9 the plane of possible initial velocities 
$(u_b, v_b)$ is divided up into four regions. For flights whose 
velocity hodograph starts in the upper region (i), the sail angle, 
$\theta$, is positive for the entire flight, $(\theta = \theta_1)$, and the acceleration is in the direction $E_2$. When the primer vector vanishes, 
the velocity vector will be directed along some line $OQ$ 
determined by the direction of $E_2$. This may be seen by remembering 
that going backwards in time from any end point on the line $OP$, 
the terminal points of the vectors $v$ and $p$ travel at constant 
velocity back along their respective hodographs. In fact, using 
the velocities implied by equations $(3.60), (3.61)$, we find that 
the line $OO$ is a continuation of the line $O'Q$ making an angle $\theta_2$ 
with the axis $Ou$. For flights whose hodograph starts in the 
middle left region (ii), $\theta$ is first negative $(\theta = -\theta_1)$ and then 
positive $(\theta = \theta_2)$ and the acceleration is first in the direction 
$E_1$ and then in the direction $E_2$. The bottom region (iii) 
corresponds to flights carried out with $\theta = -\theta_1$ so that the 
acceleration is always in the direction $E_1$. For flights whose 
initial velocity corresponds to a point in the region (iv) to the 
right of the primer locus, there is no solution of the type 
considered.

We remark here that a special solution of the type 
considered, corresponding to some initial values $(u_b, v_b)$, does 
not necessarily maximize $E_i$ absolutely for the value of $t_i$ 
appropriate to the special solution. There may be another 
programme yielding a relative maximum to $E_i$, for which the primer
locus does not pass through the origin, and this more general programme may yield the absolute maximum for the given values of $u_b, v_b, \tilde{z}$. 

We may even have to consider more than one special solution as a candidate for being the solution yielding the absolute maximum. This state of affairs is depicted in Figure 3.10 which corresponds to the case $\xi = (0, 1)$, $\sigma = 3\sqrt{3}/2$, $\theta_2 = \tan^{-1} 1/\sqrt{2}$ and for which the regions (i), (ii) and (iii) defined in Figure 3.9 overlap. In this case, if the initial values $u_b, v_b$ correspond to a point $A$ in the overlap region, as shown in the figure, then there are three possible solutions to be considered; these are numbered 1, 2, and 3 in the figure. It is not difficult to calculate the values of $E_x$ resulting from these three programmes so that, since it is found that the three manoeuvres are carried out in equal times, we may easily decide which programme yields the absolute maximum in this time.

![Figure 3.10. Overlap Condition.](image-url)
Suppose, therefore, that the initial conditions are given by

\[ u_b = -a \cos \alpha, \quad v_b = -a \sin \alpha \]

where

\[ 0 \leq \tan \alpha \leq 1/\sqrt{2}. \]

We find that the three programmes yield the following values of \( E_1 \):

\[ E_{41} = \frac{1}{2}a^2(\cos^2 \alpha - \sqrt{2} \cos \alpha \sin \alpha + \sin^2 \alpha) \quad (3.62) \]

for programme 1,

\[ E_{42} = \frac{1}{2}a^2(\sqrt{2} \sin \alpha \cos \alpha + \sin^2 \alpha) \quad (3.63) \]

for programme 2, and

\[ E_{43} = \frac{1}{2}a^2(\sqrt{2} \sin \alpha \cos \alpha - \sin^2 \alpha) \quad (3.64) \]

for programme 3.

We notice first that

\[ E_{41} - E_{43} = \frac{1}{2}a^2(\cos \alpha - \sqrt{2} \sin \alpha)^2 \quad (3.65) \]

and

\[ E_{42} - E_{43} = \frac{1}{2}a^2 \sin^2 \alpha. \quad (3.66) \]

The right hand members of equations (3.65), (3.66) are always positive so that programme 3 can never yield an absolute maximum. Also

\[ E_{41} - E_{42} = \frac{1}{2}a^2 \cos \alpha(\cos \alpha - \sqrt{2} \sin \alpha). \quad (3.67) \]
This is positive when $\alpha = 0$ and negative when $\alpha = \theta_2$. It vanishes when
\[ \tan \alpha = 1/2V2. \]

Thus, in the overlap region, $\theta = \theta_2$ maximizes $E_4$ for $0 < \tan \alpha < 1/2V2$, while $\theta = -\theta_2$ maximizes $E_4$ for $1/2V2 < \tan \alpha < 1/V2$. When $\tan \alpha = 1/2V2$, programmes 1 and 2 yield identical values for $E_4$ and both are local maxima. Thus, as in the case of the zero gravitational field, we encounter the phenomenon of having more than one relative maximum programme for certain choices of $u_b, v_b$.

We conclude this section with an example of an optimization problem for which the $\theta$-programme is such that the equations of motion (3.3)-(3.6) are integrable analytically. We will take the simplest case for which the gravitational field vanishes so that equations (3.5), (3.6) may be written
\[
\dot{u} = \sigma \cos^3 \theta, \quad (3.68) \\
\dot{v} = \sigma \cos^2 \theta \sin \theta, \quad (3.69)
\]
and we will take the initial conditions to be
\[ x = y = u = v = 0 \text{ at } t = 0. \quad (3.70) \]

Let us now try to maximize the angular momentum about the origin within a specified time $t_f$. To do this we take a payoff function $J$ given by
\[ J = -x \dot{v}, \quad (3.71) \]
where \( J \) is to be minimized.

Now \( x_i, y_i, u_i, v_i \) are not preassigned, but \( t_f \) is. We therefore have the boundary conditions

\[
\begin{align*}
\lambda_{x_f} &= v_f, \\
\lambda_{y_f} &= \lambda_{u_f} = 0, \\
\lambda_{v_f} &= x_f,
\end{align*}
\]

(from equation (1.46) of [8]). The constants \( a, b, c, d \) occurring in equations (3.13)-(3.16) may now be determined using equations (3.72)-(3.74). Substituting the values found into equation (3.17) we get the following equation determining the \( \theta \)-programme

\[
\frac{3 \cos^2 \theta - 2}{3 \sin \theta \cos \theta} = \frac{v_f}{x_f} (t_f - t).
\]

Obviously, the positive value of \( \theta \) satisfying equation (3.75) must be chosen. Figure 3.11 shows the nature of the primer locus in this case.

![Figure 3.11, Primer Locus for Maximizing Angular Momentum](image)

Differentiating equation (3.75) with respect to time we obtain
\[ \theta = \frac{v_f^2}{x_f} \frac{3 \sin^2 \theta \cos^2 \theta}{1 + \sin^2 \theta} \]  

Taking the initial conditions \((3.70)\) into account, equation \((3.69)\) may now be integrated to give

\[ v_f = \sigma \int_0^{t_f} \cos^2 \theta \sin \theta \, dt , \]
\[ = \sigma \int_0^{t_f} \frac{c^2 \sin \theta}{\theta} \, d\theta , \]
\[ = \frac{\sigma x_f}{3v_f} \int_{\theta_b}^{\theta_f} (\csc \theta + \sin \theta) d\theta , \]  

\[(3.77)\]

after using equation \((3.76)\). Also, from equation \((3.75)\) we find that \(\theta_b, \theta_f\) satisfy the equation

\[ \frac{3 \cos^2 \theta_b - 2}{3 \sin \theta_b \cos \theta_b} = \frac{v_f t_f}{x_f} , \]

\[ \cos \theta_f = \sqrt{2}/\sqrt{3} , \]  

\[(3.78)\]

\[(3.79)\]

and other trigonometric functions of \(\theta_f\) are easily determinable from equation \((3.79)\). Equation \((3.77)\) is now integrable and gives

\[ \frac{3v_f^2}{\sigma x_f} = -\sqrt{2}/\sqrt{3} - \log(\sqrt{2}/\sqrt{3}) + \cos \theta_b + \log(\csc \theta_b + \cot \theta_b) . \]  

\[(3.80)\]

Using equation \((3.76)\) again, equation \((3.68)\) may be integrated to give

\[ u = \frac{\sigma x_f}{3v_f} \int_{\theta_b}^{\theta_f} \left( \frac{\csc \theta + \cos \theta}{\sin^2 \theta + \cos \theta} \right) d\theta , \]
\[ = \frac{\sigma x_f}{3v_f} \left\{ \Phi_b + \sin \theta - \csc \theta \right\} , \]  

\[(3.81)\]

where

\[ \Phi_b = \cos \theta_b \cot \theta_b . \]  

\[(3.82)\]
Now, integrating equation (3.81), we have

\[
\frac{3v_f}{\sigma x_f} \cdot x_f = \int_0^{t_f} (R_b - \cos \theta \cot \theta)dt,
\]
so that

\[
\frac{3v_f}{\sigma} = R_b t_f - \frac{x_f}{3v_f} \int_{\theta_0}^{\theta_f} (\csc \theta + \csc^3 \theta)d\theta,
\]

(3.83)

which, substituting for \( R_b \) and \( t_f \) from equations (3.78), (3.82), reduces to

\[
9v_f^2/\sigma x_f = \cot \theta_0 \csc \theta_0 - 3 \cos \theta_0 - \int_{\theta_0}^{\theta_f} (\csc \theta + \csc^3 \theta)d\theta.
\]

(3.84)

Using the result

\[
\int \csc^3 \theta \, d\theta = -\frac{1}{2} (\csc \theta \cot \theta + \log(\csc \theta + \cot \theta))
\]

(3.85)
equation (3.84) becomes

\[
6v_f^2/\sigma x_f = \sqrt{2}/\sqrt{3} + \log(\sqrt{2}/\sqrt{3}) - \log(\csc \theta_0 + \cot \theta_0) + \frac{1}{2} \cot \theta_0 \csc \theta_0 - 2 \cos \theta_0.
\]

(3.86)

We may now eliminate \( v_f^2/\sigma x_f \) between equations (3.80) and (3.86) to obtain an equation satisfied by \( \theta_b \), viz.

\[
\frac{4}{9} \cos \theta_0 (12 - \csc^2 \theta_0) + \log(\csc \theta_0 + \cot \theta_0) = \sqrt{2}/\sqrt{3} + \log(\sqrt{2}/\sqrt{3})
\]

(3.87)

This shows that \( \theta_b \) is independent of the flight time \( t_f \). Now, substituting for \( \log(\csc \theta_0 + \cot \theta_0) \) from equation (3.87) into equation (3.80) we get

\[
27v_f^2/\sigma x_f = \cos \theta_b (\csc^2 \theta_0 - 3).
\]

(3.88)

Dividing equation (3.88) by equation (3.78) we have
\[ v_\ell = \frac{\omega t_\ell \cos^2 \theta_b}{\sin \theta_b}, \quad (3.89) \]

and substituting back for \( v_\ell \) into equation (3.78) we have

\[ x_\ell = \frac{\omega t_\ell^2 \cos^3 \theta_b}{(3 \cos^2 \theta_b - 2)}. \quad (3.90) \]

Equations (3.89), (3.90) define the required end-values \( x_\ell, v_\ell \) in terms of the parameter \( \theta_b \) which may be evaluated from equation (3.87). Substituting for \( v_\ell, x_\ell \) from equations (3.89), (3.90) into equation (3.75) we get an equation for the \( \theta \)-programme;

\[ \frac{3 \cos^2 \theta - 2}{3 \sin \theta \cos \theta} = \frac{3 \cos^2 \theta_b - 2}{3 \sin \theta_b \cos \theta_b} \left( 1 - \frac{t}{t_\ell} \right). \quad (3.91) \]

Thus, for this particular problem, the difficulty of having to satisfy split end-value conditions may be resolved analytically.
We consider the motion of a sail in the field of a gravitating and radiating body whose gravitational field and radiation intensity vary according to the inverse square of the distance from the body.

Taking an origin of coordinates, 0, coinciding with the influencing body and taking an initial line stationary in some inertial frame, we may write down the equations of motion using polar coordinates \((r, \psi)\) as in Fig 4.1.

\[
\begin{align*}
\dot{r} &= u & (4.1) \\
\dot{\psi} &= v/r & (4.2) \\
\dot{u} &= v^2/r - \gamma m/r^2 + \lambda \ell^2 \cos^3 \theta / r^2 & (4.3) \\
\dot{v} &= -uv/r + \lambda \ell^2 \cos^2 \theta \sin \theta / r^2 & (4.4)
\end{align*}
\]

where \(\gamma\) is the universal gravitational constant, \(m\) is the mass of the central body and \(\lambda \ell\) is the acceleration due to normally incident radiation at a distance \(\ell\) from 0, \(\ell\) being the semi-latus rectum of the conic from which departure is initially made. Dots
denote derivatives with respect to time.

Let

\[ \frac{G_c}{\gamma} = \frac{v^2}{c^2} \]  \hspace{1cm} (4.5)

so that

\[ \lambda_c^2/G_c = \sigma \]  \hspace{1cm} (4.6)

is the ratio of the (normal) radiational to the gravitational force on the sail vehicle at the reference distance, \( t \), from 0.

Let \( \omega \) be the angular velocity which the sail would have in the departure orbit when situated at the ends of the latus rectum. We now define the dimensionless variables \( \rho, p, q, \tau \) by the equation

\[ \rho = \omega^2 r/\gamma_c \]  \hspace{1cm} (4.7)

\[ p = \frac{u}{\omega t} \]  \hspace{1cm} (4.8)

\[ q = \frac{v}{\omega t} \]  \hspace{1cm} (4.9)

\[ \tau = \frac{\omega^3 t t/G_c}{\rho} \]  \hspace{1cm} (4.10)

Substituting for \( r, u, v, t \) from eqs. (4.7)-(4.10) into eqs. (4.1)-(4.4) we obtain

\[
\frac{d\rho}{d\tau} = p \]  \hspace{1cm} (4.11)

\[
\frac{dp}{d\tau} = \frac{q}{\rho} \]  \hspace{1cm} (4.12)

\[
\frac{dp}{d\tau} = \frac{g^2}{\rho} - \frac{1}{\rho^2} + \frac{\sigma \cos^2 \theta}{\rho^2} \]  \hspace{1cm} (4.13)

\[
\frac{dq}{d\tau} = \frac{-pa}{\rho} + \frac{\sigma \cos^2 \theta \sin \theta}{\rho^2} \]  \hspace{1cm} (4.14)
5. Boundary Conditions

From well known orbital theory and using eqs. (4.7)-(4.10) we find that the initial values of the dimensionless orbit parameters are given by

\[
\rho_b = 1/(1 + e \cos \epsilon) \quad (5.1)
\]
\[
p_b = e \sin \epsilon \quad (5.2)
\]
\[
q_b = 1 + e \cos \epsilon \quad (5.3)
\]
\[
h_b = 1 \quad (5.4)
\]
\[
\tau_b = 0 \quad (5.5)
\]
\[
\psi_b = \epsilon \quad (5.6)
\]

where \( e \) is the eccentricity of the initial conic orbit, \( \epsilon \) is the real anomaly of the departure point and \( h \) is the dimensionless angular momentum defined by

\[
h = \rho q \quad (5.7)
\]

For the case of departure from circular orbit, the eccentricity vanishes and we have the simple equations

\[
p_b = 0 \quad (5.8)
\]
\[
\rho_b = q_b = h_b = 1 \quad (5.9)
\]

In this case we may take

\[
\epsilon = 0 \quad (5.10)
\]

without loss of generality. Eqs. (5.1)-(5.6) also take particularly simple forms for the case of departure from apogee, perigee and the ends of the latus rectum.
Suppose we wish to arrive at another conic with eccentricity $e$ and semi-latus rectum $\tau$, then if our point of arrival has real anomaly $\varpi$ with respect to the major axis of the arrival conic, we easily deduce the following boundary conditions to be satisfied on arrival:

\[
\begin{align*}
\bar{\rho} &= \lambda \sqrt{1 + e \cos \varpi} \\
\bar{p} &= \lambda^{-\frac{1}{2}} e \sin \varpi \\
\bar{a} &= \lambda^{-\frac{1}{2}} (1 + e \cos \varpi) \\
\bar{h} &= \lambda^{\frac{1}{2}}
\end{align*}
\]

where

\[
\lambda = \frac{\varpi}{\epsilon}
\]

In particular, for arrival at circular orbit we have

\[
\begin{align*}
\bar{\rho} &= \lambda \\
\bar{p} &= 0 \\
\bar{a} &= \lambda^{-\frac{1}{2}} \\
\bar{h} &= \lambda^{\frac{1}{2}}
\end{align*}
\]

since in this case $\varpi$ vanishes.

We may notice immediately one result which holds for all solar sail transfers between conic orbits, where the terminal velocity of the sail matches the velocity of the planet in the destination orbit at the point of arrival, and the gravitational effects of the planet are ignored. A solar sail always arrives at its destination orbit 'from outside' that orbit, just as a sail always departs from an initial orbit along a trajectory which 'lies
outside the departure orbit near the departure point. Using capital letters for parameters pertaining to the planet and small letters for parameters pertaining to the sail, we may write according to Taylor's theorem

\[ r(\tau^+ - \delta \tau) = r(\tau^+ ) - \delta \tau \dot{r}(\tau^+) + \frac{1}{2} \delta \tau^2 \ddot{r}(\tau^+ ) + \ldots , \quad (5.20) \]

\[ R(\tau^+ - \delta \tau) = R(\tau^+) - \delta \tau \dot{R}(\tau^+) + \frac{1}{2} \delta \tau^2 \ddot{R}(\tau^+) + \ldots , \quad (5.21) \]

where \( r, R \) represent dimensionless radial distances. If \( \tau^+ \) is the time of arrival we have

\[ r(\tau^+ ) = R(\tau^+) = \bar{\rho}, \]

\[ \Omega(\tau^+ ) = P(\tau^+) = \bar{P}, \]

\[ q(\tau^+) = Q(\tau^+) = \bar{Q}, \quad (5.22) \]

for matched boundary conditions. The equation corresponding to equation (4.13) for the planet is, of course,

\[ \dot{P} = \frac{\Omega^2}{R} - \frac{1}{R^2} . \quad (5.23) \]

Now, subtracting equation (5.21) from equation (5.20) and using equations (4.13), (5.22), (5.23), we obtain

\[ r(\tau^+ - \delta \tau) = R(\tau^+ - \delta \tau) + \frac{1}{2} \delta \tau^2 \rho^2 \cos^2 \theta \rho^2 + \ldots . \quad (5.24) \]

But the last term of the right hand member of equation (5.24) is always positive for values of \( \theta \) in the permissible range \(-\pi/2 < \theta < \pi/2\); (obviously we will not have an identity \( \theta \equiv \pm \pi/2 \) near the end point). Thus

\[ r > R \quad (5.25) \]
just subsequent to the time of arrival. In a similar manner we may show that the inequality (5.25) holds for times just after the time of departure.
6. Minimum Time Flight

We will now put equations (4.11)-(4.14) in such a form that the differential equations satisfied by the Lagrange multipliers will not involve the control variable \( \theta \). To do this we multiply the equations by \( \rho^2 \) and introduce a new variable \( \eta \) defined by the equations

\[
\frac{d\eta}{d\tau} = \rho^{-2}d\tau \\
\eta = 0 \text{ at } \tau = 0
\]

Equations (4.11)-(4.14) and eq. (6.1) then become

\[
\frac{d\tau}{d\eta} = \rho^2 \\
\frac{d\rho}{d\eta} = \rho \rho^2 \\
\frac{d\psi}{d\eta} = \rho \rho \\
\frac{dp}{d\eta} = q^2 \rho - 1 + \sigma a \\
\frac{dq}{d\eta} = -pq\rho + \sigma b
\]

where

\[
a = \cos^3 \theta \\
b = \cos^2 \theta \sin \theta
\]

In the notation of the Mayer problem the Lagrange expression is given by

\[
F = -\rho^2 \lambda_x - \rho \rho^2 \lambda_y - \rho \rho \lambda_p - (q^2 \rho - 1 + \sigma a) \lambda_p - (-pq\rho + \sigma b) \lambda_q,
\]

from which we derive the Euler characteristic equations
\[ \lambda_v = 0 \] (6.11)
\[ \lambda_\rho = -2\rho\lambda_\chi - 2\rho\lambda_\rho - 2q^2\lambda_\rho + pq\lambda_q \] (6.12)
\[ \lambda_\psi = 0 \] (6.13)
\[ \lambda_p = -\rho^2\lambda_\rho + qr\lambda_q \] (6.14)
\[ \lambda_q = -r\lambda_\psi - 2q\rho\lambda_\rho + pq\lambda_q \] (6.15)
\[ 0 = \lambda_p \frac{\partial a}{\partial \theta} + \lambda_q \frac{\partial b}{\partial \theta} \] (6.16)

where dots hereafter denote differentiations with respect to \( \eta \).

If we write
\[ c = \cos \theta \] (6.17)
we find that
\[ \partial a/\partial \theta = -3\cos^2 \theta \sin \theta = -3b \] (6.18)
\[ \partial b/\partial \theta = 3\cos^3 \theta - 2\cos \theta = 3a - 2c \] (6.19)

Equation (6.16) may be written in the alternative forms
\[ \frac{\lambda_p}{\lambda_q} = \frac{3\cos^2 \theta - 2}{3\cos \theta \sin \theta} = \frac{3\cos 2\theta - 1}{3\sin 2\theta} \] (6.20)
assuming \( \cos \theta \neq 0 \). The case \( \cos \theta = 0 \) corresponds to \( \theta = \pm \pi/2 \) in which case the sail delivers zero thrust. In general this case will be eliminated by the Weierstrass condition.

Writing
\[ d = \lambda_p/\lambda_q \] (6.21)
we may expand equation (6.20) in the form
\[ 2\tan^2 \theta + 3d \tan \theta - 1 = 0. \] (6.22)
This equation yields two possible values for \( \theta \) given by

\[
\theta = \tan^{-1} \left( \frac{D \pm \sqrt{D^2 + \frac{1}{4}}}{} \right) \tag{6.23}
\]

where

\[
D = -\frac{3d}{4} \tag{6.24}
\]

and

\[-\frac{\pi}{2} < \theta < +\frac{\pi}{2}. \tag{6.25}\]

Equation (6.23) yields one positive and one negative angle. The choice between these must be made in accordance with the Weierstrass condition.

Now, if we wish to minimize the final time \( \tau_f \), we may take a payoff function, \( J \), where

\[
J = k' \tau_f \tag{6.26}
\]

and \( k' \) is a positive constant.

Since the values \( \tau_f, \psi_f \) are not specified we have the boundary conditions

\[
\lambda_{\tau_f} = -\frac{\partial J}{\partial \tau_f} \tag{6.27}
\]

\[
\lambda_{\psi_f} = -\frac{\partial J}{\partial \psi_f}. \tag{6.28}
\]

These are

\[
\lambda_{\tau_f} = -k', \tag{6.29}
\]

\[
\lambda_{\psi_f} = 0. \tag{6.30}
\]

But, from equations (6.11), (6.13) and since \( \lambda_{\tau}, \lambda_{\psi} \) must be continuous at any possible corners, we have from equations (6.29), (6.30)

\[
\lambda_{\tau} = -k' \tag{6.31}
\]
\[ \lambda_\psi = 0 \] (6.32)

for all \( \eta \).

Also, since \( \eta \) does not occur explicitly in the equations of motion (6.3)-(6.7), we know that there is a first integral of the form

\[ \lambda_i \dot{x}_i = \text{constant}, \] (6.33)

where we sum over the suffixes \( i \) corresponding to the set of state variables. Now \( \eta_k \) is not specified so that we have another boundary condition

\[ \lambda_i \dot{x}_i \eta_k = \frac{\partial J}{\partial \eta_k} = 0, \] (6.34)

showing that the constant in equation (6.33) is zero. Recalling equations (6.31), (6.32) we may now write equation (6.33) in the form

\[ -k' \rho^2 + \lambda_p \beta \rho^2 + \lambda_q \left( \alpha^2 \rho - 1 + \sigma \right) + \lambda_q ( - \rho q + \sigma b) = 0. \] (6.35)

In particular, for departure from a circular orbit, we have initially that

\[ \sigma (a_b \lambda_p \beta + b_b \lambda_q) = k'. \] (6.36)

If we are interested in weak sails for which \( \sigma \) will be small, we see from equation (6.36) that if we wish to take \( \lambda_p, \lambda_q \) to be
$O(1)$, we must choose $k'$ to be $O(\sigma)$. Therefore, we write

$$k' = \sigma k. \quad (6.37)$$

The initial condition (6.36) becomes

$$a_b \lambda_{b_b} + b_b \lambda_{q_b} = k. \quad (6.38)$$

We now return to the equations for the Lagrange multipliers (6.11)-(6.15). We already have $\lambda_t, \lambda$ given by eqs. (6.31), (6.32), (6.37). The remaining three equations may now be written

$$\lambda_\rho = -2\rho \rho \lambda_\rho - q^2 \lambda_\rho + pq \lambda_q + 2q_k \rho \quad (6.39)$$
$$\lambda_\rho = -\rho^2 \rho_\rho + q \rho \lambda_q \quad (6.40)$$
$$\lambda_\rho = -2 q \rho \lambda_\rho + q \rho \lambda_q \quad (6.41)$$

Differentiating eq. (6.40) and using eqs. (6.4), (6.7), (6.39), (6.41) we find that

$$\lambda_\rho + (\rho q)^2 \lambda_\rho = \sigma \rho (b \lambda_q - 2 k \rho^2). \quad (6.42)$$

But $\rho q$ has a physical interpretation. In fact if

$$h = \rho q \quad (6.43)$$

then $h$ is the angular momentum of the sail vehicle about the sun in dimensionless form. Thus eqs. (6.41), (6.42) may be written in operational form

$$(D^2 + h^2) \lambda_\rho = \sigma \rho (b \lambda_q - 2 k \rho^2) \quad (6.44)$$
$$(D - \rho \rho) \lambda_q = -2 h \lambda_\rho. \quad (6.45)$$
Equations (6.44), (6.45) involve $\lambda_p, \lambda_q$ alone. We may use these
equations and ignore eq. (6.39), thus eliminating the variable $\lambda_p$.

7. The Weierstrass Condition

The Weierstrass Condition requires that the function

$$P(\theta) = \lambda_p \cos^2 \theta + \lambda_q \cos^2 \theta \sin \theta$$

be maximized with respect to $\theta$ in the range

$$-\pi/2 \leq \theta \leq +\pi/2.$$  

From eq. (7.1) we have

$$P'(\theta) = \cos \theta \left\{ -3\lambda_p \cos \theta \sin \theta + \lambda_q (3\cos^2 \theta - 2) \right\}.$$  

The vanishing of $P'(\theta)$ thus requires either

$$\cos \theta = 0$$

or

$$\frac{\lambda_p}{\lambda_q} = \frac{3\cos^2 \theta - 2}{3\sin \theta \cos \theta}.$$  

These are the same conditions as those obtained from the Euler
characteristic equations, (eq. (6.20)). Since the range of $\theta$ is
bounded by condition (7.2), we must also consider values of $\theta$ on
the boundaries. However, these values are precisely those given
by eq. (7.4).

In general, eq. (7.5) is satisfied by two different
values of $\theta$, one in the first quadrant and one in the fourth quadrant. To decide whether these values are maxima or minima we calculate the second derivative of $P(\theta)$. We obtain

$$P''(\theta) = 3\lambda_p \cos \theta (3\sin^2 \theta - 1) + \lambda_q \sin \theta (9\sin^2 \theta - 7) \quad (7.6)$$

We have four values to investigate, namely, those satisfying eqs. (7.4), (7.5).

Eq. (7.4) is satisfied by $\theta = \pm \pi/2$. If

$$\theta = \pm \pi/2, \quad P(\theta) = 0, \quad P'(\theta) = 0, \quad P''(\theta) = 2\lambda_q. \quad (7.7)$$

If

$$\theta = \pm \pi/2, \quad P(\theta) = 0, \quad P'(\theta) = 0, \quad P''(\theta) = -2\lambda_q. \quad (7.8)$$

Also, substituting for $\lambda_p$ from eq. (7.5) into eq. (7.6) we get

$$P''(\theta) = -\lambda_q (1 + \sin^2 \theta) / \sin \theta \quad (7.9)$$

at the stationary points, $\theta_1$ in the first quadrant, and $\theta_4$ in the fourth quadrant.

The various cases are shown in figure 7.1.

If $\theta_1$ denotes the extremum in the first quadrant and $\theta_4$ the extremum in the fourth quadrant, we find that

\[
\begin{align*}
(\text{i}) \quad & \lambda_q > 0 \quad \implies \quad \theta = \theta_1 \text{ for a maximum} \\
(\text{ii}) \quad & \lambda_q < 0 \quad \implies \quad \theta = \theta_4 \text{ for a maximum} \\
(\text{iii}) \quad & \lambda_q = 0, \quad \lambda_p > 0 \quad \implies \theta = 0 \text{ for a maximum} \\
(\text{iv}) \quad & \lambda_q = 0, \quad \lambda_p < 0 \quad \implies \theta = \pm \pi/2 \text{ for a maximum} \\
(\text{v}) \quad & \lambda_q = 0, \quad \lambda_p = 0 \text{ does not define } \theta.
\end{align*}
\]
$$P(\Theta) = \lambda_p \cos^3 \Theta + \lambda_q \cos^2 \Theta \sin \Theta.$$ 

Figure 7.1. The Weierstrass Function $P(\Theta)$. 
In drawing the figures we note that at $\theta = 0$

$$P(0) = \lambda_p$$  \hspace{1cm} (7.11)

We note, in particular, that the Weierstrass condition demonstrates the possibility of a corner on an optimal trajectory. A corner occurs when $\lambda_q$ changes sign, $\lambda_p$ being negative. When $\lambda_q = 0$ we have from eq. (6.41) that

$$\lambda_q = -2q\rho\lambda_p.$$  \hspace{1cm} (7.12)

If $\lambda_p$ is negative then $\lambda_q$ is positive and $\lambda_q$ is passing from negative values to positive values. Thus, at a corner, $\theta$ always passes from $-\pi/2$ to $+\pi/2$ and never in the reverse direction. (We here assume that $q$ always remains positive). Also, if $\lambda_p$ is positive then $\lambda_q = 0$ corresponds to $\theta = 0$. In this case eq. (7.12) shows that $\lambda_q$ is decreasing so that $\theta$ can only pass through zero in the negative direction.

The Weierstrass condition determines the correct sign to be taken in eq. (6.23), determining $\theta$. We find that $\theta$ is given exactly by the equation

$$\theta = \tan^{-1}\left\{\frac{-3\lambda_p + \sqrt{9\lambda_p^2 + 8\lambda_q^2}}{4\lambda_q}\right\}.$$  \hspace{1cm} (7.13)

If $\lambda_q$ is positive this gives a root in the first quadrant. If $\lambda_q$ is not positive, it gives a root in the fourth quadrant. $\theta$ must, of course, satisfy the condition (7.2).
for calculating \( \theta \) when \( \theta \) is small. We know that the vanishing of \( \theta \) is determined by the vanishing of \( \lambda_p, \lambda_q \) being positive.

Under these conditions eq. (7.13) takes the improper form

\[
\tan \theta = 0/0. \tag{7.14}
\]

We therefore seek another formula to use for determining small \( \theta \).

We have from eq. (7.13), assuming \( \lambda_p \) is positive, (since we are dealing with small \( \theta \)),

\[
\tan \theta = \frac{-3\lambda_p + 3\lambda_p (1 + 8\lambda_q^2/3\lambda_p^2)^{1/4}}{4\lambda_q}
\]

Expanding the numerator by the binomial theorem we find that \( \theta \) is given in this case by

\[
\tan \theta = \frac{1}{4} \eta \{2 - \eta^2 + \eta^4 + \ldots\} \tag{7.15}
\]

where

\[
\eta = 2\lambda_q / 3\lambda_p. \tag{7.16}
\]

It is clear from eq. (7.13) that \( \lambda_p \) and \( \lambda_q \) may be multiplied by an arbitrary positive constant without changing the value of \( \theta \) determined by them. We can therefore determine 'normalized' values of \( \lambda_p, \lambda_q \) such that

\[
\lambda_p^2 + \frac{8}{3} \lambda_q^2 = 1. \tag{7.17}
\]

With these values, eq. (7.13) simplifies to give

\[
\tan \theta = \frac{3(1-\lambda_p)}{4\lambda_q}.
\]
so that
\[ \lambda_p = 1 - \frac{4}{3} \lambda_q \tan \theta. \] (7.18)

Substituting from (7.18) into (7.17) we get
\[ 0 = \lambda_q \left\{ -\frac{8}{3} \tan \theta + \frac{8}{9} \lambda_q (1 + 2 \tan^2 \theta) \right\}. \] (7.19)

Now \( \lambda_q = 0 \) corresponds to \( \theta = 0, \pm \pi/2 \). For \( \theta \) lying between these values we get from eq.(7.19)
\[ \lambda_q = \frac{3 \tan \theta}{1 + 2 \tan^2 \theta} = \frac{3 \sin \theta \cos \theta}{1 + \sin^2 \theta} \] (7.20)
whence eq.(7.18) gives
\[ \lambda_p = \frac{1 - 2 \tan^2 \theta}{1 + 2 \tan^2 \theta} = \frac{1 - 3 \sin^2 \theta}{1 + \sin^2 \theta}. \] (7.21)

We may thus consider eqs.(7.20),(7.21) as determining 'normalized' values of \( \lambda_p, \lambda_q \). Their variation with respect to \( \theta \) is shown in Fig 7.2.

\[ \text{Figure 7.2. Normalized Lagrange Parameters.} \]
The Weierstrass condition may also be used to investigate
the situation where \( \sigma \) is supposed not to be fixed but may be
varied between fixed limits. We suppose therefore that the
value of \( \sigma \) is governed by the inequalities

\[
\bar{\sigma} \geq \sigma \geq 0.
\] (7.22)

The function which must now be maximized, in accordance
with the Weierstrass condition, is

\[
P(\theta, \sigma) = \sigma (\lambda_p \cos^3 \theta + \lambda_q \cos^2 \theta \sin \theta).
\] (7.23)

Since \( \sigma \) cannot be negative, \( \theta \) is determined exactly as before.
When the maximum value of the term in brackets in the right hand
member of equation (7.23) is positive, the value of \( P \) is clearly
maximized by setting \( \sigma = \bar{\sigma} \). However, if \( \lambda_q = 0, \lambda_p < 0 \)
the right hand member of equation (7.23) is less than or equal to
zero for \( \theta \) in the allowable range. The conditions \( \lambda_q = 0, \lambda_p < 0 \)
correspond to \( \theta = \pm \pi/2 \) which results in zero thrust. Thus the
only circumstance in which intermediate values of \( \sigma \) are possible
is when \( \lambda_q = \lambda_p = 0 \), i.e. when the primer vector vanishes. In
this case \( P(\theta, \sigma) \) vanishes for all \( \theta \) and the Weierstrass condition
places no restriction on \( \sigma \).
8. Zero Order Solution

For clarity we will now replace \( \rho \) by \( r \). Also, using eq. (6.43) we will use \( h \) as a state variable in place of \( q \) and by differentiating eq. (6.6) and substituting for the first derivatives obtained we obtain a second order differential equation for \( p \).

The equations governing the system may now be written

\[
\begin{align*}
\dot{t} &= r^2 \\
\dot{r} &= pr^2 \\
\dot{\psi} &= h \\
\ddot{p} + h^2 p &= \sigma(2hb + \dot{\psi}) \\
\ddot{h} &= \sigma rb \\
\kappa_p + h^2 \lambda_p &= \sigma(r\lambda q - 2kr^3) \\
\lambda_q - pr\lambda q &= -2h\lambda p \\
r^2 \lambda q - h\lambda q - \kappa_p &= 0 \\
\ddot{p} &= \frac{h^2}{r} - 1 + \sigma a.
\end{align*}
\]

For the case of circular orbit we also have the boundary conditions

\[
\tau = 0, \ r = 1, \ \psi = 0, \ p = 0, \ h = 1 \quad \text{at} \ \eta = 0. \quad (8.10)
\]

We will take \( \sigma \) to be small and expand the functions in terms of \( \sigma \)

\[
x_1(\eta) = x_{10}(\eta) + \sigma x_{11}(\eta) + \sigma^2 x_{12}(\eta) + \ldots \quad (8.11)
\]
where $x_{1}$ represents the set of variables appearing in eqs. $(8.1)-(8.9)$. Applying the boundary conditions at $\eta = 0$, we have

$$
\begin{align*}
\tau_0 &= 0, \quad r_0 = 1, \quad \psi_0 = 0, \quad p_0 = 0, \quad h_0 = 1 \quad \{ \text{at } \eta = 0 \}.
\end{align*}
$$

Substituting eqs. $(8.11)$ into eqs. $(8.1)-(8.9)$ and equating zero-order and first order terms we get two sets of equations

$$
\begin{align*}
\dot{\tau}_0 &= \tau_0^2, \\
\dot{r}_0 &= p_0 \tau_0^2, \\
\dot{\psi}_0 &= h_0, \\
\dot{p}_0 &= h_0^2 \frac{r_0}{p_0} - 1, \\
\ddot{p}_0 + h_0^2 p_0 &= 0, \\
\dot{h}_0 &= 0, \\
\ddot{h}_0 + h_0^2 \lambda_{p_0} &= 0, \\
\lambda_{q_0} - p_0 r_0 \lambda_{q_0} &= -2h_0 \lambda_{p_0}, \\
r_0^2 \lambda_{p_0} &= h_0 \lambda_{q_0} - \lambda_{p_0} \\
\end{align*}
$$

together with

$$
\begin{align*}
\dot{\tau}_1 &= 2r_0 \tau_1, \\
\dot{r}_1 &= 2r_0 p_0 \tau_1 + r_0^2 \lambda_{p_1}, \\
\dot{\psi}_1 &= h_1, \\
\dot{\lambda}_{p_1} &= \frac{2h_0}{r_0} h_1 - \frac{h_0^2}{r_0^2} \tau_1 + a, \\
\ddot{p}_1 + h_0^2 p_1 &= 2h_0 b + \dot{a} - 2h_0 p_0 h_1, \\
\dot{h}_1 &= r_0 b, \\
\ddot{\lambda}_{p_1} + h_0^2 \lambda_{p_1} &= r_0 (h_0 \lambda_{q_0} - 2 \lambda_{p_0}) - 2h_0 \lambda_{p_0} h_1.
\end{align*}
$$
\[
\dot{\lambda}_{q_1} - p_0 r_0 \dot{\lambda}_{q_1} = -2 h_0 \lambda_{p_1} - 2 \lambda_{q_0} h_1 + p_0 \lambda_{q_0} r_0 + r_0 \lambda_{q_0} p_1 \quad (8.29)
\]
\[
r_0^2 \dot{\lambda}_{r_1} = h_0 \lambda_{q_1} + h_1 \lambda_{q_0} - \dot{\lambda}_{p_1} - 2 r_0 \lambda_{r_0} r_1 \quad (8.30)
\]

For the case of an initial circular orbit the eqs. (8.13)-(8.18) have the solution

\[
\begin{align*}
\tau_0 &= \eta \\
\rho_0 &= 1 \\
\psi_0 &= \eta \\
p_0 &= 0 \\
h_0 &= 1
\end{align*}
(8.31)
\]

Equations (8.19)-(8.21) then become

\[
\begin{align*}
\ddot{\lambda}_{p_0} + \dot{\lambda}_{p_0} &= 0 \\
\dot{\lambda}_{q_0} &= -2 \lambda_{p_0} \\
\lambda_{r_0} &= \lambda_{q_0} - \lambda_{p_0}
\end{align*}
(8.32, 8.33, 8.34)
\]

which have general solution

\[
\begin{align*}
\lambda_{p_0} &= \beta \cos(\eta + \epsilon) \\
\lambda_{q_0} &= -2 \beta \sin(\eta + \epsilon) + \alpha \\
\lambda_{r_0} &= -\beta \sin(\eta + \epsilon) + \alpha
\end{align*}
(8.35, 8.36, 8.37)
\]

Now \(\beta\) is only a scaling factor and may be put equal to unity by a suitable choice of \(k\) in eq. (6.38). Thus, we may take as the general zero order solution for the Lagrange variables
\[ \lambda_{\rho_0} = \cos(\eta + \varepsilon) \]  \hspace{1cm} (8.38)

\[ \lambda_{\eta_0} = -2\sin(\eta + \varepsilon) + A \]  \hspace{1cm} (8.39)

\[ \lambda_{\eta_0} = -\sin(\eta + \varepsilon) + A \]  \hspace{1cm} (8.40)

where \( A \) and \( \varepsilon \) are constants which must be chosen so as to satisfy the boundary conditions. We notice that the solution is periodic of period \( 2\pi \) in \( \eta \).

We may gain some idea of the types of \( \theta \)-programme which may be derived from equations (8.38), (8.39) by looking at the primer locus diagram as depicted in Fig. 8.1 and remembering that \( \theta \) may best be visualised by keeping equation (3.22) in mind together with the Weierstrass condition.

The primer locus is an ellipse with major axis of length 2 units directed along the \( 0\eta \) axis, minor axis of length 1 unit directed along the \( 0\rho \) axis and with centre at the point \((0, A)\). Thus, if \( A > 2 \) the locus lies entirely above the \( 0\rho \) axis and \( \theta \) is always positive, if \( A < -2 \) the locus lies entirely below the \( 0\rho \) axis and \( \theta \) is always negative. If \(-2 < A < 2 \) the locus
encloses the origin and there will be a corner at the point where
the locus crosses the \( O \lambda_p \) axis, \( \lambda_p \) being negative. The point \( P \),
such that \( OP \) represents the primer vector, moves in a clockwise
direction as \( \eta \) increases as shown in the figure. If \( A = 2 \), there
is a corner at which \( \theta \) transfers from 0 to \( \pi/2 \), while, if \( A = -2 \),
there is a corner at which \( \theta \) transfers from \(-\pi/2\) to 0. We note
lastly that for \( A \) such that \(-2 < A < 2\), \( \theta \) is a monotonically
decreasing function of \( \eta \) except at corners.

A family of zero order solutions has been calculated from
equations (8.38), (8.39) and (7.13) for various values of \( A \). The
results are presented in Fig. A.4 and tabulated in Table A.8. For
weak sails, (\( \sigma \) small), this solution will be a good approximation
to the exact optimal \( \theta \)-programmes. The boundary values of the
orbital parameters will determine the particular values of \( A \) and
\( \epsilon \) to be chosen in equations (8.38), (8.39).

In order to obtain an analytical estimate of the values of
\( A \) and \( \epsilon \) appropriate to the given boundary conditions we have to
return to the first order equations (8.22)-(8.27). Using the
circular orbit solution given by equation (8.31), these may be
written

\[
\begin{align*}
t'_1 &= 2r_1 , \\
\dot{r}'_1 &= p'_1 , \\
\dot{\psi}'_1 &= h'_1 , \\
\dot{p}'_1 &= 2h'_1 - r'_1 + a ,
\end{align*}
\]

\((8.41)\) \hspace{1cm} \((8.42)\) \hspace{1cm} \((8.43)\) \hspace{1cm} \((8.44)\)
\[ \ddot{p}_1 + p_1 = 2b + \dot{a} = b(2-3\dot{b}), \quad (8.45) \]
\[ \dot{h}_1 = b, \quad (8.46) \]

where we have used equation (6.18).

In order to obtain an approximate analytical solution to equation (8.45) we will restrict our attention to those programmes corresponding to large positive \( \Lambda \) in equations (8.38)-(8.40).
9. An Approximate Solution satisfying the Boundary Conditions

We will consider the flight of a weak sail departing from circular orbit. In this case a first approximation to the Lagrange variables $\lambda_p, \lambda_q$ will be given by equations (8.38), (8.39). The constants $A$ and $\epsilon$ must be chosen to satisfy the specified arrival boundary conditions. In order to obtain a situation amenable to further analytic treatment, we will assume that the arrival boundary conditions are such that $A$ is large and positive. In this case we may expand $\lambda_p/\lambda_q$ in powers of $1/A$ obtaining

\[
\frac{\lambda_p}{\lambda_q} = \frac{\cos(\eta + \epsilon)}{-2\sin(\eta + \epsilon) + A},
\]

\[= \frac{1}{A} \cos(\eta + \epsilon) + \frac{2}{A^2} \sin(\eta + \epsilon) \cos(\eta + \epsilon) + O(A^{-3}).
\] (9.1)

Equation (7.13) then becomes

\[
\theta = \tan^{-1} \left[ \frac{1}{\sqrt{2}} - \frac{3}{4A} \cos(\eta + \epsilon) - \frac{3}{2A^2} \cos(\eta + \epsilon) \cos(\eta + \epsilon)

+ \frac{9}{16A^2} \cos^2(\eta + \epsilon) + O(A^{-3}) \right]
\]

which may be expanded by Taylor's theorem to give

\[
\theta = \tan^{-1} \left[ \frac{1}{\sqrt{2}} + \frac{1}{16A^2} - \frac{1}{2A} \cos(\eta + \epsilon) - \frac{1}{2A^2} \sin^2(\eta + \epsilon)

+ \frac{1}{16A^2} \cos^2(\eta + \epsilon) + \ldots \right].
\] (9.2)

Equation (9.2) is in the form of a Fourier series with coefficients exact up to $O(A^{-2})$.

Using the value of $\theta$ determined by equation (9.2), we
may now calculate the quantities

\[ a = \cos^3 \theta, \]

\[ b = \cos^2 \theta \sin \theta, \]

as Fourier series with coefficients accurate to \( O(A^{-2}) \). We obtain

\[ a = \frac{\sqrt{3}}{3} - \frac{1}{2N6A^2} + \frac{1}{4N3A} \cos(\eta + \epsilon) \]

\[ - \frac{1}{8N6A^2} \cos2(\eta + \epsilon) + \frac{1}{4N3A} \sin2(\eta + \epsilon) + \ldots \quad (9.3) \]

\[ b = \frac{2}{3\sqrt{3}} - \frac{1}{4N3A^2} - \frac{1}{4N3A} \cos2(\eta + \epsilon) + \ldots \quad (9.4) \]

Differentiating equation (9.3) with respect to \( \eta \) we get

\[ \dot{a} = -\frac{1}{4N3A} \sin(\eta + \epsilon) + \frac{1}{4N6A^2} \sin2(\eta + \epsilon) + \frac{2}{N3A} \cos2(\eta + \epsilon) + \ldots \quad (9.5) \]

Using the approximate expressions for \( b \) and \( \dot{a} \) obtained in equations (9.4), (9.5), we may write equation (8.45), the differential equation governing \( p_1 \), in solvable form. We get

\[ (D^2 + 1)p_1 = \frac{4}{3N3} - \frac{1}{2N3A^2} - \frac{1}{4N3A} \sin(\eta + \epsilon) + \frac{1}{2N6A^2} \sin2(\eta + \epsilon) \]

\[ + \frac{\sqrt{3}}{2N3} \cos2(\eta + \epsilon). \quad (9.6) \]

This equation has the general solution

\[ p_1 = A_3 \cos \eta + B_3 \sin \eta + \frac{4}{3\sqrt{3}} - \frac{1}{2N3A^2} + \frac{1}{2N3A} \eta \cos(\eta + \epsilon) \]

\[ - \frac{1}{12N6A^2} \sin2(\eta + \epsilon) - \frac{1}{2N3A} \cos2(\eta + \epsilon). \quad (9.7) \]

Now at \( \eta = 0 \), \( p_1 = 0 \), so that from equation (9.7) we have
which determines $A_3$. Substituting for $A_3$ from equation (9.8) into equation (9.7), we have

$$P_1(\eta) = B_3 \sin \eta + \frac{1}{2N_3A^2} \eta \cos(\eta + \epsilon) + \left(\frac{1}{3\sqrt{3}} - \frac{1}{2N_3A^2}\right)(1 - \cos \eta)$$

$$+ \frac{1}{12N_6A^2} \{\cos \eta \sin 2\epsilon - \sin(\eta + \epsilon)\} + \frac{1}{2N_3A^2} \{\cos \eta \cos 2\epsilon - \cos(\eta + \epsilon)\}. \quad \ldots(9.9)$$

Again, from equations (8.46), (9.4) we have a differential equation for $h_1$,

$$Dh_1 = \frac{2}{3N_3} - \frac{1}{l_n 3A^2} - \frac{1}{l_n 3A^2} \cos 2(\eta + \epsilon) \quad (9.10)$$

and we find that the solution satisfying the initial condition $h_1 = 0$ at $\eta = 0$ is

$$h_1(\eta) = \left(\frac{2}{3\sqrt{3}} - \frac{1}{l_n 3A^2}\right)\eta + \frac{1}{8N_3A^2} \{\sin 2\epsilon - \sin(\eta + \epsilon)\}. \quad (9.11)$$

Now $r_1$ is given by equations (8.42) and (9.9). We find that the solution satisfying the initial condition $r_1 = 0$ at $\eta = 0$ is given by

$$r_1(\eta) = B_3(1 - \cos \eta) + \left(\frac{1}{3\sqrt{3}} - \frac{1}{2N_3A^2}\right)(\eta - \sin \eta)$$

$$+ \frac{1}{2N_3A^2} (\eta \sin(\eta + \epsilon) + \cos(\eta + \epsilon) - \cos \epsilon)$$

$$+ \frac{1}{12N_6A^2} (\sin \eta \sin 2\epsilon + \frac{1}{2} \cos 2(\eta + \epsilon) - \frac{1}{2} \cos 2\epsilon)$$

$$+ \frac{1}{2N_3A^2} (\sin \eta \cos 2\epsilon - \frac{1}{2} \sin 2(\eta + \epsilon) + \frac{1}{2} \sin 2\epsilon) . \quad (9.12)$$

The constant $B_3$ may now be determined by substituting from
equations (9.3), (9.9), (9.11) and (9.12) into equation (8.44). This gives

\[ B_3 = \frac{3}{2} \sqrt{\varepsilon} + \frac{1}{2\sqrt{3}A} \cos \varepsilon - \frac{1}{3\sqrt{6}A^2} \left\{ 1 - \frac{1}{3}\cos 2\varepsilon \right\}. \quad (9.13) \]

Between them, equations (9.9), (9.11)-(9.13) determine to \( O(A^{-2}) \) the trajectory which results from any particular choice of the constants \( \varepsilon \) and \( A \), \( A \) being large. We now seek to determine how these constants must be chosen in order to satisfy certain specified arrival boundary conditions.

First, however, we may notice immediately the character of the function \( p_1(\eta) \). From equations (9.9), (9.13), we have to zero order in \( A^{-1} \),

\[ p_1(\eta) = \frac{1}{3\sqrt{3}} \left( 1 - \cos \eta + \frac{1}{\sqrt{2}} \sin \eta \right). \quad (9.14) \]

Figure 9.1 shows how \( p_1(\eta) \) is obtained as the sum of two components, one proportional to \( 1 - \cos \eta \) and the other to \( \sin \eta / \sqrt{2} \).
Equation (9.15) demonstrates that to a first approximation the \( p_1(\eta) \) curve has a sinusoidal shape. Also, since \( \sqrt{3/2} \) exceeds unity, \( p_1 \) must become negative. Figure 9.1 demonstrates the feature already noted, that the outer orbit is approached from the outside, since the radial velocity is negative just before arrival. It is also simple, using equation (9.15), to compare the magnitudes of the maximum and minimum values of \( p_1 \). We get

\[
\frac{|p_{1,\text{max}}|}{p_{1,\text{min}}} = \left( \frac{\sqrt{3}}{2} + 1 \right) \left( \frac{\sqrt{3}}{2} - 1 \right)
\]

\[
= 9.397 \ldots
\]

Thus the maximum outward speed is almost exactly ten times the maximum inward speed.

Now, denoting the arrival boundary values by a bar, we have in accordance with the notation established in equation (8.11)

\[
\overline{p} = 1 + \overline{p}_1 \sigma + \ldots,
\]

\[
\overline{n} = 1 + \overline{n}_1 \sigma + \ldots,
\]

\[
\overline{p} = 0 + \overline{p}_1 \sigma + \ldots.
\]

The condition for injection into circular orbit upon arrival
is, from equations (5.17), (5.19),

\[ \bar{p} = 0 , \]
\[ \bar{n} = \bar{r}^4 . \]

Equating first order terms in the two members of these equations, we get finally

\[ \bar{p}_1 = 0 , \]
\[ \bar{r}_1 = \bar{r} , \]
\[ \bar{n}_1 = \frac{3}{2} \bar{r}_1 . \]  \hspace{1cm} (9.17)

To get an idea of the solution, we look first at the zero order terms. From equations (9.9), (9.11)-(9.13) and (9.17) we obtain

\[ 0 = \frac{4}{3\sqrt{3}} \left( 1 - \cos \bar{n} + \frac{1}{\sqrt{2}} \sin \bar{n} \right) , \]  \hspace{1cm} (9.18)
\[ \bar{r}_1 = \frac{4}{3\sqrt{3}} \bar{n} + \frac{3}{\sqrt{2}} \left( 1 - \cos \bar{n} - \sqrt{2} \sin \bar{n} \right) , \]  \hspace{1cm} (9.19)
\[ \frac{3}{2} \bar{r}_1 = \frac{2}{3\sqrt{3}} \bar{n} . \]  \hspace{1cm} (9.20)

It is easy to see that equations (9.18)-(9.20) have the solution

\[ \bar{n} = 2m\pi , \]  \hspace{1cm} (9.21)
\[ \bar{r}_1 = \frac{8}{3\sqrt{3}} m\pi , \]  \hspace{1cm} (9.22)

where \( n \) is a positive integer corresponding (roughly) to the number of circuits made round the sun before arrival. We will consider particularly the case \( n = 1 \).
In order to solve the equations more accurately and obtain relations involving $A$ and $\epsilon$, we will expand $\eta$ and $r$ in the form

$$\eta = 2\eta' + \eta''A^{-1} + \eta'''A^{-2} + \ldots ,$$  \hspace{1cm} (9.23)

$$r = \frac{8}{3\sqrt{3}} \eta' + \frac{2}{3\sqrt{3}} A^{-1} + \frac{1}{3\sqrt{3}} A^{-2} + \ldots .$$  \hspace{1cm} (9.24)

Using equations (9.23), (9.24) and equating now coefficients of $A^{-1}$ in equations (9.17) we find that

$$\eta' = -\frac{3\eta}{2N^2} \cos \epsilon ,$$  \hspace{1cm} (9.25)

$$r_1' = \frac{\eta}{N^2} \sin \epsilon ,$$  \hspace{1cm} (9.26)

$$r_1'' = \frac{1}{3\sqrt{3}} \eta' .$$  \hspace{1cm} (9.27)

And these equations lead to

$$\tan \epsilon = -\sqrt{2} .$$  \hspace{1cm} (9.28)

There are two solutions for $\epsilon$, one in the fourth quadrant and one in the second quadrant. However, these two solutions are not essentially different; a switch from one solution to the other merely corresponds to a change in the sign of the value of $A$ characterising the manoeuvre. It is most convenient to choose the solution in the second quadrant. This gives

$$\eta' = \frac{\eta}{2} \sqrt{\frac{3}{2}} ,$$  \hspace{1cm} (9.29)

$$r_1' = \frac{\eta}{3} \sqrt{2} .$$  \hspace{1cm} (9.30)
In terms of the first order analysis given above, we can now exhibit a set of approximate solutions in terms of the parameter $A$. From equations (9.23), (9.24), (9.29), (9.30) we have approximately

$$
\bar{\eta} = 2\pi + \sqrt{\frac{3}{2}} \frac{\mu}{2A},
$$

(9.31)

$$
\bar{r}_1 = \frac{8}{3\sqrt{3}} + \frac{\sqrt{2}\pi}{3A}.
$$

(9.32)

We consider the radius of the arrival orbit to be such that $A$, as defined by equation (9.32), is large; that is, the radius is approximately $1 + \frac{8\pi\sigma}{3\sqrt{3}}$. With this value of $A$, an approximation to the sail-angle programme, $\theta(\eta)$, is given by equation (9.2) and the variations of $p_1, h_1$ and $r_1$ with respect to $\eta$ are given by equations (9.9), (9.11)-(9.13). Also, from equations (9.2), (9.28), we have an equation for the initial sail angle which, to $O(A^{-1})$, is

$$
\theta_0 = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2\sqrt{3A}}.
$$

(9.33)

It is true that equation (9.28) only gives $\epsilon$ to zero order, however $\epsilon$ only occurs in the first and higher order terms in the equations for $\theta, p_1, h_1$ and $r_1$, so that it is only necessary to know $\epsilon$ to zero order in order to obtain a first order solution for the state parameters.

If we compare the above theory with the numerical results given in the Appendix, we find that there is good agreement with the zero order results, but that the numerical results
were not obtained to sufficient accuracy and for sufficiently small values of \( \sigma \) to be able to deduce meaningful values of the parameter \( A \) used in the above theory. If \( \sigma \) is not sufficiently small, the higher order terms in the expansions of the state variables in powers of \( \sigma \), which are ignored in the above work, will mask the contribution of the parameter \( A \) in the expansions obtained. The extent of agreement of the zero order theory (equations (9.21), (9.22)) with the numerical results is shown in Table 9.1. The agreement improves as \( \sigma \) decreases.

<table>
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<th>( \bar{r} )</th>
<th>( \bar{\eta} )</th>
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<td>Calculated</td>
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<td>1.024</td>
</tr>
</tbody>
</table>

Table 9.1. Comparison of Zero Order Theory Prediction with Calculated Results
10. General Conic Solution

For the case of optimal transfer between conic orbits, we can no longer assume, as we did in section 6, that the boundary conditions are independent of the angle of rotation \( \psi \). In particular, equation (6.32) will no longer be valid in general. We must now use the formulation of the Mayer problem which accommodates implicit boundary conditions.

We therefore consider the problem of transfer between two conic orbits, where the eccentricities of the departure and arrival orbits are \( e \) and \( \bar{e} \) respectively. In the dimensionless coordinates defined by equations (4.7)-(4.10), the length of the semi latus-rectum of the departure orbit is unity. Let the length of the semi latus-rectum of the arrival orbit be \( \lambda \).

Taking the major axis of the departure orbit as the initial line, we suppose that

\[
\psi = \psi_0 \quad \text{at} \quad \eta = 0,
\]

where \( \eta = 0 \) defines the moment of departure of the sail. Upon arrival at the destination orbit we take

\[
\psi = \psi_\xi \quad \text{at} \quad \eta = \eta_\xi,
\]

where \( \eta_\xi \) is not prespecified. If the major axis of the destination orbit makes an angle \( \delta \) with the major axis of the departure orbit, then the real anomaly of the sail in the destination orbit upon arrival will be given by

\[
\bar{e} = \psi_\xi - \delta.
\]
We may now write down from equations (5.1)-(5.6), (5.11)-(5.13), (10.3), a set of boundary conditions for the problem. These are

\[ \Psi_1 \equiv \tau_b = 0 \quad (10.4) \]
\[ \Psi_2 \equiv \rho_b - (1 + e \cos \psi_b)^{-1} = 0 \quad (10.5) \]
\[ \Psi_3 \equiv \rho_b - e \sin \psi_b = 0 \quad (10.6) \]
\[ \Psi_4 \equiv q_b - (1 + e \cos \psi_b) = 0 \quad (10.7) \]
\[ \Psi_5 \equiv \rho_f - \lambda(1 + e \cos(\psi_f - \delta))^{-1} = 0 \quad (10.8) \]
\[ \Psi_6 \equiv \rho_f - \lambda^{-2} e \sin(\psi_f - \delta) = 0 \quad (10.9) \]
\[ \Psi_7 \equiv q_f - \lambda^{-2}(1 + e \cos(\psi_f - \delta)) = 0 \quad (10.10) \]

These are in the form of equations (14) of Part I.

The equations of motion, equations (6.3)-(6.7) may be written in the form

\[ \Phi_i \equiv \dot{t} - \rho^2 = 0, \text{ etc.} \quad (10.11) \]

corresponding to the form of equations (13) in Part I. Now taking a payoff function \( J \) as defined by equation (6.26), so that we are again minimizing the time of transit, we find that the boundary conditions specified by equations (19)-(23) of Part I may be written

\[ \lambda_{\tau_b}(1 + e \cos \psi_b)^{-2} + \lambda_{\rho_b} \rho_b + \lambda_{q_b} q_b = 0 \quad (10.12) \]
\[ -k_1 \lambda^2 (1 + e \cos(\psi_f - \delta))^2 + \lambda_{p_f} \rho_f + \lambda_{q_f} q_f = 0 \quad (10.13) \]

\[ \lambda_{\psi_b} + e \sin \psi_b (1 + e \cos \psi_b)^{-2} \rho_b \]
\[ + \lambda_{\psi_f} e \cos \psi_b - \lambda_{q_f} e \sin \psi_b = 0 \quad (10.14) \]
\[
\lambda_{\psi} + \lambda_{\psi} \sin(\psi_{f} - \delta) \cdot (1 + \bar{e} \cos(\psi_{f} - \delta))^{-2} \lambda_{p_{f}} \\
+ \lambda_{p_{f}} \lambda_{q_{f}} \cos(\psi_{f} - \delta) - \lambda_{\psi_{f}} \lambda_{q_{f}} \sin(\psi_{f} - \delta) = 0 , \\
\lambda_{q_{f}} = -k_{f} ,
\]

after we have eliminated the parameters $\nu_{k}$. Also, from equations (18) in Part I, we obtain again the differential equations (6.11)-(6.16) for the Lagrange variables.

From equations (6.11), (10.16) we see that $\lambda_{r}$ is given by

\[
\lambda_{r} = -k_{r} ,
\]

which is identical with equation (6.31) obtained for the case of transfer between circular orbits. Also, from equation (6.13), we have

\[
\lambda_{\psi} = k_{s}, \text{ a constant.}
\]

But this constant is no longer zero, in general, as it was for the case of transfer between circular orbits. It is defined by equations (10.14) or (10.15).

Since $\eta$ does not occur explicitly in equations (6.3)-(6.7), we have a first integral given by equations (25), (26) of Part I. This equation takes the form

\[
\lambda_{r} \rho^{2} + \lambda_{p} \rho^{2} + \lambda_{q} \left( q^{2} \rho - 1 + \sigma a \right) \\
+ \lambda_{q} (-pqr + \sigma a) + \lambda_{p} q \rho = \text{constant .}
\]

Inserting the appropriate initial values into equation (10.19) and subtracting equations (10.12), (10.14) from the resulting equation,
we deduce that the constant on the right hand side of equation (10.19) is zero. Equations (10.12)-(10.16) are now seen not to be independent, for either of equations (10.13) or (10.15) can now be obtained from the other by inserting the final boundary values into the first integral, equation (10.19).

From equations (10.12), (10.17) we have

\[ k' = \sigma (1 + \cos \theta)^2 \{ \lambda_p a_b + \lambda_q b_b \} \tag{10.20} \]

so that, once again, if \( \lambda_p, \lambda_q \) are taken to be zero order quantities, then \( k' \) is \( O(\sigma) \). Thus, \( k' \), as defined by equation (6.37), is a zero order constant. Substituting for \( \lambda' \) and \( \lambda_q \) from equations (10.17), (6.37) and (10.18) into equations (6.12), (6.14) and (6.15), we obtain the following differential equations for the remaining Lagrange multipliers:

\[ \dot{\lambda}_p = 2\sigma k - 2\rho \lambda_p - k_q q - q^2 \lambda_p + \rho q \lambda_q \] \tag{10.21}
\[ \dot{\lambda}_q = -\rho^2 \lambda_q + \rho \lambda_q \] \tag{10.22}
\[ \dot{\lambda}_q = -\rho k_q - 2\rho \lambda_p + \rho \lambda_q \] \tag{10.23}

Now, differentiating equation (10.22) with respect to \( \eta \) and using equations (6.4), (6.7), (10.21), (10.23), we find that the terms involving the constant \( k_s \) vanish and we obtain

\[ \ddot{\lambda}_p + (\rho q)^2 \lambda_p = \sigma \rho (b \lambda_q - 2k \rho^2) \] \tag{10.24}

which is the same as equation (6.42). Finally, using equation (6.43), we write equations (10.23), (10.24) in operational form.
\[(D^2 + h^2) \lambda_p = \sigma \rho (b \lambda_q - 2k \rho^2), \quad (10.25)\]
\[(D-p \rho) \lambda_q = -2h \lambda_p - k_s \rho. \quad (10.26)\]

If \(\sigma\) is assumed to be small we may now proceed to obtain a zero order solution for the Lagrange multipliers \(\lambda_p\) and \(\lambda_q\) and hence for the sail angle \(\theta\). If we solve equations (8.13)-(8.18) and select the solution satisfying the initial boundary conditions (5.1)-(5.6), we obtain the zero order solution

\[h_0(\eta) = 1, \quad (10.27)\]
\[\psi_0(\eta) = \eta + \epsilon, \quad (10.28)\]
\[1/r_0(\eta) = 1 + \epsilon \cos \psi_0, \quad (10.29)\]
\[p_0(\eta) = \epsilon \sin \psi_0. \quad (10.30)\]

In the notation of section 8, equations (10.25), (10.26) may be written to zero order

\[\ddot{\lambda}_p + h_0^2 \lambda_p = 0, \quad (10.31)\]
\[\ddot{\lambda}_q - p_0 r_0 \lambda_q = -2h_0 \lambda_p - k_s r_0. \quad (10.32)\]

Since, by equation (10.27), \(h_0\) is a constant in this case also, equation (10.31) is immediately integrable to give

\[\lambda_p = \beta \cos(\eta + \epsilon_0), \quad \text{where } \beta, \epsilon_0 \text{ are constants.} \quad \text{Or, from equation (10.28),}\]
\[\lambda_p = \beta \cos(\psi_0 + \delta), \quad (10.33)\]
where
\[ \delta = \varepsilon_0 - \varepsilon. \]  \hspace{1cm} (10.34)

Also, using equation (8.14), equation (10.32) may be written in the form
\[ \lambda_0 \frac{\dot{r}_o}{r_o} \lambda_0 = -2\lambda_p - k_3 r_o . \]  \hspace{1cm} (10.35)

Equation (10.35) has an integrating factor \(1/r_o\), so that we obtain
\[ \lambda_{q_o} = r_o \left[ \alpha - k_3 \eta - 2 \int_0^\eta \frac{\lambda_p}{r_0} \, d\eta \right], \]  \hspace{1cm} (10.36)

where the constant \(\alpha\) is such that
\[ \alpha = \lambda_{q_o} (0) / r_0 (0). \]  \hspace{1cm} (10.37)

Substituting for \(r_o, \lambda_p\) from equations (10.29), (10.33) into equation (10.36) and carrying out the integration, we obtain
\[ \lambda_{q_o} = r_o \left\{ \alpha + 2\beta \sin(\epsilon + \delta) + \frac{1}{2} \beta \epsilon \sin(2\epsilon + \delta) - 2\beta \sin(\psi_o + \delta) \right. \\
- \left. \frac{1}{2} \beta \epsilon \sin(2\psi_o + \delta) - \eta (k_3 + \beta \epsilon \cos \delta) \right\}. \]  \hspace{1cm} (10.38)

Because of the term in \(\eta\), this solution is not periodic unless we have
\[ k_3 + \beta \epsilon \cos \delta = 0. \]  \hspace{1cm} (10.39)

But, since we are dealing with a zero order solution in which the effect of the sail on the motion is not taken into account, we expect the solution to be periodic. We will therefore consider programmes for which equation (10.39) holds.

In order to obtain some idea of the types of programme
which result from equations (10.33), (10.38), (10.39), sets of
θ-programmes were calculated for elliptic, parabolic and hyper-
bolic orbits. In these calculations, the value of \(k_5\) in equation
(10.39) was taken to be zero, so that \(\delta\) satisfies the equation
\[
\cos \delta = 0,
\]  
(10.40)
assuming \(\beta\) does not vanish. Hence, we have
\[
\delta = \pm \frac{1}{2} \pi.
\]  
(10.41)
The two values of \(\delta\) specified by equation (10.41) do not lead to
essentially different programmes for \(\theta\). Hence we specify, without
loss of generality, \(\delta = -\frac{1}{2} \pi\). (A choice of \(\delta = \frac{1}{2} \pi\) would merely
correspond to a change in the sign of \(\beta\).) With this choice of \(\delta\),
equations (10.33), (10.38) may be written
\[
\lambda_{p_0} = \sin \psi_0,
\]  
(10.42)
\[
\lambda_{q_0} = \left\{A + 2 \cos \psi_0 + \frac{1}{2} e \cos 2 \psi_0\right\} (1 + e \cos \psi_0)^{-1},
\]  
(10.43)
where we have divided both equations by \(\beta\) and substituted for \(r_0\)
from equation (10.29). It is clear that for the case of departure
from circular orbit, for which \(e\) vanishes, equations (10.42),
(10.43) reduce to a solution essentially identical with equations
(8.35), (8.36) already obtained for the circular orbit case.

Families of zero-order solutions for the sail angle \(\theta\) for
the cases of departure from elliptical, parabolic and hyperbolic
orbits respectively are shown in Figures A5-A7 (see Appendix).
Equation (10.43) takes a particularly simple form for large $e$. If $e \gg 4$, equation (10.43) may be written approximately (if $A$ is not large)

$$\lambda_{d_{0}} = \frac{\cos 2\psi_{0}}{\cos \psi_{0}}$$  \hspace{1cm} (10.44)

so that from equations (10.42), (10.44) we have approximately

$$\lambda_{p_{0}} / \lambda_{d_{0}} = \tan 2\psi_{0}.$$  \hspace{1cm} (10.45)

The variation in the direction of the primer vector with respect to $\psi_{0}$ is easily seen from equation (10.45). The connection between the sail angle $\theta$ and the parameter $\psi_{0}$ is illustrated in Figure 10.1. For large $e$, $\psi_{0}$ will vary approximately from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ as the sail progresses along the hyperbola, so that $\theta$, also, will vary monotonically between the approximate values $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

$$\tan \beta = 2 \tan \theta$$

![Figure 10.1. Sail Angle $\theta$ from Parameter $\psi_{0}$ for $e \gg 4$](image)

The case $e \gg 4$ corresponds physically to a high velocity 'fly-by' past a star.

There are also special values of $A$ for which the equation
(10.43) for \( \lambda_{q_0} \) simplifies considerably. We find that for values of \( A \) given by

\[ A = \frac{2 + e^2}{2e}, \]  

(10.46)

the expression for \( \lambda_{q_0} \) (equation (10.43)) factorizes to give

\[ \lambda_{q_0} = (1 + e \cos \psi_0)(e^{-1} + \cos \psi_0)(1 + e \cos \psi_0)^{-1}, \]

so that the equations for the Lagrange variables take the form

\[ \lambda_{p_0} = \sin \psi_0, \]  

(10.47)

\[ \lambda_{q_0} = e^{-1} + \cos \psi_0. \]  

(10.48)

A distinguishing property of the solutions corresponding to the special values of \( A \) given by equation (10.46) is that, for parabolic and hyperbolic orbits, \( \lambda_{q_0} \) does not tend to infinity as \( r_0 \to \infty \), and hence \( \theta \) does not tend to \( \pm \tan^{-1}1/\sqrt{2} \). This property is shown clearly in Figures A.6, A.7, which show families of \( \theta \)-programmes for parabolic \( (e = 1) \) and hyperbolic \( (e = 2) \) orbits. For these values of \( e \), the value of \( A \) given by equation (10.46) is 3/2 in both cases and the contours corresponding to \( A = 1.5 \) in Figures A.6, A.7 both show the distinguishing feature mentioned above.

Lastly, we note that for large \( e \), the special solution given by equations (10.46)-(10.48) yields approximately

\[ \lambda_{p_0}/\lambda_{q_0} = \tan \psi_0, \]  

(10.49)

so that in this case \( \theta \) is determined from \( \psi_0 \) as indicated in Figure 10.1 with the angle \( 2\psi_0 \) replaced by \( \psi_0 \).
11. The Three Dimensional Problem

So far we have only investigated the two dimensional case and have assumed that the orbits, between which transfer is made, are coplanar. We now investigate the general three dimensional situation and obtain a modified zero order solution analogous to that obtained in section 8 for the two dimensional case.

Taking spherical polar coordinates \( r, \theta, \psi \) with origin at the sun to determine the position of the sail, we resolve the vector equation of motion along the axes \( p, q, r \) as shown in Figure 11.1(i). The orientation of the sail is determined by

![Figure 11.1. Position of Sail and Sail Angle](image)

the angles \( \alpha, \beta \) made by the normal to the sail with the \( p \) and \( q \) axes as shown in Figure 11.1 (ii). With this notation the dimensionless equations of motion are

\[
\dot{r} = u, \quad (11.1)
\]

\[
\dot{\theta} = v/r, \quad (11.2)
\]

\[
\dot{\psi} = w/r \sin \theta, \quad (11.3)
\]
\[ \dot{u} = \frac{v^2 + w^2}{r} - \frac{1}{r^2} + \frac{\sigma a}{r^2}, \quad (11.4) \]
\[ \dot{v} = -\frac{uv}{r} + \frac{w^2 \cot \theta}{r} + \frac{\sigma b}{r^2}, \quad (11.5) \]
\[ \dot{w} = -\frac{uw}{r} - \frac{vw \cot \theta}{r} + \frac{\sigma c}{r^2}, \quad (11.6) \]

where

\[ a = \cos^3 \alpha, \quad (11.7) \]
\[ b = \cos^2 \alpha \sin \alpha \cos \beta, \quad (11.8) \]
\[ c = \cos^2 \alpha \sin \alpha \sin \beta, \quad (11.9) \]

and \( \sigma \) is a measure of the sail strength as given by equation \( (4.6) \).

The two dimensional case already studied is obtained by putting \( \theta = \frac{\pi}{2} \) and \( \beta = \frac{\pi}{2} \) in the above equations.

To obtain a zero order solution of the type already obtained for the two dimensional case, we will assume that \( \sigma \) is small and that the departure orbit is circular. We may then write to first order

\[ \theta = \frac{\pi}{2} n + \sigma \theta_1, \quad (11.10) \]
\[ r = 1 + \sigma r_1, \quad (11.11) \]
\[ \psi = t + \sigma \psi_1, \quad (11.12) \]
\[ u = 0 + \sigma u_1, \quad (11.13) \]
\[ v = 0 + \sigma v_1, \quad (11.14) \]
\[ w = 1 + \sigma w_1, \quad (11.15) \]

Substituting from equations \( (11.10)-(11.15) \) into the equations of motion \( (11.1)-(11.6) \) and equating the coefficients of \( \sigma \), we
obtain
\[
\begin{align*}
\dot{r}_1 &= u_1, \\
\dot{\theta}_1 &= v_1, \\
\dot{\psi}_1 &= w_1 - r_1, \\
\dot{u}_1 &= 2w_1 + r_1 + a, \\
\dot{v}_1 &= -\theta_1 + b, \\
\dot{w}_1 &= -u_1 + c.
\end{align*}
\]  

(11.16) \hspace{2cm} (11.17) \hspace{2cm} (11.18) \hspace{2cm} (11.19) \hspace{2cm} (11.20) \hspace{2cm} (11.21)

We may now write down the Lagrange expression,
\[
F = -\lambda_r u_1 - \lambda_{\theta} v_1 + \lambda_{\psi} (r_1 - w_1) - \lambda_u (2w_1 + r_1 + a) \\
+ \lambda_{\psi} (\theta_1 - b) + \lambda_w (u_1 - c),
\]
and so obtain the Euler-Lagrange equations
\[
\begin{align*}
\dot{\lambda}_r &= \lambda_{\psi} - \lambda_u, \\
\dot{\lambda}_{\theta} &= \lambda_v, \\
\dot{\lambda}_{\psi} &= 0, \\
\dot{\lambda}_u &= -\lambda_r + \lambda_w, \\
\dot{\lambda}_v &= -\lambda_{\theta}, \\
\dot{\lambda}_w &= -\lambda_{\psi} - 2\lambda_u.
\end{align*}
\]  

(11.23) \hspace{2cm} (11.24) \hspace{2cm} (11.25) \hspace{2cm} (11.26) \hspace{2cm} (11.27) \hspace{2cm} (11.28)

If the end value $\psi_f$ is not specified for a payoff function $J$ not involving $\psi_f$, we have the boundary condition
\[
\lambda_{\psi} = 0 \quad \text{at} \quad t = t_f,
\]
so that from equations (11.25), (11.29) we see that
\[ \lambda_v = 0 \quad \text{for all } t. \quad (11.30) \]

Now, using result (11.30), the characteristic equations (11.23)-(11.28) are equivalent to

\[ \ddot{\lambda}_v + \lambda_v = 0, \quad (11.31) \]
\[ \lambda_\theta = -\dot{\lambda}_v, \quad (11.32) \]
\[ \ddot{\lambda}_u + \lambda_u = 0, \quad (11.33) \]
\[ \dot{\lambda}_w = -2\lambda_u, \quad (11.34) \]
\[ \lambda_r = \lambda_w - \dot{\lambda}_u, \quad (11.35) \]

from which equations it is easy to solve for the Lagrange multipliers. In particular we have

\[ \lambda_u = A_2 \cos(t + \epsilon_2), \quad (11.36) \]
\[ \lambda_v = A_1 \cos(t + \epsilon_1), \quad (11.37) \]
\[ \lambda_w = B_2 - 2A_2 \sin(t + 2). \quad (11.38) \]

To the equations (11.23)-(11.28), we must add the equations

\[ 0 = \frac{\partial F}{\partial \alpha} = \frac{\partial F}{\partial \beta}, \]

which are respectively

\[ \frac{\lambda_u}{\lambda_v \cos \beta + \lambda_w \sin \beta} = \frac{3 \cos^2 \alpha - 2}{3 \sin \alpha \cos \alpha}, \quad (11.39) \]
\[ \frac{\lambda_w}{\lambda_v} = \tan \beta. \quad (11.40) \]

But, using equation (11.40), we find that
\[ \lambda_v \cos \beta + \lambda_w \sin \beta = \lambda_v \cos \beta \]

so that equation (11.39) may be written more simply as

\[ \frac{\lambda_u \cos \beta}{\lambda_v} = \frac{3\cos^2 \alpha - 2}{3 \sin \alpha \cos \alpha} \]  \hspace{1cm} (11.41)

Thus, the sail angles \( \alpha, \beta \) are programmed according to equations (11.40), (11.41), where the Lagrange multipliers \( \lambda_u, \lambda_v, \lambda_w \) are given by equations (11.36)-(11.38). Equations (11.40), (11.41) are valid in the general case, whereas equations (11.36)-(11.38) provide an approximate zero order solution for the case of a weak sail departing from circular orbit.

Eliminating \( \lambda_v \) between equations (11.40), (11.41) we obtain

\[ \frac{\lambda_u \sin \beta}{\lambda_w} = \frac{3\cos^2 \alpha - 2}{3 \sin \alpha \cos \alpha} \] \hspace{1cm} (11.42)

In particular, if \( \beta = \frac{1}{2} \pi \), the plane of motion remains unchanged and the equation determining \( \alpha \) is that obtained by setting \( \sin \beta = 1 \) in equation (11.42). In this case, the \( \alpha \)-programme determined by equations (11.36), (11.38), (11.42) is identical with that obtained in section 8 for the two dimensional case, (equations(8.35), (8.36)).

The set of all possible sail orientations corresponds to the limits

\[ -\frac{1}{2} \pi \leq \alpha \leq \frac{1}{2} \pi, \quad 0 \leq \beta \leq \pi \] \hspace{1cm} (11.43)

on the sail angles \( \alpha, \beta \). Within these limits, equation (11.40) yields a unique solution for \( \beta \), whereas equation (11.42) yields
two solutions for $\alpha$. To determine the correct value to take, we again apply the Weierstrass condition.

The function to be maximized according to this condition is

$$F(\alpha, \beta) = \cos^2 \alpha [\lambda_u \cos \alpha + (\lambda_v \cos \beta + \lambda_w \sin \beta) \sin \alpha]. \quad (11.44)$$

Solving this maximization problem within the limits defined by the inequalities (11.43), we obtain the following results:

If $\lambda_w \neq 0$, the angle $\beta$ is uniquely defined by the condition

$$\lambda_w : \lambda_v = \sin \beta : \cos \beta. \quad (11.45)$$

The angle $\alpha$ satisfies the condition

$$\lambda_u \sin \beta : \lambda_w = (3 \cos^2 \alpha - 2) : 3 \sin \alpha \cos \alpha, \quad (11.46)$$

where $\alpha$ is positive or negative according as $\lambda_w$ is positive or negative. If $\lambda_w = 0$, we may take $\beta = 0$. Then $\alpha$ is defined by the condition

$$\lambda_u : \lambda_v = (3 \cos^2 \alpha - 2) : 3 \sin \alpha \cos \alpha, \quad (11.47)$$

where $\alpha$ is positive or negative according as $\lambda_v$ is positive or negative. If both $\lambda_w = 0$ and $\lambda_v = 0$, then $\alpha = \pm \frac{1}{2} \pi$ if $\lambda_u$ is negative and $\alpha = 0$ if $\lambda_u$ is positive.

Conditions (11.46), (11.47) can be combined into one condition by defining a new parameter $\lambda$ by the equation

$$\lambda^2 = \lambda_w^2 + \lambda_v^2, \quad (11.48)$$
where \( \lambda \) has the same sign as \( \lambda_w \), unless \( \lambda_w = 0 \), in which case
\( \lambda \) has the same sign as \( \lambda_v \). We then find that conditions (11.46), (11.47) are equivalent to the single condition

\[
\lambda_u: \lambda = (3\cos^2\alpha - 2) \cdot 3\sin\alpha \cos\alpha, \tag{11.49}
\]

where \( \alpha \) is positive or negative according as \( \lambda \) is positive or negative. In the same way that equation (6.23) was obtained from equation (6.20), we may now obtain from condition (11.49) the equation

\[
\alpha = \tan^{-1} \left\{ \frac{2\lambda_u \pm \sqrt{9\lambda^2 + 8\lambda^2}}{4\lambda} \right\}. \tag{11.50}
\]

To gain some idea of the types of orientation programmes defined by equations (11.36)-(11.38) we can, without loss of generality, specify the values of the constants \( A_2, \epsilon_2 \). In particular, taking \( A_2 = -1, \epsilon_2 = \pi/2 \), equations (11.36)-(11.38) may be written in the form

\[
\lambda_u = \sin t, \tag{11.51}
\]
\[
\lambda_v = A \sin (t + \epsilon), \tag{11.52}
\]
\[
\lambda_w = B + 2\cos t, \tag{11.53}
\]

where \( B, A \) and \( \epsilon \) are arbitrary constants. Taking rectangular cartesian axes \( O\lambda_u, \lambda_v, \lambda_w \), equations (11.51)-(11.53) define a space curve, which we may term the primer locus. The sail angles \( \alpha, \beta \) may be determined from the three-dimensional primer locus diagram in a fairly simple manner, as illustrated in Figure 11.2. It is clear from equations (11.51)-(11.53) that
the primer locus is a closed curve contained within a rectangular block with centre \((0,0,0)\) and dimensions 2, 2, \(|A|, 4\) units respectively in the directions \(O\lambda_u, O\lambda_v, O\lambda_w\) respectively.

Certain special cases of equations (11.51)-(11.53) may be investigated further analytically. Suppose, first, that the constants \(A\) and \(B\) are much greater than unity. We may then neglect zero order terms and, altering the time origin, the behaviour of the \(\alpha\) and \(\beta\) programmes may be described approximately by the system

\[
\begin{align*}
\lambda_u &= 0 , & (11.54) \\
\lambda_v &= A \sin t , & (11.55) \\
\lambda_w &= B . & (11.56)
\end{align*}
\]

Then, from equations (11.50), (11.54)-(11.56), \(\alpha\) is given by
\[ \alpha = \pm \tan^{-1} \left( \frac{1}{\sqrt{2}} \right), \]  

(11.57)

where \( \alpha \) is positive or negative according as \( B \) is positive or negative. Also, from equation (11.45), \( \beta \) is given by

\[ \tan \beta = C \csc \theta, \]  

(11.58)

where

\[ C = \frac{B}{A}. \]  

(11.59)

Equations (11.57), (11.58) imply that the sail orientation is such as to maximize the thrust delivered along a direction \( \lambda \) lying in the Oqr plane, (the plane perpendicular to the direction of radiation), where the direction \( \lambda \) is determined by the angle \( \beta \) as in Figure 11.2.
Some numerical results were obtained for the problem of solar sail transfer between circular orbits using the exact equations. The method used for obtaining convergence will first be explained.

Let $h$ be the dimensionless angular momentum defined by the equation

$$h = \rho q.$$  \hfill (12.1)

We define a new variable $s$ by the equation

$$s = \frac{1}{2}(h^2 - \rho).$$  \hfill (12.2)

Then, for departure from circular orbit and arrival at circular orbit equations (5.8), (5.9) and equations (5.16), (5.17), (5.19) apply respectively so that we have the boundary conditions

$$s_b = p_b = s_f = p_f = 0,$$  \hfill (12.3)

where the subscripts $b$ and $f$ indicate values at $t = 0$ and $t = t_f$ respectively.

Taking rectangular axes $O_p, O_s$ we consider how the point $P$ with coordinates $p = p(\eta)$, $s = s(\eta)$ varies with respect to $\eta$. As $\eta$ varies a contour will be generated by $P$ in the $ps$-plane which, because of the boundary conditions (12.3) will take the form of a closed curve starting from the origin at $\eta = 0$ and ending at the origin at $\eta = \eta_f$.

To obtain a solution satisfying the boundary conditions, a
particular value of $\theta_b$ was chosen, where $\theta_b$ is the initial value of the sail angle. The initial values of $\lambda_p, \lambda_q$ were then taken to be defined by equations (7.20), (7.21). To proceed with the numerical integration of the governing differential equations, it was then only necessary to guess the initial value of the single variable $\lambda$, where $\lambda \equiv \dot{\lambda}_b$. The details of the integration process are set out in the appendix.

By varying $\lambda_b$ and integrating the governing equations a series of contours may be obtained in the $ps$-plane. The technique of convergence on to the correct value of $\lambda_b$ is illustrated in Figure 12.1. Contours corresponding to $\lambda_b = c_1$, $\lambda_b = c_2$ have been found straddling the origin. Another initial value, $\lambda_b = c_3$, is then chosen such that $c_3$ lies between $c_1$ and $c_2$, and a contour lying between the previous two contours is obtained. In this way a sequence of initial values $c_1, c_2, c_3, c_4, \ldots$ may be generated which will converge to the initial value required to make the contour pass through the origin and hence satisfy the boundary conditions (12.3).

![Figure 12.1. ps - Contours](image-url)
When a contour has been found which, to within some assigned degree of accuracy, may be taken as passing through the origin, the final boundary values of the variables \( \eta, \tau, \psi, \rho \) for the solution found will be given by the values of these variables corresponding to the point of closest approach to the origin of the \( ps \)-contour. The final boundary value of \( \rho \) was not taken to be preassigned in order that the simple method of obtaining specimen solutions outlined above could be adopted. Once the value of \( \rho_f \) appropriate to a given \( ps \)-contour passing through the origin has been calculated, we may say that the solution obtained constitutes a solution to the problem of minimal time transfer between circular orbits of radii \( \rho_b = 1 \) and \( \rho = \rho_f \) respectively, \( \rho_f \) being preassigned.

Some optimal sail-angle programmes obtained by the method described above are illustrated in Figure A.2, while the values of some orbital parameters at different stages of the flight are listed in Tables A.1 - A.7.

A particular property of the \( ps \)-contours which holds at \( \eta = 0 \) may be noted here. Differentiating equation (12.2) and using the results (8.2), (8.5), (8.9), we find that initially

\[
\begin{align*}
\ddot{\psi}_b &= \sigma_a b, \\
\ddot{\phi}_b &= \sigma_b b
\end{align*}
\]

(12.4) (12.5)

where we have inserted the boundary conditions (8.10). (In equations (12.4), (12.5), dots denote derivatives with respect to \( \eta \).) Thus,
from equations (12.4), (12.5), we have

\[ \frac{ds}{d\eta} \eta=0 = \frac{a}{b} \]

\[ = \frac{b}{a} \]

\[ = \tan \theta_b . \]  

(12.6)

Thus the angle made initially by the ps-contour with the Op-axis is equal to the initial solar sail angle, \( \theta_b \).

A number of solutions satisfying the boundary conditions were obtained with a choice of \( \sigma = 0.05 \) for the sail strength. Some of the sail-angle programmes obtained from these solutions are shown in Figure A.2. Figure A.3 shows the values of \( \lambda_b \), as a function of \( \theta_b \), which caused the ps-contour to pass through the origin and hence satisfied the boundary conditions. Programmes satisfying the boundary conditions were obtained only within a restricted range of values of \( \theta_b \) (approximately \( 0.26 < \theta_b < 0.53 \)). When it was attempted to apply the ps-contour convergence technique for values of \( \theta_b \) outside this range (for \( \sigma = 0.05 \)), it was found that the ps-contour always passed the origin on the same side for all the values of \( \lambda_b \) considered.

Some solutions for values of \( \sigma \) other than 0.05 were also obtained and are listed in Tables A.5, A.6. In particular, solutions for very small \( \sigma \) (\( \sigma = 0.01, 0.005 \)) were obtained for which \( \theta \) was almost constant. Such solutions are predicted by the approximate theory presented in section 9.
Appendix

1. Numerical Solutions from the Exact Equations

The state variables integrated with respect to $\eta$ in trial solutions were $r, p$ and $h$, (where we have used $r$ instead of $\rho$ as in section 8). These variables were integrated according to the equations

$$\dot{r} = pr^2, \quad (A.1)$$
$$\dot{p} = \frac{h^2}{r} - 1 + \sigma a, \quad (A.2)$$
$$\dot{h} = \sigma rb, \quad (A.3)$$

which are equations (8.2), (8.9), (8.5) respectively. Equations (8.6), (8.7) may be written

$$\dot{\lambda}_p = \lambda; \quad (A.4)$$
$$\dot{\lambda} = -h^2\lambda_p + \sigma r(\lambda_q b - 2kr^2); \quad (A.5)$$
$$\dot{\lambda}_q = pr\lambda_q - 2h\lambda_p; \quad (A.6)$$

where we have introduced the parameter $\lambda$, defined by equation (A.4). The sail-angle $\theta$ is required in order to calculate the functions $a$ and $b$ in equations (A.2), (A.3), (A.5). This was calculated, using the current values of $\lambda_p, \lambda_q$, according to equation (7.13). The initial values of $r, p$ and $h$ are given by equations (5.8), (5.9). An initial value of $\theta$ was chosen and the corresponding initial values of $\lambda_p, \lambda_q$ were then calculated according to equations (7.20), (7.21). The value of $k$ in equation (A.5) could then be calculated from the result (6.38).
The only information still required to start the integration process was the initial value of $\lambda$. This value was guessed and successive corrections were made to the value chosen in accordance with the observed behaviour of the ps-contour. The correction technique used has been described in section 12. When convergence had been obtained, provision was made to integrate the extra variables $\tau$ and $\psi$ in the final run according to the equations

$$
\dot{t} = r^2, \quad (A.7)
$$

$$
\dot{\psi} = h, \quad (A.8)
$$

which derive from equations (8.1), (8.3).

In order to apply the convergence technique described in section 12, one must be able to locate accurately the point of closest approach of the ps-contour to the origin. In order to do this, an automatic end point detection test was incorporated into the computer programme. Once an iteration cycle of numerical integration had been completed, the values of the coordinates of the point $P(p(\eta), s(\eta))$ on the ps-contour were used to calculate the current distance $d_3$ of the point $P$ from the origin in the ps-plane. The distances $d_2, d_1$ of the previous two points obtained from the origin were already stored in computer memory.
The situation which obtains in the neighbourhood of the point of closest approach is shown in Figure A.1. When the quantities \( d_3 - d_2 \) and \( d_1 - d_2 \) are both positive, it is clear that \( \eta_1 < \eta \eta_3 \), where \( \eta \eta \) is the value of \( \eta \) at the point of closest approach. The satisfying of the inequalities \( d_3 > d_2 \), \( d_1 > d_2 \) constitutes a test for the presence of the end point sought. Once the presence of the end point had been detected by the programme, control was switched to a subroutine which estimated \( \eta \), by fitting a second order polynomial through the points \( (\eta_1, d_1), (\eta_2, d_2), (\eta_3, d_3) \) and calculating the point on this polynomial at which \( d \) vanished. Lastly, the interval of integration \( \delta \eta \) was set equal to \( \eta - \eta_3 \) (a negative quantity) and one more integration step was carried out. This yielded the values of the state variables and of the parameters \( \lambda_p \), \( \lambda, \lambda_q \) at the end point.

The numerical integration technique used was the fourth order Runge-Kutta method described on pp. 110-120 of [7]. The integration interval \( \delta \eta \) could be adjusted (doubled, halved or unchanged) upon the completion of any iteration cycle. This
meant, in particular, that $\delta \eta$ could be reduced as the end-
point was approached, so that $\bar{\eta}$ could be estimated more
accurately. It was necessary to do this because the process
used to calculate $\bar{\eta}$ was only second order, whereas the
numerical integration method used was a fourth order process.
The convergence process was continued until a value of $\lambda_b$ was
obtained for which the $ps$-contour passed by the origin at a
distance of less than 0.001.

The computations involved were carried out on the IBM-1620 computer belonging to the University of Canterbury at
Christchurch, New Zealand. The programming language used was
the FORTRAN II system.

The programme used to calculate the results obtained
will now be described. It should be noted that the programme,
as it stands, is not the most efficient programme possible for
carrying out the tasks assigned to it. In particular, there
are a number of redundant variable names (such as THO, FCTRO)
occuring in the programme. The reason for this is that the
final programme was the result of successive adaptations of
previous programmes which carried out slightly different tasks
and in which the variable names, which are redundant in the
final programme, were, in fact, necessary. It was not
immediately realized that the changes made in previous pro-
grames had created redundant variables and the redundant
calculation time and storage space involved was so insignificant
that an entire retranslation of a new programme was not justified.

The significance of the variable names occurring in the programme will now be explained.

\[ Y(1) = r, \]
\[ Y(2) = p, \]
\[ Y(3) = h, \]
\[ Y(4) = \lambda_p, \]
\[ Y(5) = \lambda, \]
\[ Y(6) = \lambda_q, \]

\[ YQ(1) - YQ(6) \] are parameters occurring in the Runge-Kutta process,

\[ YK(1) - YK(6) \] are the right hand members of equations (A.1) - (A.6) respectively,

\[ A(1) - A(4), B(1) - B(4), C(1) - C(4) \] are constant coefficients occurring in the Runge-Kutta process,

\[ X(1) = \tau, \]
\[ X(2) = \psi, \]

\[ XK(1), XK(2) \] are the right hand members of equations (A.7), (A.8) respectively,

\[ XQ(1), XQ(2) \] are parameters occurring in the Runge-Kutta process,

\[ AA = a \] and is also used as a dummy variable,

\[ BA = b \] and is also used as a dummy variable,

\[ CA \] is a dummy variable,

\[ TH = \theta, \] the sail angle,

\[ THO = \theta_0, \] the initial sail angle,
SIG = $\sigma$, the sail strength,

DY = $\delta \eta$, the integration step interval,

YO = $\lambda_p$,

Y6 = $\lambda_q$,

FCTRO = $k$; calculated in the initialization process,

FCTR = $k$,

D1 = $a_1$,

D2 = $a_2$,

D3 = $d_3$; the current distance from the origin in the $ps$-plane,

USSR is a parameter; USSR = 0, continue testing for end point; USSR = 1, end point has been reached. Print final values of $p$, $s$ and $d_3$ and read new data.

L is a count parameter. L is increased by one after every iteration. When L = 5 the current parameter values are automatically printed. This enables one to keep a periodic check on the progress of the solution.

YO = $\eta$; the independent variable,

J is a subscript parameter varying from 1 to 4 and signifying the stage reached in the four cycle Runge-Kutta integration procedure.

I is a subscript parameter used in DQ-statements.

P = $p$,

S = $s$,

PRMTRS is the name of a subroutine called to calculate the parameters $\theta$, $a$ and $b$ from given values of $\lambda_p$, $\lambda_q$.

The programme itself will now be listed followed by a block diagram indicating the essential steps in the calculation.
DIMENSION BY(6), YQ(6), YK(6), A(4), B(4), C(4), X(2), XK(2), XQ(2)

COMMON BAA, BA, TH

AA = SQRTF (0.5)
A(1) = 0.5
A(2) = 1. - AA
A(3) = 1. + AA
A(4) = 1. / 6.
B(1) = 2.
B(2) = 1.
B(3) = 1.
B(4) = 2.
C(1) = 0.5
C(2) = A(2)
C(3) = A(3)
C(4) = 0.5

READ1, THO, Y(5), SIG, DY

FORMAT (F7.4, E14.8, F6.3, F5.2)

AA = SINF (THO)
BA = AA**2
CA = 1. / (1. + BA)
Y4 = CA*(1. - 3.*BA)
BA = COSF (THO)
Y6 = 3.*CA*AA*BA
CA = BA**2
FCTRO = CA*(BA*Y4 + AA*Y6)
bb102b PRINT 103,Y4,Y6,FCTR
bb103b FORMAT(3F6.3)
bb207b TH = TH0
FCTR = FCTR
Y(4) = Y4
Y(6) = Y6
bb205b PRINT 206,TH,Y(5),SIG,DY
bb206b FORMAT(4B1THB,F7.4,5HbbY5b,E14.8,6HbbSIGb,F6.3,5HbbDYa,F5.2)

D2 = -2.
D3 = -1.
USSR = 0.
L = 0
Y0 = 0.
DQ2J = 1,6
bbbb2b YQ(J) = 0.
Y(1) = 1.
Y(2) = 0.
Y(3) = 1.
DQ3I = 1,2
X(I) = 0.
bbbb3b XQ(I) = 0.
bbb15b L = L + 1
J = 1
bbb14b CALLBPRMTRS(Y(4),Y(6))
YK(1) = Y(1)**2*Y(2)
YK(2) = Y(3)*2/Y(1)-1.*AA*SIG
YK(3) = BA*SIG*Y(1)
YK(4) = Y(5)
YK(5) = -Y(3)**2*Y(4)+SIG*Y(1)*(BA*Y(6)-2.*FCTR*Y(1)**2)
YK(6) = Y(1)*Y(2)*Y(6)-2.*Y(3)*Y(4)

IF(SENSEbSWITCHb2)19,18

bb19b XK(1) = Y(1)**2
XK(2) = Y(3)

bb21b IF(J-4)23,30,30

bb30b YO = YO+DY
D1 = D2
D2 = D3
P = Y(2)
S = 0.5*(Y(3)**2-Y(1))
AA = P**2+S**2
D3 = SQRTF(AA)
IF(SENSEbSWITCHb1)40,34

L = 0
PRINT 41,P,S,TH,Y(5),Y0
FORMAT(3F7.4,2F7.3)
IF(SENSEbSWITCHb2)44,46
PRINT 45,X(1),X(2),Y(1)
FORMAT(3F8.3)
IF(SENSEbSWITCHb3)47,94

DY = 0.5*DY
PAUSE
IF(SENSEbSWITCHb3)48,94

DY = 4.0*DY

IF(USSR)95,95,111
PRINT 112,P,S,D3
FORMAT(3E14.8)
GOTO1204

IF(D1-D2)15,93,93
IF(D3-D2)15,92,92
D1 = D2-D1
D3 = D3-D2
D3 = D3-D1
D1 = D1+0.5*D3
DY = -DY*(1.0*D1/D3)
USSR = 1.
The mainline programme calls one subroutine entitled PRMTRS. Given $\lambda_p$ and $\lambda_q$ from the mainline programme, this subroutine calculates $a$, $b$, and $\theta$. The programme first performs a test for small $\theta$. If $\theta$ is small, $\left(\frac{\lambda_q}{3\lambda_p}\right)^2 < 0.01$, then the programme calculates $\theta$ according to equation (7.15). If $\theta$ is not small, it is calculated according to equation (7.13). The programme then calculates $a = \cos^3\theta$ and $b = \cos^2\theta \sin\theta$ and returns the values of $\theta$, $a$ and $b$ to the mainline programme.

```
SUBROUTINE PRMTRS(Y4P,Y6P)
    COMMON AA, BA, TH
    AA = Y4P**2 - 0.001
    IF(AA) GOTO 51, 50, 50

    IF(AA) GOTO 52, 52, 51
    F = SQRTF(9.*Y4P**2 + 8.*Y6P**2)
    TH = ATANF(0.25*(F - 3.*Y4P)/Y6P)
    GOTO 53

    F = SQRTF(9.*Y4P**2 + 8.*Y6P**2)
    TH = ATANF(0.25*(F - 3.*Y4P)/Y6P)
    GOTO 53

    AA = 2.*Y6P/(3.*Y4P)
    BA = AA**2
    CA = 0.25*AA*(2.*BA + BA**2)
    TH = ATANF(CA)

    CA = COSF(TH)
    BA = SINF(TH)
```
FIGURE A.1. P.S. CONTOUR ENDCTN PROGRAMME

BLOCK DIAGRAM.
**Fig. A.2.** Some $\Theta$-Programmes from the Exact Equations

$\sigma = 0.05$
Fig A.3. Variation of $\lambda_0$ with $\Theta_0$.
2. General Conic Case

The equations used to obtain sample \( \theta \)-programmes for the case of departure from a general conic orbit using a weak sail were

\[
\lambda_{\psi_0} = \sin \psi_0 \tag{A.9}
\]

\[
\lambda_{\varphi_0} = \left[ A + 2\cos \psi_0 + \frac{1}{2}e \cos 2\psi_0 \right] \left( 1 + e \cos \psi_0 \right)^{-1} \tag{A.10}
\]

which are equations (10.42), (10.43) respectively. Three values of \( e \), the eccentricity of the departure orbit, were considered, namely, \( e = \frac{1}{2}, 1, 2 \) corresponding to elliptic, parabolic and hyperbolic orbits respectively. In each case, solutions were computed for \( A = -3.0, -2.5, -2.0, \ldots, +3.0 \). Values of \( \theta \) were evaluated from equations (7.13), (A.9), (A.10), taking \( \psi_0 = 0.00, 0.25, 0.50, \ldots, 6.75 \), thereby covering an interval \( 2\pi \) in \( \psi_0 \).

The variable names used in the Fortran II programme which carried out the above calculations were as follows:

- \( E = e \), the eccentricity,
- \( A = A \), the constant occurring in equation (A.10),
PHI = \psi_0 , the real anomaly,

I is a count variable and subscript which collects values of \theta in blocks of seven.
P1, P2, P3 are variable names used for storing intermediate results.

\[ X_P = \lambda_{po} , \]
\[ X_Q = \lambda_{qo} , \]
\[ \text{THETA} = \theta . \]

The programme itself will now be listed.

```
Cbbbbb  GENERALbCONICbCASE
        DIMENSIONbTHETA(7)
        E = 0.25
-4-1b  E = 2.*E
        PRINT 2,E
-2b    FORMAT(F5.2)
        A = -3.5
-3b    A = A + 0.5
        PRINT2,A
bbbbb  PHI = -0.25
4b     I = 0
5b     I = I + 1
        PHI = PHI + 0.25
        P1 = SINF(PHI)
```
P2 = CQSF(Phi)
P3 = 1.29*E*P2
IF(P3)12,11,12
11b THETA(I) = 9.999
12b XP = P1
   XQ = (A+P2*(2.8+3.5*P2)-0.5*E)/P3
   IF(XQ)15,16,15
15b P1 = SQRTF(9.*XP**2+8.*XQ**2)
P2 = 0.25*(P1-3.*XP)/XQ
   THETA(I) = ATANF(P2)
13b IF(I-7)5,6,6
6b PRINT 7,THETA
7b FORMAT(7F6.3)
   IF(Phi-6.)4,8,8
8b IF(A-3.)3,9,9
9b PAUSE
10b GQbTQb1
END
Ex. A.5. Zero Order Solutions

$e = 0.5$
Fig. A.6. Zero Order Solutions

\[ e = 1 \]
Ex. A.7. Zero Order Solutions

$e = 2$
In the above programme, certain tests are carried out before calculating $\theta$ according to equation (7.13). If $\lambda_q = 0$, $\lambda_p \neq 0$, $\theta$ is determined directly according to conditions (7.10). If $\lambda_q = \lambda_p = 0$, or if $r_0$ is infinite, a code value, 9.999, is printed for $\theta$ to indicate the presence of a singular point.

A separate programme, very similar to the one given above, was used to compute sets of zero order solutions of the $\theta$-programme for the case of departure from circular orbit. For this purpose, equations (8.38), (8.39) were used with $\epsilon = 0$. The results obtained for the cases $\epsilon = 0, 0.5, 1, 2$ are plotted respectively in Figures A.4-A.7 and tabulated in Tables A.8-A.11. It will be observed that the $\theta$-programmes obtained from the exact equations for $\sigma = 0.05$ resemble quite closely the forms of $\theta$-programme obtained from the zero order equations.

3. The Three Dimensional Case

The Lagrange variables for the three dimensional case are given from zero order terms by the equations

$$\lambda_u = \sin t,$$

$$\lambda_v = A \sin(t + \epsilon),$$

$$\lambda_w = B + 2\cos t,$$

which are equations (11.51), (11.52), (11.53) respectively. The sail orientation angles, $\alpha$ and $\beta$, are then given by equations
(11.40), (11.45) respectively, where \( \lambda \) is defined by equation (11.48). A Fortran II programme was written to calculate sample \( \alpha \) and \( \beta \) joint programmes. The programme evaluated \( \alpha \) and \( \beta \) at intervals of 0.25 in \( t \) for values of \( A \) equal to \( \frac{1}{4}, 2 \) and \( 8 \), values of \( B \) equal to \( -3, -2, ..., +3 \), and values of \( \varepsilon = 0, \frac{1}{2}\pi, \frac{1}{4}\pi \). The variable names used will now be explained and the programme itself listed.

\[ S(1)-S(3), C(1)-C(3): \text{subscripted coefficients enabling } \lambda_v \text{ to be calculated by the same arithmetic statement for each of the three different values of } \varepsilon \text{ considered.} \]

\[ A = A. \]

\[ K: \text{a subscript; } K = 1 \text{ corresponds to } \varepsilon = 0, K = 2 \text{ to } \varepsilon = \frac{1}{2}\pi, K = 3 \text{ to } \varepsilon = \frac{1}{4}\pi. \]

\[ B = B. \]

\[ \text{BETA} = t. \]

\[ J: \text{a count parameter collecting values of } \alpha \text{ and } \beta \text{ for printing in batches of } 4. \]

\[ XU = \lambda_u. \]

\[ XW = \lambda_w, \text{also used as a dummy.} \]

\[ XV = \lambda_v. \]

\[ X = \lambda. \]

\[ \text{BETA} = \beta. \]

\[ \text{ALPHA} = \alpha. \]

\[ \text{Y: a dummy storage variable.} \]

\[ \text{Cbbbbb THREE\&DIMENSIONAL\&CASE} \]

\[ \text{DIMENSION S(3), C(3), ALPHA(4), BETA(4)} \]
\( S(1) = 1. \)
\( S(2) = \text{SQRTP}(0.5) \)
\( S(3) = 0. \)
\( C(1) = 0. \)
\( C(2) = S(2) \)
\( C(3) = 1. \)
\( A = 1/8. \)

---

1b \( A = 4.65^*A \)
---

PRINT 2, A
---

2b FORMAT(3HbAb,Fl.1)

\( X = 0 \)

3b \( K = K+1 \)

PRINT 4, K

4b FORMAT(3HbKb,I2)

\( B = -4. \)

5b \( B = B+1. \)

PRINT 6, B

6b FORMAT(3HbBb,Fl.1)

\( \text{ETA} = -0.25 \)

8b \( J = 0 \)

7b \( J = J+1 \)

\( \text{ETA} = \text{ETA}+0.25 \)

\( \text{XU} = \text{SINF(ETA)} \)

\( \text{XW} = \text{COSF(ETA)} \)

\( \text{XV} = A^* (S(K)^* \text{XU} + C(K)^* \text{XW}) \)

\( \text{XW} = B+2.5*\text{XW} \)
\[ X = \sqrt{R^2 + V^2} \]

15b \[ IF(XW)16,15,14 \]

17b \[ BETA(J) = 9.999 \]

18b \[ ALPHA(J) = 9.999 \]

19b \[ ALPHA(J) = 0. \]

20b \[ ALPHA(J) = 1.571 \]

16b \[ X = -X \]

14b \[ Y = \sqrt{9.000R^2 + 8.000X^2} \]

\[ Y = 0.25 \cdot (Y - 3.000X)_2 / X \]

\[ ALPHA(J) = ATANF(Y) \]

22b \[ BETA(J) = 1.571 \]

23b \[ BETA(J) = ATANF(XW/XV) \]

9b \[ PRINT 10, ALPHA, BETA \]

10b \[ FORMAT(8F7.3) \]

11b \[ IF(V-6.5)8,8,11 \]

12b \[ IF(B-3.)5,12,12 \]

13b \[ PAUSE \]
After $\lambda_u, \lambda_v$ and $\lambda_w$ have been calculated in the programme, certain tests were made to direct the further course of the calculation. If $\lambda_w, \lambda_v$ did not both vanish, $\alpha$ was calculated by the second statement after statement 14, and $\beta$ by statement 23, (or, if $\lambda_v$ vanished, by statement 22). If $\lambda_w, \lambda_v$ both vanished, $\beta$ was set equal to a code number, 9.999 by statement 17. This signified an indeterminate expression for $\beta$. $\lambda_u$ was then tested for sign; if $\lambda_u > 0$, then statement 19 set $\alpha = 0$, if $\lambda_u < 0$ then statement 20 set $\alpha = \frac{\beta}{\pi}$, while if $\lambda_u = 0$ statement 18 set $\alpha = 9.999$, implying that the expression for $\alpha$ was indeterminate.

The results obtained from the above programme are presented in Figs. A.8 - A.13. For each value of $A$, $\beta$-programmes are shown for $\epsilon = 0, \frac{\pi}{4}, \frac{\pi}{2}$, while $\alpha$-programmes are shown only for $\epsilon = \frac{\pi}{4}$. This is because the $\alpha$-programmes, unlike the $\beta$-programmes, display little significant variation with respect to changes in $\epsilon$. 
Figure A.8. Three Dimensional Case
Zero Order Solution

A = 0.5; \theta = \pi/2

B = 2

\eta = 0.0, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0
FIG. 89. THREE DIMENSIONAL CASE

ZERO ORDER SOLUTION

A = 0.5, \( \varepsilon = 0.0 \)

A = 0.5, \( \varepsilon = \frac{1}{4 \pi} \)
FIG.A10. THREE DIMENSIONAL CASE
ZERO ORDER SOLUTION

A = 2.0; \pi = \frac{1}{2} \pi
FIG. AII. THREE DIMENSIONAL CASE

ZERO ORDER SOLUTION

$A = 2.0; \varepsilon = 0$

$A = 2.0; \varepsilon = \frac{1}{4} \pi$
FIG. A12, THREE DIMENSIONAL CASE
ZERO ORDER SOLUTION

\( A = 8.0 ; \xi = \frac{1}{2} \pi \)
FIG. A13. THREE DIMENSIONAL CASE

ZERO ORDER SOLUTION

A = 8.0 ; \( \varepsilon = 0 \)

A = 8.0 ; \( \varepsilon = \frac{1}{4} \pi \)

B = 3

B = 2
<table>
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<tr>
<th>$\eta$</th>
<th>$\theta$</th>
<th>$\tau$</th>
<th>$\psi$</th>
<th>$r$</th>
<th>$p \times 10^4$</th>
<th>$q \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>7272</td>
<td>0.200</td>
<td>0.200</td>
<td>1.000</td>
<td>56</td>
<td>35</td>
</tr>
<tr>
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<td>8203</td>
<td>0.400</td>
<td>0.401</td>
<td>1.002</td>
<td>112</td>
<td>62</td>
</tr>
<tr>
<td>0.6</td>
<td>8648</td>
<td>0.602</td>
<td>0.603</td>
<td>1.005</td>
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<td>81</td>
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<td>0.805</td>
<td>1.009</td>
<td>231</td>
<td>93</td>
</tr>
<tr>
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<td>1.008</td>
<td>1.014</td>
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<td>99</td>
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<td>1.212</td>
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<td>1.417</td>
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</tr>
<tr>
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<td>1.622</td>
<td>1.039</td>
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</tr>
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<td>1.827</td>
<td>1.051</td>
<td>562</td>
<td>59</td>
</tr>
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<td>2.034</td>
<td>1.064</td>
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<td>32</td>
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<td>00</td>
</tr>
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<td>2.450</td>
<td>1.095</td>
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</tr>
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<td>2.793</td>
<td>2.659</td>
<td>1.113</td>
<td>730</td>
<td>-82</td>
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<tr>
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<td>6420</td>
<td>3.045</td>
<td>2.869</td>
<td>1.131</td>
<td>743</td>
<td>-130</td>
</tr>
<tr>
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<td>3.079</td>
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