Representation Growth of Finitely Generated Torsion-Free Nilpotent Groups: Methods and Examples

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Abstract

This thesis concerns representation growth of finitely generated torsion-free nilpotent groups. This involves counting equivalence classes of irreducible representations and embedding this counting into a zeta function. We call this the representation zeta function.

We use a new, constructive method to calculate the representation zeta functions of two families of groups, namely the Heisenberg group over rings of quadratic integers and the maximal class groups. The advantage of this method is that it is able to be used to calculate the p-local representation zeta function for all primes p. The other commonly used method, known as the Kirillov orbit method, is unable to be applied to these exceptional cases. Specifically, we calculate some exceptional p-local representation zeta functions of the maximal class groups for some well behaved exceptional primes.

Also, we describe the Kirillov orbit method and use it to calculate various examples of p-local representation zeta functions for almost all primes p.

Chapter 1

Introduction

1.1 Introduction

This thesis applies the study of asymptotic group theory to nilpotent groups, in fact finitely generated torsion-free nilpotent groups. For all $N \in \mathbb{N}$ we will count the number of (equivalence classes of) complex irreducible representations of degree N of a nilpotent group G, say $r_N(G)$. Studying the arithmetic properties of this sequence for (not necessarily nilpotent) groups, is called representation growth. To aid our study we embed these numbers as coefficients in a zeta function. The general idea of representation growth, and in fact the field of asymptotic group theory, is to relate arithmetic information associated to a group, which is possibly (but not necessarily) encoded in a zeta function (for example abscissa of convergence, zeros, poles, and functional equations) to group theoretic information (for example Hirsch length of the group and its subquotients, nilpotency class, and length of derived series).

Let G be a finitely generated torsion-free nilpotent group. We call these groups $\mathcal{T}$-groups. Let χ be a 1-dimensional complex representation and ρ an n-dimensional complex representation of G. We define the product $\chi \otimes \rho$ to be a *twist* of ρ. Two representations ρ and ρ_* are *twist-equivalent* if, for some 1-dimensional representation χ, then $\chi \otimes \rho \cong \rho_*$. This twist-equivalence is an equivalence relation on the set of irreducible representations of G. It is easy to check that the reflexive, symmetric, and transitive properties hold. In [21] Lubotzky and Magid call the equivalence classes *twist isoclasses*. We say S_ρ, the twist isoclass containing an irreducible representation ρ is of dimension n if and only if ρ is an n-dimensional representation. They also show that there are only finitely many irreducible n-dimensional complex representations up to twisting and that for each $n \in \mathbb{N}$ there is a finite quotient $G(n)$ of G such that each n-dimensional irreducible representation ρ of G is twist-equivalent to one that factors through $G(n)$; that is, for each irreducible representation ρ of dimension n there is a twist-equivalent representation $\chi : G \to GL_n(\mathbb{C})$, and homomorphism $\beta : G(n) \to GL_n(\mathbb{C})$ such that $\chi = \beta \circ \alpha$, where $\alpha : G \to G(n)$ is the canonical projection. Henceforth we call the n-dimensional complex representations of G simply representations. We denote the number of twist isoclasses of irreducible representations of dimension n by $r_n(G)$ or r_n if no confusion will arise.

Consider the formal expression

$$\zeta_G^{irr}(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}.$$

If $\zeta_G^{irr}(s)$ converges on a right half plane of $\mathbb{C}$, say D, where $D := \{s \in \mathbb{C} \mid \Re(s) > \alpha\}$ for some $\alpha \in \mathbb{R}$, we call $\zeta_G^{irr} : D \to \mathbb{C}$ the *representation zeta function* of G. For any $\mathcal{T}$-group such a D always exists [28, Lemma 2.1]. We call $\alpha_G := \alpha$ the *abscissa of convergence* of $\zeta_G^{irr}(s)$. Let $\zeta_{G,p}^{irr}(s)$, where

$$\zeta_{G,p}^{irr}(s) = \sum_{n=0}^{\infty} r_{p^n}(G) p^{-ns}$$

and p a prime, be the *p-local representation zeta functions* of $\zeta_G^{irr}(s)$. Considering the domain D of $\zeta_{G,p}^{irr}(s)$ as above, we say $\alpha_{G,p} := \alpha$ is the *p-local abscissa of convergence* of $\zeta_{G,p}^{irr}(s)$.

We know by [21, Theorem 6.6] that in each twist isoclass there exists a representation ρ such that ρ factors through a finite quotient. Since G is nilpotent, its finite quotients are nilpotent and therefore decompose as a direct product of their Sylow-p subgroups. Since the irreducible representations of direct products of finite groups are the tensor products of irreducible representations of their factors, its representation zeta function decomposes into an Eulerian product of its p-local representation zeta functions and therefore $\zeta_G^{irr}(s) = \prod_p \zeta_{G,p}^{irr}(s)$. Moreover, it was shown by Hrushovski and Martin [14] that these p-local representation zeta functions are rational functions in p^{-s}.

This thesis deals with the representation zeta functions of certain families of $\mathcal{T}$-groups. We approach the problem of constructing these zeta functions in two ways. One way, which we call the constructive method, calculates the p-local zeta functions of some $\mathcal{T}$-groups by constructing all of its twist isoclasses and then counting how many

n-dimensional twist isoclasses exist for each $n \in N$. The other way, which is known as the Kirillov orbit method, counts certain structures, explained later, relating to the Lie ring associated to a given $\mathcal{T}$-group. Both methods have their strengths and weaknesses and we compare these two methods later in this chapter.

We note that we give all presentations of a $\mathcal{T}$-group G in terms of a Mal'cev basis (see, for example, [1, End of Section 3]). As is standard in the literature, all commutators that do not follow from the ones appearing in the chosen presentation of a $\mathcal{T}$-group are assumed to be trivial. We remind the reader of this fact at various points in the thesis.

1.2 Layout of Thesis

The structure of this thesis is as follows. Chapter 1 is an introduction to the subject of representation growth and the motivation behind its study. This chapter also includes a section comparing the two different methods of calculating representation zeta functions used in this thesis.

Chapter 2 contains the preliminaries. It gives definitions of concepts used in the rest of the thesis, as well as some standard results using these concepts. Note that everything in this section is standard but is included for the sake of completeness. At the end of this chapter is a list of notation used throughout the thesis.

Chapter 3 introduces the constructive method and uses this method to calculate all p-local representation zeta functions of the Heisenberg group over the integers of a quadratic number field. Additionally in this section, we briefly mention how this result is confirmed, and generalized, in [28].

Chapter 4 uses the constructive method to calculate most p-local zeta representation functions of a family of maximal class groups of nilpotency class n, denoted M_n; more specifically, we calculate all p-local zeta functions when $p \geq n$ and, for a given prime p, the p-local zeta function of M_{p+1}. We calculate all p-local representation zeta functions for M_3 and M_4, for which the structure of the twist-equivalent irreducible representations of "small prime"-power dimension (but not the p-local representation zeta function) are of a more complicated form than in general.

Chapter 5 briefly explains the Kirillov orbit method developed by Voll in [29] to calculate p-local representation zeta functions for almost all primes p. We then calculate a number of p-local representation zeta functions of $\mathcal{T}$-groups. Most of these $\mathcal{T}$-groups are given by Lie rings that appear in [9, Chapter 2]. This is in relation to a project with

Robert Snocken to calculate the representation zeta functions of all $\mathcal{T}$-groups appearing in that chapter. Finally, and in the same vein, we briefly mention results, together with Snocken, that will appear in a future paper by us. Note that all calculations in this chapter are performed by the author of this thesis; the only collaborative results are in the final section of that chapter, which is titled appropriately.

Finally, Chapter 6 concludes the thesis. It notes some observations about work in this thesis and describes possible future work in the area of representation growth of $\mathcal{T}$-groups, notably future work in regards to the constructive method.

Before we begin the mathematical part of the thesis, we give some background on representation growth and an area that motivated its study, subgroup growth.

1.3 Subgroup Growth

Zeta functions have been used to study many different mathematical structures. Famously, Riemann zeta functions encode information about prime numbers. Zeta functions have also been used to study number fields in algebraic number theory through Dedekind zeta functions.

Zeta functions have been used to study various types of groups. For example, zeta functions were used to count maximal subgroups of classical and alternating finite simple groups by Liebeck and Shalev in [20]. In [19], by Liebeck, Martin, and Shalev, zeta functions were used to extend the results of [20] to arbitrary finite simple groups.

The idea of using zeta functions to study representations of groups is motivated by the subject of subgroup growth. In that area, one uses zeta functions to count finite index subgroups; that is, if G is a group and $a_n(G) := |\{H \leq G \mid |G : H| = n\}|$ then we can construct the subgroup zeta function:

$$\zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} a_n(G) n^{-s}. \tag{1.3.1}$$

Note that all $a_n(G)$ are finite if G is finitely generated. If the coefficients $a_n(G)$ count normal subgroups of index n instead, we say that

$$\zeta_G^{\triangleleft}(s) = \sum_{n=1}^{\infty} a_n(G) n^{-s} \tag{1.3.2}$$

is the normal subgroup zeta function of G.

Grunewald, Segal, and Smith [12] were the first to use the method of subgroup zeta functions to study $\mathcal{T}$-groups. In that paper, the authors set up some machinery and methods to calculate both normal subgroup zeta functions and subgroup zeta functions. These methods include calculating p-local zeta functions using p-adic integrals [12, Section 2] and studying (possibly normal) subgroup zeta functions by studying Lie ring zeta functions of the Lie ring associated to a given $\mathcal{T}$-group which are simpler to analyze; see [12, Sections 3-4] for details. The authors also calculate various examples of (normal) subgroup zeta functions [12, Section 8], including the normal subgroup zeta function of a family of groups for which we calculate the representation zeta function in this thesis. Also in that paper, the authors show that both p-local subgroup and normal subgroup zeta functions are rational polynomial functions in p^{-s}. Moreover, the degree of these functions are bounded independently of the prime p chosen. The result that p-local representation zeta functions of $\mathcal{T}$-groups are rational polynomial functions, proven by Hrushovski and Martin in [14, Theorem 8.4], is analogous.

Many more examples of p-local subgroup, normal subgroup, and Lie ring zeta functions have been calculated. In fact, du Sautoy and Woodward, in [9, Chapter 2] give a list of many examples of calculations performed. Most of the representation zeta functions calculated in Chapter 5 of this thesis are of groups associated to Lie rings which appear in the aforementioned chapter.

1.4 Representation Growth

The idea of using zeta functions to study representation growth was introduced in [30], in which Witten studies compact Lie groups. Later, representation zeta functions were studied in [22], where Lubotzky and Martin use representation zeta functions to study arithmetic groups, and in [15] where Jaikin gives rationality results concerning representation zeta functions of compact p-adic analytic groups with property FAb (a group has property FAb if every open subgroup has finite abelianization). This method uses Howe's work [13] on the Kirillov orbit method and the concept of p-adic integration to calculate the zeta functions.

Representation zeta functions of $\mathcal{T}$-groups were first studied by Hrushovski and Martin in [14] using model-theoretic methods. The study of representation growth of $\mathcal{T}$-groups was expanded by Voll in [29]. In that paper, Voll develops a method for calculating p-local representation zeta functions for a given $\mathcal{T}$-group G. This method, like the method that appears in [15], involves Howe's work in [13], particularly Theorem 1.a and results from p-adic integration. Note that a large part of the method that appears in [29] is also used in that paper to study other types of growth, including subgroup and subring growth. We briefly explain this method, as applied to representations of $\mathcal{T}$-groups, at the beginning of Chapter 5.

Stasinski and Voll, in [28], generalize Voll's work in [29] to $\mathcal{T}$-groups coming from unipotent group schemes. The authors also generalize the functional equation that appears in [29]. We mention more about these results later in this section.

Representation zeta functions have been used to study other classes of groups. We briefly mention some work done in these areas. In [3] Avni et al. study compact p-adic analytic groups and arithmetic groups. In [4], Avni et al. study representations of arithmetic lattices and prove a conjecture by Larsen and Lubotzky. In [2] Avni shows that arithmetic groups have representation growth with rational abscissa of convergence. Bartholdi and de la Harpe, in [5], study representation zeta functions of wreath products with finite groups. Craven, in [7], gives lower bounds for representation growth for profinite and pro-p groups. In this paper it is shown that, for a profinite group G with property FAb, $R_n(G) \geq c \log n (\log\log n)^{1-\epsilon}$ for some constant c and any $\epsilon > 0$, where $R_n(G)$ is the number of irreducible representations of G of dimension not greater than n.

Since this thesis is concerned with representation growth of $\mathcal{T}$-groups, we give a list of important results used in the study of representation zeta functions, as well as results obtained through the study of representation growth of $\mathcal{T}$-groups.

Theorem 1.4.1 ([21, Theorem 6.6])**.** *Let G be a $\mathcal{T}$-group. For every $n \in \mathbb{N}$ there is a finite quotient of G, say $G(n)$, such that each n-dimensional irreducible representation of G is twist-equivalent to one that factors through $G(n)$. In particular, the number of n-dimensional twist isoclasses is finite.*

Theorem 1.4.2 ([14, Theorem 8.4])**.** *For a given $\mathcal{T}$-group G (and also for any finitely generated nilpotent group with torsion) and for all primes p we have that $\zeta^{irr}_{G,p}(s) = \frac{A}{B}$ where A, B are polynomials in p^{-s} with coefficients, depending on p, in $\mathbb{Z}$.*

Note that this result was proved in [29, Proposition 3.1] by non-model-theoretic means, but only for almost all primes p.

Theorem 1.4.3 ([29, Theorem D])**.** *Let G be a $\mathcal{T}$-group where d' is the Hirsch length of the derived group G'. Then, for almost all primes p, $\zeta^{irr}_{G,p}(s)$ satisfies the following functional equation upon inversion of p:*

$$\zeta^{irr}_{G,p}(s)|_{p\to p^{-1}} = p^{d'} \zeta^{irr}_{G,p}(s). \tag{1.4.1}$$

This result is analogous to functional equations satisfied by (possibly normal) subgroup zeta functions, although the functional equation in those cases does not always exist and is of a slightly more complicated form. Let $* \in \{\leq, \triangleleft\}$. Then, for some (but not all) $\mathcal{T}$-groups G,

$$\zeta^*_{G,p}(s)|_{p\to p^{-1}} = (-1)^n p^{a-bs} \zeta^*_{G,p}(s) \tag{1.4.2}$$

for some $n, a, b \in \mathbb{N}$. Examples of $\mathcal{T}$-groups where these zeta functions satisfy the functional equation above, and examples where there does not exist a functional equation of this type, appear in [9, Chapter 2].

Stasinski and Voll generalize, for a restricted class of groups, both the result of Hrushovski and Martin [14, Theorem 8.4] and the functional equation of Theorem 1.4.3.

Theorem 1.4.4 ([28, Theorem A])**.** *Let $\mathbf{G}_\Lambda$ be a unipotent group scheme as in [28, Section 1.2] and $\mathcal{O}$ the ring of integers of a number field. Then there exists a finite set S of prime ideals of $\mathcal{O}$, $t \in \mathbb{N}$, and a rational function $R(X_1, \ldots, X_t, Y) \in \mathbb{Q}(X_1, \ldots, X_t, Y)$ such that for every prime ideal $\mathfrak{p} \subset \mathcal{O}$ with $\mathfrak{p} \notin S$ the following is true: there exist algebraic integers $\lambda_1, \ldots, \lambda_t$ depending on $\mathfrak{p}$ such that, for all finite extensions $\mathfrak{O}$ of $\mathcal{O}_\mathfrak{p}$ one has*

$$\zeta^{irr}_{\mathbf{G}_\Lambda(\mathfrak{O})}(s) = R(\lambda_1^f, \ldots, \lambda_t^f, q^{-fs}), \tag{1.4.3}$$

where q is the residue field cardinality of $\mathcal{O}_\mathfrak{p}$, and $f := f(\mathfrak{O}, \mathcal{O}_\mathfrak{p})$ is the relative degree of inertia. In particular, $\zeta^{irr}_{\mathbf{G}_\Lambda(\mathfrak{O})}(s)$ is a rational function in q^{-fs}. Furthermore, the following functional equation holds:

$$\zeta^{irr}_{\mathbf{G}_\Lambda(\mathfrak{O})}(s)|_{\substack{q\to q^{-1}\\ \lambda_i \to \lambda_i^{-1}}} = q^{fd} \zeta^{irr}_{\mathbf{G}_\Lambda(\mathfrak{O})}(s) \tag{1.4.4}$$

for some $d \in \mathbb{N}$ as defined in [28, Section 1.2].

Also in that paper [28, Section 2.4], the authors generalize the Kirillov orbit method introduced in [29] in the case of $\mathcal{T}$-groups of nilpotency class 2 arising from unipotent group schemes so that it can be used to calculate the p-local representation zeta function for all primes p.

In subgroup growth, it was shown in [8, Theorem 1.1] that if, for some $\mathcal{T}$-group G, α is the abscissa of convergence of a p-local (possibly normal) subgroup growth zeta function $\zeta^*_{G,p}(s)$ then $\alpha \in \mathbb{Q}^+$. While it is currently unknown whether p-local representation zeta functions of $\mathcal{T}$-groups have rational abscissas of convergence, Robert

Snocken has observed [27] that, given $\alpha \in \mathbb{Q}^+$, there exists a $\mathcal{T}$-group G (in fact of nilpotency class 2) such that, for almost all p, $\zeta^{irr}_{G,p}(s)$ has abscissa of convergence α.

Very few representation zeta functions of $\mathcal{T}$-groups appear in the literature. We now give a list of all of these functions appearing in print.

Theorem 1.4.5 ([23, Theorem 5]). *Let $H := \langle x, y, z \mid [x, y] = z\rangle$ be the Heisenberg group over the rational integers. The representation zeta function of H is*

$$\zeta^{irr}_H(s) = \frac{\zeta(s-1)}{\zeta(s)}, \tag{1.4.5}$$

where $\zeta(s)$ is the Riemann zeta function.

Theorem 1.4.6 ([10, Theorem 1.1]). *Let $H_{\sqrt{d}}$ be the Heisenberg group over the integers of some quadratic number field $\mathbb{Q}(\sqrt{d})$, as defined in Chapter 3. The representation zeta function of $H_{\sqrt{d}}$ is*

$$\zeta^{irr}_{H_{\sqrt{d}}}(s) = \frac{\zeta^{\mathbf{D}}_{\mathbb{Q}(\sqrt{d})}(s-1)}{\zeta^{\mathbf{D}}_{\mathbb{Q}(\sqrt{d})}(s)}, \tag{1.4.6}$$

where $\zeta^{\mathbf{D}}_{\mathbb{Q}(\sqrt{d})}(s)$ is the Dedekind zeta function of $\mathbb{Q}(\sqrt{d})$.

Theorem 1.4.7 ([28, Theorem B]). *Let $n \in \mathbb{N}$ and $\delta \in \{0, 1\}$. We define the following $\mathbb{Z}$-Lie lattices:*

$$\begin{aligned}
\mathcal{F}_{n,\delta} &= \langle x_1, \ldots, x_{2n+\delta}, y_{ij}, 1 \le i < j \le 2n+\delta \mid [x_i, x_j] = y_{ij}\rangle \qquad (1.4.7)\\
\mathcal{G}_n &= \langle x_1, \ldots, x_{2n}, y_{ij}, 1 \le i, j \le n \mid [x_i, x_{n+j}] = y_{ij}\rangle\\
\mathcal{H}_n &= \langle x_1, \ldots, x_{2n}, y_{ij}, 1 \le i \le j \le n \mid [x_i, x_{n+j}] = [x_j, x_{n+i}] = y_{ij}\rangle
\end{aligned}$$

with all Lie brackets not appearing above assumed to be trivial. Let $F_{n,\delta}, G_n$, and H_n be the unipotent group scheme associated to the Lie lattices above. Let $\mathcal{O}$ be the ring of integers of some number field K. Also, let m be so that $n = 2m + \varepsilon$ where $\varepsilon \in \{0, 1\}$. Then

$$\zeta^{irr}_{F_{n,\delta}(\mathcal{O})}(s) = \prod_{i=0}^{n-1} \frac{\zeta^{\mathbf{D}}_K(s - 2(n+i+\delta)+1)}{\zeta^{\mathbf{D}}_K(s-2i)} \tag{1.4.8}$$

$$\zeta^{irr}_{G_n(\mathcal{O})}(s) = \prod_{i=0}^{n-1} \frac{\zeta^{\mathbf{D}}_K(s-n-i)}{\zeta^{\mathbf{D}}_K(s-i)} \tag{1.4.9}$$

$$\zeta^{irr}_{H_n(\mathcal{O})}(s) = \frac{\zeta^{\mathbf{D}}_K(s-n)}{\zeta^{\mathbf{D}}_K(s)} \prod_{i=0}^{m-1} \frac{\zeta^{\mathbf{D}}_K(2(s-m-i-\varepsilon)-1)}{\zeta^{\mathbf{D}}_K(2(s-i-1))}. \tag{1.4.10}$$

where $\zeta_K^{\mathrm{D}}(s)$ *is the Dedekind zeta function of* K.

Note that Theorems 1.4.5 and 1.4.6 can be seen as a special case of the preceding theorem, where $K = \mathbb{Q}, \mathbb{Q}(\sqrt{d})$, respectively, and we consider the $\mathcal{T}$-group, say, $H_1(\mathcal{O})$. However, both Theorems 1.4.5 and 1.4.6 appeared chronologically before Theorem 1.4.7.

We also note that the forthcoming PhD thesis of Robert Snocken will contain a wealth of calculations of representation zeta functions of $\mathcal{T}$-groups. Noting that we do not directly appeal to p-adic integration in this thesis, we also recommend Snocken's thesis as an introduction to the techniques of p-adic integration applied to calculating representation zeta functions of $\mathcal{T}$-groups.

1.5 Kirillov Orbit Method vs. Constructive Method

In this section we compare and contrast the two different methods of calculating representation zeta functions of $\mathcal{T}$-groups: the constructive method appearing in Chapters 3 and 4, and the Kirillov orbit method, first studied in [29] and used in Chapter 5 of this thesis. Each method has its own strengths and weaknesses.

We call primes p which violate the hypotheses of Voll's Kirillov orbit method *Kirillov-exceptional primes.* Additionally, we call primes p for which the p-local representation zeta function of a group G must be calculated in a special case *constructive-exceptional primes.* If the context is clear, we shorten both to *exceptional primes.* We call any prime that is not P-exceptional a *non-P-exceptional prime,* where P is either "Kirillov" or "constructive." Again, if no confusion will arise, we shorten this to *non-exceptional prime.*

The constructive method is quite general. This method makes no extra hypotheses besides the choice of group. The techniques shown in Chapters 3 and 4 could, in principle, be used to calculate the representation zeta function of any $\mathcal{T}$-group. However, as the complexity of the eigenspace structure of the irreducible representations increases, the complexity of the calculation may increase as well. We do note that this method relies on less mathematical machinery than the Kirillov orbit method and thus can be appreciated with minimal technical background. This method explicitly constructs all twist isoclasses of dimension p^N, for some p and N, and thus it is easy to read off the coefficients $r_{p^N}(G)$ of the representation zeta function, without the need of a recursive formula.

The main benefit of the constructive method is that primes are not excluded by the method itself, as opposed to the Kirillov orbit method. While there may be special cases that occur in the calculation for certain primes, the p-local representation zeta function of these primes can still be calculated. Provided one can do the calculation, one can understand the entire representation theory of irreducibles of a $\mathcal{T}$-group by the constructive method. We are able to calculate all irreducible representations of groups M_3 and M_4 (see Chapter 4) and thus their representation zeta functions. This is not possible using the Kirillov orbit method.

While it is true that there are only finitely many of these Kirillov-exceptional primes, comparing the p-local representation zeta functions of non-exceptional primes may not be sufficient to distinguish two $\mathcal{T}$-groups from each other. The constructive method allows for the calculation of all p-local representation zeta functions, and thus one has a finer invariant of $\mathcal{T}$-groups.

The mathematically deeper methods that appear in [29] and [28] allow for easier computations in many cases since, in these methods, one counts representations without constructing them explicitly. The machinery that appears in [29], powered by deep mathematical results including Hironaka's resolution of singularities, allows one to naively calculate p-local representation zeta functions by, essentially, linear algebra. The Howe correspondence [13, Theorem 1.a] allows one, for almost all primes, to linearize the computation of calculating the number of p^N-dimensional irreducible representations. However, Voll's method does not explicitly (without using a linear recurrence relation) give the coefficients $r_{p^N}(G)$, for some non-exceptional p and some N. This is because it parameterizes representations in a way different to dimension of twist isoclass; see Chapter 5 for details.

A main strength of the Kirillov orbit method is the possibility to study p-local representation zeta functions more generally than the constructive method. Indeed, the functional equation of Theorem 1.4.3 is proved via the Kirillov orbit method. As it presently stands, the constructive method seems unable to prove such a result. In fact, using the Kirillov orbit method, one can understand much about p-local representation zeta functions by understanding antisymmetric matrices over the ring $\mathbb{Z}/p^N\mathbb{Z}$ for each N. This translates the problem of counting representations to linear algebra over the ring of p-adic integers.

Also, as shown in [28], the Kirillov orbit method is able to use number-theoretic information about a $\mathcal{T}$-group to help study the p-local representation zeta functions. Indeed, the representation zeta function of the group $H_{\sqrt{d}}$ in Chapter 3 can be fully calculated by the Kirillov orbit method that appears in [28]. The constructive method, in its current form, "forgets" any number-theoretic structure and thus treats all $\mathcal{T}$-groups the same way.

1.6 Comparing Subgroup and Representation Growth

In all examples calculated so far, we find that representation zeta functions of $\mathcal{T}$-groups are of a simpler form than those of subgroup zeta functions. This is clear even for groups with simple structure: for some $d \in \mathbb{N}$, comparing

$$\zeta^{\leq}_{\mathbb{Z}^d}(s) = \prod_{i=0}^{d-1} \zeta(s-i) \tag{1.6.1}$$

and

$$\zeta^{irr}_{\mathbb{Z}^d}(s) = 1 \tag{1.6.2}$$

one sees the difference in complexity immediately. This difference seems to increase, in an informal sense, very rapidly; let L be the Lie ring $G_{6,7}$, as defined in Table 5.1 in Chapter 5 and G the $\mathcal{T}$-group associated to L by the exp map; see [26, Chapter 6]. Then, for almost all p,

$$\zeta^{irr}_{G,p}(s) = \frac{(1-p^{-s})^2}{(1-p^{1-s})(1-p^{2-s})}. \tag{1.6.3}$$

However,

$$\begin{aligned} \zeta^{\triangleleft}_{G,p}(s) = (1-p^{-s})(1-p^{1-s})(1-p^{2-s})(1-p^{4-3s})(1-p^{3-4s}) \\ \times (1-p^{5-5s})(1-p^{6-5s})(1-p^{6-6s})(1-p^{7-7s})W \end{aligned} \tag{1.6.4}$$

where W is a polynomial in p, p^{-s} of degree 40 and with 16 terms.

It is of note that both representation zeta functions and normal subgroup zeta functions seem to be able to capture the number-theoretic information of a $\mathcal{T}$-group, at least in the case of the Heisenberg group over the integers of a number field K of degree at most 3 over $\mathbb{Q}$; that is, both the representation zeta function and normal subgroup growth zeta function can be written in terms of Riemann zeta functions, Dedekind zeta functions of K, and Euler products of polynomials in p, p^{-s}; see [9, Chapter 2], Chapter 3, and Theorem 1.4.7 for details. It is worth noting that Theorem 1.4.7 shows that this pattern holds for representation zeta functions for number fields of arbitrary degree.

1.7 Main Results

We give a list of main results of this thesis. The following chapters give the workings of these results in detail.

The main result of Chapter 3 is Theorem 1.4.6.

The following theorem is the main result of Chapter 4.

Theorem 1.7.1. *Let $M_n := \langle a_1, \ldots, a_n, b \mid [a_i, b] = a_{i+1} \rangle$ be a family of maximal class groups of nilpotency class n. Then for all $p \geq n$, for p if $n = p+1$, and for $p = 2, n = 4$ the p-local representation zeta function of M_n is*

$$\zeta^{irr}_{M_n,p}(s) = \frac{(1-p^{-s})^2}{(1-p^{(n-2)-s})(1-p^{1-s})}. \tag{1.7.1}$$

This implies that if $n \in \{2, 3, 4\}$ then

$$\zeta^{irr}_{M_n}(s) = \frac{\zeta\big(s-(n-2)\big)\zeta(s-1)}{\big(\zeta(s)\big)^2}, \tag{1.7.2}$$

where $\zeta(s)$ is the Riemann zeta function.

The main results of Chapter 5 are various examples of calculations of representation zeta functions by the Kirillov orbit method. These results are too numerous to list here; we direct the reader's attention to Section 5.3.1 of Chapter 5 and to Table 5.2.

Chapter 2

Preliminaries

2.1 Introduction

We begin by discussing the objects we intend to study. Nilpotent groups are, in some sense, a generalization of abelian groups. Informally, while two elements a_1, a_2 in some abelian group A commute "immediately", two elements n_1, n_2 in a nilpotent group N commute "eventually".

There are many open questions about nilpotent groups. For example, it is still unknown, in most cases, how many finite groups of order p^n exist, for prime p and $n \in \mathbb{N}$ (see, for example, [18]). Thus, there is still much to learn. This thesis deals with examples of infinite, yet finitely generated nilpotent groups. To learn more about $\mathcal{T}$-groups and representations, see Appendix A of this thesis.

We note a few things for the remainder of this thesis. First, until indicated otherwise, any arbitrary group G that appears in this thesis is a $\mathcal{T}$-group. Second, we always mean "complex representation" when we say "representation".

2.2 Number Theoretic Concepts

Much of the work in this thesis relies on solving polynomial equations modulo a prime power. Underlying this concept is the theory of p-adic numbers. While we do not use the concept explicitly in the thesis, we mention them here so that the reader can get a feel for the connection between our work and p-adic numbers, and in particular, p-adic integers.

Definition 2.2.1. Let p be a prime and let $A_p = \{0, 1, \dots, p-1\}$. The set of *p-adic* *numbers*, denoted $\mathbb{Q}_p$, is the set of all formal series

$$\sum_{i=k}^{\infty} a_i p^i \tag{2.2.1}$$

where $k \in \mathbb{Z}$ and $a_i \in A_p$. The set of *p-adic integers*, denoted $\mathbb{Z}_p$, is the subset of p-adic numbers such that $k \geq 0$.

The p-adic integers can also be defined algebraically in terms of an inverse limit of an inverse system. The field of fractions of $\mathbb{Z}_p$ is then $\mathbb{Q}_p$ (see, for example [11]). Also note that, for any prime p, $\mathbb{Z} \subset \mathbb{Z}_p$.

Definition 2.2.2. Let $q := \sum_{i=k}^{\infty} a_i p^i \in \mathbb{Q}_p$. The *p-adic valuation* of q, denoted $v_p(q)$, is defined by

$$v_p(q) = \min\{i \mid a_i \neq 0\} \tag{2.2.2}$$

If this does not exist, that is if $q = 0$, we say that $v_p(q) = \infty$. If q is a p-adic integer, we also say that

$$v_p(q) \bmod p^N = \min\{v_p(q), N\}. \tag{2.2.3}$$

For this thesis we only consider $v_p(q)$ for $q \in \mathbb{Z}$ so $v_p(q) < \infty$. An excellent introduction to p-adic numbers and their properties is the book *p-adic Numbers: An Introduction* by Gouvea [11].

In our work, we will have to count how many solutions of a polynomial equation exist mod prime powers p^N as N varies. A useful result for this counting is Hensel's Lemma. Indeed, we use this result in Chapter 3 to "lift" solutions of polynomial equations. For details, see, for example, [11, Chapter 3.4].

Theorem 2.2.3 (Hensel's Lemma)**.** *Let $f(x)$ be a polynomial with integer (or p-adic) coefficients and let $f'(x)$ be its formal derivative (i.e. for each non-constant term ax^n in $f(x)$ the corresponding term in $f'(x)$ is nax^{n-1} while constant terms disappear). Then, for some $s \in \mathbb{Z}$ and $k \in \mathbb{N}$, if*

$$f(s) \equiv 0 \bmod p^k \quad \textit{and} \quad f'(s) \not\equiv 0 \bmod p \tag{2.2.4}$$

then there exists an integer t such that

$$f(t) \equiv 0 \bmod p^{k+1} \quad \textit{and} \quad t \equiv s \bmod p^k \tag{2.2.5}$$

and this t is unique mod p^{k+1}. Thus, if $f(x)$ satisfying the conditions above has n solutions mod p^k, it also has n solutions mod $p^{k+\ell}$ for $\ell \in \mathbb{N}$.

We now introduce another type of zeta function from algebraic number theory. We use this in Chapter 3. For more details see, for example, [6, Section 10.5].

Definition 2.2.4. Let K be an algebraic number field and O_K its ring of integers. Then $\zeta_K^{\mathbf{D}}(s) = \sum_{I \subseteq O_K} (N_{K/\mathbb{Q}}(I))^{-s}$ is the *Dedekind zeta function* of K where I runs through the non-zero ideals of O_K and $N_{K/\mathbb{Q}}(I)$ is the norm of I with respect to $\mathbb{Q}$.

The zeta function $\zeta_K^{\mathbf{D}}(s)$ has an Euler product decomposition

$$\zeta_K^{\mathbf{D}}(s) = \prod_{P \subseteq O_K} \frac{1}{1 - (N_{K/\mathbb{Q}}(P))^{-s}},$$

where P runs over all non-zero prime ideals of O_K. This decomposition reflects the unique factorization of ideals of O_K.

We recall another standard definition; see, for example, [6, Section 3.3]. This definition can generalize to number fields of larger index, however we limit ourselves to the case of quadratic number fields.

Definition 2.2.5. Let p be a rational prime, d be a squarefree integer, O_d be the integers of a quadratic number field $\mathbb{Q}(\sqrt{d})$ and consider the ideal $(p) \lhd O_d$. If (p) is prime then p is *inert.* If (p) is the product of two distinct prime ideals then p *splits.* If (p) is the square of a prime ideal then p is *ramified.*

This definition is equivalent to the equation $x^2 - \Delta \equiv 0 \bmod p$ having $0, 2$, or 1 solution, respectively and where Δ is the discriminant of $\mathbb{Q}(\sqrt{d})$. We note that $\Delta = 4d$ if $d \equiv 2, 3 \bmod 4$ and $\Delta = d$ if $d \equiv 1 \bmod 4$. We also note that there are only a finite number of ramified primes and a prime p is ramified if and only if it divides the discriminant of the number field.

The following is a well known result.

Proposition 2.2.6. *Let p be a prime and $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then the p-local Dedekind zeta function of $\mathbb{Q}(\sqrt{d})$ is*

$$\zeta^{\mathbf{D}}_{\mathbb{Q}(\sqrt{d}),p}(s) = \begin{cases} \dfrac{1}{1-p^{-2s}} & \textit{if } p \textit{ is inert,} \\[2ex] \left(\dfrac{1}{1-p^{-s}}\right)^2 & \textit{if } p \textit{ splits,} \\[2ex] \dfrac{1}{1-p^{-s}} & \textit{if } p \textit{ is ramified.} \end{cases}$$

2.3 Representation Zeta Functions of $\mathcal{T}$-Groups

We introduce a lemma to count 1-dimensional twist isoclasses. We leave the proof as an exercise for the reader.

Lemma 2.3.1. *For a $\mathcal{T}$-group G, there is only 1 twist isoclass of dimension 1. That is $r_1 = 1$, where r_1 is the first coefficient in $\zeta^{irr}_G(s)$.*

Also, we introduce a definition that describes the form of representation zeta function in terms of how many different general types of p-local representation zeta functions occur in the Euler factorization of some representation zeta function.

Definition 2.3.2. Let $\zeta^{irr}_G(s) = \prod_p \zeta^{irr}_{G,p}(s)$ be the representation zeta function of a $\mathcal{T}$-group G. We say that $\zeta^{irr}_G(s)$ is *finitely uniform* if there exists a finite set of polynomials in two variables, say F, such that, for all primes p greater than some distinguished prime p_*, there is a $f \in F$ such that $\zeta^{irr}_{G,p} = f(p, p^{-s})$. If there is only one rational function needed, that is if $n = 1$, then we say that $\zeta^{irr}_G(s)$ is *uniform*.

Note that we show that all representation zeta functions in this thesis are finitely uniform and that all but the representation zeta function of $H_{\sqrt{d}}$ is uniform.

2.4 Non-Standard Notation

Definition 2.4.1. Let G be a $\mathcal{T}$-group of nilpotency class c. Then G is a *maximal class group* if $h(G) = c + 1$.

Since we mention roots of unity often in the course of this thesis we use the following notation.

Definition 2.4.2. Let S_p^∞ be all complex p^ℓth roots of unity for all $\ell \in \mathbb{N} \cup \{0\}$ and S_p^k be the p^kth roots of unity for $k \in \mathbb{N} \cup \{0\}$ (and note that, for $k \geq 1$, the elements of the set $S_p^k \backslash S_p^{k-1}$ are the primitive p^kth roots of unity). Define $s : S_p^\infty \to \mathbb{N}$ such that $s(\lambda) = k$ if and only if $\lambda \in S_p^k \backslash S_p^{k-1}$. If $s(\lambda) = k$ we say that λ has *depth* k.

We introduce some notation in regards to integers mod p^N for some prime p and $N \in \mathbb{N}$. Let

$$\mathcal{Z}^*_{n,p^N} = (\mathbb{Z}/p^N\mathbb{Z})^n \setminus (p\mathbb{Z}/p^N\mathbb{Z})^n; \tag{2.4.1}$$

that is, for $n \in \mathbb{N}$, the set of n-tuples of elements of $\mathbb{Z}/p^N\mathbb{Z}$ such that at least one entry is a unit.

The calculation of the irreducible representations of M_n will involve k-simplex numbers. Let $T_0(0) = 1$, $T_0(j) = 1$, and $T_j(0) = 0$ for $j \in \mathbb{N}$ and recursively define $T_k(j) = \sum_{l=1}^{j} T_{k-1}(l) = T_k(j-1) + T_{k-1}(j)$ for $k \in \mathbb{N}$. The next lemma lists some properties of these numbers that are needed in the thesis.

Lemma 2.4.3. *Let $i, j, k, b \in \mathbb{N}$, p be a prime, and $T_k(j)$ be defined as above.*

i. $T_k(i) = \binom{i+k-1}{k} = \frac{i(i+1)\ldots(i+k-1)}{k!}$.

ii. $T_k(i) - T_k(j) = (i-j)\frac{\gamma}{k!}$ *for some* $\gamma \in \mathbb{Z}$.

iii. *If* $p > k$ *then for* $b \in \mathbb{N}$ *and* $1 \leq \alpha \leq p-1$ *we have* $T_k(\alpha p^b + j) = T_k(j) \bmod p^b$.

iv. $T_k(j+1) = T_k(j) + T_{k-1}(j) + \ldots + T_0(j)$.

v. $T_k(i+j) = \sum_{l=0}^{k} T_l(i) T_{k-l}(j)$.

vi. *If* $p > k$ *then* $T_k(p^N - 1) = 0 \bmod p^N$.

Proof. i. Shown by an easy induction on $k + i$.

ii. By (i)

$$\begin{aligned} T_k(i) - T_k(j) &= \frac{i(i+1)\ldots(i+k-1) - j(j+1)\ldots(j+k-1)}{k!} \\ &= \frac{(i^k - j^k) + \beta_{k-1}(i^{k-1} - j^{k-1}) + \ldots + \beta_1(i-j)}{k!} \end{aligned}$$

for some coefficients β_ℓ. Since

$$(i^d - j^d) = (i-j) \sum_{d_1+d_2=d-1} i^{d_1} j^{d_2},$$

the numerator of $T_k(i) - T_k(j)$ does indeed have a factor of $(i-j)$ and we have proved what we wanted.

iii. Expanding out the numerator of $T_k(\alpha p^b + j) = \frac{(\alpha p^b+j)\dots(\alpha p^b+j+k-1)}{k!} \bmod p^b$, it is clear only one term, that is $j \cdot (j+1) \cdot \ldots \cdot (j+k-1)$, is not necessarily $0 \bmod p^b$.

iv. By the recursive definition of $T_k(j)$ we have that $T_k(j+1) = T_k(j) + T_{k-1}(j+1)$. The result follows immediately by induction.

v. Assume that $T_b(i+a) = \sum_{l=0}^{b} T_l(i)T_{b-l}(a)$ holds for values a, b such that $a + b < k + j$. We have by definition that $T_k(i+j) = T_k(i + (j-1)) + T_{k-1}(i+j)$. Then by inductive hypothesis

$$\begin{aligned}
T_k(i+j) &= \sum_{l=0}^{k} T_l(i)T_{k-l}(j-1) + \sum_{l=0}^{k-1} T_l(i)T_{k-1-l}(j) \\
&= T_0(i)\big(T_k(j-1) + T_{k-1}(j)\big) + T_1(i)\big(T_{k-1}(j-1) + T_{k-2}(j)\big) + \dots \\
&\quad + T_{k-1}(i)\big(T_1(j-1) + T_0(j)\big) + T_k(i)T_0(j) \\
&= T_0(i)T_k(j) + T_1(i)T_{k-1}(j) + \dots + T_{k-1}(i)T_1(j) + T_k(i)T_0(j) \\
&= \sum_{l=0}^{k} T_l(i)T_{k-l}(j).
\end{aligned}$$

vi. Consider

$$\begin{aligned}
T_k(p^N - 1) &= \frac{(p^N-1)(p^N)\dots(p^N+k-2)}{k!} \\
&= p^N \frac{(p^N-1)(p^N+1)\dots(p^N+k-2)}{k!} \\
&= \alpha p^N
\end{aligned}$$

for some α such that $p \nmid \alpha$. This is true since both $k!$ and $(p^N-1)(p^N+1)\dots(p^N+k-2)$ are units mod p^N.

□

As a corollary of (iii) we have the following.

Corollary 2.4.4. *Let p be a prime, let $k < p$, let $N \geq 1$, let $1 \leq m \leq N$, let $\alpha \in \mathbb{N}$ such that $p \nmid \alpha$, and, for $j \geq 0$, let*

$$F(k,j) = \alpha p^m T_k(j-1). \tag{2.4.2}$$

Then $F(k, \beta p^{N-m} + j + 1) = F(k, j + 1) \bmod p^N$ *for all* β *such that* $1 \leq \beta < p^m$ *and all* j *such that* $0 \leq j \leq p^{N-m} - 1$.

Proof. Consider $F(k, \beta p^{N-m} + j + 1)$. Then

$$F(k, \beta p^{N-m} + j + 1) = \alpha p^m \frac{(\beta p^{N-m} + j) \ldots (\beta p^{N-m} + (j + k - 1))}{k!} \bmod p^N \quad (2.4.3)$$

By Lemma 2.4.3(iii), and noting that $p^m(p^{N-m}) = 0 \bmod p^N$, only the term with no factor of p^{N-m} survives mod p^N; that is,

$$\alpha p^m(\beta p^{N-m} + j) \ldots (\beta p^{N-m} + (j + k - 1)) = \alpha p^m j(j + 1) \ldots (j + k - 1) \quad (2.4.4)$$

and thus $F(k, \beta p^{N-m} + j + 1) = F(k, j + 1)$. $\square$

2.5 Lie Rings and Smith Normal Form

Chapter 5 deals with the Kirillov orbit method. This method uses Lie rings associated to $\mathcal{T}$-groups to study their representation growth. We give some definitions in this respect.

Definition 2.5.1. A *Lie ring* $L := (L, +, [\cdot, \cdot])$ is a set together with two operations, addition $+$ and Lie bracket $[\cdot, \cdot]$, such that

- L is an abelian group with respect to $+$.
- $[\cdot, \cdot]$ is bilinear; that is, for $x, y, z \in L$,

$$[x + y, z] = [x, z] + [y, z] \quad \text{and} \quad [x, y + z] = [x, y] + [x, z].$$

- $[\cdot, \cdot]$ satisfies the Jacobi identity; that is, for $x, y, z \in L$,

$$[x, [y, z]\,] + [y, [z, x]\,] + [z, [x, y]\,] = 0.$$

- $[\cdot, \cdot]$ is antisymmetric; that is, for $x, y \in L$,

$$[x, y] + [y, x] = 0.$$

Analogous to groups, if L is a Lie ring and $M, N \subseteq L$ then $[M, N] := \text{span}(\{[m, n] \mid m \in M, n \in N\})$.

We note that we use the same notation for the Lie bracket as for commutators of groups. Since the elements inside the brackets are from different structures, the meaning should be clear by context. Also note that, for all Lie rings studied in this thesis, each Lie ring has a $\mathbb{Z}$-basis.

Example 2.5.2. *Let $\mathcal{H} := \langle x, y, z \mid [x, y] = z\rangle$ where all other commutators in the presentation are trivial. This is usually called the Heisenberg Lie ring. For $a_x, a_y, a_z \in \mathbb{Z}$ let $A := a_x x + a_y y + a_z z \in \mathcal{H}$ and define B and C similarly. Noting that $[A, B] = (a_x b_y - a_y b_x)z$ it is easy to check that the Jacobi identity holds for A, B, C.*

Definition 2.5.3. Define $L_0 := L$ and $L_i := [L, L_{i-1}]$. A Lie ring L is *nilpotent* if there is a $c \in \mathbb{N}$ such that $L_c = 0$. The minimal c such that this property holds is called the *nilpotency class* of L.

Example 2.5.4. *An abelian group A, with the trivial Lie bracket $[a_1, a_2] = 0$ for all $a_1, a_2 \in A$, is a nilpotent Lie ring of nilpotency class* 1. *The Heisenberg Lie ring $\mathcal{H}$ in Example 2.5.2 has nilpotency class* 2.

We introduce two concepts that are analogous to the same concepts in nilpotent groups.

Definition 2.5.5. Let L be a nilpotent Lie ring. Define the Lie ring Hirsch length to be $h(L) := \dim(L)$ and the Lie ring center to be $Z(L) := Z := \mathrm{span}\{\ell \in L \mid [\ell, L] = 0\}$.

To calculate representation zeta functions using the Kirillov orbit method we must consider the Smith Normal Form of certain matrices that encode the Lie bracket behaviour of a Lie ring L. We now give definitions in this vein.

Definition 2.5.6. Let N be an $n \times m$ matrix over a principal ideal domain (or, more generally, elementary divisor ring). There exist a $n \times n$ invertible matrix $\mathbf{S}$ and a $m \times m$

invertible matrix $\mathbf{F}$ such that

$$\mathbf{S}N\mathbf{F} = \begin{pmatrix} \alpha_1 & 0 & 0 & & \dots & 0 \\ 0 & \alpha_2 & 0 & & \dots & 0 \\ 0 & 0 & \ddots & & \dots & 0 \\ & & & \alpha_r & & \\ \vdots & & & & 0 & \vdots \\ & & & & & \ddots \\ 0 & & \dots & & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & & \dots & & \dots & 0 \end{pmatrix} \tag{2.5.1}$$

such that $r \leq \min\{n, m\}$ and $\alpha_i | \alpha_{i+1}$ for $1 \leq i \leq r$ (note in the matrix above that $n > m$; matrices for the other cases can be constructed similarly). We call $\mathbf{S}N\mathbf{F}$ the *Smith Normal Form* of N and denote this by $SNF(N)$. The α_i, which are unique up to multiplication by units, are called the *elementary divisors* of $SNF(N)$.

We define maps $[x_i, x_j]_{\mathbf{y}}$ from $L \times L$ to $\mathbb{Z}/p^N\mathbb{Z}$ in the following way: let L have a $\mathbb{Z}$-basis $\{x_1, \dots x_d, x_{d+1}, \dots, x_{d+e}\}$ and L' have a $\mathbb{Z}$-basis $\{x_{d+1}, \dots, x_{d+e}\}$. For $\mathbf{y} := (y_{d+1} \dots, y_{d+e}) \in \mathcal{Z}^*_{e,p^N}$, if $[x_i, x_j] = \sum_{k=d+1}^{d+e} \alpha_k(i,j)x_k$ with $\alpha_k(i,j) \in \mathbb{Z}$ then $[x_i, x_j]_{\mathbf{y}} = \sum_{k=d+1}^{d+e} \alpha_k(i,j)y_k$.

Definition 2.5.7. Let L be a Lie ring and let L have a basis $\{x_1, \dots, x_d, x_{d+1}, \dots, x_{d+f}, x_{d+f+1}, \dots x_{d+e}\}$ such that $\{x_{d+1}, \dots x_{d+e}\}$ is a $\mathbb{Z}$-basis for L' and $\{x_{d+f+1}, \dots x_{d+e}\}$ is a $\mathbb{Z}$-basis for $L' \cap Z(L)$. Let $\mathbf{y} := (y_{d+1}, \dots, y_{d+e}) \in \mathcal{Z}^*_{e,p^N}$. Let $\mathcal{R}(\mathbf{y})$ be the $(d+f) \times (d+f)$ matrix defined by

$$\mathcal{R}_{i,j} = [x_i, x_j]_{\mathbf{y}} \tag{2.5.2}$$

for i, j such that $1 \leq i, j \leq d+f$. We call $\mathcal{R}(\mathbf{y})$ the $\mathbf{y}$-*commutator matrix* of L (or simply commutator matrix). Note that this is all that is needed since basis elements $x_{d+f+1}, \dots x_{d+e} \in Z(L)$.

Let $\mathcal{S}(\mathbf{y})$ be the last f columns of $\mathcal{R}$; that is, the $(d+f) \times f$ matrix defined by

$$\mathcal{S}_{i,j} = [x_i, x_j]_{\mathbf{y}} \tag{2.5.3}$$

for i, j such that $1 \leq i \leq d+f$ and $d+1 \leq j \leq d+f$. We call $\mathcal{S}(\mathbf{y})$ the $\mathbf{y}$-*commutator submatrix* of L (or simply commutator submatrix). Note that if L has nilpotency class 2, then $[L, L] \subseteq Z(L)$ and thus $\mathcal{S}$ is trivial.

Note that we always consider $\mathcal{R}$ and $\mathcal{S}$ mod p^N.

The following lemma is used in Chapter 5 to determine elementary divisors of commutator matrices.

Lemma 2.5.8. *Let $N \in \mathbb{N}$, p be a prime, and $\mathbf{y} := (y_1, \ldots, y_d) \in \mathcal{Z}^*_{d,p^N}$. Let $\mathcal{R} := \mathcal{R}(\mathbf{y})$ be a $d \times d$ commutator matrix mod p^N as defined earlier and let $\mathbf{m} := (p^{m_1}, \ldots, p^{m_d})$ be its elementary divisors. Then the elements in $\mathbf{m}$ occur in pairs; that is, $m_{2i-1} = m_{2i}$ for i such that $1 \leq i \leq \lfloor d/2 \rfloor$ and, if d is odd, $p^{m_d} = 0$.*

Proof. We first explain the concept of simultaneous row and column operations. If a row operation is performed with rows r_i and r_j for some i, j, (or just r_i if the row is scaled) then the same operations are performed with columns c_i and c_j next. It can be shown that the antisymmetric property is invariant under simultaneous row and column operations.

It is well known that the determinant of a $d \times d$ antisymmetric matrix is 0 if d is odd. Thus, at least one elementary divisor must be 0 mod p^N for all N. But since elementary divisors have the property that $p^{m_i} | p^{m_{i+1}}$ it must be that $p^{m_d} = 0 \bmod p^N$ Note that it is possible that other $p^{m_i} = 0 \bmod p^N$ as well.

For any $a \in \mathbb{Z}/p^N\mathbb{Z}$ we have that $a = up^k \bmod p^N$ for some unit u and some $k \leq N$. Thus, the only possible factors of the elementary divisors are powers of p. Thus, to calculate the Smith Normal Form of $\mathcal{R}$ mod p^N, we are only concerned about the p-adic valuation $v_p(\cdot)$ mod p^N of each entry of $\mathcal{R}$. Since, for all primes p and $a \in \mathbb{Z}/p^N\mathbb{Z}$,

$$v_p(a) = v_p(-a) \tag{2.5.4}$$

$\mathcal{R}$ is symmetric entrywise by p-adic valuations of $\mathcal{R}_{i,j}$. Note that all matrices below are considered entrywise mod p^N.

Let $r_{a,b}$ be the a, b entry of $\mathcal{R}$ and choose $i, j \leq d$ such that $v_p(r_{i,j})$ is minimal. Then, by Equation 2.5.4, we have that $v_p(r_{j,i})$ is also minimal. Without loss of generality, since we can perform simultaneous row and column transpositions to shift non-diagonal entries, we can say that $i = 1$ and $j = 2$. So we have the matrix

$$\begin{pmatrix} 0 & r_{1,2} & \star & \dots & \star \\ r_{2,1} & 0 & \star & \dots & \star \\ \star & \star & 0 & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \dots & 0 \end{pmatrix}. \tag{2.5.5}$$

Now, since $v_p(r_{1,2}) = v_p(r_{2,1})$ is minimal, we can add multiples of column 1 to the other columns so that every entry of row 1 besides $r_{1,2}$ is zero and simultaneously add multiples of row 2 to the other rows so that every entry of column 1 besides $r_{2,1}$ is zero. So now we have the matrix

$$\begin{pmatrix} 0 & r_{1,2} & 0 & \dots & 0 \\ r_{2,1} & 0 & \star & \dots & \star \\ 0 & \star & 0 & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \star & \dots & 0 \end{pmatrix}. \tag{2.5.6}$$

We now can take multiples of row/column 1 and add them to the other row/columns such that each entry, besides $r_{1,2}$ and $r_{2,1}$ is 0 in the second row and second column. So now we have the matrix

$$\begin{pmatrix} 0 & r_{1,2} & 0 & \dots & 0 \\ r_{2,1} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \star & \dots & 0 \end{pmatrix}. \tag{2.5.7}$$

Finally, we transpose rows (or columns) 1 and 2. So now we have the matrix

$$\begin{pmatrix} r_{1,2} & 0 & 0 & \dots & 0 \\ 0 & r_{2,1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \star & \dots & 0 \end{pmatrix}. \tag{2.5.8}$$

Let $\mathcal{R}'$ be the $d-2 \times d-2$ matrix formed by omitting the first two rows/columns of the matrix above. We note two facts. First, that $\mathcal{R}'$ is antisymmetric; and second, that if $r'_{k,\ell}$ is an entry of $\mathcal{R}'$ such that $v_p(r'_{k,\ell})$ is minimal in $\mathcal{R}'$ then $v_p(r_{i,j}) \leq v_p(r'_{k,\ell})$. Since the elementary divisors in Smith Normal Form are invariant up to units we have determined m_1 and m_2 and $m_1 = m_2$. Thus, we can continue the process above to obtain $SNF(\mathcal{R})$ mod p^N where $m_{2i-1} = m_{2i}$ for i such that $1 \leq i \leq \lfloor d/2 \rfloor$. □

Definition 2.5.9. Let M be a matrix and let S be a $a \times a$ submatrix of M determined by a rows and a columns of M. Then we call $\det(S)$ a *a-minor* of M.

We also use the following lemma to help us determine elementary divisors of commutator matrices. Although this result is over a PID and commutator matrices are defined mod p^N, we can lift $\mathcal{R}$ to $\mathbb{Z}$, use the following lemma, and then reduce mod p^N.

Lemma 2.5.10 ([24, Chapter 6])**.** *Let M be a $n \times m$ matrix over a PID R such that $SNF(M)$ has elementary divisors $\alpha_1, \dots, \alpha_r$. For $j \leq \max\{m, n\}$, let Δ_j be the greatest common divisor of the j-minors of M and set $\Delta_0 = 1$. Then $\alpha_j = \Delta_j/\Delta_{j-1}$ for all $j \leq r$.*

2.6 List of Notation

Table 2.1 List of Notation

$GL_n(\mathbb{C})$	The n-dimensional general linear group over $\mathbb{C}$; that is, the set of invertible $n \times n$ matrices over $\mathbb{C}$
S_p^N	For $N \in \mathbb{N} \cup \{0\}$ the set of p^Nth roots of unity; that is the set $\{\lambda \in \mathbb{C} \mid \lambda^{p^N} = 1\}$. If $N = \infty$ then it is the set of all p^ℓth roots of unity for all $\ell \in \mathbb{N} \cup 0$
$s(\lambda)$	The depth of $\lambda \in S_p^\infty$; that is, $s(\lambda) = N$ iff $\lambda \in S_p^N \setminus S_p^{N-1}$
$T_k(i)$	For $i, k \in \mathbb{N}$, $\frac{i(i+1)\dots(i+k-1)}{k!}$
$[a, b]$	For $a, b \in G$ a group, the commutator $aba^{-1}b^{-1}$; for $a, b \in L$ a Lie ring, the Lie bracket of L
$[A, B]$	For $A, B \subseteq G$ a group, $\langle [a, b] \mid a \in A, b \in B \rangle$; for $A, B \subseteq L$ a Lie ring, $\mathrm{span}(\{[a, b] \mid a \in A, b \in B\})$
N'	$[N, N]$ for either a group or Lie ring N
$h(N)$	The Hirsch length of a group N or $\dim(N)$ of a Lie ring N
$Z(N)$	The center of a group or Lie ring N
$\zeta(bs - a)$	The shifted Riemann zeta function $\sum_{N=1}^{\infty} N^{a-bs},\ a, b \in \mathbb{Z}$
$\zeta_p(bs - a)$	The shifted p-local Riemann zeta function $\sum_{N=1}^{\infty} (p^N)^{a-bs}$
$SNF(M)$	The Smith Normal Form of a matrix M
$\lfloor x \rfloor$	The floor of (or greatest integer no greater than) $x \in \mathbb{Q}$
$a = b \bmod c$	The integers a and b are in the same equivalence class $\pmod{c}$
$v_p(n)$	The p-adic valuation of $n \in \mathbb{Z}$; that is, for some prime p if $n = ap^b$, where b is maximal, then $v_p(n) = b$
$\phi(n)$	The Euler phi function of n; that is, the number of $j \leq n$ co-prime to n.

Chapter 3

Representation Growth of $H_{\sqrt{d}}$

3.1 Introduction

This chapter will consist of the construction of the irreducible representations and the calculation of the representation zeta function of the Heisenberg group over the integers of a quadratic number field, denoted $H_{\sqrt{d}}$ for some square-free $d \in \mathbb{Z}$. This example generalizes the calculation of the representation zeta function of the Heisenberg group over the rational integers; this was calculated by Magid and Nunley [23] and later by Hrushovski and Martin [14].

The calculation of the representation zeta function of $H_{\sqrt{d}}$ is performed by calculating the p-local factors separately as follows: first, for a p^N-dimensional irreducible representation ρ of $H_{\sqrt{d}}$ we study the eigenspace structure of ρ. Secondly, we choose a basis for this representation and construct a canonical form for the image of each generator for every twist isoclass. Finally, we count the number of representations with different canonical forms, thus giving us the number of twist isoclasses of dimension p^N (we remind the reader that there are r_{p^N} twist isoclasses) and use the r_{p^N} as coefficients in the zeta function.

Let d be a square-free integer and define O_d to be the ring of integers of $\mathbb{Q}(\sqrt{d})$. The Heisenberg group over O_d, which we denote $H_{\sqrt{d}}$, is the group of 3×3 upper unitriangular matrices with entries in O_d.

It is easily seen that the following six matrices generate $H_{\sqrt{d}}$:

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad x_d = \begin{pmatrix} 1 & D & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad y_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & D \\ 0 & 0 & 1 \end{pmatrix}$$

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad z_d = \begin{pmatrix} 1 & 0 & D \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $D = \sqrt{d}$ if $d = 2, 3 \mod 4$ and $D = \frac{1+\sqrt{d}}{2}$ if $d = 1 \mod 4$. Note that $(1, D)$ is a $\mathbb{Z}$-basis for O_d.

A presentation for this family of groups is given by

$$\langle x, x_d, y, y_d, z, z_d \mid [x, y] = z, [x, y_d] = [x_d, y] = z_d, [x_d, y_d] = z^d \rangle$$

if $d = 2, 3 \mod 4$ and

$$\langle x, x_d, y, y_d, z, z_d \mid [x, y] = z, [x, y_d] = [x_d, y] = z_d, [x_d, y_d] = z^{\frac{d-1}{4}} z_d \rangle$$

if $d = 1 \mod 4$. We remind the reader that, by convention, commutators that do not appear among the relations are trivial. It is easy to prove that, given a particular d, the corresponding presentation is equivalent to the matrix presentation above.

The main theorem of this chapter is as follows:

Theorem 3.1.1. *Let $H_{\sqrt{d}}$ be the Heisenberg group over the ring of integers of $\mathbb{Q}(\sqrt{d})$ and $\zeta_Q^{\mathbf{D}}$ the Dedekind zeta function of a number field Q. Then the representation zeta function of $H_{\sqrt{d}}$ is*

$$\zeta^{irr}_{H_{\sqrt{d}}}(s) = \frac{\zeta^{\mathbf{D}}_{\mathbb{Q}(\sqrt{d})}(s-1)}{\zeta^{\mathbf{D}}_{\mathbb{Q}(\sqrt{d})}(s)}.$$

We remark that this theorem also holds for the Heisenberg group over the rational integers; that is, if $\mathbb{Q}(\sqrt{d})$ is replaced by $\mathbb{Q}$. See Equation 1.4.5 for this result.

3.2 The Constructive Method

In this section we introduce some general results about representations of some $\mathcal{T}$-groups. Lemma 3.2.2 is used in both this chapter and the following chapter to determine the eigenspace structure of large abelian subgroups of the images of the irreducible representations being studied.

3.2.1 Studying Eigenspaces of a Class of $\mathcal{T}$-Groups

The representation zeta function of $H_{\sqrt{d}}$ will be calculated using a very constructive approach. In general, for a $\mathcal{T}$-group G, this approach has three main components:

- Study the eigenspace structure of an irreducible representation $\rho(G)$.
- Using this knowledge of the eigenspace structure, show that all irreducible representations are of some canonical form. Additionally, show that any set of linear operators of this given form is indeed an irreducible representation of G.
- Count the number of irreducible representations constructed in the step above, up to equivalence by twisting and isomorphism.

We call this general method the *constructive method.* We note that the steps above are defined rather loosely. In future work, we plan to make the above procedure rigorous and, hopefully, algorithmic.

We introduce an important lemma that gives us much information about the eigenspace structure of representations of certain, nicely behaved, $\mathcal{T}$-groups. Before this lemma, we give a definition regarding eigenspaces of a set of linear operators.

Definition 3.2.1. Let $\mathcal{L}$ be a set of linear operators of a vector space V. If a subspace $W \subseteq V$ is an eigenspace of each $L \in \mathcal{L}$ then we say that W is a *mutual eigenspace* of $\mathcal{L}$.

Lemma 3.2.2. *For some $\alpha_{j,k} \in \mathbb{Z}$ let $G := \langle a_1, \ldots, a_n, b_1, \ldots b_m \mid [a_i, b_j] = A_{i,j} \rangle$, where $A_{i,j} := \prod_{k=i+1}^{n} a_k^{\alpha_{j,k}}$, be a $\mathcal{T}$-group and let ρ be an irreducible p^N-dimensional representation of G. Also, let $\rho(a_i) = x_i$ and $\rho(b_j) = y_j$ for all $i \leq n$ and $j \leq m$. Define $X := \{x_1, \ldots, x_n\}, Y := \{y_1 \ldots y_m\}$ and $X_{i,j} := \rho(A_{i,j})$. Then the mutual eigenspaces of X are 1-dimensional and there are p^N distinct mutual eigenspaces of X.*

Proof. Let $\mathcal{E} := \{E_1, \ldots, E_t\}$ be the set of mutual eigenspaces of X. We will show that if $\mathbf{v}$ is an eigenvector of E_{j_1} then for any $y \in Y$ the vector $y\mathbf{v}$ is an eigenvector of E_{j_2} for some $j_1, j_2 \leq t$. We then show that Y acts transitively on $\mathcal{E}$ and that all $E_j \in \mathcal{E}$ are of the same dimension, in fact a p-power. Finally, we show that each E_j is 1-dimensional, and $t = p^N$.

It is clear, by definition, that x_n commutes with all $y \in Y$. Let $E \in \mathcal{E}$ and let $\mathbf{v} \in E$. For a given $y_j \in Y$, consider $x_n y_j \mathbf{v}$. Since x_n is central,

$$x_n y_j \mathbf{v} = \lambda_{n,j} y_j \mathbf{v} \tag{3.2.1}$$

for some $\lambda_{n,j} \in \mathbb{C}^*$.

For all $k < n$, let λ_k be such that $\lambda_k \mathbf{v} = x_k \mathbf{v}$. Now, as an induction, we choose $i < n$ and assume that, for each $h > i$, $y_j \mathbf{v}$ is an eigenvector of each x_h, with eigenvalue $\lambda_{h,j}$. Then

$$\lambda_{i-1} y_j \mathbf{v} = y_j \lambda_{i-1} \mathbf{v} = y_j x_{i-1} \mathbf{v} = X_{j,i-1} x_{i-1} y_j \mathbf{v} = x_{i-1} X_{j,i-1} y_j \mathbf{v} = x_{i-1} \lambda' y_j \mathbf{v} \tag{3.2.2}$$

for some $\lambda' \in \mathbb{C}^*$. Note that the third equality is by the group relations and the final equality is by the inductive hypothesis. Thus $y_j \mathbf{v}$ is an eigenvector of x_{i-1} with eigenvalue $\lambda_{i-1}(\lambda')^{-1}$. This induction tells us that if $\mathbf{v}$ is a mutual eigenvector of X then, for all $y \in Y$, $y\mathbf{v}$ is also a mutual eigenvector of X.

Let $\mathbf{v_1}, \mathbf{v_2} \in E$. For some x_i and $\lambda_1, \lambda_2 \in \mathbb{C}^*$ let

$$x_i y_j \mathbf{v_1} = \lambda_1 y_j \mathbf{v_1} \text{ and } x_i y_j \mathbf{v_2} = \lambda_2 y_j \mathbf{v_2}. \tag{3.2.3}$$

It is clear that $\mathbf{v_1} + \mathbf{v_2} \in E$. Now consider

$$x_i y_j (\mathbf{v_1} + \mathbf{v_2}) = x_i y_j \mathbf{v_1} + x_i y_j \mathbf{v_2} = \lambda_1 y_j \mathbf{v_1} + \lambda_2 y_j \mathbf{v_2}. \tag{3.2.4}$$

Since $y_j(\mathbf{v}_1 + \mathbf{v}_2)$ must be an eigenvector of x_i we have that $\lambda_1 = \lambda_2$ and $y_j \cdot E = E_j$ for some $E_j \in \mathcal{E}$. Since y_j is invertible it preserves dimension and $\dim(E_{j_1}) = \dim(E_{j_2})$ for $j_1, j_2 \leq t$. Also, since y_j was arbitrary and ρ is irreducible, $\mathrm{span}(\langle Y \rangle \cdot E)$ must be the entire space $\mathbb{C}^{p^N}$ and thus Y must act transitively on $\mathcal{E}$. It follows, by counting, that $t = p^r$ and $\dim(E_j) = p^s$ for $j \leq t$ and r, s such that $r + s = N$.

Let $B = \langle b_1, \ldots, b_m \rangle$. For an eigenspace $E \in \mathcal{E}$ let $S = \mathrm{Stab}_B(E)$, and let $Y_* := \rho(S)$. Let $W \subseteq E$ be a S-stable subspace. Let $H = \langle S, a_1 \ldots, a_n \rangle$ and let $\eta : H \to GL_{p^s}(E)$ be the restriction of ρ to H. Since E is a mutual eigenspace of each x_i it is clear that $\eta(a_i) = \Lambda_i I$ for some scalars Λ_i and thus the η-stable subspaces are the S-stable subspaces. Consider the B-orbit of W, say O. Since ρ is irreducible then $\dim(O) = p^N$ and since W is S-stable it must be that $W = E$. Thus E has no proper stable subspaces and η is irreducible. By [21] η factors through a finite quotient up to twisting. By assumption and Schur's Lemma, since S is abelian, each $y_* \in Y_*$ must be a scalar matrix. Thus, since η is irreducible, $\dim(E) = 1$. It then follows that all mutual eigenspaces of X are 1-dimensional and $|\mathcal{E}| = p^N$. since $\langle Y \rangle$ acts transitively on $\mathcal{E}$. □

We now concentrate our efforts specifically on $H_{\sqrt{d}}$. In this regard, we have the following lemma.

Lemma 3.2.3. *Let $\rho : G \to GL_N(\mathbb{C})$ be an irreducible representation and let $J = \{x, y, x_d, y_d\}$. Then there exists a representation $\chi : H_{\sqrt{d}} \to GL_1(\mathbb{C})$ such that 1 is an eigenvalue of $\chi \otimes \rho(j)$ for each $j \in J$.*

Proof. Let $\rho : H_{\sqrt{d}} \to GL_N(\mathbb{C})$ be an irreducible representation and for each $j \in J$ let λ_j be an eigenvalue of $\rho(j)$. We can twist any irreducible representation by any 1-dimensional representation and remain in the same twist isoclass. We deduce that we can choose a 1-dimensional representation χ such that $\chi(j) = (\lambda_j)^{-1}$. □

We call a representation *good* if 1 is an eigenvalue of all of the images of the non-central generators.

We will show that the images of the generators of any irreducible representation of $H_{\sqrt{d}}$, up to twisting, can be written as matrices in a certain standard form. We also show that any set of matrices satisfying this form is, in fact, an irreducible representation of $H_{\sqrt{d}}$. Finally, if two irreducible representations are not twist-equivalent, this standard form will necessarily differ for each representation.

Let p be a prime, $n \geq 1$, and $\rho : H_{\sqrt{d}} \to GL_{p^N}(\mathbb{C})$ be a good irreducible representation. Let

$$\begin{aligned} A &:= \rho(x) & A_d &:= \rho(x_d) \\ B &:= \rho(y) & B_d &:= \rho(y_d) \\ \Lambda &:= \rho(z) & \Lambda_d &:= \rho(z_d). \end{aligned}$$

Our aim is to determine conditions for Λ and Λ_d such that ρ is irreducible. Once we establish these conditions, we choose a basis for $\mathbb{C}^{p^N}$ such that the images of our generators in $GL_{p^N}(\mathbb{C})$ are in a "nice" form. This basis will be chosen so that A and A_d are diagonal matrices, B and B_d are block permutation matrices, and Λ and Λ_d are scalar matrices. To avoid extra notation, for the rest of the chapter we do not distinguish between a linear operator and its matrix with respect to some basis.

Since z and z_d are central in $H_{\sqrt{d}}$, by Schur's lemma we must have that Λ and Λ_d are homotheties; that is, scalar multiples of the identity matrix I. By [21, Theorem 6.6], ρ, up to twisting, factors through a finite quotient of $H_{\sqrt{d}}$, say $H_{\sqrt{d}}(p^N)$. Therefore, without loss of generality, the representation ρ is such that the images of elements of $H_{\sqrt{d}}$ under ρ must have finite order. Hence, for every $g \in H_{\sqrt{d}}(p^N)$ we have $g^k = e$ for some minimal k. It then follows that $\big(\rho(g)\big)^k = I$ and the minimum polynomial of $\rho(g)$ is $x^k - 1$. Since the kth roots of unity are distinct, this polynomial factors over the complex numbers into k distinct linear factors. Thus A, A_d, B, and B_d must be diagonalizable and have eigenvalues which are roots of unity, and Λ and Λ_d must be roots of unity. It is important to note that twisting does not affect the diagonalizability of the images of the generators; twisting is simply multiplication by scalars. Also, since $[A, A_d] = I$, A and A_d are simultaneously diagonalizable.

We identify scalars with scalar matrices; in particular, for λ a root of unity, we will call the matrix λI a root of unity as well.

Since ρ is irreducible the mutual eigenspaces of A, A_d are 1-dimensional. Also, since there are p^N mutual eigenspaces it is clear that $s(\Lambda), s(\Lambda_d) \leq N$ (see Definition 2.4.2 for notation). We let $r = \max\{s(\Lambda), s(\Lambda_d)\}$. We can regard the mutual eigenspace of A and A_d as 2-tuples of the form (λ, λ_d) for eigenvalues λ of A and λ_d of A_d. We choose a mutual eigenspace $E := (\lambda, \lambda_d)$.

We calculate which choices of Λ and Λ_d are permissible such that ρ is irreducible. The structure of the calculation depends on the comparative depth of Λ and Λ_d. We break the calculation into two cases: when $s(\Lambda) \geq s(\Lambda_d)$ and when $s(\Lambda) < s(\Lambda_d)$. Let $\Lambda_{\text{deep}} = \Lambda$ in *Case 1* and $\Lambda_{\text{deep}} = \Lambda_d$ in *Case 2*. Also, let Λ_{shallow} be the other root of unity in each separate case. We write Λ_{shallow} in terms of Λ_{deep}; that is,

$$\Lambda_{\text{shallow}} = \Lambda_{\text{deep}}^{\ell} \tag{3.2.5}$$

for some ℓ. Note that $p \mid \ell$ in Case 2. Here, we remind the reader that by the group relations of $H_{\sqrt{d}}$ we have that $D = d$ if $d = 2, 3 \bmod 4$ and $D = \ell + \frac{d-1}{4}$ if $d = 1 \bmod 4$. By the group relations,

$$\begin{aligned} B \cdot (\lambda, \lambda_d) &= (\Lambda_{\text{deep}}\lambda, \Lambda_{\text{deep}}^{\ell}\lambda_d) \\ B_d \cdot (\lambda, \lambda_d) &= (\Lambda_{\text{deep}}^{\ell}\lambda, \Lambda_{\text{deep}}^{D}\lambda_d) \end{aligned} \tag{3.2.6}$$

in Case 1 and similarly

$$\begin{aligned} B \cdot (\lambda, \lambda_d) &= (\Lambda_{\text{deep}}^{\ell}\lambda, \Lambda_{\text{deep}}\lambda_d) \\ B_d \cdot (\lambda, \lambda_d) &= (\Lambda_{\text{deep}}\lambda, \Lambda_{\text{deep}}^{\ell D}\lambda_d) \end{aligned} \tag{3.2.7}$$

We twist ρ such that $E = (1, 1)$ and note that for any eigenspace $(\mu, \mu_d) \in \langle B, B_d\rangle \cdot E$ we have that $s(\mu), s(\mu_d) \leq r$.

We continue by taking the logarithm base Λ_{deep} for all eigenvalues of A and A_d. For the rest of the calculation we consider each 2-tuple mod p^r. Thus

$$\begin{aligned} B \cdot (\log(1), \log(1)) = B \cdot (0, 0) &= (1, \ell) \\ B_d \cdot (0, 0) &= (\ell, D) \end{aligned} \tag{3.2.8}$$

in Case 1 and

$$\begin{aligned} B \cdot (0, 0) &= (\ell, 1) \\ B_d \cdot (0, 0) &= (1, \ell D) \end{aligned} \tag{3.2.9}$$

in Case 2. Thus, the action

$$\langle B, B_d\rangle \cdot (0, 0) = \mathcal{E}_1 := \{(a + b\ell, a\ell + bD) \mid 0 \leq a, b, < p^r\} \tag{3.2.10}$$

in Case 1 and

$$\langle B, B_d\rangle \cdot (0, 0) = \mathcal{E}_2 := \{(a\ell + b, a + b\ell D) \mid 0 \leq a, b < p^r\} \tag{3.2.11}$$

in Case 2.

There are at most p^{2r} different 2-tuples in each case. Since we want ρ to be p^N-dimensional irreducible it follows that $r \geq N/2$ and that there are exactly p^N different 2-tuples in each case. In this vein, we have the following lemma:

Lemma 3.2.4. *In Case 1, let m be minimal such that $\ell^2 = D \bmod p^{r-m}$. Then, for each $a_1 < p^r$ there is an $a_2 < p^r$ such that*

$$\big(a_1 + (b+p^m)\ell, a_1\ell + (b+p^m)D\big) = \big(a_2 + b\ell, a_2\ell + bD\big). \tag{3.2.12}$$

Additionally, each 2-tuple $(a+b\ell, a\ell+bD)$ such that $0 \leq a < p^r, 0 \leq b < p^m$ is distinct.

In Case 2, let m be minimal such that $\ell^2 D = 1 \bmod p^{r-m}$. Then

$$\big(a_1\ell + (b+p^m), a_1 + (b+p^m)\ell D\big) = \big(a_2\ell + b, a_2 + b\ell D\big). \tag{3.2.13}$$

Additionally, each 2-tuple $(a\ell+b, a+b\ell D)$ such that $0 \leq a < p^r, 0 \leq b < p^m$ is distinct.

Thus, ρ is irreducible if and only if $N = r+m$.

Proof. We begin with Case 1. We have, by assumption, that $\ell^2 = D \bmod p^{r-m}$. This implies that

$$p^m\ell^2 = p^m D \bmod p^r. \tag{3.2.14}$$

Choose a_2 such that

$$a_2 - a_1 = p^m\ell \bmod p^r. \tag{3.2.15}$$

Then Equation 3.2.14 implies that

$$\begin{aligned} &(a_2 - a_1)\ell = p^m D \bmod p^r \\ \Rightarrow &(a_2 - a_1)\ell = (b + p^m - b)D \bmod p^r \\ \Rightarrow &a_1\ell + (b+p^m)D = a_2 + bD \bmod p^r. \end{aligned} \tag{3.2.16}$$

By Equation 3.2.15

$$\begin{aligned} &a_2 - a_1 = (b + p^m - b)\ell \bmod p^r \\ \Rightarrow &a_1 + (b+p^m)\ell = a_2 + b\ell \bmod p^r. \end{aligned} \tag{3.2.17}$$

To prove that each 2-tuple is distinct, suppose, for $0 \leq a_1, a_2 < p^r$ and $0 \leq b_1, b_2 < p^m$, that

$$(a_1 + b_1\ell, a_1\ell + b_1 D) = (a_2 + b_2\ell, a_2\ell + b_2 D). \tag{3.2.18}$$

Splitting entry-wise and rearranging, Equation 3.2.18 implies that

$$a_2 - a_1 = (b_1 - b_2)\ell \bmod p^r \tag{3.2.19}$$

$$(a_2 - a_1)\ell = (b_1 - b_2)D \bmod p^r. \tag{3.2.20}$$

Combining Equations 3.2.19 and 3.2.20

$$(b_1 - b_2)\ell^2 = (b_1 - b_2)D \bmod p^r. \tag{3.2.21}$$

For some maximal $k \leq m$ and for some α where $p \nmid \alpha$, we have that $(b_1 - b_2) = \alpha p^k$. Thus Equation 3.2.21 implies that

$$\alpha p^k \ell^2 = \alpha p^k D \bmod p^r \tag{3.2.22}$$

$$\Rightarrow \ell^2 = D \bmod p^{r-k}. \tag{3.2.23}$$

But, since, by assumption, m is minimal such that $\ell^2 = D \bmod p^{r-m}$ it must be that $k = m$ and thus $b_1 = b_2$. since $b_1, b_2 < p^m$. It then follows immediately by Equation 3.2.19 that $a_1 = a_2$.

The details of the calculation for Case 2 are similar. Note that in Case 2 the equivalent of Equation 3.2.15 is choosing a_2 such that

$$a_2 - a_1 = p^m \ell D \bmod p^r. \tag{3.2.24}$$

Assume ρ is p^N-dimensional irreducible. Then, by the argument above, we have p^{r+m} distinct 2-tuples. By Lemma 3.2.2 there are p^N distinct 2-tuples and thus $r+m = N$. Now assume $r + m = N$. Then by the argument above we have p^N distinct 2-tuples and thus, by Lemma 3.2.2, ρ must be p^N-dimensional irreducible. □

3.3 Choosing a Basis

Let ρ be an irreducible representation. We write ρ in matrix form with a standard basis. We show the construction for Case 1; Case 2 is similar.

We begin with the following lemma. This lemma will give us the basis structure for A and B.

Lemma 3.3.1. *Let $s(\Lambda) = r$. The matrices A and B have p^r eigenspaces of dimension p^m, where $r + m = N$. Similarly, B_d has, at most, p^r eigenspaces of dimension at least p^m.*

Proof. Let E_1 be an eigenspace of A with, by twisting if necessary, eigenvalue 1. Since $[A, B] = \Lambda$ it is easy to show that $B \cdot E_1$ is also an eigenspace of A. Let $J_1 = \langle B \rangle \cdot E_1$. Thus, it is clear that $|J_1| \geq p^r$. Also, since $[A, B_d] = \Lambda_d$ and $s(\Lambda_d) \leq s(\Lambda)$, we have, for $K_1 := \langle B_d \rangle \cdot E_1 \subseteq J_1$. But since ρ is irreducible the action of $\langle B \rangle$ on the eigenspaces of A must be transitive. Thus $|J_1| = p^r$. By symmetry, if E_2 is, by twisting if necessary, the eigenspace of B with eigenvalue 1, a similar argument shows that the action $J_2 := \{\langle A \rangle \cdot E_2\}$ is transitive and $|J_2| = p^r$. Noting that $s(\Lambda_d) \leq s(\Lambda)$ and $s(\Lambda^{\ell^2 - D}) \leq s(\Lambda)$ the result for B_d is similar. $\square$

The next corollary follows directly from the previous lemma:

Corollary 3.3.2. $B^{p^r} = B_d^{p^r} = I$.

Let E_{Λ^k} be the eigenspace of A with eigenvalue Λ^k. Let $\mathbf{\Theta}_{\Lambda^k}$ be a basis for E_{Λ^k} where $0 \leq k \leq p^r - 1$. We now choose a basis $\mathbf{\Theta}_1$ and construct the rest of the basis of $\mathbb{C}^{p^N}$ by letting $\mathbf{\Theta}_{\Lambda^k} = B^k \cdot \mathbf{\Theta}_1$. Therefore we have a basis $\mathbf{\Theta} = \mathbf{\Theta}_1 \cup \ldots \cup \mathbf{\Theta}_{\Lambda^{p^r-1}}$ for $\mathbb{C}^{p^N}$ with respect to which A and A_d are diagonal. Since ρ is good, 1 is an eigenvalue of A and A_d. Note, twisting ρ if necessary, that $(1, 1)$ is a mutual eigenspace of A and A_d and thus we can choose our basis such that $(A_d)_{1,1} = 1$.

In Case 1

$$A = \begin{pmatrix} I_{p^m} & & & \\ & \Lambda I_{p^m} & & \\ & & \ddots & \\ & & & \Lambda^{p^r-1} I_{p^m} \end{pmatrix}, \tag{3.3.1}$$

$$B = \begin{pmatrix} 0_{p^m} & & & P \\ I_{p^m} & \ddots & & \\ & \ddots & \ddots & \\ & & I_{p^m} & 0_{p^m} \end{pmatrix} \tag{3.3.2}$$

where I_{p^m} and 0_{p^m} are, respectively, the identity and null matrices of size p^m. By Corollary 3.3.2 we have that $B^{p^r} = I$ and thus $P = I_{p^m}$.

Since $[B, B_d] = I, [A, B_d] = \Lambda_d$, and $\Lambda_d = \Lambda^\ell$, a simple computation shows that the matrix B_d is

$$B_d = \begin{pmatrix} 0_{p^m} & & & R & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & R \\ R & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & R & & & 0_{p^m} \end{pmatrix} \tag{3.3.3}$$

for some matrix block R, and the R in the first column is in the ℓth row. We determine R later.

Finally, A_d is the matrix

$$A_d = \begin{pmatrix} J & & & \\ & \Lambda^\ell J & & \\ & & \ddots & \\ & & & \Lambda^{(p^r-1)\ell} J \end{pmatrix} \tag{3.3.4}$$

for some block J. by the relation $[A_d, B] = \Lambda_d$.

Using the relation $[A_d, B_d] = \Lambda^{\ell^2 - D}$ a straightforward computation shows that $[J, R] = \Lambda^{\ell^2 - D}$. So by Lemma 3.2.2, if E is the eigenspace of J with eigenvalue 1, then $R^k \cdot E = \Lambda^{k(\ell^2-D)} E$ for any k. But $R^k \cdot E = E$ only when $p^m \mid k$, since $\ell^2 = D \bmod p^{r-m}$ exactly. Since J and R are of p^m-dimensional, J has p^m eigenspaces, each of dimension 1, and R acts transitively, that is as a p^m-cycle, on these eigenspaces. By this result and the fact that $B_d^{p^r} = I$ by Lemma 3.3.2 we could have chosen $\mathbf{\Theta}_1$ at the beginning of the subsection such that

$$J = \begin{pmatrix} 1 & & & \\ & \Lambda^{\ell^2 - D} & & \\ & & \ddots & \\ & & & \Lambda^{(p^r-1)(\ell^2-D)} \end{pmatrix} \tag{3.3.5}$$

and

$$R = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}. \tag{3.3.6}$$

Thus, we have determined the matrices of the images of the generators of ρ under a canonical basis.

In Case 2

$$A = \begin{pmatrix} I_{p^m} & & & \\ & \Lambda_d I_{p^m} & & \\ & & \ddots & \\ & & & \Lambda_d^{p^r-1} I_{p^m} \end{pmatrix}, \tag{3.3.7}$$

$$B_d = \begin{pmatrix} 0_{p^m} & & & I_{p^m} \\ I_{p^m} & \ddots & & \\ & \ddots & \ddots & \\ & & I_{p^m} & 0_{p^m} \end{pmatrix}, \tag{3.3.8}$$

$$A_d = \begin{pmatrix} J & & & \\ & \Lambda_d^{lD} J & & \\ & & \ddots & \\ & & & \Lambda_d^{(p^r-1)lD} J \end{pmatrix}, \tag{3.3.9}$$

and

$$B = \begin{pmatrix} 0 & & & R & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & R \\ R & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & R & & & 0 \end{pmatrix} \tag{3.3.10}$$

where the block R in the first column is in the ℓth row,

$$J = \begin{pmatrix} 1 & & & \\ & \Lambda_d^{l^2D-1} & & \\ & & \ddots & \\ & & & \Lambda_d^{(p^r-1)(l^2D-1)} \end{pmatrix} \tag{3.3.11}$$

and

$$R = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}. \tag{3.3.12}$$

Now we check that all matrices of this form give us an irreducible representation of $H_{\sqrt{d}}$. This is clear; take matrices $\{A, B, A_d, B_d, \Lambda, \Lambda_d\}$ of the forms above. Then an easy calculation shows that the associated relations for $H_{\sqrt{d}}$ hold. Since z, z_d are commutators, they remain fixed under twisting. Since the matrices A, B, A_d, and B_d are determined by Λ and Λ_d, two such ordered sets of matrices define twist-equivalent representations if and only if they coincide.

3.4 Counting Twist Isoclasses

For ease of discussion we introduce some notation. We call

$$\ell^2 = D \bmod p^{r-m}, \qquad \ell^2 \neq D \bmod p^{r-m+1} \tag{3.4.1}$$

the *Case 1 Conditions* and

$$\ell^2 D = 1 \bmod p^{r-m}, \qquad \ell^2 D \neq 1 \bmod p^{r-m+1}, \qquad p|\ell \tag{3.4.2}$$

the *Case 2 Conditions*. Note that the last condition of the Case 2 Conditions is because $s(\Lambda) < s(\Lambda_d)$.

The Case 2 Conditions imply that l is invertible mod p. However, since $p|\ell$ in Case 2, there are only solutions to the Case 2 Conditions when $r = m$. Also note that, since our roots of unity are elements of S_p^N the second condition of the Case 1 and Case 2 conditions does not apply when $r = N$.

The previous section allows us to calculate each $\zeta^{irr}_{H_{\sqrt{d}},p}(s)$ by counting solutions to the Case 1 and Case 2 Conditions for each $N \geq 1$. To count the number of solutions we use Hensel's Lemma to lift solutions of the Conditions mod p if p is not ramified; if p is ramified, the computation is nevertheless straightforward. We demonstrate the computations and then summarize the results in a table. We note three things: there are always $(1-p^{-1})p^r$ choices for Λ under Case 1 and Λ_d under Case 2, under Case 2 it is easy to see there are $(1-p^{-1})p^{N-1}$ solutions when $r = m$ and 0 otherwise, and by Lemma 2.3.1 there is only 1 irreducible twist-isoclass when $N = 0$. The following cases assume $N \neq 0$.

Assume p is inert. In this case there are no solutions to the Case 1 or Case 2 Conditions unless $r = m$. Given Λ there are $p^{\frac{N}{2}}$ choices for Λ_d under Case 1. Therefore, with Case 2 contributing $(1-p^{-1})p^{\frac{N}{2}}(p^{\frac{N}{2}-1})$ terms in the even case,

$$r_{p^N} = \begin{cases} (1-p^{-1})p^{\frac{N}{2}}p^{\frac{N}{2}} + (1-p^{-1})p^{\frac{N}{2}}(p^{\frac{N}{2}-1}) & \text{for even } N \\ 0 \quad \text{for odd } N \end{cases} \tag{3.4.3}$$

and

$$\begin{aligned}
\zeta^{irr}_{H_{\sqrt{d}},p}(s) &= \sum_{N=0}^{\infty} r_{p^N} p^{-Ns} \\
&= 1 + \sum_{N=2,\ N \text{ is even}}^{\infty} (1-p^{-1})p^N + (1-p^{-1})p^{N-1} \\
&= 1 + \sum_{M=1}^{\infty} (1-p^{-2})(p^{2-2s})^M \\
&= \frac{1-p^{-2s}}{1-p^{2-2s}}.
\end{aligned} \tag{3.4.4}$$

Assume p splits. There are two solutions to the equation $l^2 = D \bmod p$ and Hensel's Lemma allows us to "lift" these solutions to solutions in $\mathbb{Z}/p^{r-m}\mathbb{Z}$, thus giving us the 2 unique solutions to $l^2 = D \bmod p^{r-m}$. When $r = N$, there are two solutions to the first condition of the Case 1 Conditions and therefore $2(1-p^{-1})p^N$ choices for the pair Λ and Λ_d under the Case 1 Conditions. If, for fixed r and m, $r > m$ and $m > 0$ then there are two solutions in $\mathbb{Z}/p^{r-m}\mathbb{Z}$ to the first condition of the Case 1 Conditions and therefore $2p^m$ solutions for $0 \leq l \leq p^r - 1$. Of these solutions, all but $2p^{m-1}$ satisfy the second condition of the Case 1 Conditions. Therefore, given Λ, there are $2(1-p^{-1})p^m$ choices for Λ_d and, in total, $2(1-p^{-1})^2p^N$ choices for the pair Λ and Λ_d in Case 1. If $r = m = \frac{N}{2}$ then there are p^m solutions to the first condition of the Case 1 Conditions, of which all but $2p^{m-1}$ satisfy the second condition of the Case 1 Conditions. Therefore there are $(1-2p^{-1})(1-p^{-1})p^N$ choices for the pair Λ and Λ_d in Case 1. Summing all cases together, and noting that the Case 2 contribution is 0 in the odd case and $(1-p^{-1})p^{N-1}$ in the even case, we have

$$r_{p^N} = \begin{cases} 2(1-p^{-1})p^N + \frac{N-1}{2}2(1-p^{-1})^2p^N & \text{if } N \text{ is odd} \\ 2(1-p^{-1})p^N + \frac{N-2}{2}2(1-p^{-1})^2p^N & \\ \quad +(1-2p^{-1})(1-p^{-1})p^N + (1-p^{-1})p^{N-1} & \text{if } N \text{ is even.} \end{cases} \tag{3.4.5}$$

Strikingly, in both cases this simplifies to

$$r_{p^N} = ((1+p^{-1}) + (1-p^{-1})N)(1-p^{-1})p^N$$

and therefore

$$\begin{aligned}
\zeta^{irr}_{H_{\sqrt{d}},p}(s) &= \sum_{N=0}^{\infty} r_{p^N} p^{-Ns} \qquad (3.4.6)\\
&= 1 + \sum_{N=1}^{\infty} (1-p^{-1})(p^{1-s})^N[(1+p^{-1}) + (1-p^{-1})N]\\
&= 1 + \sum_{N=1}^{\infty} (1-p^{-2})(p^{1-s})^N + \sum_{N=1}^{\infty} (1-p^{-1})^2 N (p^{1-s})^N\\
&= 1 + \frac{(1-p^{-2})p^{1-s}}{1-p^{1-s}} + \frac{(1-p^{-1})^2 p^{1-s}}{(1-p^{1-s})^2}\\
&= \left(\frac{1-p^{-s}}{1-p^{1-s}}\right)^2 .
\end{aligned}$$

Assume p is ramified. This is the case if $d = 0 \bmod p$ for any d or if $p = 2$ and $d = 2, 3 \bmod 4$. Then there are solutions to the Case 1 and Case 2 Conditions when $r - m = 0$ or $r - m = 1$. We break this computation into sections depending on d and p.

If $d = 2, 3 \bmod 4$, $p \neq 2$, and $d = 0 \bmod p$ then 0 is the solution to $l^2 = d \bmod p$. However, since d is squarefree, $d = kp \bmod p^2$ for some invertible $k \in \mathbb{Z}/p^2\mathbb{Z}$. Therefore $l^2 = d$ has no solutions mod p^2.

If $d = 2, 3 \bmod 4$ and $p = 2$ then $l^2 = d$ has a solution mod 2. However, since d is a quadratic non-residue mod 4, it has no solutions mod 4.

If $d = 1 \bmod 4$ and $d = 0 \bmod p$ then $l^2 - l + \frac{d-1}{4} = 0$ has the unique double solution $l = 2^{-1} \bmod p$. And $d = kp \bmod p^2$ for some invertible $k \in \mathbb{Z}/p^2\mathbb{Z}$, since d is squarefree. Then the above equation can be rearranged to the form $(2l-1)^2 = kp \bmod p^2$, which clearly has no solutions.

Thus, if $r - m = 0$, then there are p^m solutions to the first condition of the Case 1 Conditions and all but p^{m-1} of these satisfy the second. Therefore, given Λ there are $(1-p^{-1})p^m$ choices for Λ_d and $(1-p^{-1})(1-p^{-1})p^N$ choices for the pair Λ and Λ_d in Case 1 and, as usual, $(1-p^{-1})p^{N-1}$ choices in Case 2. If $r - m = 1$, a similar calculation to the ones above yield that there are $(1-p^{-1})p^N$ choices in Case 1. Therefore

$$r_{p^N} = (1-p^{-1})p^N$$

and a routine calculation gives

$$\zeta^{irr}_{H_{\sqrt{d}},p}(s) = \frac{1-p^{-s}}{1-p^{1-s}}.$$

For easy reference, we now tabulate the preceding results in Table 3.1:

Table 3.1 p-local Representation Zeta Functions of $H_{\sqrt{d}}$

prime behaviour	$r_{p^N}; N > 0$	$\zeta^{irr}_{H_{\sqrt{d}},p}(s)$
inert	$(1+p^{-1})\phi(p^N)$ for even N; 0 for odd N	$\frac{1-p^{-2s}}{1-p^{2-2s}}$
splits	$\phi(p^N)(N(1-p^{-1})+1+p^{-1})$	$\left(\frac{1-p^{-s}}{1-p^{1-s}}\right)^2$
ramified	$\phi(p^N)$	$\frac{1-p^{-s}}{1-p^{1-s}}$

It is easy to see that $\alpha_{H_{\sqrt{d}},p} = 1$ for every prime p. By Proposition 2.2.6 we can say that

$$\zeta^{irr}_{H_{\sqrt{d}}}(s) = \frac{\zeta^{\mathbf{D}}_{\mathbb{Q}[\sqrt{d}]}(s-1)}{\zeta^{\mathbf{D}}_{\mathbb{Q}[\sqrt{d}]}(s)}.$$

3.5 Results of Stasinski and Voll

Groups of the following type are an important class of examples: let S be a group scheme defined over a ring of integers O of a number field K, let O' be a finite extension of O, and let $S(O')$ be the group of O'-points of S. Suppose the group scheme S is unipotent. Stasinski and Voll, in [28, Theorem A] and [28, Remark 2.3], show that for non-zero prime ideals $P \lhd O'$ the following Euler factorization holds:

$$\zeta_{S(O')}(s) = \prod_{P} \zeta_{S(O'),P}(s) \tag{3.5.1}$$

where $\zeta_{S(O'),P}(s)$ counts continuous representations of $S(O'_P)$ and where O'_P is the completion of O' at P. For group schemes associated to a nilpotent Lie lattice the authors also show that, for almost all prime ideals, the local representation zeta functions behave uniformly under extension of scalars; that is, if O_1 and O_2 are finite extensions of O then the local representation zeta functions of $S(O_1)$ and $S(O_2)$ can be viewed as coming from a rational function whose form does not depend on the choice of extension (see [28, Theorem A] for details). Theorem A also tells us that almost all P-local representation zeta functions satisfy a functional equation which is a refinement of [29, Theorem D] (see our Theorem 3.5.1 below) and, additionally, that these local zeta functions are rational functions in q^{-fs}, where q is the cardinality of the associated residue field and f is the relative degree of inertia. Moreover, [28, Theorem B] calculates P-local representation zeta functions of three families of groups arising from unipotent group schemes.

As a special case of [28, Theorem B], Stasinski and Voll prove the following result which generalizes Theorem 1.4.6.

Theorem 3.5.1. *Let K be an arbitrary algebraic number field and O its ring of integers. Then*

$$\zeta_{H(O)}(s) = \frac{\zeta_K^{\mathbf{D}}(s-1)}{\zeta_K^{\mathbf{D}}(s)}.$$

We remark that Theorem 3.5.1 was stated as a conjecture in an earlier version of [10], before the appearance of [28].

We note that $\zeta^{irr}_{H_{\sqrt{d}},p}(s)$ does indeed satisfy the functional equation of Theorem 1.4.3 if p is not ramified. Also, these p-local zeta functions satisfy the more general functional equation, for all primes p, of Theorem 1.4.4. If the rational prime $p = \prod_{i=1}^{j} P_i$, where P_i is a prime ideal of O_d and $j \in \{1, 2\}$, is unramified, then it is easy to see that the Euler product of the p-local representation zeta functions of $H_{\sqrt{d}}$ can be refined to an Euler product of P_i-local Dedekind zeta functions by factorization.

Chapter 4

Representation Growth of M_n

4.1 Introduction

This chapter will feature the construction of the irreducible representations of a family of maximal class groups of nilpotency class n, denoted M_n. This is achieved by calculating the irreducible representations of p-power dimension. Due to the large abelian subgroup inside M_n, we can apply Lemma 3.2.2 and thus representation theory of M_n is a good candidate for calculation by the constructive method. The large abelian subgroup allows us to simultaneously diagonalize all but one element of the images of the generators and reflected by this fact is the relatively simple eigenspace structure of the irreducible representations. We note that the calculation is uniform for most primes, in fact primes p not less than the nilpotency class n; denominators that appear in the matrices of the representation are smaller than the prime considered and therefore behave as units mod p. When the prime considered is smaller than n, the calculation loses its uniformity and the structure of the matrices of the representation differs from the non-exceptional cases. We will calculate the p-power irreducible representations for certain well-behaved exceptional primes. Once we have calculated the representation theory of M_n, for almost all primes, we calculate the p-local representation zeta function. We show that for non-exceptional primes, and some exceptional ones, that the p-local zeta function does indeed satisfy the functional equation established by Voll (Theorem 1.4.3).

4.2 Basic Results

Let $M_n = \langle a_1, \ldots, a_n, b \mid [a_i, b] = a_{i+1} \rangle$. We remind the reader that all commutators that do not appear in the relations are trivial. We calculate the p-local representation zeta function of each group in this family, denoted $\zeta^{irr}_{M_n,p}(s)$, by explicitly constructing representatives of each twist isoclass. Let ρ be a p^N-dimensional irreducible representation of M_n and let $x_i = \rho(a_i)$ and $y = \rho(b)$.

We refer often to certain subgroups of M_n. It is clear that for $2 \leq k < n$ the group M_k is isomorphic to a subgroup of M_n. With a slight abuse of notation we let the subgroup $M_k = \langle a_{n-k+1}, a_{n-k+2}, \ldots, a_n, b \rangle$.

4.2.1 Determining a Basis and Standard Form for the Images of the Generators of ρ

In this section we will choose a basis for the image of ρ such that y is in the form of a p^N-cycle permutation matrix and such that each x_i is diagonal with each diagonal entry in a certain form, discussed later in the section. As in Chapter 3 it is not necessary to state a basis to understand the eigenspace structure of ρ. However, as a canonical basis is easy to determine in this case, we appeal to a basis as an indexing device on the set of mutual eigenspaces of $\{x_1, \ldots, x_n\}$. We begin by considering twisting. Since $x_2, \ldots, x_n$ are commutators they are invariant under twisting. We can twist y and x_1 by any complex number. We remind the reader that we can obtain every p^N-power irreducible representation of M_n by twisting a representative ρ from each twist isoclass.

Since all x_i commute they are all simultaneously diagonalizable. By [21, Theorem 6.6] all irreducible representations factor through a finite quotient (up to twist equivalence) and thus by Schur's lemma the central element x_n is a scalar matrix.

By the group presentation of M_n it is clear we can apply Lemma 3.2.2. Let $X = \{x_1, \ldots, x_n\}$. Then we know the mutual eigenspaces of X are 1-dimensional and that there are p^N distinct mutual eigenspaces. Also, we have that $\langle y \rangle$ must permute the eigenspaces of X transitively. Thus, y must act as a p^N-cycle on the mutual eigenspaces of X. We choose our basis, with basis vectors $\{e_1, \ldots, e_{p^N}\}$ such that the x_i are diagonal and

$$y = \begin{pmatrix} 0 & & & k \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \tag{4.2.1}$$

for some $k \in \mathbb{C}^*$. By cofactor expansion of $y - \lambda I$, the characteristic equation is $\lambda^{p^N} = k$, and thus the eigenvalues of y are all of the p^Nth roots of k. However, we have the freedom to twist y by $k^{\frac{-1}{p^N}}$ and therefore we can ensure the eigenvalues of y are all of the p^Nth roots of unity. We can choose a representative of our twist isoclass such that y is a p^N-cycle permutation matrix under some choice of basis, or equivalently, we can ensure that $k = 1$.

We set up some notation. Let $\lambda_{i,j}$ be the jth entry on the diagonal of x_i. We let $x_n = \lambda_n I$ and $\lambda_i = \lambda_{i,1}$. Note that $\lambda_{n,j} = \lambda_n$ for all j. By twisting we can ensure that $\lambda_1 = 1$. It will be shown that the $\lambda_{i,j}$, and thus ρ, are determined by the λ_ℓ, for all ℓ such that $i \le \ell \le n$.

We now determine the structure of the matrices x_i and the allowable values for the λ_i. The next lemma is the base case for the inductive lemma following it. Although we could start the induction with x_n, this case is trivial. For purposes of elucidation, we start this induction with x_{n-1}. Note that the following two lemmas are true for all primes.

Lemma 4.2.1. *The matrix x_{n-1} has the form*

$$x_{n-1} = \begin{pmatrix} \lambda_{n-1} & & & \\ & \lambda_n \lambda_{n-1} & & \\ & & \ddots & \\ & & & \lambda_n^{p^N-1}\lambda_{n-1} \end{pmatrix}. \tag{4.2.2}$$

Moreover, λ_n is a p^Nth root of unity for any prime p; that is $s(\lambda_n) \le N$.

Proof. Since by our group relations $[x_{n-1}, y] = x_n = \lambda_n$ we have that $\lambda_{n-1,j+1} = \lambda_n \lambda_{n-1,j}$ for $j = 1, \ldots, p^N - 1$ and $\lambda_{n-1,1} = \lambda_n \lambda_{n-1,p^N}$. Combining these equations, $\lambda_{n-1} = \lambda_n^{p^N} \lambda_{n-1}$ and therefore $\lambda_n \in S_p^N$. □

We remind readers of Lemma 2.4.3 for properties of numbers $T_k(i)$. We define these numbers in the paragraph directly before Lemma 2.4.3.

Lemma 4.2.2. *For $1 \leq i \leq n-1$ we have that $\lambda_{i,j} = \prod_{k=i}^{n} \lambda_k^{T_{k-i}(j-1)}$ and thus the matrix x_i has the structure*

$$x_i = \begin{pmatrix} \lambda_i & & & \\ & \prod_{k=i}^{n} \lambda_k^{T_{k-i}(1)} & & \\ & & \ddots & \\ & & & \prod_{k=i}^{n} \lambda_k^{T_{k-i}(p^N-1)} \end{pmatrix}. \tag{4.2.3}$$

Moreover

$$\lambda_i^{p^N} \prod_{k=i+1}^{n} \lambda_k^{T_{k-i}(p^N-1)} = 1. \tag{4.2.4}$$

Proof. Assume

$$x_i = \begin{pmatrix} \lambda_i & & & \\ & \prod_{k=i}^{n} \lambda_k^{T_{k-i}(1)} & & \\ & & \ddots & \\ & & & \prod_{k=i}^{n} \lambda_k^{T_{k-i}(p^N-1)} \end{pmatrix} \tag{4.2.5}$$

for some i. By the group relation $[x_{i-1}, y] = x_i$ we have, for some $j \leq p^N$-1, that

$$\lambda_{i-1,j+1} = \lambda_{i,j+1}\lambda_{i-1,j} = \left(\prod_{k=i}^{n} \lambda_k^{T_{k-i}(j)}\right) \lambda_{i-1,j} \tag{4.2.6}$$

and

$$\lambda_{i-1,1} = \lambda_i \lambda_{i-1,p^N}. \tag{4.2.7}$$

Combining the above equations for each j,

$$\lambda_{i-1,j+1} = \lambda_{i-1} \prod_{k=i}^{n} \lambda_k^{\sum_{l=1}^{j} T_{k-i}(l)} = \lambda_{i-1} \prod_{k=i}^{n} \lambda_k^{T_{k-i+1}(j)} \tag{4.2.8}$$

and

$$\lambda_{i-1,1} = \lambda_{i-1} \lambda_i^{p^N} \prod_{k=i+1}^{n} \lambda_k^{T_{k-i}(p^N-1)}. \tag{4.2.9}$$

□

We have shown that, up to twisting and isomorphism, that any irreducible representation must be of the form given above. We give this a name.

Definition 4.2.3. The matrices $x_1, \ldots, x_n, y$ are in *standard form* if the x_i are in the form of Lemma 4.2.2 and y is in the form of Equation 4.2.1. We say ρ is in *standard form* if, under a chosen basis, the matrices $x_1, \ldots, x_n, y$ are in standard form.

4.2.2 Determining Possible Stable Subspaces

In this section we determine possible stable subspaces of a representation ρ in standard form. We show that if ρ is not irreducible then it must have a certain proper stable subspace; we name this $V_{p^{N-1}}$. Thus, to determine if ρ is irreducible, we only need to check if $V_{p^{N-1}}$ is a stable subspace of ρ. In this vein, let $0 \leq k \leq N$ and let V_{p^k} be the subspace spanned by $\langle y \rangle \cdot (e_1 + e_{p^k+1} + \ldots + e_{(p^{N-k}-1)p^k+1})$. Note two things: first, V_{p^k} has dimension p^k as $\langle y \rangle \cdot (e_1 + e_{p^k+1} + \ldots + e_{(p^{N-k}-1)p^k+1})$ consists of p^k linearly independent vectors; second, if V_{p^k} is a stable subspace of ρ then so is V_{p^j} for $j \geq k$.

We define the n-tuple $\Lambda(k) := (\lambda_{1,k}, \ldots, \lambda_{n,k})$ where k is considered mod p^N.

Lemma 4.2.4. *For any k_1, k_2 if $\Lambda(k_1) = \Lambda(k_2)$ then $\Lambda(k_1 + 1) = \Lambda(k_2 + 1)$.*

Proof. By Lemma 4.2.2, $\lambda_i \in S_p^\infty$ for all i and that $\lambda_{i,j+1} = \lambda_{i+1,j+1}\lambda_{i,j}$ for all j. Consider $\Lambda(k_1 + 1)$. It is clear to see that $\lambda_{n,k_1+1} = \lambda_{n,k_2+1}$ since λ_n is central. Now, as our inductive step, choose h such that $h \leq n - 1$ and assume that $\lambda_{i,k_1+1} = \lambda_{i,k_2+1}$ for all $i > h$ Consider λ_{h,k_1+1}. Then

$$\lambda_{h,k_1+1} = \lambda_{h+1,k_1+1}\lambda_{h,k_1}. \tag{4.2.10}$$

Our inductive hypothesis holds for the first factor of the right hand side of Equation 4.2.10 and the initial assumption holds for the second factor. Thus

$$\lambda_{h,k_1+1} = \lambda_{h+1,k_1+1}\lambda_{h,k_1} = \lambda_{h+1,k_2+1}\lambda_{h,k_2} = \lambda_{h,k_2+1}. \tag{4.2.11}$$

□

Since ρ is of dimension p^N, Lemma 4.2.4 and elementary counting tells us that if, for some β_*, j, and k, $\Lambda(k) = \Lambda(\beta_* p^j + k)$ where $p \nmid \beta_*$ then $\Lambda(k) = \Lambda(\beta p^j + k)$ for all β such that $0 \leq \beta \leq p^{N-j} - 1$. This can be seen since β_* is a unit in the additive group

$\mathbb{Z}/p^{N-j}\mathbb{Z}$ and thus β_* generates all of $\mathbb{Z}/p^{N-j}\mathbb{Z}$. This argument gives us the following corollary of Lemma 4.2.4.

Corollary 4.2.5. *Let j be the smallest non-negative integer such that $\Lambda(k) = \Lambda(\beta p^j + k)$ for all β such that $0 \leq \beta \leq p^{N-j} - 1$ and for any k. Then V_{p^j} is a stable subspace of ρ and $V_{p^{j-1}}$ is not stable.*

We define notation to this effect. Let $H \leq M_n$ and let $\mathcal{V}(\rho|_H)$ be the minimal stable subspace V_{p^j}, as in Corollary 4.2.5, of $\rho|_H$. We say that $\mathcal{V}(\rho) = \mathcal{V}\big(\rho(M_n)\big)$.

We can, in fact, say more about this minimal subspace:

Corollary 4.2.6. *If j is minimal such that $\Lambda(1) = \Lambda(p^j + 1)$ then $\mathcal{V}(\rho) = V_{p^j}$.*

We attempt to use Corollary 4.2.6 as little as possible; calculations without the power of this lemma are, in most cases, not much more effort and serve to remind the reader of, and elucidate the reader to, the structure of the V_{p^j}.

Corollary 4.2.7. *Let $\rho : M_n \to GL_{p^N}(\mathbb{C})$ be a representation. Then, for $k < n$ if $\mathcal{V}(\rho|_{M_k}) = V_{p^j}$ then $\mathcal{V}(\rho) = V_{p^\ell}$ for some ℓ such that $\ell \geq k$.*

We know that if V_{p^k} is ρ-stable then so is V_{p^j} for $j \geq k$. Thus, we obtain the following corollary:

Corollary 4.2.8. *Let ρ be a representation of M_n. The representation ρ is irreducible if and only if $V_{p^{N-1}}$ is not ρ-stable.*

Throughout this chapter we use Corollary 4.2.8 to check if a representation ρ is irreducible. We use Corollary 4.2.5 to determine the number of isomorphic representations in standard form in one twist isoclass.

4.2.3 Determining Isomorphic Representations

Since representations in the same twist isoclass are equivalent under both twisting and isomorphism, we determine when two representations in standard form are isomorphic. We note that the results in this section are independent of the prime p. In this vein, we have the following proposition.

Proposition 4.2.9. *Let ρ_1, ρ_2 be p^N-dimensional irreducible representations of M_n in standard form. Then ρ_1 and ρ_2 are in the same twist isoclass if and only if there is a 1-dimensional representation χ and a permutation matrix $P \in GL_{p^N}(\mathbb{C})$ such that $\rho_1 = P\chi\rho_2 P^{-1}$.*

Since one direction is immediate, we prove the other direction with the following lemma.

Lemma 4.2.10. *For any prime p let $\rho : M_n \to GL_{p^N}(\mathbb{C})$ be irreducible and let P be a matrix such that Px_iP^{-1} is diagonal for $1 \le i \le n$ and, up to twisting, $PyP^{-1} = T_1y$ for some scalar T_1. Then $P = T_2y^m$ for some $0 \le m \le p^N - 1$ and scalar T_2. Furthermore, up to twisting, Px_iP^{-1} and PyP^{-1} are in standard form.*

Proof. Let $X = \{x_1, \ldots, x_n\}$. We will show that since all elements of X are diagonal with 1-dimensional mutual eigenspaces, P must be a generalized permutation matrix. Then we show that since, up to scalars, P commutes with y that P must be a power of y up to scalars. We then show that it follows that, up to twisting, $P\rho P^{-1}$ is in standard form.

Since all elements of X are diagonal and its mutual eigenspaces are 1-dimensional, $C_{GL_{p^N}(\mathbb{C})}(\langle X\rangle) = D$, where, for a group G, $C_G(H)$ is the centralizer of $H \subseteq G$ and $D \le GL_{p^N}(\mathbb{C})$ are the diagonal matrices. Since D is the centralizer of X and since X has 1-dimensional mutual eigenspaces D is also the centralizer of PXP^{-1}. Let $N_G(H)$ be the normalizer of H in G. It is well known that $N_{GL_{p^N}(\mathbb{C})}(D) = \mathcal{P}$ where $\mathcal{P}$ are the generalized permutation matrices; that is, matrices with precisely one non-zero entry in each row and column. And, since the mutual eigenspaces of X are all 1-dimensional, $P \in \mathcal{P}$.

If $PyP^{-1} = T_1y$ we twist ρ such that $T_1 = 1$. We know that P must be in the centralizer of y. Since y is a p^N-cycle and only powers of p^N-cycles commute with p^N-cycles in the symmetric group of p^N elements we have that $P = Ty^m$ for some diagonal matrix T and $m \in \mathbb{Z}$. But since P commutes with y and, of course, y^m commutes with y, T must as well. It follows that T must be a scalar matrix.

Conjugation of each x_i by P, which is the same as conjugating by y^m, for each i, maps λ_i to $\lambda_{i,m+1}$. We can twist by some 1-dimensional representation χ such that $\lambda_{1,m+1} = 1$ and thus by, Proposition 4.2.15 (or directly by Lemma 2.4.3(v)), $\chi P\rho P^{-1}$ is in standard form.

□

The lemma above calculates all representations in standard form which are in the same twist isoclass as ρ. Thus, we can deduce the "only if" direction of the proposition.

Remembering that representations in a twist isoclass are equivalent up to both twisting and isomorphism, we make the following definition.

Definition 4.2.11. Let ρ be irreducible and let x_i, y, for i such that $1 \leq i \leq n$, be in standard form as defined earlier in the section. A *shout* is a matrix P such that, up to twisting, PyP^{-1} and Px_iP^{-1} for $i = 1 \ldots n$ are in standard form. The representations ρ and $P\rho P^{-1}$ (note that Px_1P^{-1} may not be in standard form) are said to be equivalent under *shouting*.

We now need to count how many representations in standard form are in the same twist isoclass as ρ; that is, how many representations in standard form are twist-equivalent to ρ. We say that two representations that satisfy these conditions are equivalent under *twisting and shouting* [25]. If there are d twist-and-shout equivalent representations, and if we are just counting representations in standard form then we have overcounted by a factor of d. Thus, we must take this into account when counting twist isoclasses. In this vein, we now have the following lemma.

Lemma 4.2.12. *Let S_ρ be the twist isoclass represented by ρ and let $\mathcal{V}(\rho|_{M_{n-1}}) = V_{p^m}$. Then there are p^m representations in standard form in S_ρ that are twist-and-shout equivalent to ρ.*

Proof. By Lemma 4.2.2 the entries of the x_i are determined by the λ_i. So to determine how many representations are twist-and-shout equivalent to ρ we must count the number of choices of λ_i such that $\rho_* = \chi P\rho P^{-1}$ such that ρ_* is in standard form for some 1-dimensional representation χ.

For a shout P, let $x_i' = \chi P x_i P^{-1}$ for all $i \leq N$ and let λ_i' be the first diagonal entry of x_i'. By Lemma 4.2.10, for some $\ell \leq p^N$, $y = p^\ell$ and thus $\lambda_i' = \lambda_{i,\ell}$ for each i. Since ρ_* is in standard form it must be that we chose χ such that $\lambda_1' = 1$.

By the argument above, our choice of ℓ gives us, up to our choice of twist χ a representation that is twist-and-shout equivalent to ρ. It follows that the number of representations twist-and-shout equivalent to ρ is the size of the set $\{\Lambda'(\ell) := (\lambda_{2,\ell}, \ldots, \lambda_{n,\ell}) \mid \ell \leq p^N\}$. By Corollary 4.2.5, the size of this set is p^m. $\square$

Note two things: first, that when we reference this lemma, we say we take shouting into account; and second, since all entries of any x_i differ by products of λ_j such that $j > i$, this lemma implies that the depth of λ_2 has no effect on the number of twist-and-shout equivalent representations.

During the calculation of the p-local zeta functions that appear in this section, we break computation into various cases that depend on the depths of the λ_i that we choose. We note, without additional special mention except in one subcase, that each case is closed under shouting. For completeness, however, we have the following lemma which can be applied to the various cases to show that they are closed.

Lemma 4.2.13. *For some i and M let $s(\lambda_i) = M, s(\lambda_{i+1}), \ldots, s(\lambda_n) \leq M$. Then, for all j, $s(\lambda_{i,j}) \leq s(\lambda_i)$.*

Proof. Each $\lambda_{i,k} = \lambda_i \Lambda$ where Λ is some product of the roots of unity $\lambda_{i+1}, \ldots, \lambda_n$. Also, $s(\lambda_a \lambda_b) \leq \max\{s(\lambda_a), s(\lambda_b)\}$ for $\lambda_a, \lambda_b \in S_p^{\infty}$. The result follows immediately from these two facts. □

4.2.4 Determining When ρ is Irreducible for Non-Exceptional Primes

We note that the expressions $T_k(i)$ contain a denominator of $k!$. We note that, for representations of M_n, in standard form, it is always the case that $k \leq n-1$. For primes $p \geq n$, called the *non-exceptional primes*, the p-local zeta function will behave uniformly. In the case where $p < n$ the behaviour will be different. We call these p the *exceptional primes*. We will show some examples of this exceptional behaviour later in the chapter.

In this section we study the conditions for irreducibility of a p^N-dimensional representation ρ such that $p \geq n$. If this is the case, then the $T_k(i)$ terms that appear in the calculation of the standard forms have denominators that are all units mod p. We show that such a representation ρ is irreducible precisely when at least one of the λ_i is a primitive p^Nth root of unity.

We must determine the possible values for the λ_i. In that vein, we have the following lemma.

Lemma 4.2.14. *For non-exceptional primes $p \geq n$ we have, for all $i \leq n$, that $\lambda_i \in S_p^N$.*

Proof. Equation 4.2.4 in Lemma 4.2.2 tells us that $\lambda_i^{p^N} \prod_{k=i+1}^{n} \lambda_k^{T_{k-i}(p^N-1)} = 1$ for each i. By Lemma 2.4.3(vi), $\lambda_j^{T_k(p^N-1)} = 1$ for all $j < p$. The result then follows immediately. □

Let $x_1, \ldots, x_n, y$ be matrices in standard form and let ρ be the corresponding representation. We will show that, for non-exceptional primes, ρ is irreducible precisely when one of the x_i has all p^Nth roots of unity on its diagonal. This implies that, at least one of the λ_i, where $i \neq 1$, is in fact a primitive p^Nth root of unity. This will be shown in two stages. First, if λ_i is the "first-from-the-bottom" primitive p^Nth root of unity then x_{i-1} has every p^Nth root of unity on its diagonal. More formally, if $s(\lambda_i) = N$ for some $i \neq 1$ and $s(\lambda_k) \leq N-1$ for all k such that $i+1 \leq k \leq n$ then x_{i-1} has all p^Nth roots of unity on its diagonal. Secondly, we show a stronger result that implies that if none of the λ_i are primitive p^Nth roots of unity, that is $s(\lambda_i) \leq N-1$ for all i, then there is a proper stable subspace. We use the full strength of the second lemma later in the chapter.

We state the above as a proposition. The proof is a consequence of the two lemmas following it.

Proposition 4.2.15. *Let $p \geq n$ and ρ be a p^N-dimensional representation of M_n in standard form. Then ρ is irreducible if and only if there exists i with $2 \leq i \leq n$ such that $s(\lambda_i) = N$.*

Lemma 4.2.16. *If $s(\lambda_i) = N$ and $s(\lambda_k) \leq N-1$ for all k such that $i+1 \leq k \leq n$ then all $\lambda_{i-1,j}$, where $1 \leq j \leq p^N$, are distinct p^Nth roots of unity.*

Proof. Assume that $s(\lambda_i) = N$ and $s(\lambda_k) \leq N-1$ for $i+1 \leq k \leq n$. We can write these non-primitive λ_k, for each k, as powers of λ_i. Let $\lambda_k = \lambda_i^{\alpha_k p^{m_k}}$ such that $p \nmid \alpha_k$ and $m_k \geq 1$. For ease of display let $\alpha_i = 1$ and $m_i = 0$. Let $A_j = \sum_{k=i}^n \alpha_k p^{m_k} T_{k-i+1}(j-1)$. Then, by Lemma 4.2.2 we can say

$$\lambda_{i-1,j} = \prod_{k=i-1}^{n} \lambda_k^{T_{k-i+1}(j-1)} = \lambda_{i-1}\lambda_i^{A_j}. \tag{4.2.12}$$

We show that, for fixed i, the diagonal entries $\lambda_{i,j}$ are pairwise distinct by dividing two of them, say $\lambda_{i-1,s+1}$ and $\lambda_{i-1,t+1}$ with $s \geq t$, and showing that if $\lambda_{i-1,s+1}/\lambda_{i-1,t+1} = 1$ then $s = t$.

Consider the equation

$$\frac{\lambda_{i-1,s+1}}{\lambda_{i-1,t+1}} = \lambda_i^{A_{s+1}-A_{t+1}} = 1. \tag{4.2.13}$$

Taking the logarithm base λ_i and working mod p^N:

$$\begin{aligned} 0 &= A_{s+1} - A_{t+1} \\ &= \sum_{k=i}^{n} \alpha_k p^{m_k} T_{k-i+1}(s) - \sum_{k=i}^{n} \alpha_k p^{m_k} T_{k-i+1}(t) \\ &= [\alpha_i p^{m_i}\big(T_1(s) - T_1(t)\big) + \dots \\ &\quad + \alpha_n p^{m_n}\big(T_{n-i+1}(s) - T_{n-i+1}(t)\big)] \\ &= (s-t) + [\alpha_{i+1} p^{m_{i+1}}\big(T_2(s) - T_2(t)\big) + \dots \\ &\quad + \alpha_n p^{m_n}\big(T_{n-i+1}(s) - T_{n-i+1}(t)\big)] \bmod p^N. \end{aligned} \tag{4.2.14}$$

By Lemma 2.4.3(ii), $(s-t)$ is indeed a factor of each numerator of the right hand side of Equation 4.2.14. Therefore, remembering that $p \geq n$ and noting that all denominators are units mod p,

$$\begin{aligned} 0 \bmod p^N &= A_{s+1} - A_{t+1} \\ &= (s-t)[1 + \alpha_{i+1} p^{m_{i+1}} \frac{\gamma_{i+1}}{2!} + \dots + \alpha_n p^{m_n} \frac{\gamma_n}{(n-i+1)!}] \end{aligned} \tag{4.2.15}$$

for some γ_k. Since all $m_k \geq 1$ this implies that

$$1 + \alpha_{i+1} p^{m_{i+1}} \frac{\gamma_{i+1}}{2!} + \dots + \alpha_n p^{m_n} \frac{\gamma_n}{(n-i+1)!} \neq 0 \bmod p \tag{4.2.16}$$

and thus $s - t = 0 \bmod p^N$ and we conclude that $s = t$. We can now say that the diagonal entries of x_{i-1} are pairwise distinct. $\square$

We prove the necessity of having at least one λ_i a primitive p^N root of unity in the following lemma. The idea is to show that if no λ_i is a primitive p^Nth roots of unity then for $1 \leq \beta \leq p^k - 1$ and $1 \leq j \leq p^{N-k}$, for some k, we have that $\lambda_{i,j} = \lambda_{i,\beta p^{N-k}+j}$ for any i. Therefore V_{p^k} is in fact a proper ρ-stable subspace of $\mathbb{C}^{p^N}$. For ρ to be irreducible this cannot be the case.

Lemma 4.2.17. *Let $\lambda_* \in S_p^N \backslash S_p^{N-1}$. For each $i \geq 2$ let $\lambda_i = \lambda_*^{\alpha_i p^{m_i}}$ where $p \nmid \alpha_i$ and $m_i \geq 1$. Also, let $m_* = min\{m_i\}$. Then, for any i, $\lambda_{i,j} = \lambda_{i,\beta p^{N-m_*}+j}$ for all $1 \leq \beta \leq p^{m_*} - 1$ and $1 \leq j \leq p^{N-m_*}$.*

Proof. Consider the expression $\Lambda := \log_{\lambda_*}(\lambda_{i,\beta p^{n-m_*}+j+1})$ for some $0 \leq j \leq p^{n-m_*} - 1$ where $1 \leq \beta \leq p^{m_*}$. Then

$$\begin{aligned}\Lambda \bmod p^N &= \alpha_i p^{m_i} + \alpha_{i+1} p^{m_{i+1}} T_1(\beta p^{N-m_*} + j) + \dots \\ &\quad + \alpha_n p^{m_n} T_{n-i}(\beta p^{N-m_*} + j) \\ &= \alpha_i p^{m_i} + \alpha_{i+1} p^{m_{i+1}} (\beta p^{N-m_*} + j) + \dots \\ &\quad + \alpha_k p^{m_k} \frac{(\beta p^{N-m_*} + j)\dots(\beta p^{N-m_*} + (j+k-i-1))}{(k-i)!} + \dots \\ &\quad + \alpha_n p^{m_n} \frac{(\beta p^{N-m_*} + j)\dots(\beta p^{N-m_*} + (j+n-i-1))}{(n-i)!}\end{aligned} \tag{4.2.17}$$

where the term involving k is a typical term. By Corollary 2.4.4, since $m_* \leq m_i$ for all m_i it follows that, if each numerator was expanded, all terms are 0 mod p^N but the terms that have no factor of p^{N-m_*} in the expansion of each numerator; that is

$$\alpha_k p^{m_k} (\beta p^{N-m_*} + j) \dots (\beta p^{N-m_*} + j + k - 1) = \alpha_k p^{m_k} (j) \dots (j + k - 1) \bmod p^N \tag{4.2.18}$$

for $i \leq k \leq n$. Therefore

$$\begin{aligned}\Lambda &= \alpha_i p^{m_i} + \alpha_{i+1} p^{m_{i+1}} T_1(j) + \dots + \alpha_n p^{m_n} T_{n-i}(j) \bmod p^N \\ &= \log_{\lambda_*}(\lambda_{i,j+1}).\end{aligned} \tag{4.2.19}$$

□

This completes the proof of the proposition.

4.2.5 Counting Twist-Isoclasses of Non-Exceptional ρ

Now that we have determined all irreducible representations up to twisting and isomorphism for the non-exceptional case, we count the number of twist isoclasses. We do this by counting the number of ρ that have a basis such that they are in standard form, and taking into account representations that are isomorphic under twisting and shouting so we do not overcount. We remind the reader that one of the λ_i, for $i \geq 2$, must be a primitive p^Nth root of unity.

Regarding twist-and-shout equivalent representations, we have the following lemma that follows directly from Lemmas 4.2.12 and 4.2.17; notice the choices of p that are valid for this lemma.

Lemma 4.2.18. *For $p \geq n-1$ let ρ be an irreducible p^N-dimensional representation of M_n and let $\mathcal{V}(\rho|_{M_{n-1}}) = V_{p^k}$. Then there are p^k representations in standard form equivalent to ρ under twisting and shouting.*

We break the computation into two cases. First, assume $s(\lambda_k) = N$ for some k such that $3 \leq k \leq n$. In this case there are altogether $(1-p^{-(n-2)})p^{(n-2)N}$ choices for $\lambda_3, \dots, \lambda_n$. We can choose any p^Nth root of unity for λ_2 and therefore there are p^N choices for this. By Lemma 4.2.18 we must divide by p^N to take shouting into account.

Now assume $s(\lambda_2) = N$ and $s(\lambda_i) \leq N-1$ for $3 \leq i \leq n$. There are $(1-p^{-1})p^N$ choices for λ_2. If $\max\{s(\lambda_k)\} = \ell \neq 0$ for $3 \leq k \leq n$ then we have $(1-p^{-(n-2)})p^{(n-2)(N-\ell)}$ choices for these. By Lemma 4.2.18 we must divide by $p^{N-\ell}$. If $\max\{s(\lambda_k)\} = 0$ for $3 \leq k \leq n$ then all of the $\lambda_3, \dots, \lambda_n$ are the p^0th root of unity, namely 1. Since we have no freedom to shout in this case, we are not overcounting.

Summing these two cases together we have, for $N \geq 1$,

$$\begin{aligned} r_{p^N} = {} & (1-p^{-(n-2)})p^{(n-2)N}p^N p^{-N} \\ & + \sum_{l=1}^{N} (1-p^{-1})p^N(1-p^{-(n-2)})p^{(n-2)(N-l)}p^{-(N-l)} \\ & + (1-p^{-1})p^N \end{aligned} \tag{4.2.20}$$

and

$$\zeta^{irr}_{M_n,p}(s) = \sum_{N=0}^{\infty} r_{p^N} p^{-Ns} = 1 + \sum_{N=1}^{\infty}(1-p^{-(n-2)})p^{(n-2)N}p^{N}p^{-N}p^{-Ns} \qquad (4.2.21)$$
$$+ \sum_{N=1}^{\infty}\sum_{\ell=1}^{N-1}(1-p^{-1})p^{N}(1-p^{-(n-2)})p^{(n-2)(N-\ell)}p^{-(N-\ell)}p^{-Ns}$$
$$+ \sum_{N=1}^{\infty}(1-p^{-1})p^{N}p^{-Ns}$$
$$= 1 + (1-p^{-(n-2)})\sum_{N=1}^{\infty}(p^{(n-2)-s})^{N}$$
$$+ (1-p^{-1})(1-p^{-(n-2)})\sum_{N=1}^{\infty}p^{(n-2)N}p^{-Ns}\sum_{l=1}^{N-1}(p^{3-n})^{\ell}$$
$$+ (1-p^{-1})\sum_{N=1}^{\infty}(p^{1-s})^{N}.$$

Summing some geometric series we have

$$\zeta^{irr}_{M_n,p}(s) = 1 + (1-p^{-(n-2)})\frac{p^{(n-2)-s}}{1-p^{(n-2)-s}} \qquad (4.2.22)$$
$$+ (1-p^{-1})(1-p^{-(n-2)})\sum_{N=1}^{\infty}(p^{(n-2)-s})^{N}\frac{p^{3-n}-(p^{3-n})^{N}}{1-p^{3-n}}$$
$$+ (1-p^{-1})\frac{p^{1-s}}{1-p^{1-s}}$$
$$= 1 + (1-p^{-(n-2)})\frac{p^{(n-2)-s}}{1-p^{(n-2)-s}}$$
$$+ (1-p^{-1})(1-p^{-(n-2)})\left(\sum_{N=1}^{\infty}\frac{p^{3-n}(p^{(n-2)-s})^{N}-(p^{1-s})^{N}}{1-p^{3-n}}\right)$$
$$+ (1-p^{-1})\frac{p^{1-s}}{1-p^{1-s}}.$$

Thus

$$\begin{aligned}\zeta^{irr}_{M_n,p}(s) = {} & 1 + (1-p^{-(n-2)})\frac{p^{(n-2)-s}}{1-p^{(n-2)-s}} \\ & + \frac{(1-p^{-1})(1-p^{-(n-2)})}{1-p^{3-n}}\left(\frac{p^{1-s}}{1-p^{(n-2)-s}} - \frac{p^{1-s}}{1-p^{1-s}}\right) \\ & + (1-p^{-1})\frac{p^{1-s}}{1-p^{1-s}}\end{aligned} \tag{4.2.23}$$

and a routine calculation of Equation 4.2.23 yields that

$$\zeta^{irr}_{M_n,p}(s) = \frac{(1-p^{-s})^2}{(1-p^{(n-2)-s})(1-p^{1-s})}. \tag{4.2.24}$$

Note in particular that $\zeta^{irr}_{M_n,p}(s) \mid_{p\to p^{-1}} = p^{n-1}\zeta^{irr}_{M_n,p}(s)$ and thus this zeta function does indeed satisfy the correct functional equation as in Theorem 1.4.3. By Equation 4.2.24 we can also say that the p-local abscissa of convergence is

$$\alpha_{M_n,p} = n-2 \tag{4.2.25}$$

for $n \geq 3$. If $n = 2$, that is for the Heisenberg group $H \cong M_2$, then a factor of $(1-p^{-s})$ in the numerator cancels with the factor $(1-p^{(n-2)s})$ in the denominator. It follows that, for all primes p,

$$\alpha_{M_2,p} = 1. \tag{4.2.26}$$

4.3 Some Exceptional Prime Calculations

In the section above we dealt with the case when $p \geq n$. We remind readers that the fact that these primes did not divide the denominators of the $T_k(i)$ terms made the computation uniform across all of these primes. In this section we study some cases when $p < n$.

4.3.1 The p-local Representation Zeta Function of M_{p+1}

For a prime p, we study the p-local representations of M_n when $n = p+1$. We calculate the exceptional prime representation growth zeta function $\zeta^{irr}_{M_n,p}(s)$.

Note that, unlike the non-exceptional calculation, p **is fixed by our choice of group** for this calculation.

Let ρ be a p^N-dimensional representation. We will determine the choices of λ_i for which ρ is irreducible. By Lemma 4.2.2 we can choose a basis such that, for each x_i,

$$x_i = \begin{pmatrix} \lambda_i & & & \\ & \prod_{k=i}^{p+1} \lambda_k^{T_{k-i}(1)} & & \\ & & \ddots & \\ & & & \prod_{k=i}^{p+1} \lambda_i^{T_{k-i}(p^N-1)} \end{pmatrix}. \tag{4.3.1}$$

We divide this calculation into two cases: when $s(\lambda_{p+1}) = N$ and when $s(\lambda_{p+1}) \leq N-1$. Furthermore, we break the second case into two sub-cases: when there is a λ_i with $3 \leq i \leq p$ such that $s(\lambda_i) = N$ and when there is no such λ_i. Call these cases Case 1, and Case 2, respectively. We call Case 2's respective subcases Case 2.1 and Case 2.2. Note that, since p is not exceptional when considering $\rho|_{M_p}$ (and we remind the reader that $M_p = \langle y, x_2, \ldots, x_{p+1}\rangle$), we can apply Lemma 4.2.18 when determining the number of representations twist-and-shout equivalent to some irreducible ρ.

Case 1 Assume that $s(\lambda_{p+1}) = N$.
By [14, Theorem 8.4], $\rho|_{M_2}$ is an irreducible representation. It is clear, since $T_i(p^N-1) = 0 \bmod p^N$ for $i < p$ by Lemma 2.4.3(vi), that we can use Equation 4.2.4 in Lemma 4.2.2 to show that $s(\lambda_i) \leq N$ for $i \neq 2$ and

$$\lambda_2^{p^N} \prod_{k=3}^{p+1} \lambda_k^{T_{k-1}(p^N-1)} = 1. \tag{4.3.2}$$

Since $s(\lambda_i) \leq N$ for $3 \leq i \leq p$ by Lemma 2.4.3(vi) the preceding equation simplifies to

$$\lambda_2^{p^N} \lambda_{p+1}^{T_p(p^N-1)} = 1. \tag{4.3.3}$$

The p-simplex number

$$\begin{aligned} T_p(p^N-1) &= \frac{(p^N-1)\big(p^N\big)\ldots(p^N+p-2)}{p!} \\ &= \frac{(p^N-1)\big(p^{N-1}\big)\ldots(p^N+p-2)}{(p-1)!} \\ &= \alpha p^{N-1} \end{aligned} \tag{4.3.4}$$

for some α such that $p \nmid \alpha$. So $s(\lambda_{p+1}^{\alpha p^{N-1}}) = 1$ and $\lambda_2^{p^N} = (\lambda_{p+1}^{\alpha p^{N-1}})^{-1}$. So $s(\lambda_2) = N+1$

and there are p^N choices for λ_2 to make Equation 4.3.3 hold. So there are $(1-p^{-1})p^N$ choices for λ_{p+1} and p^N choices for each λ_i where $2 \leq i \leq p$. By Lemma 4.2.18 we must divide by p^N to take shouting into account. Therefore in this case there are

$$\begin{aligned} &(1-p^{-1})p^N p^{(p-1)N} p^{-N} \\ &= (1-p^{-1})p^{(p-1)N} \end{aligned} \tag{4.3.5}$$

twist isoclasses. Note that the right hand side of Equation 4.3.5 is also the contribution to r_{p^N} in the non-exceptional case for when $s(\lambda_{p+1}) = N$.

Case 2 Now assume $s(\lambda_{p+1}) \leq N-1$.
It is clear, since $T_i(p^N-1) = 0 \bmod p^N$ for $i < p$ by Lemma 2.4.3(vi) and since $\lambda_{p+1}^{T_p(p^N-1)} = 1$ by Equation 4.3.4, that we can say that $s(\lambda_i) \leq N$ for $2 \leq i \leq p+1$. We now break this case into subcases.

Case 2.1 For i such that $3 \leq i \leq p$, assume one of $s(\lambda_i) = N$, say λ_k. Then, since $p \geq k$, by Proposition 4.2.15 $\rho|_{M_{p+1-k+2}}$ is an irreducible representation. Thus ρ is irreducible. In this case there are $(1-p^{-(p-2)})p^{(p-2)N}$ choices for λ_i, p^N choices for λ_2, and p^{N-1} choices for λ_{p+1}. By Lemma 4.2.18 we must divide by p^N to take shouting into account. Thus there are

$$(1-p^{-(p-2)})p^{(p-2)N}p^{N-1}p^N p^{-N} = (1-p^{-(p-2)})p^{(p-1)N-1} \tag{4.3.6}$$

twist isoclasses in this case. We note that the contribution to r_{p^N} in this case is the same contribution to r_{p^N} for non-exceptional primes.

Case 2.2 Finally, assume $s(\lambda_i) \leq N-1$ where $3 \leq i \leq p$. Note that in this case $\rho|_{M_p}$ has $V_{p^{N-1}}$ as a proper stable subspace so by Lemma 4.2.8 it is not irreducible. If $s(\lambda_{p+1}) = 0$ then M_{p+1} is isomorphic to M_p and by Proposition 4.2.15 ρ is irreducible if and only if $s(\lambda_2) = N$.

Now let $s(\lambda_{p+1}) \geq 1$. We choose λ_* such that $s(\lambda_*) = N$ and write each λ_i in terms of it; that is, let $\lambda_i = \lambda_*^{\alpha_i p^{m_i}}$, $p \nmid \alpha_i$, $m_2 \geq 0$, and $m_i \geq 1$ for $3 \leq i \leq p+1$.

We appeal to Lemma 4.2.8 and determine when $\langle y, x_1 \rangle$ does not have $V_{p^{N-1}}$ as a proper stable subspace. This implies that $\lambda_{1,j} \neq \lambda_{1,\beta p^{N-1}+j}$ for some $1 \leq \beta \leq p-1$ and $1 \leq j \leq p^{N-1}$. Consider $\lambda_{1,\beta p^{N-1}+j+1}$ for $0 \leq j \leq p^N - 1$. If $N = 1$, then in order for ρ not to be trivial, $\lambda_2 = -1$ and it is easily verified that x_1 is not scalar so $V_{p^{N-1}}$ is not a stable subspace. Since we want ρ to be irreducible this must be the case. Now, for

$N \geq 2$,

$$\begin{aligned}
\Lambda &:= \log_{\lambda_*}(\lambda_{1,\beta p^{N-1}+j+1}) \\
&= \alpha_2 p^{m_2}(\beta p^{N-1}+j) + \alpha_3 p^{m_3}\frac{(\beta p^{N-1}+j)(\beta p^{N-1}+j+1)}{2} + \dots \\
&\quad + \alpha_{p+1}p^{m_{p+1}}\frac{(\beta p^{N-1}+j)\dots(\beta p^{N-1}+j+p-1)}{p!} \bmod p^N.
\end{aligned} \tag{4.3.7}$$

By Corollary 2.4.4 and keeping in mind that $m_i \geq 1$ for $3 \leq i \leq p+1$ this simplifies to the following:

$$\begin{aligned}
\Lambda = \alpha_2 p^{m_2}\beta p^{N-1} &+ \Bigg(\alpha_2 p^{m_2} j + \alpha_3 p^{m_3}\frac{j(j+1)}{2!} + \dots \\
&+ \alpha_p p^{m_p}\frac{j(j+1)\dots(j+p-2)}{(p-1)!}\Bigg) \\
&+ \alpha_{p+1}p^{m_{p+1}-1}\frac{(\beta p^{N-1}+j)\dots(\beta p^{N-1}+j+p-1)}{(p-1)!} \bmod p^N.
\end{aligned} \tag{4.3.8}$$

Note that the last term has a denominator of $(p-1)!$ since the factor of p was subtracted from m_{p+1}. We have that

$$\begin{aligned}
\Lambda = &\left(\log_{\lambda_*}(\lambda_{1,j+1}) - \alpha_{p+1}p^{m_{p+1}-1}\frac{j(j+1)\dots(j+p-1)}{(p-1)!}\right) + \Bigg[\alpha_2 p^{m_2}(\beta p^{N-1}) \\
&+ \alpha_{p+1}p^{m_{p+1}-1}\frac{(\beta p^{N-1}+j)\dots(\beta p^{N-1}+j+p-1)}{(p-1)!}\Bigg] \bmod p^N.
\end{aligned} \tag{4.3.9}$$

Let Q be the term in the square brackets above. Expanding Q

$$\begin{aligned}
Q &= \alpha_2 p^{m_2}(\beta p^{N-1}) + \alpha_{p+1}p^{m_{p+1}-1}\frac{j(j+1)\dots(j+p-1)+\Omega+\Omega'}{(p-1)!} \bmod p^N \\
&= \alpha_2 p^{m_2}(\beta p^{N-1}) + \alpha_{p+1}p^{m_{p+1}-1}\frac{j(j+1)\dots(j+p-1)+\Omega}{(p-1)!} \bmod p^N
\end{aligned} \tag{4.3.10}$$

where

$$\Omega = \beta p^{N-1}\sum_{i=0}^{p-1}\frac{j(j+1)\dots(j+p-1)}{j+i} \tag{4.3.11}$$

and Ω' is terms of higher degree than p^N (which are clearly $0 \bmod p^N$). We can say that exactly one of $j,(j+1)\dots,(j+p-1)$ is divisible by p, say for $i=\iota$. It follows that only one term of $\sum_{i=0}^{p-1} \frac{j(j+1)\dots(j+p-1)}{j+i}$ is in fact not a multiple of p. Therefore we have

$$\begin{aligned} \beta p^{N-1} \sum_{i=0}^{p-1} \frac{j(j+1)\dots(j+p-1)}{j+i} &= \beta p^{N-1} \frac{j(j+1)\dots(j+p-1)}{j+\iota} \\ &= \beta p^{N-1}(-1+ap) \bmod p^N \end{aligned} \tag{4.3.12}$$

by Wilson's Theorem, and for some a. Thus

$$\begin{aligned} Q &= \alpha_2 p^{m_2}(\beta p^{N-1}) \\ &\quad + \alpha_{p+1} p^{m_{p+1}-1} \frac{j(j+1)\dots(j+p-1) + \beta p^{N-1}(-1+ap)}{(p-1)!} \bmod p^N. \end{aligned} \tag{4.3.13}$$

Therefore

$$\begin{aligned} \Lambda = &\left(\log_{\lambda_*}(\lambda_{1,j+1}) - \alpha_{p+1} p^{m_{p+1}-1} \frac{j(j+1)\dots(j+p-1)}{(p-1)!} \right) \\ &+ \Bigg[\alpha_2 p^{m_2}(\beta p^{N-1}) \\ &+ \alpha_{p+1} p^{m_{p+1}-1} \frac{j(j+1)\dots(j+p-1) - \beta p^{N-1}}{(p-1)!} \Bigg] \bmod p^N \\ = &\log_{\lambda_*}(\lambda_{1,j+1}) + [\beta p^{N-1}(\alpha_2 p^{m_2} + \alpha_{p+1} p^{m_{p+1}-1})] \bmod p^N \end{aligned}$$

Since we want ρ to be irreducible the sum inside the round brackets in the previous equation must not be divisible by p; that is

$$\alpha_2 p^{m_2} + \alpha_{p+1} p^{m_{p+1}-1} \neq 0 \bmod p. \tag{4.3.14}$$

We now enumerate the cases when we do indeed have a factor of p. If $m_2 \geq 1$ and $m_{p+1} \geq 2$ then we clearly have a factor of p so this cannot be the case if ρ is irreducible. Now assume $m_{p+1} = 1$. Then we have a factor of p in the left hand side of Equation 4.3.14 only when $m_2 = 0$ and $\alpha_2 = -\alpha_{p+1} \bmod p$. Finally, assume $m_2 = 0$ and $m_{p+1} \neq 1$. Then it is clear that $\alpha_2 p^{m_2} + \alpha_{p+1} p^{m_{p+1}-1} \neq 0 \bmod p$. We have irreducible representations when $m_{p+1} = 1$ or $m_2 = 0$ except when $m_{p+1} = 1, m_2 = 0$, and $\alpha_2 \neq -\alpha_{p+1} \bmod p$.

We still need to take shouting into account. Therefore, by Lemma 4.2.18, we must divide our count, if we enumerated the representations in this case at this stage, by p^{m_*} where $m_* = \max\{s(\lambda_3), \dots, s(\lambda_{p+1})\}$. Note that the shouting behaviour, since p is non-exceptional when considering $\rho|_{M_p}$, is the same as in the non-exceptional case.

This ends the case distinctions.

We note that the only difference between the r_{p^N} for this exceptional prime and the r_{p^N} for non-exceptional primes is the situation when we can choose λ_2 and λ_{p+1} such that (still thinking of all λ_i written as powers of λ_*) $m_{p+1} = 1$ and $m_2 \geq 1$, which gives us additional irreducible representations, and when $m_{p+1} = 1, m_2 = 0$, and $\alpha_2 \neq -\alpha_{p+1} \bmod p$, which gives us representations that are no longer irreducible. Therefore, starting with r_{p^N} calculated for non-exceptional primes, we can add the cases where our choices of λ_i give us additional representations and subtract the cases where we lose representations.

Let C be r_{p^N} for non-exceptional primes, that is the sum in (4.2.20). The situation where $m_{p+1} = 1$ and $m_2 \geq 1$ would not correspond to irreducible representations for non-exceptional primes, but do for exceptional primes. There are $(1-p^{-1})p^{N-1}$ choices for λ_{p+1} and p^{N-1} choices for λ_2 in this case. Remembering that we assumed that $s(\lambda_i) \leq N-1$ for $3 \leq i \leq p$ then there are $p^{(p-2)(N-1)}$ choices for these λ_i. By Lemma 4.2.18 we must divide by p^{N-1} to take shouting into account. Therefore we must add

$$(1-p^{-1})p^{N-1}p^{N-1}p^{(p-2)(N-1)}p^{-(N-1)} = (1-p^{-1})p^{(p-1)(N-1)} \tag{4.3.15}$$

to C.

The situation where $m_{p+1} = 1$, $m_2 = 0$, and $\alpha_2 = -\alpha_{p+1} \bmod p$ would correspond to irreducible representations for non-exceptional primes, but not for exceptional primes. There are $(1-p^{-1})p^N$ choices for λ_2 and, given our choice for λ_2, there are p^{N-2} choices for λ_{p+1} in this case. Remembering that we assumed that $s(\lambda_i) \leq N-1$ for $3 \leq i \leq p$ then there are $p^{(p-2)(N-1)}$ choices for these λ_i. By Lemma 4.2.18 we must divide by p^{N-1} to take shouting into account. Therefore we must subtract

$$(1-p^{-1})p^N p^{N-2}p^{(p-2)(N-1)}p^{-(N-1)} = (1-p^{-1})p^{(p-1)(N-1)} \tag{4.3.16}$$

from C. Notice that (4.3.15) = (4.3.16). Therefore

$$r_{p^N} = C \tag{4.3.17}$$

and

$$\zeta^{irr}_{M_{p+1},p}(s) = \frac{(1-p^{-s})^2}{(1-p^{((p+1)-2)-s})(1-p^{1-s})} \tag{4.3.18}$$

by Equation 4.2.21.

This result, and the result from the previous section, gives us the entire irreducible representation theory, as well as the representation zeta function of M_3. In fact, we can say that

$$\zeta^{irr}_{M_3}(s) = \left(\frac{\zeta(s-1)}{\zeta(s)}\right)^2. \tag{4.3.19}$$

4.3.2 The 2-local Representation Zeta Function of M_4

We have a complete understanding of the irreducible representations of M_3. The aim of this section is to do the same for M_4. Our previous work leaves us with only one p-local zeta function to calculate; the previous section calculates the 3-local zeta function and 2 and 3 are the only exceptional primes. Therefore once we calculate the 2-local representation zeta function we have $\zeta^{irr}_{M_4}(s)$ in its entirety.

Note, for ease of computation, we calculate $r_2(M_4)$ separately later in this section. Until noted otherwise we assume the condition that $N \geq 2$.

In keeping with the style of the general cases earlier, and for elucidation if one wishes to generalize this calculation, we do not simplify the expressions $(1-2^{-1})$ to 2^{-1} as far in the calculation as possible.

Let $\rho : M_4 \to GL_{2^N}(\mathbb{C})$ be a representation. By Equation 4.2.4 in Lemma 4.2.2

$$\lambda_4^{2^N} = 1, \tag{4.3.20}$$

$$\lambda_3^{2^N}\lambda_4^{T_2(2^N-1)} = 1, \tag{4.3.21}$$

and

$$\lambda_2^{2^N}\lambda_3^{T_2(2^N-1)}\lambda_4^{T_3(2^N-1)} = 1. \tag{4.3.22}$$

Therefore, by Equation 4.3.20, $s(\lambda_4) \leq N$.

Before we begin counting twist isoclasses we must determine the possible depths of

λ_2 and λ_3. Assume $s(\lambda_4) \le N-1$. Then, by Equation 4.3.4, $\lambda_4^{T_2(2^N-1)} = \lambda_4^{T_3(2^N-1)} = 1$ and by Equation 4.3.21 $\lambda_3^{2^N} = 1$ so $s(\lambda_3) \le N$. If $s(\lambda_3) \le N-1$ then $\lambda_3^{T_2(2^N-1)} = 1$ and by Equation 4.3.22 $\lambda_2^{2^N} = 1$ so $s(\lambda_2) \le N$. If $s(\lambda_3) = N$ then, by Equation 4.3.4, $\lambda_3^{T_2(2^N-1)} = \lambda_3^{-2^{N-1}}$ and therefore $s(\lambda_3^{-2^{N-1}}) = 1$. By Equation 4.3.22 we have that $\lambda_2^{2^N} = \lambda_3^{2^{N-1}}$ and thus $\lambda_2^{2^N}$ must satisfy this equation. So $\lambda_2^{2^N} = -1$ and $s(\lambda_2) = N+1$.

Now assume $s(\lambda_4) = N$. Then, by Equation 4.3.4, $\lambda_4^{T_2(2^N-1)} = \lambda_4^{T_3(2^N-1)} = \lambda_4^{-2^{N-1}}$ and therefore $s(\lambda_4^{-2^{N-1}}) = 1$. By Equation 4.3.21,

$$\lambda_3^{2^N} = \lambda_4^{2^{N-1}} \tag{4.3.23}$$

and thus $\lambda_3^{2^N}$ must satisfy this equation. So $\lambda_3^{2^N} = -1$ and $s(\lambda_3) = N+1$. By Equation 4.3.4, $\lambda_3^{T_2(2^N-1)} = \lambda_3^{-2^{N-1}}$ and, by Equations 4.3.22 and 4.3.23, $\lambda_2^{2^N} = \lambda_3^{2^{N-1}}\lambda_4^{2^{N-1}} = \lambda_3^{2^N+2^{N-1}} = \lambda_3^{(1+2)2^{N-1}}$. Note that we leave $(1+2)$ in this form since we wish to stress that it is in fact $(1+p)$. Thus $s(\lambda_3^{(1+2)2^{N-1}}) = 2$ and $\lambda_2^{2^N}$ must satisfy Equation 4.3.22. So $\lambda_2^{2^N} = \pm\sqrt{-1}$ and $s(\lambda_2) = N+2$.

Any choices of λ_2, λ_3, and λ_4 that satisfy Equations 4.3.20, 4.3.21, and 4.3.22 are well defined representations. We now check which choices give us irreducible representations, as we want ρ to be irreducible. To do this, we break up the domain of choices into eight parts and calculate each part's contribution to $r_{2^N}(M_4)$. Table 4.1 lists this information.

Table 4.1 Table of Cases for M_4

Case	$s(\lambda_4)$	$s(\lambda_3)$	$s(\lambda_2)$	Other Conditions	No. of twist iso-classes, $N \ge 2$
1	$= N$	$= N+1$	$= N+2$		$(1-2^{-1})^4 2^{2N+3}$
2	$= N-1$	$= N$	$= N+1$	$\alpha_3 = 3 \bmod 4$	$(1-2^{-1})^3 2^{2N}$
3	$= N-1$	$\le N-1$	$\le N$		$(1-2^{-1}) 2^{2N-2}$
4	$\le N-2$	$= N$	$= N+1$		$(1-2^{-1})^2 2^{2N-1}$
5	$\le N-2$	$= N-1$	$= N$		0
6	$\le N-2$	$\le N-2$	$= N$		See Table 4.2 on page 71
7	$\le N-2$	$= N-1$	$\le N-1$		See Table 4.3 on page 71
8	$\le N-2$	$\le N-2$	$\le N-1$		0

Table 4.2 Case 6 of Table 4.1

N	Case	Relationship of $s(\lambda_3)$ and $s(\lambda_4)$	No. of twist isoclasses
$=2$	6.4	$s(\lambda_3) = s(\lambda_4) = 0$	2
$=3$	6.2	$s(\lambda_3) \leq 1, s(\lambda_4) = 1$	$(1-2^{-1})^2 2^3$
	6.4	$s(\lambda_4) = 0$	$(1-2^{-1})2^3(1+(1-2^{-1}))$
≥ 4	6.1	$s(\lambda_3) > s(\lambda_4)+1, s(\lambda_3) \geq 2, s(\lambda_4) \neq 0$	$\big[(1-2^{-1})2^N\big((2^{N-4}-1) - (1-2^{-1})(N-4)\big)\big]$
	6.2	$s(\lambda_3) < s(\lambda_4)+1, s(\lambda_4) \neq 0$	$(1-2^{-1})2^N(2^{N-3}-2^{-1})$
	6.3	$s(\lambda_3) = s(\lambda_4)+1, s(\lambda_4) \neq 0$	$(1-2^{-1})^2 2^N(2^{N-2}-2)$
	6.4	$s(\lambda_4) = 0$	$(1-2^{-1})^2 2^N(1+(1-2^{-1})(N-2))$

Table 4.3 Case 7 of Table 4.1

N	Case	No. of twist isoclasses
$=2$		1
≥ 3	7.1	$(1-2^{-1})2^{2N-4}$
	7.2	$(1-2^{-1})^2 2^{2N-2}$

Case 1: Since $s(\lambda_4) = N$ we have, by Case 1 of Section 4.3.1, that $\rho|_{M_2}$ is irreducible. Therefore ρ is indeed an irreducible representation of M_4. By Equations 4.3.20, 4.3.21, and 4.3.22 there are $(1-2^{-1})2^N$ choices for λ_4, $(1-2^{-1})2^{N+1}$ choices for λ_3, and $(1-2^{-1})^2 2^{N+2}$ choices for λ_2. Since, by Case 1 of Section 4.3.1, we know that $\rho|_{M_3}$ is irreducible, we have that $\mathcal{V}(\rho|_{M_3}) = V_{2^N}$. Thus, by Lemma 4.2.12, we must divide by 2^N to take shouting into account. So in this case we have

$$(1-2^{-1})^4 2^N 2^{N+1} 2^{N+2} 2^{-N} = (1-2^{-1})^4 2^{2N+3} \tag{4.3.24}$$

twist isoclasses.

Case 2: By Case 2.2 of Section 4.3.1, $\rho|_{M_3}$ is reducible. Appealing to Lemma 4.2.8 we must check whether $V_{2^{N-1}}$ is a stable subspace of $\langle y, x_1\rangle$. We write each root of unity in terms of a primitive 2^{N+1}th one. Let $\lambda_* = \lambda_2$, $\lambda_i = \lambda_*^{\alpha_i 2^{m_i}}$ for some α_i such that $2 \nmid \alpha_i, m_4 = 2$, $m_3 = 1$, and $i \in \{3,4\}$.

We use Corollary 4.2.6 at this point. Now, noting that $2 \cdot (2^{N-1})^2 = 0 \bmod 2^{N+1}$ for $N = 2$, consider $\lambda_{1,2^{N-1}+1}$:

$$\begin{aligned}\Lambda := {} & \log_{\lambda_*}(\lambda_{1,2^{N-1}+1}) \\ & = (2^{N-1}) + 2^{1-1}\alpha_3(2^{N-1})(2^{N-1}+1) \\ & \quad + \alpha_4 2^{2-1}\frac{(2^{N-1})(2^{N-1}+1)(2^{N-1}+2)}{3} \bmod 2^{N+1} \\ & = 2^{N-1}\left(1+\alpha_3+\alpha_4 2^1\frac{2}{3}\right) \bmod 2^{N+1} \\ & = \log_{\lambda_*}(\lambda_1) + 2^{N-1}\left[1+\alpha_3\right] \bmod 2^{N+1}\end{aligned} \tag{4.3.25}$$

So the expression in the square brackets above is a multiple of 4 if and only if $V_{2^{N-1}}$ is a $\langle y, x_1\rangle$-stable subspace. Let Q be the aforementioned expression. It is clear that $Q = 0 \bmod 4$ precisely when $\alpha_3 = 3 \bmod 4$. This means that we are only free to choose half of the elements of S_2^N/S_2^{N-1} for λ_3. Thus, there are $(1-2^{-1})2^{N-1}$ choices for λ_3, $(1-2^{-1})2^{N+1}$ choices for λ_2, and $(1-2^{-1})2^{N-1}$ choices for λ_4. Since $\rho|_{M_3}$ is not irreducible it has at least $V_{2^{N-1}}$ as a stable subspace. But since $s(\lambda_4) = N-1$, by Corollary 4.2.7, $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-1}}$. Thus, by Lemma 4.2.12 we must divide by 2^{N-1} to take shouting into account. So in this case we have

$$\begin{aligned} & (1-2^{-1})2^{N-1}(1-2^{-1})2^{N+1}(1-2^{-1})2^{N-1}2^{-(N-1)} \\ = {} & (1-2^{-1})^3 2^{2N}\end{aligned} \tag{4.3.26}$$

twist isoclasses.

Case 3: By Case 2.2 of Section 4.3.1, $\rho|_{M_3}$ is irreducible and therefore ρ is irreducible. There are $(1-2^{-1})2^{N-1}$ choices for λ_4, 2^{N-1} choices for λ_3, and 2^N choices for λ_2. By Lemma 4.2.12 we must divide by 2^N to take shouting into account. So in this case we have

$$\begin{aligned} & (1-2^{-1})2^{N-1}2^{N-1}2^N2^{-N} \\ = {} & (1-2^{-1})2^{2N-2}\end{aligned} \tag{4.3.27}$$

twist isoclasses.

Case 4: By Case 2.2 of Section 4.3.1, $\rho|_{M_3}$ is irreducible and therefore ρ is irreducible. There are 2^{N-2} choices for λ_4, $(1-2^{-1})2^N$ choices for λ_3, and $(1-2^{-1})2^{N+1}$ choices for λ_2. By Lemma 4.2.12 we must divide by 2^N to take shouting into account.

So in this case we have

$$\begin{aligned} & 2^{N-2}(1-2^{-1})2^{N}(1-2^{-1})2^{N+1}2^{-N} \\ &= (1-2^{-1})^{2}2^{2N-1} \end{aligned} \tag{4.3.28}$$

twist isoclasses.

Cases 5 and 6: We note for both cases $s(\lambda_2) = N$ and $s(\lambda_4) \leq N-2$. We have, by Case 2.2 of Section 4.3.1, that $\rho|_{M_3}$ has $V_{2^{N-1}}$ as a proper stable subspace. Appealing to Lemma 4.2.8, we check whether $V_{2^{N-1}}$ is a stable subspace of $\langle y, x_1\rangle$. We let $\lambda_* = \lambda_2$ and write each λ_i as a power of λ_*; that is, let $\lambda_4 = \lambda_*^{\alpha_4 2^{m_4}}$ and $\lambda_3 = \lambda_*^{\alpha_3 2^{m_3}}$ such that $2 \nmid \alpha_i$ $m_3 \geq 1$, $m_4 \geq 2$, and $i \in \{3,4\}$. If $m_4 = N$ then by Case 2.2 of Section 4.3.1, ρ is irreducible if and only if $m_3 \neq 1$. If $m_3 = N$ it is easy to show that $\log_{\lambda_*}(\lambda_{1,2^{N-1}+1}) \neq 1$. We leave this to the reader. Assume that $m_3, m_4 \neq N$.

Appealing to Corollary 4.2.6, consider $\lambda_{1,2^{N-1}+1}$, noting that $a(a+1)$ is even for any a and $2^{2N-2} = 0 \bmod 2^N$:

$$\begin{aligned} \Lambda &:= \log_{\lambda_*}(\lambda_{1,2^{N-1}+1}) \\ &= (2^{N-1}) + \alpha_3 2^{m_3-1}(2^{N-1})(2^{N-1}+1) \\ &\quad + \alpha_4 2^{m_4-1}\frac{2^{N-1}(2^{N-1}+1)(2^{N-1}+2)}{3} \bmod 2^N \\ &= \log_{\lambda_*}(\lambda_1) + 2^{N-1}\left[1+\alpha_3 2^{m_3-1}\right] \bmod 2^N. \end{aligned} \tag{4.3.29}$$

So when the term in the square brackets above, say Q, is not 0 mod 2 then $\lambda_1 = 1 \neq \lambda_{1,2^{N-1}+1}$. It follows that $V_{2^{N-1}}$ is not a stable subspace of ρ and therefore ρ is irreducible. Thus Q is 0 mod 2 when $m_3 = 1$; that is when $s(\lambda_3) = N-1$. So in Case 5 there are no irreducible representations.

If $m_3 \geq 2$ it is clear that $Q \neq 0 \bmod 2$. Thus, in Case 6 there are 2^{N-2} choices for λ_4, 2^{N-2} choices for λ_3, and $(1-2^{-1})2^N$ choices for λ_2.

We now need to analyze the shouting behaviour for this case. It is clear, since $\mathcal{V}(\rho|_{M_2}) = V_{2^{s(\lambda_4)}}$ by Lemma 4.2.18 and Corollary 4.2.7, that there are at least $2^{s(\lambda_4)} = 2^{N-m_4}$ twist and shout-equivalent representations. We now determine $\mathcal{V}(\rho|_{M_3})$. Let $m_4 \neq N$. We deal with the case $m_4 = N$ in the next lemma. Also, note that we use the power of Corollary 4.2.6 for this computation.

Consider, for some k such that $1 \leq k \leq m_4$,

$$\begin{aligned}\Lambda &:= \log_{\lambda_*}(\lambda_{2,2^{N-k}+1}) \\ &= \alpha_3 2^{m_3} 2^{N-k} + \alpha_4 2^{m_4-1} 2^{N-k}(2^{N-k}+1) \bmod 2^N \\ &= 2^{N-k}\left[\alpha_3 2^{m_3} + \alpha_4 2^{m_4-1}(2^{N-k}+1)\right] \bmod 2^N.\end{aligned} \tag{4.3.30}$$

We now determine $\mathcal{V}(\rho|_{M_3})$ for each possible choice of m_3 and m_4. For the following lemma let Q be the sum in the square brackets above. If $Q = 0 \bmod 2^k$ then $\lambda_{2,1} = \lambda_{2,2^{N-k}+1}$ and V_{p^k} is a proper stable subspace of $\rho|_{M_3}$.

Lemma 4.3.1. *Let $m_4 \neq N$ and let $m_* = \min\{m_3, m_4 - 1\}$. If $m_3 \neq m_4 - 1$ then $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-m_*}}$. If $m_3 = m_4 - 1$ then $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-m_4}}$.*

If $m_4 = N$ then $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-m_3}}$.

Proof. Assume $m_4 \neq N$. If $m_3 \neq m_4 - 1$ the maximum value of k such that $Q = 0 \bmod 2^k$ is $\min\{m_3, m_4 - 1\}$. If $m_3 = m_4 - 1$ then, since both terms in Q are of the same 2-adic valuation, the maximal value of k is at least m_4. However, since $\mathcal{V}(\rho|_{M_2}) = V_{2^{s(\lambda_4)}}$, by Corollary 4.2.7 it follows that $\mathcal{V}(\rho|_{M_3}) = V_{2^{s(\lambda_4)}}$.

Now let $m_4 = N$. Then,

$$\begin{aligned}\Lambda &:= \log_{\lambda_*}(\lambda_{2,2^{N-k}+1}) \\ &= \alpha_3 2^{m_3} 2^{N-k} = 0 \bmod 2^N\end{aligned} \tag{4.3.31}$$

when $k \leq m_3$. Thus k is maximal when $k = m_3$. Noting that $\mathcal{V}(\rho|_{M_2}) = V_{p^0}$, we have that $\mathcal{V}(\rho|_{M_3}) = V_{2^{s(\lambda_3)}}$ when $m_4 = N$. □

We now count the number of twist isoclasses. To do this we break the computation into four subcases. Note that, in all subcases, there are $(1-2^{-1})2^N$ choices for λ_2. For the first three subcases we assume that $s(\lambda_4) \neq 0$.

Case 6.1 For some M such that $2 \leq M \leq N-2$, let $s(\lambda_3) = M > s(\lambda_4)+1$. There are $(1-2^{-1})2^M$ choices for λ_3 and $2^{M-2}-1$ choices for λ_4. Since $s(\lambda_3) = M > s(\lambda_4)+1$ by Lemmas 4.2.12 and 4.3.1 we must divide by 2^M to take shouting into account. Thus, in this subcase there are

$$\begin{aligned}
&(1-2^{-1})2^N\sum_{M=2}^{N-2}(1-2^{-1})2^M(2^{M-2}-1)2^{-M} && (4.3.32)\\
&=(1-2^{-1})2^N\sum_{M=3}^{N-2}(1-2^{-1})(2^{M-2}-1)\\
&=(1-2^{-1})2^N(1-2^{-1})\big((2+\ldots+2^{N-4})-(N-4)\big)\\
&=(1-2^{-1})2^N\big((2^{N-4}-1)-(1-2^{-1})(N-4)\big)
\end{aligned}$$

twist isoclasses. Note that $(2^{M-2}-1)=0$ when $M=2$.

Case 6.2 For some M such that $1\leq M\leq N-2$, let $s(\lambda_4)=M$ and $s(\lambda_3)<s(\lambda_4)+1=M+1$. Note that this implies that $s(\lambda_3)\leq M$ and thus it follows that this subcase is closed under shouting. There are $(1-2^{-1})2^M$ choices for λ_4 and 2^M choices for λ_3. By Lemmas 4.2.12 and 4.3.1 we must divide by 2^{M+1} to take shouting into account. Thus in this subcase there are

$$\begin{aligned}
&(1-2^{-1})2^N\sum_{M=1}^{N-2}(1-2^{-1})2^M2^M2^{-(M+1)} && (4.3.33)\\
&=(1-2^{-1})2^N\sum_{M=1}^{N-2}(1-2^{-1})2^{M-1}\\
&=(1-2^{-1})2^N(1-2^{-1})(1+\ldots+2^{N-3})\\
&=(1-2^{-1})2^N(2^{N-3}-2^{-1})
\end{aligned}$$

twist isoclasses.

Case 6.3 For some M such that $2\leq M\leq N-2$, let $s(\lambda_3)=M=s(\lambda_4)+1$. There are $(1-2^{-1})2^M$ choices for λ_3 and $(1-2^{-1})2^{M-1}$ choices for λ_4. Since $s(\lambda_3)=M=s(\lambda_4)+1$ by Lemmas 4.2.12 and 4.3.1 we must divide by 2^{M-1} to take shouting into account. Thus, in this subcase there are

$$\begin{aligned}
&(1-2^{-1})2^N \sum_{M=2}^{N-2} (1-2^{-1})^2 2^M 2^{M-1} 2^{-(M-1)} \qquad (4.3.34)\\
&= (1-2^{-1})2^N \sum_{M=2}^{N-2} (1-2^{-1})^2 2^M \\
&= (1-2^{-1})^2 2^N (1-2^{-1})(4+\ldots+2^{N-2}) \\
&= (1-2^{-1})^2 2^N (2^{N-2}-2)
\end{aligned}$$

twist isoclasses.

Case 6.4 Assume $s(\lambda_4)=0$. Let $s(\lambda_3)=M$ for $0 \leq M \leq N-2$. If $M>0$ there are $(1-2^{-1})2^M$ choices for λ_3 and there is 1 choice for λ_3 if $M=0$. There is only 1 choice for λ_4. By Lemmas 4.2.12 and 4.3.1 we must divide by 2^M to take shouting into account. Thus in this subcase there are

$$\begin{aligned}
&(1-2^{-1})2^N \left(1+\sum_{M=1}^{N-2}(1-2^{-1})2^M 2^{-M}\right) \qquad (4.3.35)\\
&= (1-2^{-1})2^N \big(1+(1-2^{-1})(N-2)\big)
\end{aligned}$$

twist isoclasses.

This ends the subcase distinctions.

Thus, summing together all subcases, there are

$$\begin{aligned}
&(1-2^{-1})2^N\big((2^{N-4}-1)-(1-2^{-1})(N-4)\big)+(1-2^{-1})2^N(2^{N-3}-2^{-1}) \qquad (4.3.36)\\
&+(1-2^{-1})^2 2^N(2^{N-2}-2)+(1-2^{-1})2^N\big(1+(1-2^{-1})(N-2)\big)\\
&=(1-2^{-1})2^N(2^{N-2}+2^{N-4}-2^{-1})
\end{aligned}$$

twist isoclasses in Case 6 when $N \geq 4$. When $N=3$ we sum together Cases 6.2 and 6.4. Thus there are

$$\begin{aligned}
&(1-2^{-1})2^3\big((1-2^{-1})+1+(1-2^{-1})\big) \qquad (4.3.37)\\
&=8
\end{aligned}$$

twist isoclasses in Case 6. When $N = 2$ we only include Case 6.4 and thus there are

$$(1 - 2^{-1})2^2(1) = 2 \tag{4.3.38}$$

twist isoclasses in Case 6.

Cases 7 and 8: We note for both cases $s(\lambda_2) \leq N - 1$ and $s(\lambda_4) \leq N - 2$. As with the previous two cases we have, by Case 2.2 of Section 4.3.1, that $\rho|_{M_3}$ has $V_{2^{N-1}}$ as a proper stable subspace. We check whether $V_{2^{N-1}}$ is a stable subspace of $\langle y, x_1\rangle$. As usual, we choose a $\lambda_* \in S_2^N/S_2^{N-1}$ and write each λ_i as a power of λ_*; that is $\lambda_4 = \lambda_*^{\alpha_i 2^{m_i}}$ such that $2 \nmid \alpha_i$, $m_2 \geq 1$, $m_3 \geq 1$, $m_4 \geq 2$, and $i \in \{2, 3, 4\}$.

Appealing to Corollary 4.2.6, consider $\lambda_{1,2^{N-1}+1}$ noting that $a(a+1)$ is even for any a and $2^{2N-2} = 0 \bmod 2^N$:

$$\begin{aligned} \Lambda &:= \log_{\lambda_*}(\lambda_{1,2^{N-1}+1}) \\ &= \alpha_2 2^{m_2} 2^{N-1} + \alpha_3 2^{m_3-1} 2^{N-1}(2^{N-1}+1) \\ &\quad + \alpha_4 2^{m_4-1} \frac{2^{N-1}(2^{N-1}+1)(2^{N-1}+2)}{3} \bmod 2^N \\ &= \log_{\lambda_*}(\lambda_1) + 2^{N-1}[\alpha_2 2^{m_2} + \alpha_3 2^{m_3-1}] \bmod 2^N. \end{aligned} \tag{4.3.39}$$

Clearly if $m_3 \geq 2$ then the expression in the square brackets above, say C, is 0 mod 2 and $V_{2^{N-1}}$ is indeed a stable subspace of ρ. If $m_3 = 1$ then C is not 0 mod 2 and $V_{2^{N-1}}$ is not a stable subspace of ρ. Therefore ρ is irreducible. So in Case 8 there are no twist isoclasses. In Case 7 there are 2^{N-2} choices for λ_4, $(1 - 2^{-1})2^{N-1}$ choices for λ_3, and 2^{N-1} choices for λ_2.

We now determine the behaviour of shouting in this case. It is easy to see that $\mathcal{V}(\rho|_{M_2}) = V_{2^{s(\lambda_4)}}$ and thus $\mathcal{V}(\rho|_{M_3})$ is no smaller than $V_{2^{s(\lambda_4)}}$.

Let $N \geq 3$; we calculate the case when $N = 2$ separately later in the section. We write λ_3, λ_4 in terms of some $\lambda_* \in S_2^N \backslash S_2^{N-1}$ in the usual way, with $m_3 = 1$ and m_4 such that $2 \leq m_4 \leq N$. If $m_4 = N$ then it is easy to show that $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-1}}$. Now assume $m_4 \neq N$. As in Case 6, we use the power of Corollary 4.2.6. Consider $\Lambda := \log_{\lambda_*}(\lambda_{2,2^{N-k}+1})$ for k such that $1 \leq k \leq m_4$. Then

$$\begin{aligned} \Lambda &= \alpha_3 2 \cdot 2^{N-k} + \alpha_4 2^{m_4-1} 2^{N-k}(2^{N-k}+1) \bmod 2^N \\ &= 2^{N-k}[\alpha_3 2 + \alpha_4 2^{m_4-1}(2^{N-k}+1)] \end{aligned} \tag{4.3.40}$$

Let Q be the terms in the square brackets above.The expression $Q = 0 \bmod 2^k$ if and only if $\lambda_{2,1} = \lambda_{2,2^{N-k}+1}$ and thus $V_{2^{N-k}}$ is a proper stable subspace of $\langle y, x_2\rangle$. We break this computation into two subcases.

Case 7.1 Assume $m_4 > 2$

It is clear that if $m_4 > 2$ then, since $m_3 = 1$, by Equation 4.3.40 the maximal k such that $\lambda_{2,1} = \lambda_{2,2^{N-k}+1}$ is when $k = 1$. Thus $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-1}}$. Note that $V_{2^{N-1}}$ is also minimal when $m_4 = N$. Let $s(\lambda_4) = M$ where $M \leq N - 3$. In this subcase there are 2^{N-1} choices for λ_2, $(1 - 2^{-1})2^{N-1}$ choices for λ_3 and 2^{N-3} choices for λ_4. By Lemma 4.2.12 we must divide by 2^{N-1} to take shouts into account. Thus, in this subcase, there are

$$2^{N-1}(1 - 2^{-1})2^{N-1}2^{N-3}2^{-(N-1)} = (1 - 2^{-1})2^{2N-4} \tag{4.3.41}$$

twist isoclasses.

Case 7.2 Assume $m_4 = 2$

If $m_4 = 2$ then $Q = 0 \bmod 2^2$ and, since $\mathcal{V}(\rho|_{M_2}) = V_{2^{N-2}}$, then by Corollary 4.2.7 $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-2}}$. There are 2^{N-1} choices for λ_2, $(1 - 2^{-1})2^{N-1}$ choices for λ_3, and $(1-2^{-1})2^{N-2}$ choices for λ_4. By Lemma 4.2.12 we must divide by 2^{N-2} to take shouting into account. Thus, in this subcase there are

$$(1 - 2^{-1})^2 2^{N-1}2^{N-1}2^{N-2}2^{-(N-2)} = (1 - 2^{-1})^2 2^{2N-2} \tag{4.3.42}$$

twist isoclasses.

This ends the subcase distinctions.

Summing together these two subcases there are, for $N \geq 3$,

$$(1 - 2^{-1})2^{2N-4} + (1 - 2^{-1})^2 2^{2N-2} = (1 - 2^{-1})2^{2N-4}(1 + 2^2(1 - 2^{-1})) \tag{4.3.43}$$

twist isoclasses.

Now assume $N = 2$. Then $s(\lambda_4) = 0$ and thus $x_4 = I$. A short calculation shows that $\lambda_{2,1} = \lambda_{2,3}$ and, by Corollary 4.2.6, V_2 is a minimal stable subspace. By Lemma 4.2.12 we must divide by 2 to take shouting into account. There are 2^1 choices for λ_2, $(1 - 2^{-1})2^1 = 1$ choice for λ_3, and 1 choice for λ_4. Thus, in this subcase there is

$$2 \cdot 1 \cdot 1 \cdot 2^{-1} = 1 \tag{4.3.44}$$

twist isoclass.

This ends the case distinctions.

We now consider the case when $N = 1$. Note that, for clarity, we will call ι the square root of -1. By Equation 4.3.20, $s(\lambda_4) \leq 1$; that is, $\lambda_4 \in \{1, -1\}$. If $\lambda_4 = -1$, then by Equation 4.3.21 we have that $\lambda_3 \in \{\iota, -\iota\}$ and, by Equation 4.3.22, $\lambda_2 \in \{\pm\sqrt{\iota}, \pm\sqrt{\iota}\}$ such that $\lambda_2^2 = -\lambda_3$.

If $\lambda_4 = 1$ then, by Equation 4.3.21, $\lambda_3 \in \{1, -1\}$. If $\lambda_3 = 1$ then by Equation 4.3.22 and, since ρ is not the identity representation, $\lambda_2 = -1$. If $\lambda_3 = -1$ then, by Equation 4.3.22, $\lambda_2 \in \{\iota, -\iota\}$.

A set of choices of the λ_i gives us an irreducible representation if and only if $\lambda_{i,1} \neq \lambda_{i,2}$ holds for at least one $1 \leq i \leq 3$; that is, one of the following is true:

$$\lambda_4 \neq 1 \tag{4.3.45}$$

$$\lambda_3\lambda_4 \neq 1 \tag{4.3.46}$$

$$\lambda_2\lambda_3\lambda_4 \neq 1. \tag{4.3.47}$$

It is easy to see that all of our choices of sets of λ_i give us irreducible representations. For triples $(\lambda_4, \lambda_3, \lambda_2)$ it is easy to check that the pairs $\left[(-1, \iota, \sqrt{\iota}), (-1, -\iota, -\sqrt{-\iota})\right]$, $\left[(-1, \iota, -\sqrt{\iota}), (-1, -\iota, \sqrt{-\iota})\right]$, $[(1, -1, \iota), (1, -1, -\iota)]$ are twist-and-shout equivalent. Therefore we can say that

$$r_2(M_4) = 4. \tag{4.3.48}$$

We count the number of twist isoclasses for $N = 2, 3$ separately as well. Summing Cases 1 through 8 for $N = 2$ we have

$$\begin{aligned} r_4(M_4) &= (1 - 2^{-1})^4 2^7 + (1 - 2^{-1})^3 2^4 + (1 - 2^{-1})2^2 + (1 - 2^{-1})^2 2^3 + 2 + 1 \\ &= 8 + 2 + 2 + 2 + 2 + 1 \\ &= 17. \end{aligned} \tag{4.3.49}$$

For $N = 3$, also summing Cases 1 through 8, we have

$$\begin{aligned} r_8(M_4) &= (1-2^{-1})^4 2^9 + (1-2^{-1})^3 2^6 + (1-2^{-1})2^4 \\ &\quad + (1-2^{-1})^2 2^5 + 8 + (1-2^{-1})2^2(3) \\ &= 32 + 8 + 8 + 8 + 8 + 6 \\ &= 70. \end{aligned} \tag{4.3.50}$$

We can now compute the 2-local representation growth zeta function of M_4 by summing together the number of twist isoclasses from Cases 1 through 8. Keeping in mind that there is one 1-dimensional twist isoclass, four 2-dimensional twist isoclasses, 17 4-dimensional twist isoclasses, and 70 8-dimensional twist isoclasses we have the following:

$$\begin{aligned} \zeta^{irr}_{M_4,2}(s) &= 1 + 4\cdot 2^{-s} + 17\cdot 2^{-2s} + 70\cdot 2^{-3s} \\ &\quad + \sum_{N=4}^{\infty} \big[(1-2^{-1})^4 2^{2N+3} + (1-2^{-1})^3 2^{2N} \\ &\quad + (1-2^{-1})2^{2N-2} + (1-2^{-1})^2 2^{2N-1} \\ &\quad + (1-2^{-1})2^N(2^{N-2} + 2^{N-4} - 2^{-1}) \\ &\quad + (1-2^{-1})2^{2N-4}(1 + 2^2(1-2^{-1}))\big]\, 2^{-Ns}. \end{aligned} \tag{4.3.51}$$

Inputting this equation into Maple gives us that

$$\zeta^{irr}_{M_4,2}(s) = \frac{(1-2^{-s})^2}{(1-2^{1-s})(1-2^{2-s})}. \tag{4.3.52}$$

Note that this result is the same as the zeta function for non-exceptional primes in Equation 4.2.24. It is then easy to check that this does satisfy the functional equation in Theorem 1.4.3.

Now that we have the p-local representation zeta functions of M_4 we can now state the global representation zeta function:

$$\zeta^{irr}_{M_4}(s) = \frac{\zeta(s-1)\zeta(s-2)}{(\zeta(s))^2}. \tag{4.3.53}$$

This completes the proof of Theorem 1.7.1.

Chapter 5

Examples Using Kirillov Orbit Method

5.1 Introduction

In this section we calculate the representation zeta functions of various $\mathcal{T}$-groups associated to Lie rings by the Mal'cev correspondence, using techniques found in [13] and [29]. First, we explain the Kirillov orbit method and how it works. We then calculate some examples, most coming from [9, Chapter 2]. The third part gives some theoretical results calculated along with Robert Snocken.

We slightly abuse notation in this section. If L is the Lie ring associated to some group G, we "equate" G and L and, for non-Kirillov-exceptional primes, denote the p-local representation zeta function of G as $\zeta^{irr}_{L,p}(s)$.

5.2 Kirillov Orbit Method for Representations

5.2.1 Definitions and Explanation of the Method

For this section, all Lie rings are nilpotent. Also we remind the reader that $\mathcal{Z}^*_{n,p^N}$ is defined to be the set of n-tuples of elements of $\mathbb{Z}/p^N\mathbb{Z}$ such that at least one entry in the n-tuple is a unit. More precisely, let $\mathcal{Z}^*_{n,p^N} := (\mathbb{Z}/p^N\mathbb{Z})^n \setminus (p\mathbb{Z}/p^N\mathbb{Z})^n$.

Before stating results we give a brief overview of the correspondence between $\mathcal{T}$-groups and Lie rings. Let $\mathrm{Tr}_1(n, R)$ be the set of $n \times n$ upper-unitriangular matrices and let $\mathrm{Tr}_0(n, R)$ be the set of $n \times n$ strictly upper-triangular matrices over the ring R. By [26, Chapter 5] we can embed a $\mathcal{T}$-group G as a subgroup of $\mathrm{Tr}_1(n, \mathbb{Z})$ for some

n; say this subgroup is $T(G)$. Since the elements of $T(G)$ are matrices we can take the matrix logarithm map $\log : T(G) \to \mathrm{Tr}_0(n, \mathbb{Z})$ (see, for example, [26, Theorem 1, Section 6]). If $T(G)$ satisfies certain conditions, which we will discuss later, then $\log\big(T(G)\big)$ is a Lie ring and we can use this fact to study $T(G)$ by way of Howe's application [13] of the Kirillov orbit method to $\mathcal{T}$-groups. We identify $T(G)$ and G for the rest of the chapter.

In order to explain the Kirillov orbit method properly, we need a number of definitions. We give them here, before any explanation of the method.

Definition 5.2.1. Let G be a $\mathcal{T}$-group. If $L := \log(G)$ is a Lie ring we say that L is the *Lie ring associated to* G.

Definition 5.2.2. Let G be a $\mathcal{T}$-group and let $H \leq G$. We say H is *saturated* if $g^n \in H$ implies that $g \in H$ for all $g \in G$ and $n \in \mathbb{Z}$. We call the smallest saturated subgroup containing H the *isolator* of G and denote the isolator by H_s. Note that, if we regard a Lie ring L as an abelian group, we can extend these definitions to Lie rings.

Definition 5.2.3. Let G be a $\mathcal{T}$-group of nilpotency class c such that $L := \log(G)$ is a Lie ring. We say that L is *elementarily exponentiable (or e. e.)* if and only if $[L, L] \subseteq c!L$. We say G is e. e. if and only if L is e. e.

For the rest of this section let $\hat{L} := \hom\big(L, \mathbb{C}^*\big)$ be the set of abelian group homomorphisms to $\mathbb{C}^*$, where $L := \log(G)$ is the Lie ring associated to some e. e. $\mathcal{T}$-group G.

Definition 5.2.4. Let G be an e. e. $\mathcal{T}$-group. For $\psi \in \hat{L}$ we call the minimal n such that $\psi(nL) = 1$, if such an n exists, the *period* of ψ.

For the rest of this section we assume that G is e. e. We combine some results mentioned in [29, Section 3.4] into a theorem.

Theorem 5.2.5. *Let G be a $\mathcal{T}$-group. There is a Lie algebra over $\mathbb{Q}$, say $\mathcal{L} := \mathcal{L}_G(\mathbb{Q})$, of dimension $h(G)$ such that the $\mathbb{Q}$-span of $\log(G)$ is $\mathcal{L}$. Moreover, there exists a subgroup $H \leq G$ of finite index such that $L := \log H$ is a Lie subring of $\mathcal{L}$ and that L is e. e.*

By [14, Theorem 8.5] we know that for almost all primes, in fact primes not dividing $|G : H|$, twist isoclasses of irreducible representations of H of dimension p^N are in 1-1 correspondence with twist isoclasses of G of dimension p^N.

Thus, to study the twist isoclasses of a group G, we can pass to a finite index subgroup H such that $\log H$ is an e. e. Lie ring. However, this is at the cost of not being able to calculate the p-local representation function of G for finitely many primes.

For an e. e. $\mathcal{T}$-group G and its associated Lie ring L there is an adjoint action of G on L given by inner automorphisms of G: $\mathrm{Ad}(g)(\ell) = \log(g^{-1}\exp(\ell)g) = \ell + [\log g, \ell] +$ (higher terms) where the Lie terms of commutator order ≥ 3 can be computed in terms of the Baker-Campbell-Hausdorff formula. This adjoint action gives us the following co-adjoint action of G on $\hat{L}$:

$$\mathrm{Ad}^*(g)(\psi)(\ell) = \psi(\mathrm{Ad}(g)(\ell)) = \psi(\ell)\psi([\log g, \ell])\psi(\text{higher terms}). \tag{5.2.1}$$

We briefly explain the correspondence between twist isoclasses and co-adjoint orbits. See [13, Theorem 1.a] and [29, Section 3.4] for details.

Let S be a finite set of primes and let $F_S = \{n \in \mathbb{N} \mid n \text{ is divisible by some } s \in S\}$. Then the following is true: the set of all finite Ad^*-orbits of $\psi \in \hat{L}$ of period $n \in \mathbb{N} \setminus F_S$ is in bijective correspondence with the set T of twist isoclasses of irreducible representations of G such that each Ad^*-orbit U uniquely corresponds to a twist isoclass of dimension $|U|^{1/2}$. If a twist isoclass $t \in T$ is of dimension k then all twist isoclasses of dimension k appear in T. Since the set of these orbits is indeed in bijective correspondence with the corresponding set of twist isoclasses, we can count these orbits instead; this is the thrust of the Kirillov orbit method.

By [29, Section 3.4], these twist isoclasses are determined by the behaviour of $\psi(L'_s)$; that is, if $\psi_1|_{L'_s} = \psi_2|_{L'_s}$ then ψ_1 and ψ_2 are associated to the same twist isoclass (see [13, Lemmas 1-4] for details). For the remainder of this paragraph, we say $h(L'_s) = d'$. If ψ has period p^N then $\psi(\ell) \in S_p^N$ for $\ell \in L'_s$. Furthermore, there exists at least one element $\ell_* \in L'_s$ such that $\psi(\ell_*) \in S_p^N \setminus S_p^{N-1}$. We define $\Psi_{p^N} := \{\psi \in \hat{L} \mid \psi \text{ has period } p^N\}$ and, since a basis of L' contains d' elements, we can identify Ψ_{p^N} with $\mathcal{Z}^*_{d',p^N}$, by identifying S_p^N with $\mathbb{Z}/p^N\mathbb{Z}$.

Voll's method for calculating p-local representation zeta functions, based on Howe's parametrization, is as follows. We refer the reader to the preliminaries in Section 2.5 for definitions that appear in the following two paragraphs; e. g. Smith normal form, commutator (sub)matrix, and $[g, h]_{\mathbf{y}}$.

We start by giving conditions on our $\mathcal{T}$-group. Let G be a e. e. $\mathcal{T}$-group and L be its associated Lie ring with basis $\{x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+f}, x_{d+f+1}, \ldots x_{d+e}\}$ such that $\{x_{d+1}, \ldots x_{d+e}\}$ is a $\mathbb{Z}$-basis for L' and $\{x_{d+f+1}, \ldots x_{d+e}\}$ is a $\mathbb{Z}$-basis for $L' \cap Z(L)$ (without loss of generality, but at the cost of finitely many primes, we can assume that we can choose a basis this way; see [29, Proposition 3.1]).

Let $\mathcal{N}_{N,\mathbf{m},\mathbf{n}}$ be the number of $\mathbf{y} := (y_{d+1}, \ldots, y_{d+e}) \in \mathcal{Z}^*_{e,p^N}$ such that $SNF(\mathcal{R}(\mathbf{y}))$ mod p^N has elementary divisors $\mathbf{m} := (p^{m_1}, \ldots p^{m_{d+f}})$ and $SNF(\mathcal{S}(\mathbf{y}))$ mod p^N has elementary divisors $\mathbf{n} := (p^{n_{d+1}}, \ldots, p^{n_{d+f}})$, where $\mathcal{R} := \mathcal{R}(\mathbf{y})$ is the $\mathbf{y}$-commutator matrix of L and $\mathcal{S} := \mathcal{S}(\mathbf{y})$ is the $\mathbf{y}$-commutator submatrix of L.

Theorem 5.2.6 (Voll)**.** *For almost all primes p,*

$$\zeta^{irr}_{G,p}(s) = 1 + \sum_{N=1}^{\infty} \sum_{\mathbf{m}} \sum_{\mathbf{n}} \mathcal{N}_{N,\mathbf{m},\mathbf{n}} p^{A_{\mathbf{m}} s} p^{B_{\mathbf{n}}} \tag{5.2.2}$$

where $A_{\mathbf{m}} = -\sum_{i=1}^{d+f}(N - m_i)/2$, $B_{\mathbf{n}} = -\sum_{j=d+1}^{d+f}(N - n_j)$.

We use Equation 5.2.2 to calculate examples of p-local zeta functions later in the chapter. However, in order to calculate these zeta functions, we need to be able to calculate the elementary divisors of $\mathcal{R}(\mathbf{y})$ and $\mathcal{S}(\mathbf{y})$. We can use Lemmas 2.5.8 and 2.5.10 to do this calculation.

We give an example of a commutator matrix and a commutator submatrix. Let L be a Lie ring with the basis $\{x_1, x_2, x_3, x_4, w_1, w_2 \mid [x_1, x_2] = w_1, [x_3, x_4] = w_1, [x_1, y] = w_2\ \}$. Then a basis for L' is $\{w_1, w_2\}$ and a basis for $L' \cap Z(L)$ is $\{w_2\}$. We choose $\mathbf{y} := (y_1, y_2) \in \mathcal{Z}^*_{2,p^N}$. Thus $[x_1, x_2]_{\mathbf{y}} = [x_3, x_4]_{\mathbf{y}} = y_1$ and $[x_1, w_1]_{\mathbf{y}} = y_2$. The commutator matrix is

$$\mathcal{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ w_1 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & w_1 \end{array} \\ \begin{pmatrix} 0 & y_1 & 0 & 0 & y_2 \\ -y_1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & y_1 & 0 \\ 0 & 0 & -y_1 & 0 & 0 \\ -y_2 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{5.2.3}$$

and the commutator submatrix is

$$\mathcal{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} w_1 \\ \begin{pmatrix} y_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}. \tag{5.2.4}$$

We label the rows and columns of $\mathcal{R}$ and $\mathcal{S}$ in one calculation. We hope that this gives the reader a feel for constructing these matrices. Further matrices are not labeled, but the labeling should be fairly obvious from the group presentation.

As mentioned earlier, the idea is to study the representation growth of $\mathcal{T}$-groups by studying nilpotent Lie rings associated to these groups, as in Theorem 5.2.5. In fact, if we choose an e. e. Lie ring L then $[L, L] \subseteq c!L = L \bmod p^N$ for almost all p (since $c!$ would be a unit mod p^N) and for any $N \in \mathbb{N}$. By the Baker-Campbell-Hausdorff formula there is a bijection $\exp : L \to G$ for some $\mathcal{T}$-group G [26, Chapter 6]. In fact, we can assume, for any Lie ring L, that L is e. e. at the cost of being able to calculate p-local representation zeta functions for finitely many primes.

We note that Theorem 5.2.6 only holds for all but finitely many primes. We say we *lose* a prime p_* if the hypothesis of Theorem 5.2.6 does not apply for p_*; that is, if p_* is Kirillov-exceptional. We now list the ways we lose primes when calculating representation zeta functions of $\mathcal{T}$-groups using Theorem 5.2.6. First, if we pass to a finite index subgroup of our $\mathcal{T}$-group G, say H, then, by [29, Section 3.4], we lose all p_* such that $p_* \mid |G : H|$. Secondly, by Howe's parametrization [13, Theorem 1.a] we lose all primes p_* such that $p_* \mid 2|G'_s : G'|$. Next, by [29, Corollary 3.1], for all but finitely many primes $|G : G_\psi| = |L : \mathrm{Rad}_\psi|$ and $|G : G_{\psi,2}| = |L : L_{\psi,2}|$, where $G_\psi, \mathrm{Rad}_\psi, G_{\psi,2}$, and $L_{\psi,2}$ are defined in [29, Section 3.4]. Thus, we lose all primes p_* where these equalities do not hold. Also, by assuming that L has the basis structure in Theorem 5.2.6 we lose finitely many primes p_*. Finally, by [29, Section 2.2], we lose primes p_* such that, for any $\mathbf{y}$-commutator matrix $\mathcal{R}$ of G, $\mathcal{R}$ is a zero matrix mod p_*.

As a reminder, if we do not lose a prime p, we say that p is *non-exceptional*. We note that, for the following calculations, there is no discussion of which primes are exceptional primes; we feel that the work involved to determine the exceptional primes, checking each of the conditions in the previous paragraph, would be too long to include in this thesis.

5.3 Examples Using the Kirillov Orbit Method

Note that, for ease of display, in this section we let $t = p^{-s}$, $A_{\mathbf{m}} = -\sum_{i=1}^{a}(N - m_i)/2$ and $B_{\mathbf{n}} = -\sum_{i=1}^{b}(N - n_i)$. Also, for these calculations, we always consider $v_p(\cdot) \bmod p^N$.

We calculate, using the Kirillov Orbit Method, the p-local representation zeta function of a family of Lie rings that are generalizations of a family of maximal class

Lie rings (see, for example, [9, Chapter 2]). Let $k \geq 2,\ q \geq k$, and let $M_{q,k}$ be the Lie ring with basis

$$M_{q,k} := \left\{ y, x_1, \ldots, x_q, z_2, \ldots, z_{k-1} \;\middle|\; \begin{array}{l} [y, x_i] = z_{i+1} \text{ if } 1 \leq i \leq k-2, \\ \\ [y, x_i] = x_{i+1} \text{ if } k-1 \leq i \leq q-1 \end{array} \right\}. \tag{5.3.1}$$

Thus a basis for $M'_{q,k}$ is $\{z_2, \ldots, z_{k-1}, x_k, \ldots, x_q\}$ and a basis for $M'_{q,k} \cap Z(M_{q,k})$ is $\{z_2, \ldots, z_{k-1}, x_q\}$. We note, for $M_{q,k}$, the nilpotency class $c = q - k + 2,\ h(M_{q,k}) = q + k - 1,\ h(Z(M_{q,k})) = k - 1$, and $h(M'_{q,k}) = q - 1$. This calculation in the case when $k = 2$, was originally performed by Christopher Voll in personal communication with the author. We note that $M_{q,2}$ is indeed M_q as defined in Chapter 4 and that the following calculation confirms Theorem 1.7.1 for all but finitely many primes. While the details have not been checked, we believe that it can be shown that this calculation applies for all primes $p > n$, though this bound is not necessarily tight.

5.3.1 Calculation the p-local Representation Zeta Function of $M_{q,k}$

For $\mathbf{a} := (a_2, \ldots, a_q) \in \mathcal{Z}^*_{q-1,p^N}$ we state the $\mathbf{a}$-commutator matrix of $M_{q,k}$:

$$\mathcal{R} = \begin{array}{c} \\ y \\ x_1 \\ x_2 \\ \vdots \\ x_{q-1} \end{array} \begin{array}{c} \begin{array}{ccccc} y & x_1 & x_2 & \ldots & x_{q-1} \end{array} \\ \begin{pmatrix} 0 & a_2 & a_3 & \ldots & a_q \\ -a_2 & 0 & 0 & \ldots & 0 \\ -a_3 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_q & 0 & 0 & \ldots & 0 \end{pmatrix} \end{array}. \tag{5.3.2}$$

The commutator submatrix $\mathcal{S}$ is

$$\mathcal{S} = \begin{array}{c} \\ y \\ x_1 \\ x_2 \\ \vdots \\ x_{q-1} \end{array} \begin{array}{c} \begin{array}{cccc} x_k & x_{k+1} & \ldots & x_{q-1} \end{array} \\ \begin{pmatrix} a_{k+1} & a_{k+2} & \ldots & a_q \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \end{array}. \tag{5.3.3}$$

Note that if $k = q$ then $\mathcal{S}$ does not exist. This is to be expected, since $M_{q,q}$ is of nilpotency class 2.

Let $\mathbf{m} := (p^{m_1}, \ldots, p^{m_q})$ be the elementary divisors of SNF($\mathcal{R}$) and $\mathbf{n} := (p^{n_1}, \ldots, p^{n_{q-k}})$ be the elementary divisors of SNF($\mathcal{S}$). One of $a_2, \ldots, a_q$, say a_j, is a unit mod p^N and, by Lemma 2.5.8, $m_1 = m_2 = 0$. From the structure of $\mathcal{R}$ it is clear that all i-minors, where $i \geq 2$, are 0. Thus, by Lemma 2.5.10, $m_3 = m_4 = \ldots = m_q = N$. It also clear that $n_1 = \min\{v_p(a_i) \mid k+1 \leq i \leq q\}$ and $n_i = N$ for $i \geq 2$.

Since SNF($\mathcal{R}$) is invariant for any choice of $(a_2, \ldots, a_q) \in \mathcal{Z}^*_{q-1,p^N}$ we have that $A_{\mathbf{m}} s = -\sum_{i=1}^{q}(N - m_i)s/2 = \sum_{i=1}^{2}(N - m_i)s/2 = -Ns/2 - Ns/2 = -Ns$. We break up the domain in terms of $v_p(n_1)$.

First, assume $(a_{k+1}, \ldots, a_q) \in \mathcal{Z}^*_{q-k,p^N}$. Thus $n_1 = \min\{v_p(a_i) \mid k+1 \leq i \leq q\} = 0$. In this case there are $(1 - p^{-(q-k)})p^{(q-k)N}$ choices for $(a_{k+1}, \ldots, a_q)$ and $p^{(k-1)N}$ choices for $(a_2, \ldots, a_k)$. Therefore we have, in this section of the domain $\mathcal{Z}^*_{q-1,p^N}$, that $\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-(q-k)})p^{(q-1)N}$ and $B_{\mathbf{n}} = -N$.

Now assume $(a_{k+1}, \ldots, a_q) \notin \mathcal{Z}^*_{q-k,p^N}$. Thus one of $a_2, \ldots a_k$ is a unit and $\min\{v_p(a_2), \ldots, v_p(a_k)\} = 0$. Thus there are $(1-p^{-(k-1)})p^{(k-1)N}$ choices for $(a_2, \ldots, a_k)$. We break up the domain further: let $j = \min\{v_p(a_{k+1}), \ldots, v_p(a_q)\}$. Thus, for each j such that $1 \leq j \leq N-1$, there are $(1 - p^{-(q-k)})p^{(q-k)(N-j)}$ choices for $(a_{k+1}, \ldots, a_q)$. Therefore we have, in this section of the domain and for each j, $\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-(k-1)})(1 - p^{-(q-k)})p^{(k-1)N}p^{(q-k)(N-j)}$ and $B_{\mathbf{n}} = -(N - j)$.

Finally, let $v_p(a_i) = N$ for $k+1 \leq i \leq q$. There is 1 choice for $(a_{k+1}, \ldots, a_q)$. Thus, in this section of the domain, $\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-(k-1)})p^{(k-1)N}$ and $B_{\mathbf{n}} = 0$.

Combining each section of the domain

$$\begin{aligned}
\zeta^{irr}_{M_{q,k},p}(s) &= 1 + \sum_{N=1}^{\infty} \mathcal{N}_{N,\mathbf{m},\mathbf{n}} p^{A_{\mathbf{m}} s} p^{B_{\mathbf{n}}} \qquad (5.3.4)\\
&= 1 + \sum_{N=1}^{\infty} (1 - p^{-(q-k)})p^{(q-1)N} p^{-Ns} p^{-N}\\
&\quad + \sum_{N=1}^{\infty}\sum_{j=1}^{N-1} (1 - p^{-(k-1)})(1 - p^{-(q-k)})p^{(k-1)N}p^{(q-k)(N-j)}p^{-Ns}p^{-(N-j)}\\
&\quad + \sum_{N=1}^{\infty} (1 - p^{-(k-1)})p^{(k-1)N}p^{-Ns}.
\end{aligned}$$

Simplifying the above equation and substituting $t := p^{-s}$

$$\begin{aligned}\zeta^{irr}_{M_{q,k},p}(s) = 1 &+ \sum_{N=1}^{\infty}(1-p^{-(q-k)})(p^{q-2}t)^N \\ &+ \sum_{N=1}^{\infty}(1-p^{-(k-1)})(1-p^{-(q-k)})(p^{q-2}t)^N\sum_{j=1}^{N-1}(p^{-(q-k-1)})^j \\ &+ \sum_{N=1}^{\infty}(1-p^{-(k-1)})(p^{k-1}t)^N.\end{aligned} \tag{5.3.5}$$

Note that if $k = q-1$ then $q-k-1 = 0$ and the subsum in the second sum of Equation 5.3.5 cannot be expressed as a geometric progression. Assume this is not the case. Calculating geometric series and progressions and simplifying we obtain

$$\begin{aligned}\zeta^{irr}_{M_{q,k},p}(s) = 1 &+ \frac{(1-p^{-(q-k)})p^{q-2}t}{1-p^{q-2}t} \\ &+ (1-p^{-(q-k)})(1-p^{-(k-1)})\sum_{N=1}^{\infty}(p^{q-2}t)^N\frac{p^{-(q-k-1)}-(p^{-(q-k-1)})^N}{1-p^{-(q-k-1)}} \\ &+ \frac{(1-p^{-(k-1)})p^{k-1}t}{1-p^{k-1}t}.\end{aligned} \tag{5.3.6}$$

Inputting Equation 5.3.6 into Maple this simplifies to

$$\zeta^{irr}_{M_{q,k},p}(s) = \frac{(1-t)(1-p^{k-2}t)}{(1-p^{k-1}t)(1-p^{q-2}t)}. \tag{5.3.7}$$

Now assume $k = q-1$. Then, calculating geometric series and simplifying,

$$\begin{aligned}\zeta^{irr}_{M_{q,q-1},p}(s) = 1 &+ \frac{(1-p^{-1})p^{q-2}t}{1-p^{q-2}t} \\ &+ (1-p^{-1})(1-p^{-(q-2)})\sum_{N=1}^{\infty}(p^{q-2}t)^N(N-1) \\ &+ \frac{(1-p^{-(q-2)})p^{q-2}t}{1-p^{q-2}t}\end{aligned} \tag{5.3.8}$$

Continuing to simplify,

$$\begin{aligned}\zeta^{irr}_{M_{q,q-1},p}(s) = 1 &+ \frac{(1-p^{-1})p^{q-2}t}{1-p^{q-2}t} \\ &+ \frac{(1-p^{-1})(1-p^{-(q-2)})}{1-p^{-(q-k-1)}}\left(\frac{p^{k-1}t}{1-p^{q-2}t} - \frac{p^{k-1}t}{1-p^{k-1}t}\right) \\ &+ \frac{(1-p^{-(q-2)})p^{q-2}t}{1-p^{q-2}t}.\end{aligned} \tag{5.3.9}$$

Inputting Equation 5.3.9 into Maple this simplifies to

$$\zeta^{irr}_{M_{q,q-1},p}(s) = \frac{(1-t)(1-p^{q-3}t)}{(1-p^{q-2}t)^2}. \tag{5.3.10}$$

In this case $\zeta^{irr}_{M_{q,q-1},p}(s)$ has a double pole at $s = q-2$.

Note that Equation 5.3.7 with k set to $q-1$ is Equation 5.3.10 and thus we can say Equation 5.3.7 holds for all q, k as defined at the start of the section. Also note that, when $k = 2$, Equation 5.3.5 is the same as Equation 4.2.24 in Chapter 4. Finally, note that if $k = q$ then the $(1-p^{k-2}t)$ term in the numerator cancels with the $(1-p^{q-2}t)$ term in the denominator.

It is easy to see that $\alpha_{M_{q,k},p} = q-2$ if $k \neq q$ and $\alpha_{M_{q,q},p} = q-1$. Also, $\zeta^{irr}_{M_{q,k},p}(s)$ does satisfy the functional equation in Theorem 1.4.3.

5.3.2 Calculation of Some Lie Rings from Chapter 2 of du Sautoy and Woodward [9]

Table 5.1 is a list of Lie rings, and some invariants, for which we calculate the representation zeta function using the Kirillov orbit method. Note that these Lie rings appear in [9, Chapter 2]. In a future paper with Robert Snocken, we plan to give p-local representation zeta functions for all Lie rings in that chapter.

Table 5.2 is a table of the Lie rings above, their non-exceptional p-local representation zeta functions, and their local abscissas of convergence, denoted $\alpha_{L,p}$.

To calculate SNF($\mathcal{R}$) of a $r \times r$ matrix $\mathcal{R}$, by Lemma 2.5.10, we only need to know its minors. In fact, since there exist 2-minors with unit determinant in all examples computed in this section, we have $m_1 = m_2 = 0$. The commutator matrices $\mathcal{R}$ in the examples below are, at most, of dimension 5, and when they are of maximum dimension, Lemma 2.5.8 tells us that $m_5 = N$. Since $m_{2i} = m_{2i-1}$ for $i \geq 1$, for a $\mathcal{R}$ of dimension 5 we need to only compute the minimum valuation 4-minor of $\mathcal{R}$. When $\mathcal{R}$

Table 5.1 List of Lie Rings Studied

L	Presentation	$h(Z(L))$	$h(L)$	$h(L')$	c
$F_{3,2}$	$\langle x_1,\ldots,x_5 \| [x_1,x_2]=x_3, [x_1,x_3]=x_4, [x_2,x_3]=x_5\rangle$	2	5	3	3
$G_{5,3}$	$\langle x_1,\ldots,x_5 \| [x_1,x_2]=x_4, [x_1,x_4]=x_5, [x_2,x_3]=x_5\rangle$	1	5	2	3
$G_{6,7}$	$\langle x_1,\ldots,x_6 \| [x_1,x_3]=x_4, [x_1,x_4]=x_5, [x_2,x_3]=x_6\rangle$	2	6	3	3
$G_{6,12}$	$\langle x_1,\ldots x_6 \| [x_1,x_3]=x_5\ [x_1,x_5]=x_6, [x_2,x_4]=x_6\rangle$	1	6	2	3
$G_{6,14}$	$\langle x_1,\ldots,x_6 \| [x_1,x_3]=x_4, [x_1,x_4]=x_6, [x_2,x_3]=x_5,$ $[x_2,x_5]=\gamma x_6\rangle\ \gamma\in\mathbb{Z}$	1	6	3	3
G_{27A}	$\langle x_1,\ldots,x_7 \| [x_1,x_2]=x_6, [x_1,x_4]=x_7, [x_3,x_5]=x_7\rangle$	2	7	2	2
G_{37C}	$\langle x_1,\ldots,x_7 \| [x_1,x_2]=x_5, [x_2,x_3]=x_6, [x_2,x_4]=x_7,$ $[x_3,x_4]=x_5\rangle$	3	7	3	2
G_{257K}	$\langle x_1,\ldots,x_7 \| [x_1,x_2]=x_5, [x_1,x_5]=x_6, [x_2,x_5]=x_7,$ $[x_3,x_4]=x_7\rangle$	2	7	3	3

Table 5.2 p-local Representation Zeta Functions of Groups Associated to Lie Rings in Table 5.1

L	$\zeta^{irr}_{L,p}(s)$	$\alpha_{L,p}$
$F_{3,2}$	$\dfrac{(1-t)^2}{(1-pt)(1-p^2t)}$	2
$G_{5,3}$	$\dfrac{(1-t)(1-pt^2)}{(1-pt)(1-p^2t^2)}$	1
$G_{6,7}$	$\dfrac{(1-t)^2}{(1-pt)(1-p^2t)}$	2
$G_{6,12}$	$\dfrac{(1-t)(1-t^2)}{(1-pt)(1-pt^2)}$	1
$G_{6,14}$	$\dfrac{(1-t)(1-t^2)}{(1-pt^2)(1-p^2t)}$	2
G_{27A}	$\dfrac{(1-t)(1-pt^2)}{(1-pt)(1-p^2t^2)}$	1
G_{37C}	$\dfrac{(1-t)(1-p^4t^2)}{(1-p^5t^2)(1-p^2t)}$	$\frac{5}{2}$
G_{257K}	$\dfrac{(1-t)^2(1-pt^2)}{(1-pt)^2(1-p^2t^2)}$	1

is 4×4 we need only calculate $\det(\mathcal{R})$. In the examples below the size of the submatrix $\mathcal{S}$ is at most a 5×2 matrix, so calculating $\mathrm{SNF}(\mathcal{S})$ is not very difficult.

We give the explicit details of the calculations of the non-exceptional p-local representation zeta functions of two Lie rings: G_{37C} and G_{257K}. In the other examples, we state the necessary pieces of information so one could recreate the calculations without (hopefully) too much difficulty. Note that the geometric series that appear in these examples are readily calculable, so we need not appeal to p-adic integration to assist us in the calculations. This p-adic integral approach is explained, for example, in [16, Section 2.3].

5.3.3 Calculating the p-local Representation Zeta Function of G_{37C}

Let $(a_5, a_6, a_7) \in \mathcal{Z}^*_{3,p^N}$. We begin by stating the commutator matrix:

$$\mathcal{R} = \begin{pmatrix} 0 & a_5 & 0 & 0 \\ -a_5 & 0 & a_6 & a_7 \\ 0 & -a_6 & 0 & a_5 \\ 0 & -a_7 & -a_5 & 0 \end{pmatrix} \tag{5.3.11}$$

Note that the nilpotency class of G_{37C} is 2 and thus there is no commutator submatrix $\mathcal{S}$. Obviously, $\det(\mathcal{R}) = a_5^4$ and thus by Lemma 2.5.10, for $\mathrm{SNF}(\mathcal{R})$, $m_3 = m_4 = v_p(a_5^2)$. So, for $N \in \mathbb{N}$,

$$\mathrm{SNF}(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_5^2 & 0 \\ 0 & 0 & 0 & a_5^2 \end{pmatrix} \mod p^N. \tag{5.3.12}$$

We note that we must be careful here; if $v_p(a_5) \geq N/2$ then $v_p(a_5^2) = 2v_p(a_5) \geq N$ and it follows that $a_5^2 = 0 \mod p^N$. Thus, if $2v_p(a_5) \geq N$ then $m_3 = m_4 = N$.

We break up the domain of $\mathcal{Z}^*_{3,p^N}$: first, let $v_p(a_5) = 0$ and $v_p(a_6), v_p(a_7) \leq N$. In this section of the domain there are $(1-p^{-1})p^N$ choices for a_5 and p^N separate choices for both a_6 and a_7. Thus

$$\mathcal{N}_{N,\mathbf{m}} = (1-p^{-1})p^N p^N p^N = (1-p^{-1})p^{3N}. \tag{5.3.13}$$

Since $v_p(a_5) = 0$ we have that $m_3 = m_4 = 0$ and thus

$$A_{\mathbf{m}} s = -\sum_{i=1}^{4}(N - m_i)s/2 = -4(Ns/2) = -2Ns. \tag{5.3.14}$$

Now assume $(a_6, a_7) \in \mathcal{Z}^*_{2,p^N}$ and $a_5 \notin \mathcal{Z}^*_{1,p^N}$. There are $(1-p^{-2})p^{2N}$ choices for (a_6, a_7) and we further break up the domain for a_5. Let $\lfloor x \rfloor$ be the greatest integer not greater than x.

Assume $v_p(a_5) = k < N/2$. Note that this implies that $1 \le k \le \lfloor N/2 \rfloor$. Then there are $(1-p^{-1})p^{N-k}$ choices for a_5 and $m_3 = m_4 = 2k$. Thus, for each k,

$$\mathcal{N}_{N,\mathbf{m}} = (1-p^{-2})p^{2N}(1-p^{-1})p^{N-k} \tag{5.3.15}$$

and

$$A_{\mathbf{m}}s = -\sum_{i=1}^{4}(N-m_i)s/2 = -(2N+2(N-2k))s/2 = -2(N-k)s. \tag{5.3.16}$$

Now assume that $v_p(a_5) = k \ge N/2$ and $k \ne N$. Note that this implies that $\lfloor N/2 \rfloor \le k \le N-1$. Then there are $(1-p^{-1})p^{N-k}$ choices for a_5 and, since we are considering $\mathrm{SNF}(\mathcal{R})$ mod p^N and $2k \ge N$, we have that $m_3 = m_4 = N$. Thus, for each k,

$$\mathcal{N}_{N,\mathbf{m}} = (1-p^{-2})p^{2N}(1-p^{-1})p^{N-k} \tag{5.3.17}$$

and

$$A_{\mathbf{m}}s = -\sum_{i=1}^{4}(N-m_i)s/2 = -(2N+2(N-N))s/2 = -Ns. \tag{5.3.18}$$

Now assume that $v_p(a_5) = N$. Thus there is 1 choice for a_5 and

$$\mathcal{N}_{N,\mathbf{m}} = (1-p^{-2})p^{2N}. \tag{5.3.19}$$

It is clear that $m_3 = m_4 = N$ and thus

$$A_{\mathbf{m}}s = -\sum_{i=1}^{4}(N-m_i)s/2 = -(2N+2(N-N))/2s = -Ns. \tag{5.3.20}$$

Thus

$$\begin{aligned}\zeta^{irr}_{G_{37C},p}(s) &= 1 + \sum_{N=1}^{\infty}(1-p^{-1})p^{3N}p^{-2Ns} \\ &+ \sum_{N=1}^{\infty}(1-p^{-2})p^{2N}\left(\sum_{k=1}^{\lfloor\frac{N}{2}\rfloor}(1-p^{-1})p^{N-k}p^{-2(N-k)s}\right. \\ &+ \left.\sum_{k=\lfloor\frac{N}{2}\rfloor+1}^{N-1}\left((1-p^{-1})p^{N-k}p^{-Ns}\right)+p^{-Ns}\right).\end{aligned} \tag{5.3.21}$$

If N is even $\lfloor N/2\rfloor = N/2$ and if N is odd $\lfloor N/2\rfloor = (N-1)/2$. So to compute this p-local zeta function we calculate Equation 5.3.21 separately for even and odd N. We write $N = 2M$ for even N and $N = 2M-1$ for odd N, where $M \geq 1$.

Thus

$$\begin{aligned}\zeta^{irr}_{G_{37C},p}(s) &= 1 + \sum_{M=1}^{\infty}(1-p^{-1})p^{6M}p^{-4Ms} \\ &+ \sum_{M=1}^{\infty}(1-p^{-2})p^{4M}\left(\sum_{k=1}^{M}(1-p^{-1})p^{2M-k}p^{-2(2M-k)s}\right. \\ &+ \left.\left(\sum_{k=M+1}^{2M-1}(1-p^{-1})p^{2M-k}p^{-2Ms}\right)+p^{-2Ms}\right) \\ &+ \sum_{M=1}^{\infty}(1-p^{-1})p^{6M-3}p^{-(4M-2)s} \\ &+ \sum_{M=1}^{\infty}(1-p^{-2})p^{4M-2}\left(\sum_{k=1}^{M-1}(1-p^{-1})p^{(2M-1)-k}p^{-2((2M-1)-k)s}\right. \\ &+ \left.\left(\sum_{k=M}^{2M-2}(1-p^{-1})p^{(2M-1)-k}p^{-(2M-1)s}\right)+p^{-(2M-1)s}\right).\end{aligned} \tag{5.3.22}$$

Simplifying, and substituting $t := p^{-s}$

$$\begin{aligned}
\zeta^{irr}_{G_{37C},p}(s) = 1 &+ (1-p^{-1})\frac{p^6t^4}{1-p^6t^4} \\
&+ \sum_{M=1}^{\infty}\left((1-p^{-2})p^{4M}\left((1-p^{-1})p^{2M}t^{4M}\sum_{k=1}^{M}(p^{-1}t^{-2})^k\right.\right. \\
&+ \left.\left.(1-p^{-1})p^{2M}t^{2M}\sum_{k=M+1}^{2M-1}(p^{-1})^k + t^{2M}\right)\right) \\
&+ (1-p^{-1})p^{-3}t^{-2}\frac{p^6t^4}{1-p^6t^4} \\
&+ \sum_{M=1}^{\infty}\left((1-p^{-2})p^{4M-2}\left((1-p^{-1})p^{2M-1}t^{4M-2}\sum_{k=1}^{M-1}(p^{-1}t^{-2})^k\right.\right. \\
&+ \left.\left.(1-p^{-1})p^{2M-1}t^{2M-1}\sum_{k=M}^{2M-2}(p^{-1})^k + t^{2M-1}\right)\right)
\end{aligned} \tag{5.3.23}$$

Summing the geometric series of index k and rearranging we obtain

$$\begin{aligned}
\zeta^{irr}_{G_{37C},p}(s) = 1 &+ (1-p^{-1})\frac{p^6t^4}{1-p^6t^4} \\
&+ \sum_{M=1}^{\infty}\left((1-p^{-2})p^{4M}\left((1-p^{-1})(p^2t^4)^M\frac{p^{-1}t^{-2}-(p^{-1}t^{-2})^{M+1}}{1-p^{-1}t^{-2}}\right.\right. \\
&+ \left.\left.(1-p^{-1})(p^2t^2)^M\frac{(p^{-1})^{M+1}-(p^{-1})^{2M}}{1-p^{-1}} + (t^2)^M\right)\right) \\
&+ (1-p^{-1})p^{-3}t^{-2}\frac{p^6t^4}{1-p^6t^4} \\
&+ \sum_{M=1}^{\infty}\left((1-p^{-2})p^{4M}p^{-2}\left((1-p^{-1})(p^2t^4)^Mp^{-1}t^{-2}\frac{p^{-1}t^{-2}-(p^{-1}t^{-2})^{M}}{1-p^{-1}t^{-2}}\right.\right. \\
&+ \left.\left.(1-p^{-1})(p^2t^2)^Mp^{-1}t^{-1}\frac{(p^{-1})^{M}-(p^{-1})^{2M-1}}{1-p^{-1}} + (t^2)^Mt^{-1}\right)\right)
\end{aligned} \tag{5.3.24}$$

Inputting Equation 5.3.24 into Maple, this simplifies to

$$\zeta^{irr}_{G_{37C},p}(s) = \frac{(1-t)(1-p^4t^2)}{(1-p^2t)(1-p^5t^2)}. \tag{5.3.25}$$

Note that Equation 5.3.25 satisfies the functional equation of Theorem 1.4.3.

5.3.4 Calculating the p-local Representation Zeta Function of G_{257K}

This calculation is of a Lie ring of nilpotency class 3. Thus, we have a submatrix $\mathcal{S}$ whose Smith Normal Form must be calculated, as well as the Smith Normal Form of the commutator matrix $\mathcal{R}$. Like the previous examples, we will split the domain into sections and calculate each section independently. Let $(a_5, a_6, a_7) \in \mathcal{Z}^*_{3,p^N}$. We begin by stating $\mathcal{R}$:

$$\mathcal{R} = \begin{pmatrix} 0 & a_5 & 0 & 0 & a_6 \\ -a_5 & 0 & 0 & 0 & a_7 \\ 0 & 0 & 0 & a_7 & 0 \\ 0 & 0 & -a_7 & 0 & 0 \\ -a_6 & -a_7 & 0 & 0 & 0 \end{pmatrix}. \tag{5.3.26}$$

We also state the submatrix $\mathcal{S}$:

$$\mathcal{S} = \begin{pmatrix} a_6 \\ a_7 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.3.27}$$

By calculating the 4-minors of $\mathcal{R}$ we can then determine $SNF(\mathcal{R}) \bmod p^N$. By Lemma 2.5.8, $m_5 = N$ and we can use this lemma to determine the elementary divisors of $SNF(\mathcal{R})$. Thus, $\mathcal{M}_4$, the set of 4-minors of $\mathcal{R}$, calculated by taking the determinant of all 4×4 submatrices of $\mathcal{R}$, is

$$\mathcal{M}_4 = \{0, a_7^4, a_5^2 a_7^2, a_6^2 a_7^2, a_5 a_7^3, a_6 a_7^3, a_5 a_6 a_7^2\}. \tag{5.3.28}$$

If $v_p(a_7) = 0$ then $\min\{v_p(x) \mid x \in \mathcal{M}_4\} = v_p(a_7^4) = 0$. If $v_p(a_i) = 0$ where $i \in \{5, 6\}$, then $\min\{v_p(x) \mid x \in \mathcal{M}_4\} = v_p(a_i^2 a_7^2) = v_p(a_7^2)$. It then follows by Lemma 2.5.10 that $m_3 = m_4 = v_p(a_7)$.

It is easy to see, for SNF($\mathcal{S}$), that $B_{\mathbf{n}} = -(N - n_1) = -(N - \min\{v_p(a_6), v_p(a_7)\})$.

For the first section on the domain, let $v_p(a_7) = 0$. Then $m_3 = m_4 = 0$ and

$$A_{\mathbf{m}} s = -\sum_{i=1}^{4} (N - m_i)s/2 = -2Ns. \tag{5.3.29}$$

Since $v_p(a_7) = 0$ we also have that $n_1 = 0$ so

$$B_{\mathbf{n}} = -N - 0 = -N. \tag{5.3.30}$$

In this section there are $(1 - p^{-1})p^N$ choices for a_7 and p^N separate choices for each of a_5 and a_6. Thus, in this section of the domain,

$$\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-1})p^N p^N p^N = (1 - p^{-1})p^{3N}. \tag{5.3.31}$$

Now let $v_p(a_6) = 0$ and $v_p(a_7) = k$ where $1 \leq k \leq N$. For each k we have that $m_3 = m_4 = k$ and

$$A_{\mathbf{m}} s = -\sum_{i=1}^{4} (N - m_i)s/2 = -2(N + (N - k))s/2 = -(2N - k)s. \tag{5.3.32}$$

Since $v_p(a_6) = 0$ we have that $n_1 = 0$ so

$$B_{\mathbf{n}} = -N - 0 = -N. \tag{5.3.33}$$

In this section there are $(1 - p^{-1})p^N$ choices for a_6, p^N choices for a_5, and, for each $k \neq N$, $(1 - p^{-1})p^{N-k}$ choices for a_7. Thus in this section of the domain, and for each $k \neq N$, we have

$$\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-1})p^N p^N (1 - p^{-1})p^{N-k} = (1 - p^{-1})^2 p^{3N-k} \tag{5.3.34}$$

and if $k = N$ we have 1 choice for a_7 and therefore

$$\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-1})p^N p^N = (1 - p^{-1})p^{2N}. \tag{5.3.35}$$

Now let $v_p(a_5) = 0$ and $v_p(a_i)$ be such that $1 \le v_p(a_i) \le N - 1$ for $i \in \{6, 7\}$. Let $v_p(a_6) = k$ and $v_p(a_7) = \ell$. In this section of the domain, $m_3 = m_4 = \ell$ for each k and

$$A_{\mathbf{m}} s = -\sum_{i=1}^{4}(N - m_i)s/2 = -2(N + (N - \ell))s/2 = -(2N - \ell)s. \tag{5.3.36}$$

For each k and ℓ we have that $n_1 = \min\{k, \ell\}$. Thus, for each choice of k and ℓ

$$B_{\mathbf{n}} = -(N - \min\{k, \ell\}). \tag{5.3.37}$$

In this section there are $(1 - p^{-1})p^N$ choices for a_5, $(1 - p^{-1})p^{N-k}$ choices for a_6, and $(1 - p^{-1})p^{N-\ell}$ choices for a_7, for each choice of k and ℓ. Thus in this section of the domain we have that

$$\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-1})p^N(1 - p^{-1})p^{N-k}(1 - p^{-1})p^{N-\ell} = (1 - p^{-1})^3 p^{3N-k-\ell}. \tag{5.3.38}$$

Now let $v_p(a_5) = 0$, $v_p(a_6) = k$, where $1 \le k \le N - 1$, and $v_p(a_7) = N$. In this section of the domain we have that $m_3 = m_4 = N$ and thus

$$A_{\mathbf{m}} s = -\sum_{i=1}^{4}(N - m_i)s/2 = -2(N + (N - N))s/2 = -Ns. \tag{5.3.39}$$

For each k we have that $n_1 = \min\{k, N\} = k$ and thus, for each k,

$$B_{\mathbf{n}} = -(N - k). \tag{5.3.40}$$

In this section there are $(1 - p^{-1})p^N$ choices for a_5, $(1 - p^{-1})p^{N-k}$ choices for a_6, for each k, and 1 choice for a_7. Thus in this section of the domain

$$\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1 - p^{-1})p^N(1 - p^{-1})p^{N-k} = (1 - p^{-1})^2 p^{2N-k}. \tag{5.3.41}$$

Now let $v_p(a_5) = 0$, $v_p(a_6) = N$, and $v_p(a_7) = k$, where $1 \leq k \leq N-1$. In this section of the domain we have that $m_3 = m_4 = k$ and thus

$$A_{\mathbf{m}}s = -\sum_{i=1}^{4}(N - m_i)s/2 = -2(N + (N-k))s/2 = -(2N-k)s. \tag{5.3.42}$$

For each k we have that $n_1 = \min\{N, k\} = k$ and thus, for each k,

$$B_{\mathbf{n}} = -(N-k). \tag{5.3.43}$$

In this section there are $(1-p^{-1})p^N$ choices for a_5, 1 choice for a_6, and $(1-p^{-1})p^{N-k}$ choices for a_7, for each k. Thus in this section of the domain

$$\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1-p^{-1})p^N(1-p^{-1})p^{N-k} = (1-p^{-1})^2p^{2N-k}. \tag{5.3.44}$$

Finally, let $v_p(a_5) = 0$ and $v_p(a_6) = v_p(a_7) = N$. In this section of the domain $m_3 = m_4 = N$ and thus

$$A_{\mathbf{m}}s = -\sum_{i=1}^{4}(N - m_i)s/2 = -2(N + (N-N))s/2 = -Ns. \tag{5.3.45}$$

We have that $n_1 = \min\{N, N\} = N$ and thus

$$B_{\mathbf{n}} = 0. \tag{5.3.46}$$

In this section of the domain there are $(1-p^{-1})p^N$ choices for a_5 and 1 choice each for a_6 and a_7. Thus in this section of the domain

$$\mathcal{N}_{N,\mathbf{m},\mathbf{n}} = (1-p^{-1})p^N. \tag{5.3.47}$$

Combining the results for each section of the domain

$$
\begin{aligned}
\zeta^{irr}_{G_{257K},p}(s) = 1 + \sum_{N=1}^{\infty} \Bigg(& (1-p^{-1})p^{3N}p^{-2Ns}p^{-N} \\
& + \sum_{k=1}^{N-1}(1-p^{-1})^2 p^{3N-k}p^{-(2N-k)s}p^{-N} \\
& + (1-p^{-1})p^{2N}p^{-Ns}p^{-N} \\
& + \sum_{k=1}^{N-1}\sum_{\ell=1}^{N-1}(1-p^{-1})^3 p^{3N-k-\ell}p^{-(2N-\ell)s}p^{-\max\{N-k,N-\ell\}} \\
& + \sum_{k=1}^{N-1}(1-p^{-1})^2 p^{2N-k}p^{-Ns}p^{-(N-k)} \\
& + \sum_{k=1}^{N-1}(1-p^{-1})^2 p^{2N-k}p^{-(2N-k)s}p^{-(N-k)} + \\
& + (1-p^{-1})p^{N}p^{-Ns}\Bigg).
\end{aligned}
\tag{5.3.48}
$$

Simplifying and substituting $t := p^{-s}$ we have

$$
\begin{aligned}
\zeta^{irr}_{G_{257K},p}(s) = 1 + \sum_{N=1}^{\infty} \Bigg(& (1-p^{-1})(p^2t^2)^N \\
& + (1-p^{-1})^2(p^2t^2)^N \sum_{k=1}^{N-1}(p^{-1}t^{-1})^k \\
& + (1-p^{-1})(pt)^N \\
& + (1-p^{-1})^3(p^2t^2)\left[\sum_{k=1}^{N-1}\sum_{\ell=1}^{N-1} p^{-k}p^{-\ell}t^{-\ell}p^{\min\{k,\ell\}}\right] \\
& + (1-p^{-1})^2(pt)^N \sum_{k=1}^{N-1} 1 \\
& + (1-p^{-1})^2(pt^2)^N \sum_{k=1}^{N-1}(t^{-1})^k \\
& + (1-p^{-1})(pt)^N\Bigg).
\end{aligned}
\tag{5.3.49}
$$

(5.3.50)

Let Q be the double sum in square brackets. This can be broken up into three different sums depending on k and ℓ, say Q_1, Q_2, and Q_3, when $k > \ell, k < \ell$, and $k = \ell$, respectively. Then

$$\begin{aligned} Q_1 &= \sum_{k=2}^{N-1} \sum_{l=1}^{k-1} (p^{-1})^k (t^{-1})^\ell, \\ Q_2 &= \sum_{k=1}^{N-2} \sum_{\ell=k+1}^{N-1} (p^{-1}t^{-1})^\ell, \\ Q_3 &= \sum_{k=1}^{N-1} (p^{-1}t^{-1})^k. \end{aligned} \tag{5.3.51}$$

Summing the series with index ℓ we have

$$\begin{aligned} Q_1 &= \sum_{k=2}^{N-1} (p^{-1})^k \frac{t^{-1} - (t^{-1})^k}{1 - t^{-1}}, \\ Q_2 &= \sum_{k=1}^{N-2} \frac{(p^{-1}t^{-1})^{k+1} - (p^{-1}t^{-1})^N}{1 - p^{-1}t^{-1}}. \end{aligned} \tag{5.3.52}$$

Thus, substituting $Q_1 + Q_2 + Q_3$ for Q we have

$$\begin{aligned} \zeta^{irr}_{G_{257K},p}(s) = 1 + \sum_{N=1}^{\infty} \Bigg(&(1 - p^{-1})(p^2t^2)^N \\ &+ (1 - p^{-1})^2 (p^2t^2)^N \sum_{k=1}^{N-1} (p^{-1}t^{-1})^k \\ &+ (1 - p^{-1})(pt)^N \\ &+ (1 - p^{-1})^3 (p^2t^2) \left[\sum_{k=2}^{N-1} (p^{-1})^k \frac{t^{-1} - (t^{-1})^k}{1 - t^{-1}} \right. \\ &+ \left. \sum_{k=1}^{N-2} \frac{(p^{-1}t^{-1})^{k+1} - (p^{-1}t^{-1})^N}{1 - p^{-1}t^{-1}} + \sum_{k=1}^{N-1} (p^{-1}t^{-1})^k \right] \\ &+ (1 - p^{-1})^2 (pt)^N \sum_{k=1}^{N-1} 1 \\ &+ (1 - p^{-1})^2 (pt^2)^N \sum_{k=1}^{N-1} (t^{-1})^k + (1 - p^{-1})(pt)^N \Bigg). \end{aligned} \tag{5.3.53}$$

Summing the series with index k it follows that

$$
\begin{aligned}
\zeta^{irr}_{G_{257K},p}(s) = 1 + \sum_{N=1}^{\infty} \Bigg(&(1-p^{-1})(p^2t^2)^N \\
&+ (1-p^{-1})^2(p^2t^2)^N \frac{p^{-1}t^{-1} - (p^{-1}t^{-1})^N}{1-p^{-1}t^{-1}} \\
&+ (1-p^{-1})(pt)^N \\
&+ (1-p^{-1})^3(p^2t^2) \Bigg[\frac{t^{-1}}{1-t^{-1}} \cdot \frac{(p^{-1})^2 - (p^{-1})^N}{1-p^{-1}} \\
&- \frac{1}{1-t^{-1}} \cdot \frac{(p^{-1}t^{-1})^2 - (p^{-1}t^{-1})^N}{1-p^{-1}t^{-1}} \\
&+ \frac{p^{-1}t^{-1}}{1-p^{-1}t^{-1}} \cdot \frac{p^{-1}t^{-1} - (p^{-1}t^{-1})^{N-1}}{1-p^{-1}t^{-1}} \\
&-(N-2)\frac{(p^{-1}t^{-1})^N}{1-p^{-1}t^{-1}} + \frac{p^{-1}t^{-1} - (p^{-1}t^{-1})^N}{1-p^{-1}t^{-1}} \Bigg] \\
&+ (1-p^{-1})^2(pt)^N(N-1) \\
&+ (1-p^{-1})^2(pt^2)^N \frac{t^{-1} - (t^{-1})^N}{1-t^{-1}} \\
&+ (1-p^{-1})(pt)^N \Bigg).
\end{aligned}
\tag{5.3.54}
$$

Inputting Equation 5.3.54 into Maple, this simplifies to

$$
\zeta^{irr}_{G_{257K},p}(s) = \frac{(1-t)^2(1-pt^2)}{(1-pt)^2(1-p^2t^2)}. \tag{5.3.55}
$$

Note that Equation 5.3.55 satisfies the functional equation of Theorem 1.4.3.

5.3.5 Other p-local Representation Zeta Functions

The other p-local representation zeta functions are computed similarly to the two calculations above. Without going through the calculations explicitly, for each Lie ring we give the commutator matrix $\mathcal{R}$, the submatrix $\mathcal{S}$, the possible values for the m_i and n_i, as well as the final sum of $\mathcal{N}_{N,\mathbf{m},\mathbf{n}}p^{A_{\mathbf{m}}s}p^{B_{\mathbf{n}}}$ for each section of the domain. For zeta functions of Lie rings L in Table 5.1, let Q_L be such that $\zeta^{irr}_{L,p}(s) = 1 + \sum_{N=1}^{\infty} Q_L$.

$\mathbf{F_{3,2}}$

For $(a_3, a_4, a_5) \in \mathcal{Z}^*_{3,p^N}$,

$$\mathcal{R} = \begin{pmatrix} 0 & a_3 & a_4 \\ -a_3 & 0 & a_5 \\ -a_4 & -a_5 & 0 \end{pmatrix} \tag{5.3.56}$$

and

$$\mathcal{S} = \begin{pmatrix} a_4 \\ a_5 \\ 0 \end{pmatrix}. \tag{5.3.57}$$

Since $\mathcal{R}$ is a 3×3 matrix, $m_1 = m_2 = 0$ and $m_3 = N$ when we calculate SNF($\mathcal{R}$). For SNF($\mathcal{S}$), $n_1 = \min\{v_p(a_4), v_p(a_5)\}$. It follows that, for $(a_3, a_4, a_5) \in \mathcal{Z}^*_{3,p^N}$,

$$\begin{aligned} Q_{F_{3,2}} = {} & (1-p^{-2})(p^2t)^N + (1-p^{-1})^3(p^2t)^N \sum_{k=1}^{N-1}(2k-1)(p^{-1})^k \\ & + (2(1-p^{-1})(N-1)(1-p^{-1})(pt)^N + (1-p^{-1})(pt)^N \end{aligned} \tag{5.3.58}$$

The first term of $Q_{F_{3,2}}$ is the case when one of $v_p(a_4), v_p(a_5) = 0$. The second term is when $v_p(a_3) = 0$, $v_p(a_4) = k$, and $v_p(a_5) = \ell$, for some k, ℓ such that $1 \leq k, \ell \leq N-1$. When doing this calculation, note that the double sum that would be in this term can be replaced by the single sum above, due to the double sum's symmetry in k and ℓ. The third term is when $v_p(a_3) = 0$, one of $v_p(a_4), v_p(a_5) = N$, and the other $v_p(a_4), v_p(a_5) = k$ where $1 \leq k \leq N-1$. The final term is when $v_p(a_3) = 0$ and $v_p(a_4) = v_p(a_5) = N$.

Inputting Equation 5.3.58 into Maple we obtain

$$\zeta^{irr}_{F_{3,2},p}(s) = \frac{(1-t)^2}{(1-pt)(1-p^2t)}. \tag{5.3.59}$$

$\mathbf{G_{5,3}}$

For $(a_4, a_5) \in \mathcal{Z}^*_{2,p^N}$,

$$\mathcal{R} = \begin{pmatrix} 0 & a_4 & 0 & a_5 \\ -a_4 & 0 & a_5 & 0 \\ 0 & -a_5 & 0 & 0 \\ -a_5 & 0 & 0 & 0 \end{pmatrix} \tag{5.3.60}$$

and

$$\mathcal{S} = \begin{pmatrix} a_5 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.3.61}$$

We have that $\det(\mathcal{R}) = a_5^4$ and thus, for $\mathrm{SNF}(\mathcal{R})$, we have $m_3 = m_4 = 2v_p(a_5)$. For $\mathrm{SNF}(\mathcal{S})$, it is clear that $n_1 = v_p(a_5)$. It follows that, for $(a_4, a_5) \in \mathcal{Z}^*_{2,p^N}$, we have

$$\begin{aligned} Q_{G_{5,3}} = {} & (1-p^{-1})(pt^2) + (1-p^{-1})^2 p^{2M} \left(t^{4M} \sum_{k=1}^{M-1} (t^{-1})^k + t^{2M} \sum_{k=M}^{2M-1} 1 \right) \\ & + (1-p^{-1})^2 p^{2M-1} \left(t^{4M-2} \sum_{k=1}^{M-1} (t^{-1})^k + t^{2M-1} \sum_{k=M}^{2M-2} 1 \right) \\ & + (1-p^{-1})(pt)^N \end{aligned} \tag{5.3.62}$$

where $2M = N$ if N is even and $2M - 1 = N$ if N is odd. The first term is when $v_p(a_5) = 0$. The second term is when $v_p(a_4) = 0$ and $v_p(a_5) = k$ for even N, where $1 \leq k \leq N-1$. The first subterm in the second term is when $k < N/2$ and the second subterm is when $k \geq N/2$. The third term is when $v_p(a_4) = 0$ and $v_p(a_5) = k$ for odd N, where $1 \leq k \leq N-1$. The first subterm in the second term is when $k < N/2$

and the second subterm is when $k \geq N/2$. The fourth term is when $v_p(a_4) = 0$ and $v_p(a_5) = N$.

Inputting Equation 5.3.62 into Maple we obtain

$$\zeta^{irr}_{G_{5,3},p}(s) = \frac{(1-t)(1-pt^2)}{(1-pt)(1-p^2t^2)}. \tag{5.3.63}$$

$\mathbf{G_{6,7}}$

For $(a_4, a_5, a_6) \in \mathcal{Z}^*_{3,p^N}$,

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & a_4 & a_5 \\ 0 & 0 & a_6 & 0 \\ -a_4 & -a_6 & 0 & 0 \\ -a_5 & 0 & 0 & 0 \end{pmatrix} \tag{5.3.64}$$

and

$$\mathcal{S} = \begin{pmatrix} a_5 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.3.65}$$

We have that $\det(\mathcal{R}) = a_5^2 a_6^2$ and thus, for $\mathrm{SNF}(\mathcal{R})$, we have $m_3 = m_4 = v_p(a_5)+v_p(a_6)$. For $\mathrm{SNF}(\mathcal{S})$, it is clear that $n_1 = v_p(a_5)$. It follows that, for $(a_4, a_5, a_6) \in \mathcal{Z}^*_{3,p^N}$, we have

$$
\begin{aligned}
Q_{G_{6,7}} = & \left[(1-p^{-1})(p^2)^N\left((1-p^{-1})(t^2)^N\frac{1-(t^{-1})^N}{1-t^{-1}}+t^N\right)\right] \\
& + \left[(1-p^{-1})p^N\left((1-p^{-1})(pt^2)^N\frac{p^{-1}t^{-1}-(p^{-1}t^{-1})^N}{1-p^{-1}t^{-1}}+t^N\right)\right] \\
& + \left[(1-p^{-1})p^N\left((1-p^{-1})^2(pt^2)^N\left(\frac{p^{-1}t^{-1}}{1-p^{-1}t^{-1}}\cdot\frac{t^{-1}-(t^{-1})^N}{1-t^{-1}}\right.\right.\right. \\
& \left.-\frac{(p^{-1}t^{-1})^{N+1}}{1-p^{-1}t^{-1}}\cdot\frac{p-p^N}{1-p}\right)+(1-p^{-1})^2(pt)^N\left(\frac{(p^{-1})^{N+1}}{1-p^{-1}}\cdot\frac{p-p^N}{1-p}\right. \\
& \left.\left.\left.-(N-1)\frac{(p^{-1})^N}{1-p^{-1}}\right)\right)\right] \\
& + \left[(1-p^{-1})(p^2t)^Np^{-1}\right]+\left[(1-p^{-1})^2(pt)^N(N-1)\right].
\end{aligned} \tag{5.3.66}
$$

Note that we put square brackets around each term, for clarity, in the equation above. The first term is when $v_p(a_6)=0$. The first subterm of this term is when $v_p(a_5)\neq N$ and the second subterm is when $v_p(a_5)=N$. The second term is when $v_p(a_5)=0$ and $v_p(a_6)\neq 0$. The first subterm of this term is when $v_p(a_6)\neq N$ and the second subterm is when $v_p(a_6)=N$. The third term is when $v_p(a_4)=0$, $v_p(a_5)=k$, and $v_p(a_6)=\ell$ where $1\leq k,\ell,\leq N-1$. The first subterm of this term is when $v_p(a_5)+v_p(a_6)<N$ and the second subterm is when $v_p(a_5)+v_p(a_6)\geq N$. The fourth term is when $v_p(a_4)=0, v_p(a_5)=N$, and $v_p(a_6)\neq 0$. Finally, the fifth term is when $v_p(a_4)=0, v_p(a_5)=k$, and $v_p(a_6)=N$, where $1\leq k\leq N-1$.

Inputting Equation 5.3.66 into Maple we obtain the strikingly simple

$$
\zeta^{irr}_{G_{5,3},p}(s)=\frac{(1-t)(1-pt^2)}{(1-pt)(1-p^2t^2)}. \tag{5.3.67}
$$

$\mathbf{G_{6,12}}$

For $(a_5, a_6) \in \mathcal{Z}^*_{2,p^N}$,

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & a_5 & 0 & a_6 \\ 0 & 0 & 0 & a_6 & 0 \\ -a_5 & 0 & 0 & 0 & 0 \\ 0 & -a_6 & 0 & 0 & 0 \\ -a_6 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.3.68}$$

and

$$\mathcal{S} = \begin{pmatrix} a_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.3.69}$$

The 4-minors of $\mathcal{R}$ are $\mathcal{M}_4 := \{0, a_6^4, a_5^2 a_6^2, -a_5 a_6^3\}$. Thus, when $v_p(a_6) = 0$ we have that $m_3 = m_4 = 0$. When $v_p(a_5) = 0$ then $m_3 = m_4 = v_p(a_6)$. It is clear that, for SNF($\mathcal{S}$), $n_1 = v_p(a_6)$. It follows that

$$Q_{G_{6,12}} = (1-p^{-1})(pt^2)^N + (1-p^{-1})^2(pt^2)^N \frac{t^{-1} - (t^{-1})^N}{1-(t^{-1})} + (1-p^{-1})(pt)^N. \tag{5.3.70}$$

The first term is when $v_p(a_6) = 0$. The second term is when $v_p(a_5) = 0$ and $v_p(a_6) = k$ where $1 \leq k \leq N-1$. The third term is when $v_p(a_5) = 0$ and $v_p(a_6) = N$.

Inputting Equation 5.3.70 into Maple we obtain

$$\zeta^{irr}_{G_{6,12},p}(s) = \frac{(1-t)(1-t^2)}{(1-pt)(1-pt^2)}. \tag{5.3.71}$$

$\mathbf{G_{6,14}}$

We impose the additional condition on p that γ is a unit mod p. Since, when we calculate SNF($\mathcal{R}$) and SNF($\mathcal{S}$), multiplication by units does not change the valuation of a_6 and all 4-minors are monomial, we can omit γ. Thus, for $(a_4, a_5, a_6) \in \mathcal{Z}^*_{3,p^N}$,

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & a_4 & a_6 & 0 \\ 0 & 0 & a_5 & 0 & a_6 \\ -a_4 & -a_5 & 0 & 0 & 0 \\ -a_6 & 0 & 0 & 0 & 0 \\ 0 & -a_6 & 0 & 0 & 0 \end{pmatrix} \tag{5.3.72}$$

and

$$\mathcal{S} = \begin{pmatrix} a_6 & 0 \\ 0 & a_6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.3.73}$$

The 4-minors of $\mathcal{R}$ are $\mathcal{M}_4 := \{0, a_5^2a_6^2, a_6^4, a_4^2a_6^2, a_6^3a_4, -a_6^3a_5, -a_4a_5a_6^2\}$. Calculating SNF($\mathcal{R}$), if $v_p(a_4) = 0$ then $m_3 = m_4 = v_p(a_6)$. If $v_p(a_5) = 0$ then $m_3 = m_4 = v_p(a_6)$ as well. Finally, if $v_p(a_6) = 0$ then $m_3 = m_4 = 0$. It is clear, for SNF($\mathcal{S}$), that

$n_1 = n_2 = v_p(a_6)$. It follows that

$$Q_{G_{6,14}} = (1-p^{-1})(pt^2)^N + (1-p^{-2})(p^2)^N \left((1-p^{-1})(p^{-1}t^2)^N \frac{pt^{-1} - (pt^{-1})^N}{1-pt^{-1}}\right) + (1-p^{-2})(p^2t)^N. \tag{5.3.74}$$

The first term is when $v_p(a_6) = 0$. The second term is when one of $v_p(a_4), v_p(a_5) = 0$ and $v_p(a_6) = k$ where $1 \leq k \leq N-1$. The final term is when one of $v_p(a_4), v_p(a_5) = 0$ and $v_p(a_6) = N$.

Inputting Equation 5.3.74 into Maple we obtain

$$\zeta^{irr}_{G_{6,14},p}(s) = \frac{(1-t)(1-t^2)}{(1-p^2t)(1-pt^2)}. \tag{5.3.75}$$

$\mathbf{G_{27A}}$

For $(a_6, a_7) \in \mathcal{Z}^*_{2,p^N}$,

$$\mathcal{R} = \begin{pmatrix} 0 & a_6 & 0 & a_7 & 0 \\ -a_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_7 \\ -a_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_7 & 0 & 0 \end{pmatrix}. \tag{5.3.76}$$

Since G_{27A} has nilpotency class 2 there is no submatrix $\mathcal{S}$.

The 4-minors of $\mathcal{R}$ are $\mathcal{M}_4 := \{0, a_7^4, a_6^2a_7^2, -a_6a_7^3\}$. Calculating SNF($\mathcal{R}$), if $v_p(a_7) = 0$ then $m_3 = m_4 = 0$. If $v_p(a_6) = 0$ then $m_3 = m_4 = v_p(a_7)$. It follows that

$$Q_{G_{27A}} = (1-p^{-1})(p^2t^2)^N + (1-p^{-1})^2(p^2t^2)^N \frac{p^{-1}t^{-1} - (p^{-1}t^{-1})^N}{1-p^{-1}t^{-1}} + (1-p^{-1})(pt)^N. \tag{5.3.77}$$

The first term is when $v_p(a_7) = 0$. The second term is when $v_p(a_6) = 0$ and $v_p(a_7) = k$

where $1 \leq k \leq N-1$. The final term is when $v_p(a_6) = 0$ and $v_p(a_7) = N$.

Inputting this into Maple we obtain

$$\zeta^{irr}_{G_{27A},p}(s) = \frac{(1-t)(1-pt^2)}{(1-pt)(1-p^2t^2)}. \tag{5.3.78}$$

Note that, for all examples that have been calculated, the p-local representation zeta functions satisfy the functional equation of Theorem 1.4.3.

5.4 Joint Results With Robert Snocken

We briefly state some work on bounding the p-local abscissa of convergence of representation zeta functions of $\mathcal{T}$-groups that will appear in a forthcoming paper by Robert Snocken and the author of this thesis. As a reminder, the concept of saturation was defined in Definition 5.2.2.

Theorem 5.4.1. *Let L be a saturated Lie ring of nilpotency class 2 associated to some e. e. $\mathcal{T}$-group G such that L' and $Z(L)$ are also saturated. Let $h(L) = d$, $h(L') = d'$, and $h(Z) = e$. Let $L_* \subseteq L$ be the saturated abelian Lie subring of L with maximal dimension, say $h(L_*) = r$. Let $\alpha_{L,p}$ be the p-local abscissa of convergence of $\zeta^{irr}_{L,p}(s)$. Then, for almost all p,*

$$\frac{2d'}{d-e} \leq \alpha_{L,p} \leq d'. \tag{5.4.1}$$

Moreover, if $r - e \geq d - r$ then, for almost all p,

$$\alpha_{L,p} \geq \frac{d'}{d-r}. \tag{5.4.2}$$

These bounds represent another step in the process of obtaining group theoretic information from p-local representation zeta functions.

Chapter 6

Conclusion

6.1 Introduction

In this thesis we have calculated the representation zeta function of two families of $\mathcal{T}$-groups by a constructive method. This method allowed the calculation of certain p-local representation zeta functions that could not be calculated by the Kirillov orbit method. Also, we calculated the p-local representation zeta functions of various $\mathcal{T}$-groups, most of these groups appearing in [9, Chapter 2], using the Kirillov orbit method. To conclude, the final chapter of this thesis consists of two sections: first, we make some observations about the properties of representation zeta functions of $\mathcal{T}$-groups. Second, we mention some questions to be answered by future research.

6.2 Observations

Here are some observations we have made due to the research completed in this thesis.

Observation 6.2.1. There exists two non-isomorphic $\mathcal{T}$-groups, say G_1 and G_2, such that $\zeta^{irr}_{G_1}(s) = \zeta^{irr}_{G_2}(s)$.

Let H be the discrete Heisenberg group and let $G_1 = H \times H$. Since irreducible representations of direct products of finite groups are tensor products of irreducible representations of these groups,

$$\zeta^{irr}_{G_1}(s) = \left(\frac{\zeta(s-1)}{\zeta(s)}\right)^2. \tag{6.2.1}$$

by [23]. Let $G_2 = M_3$ as in Chapter 4. By Equation 4.3.19,

$$\zeta^{irr}_{G_2}(s) = \left(\frac{\zeta(s-1)}{\zeta(s)}\right)^2. \tag{6.2.2}$$

It is clear that $G_1 \ncong G_2$; for example, these two groups are of different nilpotency class.

Observation 6.2.2. We remind the reader that we call a prime p a Kirillov-exceptional prime if the hypotheses of the Kirillov orbit method does not hold for p. There exist $\mathcal{T}$-groups, say for example a group G, such that $\zeta^{irr}_{G,p}(s)$ is finitely uniform in p, p^{-s} for all primes, including Kirillov-exceptional primes; that is, as in Definition 2.3.2, $p_* = 2$.

By [13, Theorem 1.a] the prime $p = 2$ is a Kirillov-exceptional prime. By Equation 4.3.19 in Chapter 4 we know that

$$\zeta^{irr}_{M_3}(s) = \left(\frac{\zeta(s-1)}{\zeta(s)}\right)^2 \tag{6.2.3}$$

and thus the 2-local representation zeta function is of the same form as the other p-local representation zeta functions. The results calculated by the constructive method suggest that, since these exceptional p-local zeta functions are not "truly" exceptional, there may be a generalization of the Kirillov orbit method that is valid for all primes, at least for $\mathcal{T}$-groups of a relatively simple structure. In fact, Stasinski and Voll [28, Section 2.4] do this for $\mathcal{T}$-groups of nilpotency class 2.

6.3 Questions and Future Work

To date, there has not been a p-local representation zeta function calculated such that it is truly exceptional, in the sense that the p-local representation zeta function does not satisfy the usual functional equations. All p-local representation zeta functions that have been calculated do indeed satisfy the functional equations of Theorems 1.4.3 and 1.4.4. Indeed, a natural question arises:

Question 6.3.1. Does there exist a group G and a prime p such that the p-local representation zeta function of G is truly exceptional; that is, it does not satisfy the functional equation in Theorem 1.4.4?

The calculation of the p-local representation zeta functions of M_3 and M_4 suggest that it is possible that there are no truly exceptional primes; however, it may be nothing more than a fluke that the p-local representation zeta functions, for each of the groups, are uniform in p, p^{-s}. To make a conjecture either way, we suspect more examples of p-local representation zeta functions, for Kirillov-exceptional p, need to be calculated. This would most likely involve use of the constructive method.

Since representation zeta functions are built from groups it is very likely that we are able to extract group-theoretic data from representation zeta functions. In fact, from Theorem 1.4.3, we can determine the Hirsch length of the derived subgroup.

Question 6.3.2. Can we determine other group-theoretic information from representation zeta functions of $\mathcal{T}$-groups?

We suspects that we can determine much more from the form of the zeta function. In fact, the bounds on abscissas of convergence that appear in Chapter 5 are a start to discovering this hidden information. Work in representation growth of other classes of groups has given some results. For example, in the case of complex semi-simple algebraic groups, say we pick one and call it G, it was shown by Larsen and Lubotzky [17] that the abscissa of convergence of the representation zeta function of G, say α_G, has the following property:

$$\alpha_G = \frac{r}{k} \tag{6.3.1}$$

where r is the Lie rank of G and k is the number of positive roots of G.

The Kirillov orbit method is very useful for calculating the p-local representation zeta functions for almost all primes. However, for some exceptional prime p_*, if the number of p_*^N-dimensional twist isoclasses grows much faster than the number of non-exceptional p^N-dimensional twist isoclasses, we may not be able to determine the global abscissa of convergence. We ask the following question:

Question 6.3.3. For a $\mathcal{T}$-group G and a Kirillov-exceptional prime p_*, can the abscissa of convergence α_{G,p_*} be larger than p-local abscissas of non-exceptional primes; that is, larger than $\max\{\alpha_{G,p} \mid p \text{ is non-exceptional}\}$?

If it is possible, then the constructive method gains more importance to determining the overall rate of representation growth.

Of the $\mathcal{T}$-groups studied, both in this thesis and in the literature, all of the representation zeta functions are products of shifted Dedekind zeta functions and their inverses. This phenomenon does not happen subgroup growth. Therefore, it may be of interest to study groups for which their representation zeta function is as described above.

Question 6.3.4. Are all representation zeta functions products of shifted Dedekind zeta functions and their inverses? If not, which $\mathcal{T}$-groups have representation zeta functions that are of this form?

We suspect that the answer to the first question is "no". This agrees with the expectation of Stasinski and Voll in [28, Section 1.1]. The groups for which representation zeta functions have been calculated, so far, have been of a relatively simple structure. It is very possible that this behaviour does not occur for sufficiently complex groups. Assuming the answer to the first question is negative, an answer to the second question might give an interesting family of $\mathcal{T}$-groups to study. This may reflect some group-theoretic property (see Question 6.3.2).

In Chapter 4 we calculated various p-local the representation zeta function of the groups M_n. During the course of that calculation, especially the calculation of $\zeta^{irr}_{M_4,2}(s)$, we split the domain of possible choices of roots of unity into different cases. However, this sectioning was, essentially, ad-hoc; that is, there was no other reason to split the domain in this way other than ease of calculation. There may be a more natural way to break up the domain so that the calculation is more straightforward. However, we were not able to describe a better, more intuitive, sectioning of the domain.

Project 6.3.5. Find more natural case distinctions for the constructive-exceptional prime calculation of $\zeta^{irr}_{M_4,2}(s)$.

If this is possible, it may help in generalizing the constructive method to a larger class of groups. Indeed, since the "un-natural" sectioning occurs only during the calculation of exceptional cases, finding a better sectioning of the domain might lead us to a better understanding of $\mathcal{T}$-groups, and in particular, a better understanding of their finite quotients of "small prime"-power order.

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Appendix A

Elementary Concepts

We begin by defining an important concept in the study of nilpotent groups.

Definition A.6. Let G be a group. For $g, h \in G$ we call $[g, h] := ghg^{-1}h^{-1}$ the *commutator* of g and h. Let H, K be subsets of G. The commutator $[H, K]$ is defined as $\langle \{[h, k] \mid h \in H, k \in K\} \rangle$.

Definition A.7. Let G be a group. Then G is *nilpotent* if the lower central series $G_0 := G,\ G_i := [G, G_{i-1}]$ terminates in a finite number of steps; that is, there is an $c \in \mathbb{N} \cup \{0\}$ such that $G_c = 1$. The minimal c such that $G_c = 1$ is called the *nilpotency class* of G.

Example A.8. *Any abelian group A is nilpotent; since all elements of A commute, $G_1 = [A, A] = 1$ and thus A has nilpotency class* 1.

Example A.9. *The quaternion group $Q_8 := \langle -1, i, j, k \mid -1^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1 \rangle$ of order 8 is a nilpotent group of class* 2. *We have that $G_1 = \{1, -1\}$ and $G_2 = [Q_8, \{1, -1\}] = 1$.*

There are many equivalent definitions of nilpotent groups; see any graduate textbook on group theory for details. We give one equivalent definition that is important for our study.

Theorem A.10. *A finite group G is nilpotent if and only if it is the direct product of all of its Sylow subgroups.*

Now that we have introduced the idea of nilpotent groups, we can continue to define the groups studied in this thesis. We call these $\mathcal{T}$-groups.

Definition A.11. Let $G := \langle P \mid R \rangle$ be a group defined by a set of generators P and a set of relations R. We say that G is *finitely generated* if $|P| < \infty$.

Definition A.12. Let G be a group. We say that G is *torsion-free* if and only if the only element $g \in G$ such that $g^k = 1$ for any $k \in \mathbb{N}$ is 1.

Definition A.13. We say a group G is a *$\mathcal{T}$-group* if and only if G is a finitely generated torsion-free nilpotent group.

Example A.14. *The discrete Heisenberg group $H := \langle x, y, z \mid [x, y] = z \rangle$ is a $\mathcal{T}$-group.*

From now on, until indicated otherwise, any arbitrary group G that appears in this thesis is a $\mathcal{T}$-group.

We introduce an invariant of $\mathcal{T}$-groups that, informally, measures the amount of infiniteness in our group.

Definition A.15. The *Hirsch length* of a $\mathcal{T}$-group G is the number of infinite factors in a polycyclic series of G. We denote this by $h(G)$.

Example A.16. *Let $M_3 = \langle y, x_1, x_2, x_3 \mid [y, x_1] = x_2,\ [y, x_2] = x_3 \rangle$ where all other commutators in the presentation are trivial. Then M_3 has the lower central series*

$$G_0 := M_3,\ G_1 = \langle x_2, x_3 \rangle,\ G_2 = \langle x_3 \rangle,\ G_3 = 1. \tag{A.1}$$

Thus

$$G_2/G_3 = \mathbb{Z},\ G_1/G_2 = \mathbb{Z},\ G_0/G_1 = \mathbb{Z}^2 \tag{A.2}$$

and $h(M_3) = 1 + 1 + 2 = 4$.

We study nilpotent groups by way of linear algebra. For a group G the idea is to find a set of matrices (or, more generally, linear operators) $\rho(G)$ which, under multiplication, behaves the same way as G. Once we have this set, we can use linear algebraic techniques to study our group. This is the basic idea behind the general subject of representation theory.

Definition A.17. Let V be a n-dimensional vector space and G a group. The function $\rho : G \to GL(V)$, where $GL(V)$ is the general linear group of V, is a *representation* if and only if it is a homomorphism; that is, for $g, h \in G$, $\rho(gh) = \rho(g)\rho(h)$. We say that ρ has dimension n.

In this thesis, the vector space V is usually $\mathbb{C}^n$ and we denote this general linear group as $GL_n(\mathbb{C})$. In the same manner, we always mean "complex representation" when we say "representation".

Example A.18. *Let K be the Klein-4 group generated by elements a, b. Then $\rho : K \to GL_2(\mathbb{C})$ defined by*

$$\rho(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.3}$$

is a representation.

Definition A.19. Let $\rho : G \to GL_n(V)$ be a representation. We say a subspace $W \subseteq V$ is *a $\rho(G)$-stable subspace* if W is closed under the action of G; that is, $\rho(g)w \subseteq W$ for all $g \in G$ and $w \in W$.

Definition A.20. A representation $\rho : G \to GL(V)$ is *irreducible* if and only if the only $\rho(G)$-stable subspaces of V are the trivial subspace and V itself.

Note that, since a representation ρ is a homomorphism of groups, ρ is completely determined by $\rho(a)$ for all $a \in P$ where P is a generating set of the group.

Example A.21. *Let ρ be the representation in Example A.18. Since all elements of $\rho(K)$ are diagonal, the subspace spanned by the vector $\mathbf{v} := (1, 0)^T$ is ρ-stable. Indeed span$(\mathbf{v})$ is a stable subspace for any representation $\chi : A \to GL_2(\mathbb{C})$ of an abelian group A.*

The four representations $\rho_* : K \to \mathbb{C}$ such that $\rho_*(a) = \pm 1, \rho_*(b) = \pm 1$ are irreducible. In fact, it is clear that any 1-dimensional representation is irreducible.

Two results we use in this thesis are corollaries of the standard representation theoretic result Schur's Lemma. We combine the needed results into one lemma:

Lemma A.22. *Let $\rho : G \to GL_n(\mathbb{C})$ be an irreducible representation. If, for $g \in G$, $\rho(g)$ commutes with $\rho(G)$ then $\rho(g)$ must be a scalar matrix. Additionally, if $\rho(G)$ is abelian then ρ is 1-dimensional.*