Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group

by

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Contents

Chapter 1

Introduction

1.1 Introduction

The mathematical objects we investigate in this thesis are algebraic groups defined over an algebraically closed field k (for the precise definition of algebraic groups, see [Hum91]). The classical linear groups, such as the general linear group $GL_n(k)$, the special linear group $SL_n(k)$, the orthogonal group $O_n(k)$, and the symplectic group $Sp_{2n}(k)$, are examples of algebraic groups. Among many algebraic groups, a particular class of algebraic groups called reductive algebraic groups (the simplest example of them is $SL_n(k)$) is known to have very nice structure, and is much studied, see [FdV69],[Hum91],[Spr98]. In particular, reductive algebraic groups are all classified into infinite families and exceptional groups according to their corresponding root systems, and for each root system there is the associated Dynkin diagram and Cartan matrix (see [Hum72]), which encode much information about reductive algebraic groups in a combinatorial way. In this thesis, we use these combinatorial information extensively to analyze one particular property of subgroups of reductive algebraic groups called *complete reducibility*.

Let G be a reductive algebraic group over an algebraically closed field k. A subgroup H of G is said to be *G-completely reducible* (G-cr for short) if whenever H is contained in a parabolic subgroup P of G, H is contained in some Levi subgroup L of P. This is a generalization of the notion of semisimplicity in representation theory since if $G = GL_n(k)$, H is G-cr if and only if H acts completely reducibly on k^n; see [Ser98, Lecture 1].

In Lie groups and Lie algebras, we can extract lots of important information about Lie groups from the corresponding Lie algebras, see [FdV69]. A similar thing happens in algebraic groups. In our case, the notion of *G-separability* is particularly important. A subgroup H of G is said to be *G-separable* if the global centralizer of H in G agrees with the infinitesimal centralizer of H in the Lie algebra of G (see Definition 2.1.2), and it is known that several results concerning G-complete reducibility have the condition that certain subgroups are G-separable (see [BMRT10, sec. 1]) as a hypothesis. In particular, it is known that if H and M are reductive subgroups of a reductive group G such that $H < M < G$ and H is G-separable and (G, M) is a *reductive pair* (see Definition 2.1.7), then G-complete reducibility of H implies M-complete reducibility of H, (Theorem 2.1.8).

In [BMRT10, Sec. 7], Bate, Martin, Röhrle, and Tange found a pair of reductive subgroups H and M of a reductive group G such that $H < M < G$ and H is G-cr but not M-cr. The purpose of this thesis is to find a new example of subgroups with the same property. From the argument in the last paragraph, one way of finding such an example is to find a G-nonseparable and G-cr reductive subgroup H of a reductive group G first, and try to find a reductive subgroup M such that H is not M-cr. This is the method used in [BMRT10, Sec. 7]. More specifically, Bate, Martin, Röhrle, and Tange find a G-nonseparable subgroup H where H is a subgroup of a rank 1 Levi subgroup L of a group G of type G_2. The authors show that H is G-nonseparable by finding a *nilpotent witness to the G-nonseparability* of H (see Definition 2.1.3). This is the path we take in this thesis.

It is known that in characteristic zero a subgroup H of G is G-cr if and only if H is reductive (see [Ser98, Property 4] and [BMR05, Lem. 2.6]). Therefore we cannot find such examples unless k is of positive characteristic. Actually, we have stricter constraints on the characteristic of k by Theorem 2.1.6, which says that the characteristic p of k can not be *very good* (see Definition 2.1.4, 2.1.5). We eventually restrict to the cases where the characteristic of k is 2.

Our thesis splits into several chapters, and we give the outline now. In Chapter 2, we give a few definitions and relevant theorems as the preliminaries to our arguments in the following chapters. We expect some familiarity with algebraic varieties and algebraic groups, and no definition is given in our thesis. There, we define the important notion of *nilpotent witness to the G-nonseparability*, (see Definition 2.1.3). If there is such an element in the Lie algebra of G, the subgroup H is not G-separable. After that, in Chapter 3, we negatively extend the result of [BMRT10, Sec. 7]. We show that if we take the same specific form of H sitting in a rank 1 Levi subgroup as in [BMRT10, Sec. 7], there is no nilpotent witness to the G-nonseparability of H in the Lie algebra of the unipotent radical of a specific parabolic subgroup P of G containing H for any simple algebraic group G defined over an algebraically closed field of characteristics k of 2. This result suggests that we look at higher rank Levi subgroups in order to find an example we are after.

In the rest of the thesis we investigate a particular form of subgroups H sitting in higher rank Levi subgroups L in various types of groups G where H is a subgroup of the normalizer of a maximal torus T of G generated by elements corresponding to reflections in the Weyl group of G. In particular, in Section 4.1.1, we consider a group of type A_3 with A_2 Levi subgroup, and in Section 4.1.2, 4.1.3, we consider a group of type A_4 with A_2 and A_3 Levi subgroups. Then in Section 4.1.4, 4.1.5, we consider a group of type B_3 with A_2 and B_2 Levi subgroups. In these classical cases, we find that there is no nilpotent witness to the G-nonseparability in the Lie algebra of the unipotent radical of a specific parabolic subgroup P containing H. After that, in Section 4.2.1, 4.2.2, and 4.2.3, we investigate exceptional groups of type E_6, E_7, and E_8 with A_5, A_6, and A_7 Levi subgroup respectively. In these cases, we get the same result as in the classical cases. In E_8 case, we have used a computer algebra software, Maple, and we put the Maple codes and outputs in Appendix.

Then, in Section 5.2, we show a generic way to find a G-nonseparable subgroup H and a nilpotent witness to the G-nonseparability of H. To illustrate our method, we show how to get such an example in a group G of type E_7. We also show that the subgroup H is G-cr. Then in Section 5.3, we find a pair of subgroups $\tilde{H}$ and M such that $\tilde{H}$ is G-cr but not M-cr by the same method as in [BMRT10] and by using the subgroup H we found.

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Chapter 2

Preliminaries

2.1 Notation

Throughout this thesis, we work over an algebraically closed field k of positive characteristic p. We denote the multiplicative group of k by k^*. A function $\phi : G \to H$ between algebraic groups is a homomorphism if and only if ϕ is a homomorphism of abstract groups and a morphism of varieties.

By a subgroup of an algebraic group, we always mean a closed subgroup. Let H be a subgroup of an algebraic group G. We denote the derived group of H by $[H, H]$. We usually use a capital roman letter, G, H, K, etc., to represent an algebraic group, and the corresponding lowercase gothic letter, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$, etc., to represent its Lie algebra. We sometimes use another notation for Lie algebras: $\operatorname{Lie} G$, $\operatorname{Lie} H$, and $\operatorname{Lie} K$ are the Lie algebras of G, H, and K respectively. We denote the adjoint representation of G by Ad. Let H be a subgroup of G. We denote the global centralizer of H in G by $C_G(H)$, and the infinitesimal centralizer of H in $\mathfrak{g}$ by $\mathfrak{c}_{\mathfrak{g}}(H)$, that is,

Definition 2.1.1. $C_G(H) = \{g \in G \mid hgh^{-1} = g \text{ for all } h \in H\}$ and
$\mathfrak{c}_{\mathfrak{g}}(H) = \{x \in \mathfrak{g} \mid \operatorname{Ad} h(x) = x \text{ for all } h \in H\}$.

Now we give the important definitions of *G-separability*, *G-nonseparability*, and *nilpotent witness of G-nonseparability.*

Definition 2.1.2. A subgroup H of G is *G-separable* if $\operatorname{Lie} C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$ and *G-nonseparable* if $\operatorname{Lie} C_G(H) \subsetneq \mathfrak{c}_{\mathfrak{g}}(H)$.

Note that we always have $\operatorname{Lie} C_G(H) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$. We use the notion of *nilpotent witness to the G-nonseparability* repeatedly in the following sections.

Definition 2.1.3. Let H be a subgroup of G. An element $x \in \operatorname{Lie} G$ is called a *nilpotent witness to the G-nonseparability* of H if x is nilpotent and $x \in \mathfrak{c}_{\mathfrak{g}}(H)$, but $x \notin \operatorname{Lie} C_G(H)$.

We denote the maximal connected normal unipotent subgroup of G by $R_u(G)$, and call this the *unipotent radical* of G. An algebraic group G is *reductive* if $R_u(G) = \{1\}$. In particular, G is called *simple* as an algebraic group if G is connected and all proper normal subgroups of G are finite. Note that if G is simple, then G is reductive.

Throughout the thesis, G always denotes a reductive algebraic group with Lie algebra $\mathfrak{g}$. Fix a maximal torus T of G. Let $\Psi(G,T)$ denote the set of roots of G with respect to T. We sometimes write just $\Psi(G)$ instead of $\Psi(G,T)$ if no confusion arises. Let $\zeta \in \Psi(G)$, then U_ζ denotes the corresponding root subgroup of G, and $\mathfrak{u}_\zeta$ denotes the root subspace $\operatorname{Lie} U_\zeta$ of $\mathfrak{g}$. We write $\mathfrak{t}$ for the Lie algebra of T. Recall that $\mathfrak{g}$ has the root space decomposition as follows. [Hum91, Sec. 26.2]

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\zeta \in \Psi(G)} \mathfrak{u}_\zeta. \tag{2.1}$$

Also, recall that if $t \in T$, $\zeta \in \Psi(G)$, and $e_\zeta \in \mathfrak{u}_\zeta$ then

$$t \cdot e_\zeta = \operatorname{Ad} t(e_\zeta) = \zeta(t) e_\zeta.$$

A subgroup H of G is called a *regular* subgroup if H is normalized by some maximal torus T of G. Then, this yields an action of T on H by conjugation, and an action of T on $\operatorname{Lie}(H)$ by Ad. We call the subset $\Psi(H,T)$ of $\Psi(G,T)$ a *subsystem* of $\Psi(G,T)$ if $\mathfrak{u}_\zeta \subseteq \operatorname{Lie}(H)$ for each $\zeta \in \Psi(H,T)$, and we call an element in $\Psi(H,T)$ a root of H with respect to T. In this case, the Lie algebra of H has a similar decomposition to 2.1. In particular, if P is a parabolic subgroup of G, and $R_u(P)$ is the unipotent radical of P, we have

$$\operatorname{Lie}(R_u(P)) = \bigoplus_{\zeta \in \Psi(R_u(P),T)} \mathfrak{u}_\zeta.$$

We denote the semi-simple rank 1 subgroup $\langle U_\zeta, U_{-\zeta} \rangle$ of G by G_ζ, and the Lie algebra of G_ζ by $\mathfrak{g}_\zeta$ where $\zeta \in \Psi(G)$. Fix a Borel subgroup B of G containing T, and let $\Sigma(G,T)$ be the set of simple roots of $\Psi(G,T)$ defined by B. Then $\Psi(B,T) = \Psi^+(G)$ is the set of positive roots of G. We know that any $\zeta \in \Psi^+(G)$ can be written uniquely as a linear combination of the simple roots with nonnegative integer coefficients. Now we are ready to define *good* primes and *bad* primes for G.

Definition 2.1.4. For each $\zeta \in \Psi^+(G)$, let $\zeta = \sum_{\xi \in \Sigma} c_{\xi\zeta} \xi$ for some $c_{\xi\zeta} \in \mathbb{N}$. A prime p is said to be *good* for G if it does not divide nonzero $c_{\xi\zeta}$ for any $\zeta \in \Psi^+(G)$ and for any $\xi \in \Sigma(G)$, and *bad* otherwise.

In case of simple groups, it is known that 2 is a bad prime for all groups except type A_n, 3 for the exceptional groups and 5 for the groups of type E_8 (see [FdV69, Appendix. p528-p531]). Also, we use the notion of *very good* primes.

Definition 2.1.5. A prime p is said to be *very good* for G if p is good for G and also p does not divide $n+1$ for any simple component of G of type A_n.

The notion of very good primes is related to G-separability by the following theorem, [BMRT10, Thm. 1.2].

Theorem 2.1.6. *Let G be connected reductive and suppose that the characteristic of k is very good for G. Then any subgroup of G is G-separable.*

In our calculations in Chapter 3 onward, we always assume that G is simple, in particular, connected and reductive. Therefore by the theorem above, we must work in characteristic p where p is not very good in order to find a G-nonseparable subgroup H (we actually work in p=2 in most cases in the following sections). Now, we need to introduce the notion of *reductive pair* to state Theorem 2.1.8 which gives a relationship between G-separability and G-complete reducibility, [BMRT10, Thm. 1.4].

Definition 2.1.7. Let M be a reductive subgroup of a reductive group of G. We say that (G, M) is a *reductive pair* if $\operatorname{Lie} M$ is an M-module direct summand of $\mathfrak{g}$.

Theorem 2.1.8. *Suppose that (G, M) is a reductive pair. Let H be a subgroup of M such that H is a G-separable subgroup of G. If H is G-completely reducible, then it is also M-completely reducible.*

This theorem suggests that there are two strategies to find an example we are after.

1. Find a G-nonseparable and G-cr subgroup H first, and then try to find a reductive subgroup M such that $H < M < G$ and H is not M-cr.

2. Find a reductive subgroup M such that (G, M) is not a reductive pair, and then try to find a subgroup H of M such that H is G-cr but not M-cr.

In this thesis, we use the first method following [BMRT10, sec. 7].

We denote the set of cocharacters of G by $Y(G)$. The elements of $Y(G)$ are the homomorphisms of algebraic groups $k^* \to G$. Let $\zeta \in \Psi(G)$. We denote the corresponding coroot by $\zeta^\vee \in Y(G)$. Then $\zeta^\vee$ is a homomorphism of algebraic groups $k^* \to G_\zeta$. Now let $\zeta, \xi \in \Psi(G)$. Then $\xi^\vee \in Y(G)$. If we compose ζ with $\xi^\vee$, we get a homomorphism $\zeta \circ \xi^\vee : k^* \to k^*$ such that $\zeta \circ \xi^\vee(a) = a^n$ for some $n \in \mathbb{Z}$. We define $\langle \zeta, \xi^\vee \rangle = n$.

We denote by s_ξ the reflection corresponding to ξ in the Weyl group of G. Each s_ξ acts on the set of roots $\Psi(G)$ by the following formula [Spr98, Lem. 7.1.8].

$$s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee \rangle \xi. \tag{2.2}$$

For each root $\zeta \in \Psi(G)$, we define the admissible homomorphism $\epsilon_\zeta : k \to U_\zeta$ satisfying the following relationship [Car72, Prop. 6.4.2, Lem. 7.2.1]. For any $\xi \in \Psi(G)$, we have

$$n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(\pm a), \tag{2.3}$$
$$\text{where } n_\xi = \epsilon_\xi(1)\epsilon_{-\xi}(-1)\epsilon_\xi(1).$$

Now we set $\epsilon'_\zeta(0) = e_\zeta$. Then we get

$$n_\xi \cdot e_\zeta = \pm e_{s_\xi \cdot \zeta}. \tag{2.4}$$

Note that by [Car72, Lem. 7.2.1] we also have

$$n_\xi U_\zeta n_\xi^{-1} = U_{s_\xi \cdot \zeta}. \tag{2.5}$$

We use (2.2),(2.3),(2.4), and (2.5) extensively in our calculations in the following sections.

2.2 Basic properties of reductive algebraic groups and geometric invariant theory

In this subsection, we give several results about reductive algebraic groups which are useful to cut down our calculations in the following chapters. Also we give a few results concerning geometric invariant theory from [BMR05] and [BMRTar] which we use in the last chapter. Let G be a reductive algebraic group acting on an affine variety X.

Definition 2.2.1. Let $\phi : k^* \to X$ be a morphism of algebraic varieties. We say that $\lim_{t\to 0} \phi(t)$ exists if there exists a morphism $\hat{\phi} : k \to X$ (necessarily unique) whose restriction to k^* is ϕ. If this limit exists, we set $\lim_{t\to 0} \phi(t) = \hat{\phi}(0)$.

We need a characterization of a parabolic subgroup P of G, a Levi subgroup L of P, and a unipotent radical $R_u(P)$ of P in terms of a cocharacter of G, [Ric88, Sec. 2.1-2.3].

Let λ be a cocharacter of G. Then we can associate to λ a closed subgroup P_λ of G by

$$P_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}\text{exists}\}.$$

Then, P_λ is a parabolic subgroup of G, and any parabolic subgroup of G is in this form [Spr98, Prop. 8.4.5]. Also, if we define L_λ and $R_u(P_\lambda)$ by

$$L_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = g\},$$

and

$$R_u(P_\lambda) = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\},$$

then, L_λ is a Levi subgroup of P_λ, and $R_u(P_\lambda)$ is the unipotent radical of P_λ. Any Levi subgroup and the unipotent radical of any parabolic subgroup can be expressed in this form [Spr98, Prop. 8.4.5]. Note that $L_\lambda = C_G(\lambda(k^*))$, and P_λ admits a Levi decomposition $P_\lambda = R_u(P_\lambda) \rtimes L_\lambda$ [BMRT10, 2.2].

Let M be a reductive subgroup of G. Then there is a natural inclusion $Y(M) \subseteq Y(G)$ of cocharacter groups. It is obvious from our characterization of parabolic subgroups and unipotent radicals that if $\lambda \in Y(M)$, then $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$.

Definition 2.2.2. Define the map $c_\lambda : P_\lambda \to L_\lambda$ by

$$c_\lambda(g) := \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}.$$

Then this map is a surjective homomorphism of algebraic groups, [BMR05, Lem. 2.4]. We are ready to state the theorem, [Spr98, Thm. 13.4.2] , [Hum91, Sec. 28.5].

Theorem 2.2.3. *Let λ be a cocharacter of G, and let P_λ, L_λ, $R_u(P_\lambda)$ as above. Then*

(a) The product morphism $L_\lambda \times R_u(P_\lambda) \to P_\lambda$ is an isomorphism of varieties.

(b) The product morphism $R_u(P_{-\lambda}) \times P_\lambda \to G$ is an isomorphism onto an open subset of G.

From this, we get a corollary [Spr98, 13.4.5].

Corollary 2.2.4. *Let λ be a cocharacter of G. Let P_λ, L_λ, $R_u(P_\lambda)$ as above. Let H be a subgroup of L_λ. Then*

(a) The product morphism $\phi : C_{R_u(P_{-\lambda})}(H) \times C_{L_\lambda}(H) \times C_{R_u(P_\lambda)}(H) \to C_G(H)^\circ$ is an isomorphism onto an open subset of $C_G(H)^\circ$.

(b) $\dim\Big(C_G(H)\Big) = \dim\Big(C_{R_u(P_{-\lambda})}(H)\Big) + \dim\Big(C_{L_\lambda}(H)\Big) + \dim\Big(C_{R_u(P_\lambda)}(H)\Big).$

(c) The derivative of the product morphism ϕ in (a) at the identity of G, which we write $d\phi(1)$, is a Lie algebra isomorphism from $\operatorname{Lie}(C_{R_u(P_{-\lambda})}(H)) \oplus \operatorname{Lie}(C_{L_\lambda}(H)) \oplus \operatorname{Lie}(C_{R_u(P_\lambda)}(H))$ *to* $\operatorname{Lie} C_G(H)^\circ$.

(d) $$\dim\Big(\operatorname{Lie}(C_G(H))\Big) = \dim\Big(\operatorname{Lie}(C_{R_u(P_{-\lambda})}(H))\Big) + \dim\Big(\operatorname{Lie}(C_{L_\lambda}(H))\Big) + \dim\Big(\operatorname{Lie}(C_{R_u(P_\lambda)}(H))\Big).$$

Proof. Part (a) follows from Theorem 2.2.3, then part (b),(c), and (d) follow. □

We always have $\operatorname{Lie}(C_G(H)) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$ for any group G and any subgroup H of G. Therefore by part (d) of Corollary 2.2.4, in order to prove that a subgroup H of G is G-nonseparable, it is enough to show that

$$\dim(C_{R_u(P_\lambda)}(H)) < \dim(\mathfrak{c}_{\operatorname{Lie}(R_u(P_\lambda))}(H)).$$

We use Theorems 2.2.5, 2.2.6, (see [BMR05, Lem. 2.12], [BMR05, Cor. 3.22]) concerning G-complete reducibility in Chapter 5. Let $f : G_1 \to G_2$ be a homomorphism of algebraic groups. We say that f is non-degenerate if $(\ker f)^\circ$ is a torus, [BMR05, Sec. 2.4]. In particular, f is non-degenerate if f is an isogeny.

Theorem 2.2.5. *Let G_1 and G_2 be reductive groups. Let $f : G_1 \to G_2$ be an epimorphism. Let H_1 and H_2 be closed subgroups of G_1 and G_2, respectively.*

(a) If H_1 is G_1-cr, then $f(H_1)$ is G_2-cr.

(b) If f is non-degenerate, then H_1 is G_1-cr if and only if $f(H_1)$ is G_2-cr, and H_2 is G_2-cr if and only if $f^{-1}(H_2)$ is G_1-cr.

Theorem 2.2.6. *Let K be a closed subgroup of a Levi subgroup L of G. Then K is L-cr if and only if K is G-cr.*

Now we state two theorems concerning geometric invariant theory. These theorems are used in Chapter 5 to find an example we are after, [BMR05, Lem. 2.17 and Thm. 3.1], [BMRTar, Thm. 3.3].

Theorem 2.2.7. *Let H be a closed subgroup of G. Then H is G-cr if and only if for every cocharacter λ of G with $H \subseteq P_\lambda$, there exists $g \in G$ such that $c_\lambda(h) = ghg^{-1}$ for every $h \in H$.*

Theorem 2.2.8. *Suppose X is an affine G-variety and let $v \in X$. Let $\lambda \in Y(G)$ such that $v' := \lim_{t\to 0} \lambda(t) \cdot v$ exists and is G-conjugate to v. Then v' is $R_u(P_\lambda)$-conjugate to v.*

2.3 Useful results

The following results are useful when we calculate the infinitesimal and the global centralizer of a subgroup H of G in the following chapters. First, we have the following, [Spr98, Prop. 8.2.1].

Lemma 2.3.1. *Let G be a reductive algebraic group. Let P be a parabolic subgroup of G, and let $R_u(P)$ be the unipotent radical of P. Then any element u in $R_u(P)$ can be expressed uniquely as*

$$u = \prod_{\lambda \in \Psi(R_u(P))} \epsilon_\lambda(a_\lambda)$$

where the product is taken with respect to a fixed ordering of $\Psi(R_u(P))$.

We use the following lemmas repeatedly in our calculations.

Lemma 2.3.2. *Let G be a reductive algebraic group defined over an algebraically closed field k of characteristic 2. Fix a maximal torus T of G. Pick a parabolic subgroup P of G such that T is a maximal torus of P. Let H be a subgroup contained in the intersection of P and the normalizer of T generated by elements corresponding to reflections in the Weyl group of G. Pick $\zeta \in \Psi(R_u(P))$, and let H_ζ be the orbit of ζ in $\Psi(R_u(P))$. Then any element $x \in \mathfrak{c}_{\mathrm{Lie}(R_u(P))}(H)$ is of the following form.*

$$x = a\left(\sum_{\lambda \in H_\zeta} e_\lambda\right) + \sum_{\lambda \in \Psi(R_u(P)) \setminus H_\zeta} a_\lambda e_\lambda, \text{ where } a, a_\lambda \in k.$$

Proof. We know that any element $x \in \mathrm{Lie}(R_u(P))$ can be written as

$$x = \sum_{\lambda \in \Psi(R_u(P))} a_\lambda e_\lambda, \text{ where } a_\lambda \in k.$$

Since H acts transitively on H_ζ, for any $\lambda_1, \lambda_2 \in H_\zeta$ we can find $n \in H$ such that $\lambda_2 = n \cdot \lambda_1$. Then we have

$$n \cdot x = n \cdot \left(\sum_{\lambda \in \Psi(R_u(P))} a_\lambda e_\lambda\right) = n \cdot (a_{\lambda_1} e_{\lambda_1}) + n \cdot \left(\sum_{\lambda \in \Psi(R_u(P)) \setminus \{\lambda_1\}} a_\lambda e_\lambda\right)$$
$$= a_{\lambda_1} e_{n \cdot \lambda_1} + n \cdot \left(\sum_{\lambda \in \Psi(R_u(P)) \setminus \{\lambda_1\}} a_\lambda e_\lambda\right) = a_{\lambda_1} e_{\lambda_2} + n \cdot \left(\sum_{\lambda \in \Psi(R_u(P)) \setminus \{\lambda_1\}} a_\lambda e_\lambda\right).$$

If $x \in \mathfrak{c}_{\mathrm{Lie}(R_u(P))}(H)$, we must have $n \cdot x = x$. Comparing the coefficients of e_{λ_2}, we get $a_{\lambda_1} = a_{\lambda_2}$. Set $a = a_{\lambda_1}$. $\square$

The next corollary makes it easier to calculate $\mathfrak{c}_{\mathrm{Lie}(R_u(P))}(H)$

Corollary 2.3.3. *Let G be a reductive algebraic group defined over an algebraically closed field of characteristic 2. Fix a maximal torus T of G. Pick a parabolic subgroup P of G such that T is a maximal torus of P. Let H be a subgroup contained in the intersection of P and the normalizer of T generated by elements corresponding to reflections in the Weyl group of G. Let $\{H_i \mid i = 1 \ldots m\}$ be the set of orbits of the action of H on $\Psi(R_u(P))$. Then,*

$$\mathfrak{c}_{Lie(R_u(P))}(H) = \left\{ \sum_{i=1}^{n} a_i \sum_{\lambda \in H_i} e_\lambda \mid a_i \in k \right\}.$$

Proof. Pick $\zeta \in \Psi(R_u(P))$, and let H_ζ be the orbit of ζ. Choose any $n \in H$. Then,

$$\begin{aligned} n \cdot \sum_{\lambda \in H_\zeta} e_\lambda &= \sum_{\lambda \in H_\zeta} e_{n \cdot \lambda} \\ &= \sum_{\lambda \in H_\zeta} e_\lambda. \end{aligned}$$

Therefore, we have

$$\sum_{\lambda \in H_\zeta} e_\lambda \in \mathfrak{c}_{\mathrm{Lie}(R_u(P))}(H) \text{ for any } H_\zeta \in \{H_i \mid i = 1 \ldots n\}.$$

Combining this with the last lemma, and using induction on the cardinality of the set of the orbits, we get the result we want. □

The next lemma is useful when we calculate $\mathrm{Lie}(C_{R_u(P)}(H))$.

Lemma 2.3.4. *Let G be a reductive algebraic group defined over an algebraically closed field of characteristic 2. Fix a maximal torus T of G. Pick a parabolic subgroup P of G such that T is a maximal torus of P. Let H be a subgroup contained in the intersection of P and the normalizer of T generated by elements corresponding to reflections in the Weyl group of G. Pick $\zeta \in \Psi(R_u(P))$, and let H_ζ be the orbit of ζ in $\Psi(R_u(P))$. If U_{λ_1} and U_{λ_2} are commuting root subgroups of $R_u(P)$ for each $\lambda_1, \lambda_2 \in H_\zeta$, then we have*

$$\sum_{\lambda \in H_\zeta} e_\lambda \in \mathrm{Lie}(C_{R_u(P)}(H)).$$

Proof. Since U_{λ_1} and U_{λ_2} are commuting subgroups for each $\lambda_1, \lambda_2 \in H_\zeta$, the product $\prod_{\lambda \in H_\zeta} U_\lambda$ is a subgroup of G. Take the 1-dimensional subgroup $\{\prod_{\lambda \in H_\zeta} \epsilon_\lambda(a) \mid a \in k\}$ of $\prod_{\lambda \in H_\zeta} U_\lambda$. Pick any $n \in H$. We have

$$n \cdot \left(\prod_{\lambda \in H_\zeta} \epsilon_\lambda(a) \right) = \prod_{\lambda \in H_\zeta} \epsilon_{n \cdot \lambda}(a) = \prod_{\lambda \in H_\zeta} \epsilon_\lambda(a).$$

Thus we have

$$\left\{ \prod_{\lambda \in H_\zeta} \epsilon_\lambda(a) \mid a \in k \right\} \subseteq C_{R_u(P)}(H).$$

Differentiating with respect to a, and evaluating at $a = 0$, we get

$$\sum_{\lambda \in H_\zeta} e_\lambda \in \operatorname{Lie}(C_{R_u(P)}(H)).$$

□

We use the next propositions repeatedly in our calculations, [Hum91, Lem. 32.5, Prop. 33.4, Prop. 33.3].

Proposition 2.3.5. *Let G be a reductive algebraic group. Let $\alpha, \beta \in \Psi(G)$. If no positive integer linear combination of α and β is a root of G, then*

$$\epsilon_\alpha(a)\epsilon_\beta(a) = \epsilon_\beta(a)\epsilon_\alpha(a), \text{ where } a \in k.$$

Proposition 2.3.6. *Let Ψ be the root system of type B_2 with positive roots α, β, $\alpha + \beta$, and $2\alpha + \beta$ where α is short and β is long. Then the homomorphisms ϵ_α, $\epsilon_{\alpha+\beta}$, and $\epsilon_{2\alpha+\beta}$ can be chosen so that*

$$\epsilon_{\alpha+\beta}(b)\epsilon_\alpha(a) = \epsilon_\alpha(a)\epsilon_{\alpha+\beta}(b)\epsilon_{2\alpha+\beta}(2ab), \text{ where } a, b \in k.$$

Proposition 2.3.7. *Let Ψ be the root system of type A_2 with positive roots α, β, $\alpha + \beta$. Then the homomorphisms ϵ_α, ϵ_β, and $\epsilon_{\alpha+\beta}$ can be chosen so that*

$$\epsilon_\beta(b)\epsilon_\alpha(a) = \epsilon_\alpha(a)\epsilon_\beta(b)\epsilon_{\alpha+\beta}(ab), \text{ where } a, b \in k.$$

Chapter 3

The rank 1 Levi subgroup case

3.1 The negative result

In this chapter, we negatively extend the result of [BMRT10, Sec. 7]. First, we recall the result in [BMRT10, Sec. 7]. Let G be a simple reductive algebraic group of type G_2 defined over an algebraically closed field k of characteristic 2. Fix a maximal torus T of G and a Borel subgroup B of G containing T. Then the set of positive roots $\Psi^+(G)$ is $\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$ where α is a short root and β is a long root (see [Hum91, Sec. 33.5]). For each $\zeta \in \Psi(G)$, we choose an admissible isomorphism $\epsilon_\zeta : k \to U_\zeta$ and $n_\zeta = \epsilon_\zeta(1)\epsilon_{-\zeta}(-1)\epsilon_\zeta(1)$ satisfying 2.3, 2.4, and 2.5. Then n_ζ has order 2 since k is of characteristic 2, and n_ζ represents the reflection corresponding to ζ in the Weyl group $N_G(T)/T$ of G. Let $Y(T)$ be the set of coroots of G with respect to T. Pick $t \in \alpha^\vee(k^*)$ of order 3. Set

$$H = \langle n_\alpha, t\rangle.$$

Note that $S_3 \cong H \subseteq L_\alpha$ where L_α is the Levi subgroup of semi-simple rank 1 with respect to α. In [BMRT10, Sec. 7], the authors found a nilpotent witness to the G-nonseparability of H, and showed that the subgroup H is G-nonseparable [BMRT10, Prop. 7.11]. We extend this result by proving the following.

Theorem 3.1.1. *Let G be a simple algebraic group of any type except type G_2 over an algebraically closed field k of characteristic 2. Pick any root ζ of G, and choose any t of order 3 or greater in $\zeta^\vee(k^*)$. Let $H = \langle t, n_\zeta\rangle$. Then there is no nilpotent witness to the G-nonseparability of H in $\mathrm{Lie}\,(R_u(P_\zeta))$ where P_ζ is a parabolic subgroup with respect to ζ. Also, if G is a simple algebraic group of type G_2 over k of characteristic 2, and if $H = \langle t, n_\beta\rangle$ where β is a long root of G and t is an element of order 3 or greater in $\beta^\vee(k^*)$, there is no nilpotent witness to the G-nonseparability of H in $\mathrm{Lie}\,(R_u(P_\beta))$.*

Before we prove this theorem, we prove the following little lemma about the Cartan matrices and the pairings $\langle \zeta, \xi^\vee \rangle$ where $\zeta \in \Psi$, and $\xi^\vee \in Y(T)$.

Lemma 3.1.2. *Let G be a connected reductive algebraic group of any type except type G_2. Pick any root $\zeta \in \Psi$ and any coroot $\xi^\vee \in Y(T)$. Then the absolute value of $\langle \zeta, \xi^\vee \rangle$ is always less than 3. Also, the statement holds for a connected reductive algebraic group G of type G_2 if ξ is a long root of G.*

Proof. Fix the set of simple roots Σ in such a way that ξ is a simple root. We can assume that $\zeta \in \Psi^+(G)$ since $\mid \langle \zeta, \xi^\vee \rangle \mid = \mid \langle -\zeta, \xi^\vee \rangle \mid$. Then we can write ζ as a linear combination of the simple roots of G with non-negative integer coefficients. Let $\zeta = \sum_{i \in J} \lambda_i \sigma_i$ for some nonempty finite index set J where λ_i is a non-negative integer not all zero, and $\sigma_i \in \Sigma$. For each $\sigma_i \in \Sigma$, let $n(\sigma_i)$ be the node in the Dynkin diagram of G corresponding to the simple root $\sigma_i \in \Sigma$. Now, pick any simple root σ_i. We know that $\langle \sigma_i, \xi^\vee \rangle = 0$ if $n(\sigma_i)$ is not adjacent to $n(\xi)$ in the Dynkin diagram of G. We have

$$\langle \zeta, \xi^\vee \rangle = \langle \sum_{i \in J} \lambda_i \sigma_i, \xi^\vee \rangle = \sum_{i \in J} \lambda_i \langle \sigma_i, \xi^\vee \rangle$$

From a complete list of all the connected Dynkin diagrams (see [FdV69, Appendix. Table. B]), we see that $n(\sigma)$ is connected to at most 3 nodes, therefore it suffices to consider at most 3 adjacent nodes in all the Dynkin diagrams in order to get all the possible values of $\langle \zeta, \xi^\vee \rangle$. We refer [FdV69, Appendix, p528-p531] for all possible coefficients λ_i of simple roots σ_i in $\zeta \in \Psi^+(G)$. We exhaust all cases to prove the lemma. We look at one root system at a time. We consider the simply-laced root systems (i.e. ones where the Dynkin diagram has no double or triple bonds) first. These are the root systems of type E_8, E_7, E_6, D_n, and A_n. From the Cartan matrices of these root systems, we have

$$\langle \sigma_i, \xi^\vee \rangle = \begin{cases} 2 & \text{if } \sigma_i = \xi, \\ -1 & \text{if } n(\sigma_i) \text{ is adjacent to } n(\xi). \end{cases}$$

Denote by λ the coefficient of ξ in ζ. Then we need to check that for every simple root and every adjacent node to that simple root, the quantity

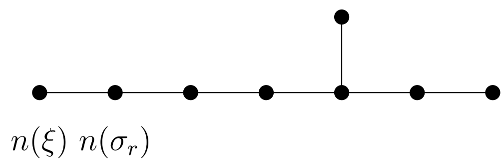
$$2\lambda - \sum_{(\sigma_i \text{ is adjacent to } \xi)} \lambda_i$$

has absolute value at most 2 in order to prove the lemma.

1. We consider the root system of type E_8. This case covers the root system of type E_7 and E_6 since E_7 and E_6 are subsystems of E_8. We divide this case into 8 subcases depending on which simple root the simple root ξ is. Let

λ_l, (λ_r, λ_a, λ_b) be the coefficient of the simple root σ_l, (σ_r, σ_a, σ_b) adjacent to ξ on the left, (right, above, below) in a root ζ in the Dynkin diagram of E_8.

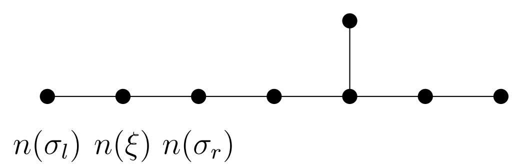
(a) $n(\xi)$ is the leftmost node in the Dynkin diagram of E_8. We list all



possible values of $(\lambda, \lambda_r, 2\lambda - \lambda_r)$ as follows.

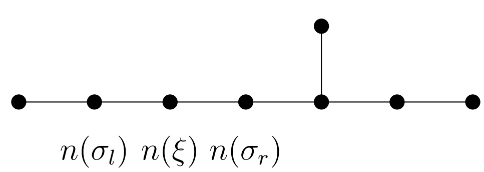
$$(0,0,0), (0,1,-1), (1,0,2), (1,1,1), (1,2,0), (1,3,-1), (2,3,1).$$

(b) $n(\xi)$ is the second node from the left in the Dynkin diagram of E_8. We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - \lambda_r)$ as follows.



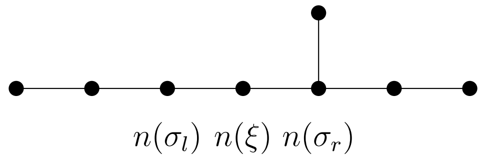
$$(0,0,0,0), (0,0,1,-1), (0,1,0,2), (0,0,1,-1), (1,1,0,1), (0,1,1,1),$$
$$(1,1,1,0), (0,1,2,0), (1,1,2,-1), (1,2,2,1), (1,2,3,0), (1,2,4,-1),$$
$$(1,3,4,1), (2,3,4,0).$$

(c) $n(\xi)$ is the third node from the left in the Dynkin diagram of E_8. We



list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - \lambda_r)$ as follows.

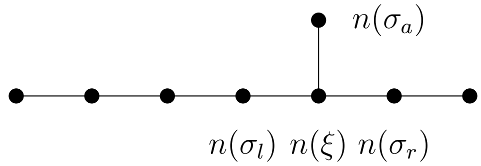
$$(0,0,0,0), (1,0,0,-1), (0,1,0,2), (0,0,1,-1), (1,1,0,1), (0,1,1,1),$$
$$(1,1,1,0), (0,1,2,0), (1,1,2,-1), (1,2,2,1), (2,2,2,0), (1,2,3,0),$$
$$(2,2,3,-1), (2,3,3,1), (2,3,4,0), (2,3,5,-1), (2,4,5,1), (3,4,5,0).$$



(d) $n(\xi)$ is the forth node from the left in the Dynkin diagram of E_8. We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - \lambda_r)$ as follows.

$$(0,0,0,0), (1,0,0,-1), (0,1,0,2), (0,0,1,-1), (1,1,0,1), (0,1,1,1),$$
$$(1,1,1,0), (0,1,2,0), (1,1,2,-1), (1,2,2,1), (2,2,2,0), (1,2,3,0),$$
$$(2,2,3,-1), (2,3,3,1), (2,3,4,0), (3,3,3,0), (3,3,4,-1), (3,4,4,1),$$
$$(3,4,5,0), (3,4,6,-1), (3,5,6,1), (4,5,6,0).$$

(e) $n(\xi)$ is at the junction in the Dynkin diagram of E_8. Note that in this case $n(\xi)$ has three adjacent notes, namely, $\sigma_l, \sigma_r, \sigma_a$. We list all

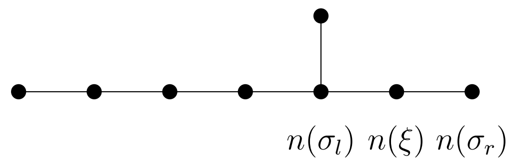


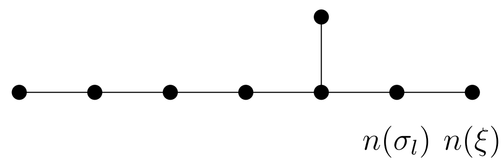
possible values of $(\lambda_l, \lambda, \lambda_r, \lambda_a, 2\lambda - \lambda_l - \lambda_r - \lambda_a)$ as follows.

$$(0,0,0,0,0), (1,0,0,0,-1), (0,1,0,0,2), (0,0,0,1,-1),$$
$$(0,0,1,0,-1), (1,1,0,0,1), (0,1,0,1,1), (0,1,1,0,1),$$
$$(1,1,0,1,0), (1,1,1,0,0), (0,1,1,1,0), (1,1,1,1,-1),$$
$$(1,2,1,1,1), (2,2,1,1,0), (1,2,2,1,0), (2,2,2,1,-1),$$
$$(2,3,2,1,1), (2,3,2,2,0), (3,3,2,1,0), (3,3,2,2,-1),$$
$$(3,4,2,2,1), (3,4,3,2,0), (4,4,2,2,0), (4,4,3,2,-1),$$
$$(4,5,3,2,1), (4,5,3,3,0), (4,5,4,2,0), (4,5,4,3,-1),$$
$$(4,6,4,3,1), (5,6,4,3,0).$$

(f) $n(\xi)$ is the second node from the right in the Dynkin diagram of E_8. We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - \lambda_r)$ as follows.

$$(0,0,0,0), (1,0,0,-1), (0,1,0,2), (0,0,1,-1), (1,1,0,1), (0,1,1,1),$$
$$(1,1,1,0), (2,1,0,0), (2,1,1,-1), (2,2,1,1), (3,2,1,0), (4,2,1,-1),$$
$$(4,3,1,1), (4,3,2,0), (5,3,1,0), (5,3,2,-1), (5,4,2,1), (6,4,2,0).$$

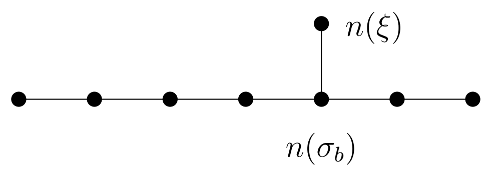




(g) $n(\xi)$ is the rightmost node in the Dynkin diagram of E_8. We list all possible values of $(\lambda_l, \lambda, 2\lambda - \lambda_l)$ as follows.

$$(0,0,0), (1,0,-1), (0,1,2), (1,1,1), (2,1,0), (3,1,-1),$$
$$(3,2,1), (4,2,0).$$

(h) $n(\xi)$ is the simple root on the short branch in the Dynkin diagram of E_8. We list all possible values of $(\lambda, \lambda_b, 2\lambda - \lambda_b)$ as follows.

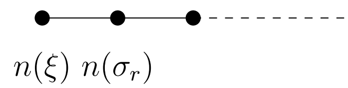


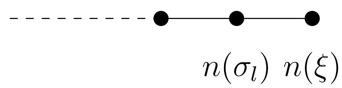
$$(0,0,0), (0,1,-1), (1,0,2), (1,1,1), (1,2,0), (1,3,-1),$$
$$(2,3,1), (2,4,0), (2,5,-1), (3,5,1), (3,6,0).$$

2. We consider the root system of type A_n where $n \geq 3$. This case covers the A_1 and A_2 root systems since A_1 and A_2 are subsystems of A_n where $n \geq 3$. We divide this case into 3 subcases depending on which simple root the simple root ξ is. Let λ_l (λ_r) be the coefficient of the simple root σ_l adjacent to the left (right) of ξ, in a root ζ in the Dynkin diagram of A_n.

   (a) $n(\xi)$ is the leftmost node in the Dynkin diagram of A_n. We list all possible values of $(\lambda, \lambda_r, 2\lambda - \lambda_r)$ as follows.

$$(0,0,0), (1,0,2), (0,1,-1), (1,1,1).$$

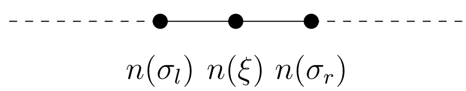




(b) $n(\xi)$ is the the rightmost node. We list all possible values of $(\lambda_l, \lambda, 2\lambda-\lambda_l)$ as follows.

$$(0,0,0), (1,0,-1), (0,1,2), (1,1,1).$$

(c) $n(\xi)$ any other node. We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda-\lambda_l-\lambda_r)$

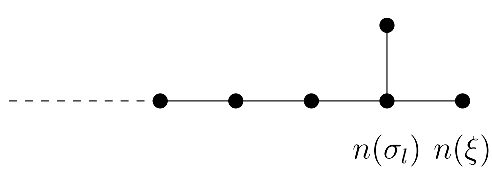


as follows.

$$(0,0,0,0), (1,0,0,-1), (0,0,1,-1), (1,1,0,1), (0,1,1,1), (1,1,1,0).$$

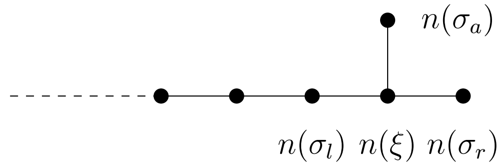
3. We consider the root system of type D_n where $n \geq 5$. This case covers the root system of type D_n of lower rank as a subsystem. We divide this case into 3 subcases depending on which simple root the simple root ξ is. Let λ_l (λ_r, λ_a, λ_b) be the coefficient of the simple root σ_l adjacent to the left (right, above, below) of ξ, in a root ζ in the Dynkin diagram of D_n.

(a) ξ is the root on one of the short branches (There are two possibilities of the choice to ξ, but each case gives the same value of $2\lambda - \sum \lambda_i$ because of the symmetry of the Dynkin diagram of type D_n). We list



all possible values of $(\lambda_l, \lambda, 2\lambda - \lambda_l)$ as follows.

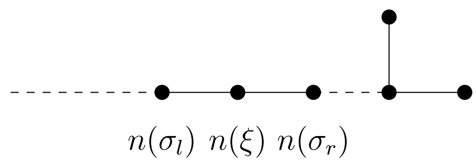
$$(0,0,0), (1,0,-1), (0,1,2), (1,1,1), (2,1,0).$$



(b) $n(\xi)$ is at the junction. We list all possible values of $(\lambda_l, \lambda, \lambda_r, \lambda_a, 2\lambda - \lambda_l - \lambda_r - \lambda_a)$ as follows.

$(0,0,0,0,0), (1,0,0,0,-1), (0,1,0,0,2), (0,0,1,0,-1), (0,0,0,1,-1),$
$(1,1,0,0,1), (0,1,1,0,1), (0,1,0,1,1), (1,1,1,0,0), (1,1,0,1,0),$
$(1,1,1,1,-1), (1,2,1,1,1), (2,2,1,1,0).$

(c) $n(\xi)$ is any other node. We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda -$



$\lambda_l - \lambda_r)$ as follows.

$(0,0,0,0), (1,0,0,-1), (0,1,0,2), (0,0,1,-1), (1,1,0,1), (0,1,1,1),$
$(1,1,1,0), (0,1,2,0), (1,1,2,-1), (1,2,2,1), (2,2,2,0).$

Now we consider the cases where there are roots of different lengths, namely, the root systems of type B_n, C_n, F_4, and G_2. We consider the root systems of type B_n, C_n, and F_4 first. Suppose that $n(\sigma_i)$ is adjacent to $n(\xi)$. From the Cartan matrices of B_n, C_n, and F_4, we have

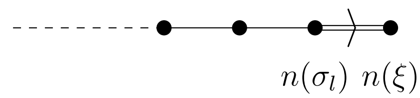
$$\langle \sigma_i, \xi^\vee \rangle = \begin{cases} 2 & \text{if } \sigma_i = \xi, \\ -1 & \text{if } \sigma_i \text{ has the same length as } \xi, \text{ or } \sigma_i \text{ is short and } \xi \text{ is long,} \\ -2 & \text{if } \sigma_i \text{ is long and } \xi \text{ is short.} \end{cases}$$

4. We consider the root system of type B_n where $n \geq 4$. This case covers the root system of type B_n of lower rank as a subsystem. We divide this case into 2 subcases depending on which simple root the simple root ξ is. Let λ_l (λ_r) be the coefficient of the simple root σ_l adjacent to the left (right) of ξ in a root ζ in the Dynkin diagram of B_n.

(a) ξ is the short root. In this case we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - 2\lambda_l.$$

We list all possible values of $(\lambda_l, \lambda, 2\lambda - 2\lambda_l)$ as follows.

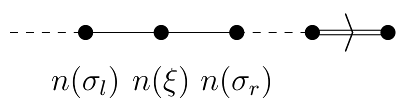


$$(0,0,0), (1,0,-2), (0,1,2), (1,1,0), (1,2,2), (2,2,0).$$

(b) ξ is a long simple root. Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_l - \lambda_r.$$

We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - \lambda_r)$ as follows.



$$(0,0,0,0), (1,0,0,-1), (0,1,0,2), (0,0,1,-1), (1,1,0,1), (0,1,1,1),$$
$$(1,1,1,0), (0,1,2,0), (1,2,2,1), (2,2,2,0).$$

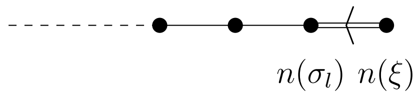
5. We consider the root system of type C_n where $n \geq 4$. This case covers the root system of type C_n of lower rank as a subsystem. We divide this case into 3 subcases depending on which simple root the simple root ξ is. Let λ_l (λ_r) be the coefficient of the simple root σ_l adjacent to the left (right) of ξ in a root ζ in the Dynkin diagram of C_n.

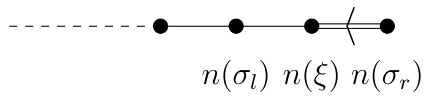
(a) $n(\xi)$ is the long simple root. Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_l.$$

We list all possible values of $(\lambda_l, \lambda, 2\lambda - \lambda_l)$ as follows.

$$(0,0,0), (1,0,-1), (0,1,2), (1,1,1), (2,1,0).$$





(b) ξ is the short simple root that is next to the long simple root. Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_l - 2\lambda_r.$$

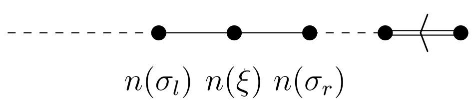
We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - 2\lambda_r)$ as follows.

$(0,0,0,0), (1,0,0,-1), (0,1,0,2), (0,0,1,-2), (1,1,0,1), (0,1,1,0),$
$(1,1,1,-1), (0,2,1,2), (1,2,1,1), (2,2,1,0).$

(c) $n(\xi)$ is one of the other short simple roots. Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_l - \lambda_r.$$

We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - \lambda_r)$ as follows.



$(0,0,0,0), (1,0,0,-1), (0,1,0,2), (0,0,1,-1), (1,1,0,1), (0,1,1,1)$
$(1,1,1,0), (0,0,2,-2), (0,1,2,0), (1,1,2,-1), (0,2,2,2), (1,2,2,1)$
$(2,2,2,0).$

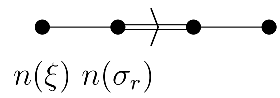
6. We consider the root system of type F_4. We divide this case into 4 subcases depending on which simple root the simple root ξ is. Let λ_l (λ_r) be the coefficient of the simple root σ_l adjacent to the left (right) of ξ in a root ζ in the Dynkin diagram of F_4.

   (a) $n(\xi)$ is the leftmost node. Then we have

   $$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_r.$$

   We list all possible values of $(\lambda, \lambda_r, 2\lambda - \lambda_r)$ as follows.

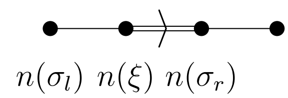
   $(0,0,0), (1,0,2), (0,1,-1), (1,1,1), (1,2,0), (1,3,-1), (2,3,1).$



(b) $n(\xi)$ is the second node from the left. Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_l - \lambda_r.$$

We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - \lambda_l - \lambda_r)$ as follows.

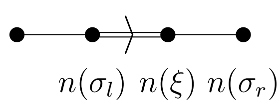


$(0,0,0,0), (0,0,1,-1), (0,1,0,2), (1,0,0,-1), (1,1,0,1), (0,1,1,1),$
$(1,1,1,0), (0,1,2,0), (1,1,2,-1), (1,2,2,1), (1,2,3,0), (1,2,4,-1),$
$(1,3,4,1), (2,3,4,0).$

(c) $n(\xi)$ is the second node from the right. Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - 2\lambda_l - \lambda_r.$$

We list all possible values of $(\lambda_l, \lambda, \lambda_r, 2\lambda - 2\lambda_l - \lambda_r)$ as follows.



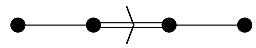
$(0,0,0,0), (0,0,1,-1), (0,1,0,2), (1,0,0,-2), (0,1,1,1), (1,1,0,0),$
$(1,1,1,-1)(1,2,0,2), (1,2,1,1), (1,2,2,0), (2,2,0,0), (2,2,1,-1),$
$(2,2,2,-2), (2,3,1,1), (2,3,2,0), (2,4,2,2), (3,4,2,0).$

(d) $n(\xi)$ is the rightmost node. Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_l.$$

We list all possible values of $(\lambda_l, \lambda, 2\lambda - \lambda_l)$ as follows.

$(0,0,0), (0,1,2), (1,0,-1), (1,1,1), (2,0,-2), (2,1,0), (2,2,2)$
$(3,1,-1), (3,2,1), (4,2,0).$



$n(\sigma_l)$ $n(\xi)$

7. We consider the root system of type G_2. We consider the case where the root ξ is long. Let λ_r be the coefficient of the simple root σ_l adjacent to the right of ξ in a root ζ in the Dynkin diagram of G_2. From the Cartan matrices of G_2, we have

$$\langle \sigma_r, \xi^\vee \rangle = \begin{cases} 2 & \text{if } \sigma_r = \xi, \\ -1 & \text{if } \sigma_r \text{ is short.} \end{cases}$$

Then we have

$$\langle \zeta, \xi^\vee \rangle = 2\lambda - \lambda_r.$$

We list all possible values of $(\lambda, \lambda_r, 2\lambda - \lambda_r)$ as follows.

$n(\xi)$ $n(\sigma_r)$

$$(1, 0, 2), (1, 1, 1), (1, 2, 0), (1, 3, -1), (2, 3, 1).$$

We have checked all possible values for $\langle \zeta, \xi^\vee \rangle$, and did not get any number with absolute value equal to 3 or greater. So the lemma is proved. □

Now, we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let G and H as in the hypotheses. Let $t = \zeta^\vee(a)$ where $a \in k^*$. Note that a has the same order as t. Suppose that x is a nilpotent witness to the G-nonseparability of H in $\mathrm{Lie}\,(R_u(P_\zeta))$. Then t centralizes x. We can write $x = \sum_{i \in I} \lambda_i e_i$ for some subset I of Ψ, $\lambda_i \in k^*$. We have

$$\begin{aligned} t \cdot x &= t \cdot (\sum_{i \in I} \lambda_i e_i) \\ &= \sum_{i \in I} \lambda_i (t \cdot e_i) \\ &= \sum_{i \in I} \lambda_i (i \circ \zeta^\vee(a) e_i) \\ &= \sum_{i \in I} \lambda_i (a^{\langle i, \zeta^\vee \rangle} e_i) \end{aligned}$$

Since t centralizes x and the order of a is 3 or greater, $\langle i, \zeta^\vee \rangle$ has to be an integer multiple of 3 or greater for each $i \in I$ from the last equation. Then $\langle i, \zeta^\vee \rangle$ has to be zero for each $i \in I$ by Lemma 3.1.2. Hence t centralizes U_i for any $i \in I$. Also, n_ζ centralizes U_i for any $i \in I$ by 2.3 since k is of characteristic 2. Therefore, we have $U_i \subseteq C_G(H)$ for any $i \in I$, then it follows that $\sum_{i \in I} \lambda_i e_i \in \mathrm{Lie}\,(C_G(H))$, which is a contradiction. We conclude that there is no nilpotent witness to the G-nonseparability of H in $\mathrm{Lie}\,(R_u(P_\zeta))$. $\square$

Chapter 4

The higher rank Levi subgroup cases

In this chapter, we try to find a nilpotent witness to the G-nonseparablity of a subgroup H of G. We always assume that H is generated by a subgroup of the normalizer of a maximal torus T of G corresponding to reflections in the Weyl group of G. Following the result of the last chapter, we consider subgroups H sitting in Levi subgroups of rank 2 or higher. We also assume that G is a simple algebraic group defined over an algebraically closed field of characteristic 2. Then, by 2.3 and 2.4 we have

$$n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(a),$$
$$\operatorname{Ad} n_\xi(e_\zeta) = n_\xi \cdot e_\zeta = e_{s_\xi \cdot \zeta} \text{ where } \zeta, \xi \in \Psi(G).$$

4.1 Classical cases

4.1.1 $G = A_3$ with A_2 Levi subgroup

Let G be a simple algebraic group of type A_3. Fix a maximal torus T of G. Pick a Borel subgroup B of G containing T. Then the set of positive roots is

$$\Psi^+(G) = \{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma\}.$$

Let $L_{\alpha\beta}$ be the Levi subgroup of type A_2 corresponding to α and β. Then

$$\Psi^+(L_{\alpha\beta}) = \{\alpha, \beta, \alpha+\beta\}.$$

Set $P_{\alpha\beta} = \langle B \cup L_{\alpha\beta} \rangle$. Then $P_{\alpha\beta}$ is a parabolic subgroup of G with

$$\Psi(R_u(P_{\alpha\beta})) = \{\gamma, \beta+\gamma, \alpha+\beta+\gamma\}.$$

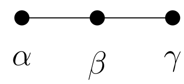


Figure 4.1: Dynkin diagram of A_3

We fix the ordering of $\Psi(R_u(P_{\alpha\beta}))$ as given above. We use round brackets for ordered n-tuples of roots. Define H by

$$H = \langle n_\alpha, n_\beta \rangle.$$

From the Cartan matrix of A_3, we have

$$\langle \alpha, \alpha^\vee \rangle = 2, \langle \beta, \alpha^\vee \rangle = -1, \langle \gamma, \alpha^\vee \rangle = 0,$$
$$\langle \alpha, \beta^\vee \rangle = -1, \langle \beta, \beta^\vee \rangle = 2, \langle \gamma, \beta^\vee \rangle = -1.$$

From this, we compute

$$\langle \beta+\gamma, \alpha^\vee \rangle = -1, \langle \alpha+\beta+\gamma, \alpha^\vee \rangle = 1,$$
$$\langle \beta+\gamma, \beta^\vee \rangle = 1, \langle \alpha+\beta+\gamma, \beta^\vee \rangle = 0.$$

These formulas tell us how n_α and n_β act on $\Psi(R_u(P_{\alpha\beta}))$.

$$n_\alpha \cdot (\gamma, \beta+\gamma, \alpha+\beta+\gamma) = (\gamma, \alpha+\beta+\gamma, \beta+\gamma),$$
$$n_\beta \cdot (\gamma, \beta+\gamma, \alpha+\beta+\gamma) = (\beta+\gamma, \gamma, \alpha+\beta+\gamma).$$

So H has the single orbit $\{\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$ in $\Psi(R_u(P_{\alpha\beta}))$. By Corollary 2.3.3, we get

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta}))}(H) = \{a(e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma}) \mid a \in k\}.$$

Now, let's check whether $e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma}$ belongs to $\mathrm{Lie}(C_{R_u(P_{\alpha\beta})}(H))$ or not. It is easy to see that no positive integral linear combinations of $\{\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$ is a root of G. By Proposition 2.3.5 and Lemma 2.3.4 we get

$$e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma} \in \mathrm{Lie}(C_{R_u(P_{\alpha\beta})}(H)).$$

So we have

$$\mathrm{Lie}(C_{R_u(P_{\alpha\beta})}(H)) \supseteq \mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta}))}(H).$$

Hence

$$\mathrm{Lie}(C_{R_u(P_{\alpha\beta})}(H)) = \mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta}))}(H).$$

Hence, we have the following.

Proposition 4.1.1. *There is no nilpotent witness to the G-nonseparability of H in* $\mathrm{Lie}\,(R_u(P_{\alpha\beta}))$.

4.1.2 $G = A_4$ with A_2 Levi subgroup

Let G be a simple algebraic group of type A_4. Fix a maximal torus T of G. Pick a Borel subgroup B of G containing T. Then the set of positive roots is

$$\Psi^+(G) = \{\alpha, \beta, \gamma, \delta, \alpha+\beta, \beta+\gamma, \gamma+\delta, \alpha+\beta+\gamma, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta\}.$$

Let $L_{\alpha\beta}$ be the Levi subgroup of type A_2 corresponding to α and β. Then

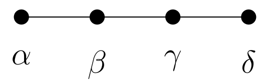


Figure 4.2: Dynkin diagram of A_4

$$\Psi^+(L_{\alpha\beta}) = \{\alpha, \beta, \alpha+\beta\}.$$

Set $P_{\alpha\beta} = \langle B \cup L_{\alpha\beta}\rangle$. Then $P_{\alpha\beta}$ is a parabolic subgroup of G with

$$\Psi(R_u(P_{\alpha\beta})) = \{\gamma, \delta, \beta+\gamma, \gamma+\delta, \alpha+\beta+\gamma, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta\}.$$

We fix the ordering of $\Psi(R_u(P_{\alpha\beta}))$ as given above. Define H by

$$H = \langle n_\alpha, n_\beta\rangle.$$

From the Cartan matrix of A_4, we get

$$\langle \alpha, \alpha^\vee\rangle = 2, \langle \beta, \alpha^\vee\rangle = -1, \langle \gamma, \alpha^\vee\rangle = 0, \langle \delta, \alpha^\vee\rangle = 0,$$
$$\langle \alpha, \beta^\vee\rangle = -1, \langle \beta, \beta^\vee\rangle = 2, \langle \gamma, \beta^\vee\rangle = -1, \langle \delta, \beta^\vee\rangle = 0.$$

From this, we compute

$$\langle \beta+\gamma, \alpha^\vee\rangle = -1, \langle \gamma+\delta, \alpha^\vee\rangle = 0, \langle \alpha+\beta+\gamma, \alpha^\vee\rangle = 1,$$
$$\langle \beta+\gamma+\delta, \alpha^\vee\rangle = -1, \langle \alpha+\beta+\gamma+\delta, \alpha^\vee\rangle = 1.$$
$$\langle \beta+\gamma, \beta^\vee\rangle = 1, \langle \gamma+\delta, \beta^\vee\rangle = -1, \langle \alpha+\beta+\gamma, \beta^\vee\rangle = 0,$$
$$\langle \beta+\gamma+\delta, \beta^\vee\rangle = 1, \langle \alpha+\beta+\gamma+\delta, \beta^\vee\rangle = 0.$$

These formulas show us how n_α and n_β act on $\Psi(R_u(P_{\alpha\beta}))$.

$$n_\alpha \cdot (\gamma, \delta, \beta+\gamma, \gamma+\delta, \alpha+\beta+\gamma, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta)$$
$$= (\gamma, \delta, \alpha+\beta+\gamma, \gamma+\delta, \beta+\gamma, \alpha+\beta+\gamma+\delta, \beta+\gamma+\delta).$$
$$n_\beta \cdot (\gamma, \delta, \beta+\gamma, \gamma+\delta, \alpha+\beta+\gamma, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta)$$
$$= (\beta+\gamma, \delta, \gamma, \beta+\gamma+\delta, \alpha+\beta+\gamma, \gamma+\delta, \alpha+\beta+\gamma+\delta).$$

It is easy to see that H has three orbits $\{\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$, $\{\gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta\}$, and $\{\delta\}$. By Corollary 2.3.3, we get

$$\begin{aligned}\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta}))}(H) =& \{a(e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma}) + b(e_{\gamma+\delta} + e_{\beta+\gamma+\delta} + e_{\alpha+\beta+\gamma+\delta}) + ce_\delta \\ & \mid a, b, c \in k\}.\end{aligned}$$

In this case, $\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta}))}(H)$ is 3-dimensional, and we want to show that at least one of $e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma}$, $e_{\gamma+\delta} + e_{\beta+\gamma+\delta} + e_{\alpha+\beta+\gamma+\delta}$ or e_δ does not belong to $\mathrm{Lie}(C_{R_u(P_{\alpha\beta})}(H))$ in order to prove that H is G-nonseparable. It is easy to see that no positive integral combinations of roots in the first two orbits of H is a root of G. By Proposition 2.3.5 and Lemma 2.3.4, we get

$$e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma},\ e_{\gamma+\delta} + e_{\beta+\gamma+\delta} + e_{\alpha+\beta+\gamma+\delta},\ e_\delta \in \mathrm{Lie}\ (C_{R_u(P_{\alpha\beta})}(H)).$$

So we have

$$\mathrm{Lie}\ (C_{R_u(P_{\alpha\beta})}(H)) = \mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta}))}(H).$$

Hence we have the following.

Proposition 4.1.2. *There is no nilpotent witness to the G-nonseparability of H in* $\mathrm{Lie}\,(R_u(P_{\alpha\beta}))$.

4.1.3 $G = A_4$ with A_3 Levi subgroup

Let G be a simple algebraic group of type A_4. Fix a maximal torus T of G. Pick a Borel subgroup B of G containing T. Then the set of positive roots is

$$\Psi^+(G) = \{\alpha, \beta, \gamma, \delta, \alpha+\beta, \beta+\gamma, \gamma+\delta, \alpha+\beta+\gamma, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta\}.$$

Let $L_{\alpha\beta\gamma}$ be a Levi subgroup of type A_3 corresponding to α, β, and γ. Then

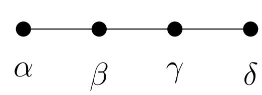


Figure 4.3: Dynkin diagram of A_4

$$\Psi^+(L_{\alpha\beta\gamma}) = \{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma\}.$$

Set $P_{\alpha\beta\gamma} = \langle B \cup L_{\alpha\beta\gamma} \rangle$. Then $P_{\alpha\beta\gamma}$ is a parabolic subgroup of G with

$$\Psi(R_u(P_{\alpha\beta\gamma})) = \{\delta, \gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta\}.$$

We fix the ordering of $\Psi(R_u(P_{\alpha\beta\gamma}))$ as given above. Define H by

$$H = \langle n_\alpha, n_\beta, n_\gamma \rangle. \tag{4.1}$$

From the Cartan matrix of A_4, we have

$$\begin{gathered}\langle \alpha, \alpha^\vee \rangle = 2, \langle \beta, \alpha^\vee \rangle = -1, \langle \gamma, \alpha^\vee \rangle = 0, \langle \delta, \alpha^\vee \rangle = 0,\\ \langle \alpha, \beta^\vee \rangle = -1, \langle \beta, \beta^\vee \rangle = 2, \langle \gamma, \beta^\vee \rangle = -1, \langle \delta, \beta^\vee \rangle = 0,\\ \langle \alpha, \gamma^\vee \rangle = 0, \langle \beta, \gamma^\vee \rangle = -1, \langle \gamma, \gamma^\vee \rangle = 2, \langle \delta, \gamma^\vee \rangle = -1.\end{gathered}$$

From this, we compute

$$\begin{gathered}\langle \gamma+\delta, \alpha^\vee \rangle = 0, \langle \beta+\gamma+\delta, \alpha^\vee \rangle = -1, \langle \alpha+\beta+\gamma+\delta, \alpha^\vee \rangle = 1,\\ \langle \gamma+\delta, \beta^\vee \rangle = -1, \langle \beta+\gamma+\delta, \beta^\vee \rangle = 1, \langle \alpha+\beta+\gamma+\delta, \beta^\vee \rangle = 0,\\ \langle \gamma+\delta, \gamma^\vee \rangle = 1, \langle \beta+\gamma+\delta, \gamma^\vee \rangle = 0, \langle \alpha+\beta+\gamma+\delta, \gamma^\vee \rangle = 0,\end{gathered}$$

These calculations show us how n_α, n_β, and n_γ act on $\Psi(R_u(P_{\alpha\beta\gamma}))$.

$$\begin{aligned}n_\alpha \cdot (\delta, \gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta) &= (\delta, \gamma+\delta, \alpha+\beta+\gamma+\delta, \beta+\gamma+\delta).\\ n_\beta \cdot (\delta, \gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta) &= (\delta, \beta+\gamma+\delta, \gamma+\delta, \alpha+\beta+\gamma+\delta).\\ n_\gamma \cdot (\delta, \gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta) &= (\gamma+\delta, \delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta).\end{aligned}$$

From this, we see that H has the single orbit $\{\delta, \gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta\}$. By Corollary 2.3.3 we get

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma}))}(H) = \{a(e_\delta + e_{\gamma+\delta} + e_{\beta+\gamma+\delta} + e_{\alpha+\beta+\gamma+\delta}) \mid a \in k\}.$$

It is not difficult to see that no positive integral combination of the roots in the orbit of H is a root of G. By Proposition 2.3.5 and Lemma 2.3.4 we get

$$\mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma})}(H)) = \mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma})}(H).$$

Hence we have the following.

Proposition 4.1.3. *There is no nilpotent witness to the G-nonseparability of H in* $\mathrm{Lie}\,(R_u(P_{\alpha\beta\gamma}))$.

4.1.4 $G = B_3$ with A_2 Levi subgroup

Let G be a simple algebraic group of type B_3. Fix a maximal torus T of G. Take a Borel subgroup B of G containing T. Then the set of positive roots is

$$\Psi^+(G) = \{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma, \beta+2\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma\}.$$

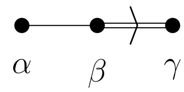


Figure 4.4: Dynkin diagram of B_3

Let $L_{\alpha\beta}$ be the Levi subgroup of type A_2 corresponding to α and β. Then

$$\Psi^+(L_{\alpha\beta}) = \{\alpha, \beta, \alpha+\beta\}.$$

Set $P_{\alpha\beta} = \langle B \cup L_{\alpha\beta} \rangle$. Then $P_{\alpha\beta}$ is a parabolic subgroup of G with

$$\Psi(R_u(P_{\alpha\beta})) = \{\gamma, \beta+\gamma, \alpha+\beta+\gamma, \beta+2\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma\}.$$

We fix the ordering of $\Psi(R_u(P_{\alpha\beta}))$ as given above. Now, define H by

$$H = \langle n_\alpha, n_\beta \rangle.$$

From the Cartan matrix for B_3, we get

$$\begin{gathered}\langle \alpha, \alpha^\vee \rangle = 2, \langle \beta, \alpha^\vee \rangle = -1, \langle \gamma, \alpha^\vee \rangle = 0, \\ \langle \alpha, \beta^\vee \rangle = -1, \langle \beta, \beta^\vee \rangle = 2, \langle \gamma, \beta^\vee \rangle = -1.\end{gathered}$$

From this, we compute

$$\begin{gathered}\langle \beta+\gamma, \alpha^\vee \rangle = -1, \langle \alpha+\beta+\gamma, \alpha^\vee \rangle = 1, \langle \beta+2\gamma, \alpha^\vee \rangle = -1, \\ \langle \alpha+\beta+2\gamma, \alpha^\vee \rangle = 1, \langle \alpha+2\beta+2\gamma, \alpha^\vee \rangle = 0 \\ \langle \beta+\gamma, \beta^\vee \rangle = 1, \langle \alpha+\beta+\gamma, \beta^\vee \rangle = 0, \langle \beta+2\gamma, \beta^\vee \rangle = 0, \\ \langle \alpha+\beta+2\gamma, \beta^\vee \rangle = -1, \langle \alpha+2\beta+2\gamma, \beta^\vee \rangle = 1.\end{gathered}$$

These formulas show us how n_α and n_β act on $\Psi(R_u(P_{\alpha\beta}))$.

$$\begin{gathered}n_\alpha \cdot (\gamma, \beta+\gamma, \alpha+\beta+\gamma, \beta+2\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma) \\ = (\gamma, \alpha+\beta+\gamma, \beta+\gamma, \alpha+\beta+2\gamma, \beta+2\gamma, \alpha+2\beta+2\gamma). \\ n_\beta \cdot (\gamma, \beta+\gamma, \alpha+\beta+\gamma, \beta+2\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma) \\ = (\beta+\gamma, \gamma, \alpha+\beta+\gamma, \beta+2\gamma, \alpha+2\beta+2\gamma, \alpha+\beta+2\gamma).\end{gathered}$$

From this, it is not difficult to see that H has 2 orbits $\{\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$ and $\{\beta+2\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma\}$. By Corollary 2.3.3 we find

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta}))}(H) = \{a(e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma}) + b(e_{\beta+2\gamma} + e_{\alpha+\beta+2\gamma} + e_{\alpha+2\beta+2\gamma}) \mid a, b \in k\}.$$

Note that $\{\beta, \gamma \beta + \gamma, \beta + 2\gamma\}$ is the root system of type B_2 where γ and $\beta + \gamma$ are short. So, by Proposition 2.3.6, we have

$$\begin{aligned}\epsilon_{\beta+\gamma}(a)\epsilon_{\gamma}(b) &= \epsilon_{\gamma}(b)\epsilon_{\beta+\gamma}(a)\epsilon_{\beta+2\gamma}(2ab) \\ &= \epsilon_{\gamma}(b)\epsilon_{\beta+\gamma}(a), \text{ where } a, b \in k.\end{aligned}$$

The last equality holds because we work in characteristic 2, and this tells us that U_γ and $U_{\beta+\gamma}$ are commuting subgroups of G. Then,

$$\begin{aligned}&\epsilon_{\beta+\gamma}(a)\epsilon_{\gamma}(b) = \epsilon_{\gamma}(b)\epsilon_{\beta+\gamma}(a) \\ \Leftrightarrow\ & n_\alpha \cdot (\epsilon_{\beta+\gamma}(a)\epsilon_{\gamma}(b)) = n_\alpha \cdot (\epsilon_{\gamma}(b)\epsilon_{\beta+\gamma}(a)) \\ \Leftrightarrow\ & \epsilon_{\alpha+\beta+\gamma}(a)\epsilon_{\gamma}(b) = \epsilon_{\gamma}(b)\epsilon_{\alpha+\beta+\gamma}(a) \\ \Leftrightarrow\ & n_\beta \cdot (\epsilon_{\alpha+\beta+\gamma}(a)\epsilon_{\gamma}(b)) = n_\beta \cdot (\epsilon_{\gamma}(b)\epsilon_{\alpha+\beta+\gamma}(a)) \\ \Leftrightarrow\ & \epsilon_{\alpha+\beta+\gamma}(a)\epsilon_{\beta+\gamma}(b) = \epsilon_{\beta+\gamma}(b)\epsilon_{\alpha+\beta+\gamma}(a)\end{aligned}$$

The third and the last equation show us that $U_{\alpha+\beta+\gamma}$ commutes with U_γ and $U_{\beta+\gamma}$. Therefore U_γ, $U_{\beta+\gamma}$ and $U_{\alpha+\beta+\gamma}$ are mutually commuting subgroups of G. It is easy to see that no positive integral combinations of the roots in the second orbit $\{\beta + 2\gamma, \alpha + \beta + 2\gamma, \alpha + 2\beta + 2\gamma\}$ is a root of G, so by Proposition 2.3.5 and Lemma 2.3.4 we find that

$$\mathrm{Lie}\,(C_{R_u(P_{\alpha\beta})}(H)) = \{a(e_\gamma + e_{\beta+\gamma} + e_{\alpha+\beta+\gamma}) + b(e_{\beta+2\gamma} + e_{\alpha+\beta+2\gamma} + e_{\alpha+2\beta+2\gamma}) \mid a, b \in k\}.$$

So we have the following.

Proposition 4.1.4. *There is no nilpotent witness to the G-nonseparability of H in* $\mathrm{Lie}\,(R_u(P_{\alpha\beta}))$.

4.1.5 $G = B_3$ with B_2 Levi subgroup

Let G be a simple algebraic group of type B_3. Fix a maximal torus T of G. Pick a Borel subgroup B containing T. Then the set of positive roots is

$$\Psi^+(G) = \{\alpha, \beta, \gamma, \alpha + \beta, \beta + \gamma, \alpha + \beta + \gamma, \beta + 2\gamma, \alpha + \beta + 2\gamma, \alpha + 2\beta + 2\gamma\}.$$

Let $L_{\beta\gamma}$ be a Levi subgroup of G corresponding to β and γ. Then

$$\Psi(L_{\beta\gamma}) = \{\beta, \gamma, \beta + \gamma, \beta + 2\gamma\}.$$

Set $P_{\beta\gamma} = \langle B \cup L_{\beta\gamma} \rangle$. Then $P_{\beta\gamma}$ is a parabolic subgroup of G with

$$\Psi(R_u(P_{\beta\gamma})) = \{\alpha, \alpha + \beta, \alpha + \beta + \gamma, \alpha + \beta + 2\gamma, \alpha + 2\beta + 2\gamma\}.$$

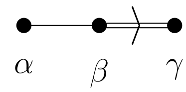


Figure 4.5: Dynkin diagram of B_3

We fix the ordering of $\Psi(R_u(P_{\beta\gamma}))$ as given above. Define H by

$$H = \langle n_\beta, n_\gamma \rangle$$

From the Cartan matrix of B_3, we get

$$\langle \alpha, \beta^\vee \rangle = -1, \langle \beta, \beta^\vee \rangle = 2, \langle \gamma, \beta^\vee \rangle = -1,$$
$$\langle \alpha, \gamma^\vee \rangle = 0, \langle \beta, \gamma^\vee \rangle = -2, \langle \gamma, \beta^\vee \rangle = 2.$$

From this, we compute

$$\langle \alpha + \beta, \beta^\vee \rangle = 1, \langle \alpha + \beta + \gamma, \beta^\vee \rangle = 0,$$
$$\langle \alpha + \beta + 2\gamma, \beta^\vee \rangle = -1, \langle \alpha + 2\beta + 2\gamma, \beta^\vee \rangle = 1,$$
$$\langle \alpha + \beta, \gamma^\vee \rangle = -2, \langle \alpha + \beta + \gamma, \gamma^\vee \rangle = 0,$$
$$\langle \alpha + \beta + 2\gamma, \gamma^\vee \rangle = 2, \langle \alpha + 2\beta + 2\gamma, \gamma^\vee \rangle = 0.$$

These formulas show us how n_β and n_γ act on $\Psi(R_u(P_{\beta\gamma}))$.

$$\begin{aligned} & n_\beta \cdot (\alpha, \alpha+\beta, \alpha+\beta+\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma) \\ & = (\alpha+\beta, \alpha, \alpha+\beta+\gamma, \alpha+2\beta+2\gamma, \alpha+\beta+2\gamma). \\ & n_\gamma \cdot (\alpha, \alpha+\beta, \alpha+\beta+\gamma, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma) \\ & = (\alpha, \alpha+\beta+2\gamma, \alpha+\beta+\gamma, \alpha+\beta, \alpha+2\beta+2\gamma). \end{aligned}$$

From this, it is easy to see that H has two orbits $\{\alpha, \alpha+\beta, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma, \}$ and $\{\alpha+\beta+\gamma\}$. By Corollary 2.3.3 we get

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\beta\gamma}))}(H) = \{a(e_\alpha + e_{\alpha+\beta} + e_{\alpha+\beta+2\gamma} + e_{\alpha+2\beta+2\gamma}) + be_{\alpha+\beta+\gamma} \mid a, b \in k\}.$$

Now let us calculate $C_{R_u(P_{\beta\gamma})}(H)$. Note that no non-negative integer linear combination of the roots in the first orbit $\{\alpha, \alpha+\beta, \alpha+\beta+2\gamma, \alpha+2\beta+2\gamma, \}$ is a root of G. So by Proposition 2.3.5 and Lemma 2.3.4 we find

$$\mathrm{Lie}\ (C_{R_u(P_{\beta\gamma})}(H)) = \mathfrak{c}_{\mathrm{Lie}(R_u(P_{\beta\gamma}))}(H).$$

Then we have the following.

Proposition 4.1.5. *There is no nilpotent witness to the G-nonseparability of H in* Lie $(R_u(P_{\beta\gamma}))$.

4.2 Exceptional cases

4.2.1 $G = E_6$ with A_5 Levi subgroup

Let G be a simple algebraic group of type E_6. Fix a maximal torus T of G. Pick a Borel subgroup B of G containing T. Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \sigma\}$ be the set of simple roots of G corresponding to B and T. The next figure defines how each simple root of G corresponds to each node in the Dynkin diagram of E_6. From

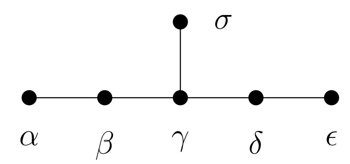


Figure 4.6: Dynkin diagram of E_6

[FdV69, Appendix, Table B], we have the list of the coefficients of all positive roots of G. We label all positive roots of G as in Table 4.1 from 1 to 36. For example, root 1 corresponds to the root σ, and root 5 to the root $\alpha + \beta + \gamma + \sigma$, and root 21 to the root $\alpha + 2\beta + 3\gamma + 2\delta + \epsilon + 2\sigma$, and so on. Then the set of positive roots is

$$\Psi^+(G) = \{1, 2, \cdots, 36\}.$$

We fix the ordering of the positive roots in the natural order. Let $L_{\alpha\beta\gamma\delta\epsilon}$ be the Levi subgroup of type A_5 corresponding to $\alpha, \cdots, \epsilon$. Then

$$\Psi^+(L_{\alpha\beta\gamma\delta\epsilon}) = \{22, \cdots, 36\}.$$

Let $P_{\alpha\beta\gamma\delta\epsilon} = \langle L_{\alpha\beta\gamma\delta\epsilon} \cup B \rangle$. Then $P_{\alpha\beta\gamma\delta\epsilon}$ is the parabolic subgroup of G with

$$\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon})) = \{1, \cdots, 21\}.$$

Note that these are precisely the roots in E_6 such that the coefficient of σ is 1 or 2. We call the roots of the first type *weight* 1 *roots*, and the second type *weight* 2 *roots*. Define H by

$$H = \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon \rangle.$$

① $\begin{smallmatrix}&&1&&\\0&0&0&0&0\end{smallmatrix}$	② $\begin{smallmatrix}&&1&&\\0&0&1&0&0\end{smallmatrix}$	③ $\begin{smallmatrix}&&1&&\\0&1&1&0&0\end{smallmatrix}$	④ $\begin{smallmatrix}&&1&&\\0&0&1&1&0\end{smallmatrix}$
⑤ $\begin{smallmatrix}&&1&&\\1&1&1&0&0\end{smallmatrix}$	⑥ $\begin{smallmatrix}&&1&&\\0&1&1&1&0\end{smallmatrix}$	⑦ $\begin{smallmatrix}&&1&&\\0&0&1&1&1\end{smallmatrix}$	⑧ $\begin{smallmatrix}&&1&&\\1&1&1&1&0\end{smallmatrix}$
⑨ $\begin{smallmatrix}&&1&&\\0&1&1&1&1\end{smallmatrix}$	⑩ $\begin{smallmatrix}&&1&&\\0&1&2&1&0\end{smallmatrix}$	⑪ $\begin{smallmatrix}&&1&&\\1&1&1&1&1\end{smallmatrix}$	⑫ $\begin{smallmatrix}&&1&&\\1&1&2&1&0\end{smallmatrix}$
⑬ $\begin{smallmatrix}&&1&&\\0&1&2&1&1\end{smallmatrix}$	⑭ $\begin{smallmatrix}&&1&&\\1&1&2&1&1\end{smallmatrix}$	⑮ $\begin{smallmatrix}&&1&&\\1&2&2&1&0\end{smallmatrix}$	⑯ $\begin{smallmatrix}&&1&&\\0&1&2&2&1\end{smallmatrix}$
⑰ $\begin{smallmatrix}&&1&&\\1&2&2&1&1\end{smallmatrix}$	⑱ $\begin{smallmatrix}&&1&&\\1&1&2&2&1\end{smallmatrix}$	⑲ $\begin{smallmatrix}&&1&&\\1&2&2&2&1\end{smallmatrix}$	⑳ $\begin{smallmatrix}&&1&&\\1&2&3&2&1\end{smallmatrix}$
㉑ $\begin{smallmatrix}&&2&&\\1&2&3&2&1\end{smallmatrix}$	㉒ $\begin{smallmatrix}&&0&&\\1&0&0&0&0\end{smallmatrix}$	㉓ $\begin{smallmatrix}&&0&&\\0&1&0&0&0\end{smallmatrix}$	㉔ $\begin{smallmatrix}&&0&&\\0&0&1&0&0\end{smallmatrix}$
㉕ $\begin{smallmatrix}&&0&&\\0&0&0&1&0\end{smallmatrix}$	㉖ $\begin{smallmatrix}&&0&&\\0&0&0&0&1\end{smallmatrix}$	㉗ $\begin{smallmatrix}&&0&&\\1&1&0&0&0\end{smallmatrix}$	㉘ $\begin{smallmatrix}&&0&&\\0&1&1&0&0\end{smallmatrix}$
㉙ $\begin{smallmatrix}&&0&&\\0&0&1&1&0\end{smallmatrix}$	㉚ $\begin{smallmatrix}&&0&&\\0&0&0&1&1\end{smallmatrix}$	㉛ $\begin{smallmatrix}&&0&&\\1&1&1&0&0\end{smallmatrix}$	㉜ $\begin{smallmatrix}&&0&&\\0&1&1&1&0\end{smallmatrix}$
㉝ $\begin{smallmatrix}&&0&&\\0&0&1&1&1\end{smallmatrix}$	㉞ $\begin{smallmatrix}&&0&&\\1&1&1&1&0\end{smallmatrix}$	㉟ $\begin{smallmatrix}&&0&&\\0&1&1&1&1\end{smallmatrix}$	㊱ $\begin{smallmatrix}&&0&&\\1&1&1&1&1\end{smallmatrix}$

Table 4.1: The set of positive roots of G

Let ζ_1, ζ_2 be simple roots of G. From the Cartan matrix of E_6 [Hum91, Sec. 11.4], we have

$$\langle \zeta_1, \zeta_2^\vee \rangle = \begin{cases} 2 & \text{if } \zeta_1 = \zeta_2. \\ -1 & \text{if the nodes corresponding to } \zeta_1, \zeta_2 \text{ are adjacent to each other} \\ & \text{in the Dynkin diagram.} \\ 0 & \text{if the nodes corresponding to } \zeta_1, \zeta_2 \text{ are NOT adjacent to each} \\ & \text{other in the Dynkin diagram.} \end{cases}$$

From this, it is not difficult to calculate $\langle \xi, \zeta^\vee \rangle$ for all $\xi \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon}))$ and $\zeta \in \Sigma$. For example,

$$\begin{aligned} &\langle \alpha + \beta + 2\gamma + \delta + \epsilon + \sigma, \gamma^\vee \rangle \\ &= \langle \alpha, \gamma^\vee \rangle + \langle \beta, \gamma^\vee \rangle + 2\langle \gamma, \gamma^\vee \rangle + \langle \delta, \gamma^\vee \rangle + \langle \epsilon, \gamma^\vee \rangle + \langle \sigma, \gamma^\vee \rangle \\ &= 0 + (-1) + 4 + (-1) + 0 + (-1) = 1. \end{aligned}$$

These calculations show how n_α, n_β, n_γ, n_δ, and n_ϵ act on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon})) = \{1, \cdots, 21\}$. Let π be the corresponding representation of H on

$Sym(\Psi(R_u(P_{\alpha,\beta\gamma\delta\epsilon}))) \cong S_{21}$. We get Table 4.2. So, for example, n_α maps the

$$\begin{array}{rcl}
\pi(n_\alpha) & = & (3\ 5)(6\ 8)(9\ 11)(10\ 12)(13\ 14)(16\ 18) \\
\pi(n_\beta) & = & (2\ 3)(4\ 6)(7\ 9)(12\ 15)(14\ 17)(18\ 19) \\
\pi(n_\gamma) & = & (1\ 2)(6\ 10)(8\ 12)(9\ 13)(11\ 14)(19\ 20) \\
\pi(n_\delta) & = & (2\ 4)(3\ 6)(5\ 8)(13\ 16)(14\ 18)(17\ 19) \\
\pi(n_\epsilon) & = & (4\ 7)(6\ 9)(8\ 11)(10\ 13)(12\ 14)(15\ 17)
\end{array}$$

Table 4.2: Action of n_τ's on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon}))$

root 3 to the root 5, and the root 6 to the root 8, and fixes the root 4, etc. It is not difficult to see that H has two orbits on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon}))$. These are

$$O_1 = \{1, \cdots, 20\},$$
$$\text{and } \{21\}.$$

Then, by Corollary 2.3.3, we have

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon}))}(H) = \{a(\sum_{\lambda \in O_1} e_\lambda) + be_{21} \mid a, b \in k\}.$$

Now, let us calculate $\mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H))$. We prove the following claim first.

Claim 4.2.1. *Any element u in $C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H)$ can be expressed uniquely as*

$$u = \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a)\right) \epsilon_{21}(a_{21}), \text{ where } a, a_{21} \in k$$

Proof. By Lemma 2.3.1, any element u in $R_u(P_{\alpha\beta\gamma\delta\epsilon})$ can be written uniquely as

$$u = \prod_{\lambda \in \{1, \cdots, 21\}} \epsilon_\lambda(a_\lambda), \text{ where } a_\lambda \in k.$$

Pick any two roots $i, j \in \{1, \cdots, 20\}$ such that $i < j$ in the given ordering. Since H acts transitively on $\{1, \cdots, 20\}$, we can find $n \in H$ such that $n \cdot j = i$. We compute

$$\begin{gathered}
n \cdot u = n \cdot (\epsilon_1(a_1) \cdots \epsilon_i(a_i) \cdots \epsilon_j(a_j) \cdots \epsilon_{21}(a_{21})) \\
= \epsilon_{n\cdot 1}(a_1) \cdots \epsilon_{n\cdot i}(a_i) \cdots \epsilon_{n\cdot j}(a_j) \cdots \epsilon_{n\cdot 21}(a_{21}) \\
= \epsilon_1(a_{n^{-1}\cdot 1}) \cdots \epsilon_i(a_{n^{-1}\cdot i}) \cdots \epsilon_j(a_{n^{-1}\cdot j}) \cdots \epsilon_{21}(a_{21} + f(a_1, \cdots, a_{20})) \\
\text{for some } f \in k[a_1, \cdots, a_{20}].
\end{gathered}$$

To get the last equality, we reordered the terms in the following way using Propositions 2.3.5 and 2.3.7. Note that given $l, m \in \{1, \cdots, 20\}$, either U_l and U_m commute or we get the $l\tilde{+}m = 21$ and $\{l, m, 21\}$ forms an A_2 subsystem. (We use $\tilde{+}$ to represent the sum of roots, not the sum of the labels of the roots.) In the latter case, by Proposition 2.3.7 we have

$$\epsilon_l(a)\epsilon_m(b) = \epsilon_m(b)\epsilon_l(a)\epsilon_{21}(ab).$$

Also note that in this case there is only one weight-2 root, which is root 21.

($\star$) Move ϵ_1 to the left, and if an ϵ_{21} term occurs, this can be moved directly to the right since a ϵ_{21} term commutes with any other terms by Proposition 2.3.5. Then move ϵ_2 term to the left until it appears right after ϵ_1 term. Continue with this process until all ϵ_i are rearranged into the natural order.

If $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H)$, we must have $n \cdot u = u$. Comparing the arguments of the ϵ_i, we get $a_i = a_{n^{-1} \cdot i}$, so we have $a_i = a_j$. $\square$

Now, pick $a, a_{21} \in k$, and let

$$u = \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}).$$

We list all pairs of non-commuting root subgroups of $R_u(P_{\alpha\beta\gamma\delta\epsilon})$ in Table 4.3. They are non-commuting pairs because for each pair the set of the corresponding

$\{U_1, U_{20}\}$	$\{U_2, U_{19}\}$	$\{U_3, U_{18}\}$	$\{U_4, U_{17}\}$
$\{U_5, U_{16}\}$	$\{U_6, U_{14}\}$	$\{U_7, U_{15}\}$	$\{U_8, U_{13}\}$
$\{U_9, U_{12}\}$	$\{U_{10}, U_{11}\}$		

Table 4.3: Non-commuting pairs of root subgroups of $R_u(P_{\alpha\beta\gamma\delta\epsilon})$ in E_6

roots, say, root l and root m, together with root 21, is the root system of type A_2, and by Proposition 2.3.7, we have

$$\epsilon_l(a)\epsilon_m(a) = \epsilon_m(a)\epsilon_l(a)\epsilon_{21}(a^2).$$

Now, we compute

$$\begin{aligned}
n_\alpha \cdot u =& n_\alpha \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\alpha \cdot \lambda}(a) \right) \epsilon_{n_\alpha \cdot 21}(a_{21}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21} + 2a^2) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}).
\end{aligned}$$

The last equation holds since k is of characteristic 2. To get the second last equation, we have used Proposition 2.3.5, Proposition 2.3.7 and $(\star)$ in the following way. Note that n_α permutes roots within each set of roots $\{1, \cdots, 8\}$, $\{9, \cdots, 12\}$ and $\{13, \cdots, 18\}$. Any two roots in $\{1, \cdots, 8\}$ commute, and any two roots in $\{13, \cdots, 18\}$ commute, so it suffices to consider $\{9, \cdots, 12\}$. We compute

$$\begin{aligned}
n_\alpha \cdot (\epsilon_9(a)\epsilon_{10}(a)\epsilon_{11}(a)\epsilon_{12}(a)) =& \epsilon_{n_\alpha \cdot 9}(a)\epsilon_{n_\alpha \cdot 10}(a)\epsilon_{n_\alpha \cdot 11}(a)\epsilon_{n_\alpha \cdot 12}(a) \\
=& \epsilon_{11}(a)\epsilon_{12}(a)\epsilon_9(a)\epsilon_{10}(a) \\
=& \epsilon_{11}(a)\epsilon_9(a)\epsilon_{12}(a)\epsilon_{21}(a^2)\epsilon_{10}(a) \\
=& \epsilon_{11}(a)\epsilon_9(a)\epsilon_{12}(a)\epsilon_{10}(a)\epsilon_{21}(a^2) \\
=& \epsilon_9(a)\epsilon_{11}(a)\epsilon_{12}(a)\epsilon_{10}(a)\epsilon_{21}(a^2) \\
=& \epsilon_9(a)\epsilon_{11}(a)\epsilon_{10}(a)\epsilon_{12}(a)\epsilon_{21}(a^2) \\
=& \epsilon_9(a)\epsilon_{10}(a)\epsilon_{11}(a)\epsilon_{21}(a^2)\epsilon_{12}(a)\epsilon_{21}(a^2) \\
=& \epsilon_9(a)\epsilon_{10}(a)\epsilon_{11}(a)\epsilon_{12}(a)\epsilon_{21}(2a^2)
\end{aligned}$$

Note that we get $2a^2$ as the argument of ϵ_{21} because the action of n_α on u swaps the order of two non-commuting pairs, namely, $\{\epsilon_9, \epsilon_{12}\}$ and $\{\epsilon_{10}, \epsilon_{11}\}$.

We calculate the action of $n_\beta, n_\gamma, n_\delta, n_\epsilon$ on u in the similar way.

$$\begin{aligned}
n_\beta \cdot u =& n_\beta \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\beta \cdot \lambda}(a) \right) \epsilon_{n_\beta \cdot 21}(a_{21}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}).
\end{aligned}$$

In this case, no non-commuting pair has their order swapped by the action of n_β on u.

$$
\begin{aligned}
n_\gamma \cdot u =& n_\gamma \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\gamma \cdot \lambda}(a) \right) \epsilon_{n_\gamma \cdot 21}(a_{21}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21} + 2a^2) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}).
\end{aligned}
$$

In this case, two non-commuting pairs, namely, $\{\epsilon_8, \epsilon_{13}\}$ and $\{\epsilon_9, \epsilon_{12}\}$, have their order swapped by the action of n_γ.

$$
\begin{aligned}
n_\delta \cdot u =& n_\delta \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\delta \cdot \lambda}(a) \right) \epsilon_{n_\delta \cdot 21}(a_{21}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}).
\end{aligned}
$$

In this case, no non-commuting pair has their order swapped by the action of n_δ on u.

$$
\begin{aligned}
n_\epsilon \cdot u =& n_\epsilon \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\epsilon \cdot \lambda}(a) \right) \epsilon_{n_\epsilon \cdot 21}(a_{21}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21} + 2a^2) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{21}(a_{21}).
\end{aligned}
$$

In this case, two non-commuting pairs, namely, $\{\epsilon_8, \epsilon_{13}\}$ and $\{\epsilon_{10}, \epsilon_{11}\}$, have their order swapped by the action of s_γ.

Since the set $\{n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon\}$ generates H, these calculations show that

$$C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H) = \{(\prod_{\lambda \in O_1} \epsilon_\lambda(a))\epsilon_{21}(a_{21}) \mid a, a_{21} \in k\}. \tag{4.2}$$

In (4.2), setting $a = 0$, and differentiating with respect to a_{21}, and evaluating at $a_{21} = 0$, we get

$$e_{21} \in \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H)).$$

Again, in (4.2), if we set $a_{21} = 0$, differentiate with respect to a, and evaluate at $a = 0$, we get

$$\sum_{\lambda \in O_1} e_\lambda \in \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H)).$$

So we have

$$\{a(\sum_{\lambda \in O_1} e_\lambda) + be_{21} \mid a, b \in k\} \subseteq \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H)).$$

Hence

$$\mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon})}(H)) = \mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon}))}(H).$$

Thus, we have the following.

Proposition 4.2.2. *There is no nilpotent witness to the G-nonseparability of H in* $\mathrm{Lie}\,(R_u(P_{\alpha\beta\gamma\delta\epsilon}))$.

4.2.2 $G = E_7$ with A_6 Levi subgroup

Let G be a simple algebraic group of type E_7. Fix a maximal torus T of G. Pick a Borel subgroup B of G containing T. Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma\}$ be the set of simple roots of G. The next figure defines how each simple root of G corresponds to each node in the Dynkin diagram of E_7. From [FdV69, Appendix,

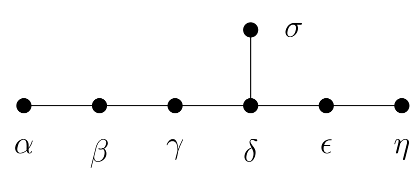


Figure 4.7: Dynkin diagram of E_7

Table B], we have the coefficients of the simple roots in all positive roots of G. We label all positive roots of G in Table 4.4. Then the set of positive roots is

$$\Psi^+(G) = \{1, 2, \cdots, 63\}.$$

①	$\begin{smallmatrix} &&&1&& \\ 0&0&0&0&0&0 \end{smallmatrix}$	②	$\begin{smallmatrix} &&&1&& \\ 0&0&0&1&0&0 \end{smallmatrix}$	③	$\begin{smallmatrix} &&&1&& \\ 0&0&1&1&0&0 \end{smallmatrix}$	④	$\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&0 \end{smallmatrix}$
⑤	$\begin{smallmatrix} &&&1&& \\ 0&1&1&1&0&0 \end{smallmatrix}$	⑥	$\begin{smallmatrix} &&&1&& \\ 0&0&1&1&1&0 \end{smallmatrix}$	⑦	$\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&1 \end{smallmatrix}$	⑧	$\begin{smallmatrix} &&&1&& \\ 1&1&1&1&0&0 \end{smallmatrix}$
⑨	$\begin{smallmatrix} &&&1&& \\ 0&1&1&1&1&0 \end{smallmatrix}$	⑩	$\begin{smallmatrix} &&&1&& \\ 0&0&1&1&1&1 \end{smallmatrix}$	⑪	$\begin{smallmatrix} &&&1&& \\ 0&0&1&2&1&0 \end{smallmatrix}$	⑫	$\begin{smallmatrix} &&&1&& \\ 1&1&1&1&1&0 \end{smallmatrix}$
⑬	$\begin{smallmatrix} &&&1&& \\ 0&1&1&2&1&0 \end{smallmatrix}$	⑭	$\begin{smallmatrix} &&&1&& \\ 0&1&1&1&1&1 \end{smallmatrix}$	⑮	$\begin{smallmatrix} &&&1&& \\ 0&0&1&2&1&1 \end{smallmatrix}$	⑯	$\begin{smallmatrix} &&&1&& \\ 1&1&1&2&1&0 \end{smallmatrix}$
⑰	$\begin{smallmatrix} &&&1&& \\ 1&1&1&1&1&1 \end{smallmatrix}$	⑱	$\begin{smallmatrix} &&&1&& \\ 0&1&2&2&1&0 \end{smallmatrix}$	⑲	$\begin{smallmatrix} &&&1&& \\ 0&1&1&2&1&1 \end{smallmatrix}$	⑳	$\begin{smallmatrix} &&&1&& \\ 0&0&1&2&2&1 \end{smallmatrix}$
㉑	$\begin{smallmatrix} &&&1&& \\ 1&1&2&2&1&0 \end{smallmatrix}$	㉒	$\begin{smallmatrix} &&&1&& \\ 1&1&1&2&1&1 \end{smallmatrix}$	㉓	$\begin{smallmatrix} &&&1&& \\ 0&1&2&2&1&1 \end{smallmatrix}$	㉔	$\begin{smallmatrix} &&&1&& \\ 0&1&1&2&2&1 \end{smallmatrix}$
㉕	$\begin{smallmatrix} &&&1&& \\ 1&2&2&2&1&0 \end{smallmatrix}$	㉖	$\begin{smallmatrix} &&&1&& \\ 1&1&2&2&1&1 \end{smallmatrix}$	㉗	$\begin{smallmatrix} &&&1&& \\ 1&1&1&2&2&1 \end{smallmatrix}$	㉘	$\begin{smallmatrix} &&&1&& \\ 0&1&2&2&2&1 \end{smallmatrix}$
㉙	$\begin{smallmatrix} &&&1&& \\ 1&2&2&2&1&1 \end{smallmatrix}$	㉚	$\begin{smallmatrix} &&&1&& \\ 1&1&2&2&2&1 \end{smallmatrix}$	㉛	$\begin{smallmatrix} &&&1&& \\ 0&1&2&3&2&1 \end{smallmatrix}$	㉜	$\begin{smallmatrix} &&&1&& \\ 1&2&2&2&2&1 \end{smallmatrix}$
㉝	$\begin{smallmatrix} &&&1&& \\ 1&1&2&3&2&1 \end{smallmatrix}$	㉞	$\begin{smallmatrix} &&&1&& \\ 1&2&2&3&2&1 \end{smallmatrix}$	㉟	$\begin{smallmatrix} &&&1&& \\ 1&2&3&3&2&1 \end{smallmatrix}$	㊱	$\begin{smallmatrix} &&&2&& \\ 0&1&2&3&2&1 \end{smallmatrix}$
㊲	$\begin{smallmatrix} &&&2&& \\ 1&1&2&3&2&1 \end{smallmatrix}$	㊳	$\begin{smallmatrix} &&&2&& \\ 1&2&2&3&2&1 \end{smallmatrix}$	㊴	$\begin{smallmatrix} &&&2&& \\ 1&2&3&3&2&1 \end{smallmatrix}$	㊵	$\begin{smallmatrix} &&&2&& \\ 1&2&3&4&2&1 \end{smallmatrix}$
㊶	$\begin{smallmatrix} &&&2&& \\ 1&2&3&4&3&1 \end{smallmatrix}$	㊷	$\begin{smallmatrix} &&&2&& \\ 1&2&3&4&3&2 \end{smallmatrix}$	㊸	$\begin{smallmatrix} &&&0&& \\ 1&0&0&0&0&0 \end{smallmatrix}$	㊹	$\begin{smallmatrix} &&&0&& \\ 0&1&0&0&0&0 \end{smallmatrix}$
㊺	$\begin{smallmatrix} &&&0&& \\ 0&0&1&0&0&0 \end{smallmatrix}$	㊻	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&0&0 \end{smallmatrix}$	㊼	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&1&0 \end{smallmatrix}$	㊽	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&0&1 \end{smallmatrix}$
㊾	$\begin{smallmatrix} &&&0&& \\ 1&1&0&0&0&0 \end{smallmatrix}$	㊿	$\begin{smallmatrix} &&&0&& \\ 0&1&1&0&0&0 \end{smallmatrix}$	51	$\begin{smallmatrix} &&&0&& \\ 0&0&1&1&0&0 \end{smallmatrix}$	52	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&1&0 \end{smallmatrix}$
53	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&1&1 \end{smallmatrix}$	54	$\begin{smallmatrix} &&&0&& \\ 1&1&1&0&0&0 \end{smallmatrix}$	55	$\begin{smallmatrix} &&&0&& \\ 0&1&1&1&0&0 \end{smallmatrix}$	56	$\begin{smallmatrix} &&&0&& \\ 0&0&1&1&1&0 \end{smallmatrix}$
57	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&1&1 \end{smallmatrix}$	58	$\begin{smallmatrix} &&&0&& \\ 1&1&1&1&0&0 \end{smallmatrix}$	59	$\begin{smallmatrix} &&&0&& \\ 0&1&1&1&1&0 \end{smallmatrix}$	60	$\begin{smallmatrix} &&&0&& \\ 0&0&1&1&1&1 \end{smallmatrix}$
61	$\begin{smallmatrix} &&&0&& \\ 1&1&1&1&1&0 \end{smallmatrix}$	62	$\begin{smallmatrix} &&&0&& \\ 0&1&1&1&1&1 \end{smallmatrix}$	63	$\begin{smallmatrix} &&&0&& \\ 1&1&1&1&1&1 \end{smallmatrix}$		

Table 4.4: The set of positive roots of G

We fix the ordering of $\Psi^+(G)$ in the natural order. Let $L_{\alpha\beta\gamma\delta\epsilon\eta}$ be the Levi subgroup of type A_6 corresponding to $\alpha, \cdots, \eta$. Then

$$\Psi^+(L_{\alpha\beta\gamma\delta\epsilon\eta}) = \{43, \cdots, 63\}.$$

Let $P_{\alpha\beta\gamma\delta\epsilon\eta} = \langle L_{\alpha\beta\gamma\delta\epsilon\eta} \cup B\rangle$. Then $P_{\alpha\beta\gamma\delta\epsilon\eta}$ is a parabolic subgroup of G with

$$\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})) = \{1, \cdots, 42\}.$$

Note that $\{1, \cdots, 35\}$ and $\{36, \cdots, 42\}$ are precisely the roots in E_7 such that the coefficient of σ is 1 and 2 respectively. We call the roots of the first type *weight-1 roots* and the second type *weight-2 roots*. Define H by

$$H = \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta\rangle.$$

Let ζ_1, ζ_2 be simple roots of G. From the Cartan matrix of E_7 [Hum91, Sec. 11.4], we have

$$\langle \zeta_1, \zeta_2^\vee\rangle = \begin{cases} 2, & \text{if } \zeta_1 = \zeta_2. \\ -1, & \text{if the nodes corresponding to } \zeta_1, \zeta_2 \\ & \text{are adjacent to each other in the Dynkin diagram.} \\ 0, & \text{if the nodes corresponding to } \zeta_1, \zeta_2 \\ & \text{are NOT adjacent to each other in the Dynkin diagram,} \\ & \text{and } \zeta_1 \neq \zeta_2. \end{cases}$$

From this, it is not difficult to calculate $\langle \xi, \zeta^\vee\rangle$ for all $\xi \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ and $\zeta \in \Sigma$. Then, these calculations show how n_α, n_β, n_γ, n_δ, n_ϵ, and n_η act on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})) = \{1, \cdots, 42\}$. Let π be the corresponding representation of H on $Sym(\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))) \cong S_{42}$. Then we get Table 4.5. It is not difficult to see that H has two orbits,

$$\begin{aligned} O_1 =& \{1, \cdots, 35\}, \\ O_{36} =& \{36, \cdots, 42\}. \end{aligned}$$

Then by Corollary 2.3.3, we have

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(H) = \{a(\sum_{\lambda\in O_1} e_\lambda) + b(\sum_{\lambda\in O_{36}} e_\lambda) \mid a, b \in k\}.$$

Now, we calculate $\mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H))$. By Proposition 2.3.7, a pair of root subgroups $\{U_l, U_m\}$ such that $l\tilde{+}m$ is a root of $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})$ is a non-commuting pair of root subgroups. We list all such pairs $\{U_l, U_m\}$ with root $l\tilde{+}m$ in Table 4.6. Note that $l, m, l\tilde{+}m$ form an A_2 subsystem. Now, we prove the following claim.

$$
\begin{array}{lcl}
\pi(n_\alpha) & = & (5\ 8)(9\ 12)(13\ 16)(14\ 17)(18\ 21)(19\ 22)(23\ 26)(24\ 27)(28\ 30)(31\ 33) \\
 & & (36\ 37) \\
\pi(n_\beta) & = & (3\ 5)(6\ 9)(10\ 14)(11\ 13)(15\ 19)(20\ 24)(21\ 25)(26\ 29)(30\ 32)(33\ 34) \\
 & & (37\ 38) \\
\pi(n_\gamma) & = & (2\ 3)(4\ 6)(7\ 10)(13\ 18)(16\ 21)(19\ 23)(22\ 26)(24\ 28)(27\ 30)(34\ 35) \\
 & & (38\ 39) \\
\pi(n_\delta) & = & (1\ 2)(6\ 11)(9\ 13)(10\ 15)(12\ 16)(14\ 19)(17\ 22)(28\ 31)(30\ 33)(32\ 34) \\
 & & (39\ 40) \\
\pi(n_\epsilon) & = & (2\ 4)(3\ 6)(5\ 9)(8\ 12)(15\ 20)(19\ 24)(22\ 27)(23\ 28)(26\ 30)(29\ 32) \\
 & & (40\ 41) \\
\pi(n_\eta) & = & (4\ 7)(6\ 10)(9\ 14)(11\ 15)(12\ 17)(13\ 19)(16\ 22)(18\ 23)(21\ 26)(25\ 29) \\
 & & (41\ 42)
\end{array}
$$

Table 4.5: Action of n_τ's on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$

Claim 4.2.3. *Any element u in $C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)$ can be expressed uniquely as*

$$u = \left(\prod_{\lambda\in O_1} \epsilon_\lambda(a)\right)\left(\prod_{\lambda\in O_{36}} \epsilon_\lambda(a_\lambda)\right), \text{ where } a, a_\lambda \in k$$

Proof. By Lemma 2.3.1 any element u in $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})$ can be written uniquely as

$$u = \prod_{\lambda\in\{1,\cdots,42\}} \epsilon_\lambda(a_\lambda), \text{ where } a_\lambda \in k.$$

Pick any two roots $i, j \in \{1, \cdots, 35\}$ such that $i < j$ in the given ordering. Since H acts transitively on $\{1, \cdots, 35\}$, we can find $n \in H$ such that $n \cdot j = i$. We compute

$$
\begin{aligned}
n \cdot u =& n \cdot \left(\epsilon_1(a_1)\cdots\epsilon_i(a_i)\cdots\epsilon_j(a_j)\cdots \prod_{\lambda\in O_{36}} \epsilon_\lambda(a_\lambda)\right) \\
=& \epsilon_{n\cdot 1}(a_1)\cdots\epsilon_{n\cdot i}(a_i)\cdots\epsilon_{n\cdot j}(a_j)\cdots \prod_{\lambda\in O_{36}} \epsilon_{n\cdot\lambda}(a_\lambda) \\
=& \epsilon_1(a_{n^{-1}\cdot 1})\cdots\epsilon_i(a_{n^{-1}\cdot i})\cdots\epsilon_j(a_{n^{-1}\cdot j})\cdots \\
& \prod_{\lambda\in O_{36}} \epsilon_\lambda(a_{n^{-1}\cdot\lambda} + f_\lambda(a_1, \cdots, a_{35})) \\
& \text{for some } f_\lambda \in k[a_1, \cdots, a_{35}].
\end{aligned}
$$

To get the last equality, we reordered the terms in the following way using Proposition 2.3.5 and Proposition 2.3.7. Note that we get a root subsystem of type A_2

$\{U_1, U_{31}, 36\}$	$\{U_2, U_{28}, 36\}$	$\{U_3, U_{24}, 36\}$	$\{U_4, U_{23}, 36\}$
$\{U_5, U_{20}, 36\}$	$\{U_6, U_{19}, 36\}$	$\{U_7, U_{18}, 36\}$	$\{U_9, U_{15}, 36\}$
$\{U_{10}, U_{13}, 36\}$	$\{U_{11}, U_{14}, 36\}$		
$\{U_1, U_{33}, 37\}$	$\{U_2, U_{30}, 37\}$	$\{U_3, U_{27}, 37\}$	$\{U_4, U_{26}, 37\}$
$\{U_6, U_{22}, 37\}$	$\{U_7, U_{21}, 37\}$	$\{U_8, U_{20}, 37\}$	$\{U_{10}, U_{16}, 37\}$
$\{U_{11}, U_{17}, 37\}$	$\{U_{12}, U_{15}, 37\}$		
$\{U_1, U_{34}, 38\}$	$\{U_2, U_{32}, 38\}$	$\{U_4, U_{29}, 38\}$	$\{U_5, U_{27}, 38\}$
$\{U_7, U_{25}, 38\}$	$\{U_8, U_{24}, 38\}$	$\{U_9, U_{22}, 38\}$	$\{U_{12}, U_{19}, 38\}$
$\{U_{13}, U_{17}, 38\}$	$\{U_{14}, U_{16}, 38\}$		
$\{U_1, U_{35}, 39\}$	$\{U_3, U_{32}, 39\}$	$\{U_5, U_{30}, 39\}$	$\{U_6, U_{29}, 39\}$
$\{U_8, U_{28}, 39\}$	$\{U_9, U_{26}, 39\}$	$\{U_{10}, U_{25}, 39\}$	$\{U_{12}, U_{23}, 39\}$
$\{U_{14}, U_{21}, 39\}$	$\{U_{17}, U_{18}, 39\}$		
$\{U_2, U_{35}, 40\}$	$\{U_3, U_{34}, 40\}$	$\{U_5, U_{33}, 40\}$	$\{U_8, U_{31}, 40\}$
$\{U_{11}, U_{29}, 40\}$	$\{U_{13}, U_{26}, 40\}$	$\{U_{15}, U_{25}, 40\}$	$\{U_{16}, U_{23}, 40\}$
$\{U_{18}, U_{22}, 40\}$	$\{U_{19}, U_{21}, 40\}$		
$\{U_4, U_{35}, 41\}$	$\{U_6, U_{34}, 41\}$	$\{U_9, U_{33}, 41\}$	$\{U_{11}, U_{32}, 41\}$
$\{U_{12}, U_{31}, 41\}$	$\{U_{13}, U_{30}, 41\}$	$\{U_{16}, U_{28}, 41\}$	$\{U_{18}, U_{27}, 41\}$
$\{U_{20}, U_{25}, 41\}$	$\{U_{21}, U_{24}, 41\}$		
$\{U_7, U_{35}, 42\}$	$\{U_{10}, U_{34}, 42\}$	$\{U_{14}, U_{33}, 42\}$	$\{U_{15}, U_{32}, 42\}$
$\{U_{17}, U_{31}, 42\}$	$\{U_{19}, U_{30}, 42\}$	$\{U_{20}, U_{29}, 42\}$	$\{U_{22}, U_{28}, 42\}$
$\{U_{23}, U_{27}, 42\}$	$\{U_{24}, U_{26}, 42\}$		

Table 4.6: Non-commuting pair of root subgroups U_l, U_m with root $l\tilde{+}m$

when we choose two weight-1 roots, say, l and m, such that $l\tilde{+}m$ is a weight 2 root. Also note that in this case the weight-2 root in A_2 subsystem can be any one of seven weight-2 roots of $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ depending on the choice of two weight-1 roots.

($\star\star$) Move ϵ_1 to the leftmost, and if a weight-2 term, say, ϵ_i, occurs, this can be moved to the rightmost since weight-2 terms commute with any other term by Proposition 2.3.5. Then move the ϵ_2 term to the left until it appears immediately after ϵ_1 term. Continue with this process until all terms corresponding to weight-1 roots are rearranged into the natural order.

Then reorder the weight-2 terms. This is easy since all weight-2 terms mutually commute by Proposition 2.3.5.

If $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)$, we must have $n \cdot u = u$. Comparing the arguments of ϵ_i, we get $a_i = a_{n^{-1}.i}$, so we have $a_i = a_j$. $\square$

Now, pick $a, a_\lambda \in k$ where $\lambda \in \{36, \cdots, 42\}$, and let

$$u = \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a)\right)\left(\prod_{\lambda \in O_{36}} \epsilon_\lambda(a_\lambda)\right).$$

By Proposition 2.3.7, for any pair of non-commuting 1-dimensional root subgroups, we have

$$\epsilon_l(a)\epsilon_m(a) = \epsilon_m(a)\epsilon_l(a)\epsilon_{l+m}(a^2).$$

Now, we compute

$$\begin{aligned}
n_\alpha \cdot u =& n_\alpha \cdot \left(\left(\prod_{\lambda \in \{1,\cdots,35\}} \epsilon_\lambda(a)\right)\epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\right.\\
&\left.\epsilon_{42}(a_{42})\right)\\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\alpha\cdot\lambda}(a)\right)\epsilon_{n_\alpha\cdot 36}(a_{36})\epsilon_{n_\alpha\cdot 37}(a_{37})\epsilon_{n_\alpha\cdot 38}(a_{38})\epsilon_{n_\alpha\cdot 39}(a_{39})\epsilon_{n_\alpha\cdot 40}(a_{40})\\
&\epsilon_{n_\alpha\cdot 41}(a_{41})\epsilon_{n_\alpha\cdot 42}(a_{42})\\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\alpha\cdot\lambda}(a)\right)\epsilon_{37}(a_{36})\epsilon_{36}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42})\\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a)\right)\epsilon_{36}(a_{37})\epsilon_{37}(a_{36})\epsilon_{38}(a_{38}+2a^2)\epsilon_{39}(a_{39})\epsilon_{40}(a_{40}+2a^2)\epsilon_{41}(a_{41})\\
&\epsilon_{42}(a_{42}+2a^2)\\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a)\right)\epsilon_{36}(a_{37})\epsilon_{37}(a_{36})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}).
\end{aligned}$$

The last equation holds since k is of characteristic 2. To get the second last equation, we have used Proposition 2.3.5, Proposition 2.3.7, and $(\star\star)$. We have $2a^2$ in the argument of ϵ_{38} because two non-commuting pairs contributing to the ϵ_{38} term, namely $\{\epsilon_{13}, \epsilon_{17}\}$ and $\{\epsilon_{14}, \epsilon_{16}\}$, have their order swapped by the action of n_α on u.

We calculate the action of $n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta$ on u in a similar way. Also, if $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)$, then we must have $u = n_\alpha \cdot u = n_\beta \cdot u = n_\gamma \cdot u = n_\delta \cdot u = n_\epsilon \cdot u = n_\eta \cdot u$. Comparing the coefficients of each term in each case, we get a system of simultaneous equations. From the calculation of the action of n_α on u we have

$$a_{36} = a_{37}. \tag{4.3}$$

We compute

$$\begin{aligned}
n_\beta \cdot u =& n_\beta \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\beta \cdot \lambda}(a) \right) \epsilon_{n_\beta \cdot 36}(a_{36})\epsilon_{n_\beta \cdot 37}(a_{37})\epsilon_{n_\beta \cdot 38}(a_{38})\epsilon_{n_\beta \cdot 39}(a_{39})\epsilon_{n_\beta \cdot 40}(a_{40}) \\
& \epsilon_{n_\beta \cdot 41}(a_{41})\epsilon_{n_\beta \cdot 42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\beta \cdot \lambda}(a) \right) \epsilon_{36}(a_{36})\epsilon_{38}(a_{37})\epsilon_{37}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36} + 2a^2)\epsilon_{37}(a_{38})\epsilon_{38}(a_{37})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41} + 2a^2) \\
& \epsilon_{42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{38})\epsilon_{38}(a_{37})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}).
\end{aligned}$$

We get

$$a_{37} = a_{38}. \tag{4.4}$$

$$
\begin{aligned}
n_\gamma \cdot u =& n_\gamma \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\gamma \cdot \lambda}(a) \right) \epsilon_{n_\gamma \cdot 36}(a_{36})\epsilon_{n_\gamma \cdot 37}(a_{37})\epsilon_{n_\gamma \cdot 38}(a_{38})\epsilon_{n_\gamma \cdot 39}(a_{39})\epsilon_{n_\gamma \cdot 40}(a_{40}) \\
& \epsilon_{n_\gamma \cdot 41}(a_{41})\epsilon_{n_\gamma \cdot 42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\gamma \cdot \lambda}(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{39}(a_{38})\epsilon_{38}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{39}+a^2)\epsilon_{39}(a_{38}+a^2)\epsilon_{40}(a_{40}+2a^2) \\
& \epsilon_{41}(a_{41})\epsilon_{42}(a_{42}+2a^2) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{39}+a^2)\epsilon_{39}(a_{38}+a^2)\epsilon_{40}(a_{40})\epsilon_{41}(a_{41}) \\
& \epsilon_{42}(a_{42}).
\end{aligned}
$$

We get

$$
a_{38} = a_{39} + a^2. \tag{4.5}
$$

$$
\begin{aligned}
n_\delta \cdot u =& n_\delta \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\delta \cdot \lambda}(a) \right) \epsilon_{n_\delta \cdot 36}(a_{36})\epsilon_{n_\delta \cdot 37}(a_{37})\epsilon_{n_\delta \cdot 38}(a_{38})\epsilon_{n_\delta \cdot 39}(a_{39})\epsilon_{n_\delta \cdot 40}(a_{40}) \\
& \epsilon_{n_\delta \cdot 41}(a_{41})\epsilon_{n_\delta \cdot 42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\delta \cdot \lambda}(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{40}(a_{39})\epsilon_{39}(a_{40})\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36}+2a^2)\epsilon_{37}(a_{37}+2a^2)\epsilon_{38}(a_{38}+2a^2)\epsilon_{39}(a_{40}+a^2) \\
& \epsilon_{40}(a_{39}+a^2)\epsilon_{41}(a_{41})\epsilon_{42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{40}+a^2)\epsilon_{40}(a_{39}+a^2)\epsilon_{41}(a_{41}) \\
& \epsilon_{42}(a_{42}).
\end{aligned}
$$

We get

$$a_{39} = a_{40} + a^2. \tag{4.6}$$

$$\begin{aligned}
n_\epsilon \cdot u =& n_\epsilon \cdot \left(\left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36}) \epsilon_{37}(a_{37}) \epsilon_{38}(a_{38}) \epsilon_{39}(a_{39}) \epsilon_{40}(a_{40}) \epsilon_{41}(a_{41}) \epsilon_{42}(a_{42}) \right) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\epsilon \cdot \lambda}(a) \right) \epsilon_{n_\epsilon \cdot 36}(a_{36}) \epsilon_{n_\epsilon \cdot 37}(a_{37}) \epsilon_{n_\epsilon \cdot 38}(a_{38}) \epsilon_{n_\epsilon \cdot 39}(a_{39}) \epsilon_{n_\epsilon \cdot 40}(a_{40}) \\
& \epsilon_{n_\epsilon \cdot 41}(a_{41}) \epsilon_{n_\epsilon \cdot 42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_{n_\epsilon \cdot \lambda}(a) \right) \epsilon_{36}(a_{36}) \epsilon_{37}(a_{37}) \epsilon_{38}(a_{38}) \epsilon_{39}(a_{39}) \epsilon_{41}(a_{40}) \epsilon_{40}(a_{41}) \epsilon_{42}(a_{42}) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36}) \epsilon_{37}(a_{37}) \epsilon_{38}(a_{38}) \epsilon_{39}(a_{39}) \epsilon_{40}(a_{41} + a^2) \epsilon_{41}(a_{40} + a^2) \\
& \epsilon_{42}(a_{42} + 2a^2) \\
=& \left(\prod_{\lambda \in O_1} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36}) \epsilon_{37}(a_{37}) \epsilon_{38}(a_{38}) \epsilon_{39}(a_{39}) \epsilon_{40}(a_{41} + a^2) \epsilon_{41}(a_{40} + a^2) \\
& \epsilon_{42}(a_{42}).
\end{aligned}$$

We get

$$a_{40} = a_{41} + a^2. \tag{4.7}$$

$$
\begin{aligned}
n_\eta \cdot u =& n_\eta \cdot \left(\left(\prod_{\lambda \in \{1,\cdots,35\}} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{41}) \right.\\
& \left. \epsilon_{42}(a_{42}) \right)\\
=& \left(\prod_{\lambda \in \{1,\cdots,35\}} \epsilon_{n_\eta \cdot \lambda}(a) \right) \epsilon_{n_\eta \cdot 36}(a_{36})\epsilon_{n_\eta \cdot 37}(a_{37})\epsilon_{n_\eta \cdot 38}(a_{38})\epsilon_{n_\eta \cdot 39}(a_{39})\epsilon_{n_\eta \cdot 40}(a_{40})\\
& \epsilon_{n_\eta \cdot 41}(a_{41})\epsilon_{n_\eta \cdot 42}(a_{42})\\
=& \left(\prod_{\lambda \in \{1,\cdots,35\}} \epsilon_{n_\eta \cdot \lambda}(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{42}(a_{41})\\
& \epsilon_{41}(a_{42})\\
=& \left(\prod_{\lambda \in \{1,\cdots,35\}} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36}+2a^2)\epsilon_{37}(a_{37}+2a^2)\epsilon_{38}(a_{38}+2a^2)\epsilon_{39}(a_{39})\\
& \epsilon_{40}(a_{40}+2a^2)\epsilon_{41}(a_{42}+a^2)\epsilon_{42}(a_{41}+a^2)\\
=& \left(\prod_{\lambda \in \{1,\cdots,35\}} \epsilon_\lambda(a) \right) \epsilon_{36}(a_{36})\epsilon_{37}(a_{37})\epsilon_{38}(a_{38})\epsilon_{39}(a_{39})\epsilon_{40}(a_{40})\epsilon_{41}(a_{42}+a^2)\\
& \epsilon_{42}(a_{41}+a^2).
\end{aligned}
$$

We get

$$
a_{41} = a_{42} + a^2. \tag{4.8}
$$

Since the set $\{n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon\, n_\eta\}$ generates H, equations (4.3), (4.4), (4.5), (4.6), (4.7), and (4.8) give

$$
\begin{aligned}
C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H) =& \{ (\prod_{\lambda \in O_1} \epsilon_\lambda(a))\epsilon_{36}(b)\epsilon_{37}(b)\epsilon_{38}(b)\epsilon_{39}(a^2+b)\epsilon_{40}(b)\epsilon_{41}(a^2+b) \\
& \epsilon_{42}(b) \mid a, b \in k \}.
\end{aligned} \tag{4.9}
$$

In (4.9), set $b = 0$, and differentiate with respect to a, and evaluate at $a = 0$, we get

$$
\sum_{\lambda \in O_1} e_\lambda \in \operatorname{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)).
$$

By Proposition 2.3.5 all weight-2 terms mutually commute , so by Lemma 2.3.4 we have

$$
\sum_{\lambda \in O_{36}} e_\lambda \in \operatorname{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)).
$$

So we have

$$\{a(\sum_{\lambda\in O_1} e_\lambda) + b(\sum_{\lambda\in O_{36}} e_\lambda) \mid a,b\in k\} \subseteq \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)).$$

We already know that

$$\mathrm{Lie}\,(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)) \subseteq \mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(H) = \{a(\sum_{\lambda\in\{1,\cdots,35\}} e_\lambda) + b(\sum_{\lambda\in\{36,\cdots,42\}} e_\lambda) \mid a,b\in k\}.$$

Therefore we have shown that

$$\mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)) = \mathfrak{c}_{\mathrm{Lie}(R_u(P\alpha\beta\gamma\delta\epsilon\eta))}(H).$$

Thus, we have the following.

Proposition 4.2.4. *There is no nilpotent witness to the G-nonseparability of H in* $\mathrm{Lie}\,(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$.

4.2.3 $G = E_8$ with A_7 Levi subgroup

Let G be a simple algebraic group of type E_8. Fix a maximal torus T of G. Pick a Borel subgroup B of G containing T. Let $\Sigma = \{\alpha,\beta,\gamma,\delta,\epsilon,\eta,\xi,\sigma\}$ be the set of simple roots of G. The next figure defines how each simple root of G corresponds to each node in the Dynkin diagram of E_8. From [FdV69, Appendix,

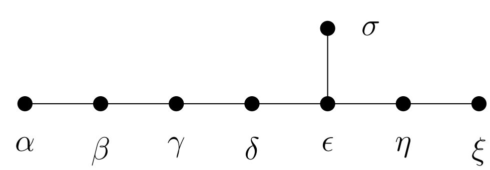


Figure 4.8: Dynkin diagram of E_8

Table B], we have the list of the coefficients of the simple roots for all positive roots of G. We label all positive roots of G in Table 4.7, and Table 4.8.

1	$\begin{smallmatrix}&&&&0&&\\1&0&0&0&0&0&0\end{smallmatrix}$	2	$\begin{smallmatrix}&&&&0&&\\0&1&0&0&0&0&0\end{smallmatrix}$	3	$\begin{smallmatrix}&&&&0&&\\0&0&1&0&0&0&0\end{smallmatrix}$	4	$\begin{smallmatrix}&&&&0&&\\0&0&0&1&0&0&0\end{smallmatrix}$
5	$\begin{smallmatrix}&&&&0&&\\0&0&0&0&1&0&0\end{smallmatrix}$	6	$\begin{smallmatrix}&&&&0&&\\0&0&0&0&0&1&0\end{smallmatrix}$	7	$\begin{smallmatrix}&&&&0&&\\0&0&0&0&0&0&1\end{smallmatrix}$	8	$\begin{smallmatrix}&&&&0&&\\1&1&0&0&0&0&0\end{smallmatrix}$
9	$\begin{smallmatrix}&&&&0&&\\0&1&1&0&0&0&0\end{smallmatrix}$	10	$\begin{smallmatrix}&&&&0&&\\0&0&1&1&0&0&0\end{smallmatrix}$	11	$\begin{smallmatrix}&&&&0&&\\0&0&0&1&1&0&0\end{smallmatrix}$	12	$\begin{smallmatrix}&&&&0&&\\0&0&0&0&1&1&0\end{smallmatrix}$
13	$\begin{smallmatrix}&&&&0&&\\0&0&0&0&0&1&1\end{smallmatrix}$	14	$\begin{smallmatrix}&&&&0&&\\1&1&1&0&0&0&0\end{smallmatrix}$	15	$\begin{smallmatrix}&&&&0&&\\0&1&1&1&0&0&0\end{smallmatrix}$	16	$\begin{smallmatrix}&&&&0&&\\0&0&1&1&1&0&0\end{smallmatrix}$
17	$\begin{smallmatrix}&&&&0&&\\0&0&0&1&1&1&0\end{smallmatrix}$	18	$\begin{smallmatrix}&&&&0&&\\0&0&0&0&1&1&1\end{smallmatrix}$	19	$\begin{smallmatrix}&&&&0&&\\1&1&1&1&0&0&0\end{smallmatrix}$	20	$\begin{smallmatrix}&&&&0&&\\0&1&1&1&1&0&0\end{smallmatrix}$
21	$\begin{smallmatrix}&&&&0&&\\0&0&1&1&1&1&0\end{smallmatrix}$	22	$\begin{smallmatrix}&&&&0&&\\0&0&0&1&1&1&1\end{smallmatrix}$	23	$\begin{smallmatrix}&&&&0&&\\1&1&1&1&1&0&0\end{smallmatrix}$	24	$\begin{smallmatrix}&&&&0&&\\0&1&1&1&1&1&0\end{smallmatrix}$
25	$\begin{smallmatrix}&&&&0&&\\0&0&1&1&1&1&1\end{smallmatrix}$	26	$\begin{smallmatrix}&&&&0&&\\1&1&1&1&1&1&0\end{smallmatrix}$	27	$\begin{smallmatrix}&&&&0&&\\0&1&1&1&1&1&1\end{smallmatrix}$	28	$\begin{smallmatrix}&&&&0&&\\1&1&1&1&1&1&1\end{smallmatrix}$

Table 4.7: The set of positive roots of $L_{\alpha\beta\gamma\delta\epsilon\eta\xi}$

Then the set of positive roots is

$$\Psi^+(G) = \{1, 2, \cdots, 120\}.$$

Let $L_{\alpha\beta\gamma\delta\epsilon\eta\xi}$ be the Levi subgroup of type A_7 corresponding to $\alpha, \cdots, \xi$. Then

$$\Psi^+(L_{\alpha\beta\gamma\delta\epsilon\eta\xi}) = \{1, \cdots, 28\}.$$

Let $P_{\alpha\beta\gamma\delta\epsilon\eta\xi} = \langle L_{\alpha\beta\gamma\delta\epsilon\eta\xi} \cup B\rangle$. Then $P_{\alpha\beta\gamma\delta\epsilon\eta\xi}$ is the parabolic subgroup of G with

$$\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})) = \{29, \cdots, 120\}.$$

Note that $\{29, \cdots, 84\}$, $\{85, \cdots, 112\}$, and $\{113, \cdots, 120\}$ are precisely the roots in E_8 such that the coefficient of σ is 1, 2, and 3 respectively. We call the roots of the first, second, and third type, *weight-1 roots*, *weight-2 roots*, and *weight-3 roots*, respectively. Define H by

$$H = \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta, n_\xi\rangle.$$

Let ζ_1, ζ_2 be simple roots of G. From the Cartan matrix of E_8 [Hum91, Sec. 11.4], we have

$$\langle \zeta_1, \zeta_2^\vee\rangle = \begin{cases} 2, & \text{if } \zeta_1 = \zeta_2. \\ -1, & \text{if the nodes corresponding to } \zeta_1, \zeta_2 \\ & \text{are adjacent to each other in the Dynkin diagram.} \\ 0, & \text{if the nodes corresponding to } \zeta_1, \zeta_2 \\ & \text{are NOT adjacent to each other in the Dynkin diagram,} \\ & \text{and } \zeta_1 \neq \zeta_2. \end{cases}$$

29	$\begin{smallmatrix}&&&&1&&\\0&0&0&0&0&0&0\end{smallmatrix}$	30	$\begin{smallmatrix}&&&&1&&\\0&0&0&0&1&0&0\end{smallmatrix}$	31	$\begin{smallmatrix}&&&&1&&\\0&0&0&1&1&0&0\end{smallmatrix}$	32	$\begin{smallmatrix}&&&&1&&\\0&0&0&0&1&1&0\end{smallmatrix}$
33	$\begin{smallmatrix}&&&&1&&\\0&0&1&1&1&0&0\end{smallmatrix}$	34	$\begin{smallmatrix}&&&&1&&\\0&0&0&1&1&1&0\end{smallmatrix}$	35	$\begin{smallmatrix}&&&&1&&\\0&0&0&0&1&1&1\end{smallmatrix}$	36	$\begin{smallmatrix}&&&&1&&\\0&1&1&1&1&0&0\end{smallmatrix}$
37	$\begin{smallmatrix}&&&&1&&\\0&0&1&1&1&1&0\end{smallmatrix}$	38	$\begin{smallmatrix}&&&&1&&\\0&0&0&1&2&1&0\end{smallmatrix}$	39	$\begin{smallmatrix}&&&&1&&\\0&0&0&1&1&1&1\end{smallmatrix}$	40	$\begin{smallmatrix}&&&&1&&\\1&1&1&1&1&0&0\end{smallmatrix}$
41	$\begin{smallmatrix}&&&&1&&\\0&1&1&1&1&1&0\end{smallmatrix}$	42	$\begin{smallmatrix}&&&&1&&\\0&0&1&1&2&1&0\end{smallmatrix}$	43	$\begin{smallmatrix}&&&&1&&\\0&0&1&1&1&1&1\end{smallmatrix}$	44	$\begin{smallmatrix}&&&&1&&\\0&0&0&1&2&1&1\end{smallmatrix}$
45	$\begin{smallmatrix}&&&&1&&\\1&1&1&1&1&1&0\end{smallmatrix}$	46	$\begin{smallmatrix}&&&&1&&\\0&1&1&1&2&1&0\end{smallmatrix}$	47	$\begin{smallmatrix}&&&&1&&\\0&1&1&1&1&1&1\end{smallmatrix}$	48	$\begin{smallmatrix}&&&&1&&\\0&0&1&2&2&1&0\end{smallmatrix}$
49	$\begin{smallmatrix}&&&&1&&\\0&0&1&1&2&1&1\end{smallmatrix}$	50	$\begin{smallmatrix}&&&&1&&\\0&0&0&1&2&2&1\end{smallmatrix}$	51	$\begin{smallmatrix}&&&&1&&\\1&1&1&1&2&1&0\end{smallmatrix}$	52	$\begin{smallmatrix}&&&&1&&\\1&1&1&1&1&1&1\end{smallmatrix}$
53	$\begin{smallmatrix}&&&&1&&\\0&1&1&2&2&1&0\end{smallmatrix}$	54	$\begin{smallmatrix}&&&&1&&\\0&1&1&1&2&1&1\end{smallmatrix}$	55	$\begin{smallmatrix}&&&&1&&\\0&0&1&2&2&1&1\end{smallmatrix}$	56	$\begin{smallmatrix}&&&&1&&\\0&0&1&1&2&2&1\end{smallmatrix}$
57	$\begin{smallmatrix}&&&&1&&\\1&1&1&2&2&1&0\end{smallmatrix}$	58	$\begin{smallmatrix}&&&&1&&\\1&1&1&1&2&1&1\end{smallmatrix}$	59	$\begin{smallmatrix}&&&&1&&\\0&1&2&2&2&1&0\end{smallmatrix}$	60	$\begin{smallmatrix}&&&&1&&\\0&1&1&2&2&1&1\end{smallmatrix}$
61	$\begin{smallmatrix}&&&&1&&\\0&1&1&1&2&2&1\end{smallmatrix}$	62	$\begin{smallmatrix}&&&&1&&\\0&0&1&2&2&2&1\end{smallmatrix}$	63	$\begin{smallmatrix}&&&&1&&\\1&1&2&2&2&1&0\end{smallmatrix}$	64	$\begin{smallmatrix}&&&&1&&\\1&1&1&2&2&1&1\end{smallmatrix}$
65	$\begin{smallmatrix}&&&&1&&\\1&1&1&1&2&2&1\end{smallmatrix}$	66	$\begin{smallmatrix}&&&&1&&\\0&1&2&2&2&1&1\end{smallmatrix}$	67	$\begin{smallmatrix}&&&&1&&\\0&1&1&2&2&2&1\end{smallmatrix}$	68	$\begin{smallmatrix}&&&&1&&\\0&0&1&2&3&2&1\end{smallmatrix}$
69	$\begin{smallmatrix}&&&&1&&\\1&2&2&2&2&1&0\end{smallmatrix}$	70	$\begin{smallmatrix}&&&&1&&\\1&1&2&2&2&1&1\end{smallmatrix}$	71	$\begin{smallmatrix}&&&&1&&\\1&1&1&2&2&2&1\end{smallmatrix}$	72	$\begin{smallmatrix}&&&&1&&\\0&1&2&2&2&2&1\end{smallmatrix}$
73	$\begin{smallmatrix}&&&&1&&\\0&1&1&2&3&2&1\end{smallmatrix}$	74	$\begin{smallmatrix}&&&&1&&\\1&2&2&2&2&1&1\end{smallmatrix}$	75	$\begin{smallmatrix}&&&&1&&\\1&1&2&2&2&2&1\end{smallmatrix}$	76	$\begin{smallmatrix}&&&&1&&\\1&1&1&2&3&2&1\end{smallmatrix}$
77	$\begin{smallmatrix}&&&&1&&\\0&1&2&2&3&2&1\end{smallmatrix}$	78	$\begin{smallmatrix}&&&&1&&\\1&2&2&2&2&2&1\end{smallmatrix}$	79	$\begin{smallmatrix}&&&&1&&\\1&1&2&2&3&2&1\end{smallmatrix}$	80	$\begin{smallmatrix}&&&&1&&\\0&1&2&3&3&2&1\end{smallmatrix}$
81	$\begin{smallmatrix}&&&&1&&\\1&2&2&2&3&2&1\end{smallmatrix}$	82	$\begin{smallmatrix}&&&&1&&\\1&1&2&3&3&2&1\end{smallmatrix}$	83	$\begin{smallmatrix}&&&&1&&\\1&2&2&3&3&2&1\end{smallmatrix}$	84	$\begin{smallmatrix}&&&&1&&\\1&2&3&3&3&2&1\end{smallmatrix}$
85	$\begin{smallmatrix}&&&&2&&\\0&0&1&2&3&2&1\end{smallmatrix}$	86	$\begin{smallmatrix}&&&&2&&\\0&1&1&2&3&2&1\end{smallmatrix}$	87	$\begin{smallmatrix}&&&&2&&\\1&1&1&2&3&2&1\end{smallmatrix}$	88	$\begin{smallmatrix}&&&&2&&\\0&1&2&2&3&2&1\end{smallmatrix}$
89	$\begin{smallmatrix}&&&&2&&\\1&1&2&2&3&2&1\end{smallmatrix}$	90	$\begin{smallmatrix}&&&&2&&\\0&1&2&3&3&2&1\end{smallmatrix}$	91	$\begin{smallmatrix}&&&&2&&\\1&2&2&2&3&2&1\end{smallmatrix}$	92	$\begin{smallmatrix}&&&&2&&\\1&1&2&3&3&2&1\end{smallmatrix}$
93	$\begin{smallmatrix}&&&&2&&\\0&1&2&3&4&2&1\end{smallmatrix}$	94	$\begin{smallmatrix}&&&&2&&\\1&2&2&3&3&2&1\end{smallmatrix}$	95	$\begin{smallmatrix}&&&&2&&\\1&1&2&3&4&2&1\end{smallmatrix}$	96	$\begin{smallmatrix}&&&&2&&\\0&1&2&3&4&3&1\end{smallmatrix}$
97	$\begin{smallmatrix}&&&&2&&\\1&2&3&3&3&2&1\end{smallmatrix}$	98	$\begin{smallmatrix}&&&&2&&\\1&2&2&3&4&2&1\end{smallmatrix}$	99	$\begin{smallmatrix}&&&&2&&\\1&1&2&3&4&3&1\end{smallmatrix}$	100	$\begin{smallmatrix}&&&&2&&\\0&1&2&3&4&3&2\end{smallmatrix}$
101	$\begin{smallmatrix}&&&&2&&\\1&2&3&3&4&2&1\end{smallmatrix}$	102	$\begin{smallmatrix}&&&&2&&\\1&2&2&3&4&3&1\end{smallmatrix}$	103	$\begin{smallmatrix}&&&&2&&\\1&1&2&3&4&3&2\end{smallmatrix}$	104	$\begin{smallmatrix}&&&&2&&\\1&2&3&4&4&2&1\end{smallmatrix}$
105	$\begin{smallmatrix}&&&&2&&\\1&2&3&3&4&3&1\end{smallmatrix}$	106	$\begin{smallmatrix}&&&&2&&\\1&2&2&3&4&3&2\end{smallmatrix}$	107	$\begin{smallmatrix}&&&&2&&\\1&2&3&4&4&3&1\end{smallmatrix}$	108	$\begin{smallmatrix}&&&&2&&\\1&2&3&3&4&3&2\end{smallmatrix}$
109	$\begin{smallmatrix}&&&&2&&\\1&2&3&4&5&3&1\end{smallmatrix}$	110	$\begin{smallmatrix}&&&&2&&\\1&2&3&4&4&3&2\end{smallmatrix}$	111	$\begin{smallmatrix}&&&&2&&\\1&2&3&4&5&3&2\end{smallmatrix}$	112	$\begin{smallmatrix}&&&&2&&\\1&2&3&4&5&4&2\end{smallmatrix}$
113	$\begin{smallmatrix}&&&&3&&\\1&2&3&4&5&3&1\end{smallmatrix}$	114	$\begin{smallmatrix}&&&&3&&\\1&2&3&4&5&3&2\end{smallmatrix}$	115	$\begin{smallmatrix}&&&&3&&\\1&2&3&4&5&4&2\end{smallmatrix}$	116	$\begin{smallmatrix}&&&&3&&\\1&2&3&4&6&4&2\end{smallmatrix}$
117	$\begin{smallmatrix}&&&&3&&\\1&2&3&5&6&4&2\end{smallmatrix}$	118	$\begin{smallmatrix}&&&&3&&\\1&2&4&5&6&4&2\end{smallmatrix}$	119	$\begin{smallmatrix}&&&&3&&\\1&3&4&5&6&4&2\end{smallmatrix}$	120	$\begin{smallmatrix}&&&&3&&\\2&3&4&5&6&4&2\end{smallmatrix}$

Table 4.8: The set of positive roots of $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})$

From this, it is not difficult to calculate $\langle \xi, \zeta^\vee \rangle$ for all $\xi \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi}))$ and $\zeta \in \Sigma$. Then, these calculations show how n_α, n_β, n_γ, n_δ, n_ϵ, n_η, and n_ξ act on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})) = \{29, \cdots, 120\}$. Let π be the corresponding representation of H on $Sym(\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi}))) \cong S_{92}$. Then we get Table 4.9 It is not difficult to

$$
\begin{aligned}
\pi(n_\alpha) \quad = \quad & (36\ 40)(41\ 45)(46\ 51)(47\ 52)(53\ 57)(54\ 58)(59\ 63)(60\ 64)(61\ 65) \\
& (66\ 70)(67\ 71)(72\ 75)(73\ 76)(77\ 79)(80\ 82)(86\ 87)(88\ 89)(90\ 92) \\
& (93\ 95)(96\ 99)(100\ 103)(119\ 120) \\
\\
\pi(n_\beta) \quad = \quad & (33\ 36)(37\ 41)(42\ 46)(43\ 47)(48\ 53)(49\ 54)(55\ 60)(56\ 61)(62\ 67) \\
& (63\ 69)(68\ 73)(70\ 74)(75\ 78)(79\ 81)(82\ 83)(85\ 86)(89\ 91)(92\ 94) \\
& (95\ 98)(99\ 102)(103\ 106)(118\ 119) \\
\\
\pi(n_\gamma) \quad = \quad & (31\ 33)(34\ 37)(38\ 42)(39\ 43)(44\ 49)(50\ 56)(53\ 59)(57\ 63)(60\ 66) \\
& (64\ 70)(67\ 72)(71\ 75)(73\ 77)(76\ 79)(83\ 84)(86\ 88)(87\ 89)(94\ 97) \\
& (98\ 101)(102\ 105)(106\ 108)(117\ 118) \\
\\
\pi(n_\delta) \quad = \quad & (30\ 31)(32\ 34)(35\ 39)(42\ 48)(46\ 53)(49\ 55)(51\ 57)(54\ 60)(56\ 62) \\
& (58\ 64)(61\ 67)(65\ 71)(77\ 80)(79\ 82)(81\ 83)(88\ 90)(89\ 92)(91\ 94) \\
& (101\ 104)(105\ 107)(108\ 110)(116\ 117) \\
\\
\pi(n_\epsilon) \quad = \quad & (29\ 30)(34\ 38)(37\ 42)(39\ 44)(41\ 46)(43\ 49)(45\ 51)(47\ 54)(52\ 58) \\
& (62\ 68)(67\ 73)(71\ 76)(72\ 77)(75\ 79)(78\ 81)(90\ 93)(92\ 95)(94\ 98) \\
& (97\ 101)(107\ 109)(110\ 111)(115\ 116) \\
\\
\pi(n_\eta) \quad = \quad & (30\ 32)(31\ 34)(33\ 37)(36\ 41)(40\ 45)(44\ 50)(49\ 56)(54\ 61)(55\ 62) \\
& (58\ 65)(60\ 67)(64\ 71)(66\ 72)(70\ 75)(74\ 78)(93\ 96)(95\ 99)(98\ 102) \\
& (101\ 105)(104\ 107)(111\ 112)(114\ 115) \\
\\
\pi(n_\xi) \quad = \quad & (32\ 35)(34\ 39)(37\ 43)(38\ 44)(41\ 47)(42\ 49)(45\ 52)(46\ 54)(48\ 55) \\
& (51\ 58)(53\ 60)(57\ 64)(59\ 66)(63\ 70)(69\ 74)(96\ 100)(99\ 103) \\
& (102\ 106)(105\ 108)(107\ 110)(109\ 111)(113\ 114)
\end{aligned}
$$

Table 4.9: Action of n_τ's on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi}))$

see that H has three orbits,

$$
\begin{aligned}
O_{29} &= \{29, \cdots, 84\}, \\
O_{85} &= \{85, \cdots, 112\}, \\
O_{113} &= \{113, \cdots, 120\}.
\end{aligned}
$$

Then by Corollary 2.3.3, we have

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi}))}(H) = \{a(\sum_{\lambda\in O_{29}} e_\lambda) + b(\sum_{\lambda\in O_{85}} e_\lambda) + c(\sum_{\lambda\in O_{113}} e_\lambda) \\ \mid a, b, c \in k\}.$$

Now, we calculate $\mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H))$. Fix an ordering of $\Psi(R_u(P))$ in the given order. By a similar argument as in the E_6 and E_7 cases, it is not difficult to establish the following claim.

Claim 4.2.5. *Any element u in $C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)$ can be expressed uniquely as*

$$u = \left(\prod_{\lambda\in O_{29}} \epsilon_\lambda(a)\right)\left(\prod_{\lambda\in O_{85}\cup O_{113}} \epsilon_\lambda(a_\lambda)\right), \text{ where } a, a_\lambda \in k.$$

Now, pick $a, a_\lambda \in k$ for $\lambda \in \{85, \cdots, 120\}$, and let

$$u = \left(\prod_{\lambda\in O_{29}} \epsilon_\lambda(a)\right)\left(\prod_{\lambda\in O_{85}\cup O_{113}} \epsilon_\lambda(a_\lambda)\right).$$

Then we have

$$n_\alpha \cdot u = \left(n_\alpha \cdot \prod_{\lambda\in O_{29}} (\epsilon_\lambda(a))\right)\left(n_\alpha \cdot \prod_{\lambda\in O_{85}\cup O_{113}} \epsilon_\lambda(a_\lambda)\right).$$

We use Maple to reorder $n_\zeta \cdot \prod_{\lambda\in O_{29}} \epsilon_\lambda(a)$ for $\zeta \in \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \xi\}$ (see Maple output in Appendix A). We have

$$
\begin{aligned}
n_\alpha \cdot u = &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{86}(a^2)\epsilon_{87}(a^2)\epsilon_{90}(a^2)\epsilon_{91}(2a^2)\epsilon_{92}(a^2)\epsilon_{96}(a^2)\epsilon_{98}(2a^2)\epsilon_{99}(a^2) \\
&\epsilon_{104}(2a^2)\epsilon_{105}(2a^2)\epsilon_{106}(2a^2)\epsilon_{110}(2a^2)\epsilon_{112}(2a^2)\epsilon_{113}(4a^3)\epsilon_{114}(6a^3)\epsilon_{115}(6a^2) \\
&\epsilon_{116}(6a^2)\epsilon_{117}(8a^2)\epsilon_{118}(4a^2)\epsilon_{119}(10a^2)\epsilon_{120}(10a^2)\epsilon_{85}(a_{85})\epsilon_{86}(a_{87})\epsilon_{87}(a_{86}) \\
&\epsilon_{88}(a_{89})\epsilon_{89}(a_{88})\epsilon_{90}(a_{92})\epsilon_{91}(a_{91})\epsilon_{92}(a_{90})\epsilon_{93}(a_{95})\epsilon_{94}(a_{94})\epsilon_{95}(a_{93})\epsilon_{96}(a_{99}) \\
&\epsilon_{97}(a_{97})\epsilon_{98}(a_{98})\epsilon_{99}(a_{96})\epsilon_{100}(a_{103})\epsilon_{101}(a_{101})\epsilon_{102}(a_{102})\epsilon_{103}(a_{100})\epsilon_{104}(a_{104}) \\
&\epsilon_{105}(a_{105})\epsilon_{106}(a_{106})\epsilon_{107}(a_{107})\epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(a_{110})\epsilon_{111}(a_{111}) \\
&\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{115})\epsilon_{116}(a_{116})\epsilon_{117}(a_{117})\epsilon_{118}(a_{118}) \\
&\epsilon_{119}(a_{120})\epsilon_{120}(a_{119}) \\
= &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a^2 + a_{87})\epsilon_{87}(a^2 + a_{86})\epsilon_{88}(a_{89})\epsilon_{89}(a_{88}) \\
&\epsilon_{90}(a^2 + a_{92})\epsilon_{91}(2a^2 + a_{91})\epsilon_{92}(a^2 + a_{90})\epsilon_{93}(a_{95})\epsilon_{94}(a_{94})\epsilon_{95}(a_{93}) \\
&\epsilon_{96}(a^2 + a_{99})\epsilon_{97}(a_{97})\epsilon_{98}(2a^2 + a_{98})\epsilon_{99}(a^2 + a_{96})\epsilon_{100}(a_{103})\epsilon_{101}(a_{101}) \\
&\epsilon_{102}(a_{102})\epsilon_{103}(a_{100})\epsilon_{104}(2a^2 + a_{104})\epsilon_{105}(2a^2 + a_{105})\epsilon_{106}(2a^2 + a_{106}) \\
&\epsilon_{107}(a_{107})\epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(2a^2 + a_{110})\epsilon_{111}(a_{111})\epsilon_{112}(2a^2 + a_{112}) \\
&\epsilon_{113}(4a^3 + a_{113})\epsilon_{114}(6a^3 + a_{114})\epsilon_{115}(6a^2 + a_{115})\epsilon_{116}(6a^2 + a_{116}) \\
&\epsilon_{117}(8a^2 + a_{117})\epsilon_{118}(4a^2 + a_{118})\epsilon_{119}(10a^2 + a_{120})\epsilon_{120}(10a^2 + a_{119}) \\
= &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a^2 + a_{87})\epsilon_{87}(a^2 + a_{86})\epsilon_{88}(a_{89})\epsilon_{89}(a_{88}) \\
&\epsilon_{90}(a^2 + a_{92})\epsilon_{91}(a_{91})\epsilon_{92}(a^2 + a_{90})\epsilon_{93}(a_{95})\epsilon_{94}(a_{94})\epsilon_{95}(a_{93})\epsilon_{96}(a^2 + a_{99}) \\
&\epsilon_{97}(a_{97})\epsilon_{98}(a_{98})\epsilon_{99}(a^2 + a_{96})\epsilon_{100}(a_{103})\epsilon_{101}(a_{101})\epsilon_{102}(a_{102})\epsilon_{103}(a_{100}) \\
&\epsilon_{104}(a_{104})\epsilon_{105}(a_{105})\epsilon_{106}(a_{106})\epsilon_{107}(a_{107})\epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(a_{110}) \\
&\epsilon_{111}(a_{111})\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{115})\epsilon_{116}(a_{116})\epsilon_{117}(a_{117}) \\
&\epsilon_{118}(a_{118})\epsilon_{119}(a_{120})\epsilon_{120}(a_{119}).
\end{aligned}
$$

We have used Proposition 2.3.5, Proposition 2.3.7, and a similar argument to ($\star\star$) in the E_7 case. Note that in this case, we see some cubic terms a^3 in the arguments of ϵ_λ during our calculation. We explain why that happens, and this is one of the reasons why calculations for this case are much more complicated than those from the E_6 and E_7 cases. Suppose we have some $n \in H$, $m_1, m_2, m_3 \in \{29, \cdots, 84\}$ such that $m_1 < m_2 < m_3$, $n \cdot m_2 < n \cdot m_1$ and $n \cdot m_3 < n \cdot m_1 \tilde{+} n \cdot m_2$ with

$[U_{n\cdot m_1}, U_{n\cdot m_2}] \neq 0$ and $[U_{n\cdot m_1+n\cdot m_2}, U_{n\cdot m_3}] \neq 0$. Then we have

$$\begin{aligned}
n\cdot\prod_{\lambda\in O_{29}}\epsilon_\lambda(a) =&\epsilon_{n\cdot 1}(a)\epsilon_{n\cdot 2}(a)\cdots\epsilon_{n\cdot m_1}(a)\cdots\epsilon_{n\cdot m_2}(a)\cdots\epsilon_{n\cdot m_3}(a)\cdots\epsilon_{n\cdot 83}(a)\epsilon_{n\cdot 84}(a)\\
=&\epsilon_{n\cdot 1}(a)\epsilon_{n\cdot 2}(a)\cdots\epsilon_{n\cdot m_2}(a)\cdots\epsilon_{n\cdot m_1}(a)\cdots\epsilon_{n\cdot m_1\tilde{+}n\cdot m_2}(a^2)\\
&\cdots\epsilon_{n\cdot m_3}(a)\cdots\epsilon_{n\cdot 83}(a)\epsilon_{n\cdot 84}(a)\\
=&\epsilon_{n\cdot 1}\epsilon_{n\cdot 2}\cdots\epsilon_{n\cdot m_2}(a)\cdots\epsilon_{n\cdot m_1}(a)\cdots\epsilon_{n\cdot m_3}(a)\cdots\epsilon_{n\cdot l\tilde{+}n\cdot m}(a^2)\\
&\cdots\epsilon_{n\cdot m_1\tilde{+}n\cdot m_2\tilde{+}n\cdot m_3}(a^3)\cdots\epsilon_{n\cdot 83}(a)\epsilon_{n\cdot 84}(a)\\
=&\cdots.
\end{aligned}$$

If $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)$, we must have $n_\alpha \cdot u = u$. Equating variables in each term, we get

$$\begin{gathered}
a_{86} = a_{87} + a^2, a_{88} = a_{89}, a_{90} = a_{92} + a^2, a_{93} = a_{95}, a_{96} = a_{99} + a^2,\\
a_{100} = a_{103}, a_{119} = a_{120}.
\end{gathered}\tag{4.10}$$

Similarly, we calculate $n_\lambda \cdot u$ for $\lambda \in \{\beta, \gamma, \delta, \epsilon, \eta, \xi\}$.

$$
\begin{aligned}
n_\beta \cdot u = & \left(\prod_{\lambda \in \{29, \cdots, 84\}} \epsilon_\lambda(a) \right) \epsilon_{88}(2a^2)\epsilon_{92}(a^2)\epsilon_{93}(2a^2)\epsilon_{94}(a^2)\epsilon_{99}(a^2)\epsilon_{100}(2a^2)\epsilon_{102}(a^2) \\
& \epsilon_{107}(2a^2)\epsilon_{109}(2a^2)\epsilon_{111}(2a^2)\epsilon_{113}(2a^3)\epsilon_{114}(4a^3)\epsilon_{115}(4a^3)\epsilon_{116}(4a^3)\epsilon_{117}(6a^3) \\
& \epsilon_{118}(8a^3)\epsilon_{119}(8a^3)\epsilon_{120}(4a^3)\epsilon_{85}(a_{86})\epsilon_{86}(a_{85})\epsilon_{87}(a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{91})\epsilon_{90}(a_{90}) \\
& \epsilon_{91}(a_{89})\epsilon_{92}(a_{94})\epsilon_{93}(a_{93})\epsilon_{94}(a_{92})\epsilon_{95}(a_{98})\epsilon_{96}(a_{96})\epsilon_{97}(a_{97})\epsilon_{98}(a_{95})\epsilon_{99}(a_{102}) \\
& \epsilon_{100}(a_{100})\epsilon_{101}(a_{101})\epsilon_{102}(a_{99})\epsilon_{103}(a_{106})\epsilon_{104}(a_{104})\epsilon_{105}(a_{105})\epsilon_{106}(a_{103})\epsilon_{107}(a_{107}) \\
& \epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(a_{110})\epsilon_{111}(a_{111})\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114}) \\
& \epsilon_{115}(a_{115})\epsilon_{116}(a_{116})\epsilon_{117}(a_{117})\epsilon_{118}(a_{119})\epsilon_{119}(a_{118})\epsilon_{120}(a_{120}) \\
= & \left(\prod_{\lambda \in \{29, \cdots, 84\}} \epsilon_\lambda(a) \right) \epsilon_{85}(a_{86})\epsilon_{86}(a_{85})\epsilon_{87}(a_{87})\epsilon_{88}(2a^2 + a_{88})\epsilon_{89}(a_{91})\epsilon_{90}(a_{90}) \\
& \epsilon_{91}(a_{89})\epsilon_{92}(a^2 + a_{94})\epsilon_{93}(2a^2 + a_{93})\epsilon_{94}(a^2 + a_{92})\epsilon_{95}(a_{98})\epsilon_{96}(a_{96})\epsilon_{97}(a_{97}) \\
& \epsilon_{98}(a_{95})\epsilon_{99}(a^2 + a_{102})\epsilon_{100}(a^2 + a_{100})\epsilon_{101}(a_{101})\epsilon_{102}(a^2 + a_{99})\epsilon_{103}(a_{106}) \\
& \epsilon_{104}(a_{104})\epsilon_{105}(a_{105})\epsilon_{106}(a_{103})\epsilon_{107}(2a^2 + a_{107})\epsilon_{108}(a_{108})\epsilon_{109}(2a^2 + a_{109}) \\
& \epsilon_{110}(a_{110})\epsilon_{111}(2a^2 + a_{111})\epsilon_{112}(a_{112})\epsilon_{113}(2a^3 + a_{113})\epsilon_{114}(4a^3 + a_{114}) \\
& \epsilon_{115}(4a^3 + a_{115})\epsilon_{116}(4a^3 + a_{116})\epsilon_{117}(6a^3 + a_{117})\epsilon_{118}(8a^3 + a_{119}) \\
& \epsilon_{119}(8a^3 + a_{118})\epsilon_{120}(4a^3 + a_{120}) \\
= & \left(\prod_{\lambda \in \{29, \cdots, 84\}} \epsilon_\lambda(a) \right) \epsilon_{85}(a_{86})\epsilon_{86}(a_{85})\epsilon_{87}(a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{91})\epsilon_{90}(a_{90}) \\
& \epsilon_{91}(a_{89})\epsilon_{92}(a^2 + a_{94})\epsilon_{93}(a_{93})\epsilon_{94}(a^2 + a_{92})\epsilon_{95}(a_{98})\epsilon_{96}(a_{96})\epsilon_{97}(a_{97})\epsilon_{98}(a_{95}) \\
& \epsilon_{99}(a^2 + a_{102})\epsilon_{100}(a_{100})\epsilon_{101}(a_{101})\epsilon_{102}(a^2 + a_{99})\epsilon_{103}(a_{106})\epsilon_{104}(a_{104})\epsilon_{105}(a_{105}) \\
& \epsilon_{106}(a_{103})\epsilon_{107}(a_{107})\epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(a_{110})\epsilon_{111}(a_{111})\epsilon_{112}(a_{112}) \\
& \epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{115})\epsilon_{116}(a_{116})\epsilon_{117}(a_{117})\epsilon_{118}(a_{119})\epsilon_{119}(a_{118}) \\
& \epsilon_{120}(a_{120}).
\end{aligned}
$$

Then $n_\beta \cdot u = u$ gives

$$
\begin{gathered}
a_{85} = a_{86}, a_{89} = a_{91}, a_{92} = a^2 + a_{94}, a_{95} = a_{98}, a_{99} = a^2 + a_{102}, \\
a_{103} = a_{106}, a_{118} = a_{119}.
\end{gathered} \tag{4.11}
$$

$$
\begin{aligned}
n_\gamma \cdot u = &\left(\prod_{\lambda\in\{29,\cdots,84\}} \epsilon_\lambda(a)\right) \epsilon_{85}(2a^2)\epsilon_{87}(a^2)\epsilon_{89}(a^2)\epsilon_{96}(2a^2)\epsilon_{98}(a^2)\epsilon_{101}(a^2)\epsilon_{102}(a^2) \\
&\epsilon_{104}(2a^2)\epsilon_{105}(a^2)\epsilon_{106}(a^2)\epsilon_{108}(a^2)\epsilon_{110}(2a^2)\epsilon_{112}(2a^2)\epsilon_{113}(4a^3)\epsilon_{114}(4a^3) \\
&\epsilon_{115}(4a^3)\epsilon_{116}(6a^3)\epsilon_{117}(6a^3)\epsilon_{118}(6a^3)\epsilon_{119}(6a^3)\epsilon_{120}(8a^3)\epsilon_{85}(a_{85})\epsilon_{86}(a_{88}) \\
&\epsilon_{87}(a_{89})\epsilon_{88}(a_{86})\epsilon_{89}(a_{87})\epsilon_{90}(a_{90})\epsilon_{91}(a_{91})\epsilon_{92}(a_{92})\epsilon_{93}(a_{93})\epsilon_{94}(a_{97})\epsilon_{95}(a_{95}) \\
&\epsilon_{96}(a_{96})\epsilon_{97}(a_{94})\epsilon_{98}(a_{101})\epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a_{98})\epsilon_{102}(a_{105})\epsilon_{103}(a_{103}) \\
&\epsilon_{104}(a_{104})\epsilon_{105}(a_{102})\epsilon_{106}(a_{108})\epsilon_{107}(a_{107})\epsilon_{108}(a_{106})\epsilon_{109}(a_{109})\epsilon_{110}(a_{110}) \\
&\epsilon_{111}(a_{111})\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{115})\epsilon_{116}(a_{116})\epsilon_{117}(a_{118}) \\
&\epsilon_{118}(a_{117})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}) \\
= &\left(\prod_{\lambda\in\{29,\cdots,84\}} \epsilon_\lambda(a)\right) \epsilon_{85}(2a^2+a_{85})\epsilon_{86}(a_{88})\epsilon_{87}(a^2+a_{89})\epsilon_{88}(a_{86}) \\
&\epsilon_{89}(a^2+a_{87})\epsilon_{90}(a_{90})\epsilon_{91}(a_{91})\epsilon_{92}(a_{92})\epsilon_{93}(a_{93})\epsilon_{94}(a_{97})\epsilon_{95}(a_{95})\epsilon_{96}(2a^2+a_{96}) \\
&\epsilon_{97}(a_{94})\epsilon_{98}(a^2+a_{101})\epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a^2+a_{98})\epsilon_{102}(a^2+a_{105}) \\
&\epsilon_{103}(a_{103})\epsilon_{104}(2a^2+a_{104})\epsilon_{105}(a^2+a_{102})\epsilon_{106}(a^2+a_{108})\epsilon_{107}(a_{107}) \\
&\epsilon_{108}(a^2+a_{106})\epsilon_{109}(a_{109})\epsilon_{110}(2a^2+a_{110})\epsilon_{111}(a_{111})\epsilon_{112}(2a^2+a_{112}) \\
&\epsilon_{113}(4a^3+a_{113})\epsilon_{114}(4a^3+a_{114})\epsilon_{115}(4a^3+a_{115})\epsilon_{116}(6a^3+a_{116}) \\
&\epsilon_{117}(6a^3+a_{118})\epsilon_{118}(6a^3+a_{117})\epsilon_{119}(6a^3+a_{119})\epsilon_{120}(8a^3+a_{120}) \\
= &\left(\prod_{\lambda\in\{29,\cdots,84\}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a_{88})\epsilon_{87}(a^2+a_{89})\epsilon_{88}(a_{86})\epsilon_{89}(a^2+a_{87}) \\
&\epsilon_{90}(a_{90})\epsilon_{91}(a_{91})\epsilon_{92}(a_{92})\epsilon_{93}(a_{93})\epsilon_{94}(a_{97})\epsilon_{95}(a_{95})\epsilon_{96}(a_{96})\epsilon_{97}(a_{94}) \\
&\epsilon_{98}(a^2+a_{101})\epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a^2+a_{98})\epsilon_{102}(a^2+a_{105})\epsilon_{103}(a_{103}) \\
&\epsilon_{104}(a_{104})\epsilon_{105}(a^2+a_{102})\epsilon_{106}(a^2+a_{108})\epsilon_{107}(a_{107})\epsilon_{108}(a^2+a_{106})\epsilon_{109}(a_{109}) \\
&\epsilon_{110}(a_{110})\epsilon_{111}(a_{111})\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{115})\epsilon_{116}(a_{116}) \\
&\epsilon_{117}(a_{118})\epsilon_{118}(a_{117})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}).
\end{aligned}
$$

Then $n_\gamma \cdot u = u$ gives

$$
\begin{gathered}
a_{86} = a_{88}, a_{87} + a^2 = a_{89}, a_{94} = a_{97}, a_{98} + a^2 = a_{101}, a_{102} + a^2 = a_{105}, \\
a_{106} + a^2 = a_{108}, a_{117} = a_{118}.
\end{gathered} \tag{4.12}
$$

Next, we calculate

$$
\begin{aligned}
n_\delta \cdot u = & \left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{88}(a^2)\epsilon_{90}(a^2)\epsilon_{91}(a^2)\epsilon_{93}(2a^2)\epsilon_{94}(a^2)\epsilon_{95}(2a^2)\epsilon_{98}(2a^2)\epsilon_{99}(2a^2) \\
& \epsilon_{100}(2a^2)\epsilon_{101}(a^2)\epsilon_{103}(2a^2)\epsilon_{104}(a^2)\epsilon_{105}(a^2)\epsilon_{106}(2a^2)\epsilon_{107}(a^2)\epsilon_{108}(a^2)\epsilon_{110}(a^2) \\
& \epsilon_{113}(2a^3)\epsilon_{114}(4a^3)\epsilon_{115}(4a^3)\epsilon_{116}(9a^3)\epsilon_{117}(9a^3)\epsilon_{118}(12a^3)\epsilon_{119}(12a^3)\epsilon_{120}(14a^3) \\
& \epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a_{90})\epsilon_{89}(a_{92})\epsilon_{90}(a_{88})\epsilon_{91}(a_{94})\epsilon_{92}(a_{89})\epsilon_{93}(a_{93}) \\
& \epsilon_{94}(a_{91})\epsilon_{95}(a_{95})\epsilon_{96}(a_{96})\epsilon_{97}(a_{97})\epsilon_{98}(a_{98})\epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a_{104})\epsilon_{102}(a_{102}) \\
& \epsilon_{103}(a_{103})\epsilon_{104}(a_{101})\epsilon_{105}(a_{107})\epsilon_{106}(a_{106})\epsilon_{107}(a_{105})\epsilon_{108}(a_{110})\epsilon_{109}(a_{109}) \\
& \epsilon_{110}(a_{108})\epsilon_{111}(a_{111})\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{115})\epsilon_{116}(a_{117}) \\
& \epsilon_{117}(a_{116})\epsilon_{118}(a_{118})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}) \\
= & \left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a^2 + a_{90})\epsilon_{89}(a_{92})\epsilon_{90}(a^2 + a_{88}) \\
& \epsilon_{91}(a^2 + a_{94})\epsilon_{92}(a_{89})\epsilon_{93}(2a^2 + a_{93})\epsilon_{94}(a^2 + a_{91})\epsilon_{95}(2a^2 + a_{95})\epsilon_{96}(a_{96}) \\
& \epsilon_{97}(a_{97})\epsilon_{98}(2a^2 + a_{98})\epsilon_{99}(2a^2 + a_{99})\epsilon_{100}(2a^2 + a_{100})\epsilon_{101}(a^2 + a_{104}) \\
& \epsilon_{102}(a_{102})\epsilon_{103}(2a^2 + a_{103})\epsilon_{104}(a^2 + a_{101})\epsilon_{105}(a^2 + a_{107})\epsilon_{106}(2a^2 + a_{106}) \\
& \epsilon_{107}(a^2 + a_{105})\epsilon_{108}(a^2 + a_{110})\epsilon_{109}(a_{109})\epsilon_{110}(a^2 + a_{108})\epsilon_{111}(a_{111})\epsilon_{112}(a_{112}) \\
& \epsilon_{113}(2a^3 + a_{113})\epsilon_{114}(4a^3 + a_{114})\epsilon_{115}(4a^3 + a_{115})\epsilon_{116}(9a^3 + a_{117}) \\
& \epsilon_{117}(9a^3 + a_{116})\epsilon_{118}(12a^3 + a_{118})\epsilon_{119}(12a^3 + a_{119})\epsilon_{120}(14a^3 + a_{120}) \\
= & \left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a^2 + a_{90})\epsilon_{89}(a_{92})\epsilon_{90}(a^2 + a_{88}) \\
& \epsilon_{91}(a^2 + a_{94})\epsilon_{92}(a_{89})\epsilon_{93}(a_{93})\epsilon_{94}(a^2 + a_{91})\epsilon_{95}(a_{95})\epsilon_{96}(a_{96})\epsilon_{97}(a_{97})\epsilon_{98}(a_{98}) \\
& \epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a^2 + a_{104})\epsilon_{102}(a_{102})\epsilon_{103}(a_{103})\epsilon_{104}(a^2 + a_{101}) \\
& \epsilon_{105}(a^2 + a_{107})\epsilon_{106}(a_{106})\epsilon_{107}(a^2 + a_{105})\epsilon_{108}(a^2 + a_{110})\epsilon_{109}(a_{109}) \\
& \epsilon_{110}(a^2 + a_{108})\epsilon_{111}(a_{111})\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{115}) \\
& \epsilon_{116}(a^3 + a_{117})\epsilon_{117}(a^3 + a_{116})\epsilon_{118}(a_{118})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}).
\end{aligned}
$$

Then $n_\delta \cdot u = u$ gives

$$
\begin{gathered}
a_{88} + a^2 = a_{90}, a_{89} = a_{92}, a_{91} + a^2 = a_{94}, a_{101} + a^2 = a_{104}, a_{105} + a^2 = a_{107}, \\
a_{108} + a^2 = a_{110}, a_{116} + a^3 = a_{117}.
\end{gathered} \tag{4.13}
$$

Next, we calculate $n_\epsilon \cdot u$.

$$
\begin{aligned}
n_\epsilon \cdot u = &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(2a^2)\epsilon_{86}(2a^2)\epsilon_{88}(2a^2)\epsilon_{89}(2a^2)\epsilon_{90}(a^2)\epsilon_{91}(2a^2)\epsilon_{93}(a^2) \\
&\epsilon_{94}(a^2)\epsilon_{98}(a^2)\epsilon_{110}(a^2)\epsilon_{111}(a^2)\epsilon_{112}(2a^2)\epsilon_{113}(10a^3)\epsilon_{114}(12a^3)\epsilon_{115}(12a^3) \\
&\epsilon_{116}(12a^3)\epsilon_{117}(8a^3)\epsilon_{118}(8a^3)\epsilon_{119}(10a^3)\epsilon_{120}(6a^3)\epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87}) \\
&\epsilon_{88}(a_{88})\epsilon_{89}(a_{89})\epsilon_{90}(a_{93})\epsilon_{91}(a_{91})\epsilon_{92}(a_{95})\epsilon_{93}(a_{90})\epsilon_{94}(a_{98})\epsilon_{95}(a_{92})\epsilon_{96}(a_{96}) \\
&\epsilon_{97}(a_{101})\epsilon_{98}(a_{94})\epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a_{97})\epsilon_{102}(a_{102})\epsilon_{103}(a_{103})\epsilon_{104}(a_{104}) \\
&\epsilon_{105}(a_{105})\epsilon_{106}(a_{106})\epsilon_{107}(a_{109})\epsilon_{108}(a_{108})\epsilon_{109}(a_{107})\epsilon_{110}(a_{111})\epsilon_{111}(a_{110}) \\
&\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{116})\epsilon_{116}(a_{115})\epsilon_{117}(a_{117})\epsilon_{118}(a_{118}) \\
&\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}) \\
= &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(2a^2 + a_{85})\epsilon_{86}(2a^2 + a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(2a^2 + a_{88}) \\
&\epsilon_{89}(2a^2 + a_{89})\epsilon_{90}(a^2 + a_{93})\epsilon_{91}(2a^2 + a_{91})\epsilon_{92}(a_{95})\epsilon_{93}(a^2 + a_{90})\epsilon_{94}(a^2 + a_{98}) \\
&\epsilon_{95}(a_{92})\epsilon_{96}(a_{96})\epsilon_{97}(a_{101})\epsilon_{98}(a^2 + a_{94})\epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a_{97})\epsilon_{102}(a_{102}) \\
&\epsilon_{103}(a_{103})\epsilon_{104}(a_{104})\epsilon_{105}(a_{105})\epsilon_{106}(a_{106})\epsilon_{107}(a_{109})\epsilon_{108}(a_{108})\epsilon_{109}(a_{107}) \\
&\epsilon_{110}(a^2 + a_{111})\epsilon_{111}(a^2 + a_{110})\epsilon_{112}(2a^2 + a_{112})\epsilon_{113}(10a^3 + a_{113}) \\
&\epsilon_{114}(12a^3 + a_{114})\epsilon_{115}(12a^3 + a_{116})\epsilon_{116}(12a^3 + a_{115})\epsilon_{117}(8a^3 + a_{117}) \\
&\epsilon_{118}(8a^3 + a_{118})\epsilon_{119}(10a^3 + a_{119})\epsilon_{120}(6a^3 + a_{120}) \\
= &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{89})\epsilon_{90}(a^2 + a_{93}) \\
&\epsilon_{91}(a_{91})\epsilon_{92}(a_{95})\epsilon_{93}(a^2 + a_{90})\epsilon_{94}(a^2 + a_{98})\epsilon_{95}(a_{92})\epsilon_{96}(a_{96})\epsilon_{97}(a_{101}) \\
&\epsilon_{98}(a^2 + a_{94})\epsilon_{99}(a_{99})\epsilon_{100}(a_{100})\epsilon_{101}(a_{97})\epsilon_{102}(a_{102})\epsilon_{103}(a_{103})\epsilon_{104}(a_{104}) \\
&\epsilon_{105}(a_{105})\epsilon_{106}(a_{106})\epsilon_{107}(a_{109})\epsilon_{108}(a_{108})\epsilon_{109}(a_{107})\epsilon_{110}(a^2 + a_{111}) \\
&\epsilon_{111}(a^2 + a_{110})\epsilon_{112}(a_{112})\epsilon_{113}(a_{113})\epsilon_{114}(a_{114})\epsilon_{115}(a_{116})\epsilon_{116}(a_{115})\epsilon_{117}(a_{117}) \\
&\epsilon_{118}(a_{118})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}).
\end{aligned}
$$

Then $n_\epsilon \cdot u = u$ gives

$$
\begin{gathered}
a_{90} + a^2 = a_{93}, a_{92} = a_{95}, a_{94} + a^2 = a_{98}, a_{97} = a_{101}, a_{107} = a_{109}, \\
a_{110} + a^2 = a_{111}, a_{115} = a_{116}.
\end{gathered} \tag{4.14}
$$

Now we calculate $n_\eta \cdot u$.

$$\begin{aligned}
n_\eta \cdot u = &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{87}(2a^2)\epsilon_{93}(a^2)\epsilon_{96}(a^2)\epsilon_{98}(a^2)\epsilon_{100}(2a^2)\epsilon_{101}(a^2)\epsilon_{102}(a^2) \\
&\epsilon_{103}(2a^2)\epsilon_{104}(a^2)\epsilon_{105}(a^2)\epsilon_{106}(2a^2)\epsilon_{107}(a^2)\epsilon_{110}(2a^2)\epsilon_{111}(a^2)\epsilon_{112}(a^2)\epsilon_{113}(2a^3) \\
&\epsilon_{114}(2a^3)\epsilon_{115}(2a^3)\epsilon_{116}(4a^3)\epsilon_{117}(6a^3)\epsilon_{118}(6a^3)\epsilon_{119}(8a^3)\epsilon_{120}(10a^3)\epsilon_{85}(a_{85}) \\
&\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{89})\epsilon_{90}(a_{90})\epsilon_{91}(a_{91})\epsilon_{92}(a_{92})\epsilon_{93}(a_{96})\epsilon_{94}(a_{94}) \\
&\epsilon_{95}(a_{99})\epsilon_{96}(a_{93})\epsilon_{97}(a_{97})\epsilon_{98}(a_{102}) \\
&\epsilon_{99}(a_{95})\epsilon_{100}(a_{100})\epsilon_{101}(a_{105})\epsilon_{102}(a_{98})\epsilon_{103}(a_{103})\epsilon_{104}(a_{107})\epsilon_{105}(a_{101})\epsilon_{106}(a_{106}) \\
&\epsilon_{107}(a_{104})\epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(a_{110})\epsilon_{111}(a_{112})\epsilon_{112}(a_{111})\epsilon_{113}(a_{113}) \\
&\epsilon_{114}(a_{115})\epsilon_{115}(a_{114})\epsilon_{116}(a_{116})\epsilon_{117}(a_{117})\epsilon_{118}(a_{118})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}) \\
= &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(2a^2 + a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{89})\epsilon_{90}(a_{90}) \\
&\epsilon_{91}(a_{91})\epsilon_{92}(a_{92})\epsilon_{93}(a^2 + a_{96})\epsilon_{94}(a_{94})\epsilon_{95}(a_{99})\epsilon_{96}(a^2 + a_{93})\epsilon_{97}(a_{97}) \\
&\epsilon_{98}(a^2 + a_{102})\epsilon_{99}(a_{95})\epsilon_{100}(2a^2 + a_{100})\epsilon_{101}(a^2 + a_{105})\epsilon_{102}(a^2 + a_{98}) \\
&\epsilon_{103}(2a^2 + a_{103})\epsilon_{104}(a^2 + a_{107})\epsilon_{105}(a^2 + a_{101})\epsilon_{106}(2a^2 + a_{106}) \\
&\epsilon_{107}(a^2 + a_{104})\epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(2a^2 + a_{110})\epsilon_{111}(a^2 + a_{112}) \\
&\epsilon_{112}(a^2 + a_{111})\epsilon_{113}(2a^3 + a_{113})\epsilon_{114}(2a^3 + a_{115})\epsilon_{115}(2a^3 + a_{114}) \\
&\epsilon_{116}(4a^3 + a_{116})\epsilon_{117}(6a^3 + a_{117})\epsilon_{118}(6a^3 + a_{118})\epsilon_{119}(8a^3 + a_{119}) \\
&\epsilon_{120}(10a^3 + a_{120}) \\
= &\left(\prod_{\lambda \in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{89})\epsilon_{90}(a_{90})\epsilon_{91}(a_{91}) \\
&\epsilon_{92}(a_{92})\epsilon_{93}(a^2 + a_{96})\epsilon_{94}(a_{94})\epsilon_{95}(a_{99})\epsilon_{96}(a^2 + a_{93})\epsilon_{97}(a_{97})\epsilon_{98}(a^2 + a_{102}) \\
&\epsilon_{99}(a_{95})\epsilon_{100}(a_{100})\epsilon_{101}(a^2 + a_{105})\epsilon_{102}(a^2 + a_{98})\epsilon_{103}(a_{103})\epsilon_{104}(a^2 + a_{107}) \\
&\epsilon_{105}(a^2 + a_{101})\epsilon_{106}(a_{106})\epsilon_{107}(a^2 + a_{104})\epsilon_{108}(a_{108})\epsilon_{109}(a_{109})\epsilon_{110}(a_{110}) \\
&\epsilon_{111}(a^2 + a_{112})\epsilon_{112}(a^2 + a_{111})\epsilon_{113}(a_{113})\epsilon_{114}(a_{115})\epsilon_{115}(a_{114})\epsilon_{116}(a_{116}) \\
&\epsilon_{117}(a_{117})\epsilon_{118}(a_{118})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}).
\end{aligned}$$

Then $n_\eta \cdot u = u$ gives

$$\begin{gathered}
a_{93} + a^2 = a_{96}, a_{95} = a_{99}, a_{98} + a^2 = a_{102}, a_{101} + a^2 = a_{105}, a_{104} + a^2 = a_{107}, \\
a_{111} + a^2 = a_{112}, a_{114} = a_{115}.
\end{gathered} \tag{4.15}$$

Finally, we calculate $n_\xi \cdot u$.

$$\begin{aligned}
n_\xi \cdot u = &\left(\prod_{\lambda\in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(2a^2)\epsilon_{86}(2a^2)\epsilon_{88}(2a^2)\epsilon_{89}(2a^2)\epsilon_{91}(2a^2)\epsilon_{92}(2a^2)\epsilon_{93}(2a^2)\\
&\epsilon_{95}(2a^2)\epsilon_{96}(a^2)\epsilon_{98}(2a^2)\epsilon_{100}(a^2)\epsilon_{102}(a^2)\epsilon_{104}(2a^2)\epsilon_{105}(a^2)\epsilon_{106}(a^2)\epsilon_{107}(a^2)\\
&\epsilon_{108}(a^2)\epsilon_{110}(a^2)\epsilon_{113}(11a^3)\epsilon_{114}(11a^3)\epsilon_{115}(12a^3)\epsilon_{116}(16a^3)\epsilon_{117}(14a^3)\\
&\epsilon_{118}(14a^3)\epsilon_{119}(14a^3)\epsilon_{120}(14a^3)\epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{89})\\
&\epsilon_{90}(a_{90})\epsilon_{91}(a_{91})\epsilon_{92}(a_{92})\epsilon_{93}(a_{93})\epsilon_{94}(a_{94})\epsilon_{95}(a_{95})\epsilon_{96}(a_{100})\epsilon_{97}(a_{97})\epsilon_{98}(a_{98})\\
&\epsilon_{99}(a_{103})\epsilon_{100}(a_{96})\epsilon_{101}(a_{101})\epsilon_{102}(a_{106})\epsilon_{103}(a_{99})\epsilon_{104}(a_{104})\epsilon_{105}(a_{108})\epsilon_{106}(a_{102})\\
&\epsilon_{107}(a_{110})\epsilon_{108}(a_{105})\epsilon_{109}(a_{111})\epsilon_{110}(a_{107})\epsilon_{111}(a_{109})\epsilon_{112}(a_{112})\epsilon_{113}(a_{114})\\
&\epsilon_{114}(a_{113})\epsilon_{115}(a_{115})\epsilon_{116}(a_{116})\epsilon_{117}(a_{117})\epsilon_{118}(a_{118})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120})\\
= &\left(\prod_{\lambda\in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(2a^2+a_{85})\epsilon_{86}(2a^2+a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(2a^2+a_{88})\\
&\epsilon_{89}(2a^2+a_{89})\epsilon_{90}(a_{90})\epsilon_{91}(2a^2+a_{91})\epsilon_{92}(2a^2+a_{92})\epsilon_{93}(2a^2+a_{93})\epsilon_{94}(a_{94})\\
&\epsilon_{95}(2a^2+a_{95})\epsilon_{96}(a^2+a_{100})\epsilon_{97}(a_{97})\epsilon_{98}(2a^2+a_{98})\epsilon_{99}(a_{103})\epsilon_{100}(a^2+a_{96})\\
&\epsilon_{101}(a_{101})\epsilon_{102}(a^2+a_{106})\epsilon_{103}(a_{99})\epsilon_{104}(2a^2+a_{104})\epsilon_{105}(a^2+a_{108})\\
&\epsilon_{106}(a^2+a_{102})\epsilon_{107}(a^2+a_{110})\epsilon_{108}(a^2+a_{105})\epsilon_{109}(a_{111})\epsilon_{110}(a^2+a_{107})\\
&\epsilon_{111}(a_{109})\epsilon_{112}(a_{112})\epsilon_{113}(11a^3+a_{114})\epsilon_{114}(11a^3+a_{113})\epsilon_{115}(12a^3+a_{115})\\
&\epsilon_{116}(16a^3+a_{116})\epsilon_{117}(14a^3+a_{117})\epsilon_{118}(14a^3+a_{118})\epsilon_{119}(14a^3+a_{119})\\
&\epsilon_{120}(14a^3+a_{120})\\
= &\left(\prod_{\lambda\in O_{29}} \epsilon_\lambda(a)\right) \epsilon_{85}(a_{85})\epsilon_{86}(a_{86})\epsilon_{87}(a_{87})\epsilon_{88}(a_{88})\epsilon_{89}(a_{89})\epsilon_{90}(a_{90})\epsilon_{91}(a_{91})\\
&\epsilon_{92}(a_{92})\epsilon_{93}(a_{93})\epsilon_{94}(a_{94})\epsilon_{95}(a_{95})\epsilon_{96}(a^2+a_{100})\epsilon_{97}(a_{97})\epsilon_{98}(a_{98})\epsilon_{99}(a_{103})\\
&\epsilon_{100}(a^2+a_{96})\epsilon_{101}(a_{101})\epsilon_{102}(a^2+a_{106})\epsilon_{103}(a_{99})\epsilon_{104}(a_{104})\epsilon_{105}(a^2+a_{108})\\
&\epsilon_{106}(a^2+a_{102})\epsilon_{107}(a^2+a_{110})\epsilon_{108}(a^2+a_{105})\epsilon_{109}(a_{111})\epsilon_{110}(a^2+a_{107})\\
&\epsilon_{111}(a_{109})\epsilon_{112}(a_{112})\epsilon_{113}(a^3+a_{114})\epsilon_{114}(a^3+a_{113})\epsilon_{115}(a_{115})\epsilon_{116}(a_{116})\\
&\epsilon_{117}(a_{117})\epsilon_{118}(a_{118})\epsilon_{119}(a_{119})\epsilon_{120}(a_{120}).
\end{aligned}$$

Then $n_\xi \cdot u = u$ gives

$$\begin{gathered}
a_{96}+a^2=a_{100}, a_{99}=a_{103}, a_{102}+a^2=a_{106}, a_{105}+a^2=a_{108}, a_{107}+a^2=a_{110},\\
a_{109}=a_{111}, a_{113}+a^3=a_{114}.
\end{gathered} \tag{4.16}$$

Solving (4.10),(4.11),(4.12),(4.13),(4.14),(4.15), and (4.16) simultaneously, we get

$$a_{85} = a_{86} = a_{88} = a_{89} = a_{91} = a_{92} = a_{93} = a_{95} = a_{98} = a_{99} = a_{100} = a_{103}$$
$$= a_{104} = a_{105} = a_{106} = a_{110} = a_{112}.$$
$$a_{87} = a_{85} + a^2.$$
$$a_{87} = a_{90} = a_{94} = a_{96} = a_{97} = a_{101} = a_{102} = a_{107} = a_{108} = a_{109} = a_{111}.$$
$$a_{113} = a_{117} = a_{118} = a_{119} = a_{120}.$$
$$a_{114} = a_{113} + a^3.$$
$$a_{114} = a_{115} = a_{116}.$$

Since the set $\{n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon\, n_\eta, n_\xi\}$ generates H, we have

$$\begin{aligned}C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H) =&\Bigg\{\left(\prod_{\lambda\in O_{29}} \epsilon_\lambda(a)\right)\epsilon_{85}(b)\epsilon_{86}(b)\epsilon_{87}(a^2+b)\epsilon_{88}(b)\epsilon_{89}(b)\\ &\epsilon_{90}(a^2+b)\epsilon_{91}(b)\epsilon_{92}(b)\epsilon_{93}(b)\epsilon_{94}(a^2+b)\epsilon_{95}(b)\epsilon_{96}(a^2+b)\\ &\epsilon_{97}(a^2+b)\epsilon_{98}(b)\epsilon_{99}(b)\epsilon_{100}(b)\epsilon_{101}(a^2+b)\epsilon_{102}(a^2+b)\epsilon_{103}(b)\\ &\epsilon_{104}(b)\epsilon_{105}(b)\epsilon_{106}(b)\epsilon_{107}(a^2+b)\epsilon_{108}(a^2+b)\epsilon_{109}(a^2+b)\epsilon_{110}(b)\\ &\epsilon_{111}(a^2+b)\epsilon_{112}(b)\epsilon_{113}(c)\epsilon_{114}(a^3+c)\epsilon_{115}(a^3+c)\epsilon_{116}(a^3+c)\\ &\epsilon_{117}(c)\epsilon_{118}(c)\epsilon_{119}(c)\epsilon_{120}(c) \mid a,b,c\in k\Bigg\}. \qquad (4.17)\end{aligned}$$

In (4.17), set $b = 0$, $c = 0$, and differentiate with respect to a, and evaluate at $a = 0$, we get

$$\sum_{\lambda\in O_{29}} e_\lambda \in \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)).$$

Note that all weight-2 roots mutually commute by Proposition 2.3.5. So, by Lemma 2.3.4, we get

$$\sum_{\lambda\in O_{85}} e_\lambda \in \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)).$$

Similarly, all weight-3 roots mutually commute, so we get

$$\sum_{\lambda\in O_{113}} e_\lambda \in \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)).$$

So we have

$$\Big\{a(\sum_{\lambda\in O_{29}} e_\lambda) + b(\sum_{\lambda\in O_{85}} e_\lambda) + c(\sum_{\lambda\in O_{113}} e_\lambda) \mid a,b\in k\Big\} \subseteq \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)).$$

We already know that

$$\operatorname{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)) \subseteq \mathfrak{c}_{\operatorname{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi}))}(H) =$$
$$\Big\{a(\sum_{\lambda\in O_{29}} e_\lambda) + b(\sum_{\lambda\in O_{85}} e_\lambda) + c(\sum_{\lambda\in O_{113}} e_\lambda) \mid a, b \in k\Big\}.$$

Therefore we have shown that

$$\operatorname{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)) = \mathfrak{c}_{\operatorname{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi}))}(H).$$

Thus, we have the following.

Proposition 4.2.6. *There is no nilpotent witness to the G-nonseparability of H in* $\operatorname{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi}))$.

Chapter 5

A positive example

5.1 Introduction

Let G be a simple algebraic group of type E_7. We use the same notation as in the previous chapters. In the next section we find an example of a G-nonseparable subgroup H of G. Here, we describe a generic way to find such a subgroup H of G. Consider a subgroup H with two generators, say, P_1 and P_2, where P_1 and P_2 are products of n_τ satisfying the following two conditions.

1. One of the orbits of $\langle P_1, P_2 \rangle$ in $\Psi(R_u(P))$ contains an odd number of pairs of roots whose corresponding root-subgroups are non-commuting and contributing to a weight-2 root-subgroup, say, U_ζ, and get their order swapped by the action of $\langle P_1, P_2 \rangle$.

2. P_1 centralizes a weight-2 root ζ.

We show how this method works by finding a nilpotent witness to the G-nonseparability of H. Then we find an example we are after in the following way. First, we check that H is G-cr. Then we conjugate H by some particular element $u(a)$ of G, following the method in [BMRT10, Sec. 7]. We explain the reason for this particular choice of $u(a)$, which is not explicitly mentioned in [BMRT10, Sec. 7]. Then, we find a reductive subgroup M of G containing $u(a)Hu(a)^{-1}$ such that $u(a)Hu(a)^{-1}$ is NOT M-cr using the results from geometric invariant theory.

5.2 A G-nonseparable subgroup H

Let G be a simple algebraic group of type E_7. We use the same Levi subgroup $L_{\alpha\beta\gamma\delta\epsilon\eta}$ of type A_6, and the same parabolic subgroup $P_{\alpha\beta\gamma\delta\epsilon\eta}$ of G, as in Section 4.2.2. We have $\Psi^+(G)$, $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$, and $\Psi^+(L_{\alpha\beta\gamma\delta\epsilon\eta})$ as follows.

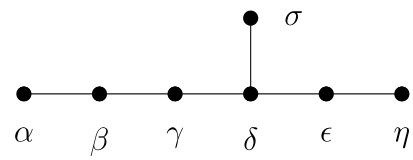


Figure 5.1: Dynkin diagram of E_7

$$\begin{aligned}\Psi^+(G) =&\{1, \cdots, 63\},\\ \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})) =&\{1, \cdots, 42\},\\ \Psi^+(L_{\alpha\beta\gamma\delta\epsilon\eta}) =&\{43, \cdots, 63\}.\end{aligned}$$

We fix an ordering of $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ in the natural order. Define P_1 and P_2 as follows.

$$\begin{aligned}P_1 =&n_\epsilon n_\beta n_\gamma n_\alpha n_\beta,\\ P_2 =&n_\epsilon n_\beta n_\gamma n_\alpha n_\beta n_\eta n_\delta n_\beta.\end{aligned}$$

Now let

$$H = \langle P_1, P_2\rangle.$$

It is not difficult to calculate how P_1 and P_2 act on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$. Let π_1 be the corresponding representation of H on $Sym(\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})))$. Then we have

$$\begin{aligned}\pi_1(P_1) =&(2\ 9)(3\ 12)(4\ 5)(6\ 8)(7\ 14)(10\ 17)(11\ 25)(15\ 32)(16\ 18)(19\ 24)(20\ 29)\\ &(22\ 28)(23\ 27)(26\ 30)(31\ 34)(33\ 35)(36\ 38)(37\ 39)(40\ 41),\\ \pi_1(P_2) =&(1\ 9\ 32\ 35\ 33\ 15\ 2)(3\ 4\ 14\ 25\ 30\ 31\ 22)(5\ 12\ 28\ 34\ 26\ 11\ 7)\\ &(6\ 24\ 29\ 21\ 20\ 19\ 8)(10\ 13\ 17\ 18\ 27\ 23\ 16)(36\ 38\ 39\ 41\ 42\ 40\ 37).\end{aligned}$$

From this, we calculate all orbits of H in $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$.

$$\begin{aligned}O_1 &= \{1, 2, 9, 15, 32, 33, 35\},\\ O_3 &= \{3, 4, 5, 7, 11, 12, 14, 22, 25, 26, 28, 30, 31, 34\},\\ O_6 &= \{6, 8, 19, 20, 21, 24, 29\},\\ O_{10} &= \{10, 13, 16, 17, 18, 23, 27\},\\ O_{36} &= \{36, 37, 38, 39, 40, 41, 42\}.\end{aligned}$$

Then by Corollary 2.3.3, we have

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(H) = \{a(\sum_{\lambda\in O_1} e_\lambda) + b(\sum_{\lambda\in O_3} e_\lambda) + c(\sum_{\lambda\in O_6} e_\lambda) + d(\sum_{\lambda\in O_{10}} e_\lambda) \\ + m(\sum_{\lambda\in O_{36}} e_\lambda) \mid a,b,c,d,m \in k\}.$$

Lemma 5.2.1. $\sum_{\lambda\in O_6} e_\lambda \notin \mathrm{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H))$.

Proof. Suppose that $\sum_{\lambda\in O_6} e_\lambda \in \mathrm{Lie}(C_{R_u(P)}(H))$. By Corollary [Spr98, Cor. 14.2.7], $C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)^\circ$ is isomorphic as a variety to k^n for some $n \in \mathbb{N}$. Therefore there exists a morphism of varieties $u : k \to C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)^\circ$ such that $u(0) = 1$ and $u'(0) = \sum_{\lambda\in O_6} e_\lambda$. By Lemma 2.3.4, $u(a)$ can be expressed uniquely as

$$u(a) = \prod_{\lambda\in\{1,\cdots,42\}} \epsilon_\lambda(f_\lambda(a)), \tag{5.1}$$
$$\text{where } f_\lambda \in k[a].$$

Note that any f_λ cannot contain any nonzero constant term since if $c_\lambda = f_\lambda(0)$ then we have $u(0) = \prod_{\lambda\in\{1,\cdots,42\}} \epsilon_\lambda(c_\lambda) = 1$, which is possible only when $c_\lambda = 0$ for all $\lambda \in \{1, \cdots, 42\}$. Differentiating 5.1, and evaluating at $a = 0$, we get

$$u'(0) = \sum_{\lambda\in\{1,\cdots,42\}} (f_\lambda)'(0)e_\lambda.$$

Since $u'(0) = \sum_{\lambda\in O_6} e_\lambda$, we get

$$(f_\lambda)'(0) = \begin{cases} 1 & \text{if } \lambda \in O_6. \\ 0 & \text{if } \lambda \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))\backslash O_6. \end{cases}$$

Then we have

$$f_\lambda(a) = \begin{cases} a + g_\lambda(a) & \text{if } \lambda \in O_6 \\ g_\lambda(a) & \text{if } \lambda \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))\backslash O_6 \end{cases}$$
$$\text{where } g_\lambda \in k[a] \text{ has no constant or linear term.}$$

Now we have

$$
\begin{aligned}
P_1 \cdot u(a) =& P_1 \cdot \left(\prod_{\lambda \in \{1,\cdots,42\}} \epsilon_\lambda(f_\lambda(a)) \right) \\
=& P_1 \cdot \Bigg[\left(\prod_{\lambda \in \{1,\cdots,5\}} \epsilon_\lambda(g_\lambda(a)) \right) \epsilon_6(a + g_6(a))\epsilon_7(g_7(a))\epsilon_8(a + g_8(a)) \\
& \left(\prod_{\lambda \in \{9,\cdots,18\}} \epsilon_\lambda(g_\lambda(a)) \right) \epsilon_{19}(a + g_{19}(a))\epsilon_{20}(a + g_{20}(a))\epsilon_{21}(a + g_{21}(a)) \\
& \epsilon_{22}(g_{22}(a))\epsilon_{23}(g_{23}(a))\epsilon_{24}(a + g_{24}(a)) \left(\prod_{\lambda \in \{25,\cdots,28\}} \epsilon_\lambda(g_\lambda(a)) \right) \\
& \epsilon_{29}(a + g_{29}(a)) \left(\prod_{\lambda \in \{30,\cdots,41\}} \epsilon_\lambda(g_\lambda(a)) \right) \epsilon_{42}(g_{42}(a)) \Bigg] \\
=& \left(\prod_{\lambda \in \{1,\cdots,5\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \epsilon_{P_1\cdot 6}(a + g_6(a))\epsilon_{P_1\cdot 7}(g_7(a))\epsilon_{P_1\cdot 8}(a + g_8(a)) \\
& \left(\prod_{\lambda \in \{9,\cdots,18\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \epsilon_{P_1\cdot 19}(a + g_{19}(a))\epsilon_{P_1\cdot 20}(a + g_{20}(a)) \\
& \epsilon_{P_1\cdot 21}(a + g_{21}(a))\epsilon_{P_1\cdot 22}(g_{22}(a))\epsilon_{P_1\cdot 23}(g_{23}(a))\epsilon_{P_1\cdot 24}(a + g_{24}(a)) \\
& \left(\prod_{\lambda \in \{25,\cdots,28\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \epsilon_{P_1\cdot 29}(a + g_{29}(a)) \left(\prod_{\lambda \in \{30,\cdots,41\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \\
& \epsilon_{P_1\cdot 42}(g_{42}(a)) \\
=& \left(\prod_{\lambda \in \{1,\cdots,5\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \epsilon_8(a + g_6(a))\epsilon_{P_1\cdot 7}(g_7(a))\epsilon_6(a + g_8(a)) \\
& \left(\prod_{\lambda \in \{9,\cdots,18\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \epsilon_{24}(a + g_{19}(a))\epsilon_{29}(a + g_{20}(a))\epsilon_{21}(a + g_{21}(a)) \\
& \epsilon_{P_1\cdot 22}(g_{22}(a))\epsilon_{P_1\cdot 23}(g_{23}(a))\epsilon_{19}(a + g_{24}(a)) \left(\prod_{\lambda \in \{25,\cdots,28\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \\
& \epsilon_{20}(a + g_{29}(a)) \left(\prod_{\lambda \in \{30,\cdots,41\}} \epsilon_{P_1\cdot\lambda}(g_\lambda(a)) \right) \epsilon_{P_1\cdot 42}(g_{42}(a)).
\end{aligned} \tag{5.2}
$$

Now, we reorder the terms in the last expression of 5.2 using Proposition 2.3.5, Proposition 2.3.7, and ($\star\star$). We focus on new a^2 terms contributing to the ϵ_{42} term that occur during reordering. First, note that $\epsilon_{P_1\cdot 42}(g_{42}) = \epsilon_{42}(g_{42})$. We get a new contribution to the ϵ_{42} term when we swap a non-commuting pair ϵ_i, ϵ_j such that $i\tilde{+}j = 42$ by Proposition 2.3.7. Note that when this happens, we have

$$\epsilon_i(f_i(a))\epsilon_j(f_j(a)) = \epsilon_j(f_j(a))\epsilon_i(f_i(a))\epsilon_{42}(f_i(a)f_j(a))$$

Note that each of f_i and f_j is the zero polynomial, or has lowest degree term a, or has lowest degree term of degree greater or equal to 2. We divide into 3 cases depending on the type of the lowest degree term in $f_i f_j$.

1. If $f_i = 0$ or f_j=0, then $f_i f_j = 0$.
2. If both of f_i and f_j have lowest degree term a, then the lowest degree term of $f_i f_j$ is a^2.
3. Otherwise, the lowest degree term of $f_i f_j$ has degree 3 or greater.

Therefore during the reordering, we get a new a^2 term contributing to the ϵ_{42} term only when we swap a non-commuting pair $\{\epsilon_{P_1 \cdot i}(a + g_i(a)), \epsilon_{P_1 \cdot j}(a + g_j(a))\}$ such that $(P_1 \cdot i)\tilde{+}(P_1 \cdot j) = 42$. Here, we list of all non-commuting root subgroups U_i, U_j with $i\tilde{+}j = 42$.

$\{U_7, U_{35}\}$	$\{U_{10}, U_{34}\}$	$\{U_{14}, U_{33}\}$	$\{U_{15}, U_{32}\}$
$\{U_{17}, U_{31}\}$	$\{U_{19}, U_{30}\}$	$\{U_{20}, U_{29}\}$	$\{U_{22}, U_{28}\}$
$\{U_{23}, U_{27}\}$	$\{U_{24}, U_{26}\}$		

Table 5.1: Non-commuting pairs of root subgroups U_i, U_j in $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})$

From the last equation and Table 5.1, it is easy to see that during the reordering we need to swap only one such non-commuting pair of ϵ_i's whose lowest degree term is a, namely, $\epsilon_{20}(a + g_{29}(a))$ and $\epsilon_{29}(a + g_{20}(a))$. Therefore after reordering, we have

$$P_1 \cdot u(a) = \left(\prod_{\lambda \in \{1, \cdots, 41\}} \epsilon_\lambda(h_\lambda(a)) \right) \epsilon_{42}(a^2 + h_{42}(a) + g_{42}(a)),$$

where $h_\lambda, h_{42}, g_{42} \in k[a]$ with h_{42} having lowest degree term at least 3 or $h_{42} = 0$.

If $u(a) \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(H)$, we must have $u = P_1 \cdot u$. So, in particular, we must have

$$g_{42}(a) = a^2 + h_{42}(a) + g_{42}(a)$$
$$\Leftrightarrow a^2 + h_{42}(a) = 0.$$

Since h_{42} is the zero polynomial or a polynomial of the degree of the lowest degree term at least 3, the last equation can not hold for all $a \in k$. This is a contradiction. $\square$

Proposition 5.2.2. *H is non-separable in G.*

Proof. This is the consequence of Lemma 5.2.1 and the form of $\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(H)$ as given before the lemma. $\square$

Proposition 5.2.3. *H is G-cr.*

Proof. By Theorem 2.2.6, it it enough to show that H is L-cr. We know that H is L-cr if and only if H is $[L, L]$-cr from Theorem 2.2.5. We also know that $L_{\alpha\beta\gamma\delta\epsilon\eta}$ is of type A_6, so $[L_{\alpha\beta\gamma\delta\epsilon\eta}, L_{\alpha\beta\gamma\delta\epsilon\eta}] \cong SL_7(k)$ or $PGL_7(k)$. From Theorem 2.2.5, we can assume that $[L_{\alpha\beta\gamma\delta\epsilon\eta}, L_{\alpha\beta\gamma\delta\epsilon\eta}] \cong SL_7(k)$ in order to prove that H is $[L, L]$-cr. From [Spr98, Sec. 9.2.2], there is an isomorphism π_2 from $[L_{\alpha\beta\gamma\delta\epsilon\eta}, L_{\alpha\beta\gamma\delta\epsilon\eta}]$ to $SL_7(k)$ which satisfies the followings.

$$\pi_2(n_\alpha) = \begin{bmatrix} 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \pi_2(n_\beta) = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$\pi_2(n_\gamma) = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \pi_2(n_\delta) = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$\pi_2(n_\epsilon) = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \pi_2(n_\eta) = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{bmatrix}.$$

Consider the permutation representation π_3 of $\langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\epsilon\rangle$ on S_7. We have

$$\pi_3(n_\alpha) = (1\ 2),\ \pi_3(n_\beta)= (2\ 3),\ \pi_3(n_\gamma) = (3\ 4),$$
$$\pi_3(n_\delta) = (4\ 5),\ \pi_3(n_\epsilon)= (5\ 6),\ \pi_3(n_\eta) = (6\ 7).$$

Then we get

$$\pi_3(P_1) = (5\ 6)(2\ 3)(3\ 4)(1\ 2)(2\ 3) = (1\ 3)(2\ 4)(5\ 6),$$
$$\pi_3(P_2) = (5\ 6)(2\ 3)(3\ 4)(1\ 2)(2\ 3)(6\ 7)(4\ 5)(2\ 3) = (1\ 3\ 4\ 6\ 7\ 5\ 2).$$

Note that $\pi_3(P_1)$ and $\pi_3(P_2)$ satisfy the following equations.

$$(\pi_3(P_1))^2 = (\pi_3(P_2))^7 = 1,\ \ \pi_3(P_1)\pi_3(P_2)(\pi_3(P_1))^{-1} = (\pi_3(P_2))^{-1}.$$

Thus it is easy to see that $H \cong D_{14}$. After relabeling, $\pi_3(P_2)$ can be expressed as

$$\pi_3(P_2) = (1\ 2\ 3\ 4\ 5\ 6\ 7).$$

or, in matrix form, we have

$$\pi_2(P_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda^7 - 1$. Let ω be a 7th root of 1. Then, choosing the suitable basis, the matrix can be diagonalized as

$$\pi_2(P_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^{-3} \end{bmatrix}.$$

Let the 1-dimensional eigenspaces of $\pi_2(P_2)$ be $V_1, V_\omega, V_{\omega^{-1}}, V_{\omega^2}, V_{\omega^{-2}}, V_{\omega^3},$ and $V_{\omega^{-3}}$ where the subscripts represent the corresponding eigenvalues. We have

$$\pi_2(P_1)\pi_2(P_2)(\pi_2(P_1))^{-1} = (\pi_2(P_2))^{-1},$$

so it is easy to see that $\pi_2(P_1)$ fixes V_1, and permutes V_{ω^n} with $V_{\omega^{-n}}$ for $n \in \{1,2,3\}$. Therefore $\pi_2(P_1)$ must look like

$$\pi_2(P_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & c^{-1} & 0 \end{bmatrix}, \text{ for some } a,b,c \in k^*.$$

Normalizing the basis, we can assume that $a = b = c = 1$, that is

$$\pi_2(P_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let ρ be the restriction of π_2 to H. For each $a \in \{1,2,3\}$, define the representation $\rho_a : H \to GL_2(k)$ such that

$$\rho_a(P_2) = \begin{bmatrix} \omega^a & 0 \\ 0 & \omega^{-a} \end{bmatrix}.$$

$$\rho_a(P_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

From the matrices of P_1 and P_2, it is easy to see that

$$\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_0,$$

where ρ_0 is 1-dimensional trivial representation.

It is obvious that ρ_a is irreducible for any $a \in \{1,2,3\}$. Therefore H is $[L_{\alpha\beta\gamma\delta\epsilon\eta}, L_{\alpha\beta\gamma\delta\epsilon\eta}]$-cr, and we are done. □

5.3 A positive example

In this last section, we find a pair of reductive subgroups H and M of G such that $H < M < G$ and H is G-cr but NOT M-cr. We use the same notation and the same $H = \langle P_1, P_2 \rangle$ and G as in Section 5.2. We define

$$G_i := \langle U_i \cup U_{-i} \rangle, \text{ where } i \in \{1, \cdots, 42\}.$$

Pick any $a \in k^*$, and define

$$u(a) = \prod_{i \in O_6} \epsilon_i(a), \text{ where } O_6 = \{6, 8, 19, 20, 21, 24, 29\}.$$

Set

$$H_a = u(a)Hu(a)^{-1} = \langle u(a)P_1u(a)^{-1}, u(a)P_2u(a)^{-1}\rangle.$$

Set

$$M = \langle L_{\alpha\beta\gamma\delta\epsilon\eta}, G_{36}, G_{37}, \cdots, G_{42}\rangle.$$

Then M is a reductive group containing T such that

$$\Psi(M) = \{\pm 36, \cdots, \pm 63\}$$

because $\{\pm 36, \cdots, \pm 63\}$ is a closed subsystem of $\Psi(G)$, that is, it is closed under taking negatives and has the property that for any $\alpha, \beta \in \Psi(M)$, if $m, n \in \mathbb{Z}_{\geq 0}$ and $m\alpha + n\beta \in \Psi(G)$ then $m\alpha + n\beta \in \Psi(M)$.

Note that $P_1 = P_1^{-1}$. So, we have

$$\begin{aligned}
u(a)P_1u(a)^{-1} =& P_1P_1u(a)P_1u(a)^{-1} \\
=& P_1(P_1 \cdot u(a))u(a)^{-1} \\
=& P_1 \left(P_1 \cdot (\prod_{i \in O_6} \epsilon_i(a))\right) \left(\prod_{i \in O_6} \epsilon_i(a)\right)^{-1} \\
=& P_1 \left(\prod_{i \in O_6} \epsilon_{P_1 \cdot i}(a)\right) \left(\prod_{i \in O_6} \epsilon_i(a)\right)^{-1} \\
=& P_1 \Big(\epsilon_8(a)\epsilon_6(a)\epsilon_{24}(a)\epsilon_{29}(a)\epsilon_{21}(a)\epsilon_{19}(a)\epsilon_{20}(a)\Big)\epsilon_{29}(a)^{-1}\epsilon_{24}(a)^{-1} \\
& \epsilon_{21}(a)^{-1}\epsilon_{20}(a)^{-1}\epsilon_{19}(a)^{-1}\epsilon_8(a)^{-1}\epsilon_6(a)^{-1} \\
=& P_1 \Big(\epsilon_6(a)\epsilon_8(a)\epsilon_{19}(a)\epsilon_{20}(a)\epsilon_{21}(a)\epsilon_{24}(a)\epsilon_{29}(a)\Big) \\
& \left(\prod_{i \in \{36 \cdots 41\}} \epsilon_i(f_i(a))\right)\epsilon_{42}(a^2)\epsilon_{29}(a)^{-1}\epsilon_{24}(a)^{-1}\epsilon_{21}(a)^{-1}\epsilon_{20}(a)^{-1} \\
& \epsilon_{19}(a)^{-1}\epsilon_8(a)^{-1}\epsilon_6(a)^{-1} \\
=& P_1 \left(\prod_{i \in \{36 \cdots 41\}} \epsilon_i(f_i(a))\right) \epsilon_{42}(a^2), \text{ where } f_i \in k[a].
\end{aligned} \tag{5.3}$$

We have used the same argument in the proof of 5.2.1 to reorder terms. Also note that

$$\begin{aligned}\pi_1(P_2^{-1}) =&(1\ 2\ 15\ 33\ 35\ 32\ 9)(3\ 22\ 31\ 30\ 25\ 14\ 4)(5\ 7\ 11\ 26\ 34\ 28\ 12)\\ &(6\ 8\ 19\ 20\ 21\ 29\ 24)(10\ 16\ 23\ 27\ 18\ 17\ 13)(36\ 37\ 40\ 42\ 41\ 39\ 38).\end{aligned}$$

Then similarly we have

$$\begin{aligned}u(a)P_2u(a)^{-1} =&P_2P_2^{-1}u(a)P_2u(a)^{-1}\\ =&P_2(P_2^{-1}\cdot u(a))u(a)^{-1}\\ =&P_2\left(P_2^{-1}\cdot (\prod_{i\in O_6}\epsilon_i(a))\right)\left(\prod_{i\in O_6}\epsilon_i(a)\right)^{-1}\\ =&P_2\left(\prod_{i\in O_6}\epsilon_{(P_2)^{-1}\cdot i}(a)\right)\left(\prod_{i\in O_6}\epsilon_i(a)\right)^{-1}\\ =&P_2\Big(\epsilon_8(a)\epsilon_{19}(a)\epsilon_{20}(a)\epsilon_{21}(a)\epsilon_{29}(a)\epsilon_6(a)\epsilon_{24}(a)\Big)\epsilon_{29}(a)^{-1}\epsilon_{24}(a)^{-1}\\ &\epsilon_{21}(a)^{-1}\epsilon_{20}(a)^{-1}\epsilon_{19}(a)^{-1}\epsilon_8(a)^{-1}\epsilon_6(a)^{-1}\\ =&P_2\Big(\epsilon_6(a)\epsilon_8(a)\epsilon_{19}(a)\epsilon_{20}(a)\epsilon_{21}(a)\epsilon_{24}(a)\epsilon_{29}(a)\Big)\\ &\left(\prod_{i\in\{36\cdots 42\}}\epsilon_i(f_i(a))\right)\\ &\epsilon_{29}(a)^{-1}\epsilon_{24}(a)^{-1}\epsilon_{21}(a)^{-1}\\ &\epsilon_{20}(a)^{-1}\epsilon_{19}(a)^{-1}\epsilon_8(a)^{-1}\epsilon_6(a)^{-1}\\ =&P_2\left(\prod_{i\in\{36\cdots 42\}}\epsilon_i(f_i(a))\right),\text{ where } f_i\in k[a].\end{aligned}$$

Then it is easy to see that $u(a)P_1u(a)^{-1} = P_1v_1(a) \in M$ and $u(a)P_2u(a)^{-1} = P_2v_2(a) \in M$, where each of $v_1(a)$ and $v_2(a)$ is the product of some $\epsilon_i(a)$'s corresponding to weight-2 roots. Therefore we have

$$H_a < M < G.$$

Note that $P_1v_1(a) \notin H$ because $v_1(a)$ contains $\epsilon_{42}(a^2)$ which is a nontrivial weight-2 term.

Theorem 5.3.1. *Let $a \in k^*$. Then H_a is G-cr, but not M-cr.*

Proof. In Section 5.2 we have shown that H is G-cr. Now H_a is G-conjugate to H, therefore, H_a is G-cr. Let

$$\lambda = 3\alpha^\vee + 6\beta^\vee + 9\gamma^\vee + 12\delta^\vee + 8\epsilon^\vee + 4\eta^\vee + 7\sigma^\vee.$$

Then we have

$$\begin{gathered}\langle\alpha,\lambda\rangle = 0,\ \langle\beta,\lambda\rangle = 0,\ \langle\gamma,\lambda\rangle = 0,\\ \langle\delta,\lambda\rangle = 0, \langle\epsilon,\lambda\rangle = 0, \langle\eta,\lambda\rangle = 0,\\ \langle\sigma,\lambda\rangle = 2.\end{gathered}$$

Therefore we have

$$P_{\alpha\beta\gamma\delta\epsilon\eta} = P_\lambda, L_{\alpha\beta\gamma\delta\epsilon\eta} = L_\lambda.$$

We have a homomorphism $c_\lambda : P_\lambda \to L_\lambda$ (see Definition 2.2.2). By Theorem 2.2.7, it is enough to find an element $h \in H_a$ such that $c_\lambda(h)$ exists and $c_\lambda(h)$ is not M- conjugate to h in order to prove that H_a is not M-cr. Set $h := u(a)P_1u(a)^{-1}$. We know that $u(a)P_1u(a)^{-1} \in H_a$. Also we have

$$\begin{aligned}c_\lambda(u(a)P_1u(a)^{-1}) =& \lim_{x\to 0}(\lambda(x)u(a)P_1u(a)^{-1}\lambda(x)^{-1})\\ =& \lim_{x\to 0}(\lambda(x)P_1v_1(a)\lambda(x)^{-1})\\ =& \lim_{x\to 0}(P_1\lambda(x)v_1(a)\lambda(x)^{-1})\\ =& P_1.\end{aligned}$$

The last equation holds since $v_1(a) \in R_u(P_\lambda)$. Therefore $c_\lambda(u(a)P_1u(a)^{-1})$ exists. By Theorem 2.2.8, it suffices to show that $c_\lambda(u(a)P_1u(a)^{-1})$ is not $R_u(P_\lambda(M))$-conjugate to $u(a)P_1u(a)^{-1}$ in order to prove that H_a is not M-cr. Suppose the contrary. Then there exists $m \in R_u(P_\lambda(M))$ such that

$$u(a)P_1u(a)^{-1} = mP_1m^{-1} \tag{5.4}$$

Note that we have

$$\Psi(R_u(P_\lambda(M))) = \{36, \cdots, 42\}.$$

So, by Lemma 2.3.1, m can be expressed uniquely as

$$m := \prod_{i\in\{36,\cdots,42\}} \epsilon_i(a_i),\ \text{for some } a_i \in k.$$

Then by 5.4 we have

$$\begin{aligned}
u(a)P_1u(a)^{-1} =& mP_1m^{-1} \\
=& P_1P_1\left(\prod_{i\in\{36,\cdots,42\}}\epsilon_i(a_i)\right)P_1m^{-1} \\
=& P_1\left(\prod_{i\in\{36,\cdots,42\}}\epsilon_{P_1\cdot i}(a_i)\right)m^{-1} \\
=& P_1\Big(\epsilon_{38}(a_{36})\epsilon_{39}(a_{37})\epsilon_{36}(a_{38})\epsilon_{37}(a_{39})\epsilon_{41}(a_{40})\epsilon_{40}(a_{41})\epsilon_{42}(a_{42})\Big) \\
&\Big(\epsilon_{42}(a_{42})\epsilon_{41}(a_{41})\epsilon_{40}(a_{40})\epsilon_{39}(a_{39})\epsilon_{38}(a_{38})\epsilon_{37}(a_{37})\epsilon_{36}(a_{36})\Big) \\
=& P_1\Big(\epsilon_{36}(a_{36}+a_{38})\epsilon_{37}(a_{37}+a_{39})\epsilon_{38}(a_{36}+a_{38}) \\
&\epsilon_{39}(a_{37}+a_{39})\epsilon_{40}(a_{40}+a_{41})\epsilon_{41}(a_{40}+a_{41})\Big).
\end{aligned}$$

This is a contradiction because $u(a)P_1u(a)^{-1}$ contains a nontrivial ϵ_{42} term by 5.3. Therefore $c_\lambda(u(a)P_1u(a)^{-1})$ is not $R_u(P_\lambda(M))$-conjugate to $u(a)P_1u(a)^{-1}$, and we are done. $\square$

Appendix A

Calculation in E_8 with Maple

We use Maple to calculate $C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})}(H)$. Here, $M[29], \cdots, M[120]$ represent the roots of $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta\xi})$, and the vector $M[i]$ gives the coefficients of the root i with respect to $\sigma, \alpha, \beta, \gamma, \delta, \epsilon, \eta, \xi$ in this order. For example, $M[29]$ represents σ, $M[48]$ represents $\sigma + \gamma + 2\delta + 2\epsilon + 2\eta$, and $M[100]$ represents $2\sigma + \beta + 2\gamma + 3\delta + 4\epsilon + 3\eta + 2\xi$, etc.

```
> M[29]:=[1,0,0,0,0,0,0,0]:M[30]:=[1,0,0,0,0,1,0,0]:
> M[31]:=[1,0,0,0,1,1,0,0]:M[32]:=[1,0,0,0,0,1,1,0]:
> M[33]:=[1,0,0,1,1,1,0,0]:M[34]:=[1,0,0,0,1,1,1,0]:
> M[35]:=[1,0,0,0,0,1,1,1]:M[36]:=[1,0,1,1,1,1,0,0]:
> M[37]:=[1,0,0,1,1,1,1,0]:M[38]:=[1,0,0,0,1,2,1,0]:
> M[39]:=[1,0,0,0,1,1,1,1]:M[40]:=[1,1,1,1,1,1,0,0]:
> M[41]:=[1,0,1,1,1,1,1,0]:M[42]:=[1,0,0,1,1,2,1,0]:
> M[43]:=[1,0,0,1,1,1,1,1]:M[44]:=[1,0,0,0,1,2,1,1]:
> M[45]:=[1,1,1,1,1,1,1,0]:M[46]:=[1,0,1,1,1,2,1,0]:
> M[47]:=[1,0,1,1,1,1,1,1]:M[48]:=[1,0,0,1,2,2,1,0]:
> M[49]:=[1,0,0,1,1,2,1,1]:M[50]:=[1,0,0,0,1,2,2,1]:
> M[51]:=[1,1,1,1,1,2,1,0]:M[52]:=[1,1,1,1,1,1,1,1]:
> M[53]:=[1,0,1,1,2,2,1,0]:M[54]:=[1,0,1,1,1,2,1,1]:
> M[55]:=[1,0,0,1,2,2,1,1]:M[56]:=[1,0,0,1,1,2,2,1]:
> M[57]:=[1,1,1,1,2,2,1,0]:M[58]:=[1,1,1,1,1,2,1,1]:
> M[59]:=[1,0,1,2,2,2,1,0]:M[60]:=[1,0,1,1,2,2,1,1]:
> M[61]:=[1,0,1,1,1,2,2,1]:M[62]:=[1,0,0,1,2,2,2,1]:
> M[63]:=[1,1,1,2,2,2,1,0]:M[64]:=[1,1,1,1,2,2,1,1]:
> M[65]:=[1,1,1,1,1,2,2,1]:M[66]:=[1,0,1,2,2,2,1,1]:
> M[67]:=[1,0,1,1,2,2,2,1]:M[68]:=[1,0,0,1,2,3,2,1]:
> M[69]:=[1,1,2,2,2,2,1,0]:M[70]:=[1,1,1,2,2,2,1,1]:
> M[71]:=[1,1,1,1,2,2,2,1]:M[72]:=[1,0,1,2,2,2,2,1]:
```

```
> M[73]:=[1,0,1,1,2,3,2,1]:M[74]:=[1,1,2,2,2,2,1,1]:
> M[75]:=[1,1,1,2,2,2,2,1]:M[76]:=[1,1,1,1,2,3,2,1]:
> M[77]:=[1,0,1,2,2,3,2,1]:M[78]:=[1,1,2,2,2,2,2,1]:
> M[79]:=[1,1,1,2,2,3,2,1]:M[80]:=[1,0,1,2,3,3,2,1]:
> M[81]:=[1,1,2,2,2,3,2,1]:M[82]:=[1,1,1,2,3,3,2,1]:
> M[83]:=[1,1,2,2,3,3,2,1]:M[84]:=[1,1,2,3,3,3,2,1]:
> M[85]:=[2,0,0,1,2,3,2,1]:M[86]:=[2,0,1,1,2,3,2,1]:
> M[87]:=[2,1,1,1,2,3,2,1]:M[88]:=[2,0,1,2,2,3,2,1]:
> M[89]:=[2,1,1,2,2,3,2,1]:M[90]:=[2,0,1,2,3,3,2,1]:
> M[91]:=[2,1,2,2,2,3,2,1]:M[92]:=[2,1,1,2,3,3,2,1]:
> M[93]:=[2,0,1,2,3,4,2,1]:M[94]:=[2,1,2,2,3,3,2,1]:
> M[95]:=[2,1,1,2,3,4,2,1]:M[96]:=[2,0,1,2,3,4,3,1]:
> M[97]:=[2,1,2,3,3,3,2,1]:M[98]:=[2,1,2,2,3,4,2,1]:
> M[99]:=[2,1,1,2,3,4,3,1]:M[100]:=[2,0,1,2,3,4,3,2]:
> M[101]:=[2,1,2,3,3,4,2,1]:M[102]:=[2,1,2,2,3,4,3,1]:
> M[103]:=[2,1,1,2,3,4,3,2]:M[104]:=[2,1,2,3,4,4,2,1]:
> M[105]:=[2,1,2,3,3,4,3,1]:M[106]:=[2,1,2,2,3,4,3,2]:
> M[107]:=[2,1,2,3,4,4,3,1]:M[108]:=[2,1,2,3,3,4,3,2]:
> M[109]:=[2,1,2,3,4,5,3,1]:M[110]:=[2,1,2,3,4,4,3,2]:
> M[111]:=[2,1,2,3,4,5,3,2]:M[112]:=[2,1,2,3,4,5,4,2]:
> M[113]:=[3,1,2,3,4,5,3,1]:M[114]:=[3,1,2,3,4,5,3,2]:
> M[115]:=[3,1,2,3,4,5,4,2]:M[116]:=[3,1,2,3,4,6,4,2]:
> M[117]:=[3,1,2,3,5,6,4,2]:M[118]:=[3,1,2,4,5,6,4,2]:
> M[119]:=[3,1,3,4,5,6,4,2]:M[120]:=[3,2,3,4,5,6,4,2]:
```

The next code lists all roots $\{i, j, i\tilde{+}j\}$ such that $[U_i, U_j] \neq 0$. For example, the output $[29, 68, 85]$ means $[U_{29}, U_{68}] \neq 0$ and $29\tilde{+}68 = 85$.

```
> S:=NULL:
> for i from 29 to 120
> do
> for j from i+1 to 120
> do
> for k from 29 to 120
> do
> if M[i]+M[j]=M[k]
> then S:=S,[i,j,k]:
> end if
> end do
> end do
> end do;
```

```
> S;
```
We get

$$[29,68,85],[29,73,86],[29,76,87],[29,77,88],[29,79,89],[29,80,90],$$
$$[29,81,91],[29,82,92],[29,83,94],[29,84,97],[29,109,113],[29,111,114],$$
$$[29,112,115],[30,62,85],[30,67,86],[30,71,87],[30,72,88],[30,75,89],$$
$$[30,78,91],[30,80,93],[30,82,95],[30,83,98],[30,84,101],[30,107,113],$$
$$[30,110,114],[30,112,116],[31,56,85],[31,61,86],[31,65,87],[31,72,90],$$
$$[31,75,92],[31,77,93],[31,78,94],[31,79,95],[31,81,98],[31,84,104],$$
$$[31,105,113],[31,108,114],[31,112,117],[32,55,85],[32,60,86],[32,64,87],$$
$$[32,66,88],[32,70,89],[32,74,91],[32,80,96],[32,82,99],[32,83,102],$$
$$[32,84,105],[32,104,113],[32,110,115],[32,111,116],[33,50,85],[33,61,88],$$
$$[33,65,89],[33,67,90],[33,71,92],[33,73,93],[33,76,95],[33,78,97],$$
$$[33,81,101],[33,83,104],[33,102,113],[33,106,114],[33,112,118],[34,49,85],$$
$$[34,54,86],[34,58,87],[34,66,90],[34,70,92],[34,74,94],[34,77,96],$$
$$[34,79,99],[34,81,102],[34,84,107],[34,101,113],[34,108,115],[34,111,117],$$
$$[35,48,85],[35,53,86],[35,57,87],[35,59,88],[35,63,89],[35,69,91],$$
$$[35,80,100],[35,82,103],[35,83,106],[35,84,108],[35,104,114],[35,107,115],$$
$$[35,109,116],[36,50,86],[36,56,88],[36,62,90],[36,65,91],[36,68,93],$$
$$[36,71,94],[36,75,97],[36,76,98],[36,79,101],[36,82,104],[36,99,113],$$
$$[36,103,114],[36,112,119],[37,44,85],[37,54,88],[37,58,89],[37,60,90],$$
$$[37,64,92],[37,73,96],[37,74,97],[37,76,99],[37,81,105],[37,83,107],$$
$$[37,98,113],[37,106,115],[37,111,118],[38,43,85],[38,47,86],[38,52,87],$$
$$[38,66,93],[38,70,95],[38,72,96],[38,74,98],[38,75,99],[38,78,102],$$
$$[38,84,109],[38,97,113],[38,108,116],[38,110,117],[39,42,85],[39,46,86],$$
$$[39,51,87],[39,59,90],[39,63,92],[39,69,94],[39,77,100],[39,79,103],$$
$$[39,81,106],[39,84,110],[39,101,114],[39,105,115],[39,109,117],[40,50,87],$$
$$[40,56,89],[40,61,91],[40,62,92],[40,67,94],[40,68,95],[40,72,97],[40,73,98],$$
$$[40,77,101],[40,80,104],[40,96,113],[40,100,114],[40,112,120],[41,44,86],$$
$$[41,49,88],[41,55,90],[41,58,91],[41,64,94],[41,68,96],[41,70,97],$$
$$[41,76,102],[41,79,105],[41,82,107],[41,95,113],[41,103,115],[41,111,119],$$
$$[42,47,88],[42,52,89],[42,60,93],[42,64,95],[42,67,96],[42,71,99],$$
$$[42,74,101],[42,78,105],[42,83,109],[42,94,113],[42,106,116],[42,110,118],$$

$[43, 46, 88], [43, 51, 89], [43, 53, 90], [43, 57, 92], [43, 69, 97], [43, 73, 100],$
$[43, 76, 103], [43, 81, 108], [43, 83, 110], [43, 98, 114], [43, 102, 115], [43, 109, 118],$
$[44, 45, 87], [44, 59, 93], [44, 63, 95], [44, 69, 98], [44, 72, 100], [44, 75, 103],$
$[44, 78, 106], [44, 84, 111], [44, 97, 114], [44, 105, 116], [44, 107, 117], [45, 49, 89],$
$[45, 54, 91], [45, 55, 92], [45, 60, 94], [45, 66, 97], [45, 68, 99], [45, 73, 102],$
$[45, 77, 105], [45, 80, 107], [45, 93, 113], [45, 100, 115], [45, 111, 120], [46, 52, 91],$
$[46, 55, 93], [46, 62, 96], [46, 64, 98], [46, 70, 101], [46, 71, 102], [46, 75, 105],$
$[46, 82, 109], [46, 92, 113], [46, 103, 116], [46, 110, 119], [47, 48, 90], [47, 51, 91],$
$[47, 57, 94], [47, 63, 97], [47, 68, 100], [47, 76, 106], [47, 79, 108], [47, 82, 110],$
$[47, 95, 114], [47, 99, 115], [47, 109, 119], [48, 52, 92], [48, 54, 93], [48, 58, 95],$
$[48, 61, 96], [48, 65, 99], [48, 74, 104], [48, 78, 107], [48, 81, 109], [48, 91, 113],$
$[48, 106, 117], [48, 108, 118], [49, 53, 93], [49, 57, 95], [49, 67, 100], [49, 69, 101],$
$[49, 71, 103], [49, 78, 108], [49, 83, 111], [49, 94, 114], [49, 102, 116], [49, 107, 118],$
$[50, 59, 96], [50, 63, 99], [50, 66, 100], [50, 69, 102], [50, 70, 103], [50, 74, 106],$
$[50, 84, 112], [50, 97, 115], [50, 101, 116], [50, 104, 117], [51, 55, 95], [51, 60, 98],$
$[51, 62, 99], [51, 66, 101], [51, 67, 102], [51, 72, 105], [51, 80, 109], [51, 90, 113],$
$[51, 100, 116], [51, 110, 120], [52, 53, 94], [52, 59, 97], [52, 68, 103], [52, 73, 106],$
$[52, 77, 108], [52, 80, 110], [52, 93, 114], [52, 96, 115], [52, 109, 120], [53, 56, 96],$
$[53, 58, 98], [53, 65, 102], [53, 70, 104], [53, 75, 107], [53, 79, 109], [53, 89, 113],$
$[53, 103, 117], [53, 108, 119], [54, 57, 98], [54, 62, 100], [54, 63, 101], [54, 71, 106],$
$[54, 75, 108], [54, 82, 111], [54, 92, 114], [54, 99, 116], [54, 107, 119], [55, 61, 100],$
$[55, 65, 103], [55, 69, 104], [55, 78, 110], [55, 81, 111], [55, 91, 114], [55, 102, 117],$
$[55, 105, 118], [56, 57, 99], [56, 60, 100], [56, 64, 103], [56, 69, 105], [56, 74, 108],$
$[56, 83, 112], [56, 94, 115], [56, 98, 116], [56, 104, 118], [57, 61, 102], [57, 66, 104],$
$[57, 72, 107], [57, 77, 109], [57, 88, 113], [57, 100, 117], [57, 108, 120], [58, 59, 101],$
$[58, 62, 103], [58, 67, 106], [58, 72, 108], [58, 80, 111], [58, 90, 114], [58, 96, 116],$
$[58, 107, 120], [59, 64, 104], [59, 65, 105], [59, 71, 107], [59, 76, 109], [59, 87, 113],$
$[59, 103, 118], [59, 106, 119], [60, 63, 104], [60, 65, 106], [60, 75, 110], [60, 79, 111],$
$[60, 89, 114], [60, 99, 117], [60, 105, 119], [61, 63, 105], [61, 64, 106], [61, 70, 108],$
$[61, 82, 112], [61, 92, 115], [61, 95, 116], [61, 104, 119], [62, 69, 107], [62, 74, 110],$
$[62, 81, 112], [62, 91, 115], [62, 98, 117], [62, 101, 118], [63, 67, 107], [63, 73, 109],$
$[63, 86, 113], [63, 100, 118], [63, 106, 120], [64, 72, 110], [64, 77, 111], [64, 88, 114],$

$$[64,96,117],[64,105,120],[65,66,108],[65,80,112],[65,90,115],[65,93,116],$$
$$[65,104,120],[66,71,110],[66,76,111],[66,87,114],[66,99,118],[66,102,119],$$
$$[67,70,110],[67,79,112],[67,89,115],[67,95,117],[67,101,119],[68,69,109],$$
$$[68,74,111],[68,78,112],[68,91,116],[68,94,117],[68,97,118],[69,85,113],$$
$$[69,100,119],[69,103,120],[70,73,111],[70,86,114],[70,96,118],[70,102,120],$$
$$[71,77,112],[71,88,115],[71,93,117],[71,101,120],[72,76,112],[72,87,115],$$
$$[72,95,118],[72,98,119],[73,75,112],[73,89,116],[73,92,117],[73,97,119],$$
$$[74,85,114],[74,96,119],[74,99,120],[75,86,115],[75,93,118],[75,98,120],$$
$$[76,88,116],[76,90,117],[76,97,120],[77,87,116],[77,92,118],[77,94,119],$$
$$[78,85,115],[78,93,119],[78,95,120],[79,86,116],[79,90,118],[79,94,120],$$
$$[80,87,117],[80,89,118],[80,91,119],[81,85,116],[81,90,119],[81,92,120],$$
$$[82,86,117],[82,88,118],[82,91,120],[83,85,117],[83,88,119],[83,89,120],$$
$$[84,85,118],[84,86,119],[84,87,120]$$

In the next code, we input $\prod_{\lambda\in\{29,\cdots,84\}}\epsilon_{n_\zeta\cdot\lambda}(a)$ for $\zeta\in\{\alpha,\beta,\gamma,\delta,\epsilon,\eta,\xi\}$. Here, "`alpha`" represent $\prod_{\lambda\in\{29,\cdots,84\}}\epsilon_{n_\alpha\cdot\lambda}(a)$, and `[29,a]` in "`alpha`" represents $\epsilon_{29}(a)$.

```
> alpha:=[[29,a],[30,a],[31,a],[32,a],[33,a],[34,a],[35,a],[40,a],
> [37,a],[38,a],[39,a],[36,a],[45,a],[42,a],[43,a],[44,a],[41,a],
> [51,a],[52,a],[48,a],[49,a],[50,a],[46,a],[47,a],[57,a],[58,a],
> [55,a],[56,a],[53,a],[54,a],[63,a],[64,a],[65,a],[62,a],[59,a],
> [60,a],[61,a],[70,a],[71,a],[68,a],[69,a],[66,a],[67,a],[75,a],
> [76,a],[74,a],[72,a],[73,a],[79,a],[78,a],[77,a],[82,a],[81,a],
> [80,a],[83,a],[84,a]];

> beta:=[[29,a],[30,a],[31,a],[32,a],[36,a],[34,a],[35,a],[33,a],
> [41,a],[38,a],[39,a],[40,a],[37,a],[46,a],[47,a],[44,a],[45,a],
> [42,a],[43,a],[53,a],[54,a],[50,a],[51,a],[52,a],[48,a],[49,a],
> [60,a],[61,a],[57,a],[58,a],[59,a],[55,a],[56,a],[67,a],[69,a],
> [64,a],[65,a],[66,a],[62,a],[73,a],[63,a],[74,a],[71,a],[72,a],
> [68,a],[70,a],[78,a],[76,a],[77,a],[75,a],[81,a],[80,a],[79,a],
> [83,a],[82,a],[84,a]];

> Gamma:=[[29,a],[30,a],[33,a],[32,a],[31,a],[37,a],[35,a],[36,a],
> [34,a],[42,a],[43,a],[40,a],[41,a],[38,a],[39,a],[49,a],[45,a],
> [46,a],[47,a],[48,a],[44,a],[56,a],[51,a],[52,a],[59,a],[54,a],
> [55,a],[50,a],[63,a],[58,a],[53,a],[66,a],[61,a],[62,a],[57,a],
> [70,a],[65,a],[60,a],[72,a],[68,a],[69,a],[64,a],[75,a],[67,a],
```

```
> [77,a],[74,a],[71,a],[79,a],[73,a],[78,a],[76,a],[80,a],[81,a],
> [82,a],[83,a],[84,a]];

> Delta:=[[29,a],[31,a],[30,a],[34,a],[33,a],[32,a],[39,a],[36,a],
> [37,a],[38,a],[35,a],[40,a],[41,a],[48,a],[43,a],[44,a],[45,a],
> [53,a],[47,a],[42,a],[55,a],[50,a],[57,a],[52,a],[46,a],[60,a],
> [49,a],[62,a],[51,a],[64,a],[59,a],[54,a],[67,a],[56,a],[63,a],
> [58,a],[71,a],[66,a],[61,a],[68,a],[69,a],[70,a],[65,a],[72,a],
> [73,a],[74,a],[75,a],[76,a],[80,a],[78,a],[82,a],[77,a],[83,a],
> [79,a],[81,a],[84,a]];

> epsilon:=[[30,a],[29,a],[31,a],[32,a],[33,a],[38,a],[35,a],[36,a],
> [42,a],[34,a],[44,a],[40,a],[46,a],[37,a],[49,a],[39,a],[51,a],
> [41,a],[54,a],[48,a],[43,a],[50,a],[45,a],[58,a],[53,a],[47,a],
> [55,a],[56,a],[57,a],[52,a],[59,a],[60,a],[61,a],[68,a],[63,a],
> [64,a],[65,a],[66,a],[73,a],[62,a],[69,a],[70,a],[76,a],[77,a],
> [67,a],[74,a],[79,a],[71,a],[72,a],[81,a],[75,a],[80,a],[78,a],
> [82,a],[83,a],[84,a]];

> eta:=[[29,a],[32,a],[34,a],[30,a],[37,a],[31,a],[35,a],[41,a],
> [33,a],[38,a],[39,a],[45,a],[36,a],[42,a],[43,a],[50,a],[40,a],
> [46,a],[47,a],[48,a],[56,a],[44,a],[51,a],[52,a],[53,a],[61,a],
> [62,a],[49,a],[57,a],[65,a],[59,a],[67,a],[54,a],[55,a],[63,a],
> [71,a],[58,a],[72,a],[60,a],[68,a],[69,a],[75,a],[64,a],[66,a],
> [73,a],[78,a],[70,a],[76,a],[77,a],[74,a],[79,a],[80,a],[81,a],
> [82,a],[83,a],[84,a]];

> xi:=[[29,a],[30,a],[31,a],[35,a],[33,a],[39,a],[32,a],[36,a],
> [43,a],[44,a],[34,a],[40,a],[47,a],[49,a],[37,a],[38,a],[52,a],
> [54,a],[41,a],[55,a],[42,a],[50,a],[58,a],[45,a],[60,a],[46,a],
> [48,a],[56,a],[64,a],[51,a],[66,a],[53,a],[61,a],[62,a],[70,a],
> [57,a],[65,a],[59,a],[67,a],[68,a],[74,a],[63,a],[71,a],[72,a],
> [73,a],[69,a],[75,a],[76,a],[77,a],[78,a],[79,a],[80,a],[81,a],
> [82,a],[83,a],[84,a]];
```

The next code checks whether $\{U_i, U_j\}$ is a non-commuting pair of root subgroups or not, and if so, gives $[true, i+j]$.

```
> IsInList:=proc(a,b,F)
> local i, x, K;
> for i from 1 to nops(F) do
> x:=[op(1,op(i,F)),op(2,op(i,F))];
```

```
> if [a,b]=x then
> K:=[true,op(3,op(i,F))];
> end if;
> end do;
> K;
> end proc;
```

The next procedure reorders $\prod_{\lambda\in\{29,\cdots,84\}} \epsilon_{n_\zeta\cdot\lambda}(a)$ for $\zeta \in \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \xi\}$ and produces extra terms if they occur.

```
> Reorder:=proc(W)
> local i,j,z,s,t,M,L;
> L:=W:
> for i from 1 to 200 do
> for j from 1 to nops(L)-1 do
> if op(1, op(j,L)) > op(1, op(j+1,L)) then
> z:=IsInList(op(1,op(j+1,L)),op(1,op(j,L)),[S]);
> if z[1] = true then
> s:=op(2,op(j,L));
> t:=op(2,op(j+1,L));
> M:=op(1..j-1,L),op(j+1,L),op(j,L),[z[2],s*t],op(j+2..nops(L),L);
> L:=[M];
> else M:=op(1..j-1,L),op(j+1,L),op(j,L),op(j+2..nops(L),L);
> L:=[M];
> end if;
> end if;
> end do;
> end do;
> print(L);
> end proc;
```

Using this procedure, we get

```
> Reorder(alpha);
```

$$
\begin{gathered}
[[29,a],[30,a],[31,a],[32,a],[33,a],[34,a],[35,a],[36,a],[37,a],[38,a],[39,a],[40,a],\\
[41,a],[42,a],[43,a],[44,a],[45,a],[46,a],[47,a],[48,a],[49,a],[50,a],[51,a],[52,a],\\
[53,a],[54,a],[55,a],[56,a],[57,a],[58,a],[59,a],[60,a],[61,a],[62,a],[63,a],[64,a],\\
[65,a],[66,a],[67,a],[68,a],[69,a],[70,a],[71,a],[72,a],[73,a],[74,a],[75,a],[76,a],\\
[77,a],[78,a],[79,a],[80,a],[81,a],[82,a],[83,a],[84,a],[86,a^2],[87,a^2],[90,a^2],\\
[91,a^2],[91,a^2],[92,a^2],[96,a^2],[98,a^2],[98,a^2],[99,a^2],[104,a^2],[104,a^2],\\
[105,a^2],[105,a^2],[106,a^2],[106,a^2],[110,a^2],[110,a^2],[112,a^2],[112,a^2],\\
[113,a^3],[113,a^3],[113,a^3],[113,a^3],[114,a^3],[114,a^3],[114,a^3],[114,a^3],\\
[114,a^3],[114,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3],\\
[116,a^3],[116,a^3],[116,a^3],[116,a^3],[116,a^3],[116,a^3],[117,a^3],[117,a^3],\\
[117,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3],[118,a^3],[118,a^3],\\
[118,a^3],[118,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],\\
[119,a^3],[119,a^3],[119,a^3],[119,a^3],[120,a^3],[120,a^3],[120,a^3],[120,a^3],\\
[120,a^3],[120,a^3],[120,a^3],[120,a^3],[120,a^3],[120,a^3]]
\end{gathered}
$$

```
> Reorder(beta);
```

$$
\begin{gathered}
[[29,a],[30,a],[31,a],[32,a],[33,a],[34,a],[35,a],[36,a],[37,a],[38,a],[39,a],[40,a],\\
[41,a],[42,a],[43,a],[44,a],[45,a],[46,a],[47,a],[48,a],[49,a],[50,a],[51,a],[52,a],\\
[53,a],[54,a],[55,a],[56,a],[57,a],[58,a],[59,a],[60,a],[61,a],[62,a],[63,a],[64,a],\\
[65,a],[66,a],[67,a],[68,a],[69,a],[70,a],[71,a],[72,a],[73,a],[74,a],[75,a],[76,a],\\
[77,a],[78,a],[79,a],[80,a],[81,a],[82,a],[83,a],[84,a],[88,a^2],[88,a^2],[92,a^2],\\
[93,a^2],[93,a^2],[94,a^2],[99,a^2],[100,a^2],[100,a^2],[102,a^2],[107,a^2],[107,a^2],\\
[109,a^2],[109,a^2],[111,a^2],[111,a^2],[113,a^3],[113,a^3],[114,a^3],[114,a^3],\\
[114,a^3],[114,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3],[116,a^3],[116,a^3],\\
[116,a^3],[116,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3],\\
[118,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3],\\
[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],\\
[120,a^3],[120,a^3],[120,a^3],[120,a^3]]
\end{gathered}
$$

```
> Reorder(Gamma);
```

$$[[29, a], [30, a], [31, a], [32, a], [33, a], [34, a], [35, a], [36, a], [37, a], [38, a], [39, a], [40, a], [41, a], [42, a], [43, a], [44, a], [45, a], [46, a], [47, a], [48, a], [49, a], [50, a], [51, a], [52, a], [53, a], [54, a], [55, a], [56, a], [57, a], [58, a], [59, a], [60, a], [61, a], [62, a], [63, a], [64, a], [65, a], [66, a], [67, a], [68, a], [69, a], [70, a], [71, a], [72, a], [73, a], [74, a], [75, a], [76, a], [77, a], [78, a], [79, a], [80, a], [81, a], [82, a], [83, a], [84, a], [85, a^2], [85, a^2], [87, a^2], [89, a^2], [96, a^2], [96, a^2], [98, a^2], [101, a^2], [102, a^2], [104, a^2], [104, a^2], [105, a^2], [106, a^2], [108, a^2], [110, a^2], [110, a^2], [112, a^2], [112, a^2], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3]]$$

```
> Reorder(Delta);
```

$$[[29, a], [30, a], [31, a], [32, a], [33, a], [34, a], [35, a], [36, a], [37, a], [38, a], [39, a], [40, a], [41, a], [42, a], [43, a], [44, a], [45, a], [46, a], [47, a], [48, a], [49, a], [50, a], [51, a], [52, a], [53, a], [54, a], [55, a], [56, a], [57, a], [58, a], [59, a], [60, a], [61, a], [62, a], [63, a], [64, a], [65, a], [66, a], [67, a], [68, a], [69, a], [70, a], [71, a], [72, a], [73, a], [74, a], [75, a], [76, a], [77, a], [78, a], [79, a], [80, a], [81, a], [82, a], [83, a], [84, a], [88, a^2], [90, a^2], [91, a^2], [93, a^2], [93, a^2], [94, a^2], [95, a^2], [95, a^2], [98, a^2], [98, a^2], [99, a^2], [99, a^2], [100, a^2], [100, a^2], [101, a^2], [103, a^2], [103, a^2], [104, a^2], [105, a^2], [106, a^2], [106, a^2], [107, a^2], [108, a^2], [110, a^2], [113, a^3], [113, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3]]$$

```
> Reorder(epsilon);
```

$$[[29,a],[30,a],[31,a],[32,a],[33,a],[34,a],[35,a],[36,a],[37,a],[38,a],[39,a],[40,a], [41,a],[42,a],[43,a],[44,a],[45,a],[46,a],[47,a],[48,a],[49,a],[50,a],[51,a],[52,a], [53,a],[54,a],[55,a],[56,a],[57,a],[58,a],[59,a],[60,a],[61,a],[62,a],[63,a],[64,a], [65,a],[66,a],[67,a],[68,a],[69,a],[70,a],[71,a],[72,a],[73,a],[74,a],[75,a],[76,a], [77,a],[78,a],[79,a],[80,a],[81,a],[82,a],[83,a],[84,a],[85,a^2],[85,a^2],[86,a^2], [86,a^2],[88,a^2],[88,a^2],[89,a^2],[89,a^2],[90,a^2],[91,a^2],[91,a^2],[93,a^2], [94,a^2],[98,a^2],[110,a^2],[111,a^2],[112,a^2],[112,a^2],[113,a^3],[113,a^3], [113,a^3],[113,a^3],[113,a^3],[113,a^3],[113,a^3],[113,a^3],[113,a^3],[113,a^3], [114,a^3],[114,a^3],[114,a^3],[114,a^3],[114,a^3],[114,a^3],[114,a^3],[114,a^3], [114,a^3],[114,a^3],[114,a^3],[114,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3], [115,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3],[115,a^3], [116,a^3],[116,a^3],[116,a^3],[116,a^3],[116,a^3],[116,a^3],[116,a^3],[116,a^3], [116,a^3],[116,a^3],[116,a^3],[116,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3], [117,a^3],[117,a^3],[117,a^3],[117,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3], [118,a^3],[118,a^3],[118,a^3],[118,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3], [119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[120,a^3],[120,a^3], [120,a^3],[120,a^3],[120,a^3],[120,a^3]]$$

```
> Reorder(eta);
```

$$[[29,a],[30,a],[31,a],[32,a],[33,a],[34,a],[35,a],[36,a],[37,a],[38,a],[39,a],[40,a], [41,a],[42,a],[43,a],[44,a],[45,a],[46,a],[47,a],[48,a],[49,a],[50,a],[51,a],[52,a], [53,a],[54,a],[55,a],[56,a],[57,a],[58,a],[59,a],[60,a],[61,a],[62,a],[63,a],[64,a], [65,a],[66,a],[67,a],[68,a],[69,a],[70,a],[71,a],[72,a],[73,a],[74,a],[75,a],[76,a], [77,a],[78,a],[79,a],[80,a],[81,a],[82,a],[83,a],[84,a],[87,a^2],[87,a^2],[93,a^2], [96,a^2],[98,a^2],[100,a^2],[100,a^2],[101,a^2],[102,a^2],[103,a^2],[103,a^2], [104,a^2],[105,a^2],[106,a^2],[106,a^2],[107,a^2],[110,a^2],[110,a^2],[111,a^2], [112,a^2],[113,a^3],[113,a^3],[114,a^3],[114,a^3],[115,a^3],[115,a^3],[116,a^3], [116,a^3],[116,a^3],[116,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3],[117,a^3], [117,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3],[118,a^3],[119,a^3], [119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[119,a^3],[120,a^3], [120,a^3],[120,a^3],[120,a^3],[120,a^3],[120,a^3],[120,a^3],[120,a^3],[120,a^3], [120,a^3]]$$

```
> Reorder(xi);
```

$$[[29, a], [30, a], [31, a], [32, a], [33, a], [34, a], [35, a], [36, a], [37, a], [38, a], [39, a], [40, a], [41, a], [42, a], [43, a], [44, a], [45, a], [46, a], [47, a], [48, a], [49, a], [50, a], [51, a], [52, a], [53, a], [54, a], [55, a], [56, a], [57, a], [58, a], [59, a], [60, a], [61, a], [62, a], [63, a], [64, a], [65, a], [66, a], [67, a], [68, a], [69, a], [70, a], [71, a], [72, a], [73, a], [74, a], [75, a], [76, a], [77, a], [78, a], [79, a], [80, a], [81, a], [82, a], [83, a], [84, a], [85, a^2], [85, a^2], [86, a^2], [86, a^2], [88, a^2], [88, a^2], [89, a^2], [89, a^2], [91, a^2], [91, a^2], [92, a^2], [92, a^2], [93, a^2], [93, a^2], [95, a^2], [95, a^2], [96, a^2], [98, a^2], [98, a^2], [100, a^2], [102, a^2], [104, a^2], [104, a^2], [105, a^2], [106, a^2], [107, a^2], [108, a^2], [110, a^2], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [113, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [114, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [115, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [116, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [117, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [118, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [119, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3], [120, a^3]]$$

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