The Stabilized Poincare-Heisenberg algebra: a Clifford algebra viewpoint

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Abstract. The stabilized Poincare-Heisenberg algebra (SPHA) is the Lie algebra of quantum relativistic kinematics generated by fifteen generators. It is obtained from imposing stability conditions after attempting to combine the Lie algebras of quantum mechanics and relativity which by themselves are stable, however not when combined. In this paper we show how the sixteen dimensional Clifford algebra $C\ell(1,3)$ can be used to generate the SPHA. The Clifford algebra path to the SPHA avoids the traditional stability considerations, relying instead on the fact that $C\ell(1,3)$ is a semi-simple algebra and therefore stable. It is therefore conceptually easier and more straightforward to work with a Clifford algebra. The Clifford algebra path suggests the next evolutionary step toward a theory of physics at the interface of GR and QM might be to depart from working in space-time and instead to work in space-time-momentum.

1. Introduction

Physical theories are merely approximations to the natural world and the physical constants involved cannot be known without some degree of uncertainty. Properties of a model that are sensitive to small changes in the model, in particular changes in the values of the parameters, are unlikely to be observed. It can thus be reasoned that one should search for physical theories which do not change in a qualitative matter under a small change of the parameters. Such theories are said to be physically *stable*.

This concept of the physical stability of a theory can be given a mathematical meaning as follows. A mathematical structure is said to be mathematically stable for a class of deformations if any deformation in this class leads to an isomorphic structure. More precisely, a Lie algebra is said to be stable if small perturbations in its structure constants lead to isomorphic Lie algebras. The idea of mathematical stability provides insight into the validity of a physical theory or the need for a generalization of the theory. If a theory is not stable, one might choose to deform it until a stable theory is reached. Such a stable theory is likely to be a generalization of wider validity compared to the original unstable theory.

Lie algebraic deformation theory has been historically successful. Snyder [1] in 1947 showed that the assumption that spacetime be a continuum is not required for Lorentz invariance. Snyder's framework however leads to a lack of translational invariance, which later in the same year, Yang [2] showed can be corrected if one allows for spacetime to be curved. Yang, in the same paper also presented the complete Lie algebra associated with the suggested corrections. It was Mendes [3] who in the last decade concluded that when one considers the Poincare and Heisenberg algebras together, the resultant Poincare-Heisenberg algebra is not a stable Lie algebra. Mendes showed however that the algebra can be stabilized, requiring two additional length scales. The stabilized algebra is the same as the algebra obtained by Yang in 1947. It was Faddeev [4] and Mendes who noted that, in hindsight, stability considerations could have predicted the relativistic and quantum revolutions of the last century. Chryssomalakos and Okon [5, 6] showed that by a suitable identification of the generators, triply special relativity proposed by Kowalski-Glikman and Smolin [20] can be brought to a linear form and that the resulting algebra is again the same as Yang's algebra.

More recently, stability considerations have led to the stabilized Poincare-Heisenberg algebra (SPHA) as the favorite candidate for the Lie algebra describing physics at the interface of GR and QM. Chryssomalakos and Okon [5] showed uniqueness of the SPHA. Incorporating gravitational effects in quantum measurement of spacetime events renders spacetime non-commutative and leads to modifications in the fundamental commutators [9]. In 2005, Ahluwalia-Khalilova [7,] showed that the fact that the Heisenberg fundamental commutator, $[X_\mu, P_\mu] = i\hbar$, undergoes non-trivial modifications at the interface of GR and QM, suggests quantum mechanics and relativity will undergo numerous corrections and modification including modification of the position-momentum Heisenberg uncertainty relations. On top of this, spacetime in SPHA requires two new length scales, one in the extreme short distance range, the other on the cosmological scale. We denote these new length scales ℓ_p and ℓ_c respectively (following the notation used in [7]).

In addition, SPHA points to the existence of another dimensionless constant β (or α_3 in [5]) which, if nonzero, will radically affect some of the quantum relativistic notions. The presence of β has been noted in [3, 5, 12] with different emphasis. Chryssomalakos and Okon [5] show that it is always possible to gauge away this dimensionless constant by a suitable redefinition of the generators. The overall meaning of β seems to not be well understood in the literature. We show that Clifford algebra allows one to obtain an explicit expression for β in terms of an angle parameter φ. This is a step toward understanding how β will affect various quantum relativistic notions.

Our approach to the SPHA is quite different in that we adopt a Clifford algebra perspective. A physical theory should have its roots in the geometry underlying

the theory and be represented in a way that captures this geometry accordingly. Clifford (geometric) algebra is the appropriate tool that allows us to naturally encompass the underlying geometry of the space we are working in. Although a brief overview of Clifford algebra will be presented in section 3 of this paper, we omit any in-depth discussion of Clifford algebra and how it arises from geometry. The reader is instead referred to the following texts by Lounesto [14], Hestenes [15, 16] and Doran and Lasenby [13].

The SPHA has fifteen generators. We wish to represent these generators by elements of the Clifford algebra $C\ell(1,3)$ which is a sixteen dimensional algebra. The non-scalar basis elements of $C\ell(1,3)$ can be used to generate the SPHA by taking commutators with the Clifford product. ($C\ell(1,3)$ is chosen over $C\ell(3,1)$ or other sixteen dimensional Clifford algebras for reasons explained in [17, 18].) It is interesting to note that using this Clifford algebra, one avoids all stability considerations. This is because the Clifford algebra $C\ell(1,3)$ is semi-simple and therefore stable. Some further discussion on this is reserved for the next section. Ahluwalia-Khalilova [9] noted that incorporating gravitational effects in quantum measurement of spacetime events renders spacetime non-commutative and leads to modifications in the fundamental commutators. This non-commutativity of space-time and modifications to the fundamental commutators arise naturally from Clifford algebra. There thus seems to be a number of reasons why one should consider using Clifford algebra in theories of quantum gravity.

Since we are logically led to consider two additional length scales as well as a dimensionless constant in the proposed new algebra for kinematics at the interface of GR and QM, it is important to note that on heuristic grounds Amelino-Camelia [19] and Smolin [20] have also considered such a path. However, in the work presented here such resulting modifications to the quantum and spacetime structure arise not as an ad hoc speculation but from the Principle of Lie Algebraic stability. The reader is referred to [21, 22, 23, 5, 7, 24] for additional discussion of these and related issues.

2.   Stability Theory

As noted by Mendes [3] and Faddeev [4], in hindsight, the paradigm of algebraic deformations to obtain stable theories has the power to have predicted the relativistic and the quantum revolutions of the last century. This section will show how algebraic deformations lead from the Poisson algebra of classical mechanics to the Heisenberg algebra of quantum mechanics and from the Galilean algebra of Galilean relativity to the Poincare algebra of special relativity. The theory of Lie-algebraic deformations is not discussed in detail here. For a thorough and complete treatment the reader is referred to Gerstenhaber [11], Nijenhuis and Richardson [10] and Chryssomalakos and Okon [5].

When considering the Poisson and Galilean algebras, one finds that the algebraic structures are unstable. It is however possible to stabilize both of the algebras. Doing so requires two deformation parameters. These turn out to be the physical constants $\frac{1}{c^2}$ and $\hbar$ for the Galilean algebra and Poisson algebra respectively. Chryssomalakos and Okon [5] explain that both the Galilean and Poisson algebra cases, the deformed algebras are isomorphic for all non-zero values of $\frac{1}{c^2}$ and $\hbar$. The values of these deformation parameters are determined by experiment.

Neither the Heisenberg nor the Poincare algebras preserve their stability at the interface of GR and QM. A first attempt to find an algebra describing physics at the interface of GR and QM may be to take the direct sum of the Poincare and Heisenberg algebras to give the Poincare-Heisenberg algebra but this algebra however is not stable. Mendes [5] and Chryssomalakos and Okon [5] have both emphasised that one of the important criteria to consider for a theory to be physically viable, is the stability of the underlying Lie algebras, and so the Poincare-Heisenberg algebra fails to be the algebra we desire.

The Poincare-Heisenberg algebra can however be stabilized, requiring two additional length scales and a dimensionless constant in the same manner that the stabilization of the algebras of Galilean and classical kinematics requires two constants $\frac{1}{c^2}$ and $\hbar$. The resulting algebra is the stabilized Poincare-Heisenberg algebra (SPHA), the algebra of kinematics at the interface of GR and QM. It will be shown later that the SPHA is precisely the algebra obtained by taking commutators of the elements of the Clifford space-time algebra $C\ell(1,3)$.

3.   The Stabilised Poincare-Heisenberg Algebra

As noted above, the stabilised Poincare-Heisenberg algebra comes from combining quantum mechanics and relativity to get a stable theory of kinematics at the interface of GR and QM. The algebra is given below in a similar form to that of Chryssomalakos and Okon [5] (which in turn is a somewhat more abstract and mathematical version than the one given by Ahluwalia-Khalilova [7] which focuses more on the physical aspects of the algebra).

$$[iJ_{\mu\nu}, iJ_{\rho\sigma}] = -(\eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}) \tag{1}$$

$$[iJ_{\mu\nu}, P_\lambda] = -(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu) \tag{2}$$

$$[iJ_{\mu\nu}, X_\lambda] = -\hbar(\eta_{\nu\lambda}X_\mu - \eta_{\mu\lambda}X_\nu) \tag{3}$$

$$[P_\mu, P_\nu] = q\alpha_1 J_{\mu\nu} \tag{4}$$

$$[X_\mu, X_\nu] = q\alpha_2 J_{\mu\nu} \tag{5}$$

$$[P_\mu, X_\nu] = q\eta_{\mu\nu}M + q\alpha_3 J_{\mu\nu} \tag{6}$$

$$[P_\mu, iM] = -\alpha_1 X_\mu + \alpha_3 P_\mu \tag{7}$$

$$[X_\mu, iM] = -\alpha_3 X_\mu + \alpha_2 P_\mu \tag{8}$$

$$[iJ_{\mu\nu}, iM] = 0 \tag{9}$$

This paper is primarily concerned with finding a Clifford representation of the SPHA. For this reason we focus on the mathematical theory and therefore adopt the notation used by Chryssomalakos and Okon [5] instead of the notation used by Ahluwalia-Khalilova [7], with the exception of the part of section 5.

4.   The Clifford Algebra $C\ell(1,3)$

The Clifford algebra $C\ell(1,3)$ is a 16 dimensional associative algebra with a basis consisting of one scalar 1, four vectors e_μ, six bivectors $e_{\mu\nu}$, three trivectors $e_{\mu\nu\rho}$ and a pseudoscalar $e_{\mu\nu\rho\sigma} = e_{0123}$ which for brevity we write e, together with the Clifford product (geometric product) used to multiply elements of the algebra. Duals of elements of the Clifford algebra are found by multiplying through by the pseudoscalar $e = e_{0123}$. Trivectors are thus dual to vectors. Instead of writing trivectors as $e_{\mu\nu\rho}$, we write them as ee_μ for reasons which will become clear later.

As noted earlier, we do not have to concern ourselves with the issue of stability when using a Clifford algebra. All Clifford algebras are stable because the diagonal entries of the metric $\eta_{\mu\nu}$ are non-zero and therefore under a small perturbation, the signature remains unchanged.

We are looking for a Clifford representation of the stabilized Poincare-Heisenberg algebra. This means that X_μ, P_μ, $iJ_{\mu\nu}$ and iM are to be represented by elements of a real Clifford algebra. The stabilized Poincare-Heisenberg algebra is a Lie algebra so that we will require the Lie product to be a commutator in the Clifford algebra, i.e. $[x, y] = xy - yx$. The scalar element of the Clifford algebra commutes with every element leaving us with 15 generators. We will start with $\omega_1 e_\mu$, $\omega_2 ee_\mu$, $\omega_3 e_{\mu\nu}$ and $\omega_4 e$, where $\omega_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, and calculate their commutators.

$$
\begin{aligned}
[\omega_3 e_{\mu\nu}, \omega_3 e_{\rho\sigma}] &= 2\omega_3^2(\eta_{\nu\rho}e_{\mu\sigma} + \eta_{\mu\sigma}e_{\nu\rho} - \eta_{\mu\rho}e_{\nu\sigma} - \eta_{\nu\sigma}e_{\mu\rho}) & (10) \\
[\omega_3 e_{\mu\nu}, \omega_2 ee_\rho] &= 2\omega_2\omega_3(\eta_{\nu\rho}ee_\mu - \eta_{\mu\rho}ee_\nu) & \\
&= 2\omega_3(\eta_{\nu\rho}(\omega_2 ee_\mu) - \eta_{\mu\rho}(\omega_2 ee_\nu)) & (11) \\
[\omega_3 e_{\mu\nu}, \omega_1 e_\rho] &= 2\omega_1\omega_3(\eta_{\nu\rho}e_\mu - \eta_{\mu\rho}e_\nu) & \\
&= 2\omega_2(\eta_{\nu\rho}(\omega_1 e_\mu) - \eta_{\mu\rho}(\omega_1 e_\nu)) & (12)
\end{aligned}
$$

$$
\begin{aligned}
[\omega_2 e e_\mu, \omega_2 e e_\nu] &= 2\omega_2^2 e_{\mu\nu} \\
&= 2\frac{\omega_2^2}{\omega_3}(\omega_3 e_{\mu\nu}) && (13) \\
[\omega_1 e_\mu, \omega_1 e_\nu] &= 2\omega_1^2 e_{\mu\nu} \\
&= 2\frac{\omega_1^2}{\omega_3}(\omega_3 e_{\mu\nu}) && (14) \\
[\omega_2 e e_\mu, \omega_1 e_\nu] &= 2\omega_1 \omega_2 \eta_{\mu\nu} e \\
&= 2\eta_{\mu\nu}\frac{\omega_1 \omega_2}{\omega_4}(\omega_4 e) && (15) \\
[\omega_2 e e_\mu, \omega_4 e] &= 2\omega_2 \omega_4 e_\mu \\
&= 2\frac{\omega_2 \omega_4}{\omega_1}(\omega_1 e_\mu) && (16) \\
[\omega_1 e_\mu, \omega_4 e] &= -2\omega_1 \omega_4 e e_\mu \\
&= -2\frac{\omega_1 \omega_4}{\omega_2}(\omega_2 e e_\mu) && (17) \\
[\omega_3 e_{\mu\nu}, \omega_4 e] &= 0 && (18)
\end{aligned}
$$

By defining the 15 operators

$$
X_\mu = \omega_1 e_\mu = \frac{1}{2}\sqrt{-q\alpha_2}e_\mu \tag{19}
$$

$$
P_\mu = \omega_2 e e_\mu = \frac{1}{2}\sqrt{-q\alpha_1}e e_\mu \tag{20}
$$

$$
iJ_{\mu\nu} = \omega_3 e_{\mu\nu} = \frac{-1}{2}e_{\mu\nu} \tag{21}
$$

$$
iM = \omega_4 e = \frac{-1}{2}\sqrt{\alpha_1 \alpha_2}e \tag{22}
$$

we obtain the stabilized Poincare-Heisenberg algebra for the special case where α_3 is equal to zero. (We discuss this point in the next section.) We call X_μ, P_μ, $iJ_{\mu\nu}$ and iM above the Clifford generators of the stabilised Poincare-Heisenberg algebra. As in [17, 18], we interpret X_μ to be the position vectors, P_μ to be the momenta and $iJ_{\mu\nu}$ to be the rotations and boosts.

5.   Clifford representation with $\alpha_3 \neq 0$

In this section we will transform the Clifford representation in such a way that the transformed Clifford generators will generate the entire stabilized Poincare-Heisenberg algebra rather than just the special case where α_3 is equal to zero. The physical interpretation of this transformation will be discussed in the following section and gives rise to a new concept in physics.

We start by defining by redefining X_μ as

$$X_\mu = a\,e_\mu + b\,ee_\mu \tag{23}$$

giving

$$X_\mu X_\nu = (a^2 + b^2)e_{\mu\nu}$$

and therefore

$$[X_\mu, X_\nu] = 2(a^2 + b^2)\,e_{\mu\nu}, \qquad \mu \neq \nu \tag{24}$$

What we require is;

$$[X_\mu, X_\nu] = i\,q\,\alpha_2\,J_{\mu\nu}$$

To get this we must thus define;

$$iJ_{\mu\nu} = \frac{2(a^2 + b^2)}{q\,\alpha_2}\,e_{\mu\nu}, \qquad \mu \neq \nu \tag{25}$$

Similarly, define P_μ as

$$P_\mu = d\,ee_\mu + c\,e_\mu \tag{26}$$

giving

$$P_\mu P_\nu = (c^2 + d^2)e_{\mu\nu}$$

and

$$[P_\mu, P_\nu] = 2(c^2 + d^2)\,e_{\mu\nu}, \qquad \mu \neq \nu \tag{27}$$

We want this to be equal to

$$[P_\mu, P_\nu] = i\,q\,\alpha_1\,J_{\mu\nu}$$

which requires

$$iJ_{\mu\nu} = \frac{2(c^2 + d^2)}{q\,\alpha_1}\,e_{\mu\nu}, \qquad \mu \neq \nu \tag{28}$$

Note that consistency in the definition for $J_{\mu\nu}$ implies that

$$\frac{a^2 + b^2}{\alpha_2} = \frac{c^2 + d^2}{\alpha_1}$$

The above transformed definitions for X_μ and P_μ can be written in the form

$$\begin{bmatrix} X_\mu \\ P_\mu \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_\mu \\ ee_\mu \end{bmatrix}$$

and therefore

$$\begin{bmatrix} e_\mu \\ ee_\mu \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} X_\mu \\ P_\mu \end{bmatrix}$$

or

$$e_\mu = \frac{1}{\Delta}(dX_\mu - bP_\mu) \tag{29}$$

$$ee_\mu = \frac{1}{\Delta}(-cX_\mu + aP_\mu) \tag{30}$$

where $\Delta = ad - bc$ is the determinant of the matrix.

Working out the commutator $[P_\mu, X_\nu]$ with the new transformed expressions for X_μ and P_μ, we get

$$\begin{aligned} P_\mu X_\nu &= (dee_\mu + ce_\mu)(ae_\nu + bee_\nu) \\ &= (ac + bd)e_{\mu\nu} + (ad - bc)ee_{\mu\nu} \end{aligned}$$

while

$$X_\nu P_\mu = (ac + bd)e_{\nu\mu} - (ad - bc)ee_{\nu\mu}$$

So, if $\mu = \nu$,

$$\begin{aligned} [P_\mu, X_\nu] &= P_\mu X_\mu - X_\mu P_\mu \\ &= (ac + bd)\eta_{\mu\nu} + (ad - bc)\eta_{\mu\nu}\, e \\ &\quad -(ac + bd)\eta_{\mu\nu} + (ad - bc)\eta_{\mu\nu}\, e \\ &= 2(ad - bc)\eta_{\mu\nu}\, e \end{aligned}$$

while for $\mu \neq \nu$,

$$[P_\mu, X_\nu] = 2(ac + bd)e_{\mu\nu}$$

We want this to be equal to

$$[P_\mu, X_\nu] = i\, q\eta_{\mu\nu}M + i\, q\alpha_3 J_{\mu\nu}$$

so that we have

$$(\mu = \nu) \quad i\,q\,M = 2(ad - bc)e \tag{31}$$

and

$$(\mu \neq \nu) \quad i\,q\alpha_3 J_{\mu\nu} = 2(ac + bd)e_{\mu\nu} \tag{32}$$

i.e.

$$iM = \frac{2(ad - bc)}{q}\,e \tag{33}$$

and

$$iJ_{\mu\nu} = \frac{2(ac + bd)}{q\,\alpha_3}\,e_{\mu\nu} \tag{34}$$

Note that consistency now requires

$$\frac{c^2 + d^2}{\alpha_1} = \frac{a^2 + b^2}{\alpha_2} = \frac{ac + bd}{\alpha_3}$$

Next consider $[P_\mu, iM] = \alpha_3 P_\mu - \alpha_1 X_\mu$. We have

$$
\begin{aligned}
{[P_\mu, iM]} &= \left[dee_\mu + ce_\mu, \frac{2(ad - bc)}{q}\,e\right] \\
&= \frac{2(ad - bc)}{q}\,[dee_\mu + ce_\mu, e] \\
&= \frac{2(ad - bc)}{q}(dee_\mu e + ce_\mu e - de^2 e_\mu - cee_\mu) \\
&= \frac{2(ad - bc)}{q}(de_\mu - ce_\mu e - de_\mu - cee_\mu) \\
&= \frac{4(ad - bc)}{q}(de_\mu - cee_\mu) \\
&= \frac{4\Delta}{q}\left(\frac{d}{\Delta}(dX_\mu - bP_\mu) - \frac{c}{\Delta}(-cX_\mu + aP_\mu)\right) \\
&= \frac{4}{q}(d^2 X_\mu - bdP_\mu + c^2 X_\mu - acP_\mu) \\
&= -\frac{4}{q}(ac + bd)P_\mu + \frac{4}{q}(c^2 + d^2)X_\mu \tag{35}
\end{aligned}
$$

For this to be equal to

$$[P_\mu, iM] = \alpha_3 P_\mu - \alpha_1 X_\mu$$

we must have

$$\alpha_3 = -\frac{4}{q}(ac + bd) \tag{36}$$

$$\alpha_1 = -\frac{4}{q}(c^2 + d^2) \tag{37}$$

and the consistency conditions must now be extended to read

$$\frac{c^2 + d^2}{\alpha_1} = \frac{a^2 + b^2}{\alpha_2} = \frac{ac + bd}{\alpha_3} = -\frac{q}{4} \tag{38}$$

We also need to check $[X_\mu, iM] = \alpha_2 P_\mu - \alpha_3 X_\mu$. We have

$$\begin{aligned}
[X_\mu, iM] &= \left[ae_\mu + bee_\mu, \frac{2(ad - bc)}{q} e \right] \\
&= \frac{2(ad - bc)}{q}(ae_\mu e + bee_\mu e - aee_\mu - be^2 e_\mu) \\
&= \frac{2(ad - bc)}{q}(-2aee_\mu + 2be_\mu) \\
&= \frac{4\Delta}{q}(be_\mu - aee_\mu) \\
&= \frac{4\Delta}{q}\left(\frac{b}{\Delta}(dX_\mu - bP_\mu) - \frac{a}{\Delta}(-cX_\mu + aP_\mu) \right) \\
&= \frac{4}{q}(bdX_\mu - b^2 P_\mu + acX_\mu - a^2 P_\mu) \\
&= -\frac{4}{q}(a^2 + b^2)P_\mu + \frac{4}{q}(ac + bd)X_\mu \tag{39}
\end{aligned}$$

So we need

$$\alpha_2 = -\frac{4}{q}(a^2 + b^2) \tag{40}$$

$$\alpha_3 = -\frac{4}{q}(ac + bd) \tag{41}$$

which hold by the consistency equations.

To satisfy the consistency equation (38), we may define

$$a = \sqrt{-\frac{q\,\alpha_2}{4}}\ \cos\theta, \qquad b = \sqrt{-\frac{q\,\alpha_2}{4}}\ \sin\theta$$

and

$$c = \sqrt{-\frac{q\,\alpha_1}{4}}\ \sin\phi, \qquad d = -\sqrt{-\frac{q\,\alpha_1}{4}}\ \cos\phi$$

and since also $\dfrac{ac + bd}{\alpha_3} = -\dfrac{q}{4}$, we obtain

$$\sin(\theta - \phi) = \frac{\alpha_3}{\sqrt{\alpha_1 \alpha_2}} \tag{42}$$

Comparing these results with those of Ahluwalia-Khalilova [7], we find that [1]

$$q\alpha_1 = \frac{1}{\ell_c^2}, \qquad q\alpha_2 = \ell_p^2 \tag{43}$$

This in turn tells us that

$$\alpha_3 = \frac{\ell_p}{\ell_c} \sin(\varphi) \tag{44}$$

where $\varphi = \theta - \phi$. In this section we have transformed the Clifford generators of the representation to obtain a representation where α_3 is not equal to zero. This is going in the opposite direction to Chryssomalakos and Okon [5] who begin with a representation where α_3 is not necessarily zero and then show that there always exists a representation in the α_1-α_2 plane with α_3 equal to zero by performing a linear redefinition of the generators. The Clifford algebra approach is thus consistent with [5].

6.   Physical Interpretation of Transformation

Chryssomalakos and Okon, [5], comment that physicists in particular may frown upon the idea of working with arbitrary linear combinations of momenta and positions. For this reason it is important that we interpret what the transformation in the previous section might mean physically.

The transformation

$$\omega_1 e_\mu \mapsto \sqrt{\frac{-q\alpha_2}{4}} \cos\theta e_\mu + \sqrt{\frac{-q\alpha_2}{4}} \sin\theta ee_\mu$$

---

[1]In addition to the two conversions in (43) above, there are two more conversions, one between M and F and one between α_3 and β. A comparison reveals that consistency between [5] and [7] requires that;

$$
\begin{aligned}
qM &= F \\
q\alpha_3 &= \beta \\
q\alpha_2 &= \frac{1}{\ell_c^2} \\
q\alpha_1 &= \ell_p^2
\end{aligned}
$$

and

$$\omega_2 \mathbf{e}\mathbf{e}_\mu \mapsto \sqrt{\frac{-q\alpha_1}{4}} \cos \phi ee_\mu + \sqrt{\frac{-q\alpha_1}{4}} \sin \phi e_\mu$$

is equivalent to adding some momentum to the position vector and vice versa. The magnitude of the vector is invariant. The transformation looks like a rotation in the position-momentum plane however $\omega_1 e_\mu$ and $\omega_2 ee_\mu$ are rotated by different angles.

What is the physical interpretation of this transformation? In a Newtonian mindset we consider time and space to be disjoint and one can determine the absolute time and position of an event. Switching to a relativistic mindset, we know that we can no longer treat space and time separately but that in fact we have to consider them together in what we call spacetime. It is no longer possible to determine the time and position of an event absolutely.

Similarly, in a Newtonian mindset, we can treat position and momentum separately and thus the above transformation may seem unphysical. In a quantum mechanical frame of mind however, we cannot measure position X_μ and momentum P_μ with absolute certainty. The more accurately we know one, the less accurately we know the other as is described by the Heisenberg uncertainty relationship $\Delta X_\mu \Delta P_\mu \geq \frac{\hbar}{2}$. This suggests that we cannot just think about position or momentum without considering the other. Furthermore, making a measurement of the position of some particle will in itself affect the position. The photon used to measure the particle's position will give the particle some momentum. On the quantum scale therefore a linear combination of position and momentum does make sense and in fact treating position and momentum separately as in Newtonian physics may no longer be desirable.

At the interface of GR and QM one combines quantum mechanics and relativity and therefore should consider spacetime-momentum instead of spacetime alone [9].

7.   Acknowledgments

Bibliography

1. H. S. Snyder, "Quantized space-time," Phys. Rev. **71** (1947) 38.
2. C. N. Yang, "On quantized space-time," Phys. Rev. **72** (1947) 874.

3. R. Vilela Mendes, "Deformations, stable theories and fundamental constants," J. Phys. A **27** (1994) 8091.

4. L. D. Faddeev 1989, "Mathematician's view on the development of physics, Frontiers in Physics: High technology and mathematics" ed H. A. Cerdeira and S. Lundqvist (Singapore: Word Scientific, 1989) pp. 238-46

5. C. Chryssomalakos and E. Okon, "Generalized quantum relativistic kinematics: A stability point of view," Int. J. Mod. Phys. D **13** (2004) 2003 [arXiv:hep-th/0410212].

6. C. Chryssomalakos and E. Okon, "Linear Form of 3-scale Relativity Algebra and the Relevance of Stability," Int. J. Mod. Phys. D **13** (2004) 1817 [arXiv:hep-th/0407080].

7. D. V. Ahluwalia-Khalilova, "A freely falling frame at the interface of gravitational and quantum realms," Class. Quant. Grav. **22** (2005) 1433 [arXiv:hep-th/0503141].

8. D. V. Ahluwalia-Khalilova, "Minimal spatio-temporal extent of events, neutrinos, and the cosmological constant problem," Int. J. Mod. Phys. D **14** (2005) 2151 [arXiv:hep-th/0505124].

9. D. V. Ahluwalia, "Quantum Measurements, Gravitation, And Locality," Phys. Lett. B **339** (1994) 301 [arXiv:gr-qc/9308007].

10. A. Nijenhuis and R. W. Richardson 1967 "Deformations of Lie algebra structures" J. Math. Mech. **17** 89

11. M. Gerstenhaber 1964 "On the Deformation of Rings and Algebras" Ann. Math. **79** 59-103

12. V. V. Khruschev and A. N. Leznov, "Relativistically invariant Lie algebras for kinematic observables in quantum space-time," Grav. Cosmol. **9** (2003) 159 [arXiv:hep-th/0207082].

13. C. Doran and A. Lasenby 2003 "Geometric Algebra for Physicists" (Cambridge University Press)

14. P. Lounesto 1997 "Clifford Algebras and Spinors", London Mathematical Society Lecture Note Series 239 (Cambridge University Press)

15. D. Hestenes 1966 "Space–Time Algebra" (Gordon and Breach, New York)

16. D. Hestenes 2003 "Spacetime physics with geometric algebra", Am. J. Phys. **71** (7), 691–714

17. W. P. Joyce and P. H. Butler 2002 "The Geometric Associative Algebras of Euclidean Space" Adv. Appl. Clifford Alg. **12** (2), 195–233

18. P. Butler and L. McAven 1998 "Space: The Anti-Euclidean Metric from the Structure of Rotations", In Proceedings of the XII International Colloquium on Group Theoretical Methods in Physics, 494–498 (International Press)

19. G. Amelino-Camelia, "Proposal of a second generation of quantum-gravity-motivated Lorentz-symmetry tests: Sensitivity to effects suppressed quadratically by the Planck scale," Int. J. Mod. Phys. D **12** (2003) 1633 [arXiv:gr-qc/0305057].

20. J. Kowalski-Glikman and L. Smolin, "Triply special relativity," Phys. Rev. D **70** (2004) 065020 [arXiv:hep-th/0406276].

21. S. Judes and M. Visser, "Conservation Laws in Doubly Special Relativity," Phys. Rev. D **68** (2003) 045001

22. D. Grumiller, W. Kummer and D. V. Vassilevich, "A note on the triviality of kappa-deformations of gravity," Ukr. J. Phys. **48** (2003) 329 [arXiv:hep-th/0301061].

23. R. Schutzhold and W. G. Unruh, "Problems of doubly special relativity with variable speed of light," JETP Lett. **78** (2003) 431 [Pisma Zh. Eksp. Teor. Fiz. **78** (2003) 899] [arXiv:gr-qc/0308049].

24. J. Christian, "Absolute being vs relative becoming," arXiv:gr-qc/0610049.

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