The Constructive Theory of Riesz Spaces and Applications in Mathematical Economics

A thesis
submitted in partial fulfilment
of the requirements for
the Degree of Doctor of Philosophy in Mathematics
at the
University of Canterbury
by

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2004

Abstract

This thesis is an introduction to a constructive development of the theory of ordered vector spaces. Order structures are examined constructively; that is, with intuitionistic logic.

Since the least–upper–bound principle does not hold constructively, some problems that are classically trivial are much more difficult from a constructive standpoint. The first problem in a constructive development of a theory is to find appropriate counterparts of the classical notions. We introduce a positive definition of an ordered vector space and we extend the constructive notions of supremum, order locatedness, and Dedekind completeness from the real number line to arbitrary partially ordered sets. As a main result, we prove that the supremum of a subset S exists if and only if S is upper located and has a weak supremum—that is, the classical least upper bound.

We investigate ordered vector spaces and, in particular, Riesz spaces with order units and their order duals. For an Archimedean space, we obtain several constructive counterparts of a classical theorem that links order units and Minkowski functionals. We also examine linearly ordered vector spaces; it turns out that, as in the classical case, any nontrivial Archimedean space with a linear order is isomorphic to $\mathbf{R}$.

Various notions of monotonicity for mappings and for preference relations are discussed. In particular, we examine positive operators and highlight the relationship between strong extensionality and strong positivity—a stronger counterpart of the classical positivity.

The last chapter is dedicated to applications in mathematical economics. We deal with the problem of the representation of a preference by a continuous utility function. Since strong extensionality is a necessary condition for such a representation, we examine in detail the relationship between continuity and strong extensionality and we obtain sufficient conditions for the latter property. We apply these results to obtain a theorem of representation for preferences with unit elements.

Acknowledgements

First of all, I would like to thank Professor Douglas Bridges, my supervisor, for his guidance and support.

This research would not have been possible without the financial support offered by the University of Canterbury and its Department of Mathematics and Statistics. I would also like to thank the New Zealand Mathematical Society and the New Zealand Mathematics Research Institute for conference and travel support.

I am most grateful to Professor Cristian Calude for his inspiring advice and continuous encouragement.

Special thanks are due to my professors from the University of Bucharest, Romania: to Professor Romulus Cristescu who introduced me in the realm of ordered vector spaces, and to Professor Solomon Marcus who encouraged me to undertake this doctoral research.

I had a great time at the phylogenetics meetings in Kaikoura and Whakapapa. Warm thanks to Professor Mike Steel and Dr Charles Semple. Thanks to all the people in the Department of Mathematics and Statistics for providing a friendly environment.

I am very grateful to Neville and Ella Bolsover, Douglas and Vivien Bridges, and all the friends who made me and my family feel at home in New Zealand.

A word of gratitude to my friends in Romania—Cristina Costache, Elena Costache, and Cătălina Iticescu—for their kind and efficient help.

I wish to thank my family: my wife, Mihaela, for her patience during three years of sharing the same office, and my daughter, Andreea, who still wonders what this thesis is useful for.

I am forever grateful to my parents, Marian and Virginia Baroni, for their love and support over the years. This thesis is dedicated to their memory.

Contents

Chapter 1

Introduction

This work is a first step towards a constructive development of the theory of ordered vector spaces. Almost every vector space over the real field has a natural ordering that is compatible in a certain way with the algebraic structure of the space. However, although the classical theory has been considerably developed in the last decades, constructive mathematics has paid much less attention than its classical counterpart to the order structures of vector spaces.

Before starting our constructive examination of ordered vector spaces, we should clarify its setting. By constructive mathematics, we mean Bishop–style mathematics which enables one to interpret the results both in classical mathematics and in other varieties of constructivism. We adopt Fred Richman's viewpoint [60]: constructive mathematics is simply mathematics carried out with intuitionistic logic.

1.1 Intuitionistic logic

When is a mathematical statement accepted as valid? For the traditional point of view, a statement is true whenever is not contradictory. Although certain mathematicians had expressed dissatisfaction regarding the idealistic content of mathematics, nonconstructive methods became standard at the end of the nineteenth century.

It was the Dutch mathematician Luitzen Egbertus Jan Brouwer (1881–1966) who attempted a fully constructive approach to mathematics. Brouwer's views about mathematics were presented first in his 1907 doctoral thesis [25]. In Brouwer's philosophy, known as "intuitionism", mathematics is a free creation of the human mind, and an object exists only if it can be (mentally) constructed.

When we work constructively, the logical connectives and quantifiers should be interpreted in a different manner.

- To prove $P \wedge Q$, we need a proof of P and a proof of Q.
- To prove $P \vee Q$, we must have either a proof of P or a proof of Q. Unlike in classical logic, it is not enough to prove the impossibility of $\neg P \wedge \neg Q$.
- To prove $P \Rightarrow Q$, we have to provide an algorithm that converts a proof of P into a proof of Q.[1]
- To prove $\neg P$, we must show that P implies $0 = 1$.
- To prove $\exists x \in A\ P(x)$, we require an algorithm that produces an object x and a proof that $P(x)$ holds.
- To prove $\forall x \in A\ P(x)$, we need an algorithm that, applied to an object x and a proof that $x \in A$, proves that $P(x)$ holds.

In view of the above interpretation of disjunction, the universal validity of the Aristotelian principle *Tertium non datur* (the **law of excluded middle**),

$$P \vee \neg P,$$

should be questioned. (We shall return to this subject in Section 1.3.) It was Brouwer [26] the first who observed that the law of excluded middle was extended without justification to statements about infinite sets.

[1] A proof of P is not required. However, if P has a proof, we should be able to obtain a proof of Q.

The analysis of the logical principles used in constructive proofs led Heyting[2] to the axioms of **intuitionistic logic**. Both Brouwer's intuitionistic mathematics and another variety of constructivism, the recursive constructive mathematics initiated by A.A. Markov, are based on intuitionistic logic together with some additional principles.[3] On the other hand, classical logic can be obtained from the axioms of intuitionistic logic by adding the law of excluded middle.

1.2 Bishop's constructive mathematics

In the 1960s, although intuitionism and Markov's constructive mathematics had inspired much work in logic and metamathematics, it was not at all clear that classical mathematics had a satisfactory alternative. The common viewpoint was still that of Hilbert [35]:

> *Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists.*

The publication of the monograph *Foundations of Constructive Analysis* [8] by Errett Bishop (1928–1983) changed the situation dramatically. As observed in one of the reviews [68] of the book, Bishop showed "that to replace the classical system by the constructive one does not in any way mutilate the great classical theories of mathematics. Not at all. If anything, it strenghthens them, and shows them, in a truer light, to be far grander than we had known."

Bishop showed that, contrary to the common view, a constructive development of a large part of analysis was possible. Not only was the content important but also the mathematics was written in a normal mathematical style. Although most

[2] Arend Heyting (1898–1980) was Brouwer's most famous pupil. His system of axioms, published first in 1930, can be found in the book [34].

[3] Instead of going into details about these varieties of constructive mathematics, we refer the reader to the books [6, 23, 69]. A survey, including Martin–Löf's constructive type theory, can be found in [18].

mathematicians have not been persuaded to work constructively, there have been important developments in constructive mathematics since 1967.

Why should a mathematician choose to work in this manner? First of all, Bishop–style mathematics is more general [58]. The theorems have more models, including intuitionism, recursive constructive mathematics and even classical mathematics. In contrast, the other varieties of constructivism are not consistent with classical mathematics. To illustrate this, see the continuity theorems of intuitionism and Markov's constructive mathematics [23].

Secondly, when working constructively we are able to make distinctions that are obscured in classical mathematics. According to Bishop [10], "meaningful distinctions deserve to be maintained." For example, pointwise continuity, sequential continuity, near continuity and nondiscontinuity of functions between metric spaces are classically equivalent but not constructively [24, 42]. See also Section 1.4 for a distinction between two classically equivalent notions of supremum. More details about constructive mathematics can be found in the expository papers [16, 57, 61, 66]. Bishop presented his views in the first chapter of his book, as well as in [9, 10]. For Bishop's philosophy of mathematics the reader is referred to [7, 31].

To end this section, let us see the positions of constructivists with respect to classical mathematics. Although Brouwer and other radical constructive mathematicians considered classical mathematics as illegitimate, liberal constructivists have a more flexible position.[4] The "only real requirement is that the use of the law of excluded middle be recognized and recorded" [58]. In our opinion, the liberalism of Douglas Bridges [22], Fred Richman, and others, is characterized mainly by their conception about constructive mathematics—as mathematics carried out with intuitionistic logic—and about mathematical objects. As pointed out by Richman [58, 59, 60], constructive mathematics is the constructive study—that is, with intuitionistic logic—of any mathematical objects, rather than the study of so–called constructive objects.

[4] The terms "radical" and "liberal" are used by Helen Billinge [7], to describe the constructivists' views about common (classical) mathematical practice.

1.3 Sources of nonconstructivity

Several consequences of the law of excluded middle are not accepted in Bishop's constructivism. We will briefly discuss four such nonconstructive principles.

- The **limited principle of omniscience** (**LPO**): for every binary sequence (a_n) either $a_n = 0$ for all n, or else there exists n such that $a_n = 1$.
- The **weak limited principle of omniscience** (**WLPO**): for every binary sequence (a_n) either $a_n = 0$ for all n, or it is contradictory that $a_n = 0$ for all n.
- The **lesser limited principle of omniscience** (**LLPO**): if (a_n) is a binary sequence containing at most one term equal to 1, then either $a_{2n} = 0$ for all n, or else $a_{2n+1} = 0$ for all n.
- **Markov's principle** (**MP**): if (a_n) is a binary sequence and $\neg\forall n\ (a_n = 0)$, then there exists n such that $a_n = 1$.

One can prove that LPO entails WLPO and the latter entails LLPO. These three principles are proved to be false both in Brouwer's intuitionism and in Markov's constructive mathematics [23, 69], and are regarded as essentially nonconstructive in Bishop's constructive mathematics. Although accepted in recursive constructive mathematics, MP is rejected in Bishop's constructivism.

LPO is equivalent to the decidability of equality on the real number line[5]

$$\forall x \in \mathbf{R}\ (x = 0 \vee x \neq 0),$$

to the law of trichotomy

$$\forall x \in \mathbf{R}\ (x < 0 \vee x = 0 \vee x > 0),$$

and to the statement

$$\forall x \in \mathbf{R}\ (x \leq 0 \vee x > 0).$$

[5] We consider the real number set $\mathbf{R}$ as defined in [8] or [11], or presented axiomatically as in [17] or [22]. A detailed constructive study of $\mathbf{R}$ can be found in [15].

WLPO is equivalent to the condition

$$\forall x \in \mathbf{R}\ (x \leq 0 \vee \neg(x \leq 0)).$$

LLPO is equivalent to the statement

$$\forall x \in \mathbf{R}\ (x \leq 0 \vee x \geq 0).$$

Markov's principle is equivalent to the statement

$$\forall x \in \mathbf{R}\ (\neg(x \leq 0) \Rightarrow x > 0).$$

A **Brouwerian counterexample** to a proposition P is a (constructive) proof that P implies some nonconstructive principle. This technique, due to Brouwer, enables one to show that certain propositions are nonconstructive. More details about nonconstructive principles and various classical theorems that are not constructively valid can be found in [41].

As we have already seen, the law of excluded middle can be regarded as the main source of nonconstructivity. Since the axiom of choice implies the law of excluded middle [32], it follows that the former is not constructive. Nevertheless, Bishop's mathematics accepts the following **axiom of dependent choice**:

> *If $a \in A$ and S is a subset of $A \times A$ such that for each $x \in A$ there exists $y \in A$ with $(x, y) \in S$, then there exists a sequence (a_n) of elements of A with $a_1 = a$ and $(a_n, a_{n+1}) \in S$ for all n.*

An alternative way to do constructive mathematics without the axiom of countable choice, a consequence of the axiom of dependent choice, is presented in [62].

1.4 An example

To illustrate the distinctive features of constructive mathematics, let us examine an example: the existence of the supremum of a subset of $\mathbf{R}$. Classically, according to the least–upper–bound principle, every nonempty subset of $\mathbf{R}$ that is bounded above has a supremum. The situation is much more complex from a constructive standpoint.

First, we have at least two options for the definition of supremum. An alternative is to consider the supremum of a set S as the least upper bound of S. In other words, a real number s is the supremum of S if it is an upper bound of S and $s \leq b$ whenever b is an upper bound of S. The most important virtue of this definition is that it can be used in a general context: the supremum of a subset of an arbitrary partially ordered set can be defined exactly in the same way. For real numbers we can use another definition: an upper bound s of S is the supremum of S if for each $x < s$ there exists an element a of S such that $x < a$. Classically, the two definitions are equivalent, but this does not hold constructively. Clearly, any supremum in the latter sense is a least upper bound; but the converse implication entails LPO [50].

When we work constructively, the latter supremum is in general more useful: it enables one to obtain stronger results. Since the notion of supremum is almost ubiquitous in the theory of ordered vector spaces, to develop a constructive theory we need a similar supremum, stronger than the classical least upper bound. As a consequence, the first major problem of this thesis was to obtain an appropriate generalization. Suprema of subsets of a lattice are characterized by Proposition 2.5.2. More generally, the supremum of a subset of an arbitrary partially ordered set is defined in Section 2.4. Nevertheless, both variants of supremum are considered in this work. As a consequence, each definition based on a supremum has in general a weaker counterpart based on the weak supremum—that is, the least upper bound.

Another problem that arises in the constructive theory is that of the Dedekind completeness, the completeness of a partially ordered set with respect to its ordering. The least–upper–bound principle is not valid from the constructive point of view, either in the stronger form (with suprema), or in the weaker version (with weak suprema). To see this consider an arbitrary binary sequence (a_n). If the set $\{a_n : n = 1, 2, \dots\}$ has either a supremum or a weak supremum s, then $s < 1$ or $s > 0$. In the former case, $a_n = 0$ for all n. If $s > 0$ and s satisfies the stronger condition of supremum, then there exists n with $a_n > 0$. When s is the weak supremum, the condition $s > 0$ implies that $\neg\ (\forall n(a_n = 0))$. Therefore the stronger version of the least–upper–bound principle implies LPO[6], and the weaker one entails WLPO.

Since the least–upper–bound principle is nonconstructive and, on the other hand, the Dedekind completeness of **R** is a reasonable requirement, it follows that an alternative definition should be given. For real numbers, there is a satisfactory property: the existence of the supremum is equivalent to a certain type of order locatedness (Proposition 4.3 in Chapter 2 of [11]). In Chapter 3 we will present a generalization of this order locatedness that enables us to define Dedekind completeness for arbitrary partially ordered sets.

1.5 An overview of the thesis

Following this introductory chapter, there are five chapters, a list of references, a list of symbols, and an index for quick referencing.

In Chapter 2 we begin the constructive examination of ordered vector spaces. Although the notion of partial order is fundamental in the classical theory, from a constructive standpoint this is a negative concept that should be replaced by a positive, stronger notion. Jan von Plato's excess relation [55] provides a constructive alternative and we consider only partially ordered sets as defined by von Plato.[7] Our definition[8] of an ordered vector space, as well as the generalization of the constructive supremum, use an excess relation rather than a partial order. Classically, given a pointed cone in a vector space, we obtain an ordered vector space in a standard way. We prove in Chapter 2 a constructive counterpart of this theorem.

In Chapter 3 we extend the notion of order locatedness from subsets of **R** to subsets of an arbitrary partially ordered set. As a main result, we prove that the supremum of a subset S exists if and only if S is upper located and has a least upper

[6] This was explained in detail in the first chapter of [8].

[7] Linearly and partially ordered sets are discussed in the first two sections of Chapter 2.

[8] We should emphasize that whenever a constructive definition has a classical counterpart, the two definitions are classically equivalent. To choose appropriate counterparts of the classical definitions is a major problem in constructive mathematics.

bound (weak supremum). A Dedekind complete set is defined in a natural way and, in particular, we prove that the spaces $\mathbf{R}^n$ are Dedekind complete with respect to the standard product order. We also obtain several equivalent conditions for the existence of the supremum, respectively weak supremum, of a subset of $\mathbf{R}^n$.

Chapter 4 deals with Archimedean spaces. We prove that, as in the classical case, a nontrivial Archimedean space X is a linearly ordered vector space if and only if $X = \mathbf{R}u$ for some vector $u \neq 0$. A classical result states that if the Archimedean space X has an order unit e, then the Minkowski functional of the order interval $[-e, e]$ is a norm. Ishihara's results on the Minkowski functional enable us to prove constructive counterparts of this theorem.

Various notions of positivity for operators between ordered vector spaces are investigated in Chapter 5. Classically, the order dual of a Riesz space is also a Riesz space, but we cannot prove constructively that every linear functional on a Riesz space has a modulus. However, we show that if X is a Riesz space with an order unit e, and φ is a linear functional on X, then $|\varphi|(x)$ is defined for all x whenever $|\varphi|(e)$ can be computed.

Chapter 6 is dedicated to applications in mathematical economics. We define a weak excess relation, a generalization of a (weak) preference relation, and we introduce several notions of monotonicity for such a relation. In particular, we obtain properties of monotonicity for preferences on a partially ordered set and for mappings between two partially ordered sets. We then examine various notions of continuity for a preference and we show that each type of continuity corresponds in a certain way to a notion of continuity of functions.

An important problem of microeconomics is to find sufficient conditions which guarantee the representation of a preference by a continuous utility function. Since strong extensionality is a necessary condition for such a representation, we examine in detail the relationship between continuity and strong extensionality. We show that even near continuity, a weak notion of continuity for preferences ensures strong extensionality. We also provide another sufficient condition—hyperextensionality, for the strong extensionality of a (weak) preference. We apply these results to obtain a theorem of representation for preference relations with unit elements.

Chapter 2

Ordered vector spaces

2.1 Preliminaries

As pointed out in [23], to define a set X we have to explain how to construct elements of X and to describe the equality between elements of X. We will consider every set X as endowed with a binary relation $=$ that satisfies the axioms of an **equivalence relation**:

E1 $x = x$;

E2 $x = y \Rightarrow y = x$;

E3 $(x = y \wedge y = z) \Rightarrow x = z$.

The relation $=$ is called the **equality** of X. We assume that each property P which is applicable to the elements of a set X is **extensional**, in the sense that for each pair x, x' of elements of X, $P(x)$ and $P(x')$ are equivalent whenever $x = x'$.

Let X be a nonempty[1] set. An **apartness relation**, as defined by Heyting, is a binary relation $\neq$ on X that satisfies the axioms of **irreflexivity**, **symmetry**, and **cotransitivity**:

[1] By "nonempty" we mean "inhabited"; we can construct an element of the set.

A1 $\neg(x \nleq x)$;

A2 $x \neq y \Rightarrow y \neq x$;

A3 $x \neq y \Rightarrow \forall z \in X\ (x \neq z \vee z \neq y)$.

A binary relation $>$ on X is called a **linear order** if it satisfies the following conditions:

LO1 $x > y \Rightarrow \neg(y > x)$;

LO2 $x > y \Rightarrow \forall z \in X\ (x > z \vee z > y)$;

LO3 $\neg(x > y \vee y > x) \Rightarrow x = y$.

In this case, we say that the set X is **linearly ordered** by $>$. As shown in [67], we obtain an apartness relation $\neq$ by the following definition:

$$x \neq y \Leftrightarrow (x > y \vee y > x)\,.$$

It follows from LO3 that the apartness is **tight** in the sense that $x = y$ whenever $x \neq y$ is impossible.

The linear order $>$ is called **dense** if for each pair x, y of elements of X with $x > y$, there exists $z \in X$ such that $x > z$ and $z > y$. For instance, the relation "greater than" on the real number set $\mathbf{R}$ is a dense linear order.

2.2 Partially ordered sets

To develop a constructive theory of ordered vector spaces, we need first to clarify the notion of a partially ordered set. From a constructive point of view, the partial order is a negative concept and, consequently, its role as a primary relation should be replaced by an affirmative, stronger relation. As shown by von Plato [55], a positive partial order, a generalization of the linear order, can be used to define a partially ordered set in a constructive manner.

Let X be a nonempty set. A binary relation $\not\leq$ on X is called a **positive partial order** if it satisfies the following axioms:

PO1 $\neg(x \not\leq x)$;

PO2 $x \not\leq y \Rightarrow \forall z \in X \ (x \not\leq z \vee z \not\leq y)$;

PO3 $\neg(x \not\leq y \vee y \not\leq x) \Rightarrow x = y$.

In this case, we say that X is a **partially ordered set**. Clearly, every linearly ordered set is also a partially ordered set. More precisely, a positive partial order is a linear order if it satisfies the condition LO1—**asymmetry**, instead of the weaker condition PO1.

Following von Plato, we say that $\not\leq$ is an **excess relation** if it satisfies the axioms PO1 and PO2. It is said that x **exceeds** y whenever $x \not\leq y$. As shown in [55], from an excess relation $\not\leq$ we obtain an apartness relation $\neq$ and an equality $=$ by the following definitions:

$$x \neq y \Leftrightarrow (x \not\leq y \vee y \not\leq x)\,;$$
$$x = y \Leftrightarrow \neg\,(x \neq y)\,.$$

Therefore an excess relation $\not\leq$ defines a positive partial order on X if and only if the equality of X coincides with the one obtained, as above, from $\not\leq$. In this case, as proved in [55], the relation $\leq$, defined by

$$x \leq y \Leftrightarrow \neg\,(x \not\leq y)\,,$$

is **reflexive**, **transitive**, and **antisymmetric**; that is, $\leq$ is a **partial order**. We then obtain a **strict partial order** in the standard way:

$$x < y \Leftrightarrow (x \leq y \wedge x \neq y)\,.$$

If an apartness and a partial order are considered as basic relations, the transitivity of strict order cannot be obtained. (A proof based on Kripke models is given by Greenleaf in [33].) In contrast, an excess relation as a primary relation enables us to prove this property. Moreover, it is straightforward to see that

$$(x \leq y \wedge y < z) \vee (x < y \wedge y \leq z) \Rightarrow x < z.$$

Clearly, given an excess relation $\nleq$, we can define its **dual excess relation** $\ngeq$ by

$$x \ngeq y \Leftrightarrow y \nleq x.$$

Both excess relations lead to the same apartness and therefore to the same equality. The partial order and the strict partial order obtained from $\ngeq$ are the relations $\geq$ and $>$. As expected:

$$x \geq y \Leftrightarrow y \leq x,$$

$$x > y \Leftrightarrow y < x.$$

The positive partial order $\nleq$ is a linear order if and only if it coincides with the corresponding strict partial order $>$.

To give an example, let us consider a set X of real-valued functions defined on a nonempty set S, and let $\nleq$ be the relation on X defined by $f \nleq g$ if there exists x in S such that $g(x) < f(x)$. Clearly, this is an excess relation whose corresponding partial order relation is the pointwise ordering of X. When $S = \{1, 2, \ldots, n\}$, we may view the set of all real-valued functions on S as the Cartesian product $\mathbf{R}^n$. From now on, unless otherwise stated, $\mathbf{R}^n$ will be considered as a partially ordered set with respect to this excess relation.

For an arbitrary partially ordered set X, we cannot expect to prove constructively any of the next four properties. It suffices to consider the linear ordering of $\mathbf{R}$ to observe that each condition entails the nonconstructive principle on the right–hand side.

- $\forall a, b \in X(a \leq b \vee a \nleq b)$ (LPO)
- $\forall a, b \in X(a \leq b \vee \neg(a \leq b))$ (WLPO)

- $\forall a, b \in X(a \leq b \vee\ b \leq a \vee (a \not\leq b \wedge b \not\leq a))$ (LLPO)
- $\forall a, b \in X(\neg(a \leq b) \Rightarrow a \not\leq b)$ (MP)

The next five statements hold whenever the order on X is linear, but cannot be proved for an arbitrary partially ordered set. If $X = \mathbf{R}^2$, then each condition implies a nonconstructive principle, as pointed out on the right of each line.

- $\forall a, b \in X\ (a \neq b \wedge \neg(a \leq b)) \Rightarrow a \not\leq b$ (MP)
- $\forall a, b \in X\ a \neq b \Rightarrow (a < b \vee a \not\leq b)$ (LPO)
- $\forall a, b \in X\ a \neq b \Rightarrow (a < b \vee \neg(a \leq b))$ (WLPO)
- $\forall a, b \in X\ a \neq b \Rightarrow ((a \not\leq b \wedge b \not\leq a) \vee a < b \vee b < a)$ (LPO)
- $\forall a, b \in X\ a \neq b \Rightarrow ((\neg(a \leq b) \wedge \neg(b \leq a)) \vee a < b \vee b < a)$ (WLPO)

To prove that the first statement entails Markov's principle, let x be any real number such that $\neg(x = 0)$, let $a = (|x|, 0)$, and let $b = (0, 1)$. Then $a \neq b$, $\neg(a \leq b)$, and $a \not\leq b$ if and only if $x \neq 0$. For the remaining four statements, let x be an arbitrary real number, $a = (0, 0)$, and $b = (1, x)$. Then, b exceeds a and, as a consequence, $a \neq b$, $\neg(b \leq a)$, and $\neg(b < a)$. It is sufficient now to observe that the conditions $a \not\leq b$, $a < b$, and $\neg(a \leq b)$ are equivalent to $x < 0$, $x \geq 0$, and $\neg(x \geq 0)$, respectively.

2.3 Ordered vector spaces

The classical theory of ordered vector spaces was founded, independently, by F. Riesz [63, 64], H. Freudenthal [30], and L.V. Kantorovich [45]. Classically, an ordered vector space is a real vector space equipped with a partial order relation that is invariant under translation and multiplication by positive scalars.[2] It seems unlikely that a constructive theory of ordered vector spaces can be developed with this

[2] For the classical theory of ordered vector spaces and Riesz spaces we refer the reader to the books [28, 48, 70, 73].

definition based on the negative relation $\leq$. In [4], we used the excess relation as a primary relation to define ordered vector spaces in a positive way.

Let X be a nonempty set that is partially ordered by the excess relation $\nleq$. Assume that X is endowed with an addition and a multiplication by real scalars satisfying the axioms of a real vector space, the equality being the one given by $\nleq$. The vector space X is called an **ordered vector space** if the following axioms are satisfied for all x,y in X, and α in $\mathbf{R}$:

O1 $\alpha x \nleq 0 \Rightarrow (\alpha > 0 \wedge x \nleq 0) \vee (\alpha < 0 \wedge 0 \nleq x)$;

O2 $x \nleq y \Rightarrow \forall z \in X\ (x + z \nleq y + z)$.

For instance, let us consider the vector space $\mathbf{R}^2$. The excess relation defined in Section 2.2 leads to the usual apartness relation: $(x_1, x_2) \neq (y_1, y_2)$ if $x_1 \neq y_1$ or $x_2 \neq y_2$. Consider now the excess relation defined by $(x_1, x_2) \nleq (y_1, y_2)$ if $x_1 \neq y_1$ or $y_2 < x_2$. By symmetrization, we obtain again the usual apartness. However, the partial order relations induced by the two excess relations are different [55]. It can be easily verified that in both cases $\mathbf{R}^2$ is an ordered vector space.

Clearly, the positive definition of an ordered vector space is classically equivalent to the classical one. Moreover, it can be proved constructively that the conditions O1 and O2 entail the classical axioms.

Proposition 2.3.1. *If X is an ordered vector space, then the following statements hold for all x,y in X and α in $\mathbf{R}$.*

(i) $\alpha \geq 0 \wedge x \leq y \Rightarrow \alpha x \leq \alpha y$;

(ii) $x \leq y \Rightarrow \forall z \in X\ (x + z \leq y + z)$.

Proof. (i) Firstly, we will prove that $\alpha x \leq \alpha y$ whenever $\alpha > 0$ and $x \leq y$. Indeed, assuming that $\alpha x \nleq \alpha y$, we see from O2 that $\alpha(x - y) \nleq 0$ and, from O1, that $x - y \nleq 0$. Hence $x \nleq y$, a contradiction.

Let us consider now $\alpha \geq 0$, $x, y \in X$ such that $x \leq y$, and assume that $\alpha x \nleq \alpha y$. If $\alpha > 0$, then $\alpha x \leq \alpha y$, a contradiction. Therefore $\alpha = 0$ and $\alpha x = 0 = \alpha y$, contradictory to $\alpha x \nleq \alpha y$. It follows that $\alpha x \leq \alpha y$.

(ii) Taking into account O2, we see that $x + z \nleq y + z$ implies $x \nleq y$. It now suffices to observe that (ii) is the contrapositive of this implication. □

Consider now the following implications:

O3 $\alpha > 0 \wedge x \nleq 0 \Rightarrow \alpha x \nleq 0$;

O4 $\alpha x \nleq 0 \Rightarrow \alpha \neq 0$;

O5 $\alpha x \neq 0 \Rightarrow \alpha \neq 0$.

It is easily to verify that X is an ordered vector space if and only if it satisfies the conditions O2, O3, and O4 or, equivalently, O2, O3, and O5.

Lemma 2.3.2. *If X is an ordered vector space, then the following conditions are satisfied for all x, y in X and α in* **R**.

(i) $x \neq y \Leftrightarrow x - y \neq 0$;

(ii) $\alpha x \neq 0 \Leftrightarrow (\alpha \neq 0 \wedge x \neq 0)$.

Proof. (i) This follows from O2.

(ii) If $\alpha x \neq 0$, then either $\alpha x \nleq 0$ or $-\alpha x \nleq 0$. It follows now from O1 that $\alpha \neq 0$ and $x \neq 0$. To prove the converse implication, let us observe that either α or $-\alpha$ is strictly positive and similarly that x or $-x$ exceeds 0. As a consequence of O3, either $\alpha x \nleq 0$ or $-\alpha x \nleq 0$. Therefore $\alpha x \neq 0$. □

Vector addition and multiplication by scalars are strongly extensional.

Corollary 2.3.3. *In each ordered vector space the following implications hold:*

(i) $x_1 + y_1 \neq x_2 + y_2 \Rightarrow x_1 \neq x_2 \vee y_1 \neq y_2$;

(ii) $\alpha x \neq \beta y \Rightarrow \alpha \neq \beta \vee x \neq y$.

Proof. (i) If $x_1 + y_1 \neq x_2 + y_2$, then either $x_1 + y_1 \neq x_2 + y_1$ or $x_2 + y_1 \neq x_2 + y_2$. It follows from Lemma 2.3.2(i) that $x_1 - x_2 \neq 0$ or $y_1 - y_2 \neq 0$. Therefore $x_1 \neq x_2$ or $y_1 \neq y_2$.

(ii) If $\alpha x \neq \beta y$, then either $\alpha x \neq \beta x$ or $\beta x \neq \beta y$, and hence $(\alpha - \beta)x \neq 0$ or $\beta(x - y) \neq 0$. It now follows from Lemma 2.3.2(ii) that $\alpha - \beta \neq 0$ or $x - y \neq 0$; that is, $\alpha \neq \beta$ or $x \neq y$. □

Corollary 2.3.4. *Let X be an ordered vector space and x,y in X such that $x < y$. Then $x + z < y + z$ for all z in X*

Proof. The hypothesis $x < y$ is equivalent to $x \leq y$ and $x \neq y$. According to Lemma 2.3.2(i), $0 \neq x - y = x + z - (y + z)$; whence $x + z \neq y + z$. To end the proof it suffices to observe that $x + z \leq y + z$ because $x \leq y$. □

In an ordered vector space X, the set

$$X^+ = \{x \in X : 0 \leq x\}$$

is called the **positive cone** of X, and its elements are said to be **positive**. Each vector $x > 0$ is called **strictly positive**.

Clearly, X^+ is a cone—that is, for all x, y in X^+ and $\alpha \geq 0$, the vectors $x + y$ and αx belong to X^+. Moreover,

$$-X^+ \cap X^+ = \{0\}$$

—that is, the cone X^+ is **pointed**. Conversely, if C is a pointed cone in the vector space X, then a partial order relation on X can be obtained in a natural way, by setting $x \leq y$ if $y - x \in C$. The space X satisfies the classical axioms of an ordered vector space with respect to this partial order. If we consider the set

$$X^e = \{x \in X : 0 \not\leq x\}$$

instead of X^+, we can obtain a similar result for constructive ordered vector spaces. Classically, X^e is the complement of X^+. Constructively, a subset S of X has two natural complementary subsets:

- the **logical complement**

  $\neg S = \{x \in X : \forall y \in S \neg(x = y)\}$,

- the **complement**

  $\sim S = \{x \in X : \forall y \in S\ (x \neq y)\}$.

It is straightforward to see that

$$\sim X^+ \subseteq \{x \in X : x \neq 0 \wedge \neg(0 \leq x)\} \subseteq \neg X^+.$$

However, we cannot expect to prove the converse inclusions constructively. For $X = \mathbf{R}$, the condition

$$\neg X^+ \subseteq \{x \in X : x \neq 0\}$$

entails Markov's principle. Furthermore, when $X = \mathbf{R}^2$, the Markov's principle is also a consequence of the condition $\{x \in X : x \neq 0 \wedge \neg(0 \leq x)\} \subseteq \sim X^+$. To prove this, let α be a real number such that $\neg(\alpha = 0)$ and $x = (-|\alpha|, 1)$. Then, $x \neq 0$, $\neg(0 \leq x)$; but $x \in \sim X^+$ entails $x \neq (0, 1)$ and therefore $\alpha \neq 0$.

Proposition 2.3.5. *If X is an ordered vector space, then $X^+ = \neg X^e, X^e \subseteq \sim X^+$, and the following conditions are satisfied for all x, y in X and $\alpha > 0$.*

(i) $x + y \in X^e \Rightarrow x \in X^e \vee y \in X^e$;

(ii) $\alpha > 0 \wedge x \in X^e \Rightarrow \alpha x \in X^e$;

(iii) $x \neq 0 \Leftrightarrow -x \in X^e \vee x \in X^e$.

Proof. Clearly, $X^+ = \neg X^e$. If $x \in X^e$ and $y \in X^+$, then y exceeds x. It follows that $X^e \subseteq \sim X^+$.

(i) If $0 \not\leq x + y$, then either $0 \not\leq x$ or $x \not\leq x + y$. In the former case $x \in X^e$ and in the latter $y \in X^e$.

(ii) If $\alpha > 0$ and $0 \nleq x$, then $0 \nleq \alpha x$.

(iii) This follows from the definition of X^{e}, taking into account that the relation $\neq$ is the apartness relation associated to $\nleq$. □

To obtain a reciprocal theorem, we will consider a vector space X with a tight apartness relation and we will assume that the algebraic operations are strongly extensional. As in the classical case, where each pointed cone is the positive cone of a certain ordered vector space, a subset S satisfying the properties (i)–(iii) leads us to an excess relation with respect to which X is an ordered vector space and $S = X^{\mathrm{e}}$.

Proposition 2.3.6. *Let X be a vector space with a tight apartness relation $\neq$, such that $\alpha \neq 0$ whenever $\alpha x \neq 0$, and $x \neq y$ if and only if $x - y \neq 0$. Let S be a subset of X satisfying the properties:*

(1) $x + y \in S \Rightarrow x \in S \vee y \in S$;

(2) $\alpha > 0 \wedge x \in S \Rightarrow \alpha x \in S$;

(3) $x \neq 0 \Leftrightarrow -x \in S \vee x \in S$.

If $\nleq$ is a binary relation on X defined by $x \nleq y$ if $y - x \in S$, then the following statements hold.

(i) *The relation $\nleq$ is an excess relation on X and $\neq$ is its corresponding apartness relation.*

(ii) *The vector space X is an ordered vector space with respect to the relation $\nleq$, $X^+ = \neg S$, and $S \subseteq_{\sim} X^+$.*

Proof. (i) From the condition (3) it follows that $\neg(0 \in S)$; therefore for all x in X we have $\neg(x \nleq x)$. Assume now that $x \nleq y$, and let z be an arbitrary element of X. Since $y - x \in S$ we have $y - z + z - x \in S$ and, according to (1), either $y - z$ or $z - x$ belongs to S. This proves the cotransitivity of $\nleq$.

To prove that the apartness relation defined by $\not\leq$ is identical to the relation $\neq$, let us consider x, y in X such that $x \neq y$. This is equivalent to $x - y \neq 0$ and, from (3), to the condition $x - y \in S$ or $y - x \in S$; that is, $x \not\leq y$ or $y \not\leq x$.

(ii) The vector space X satisfies the condition O5 ($\alpha x \neq 0 \Rightarrow \alpha \neq 0$). We will prove the conditions O2 and O3. If $x \not\leq y$, then for all z, $y + z - (x + z) \in S$ hence $x + z \not\leq y + z$. Consider now $\alpha > 0$ and $x \not\leq 0$. It follows that $-x \in S$ and, from (2), that $-\alpha x \in S$; that is, $\alpha x \not\leq 0$ and so condition O3 is proved. Consequently, X is an ordered vector space. Clearly, $x \in S$ if and only if $0 \not\leq x$; whence $S = X^e$ and, from Proposition 2.3.5, $X^+ = \neg S$ and $S \subseteq_\sim X^+$. □

When S is the empty subset of X, we obtain the null space. Indeed, the excess relation induced by S is the empty subset of $X \times X$ and, as a consequence, $x = y$ for each pair x, y of elements of X.

To end this section, let us consider a nontrivial ordered vector space: the space $C[0,1]$ of the real-valued continuous functions on $[0,1]$. The usual apartness is given by $f \neq g$ if $f(x) \neq g(x)$ for some x. If

$$S = \{f \in C[0,1] : \exists x\ (f(x) < 0)\},$$

then S satisfies the hypotheses of Proposition 2.3.6. The excess relation induced by S is the excess relation defined in Section 2.2: $f \not\leq g$ if $g(x) < f(x)$ for some x.

2.4 Suprema and infima

As in the classical case, a nonempty subset S of a partially ordered set X is said to be **bounded above** if there exists an element b of X such that $a \leq b$ for all a in S. In this case, b is called an **upper bound** for S. A **bounded below** subset and a **lower bound** are defined similarly, as expected. It is said that S is **order bounded** if it is bounded above and below. If a, b are elements of X, then the **order interval** $\{x \in X : a \leq x \leq b\}$ will be denoted by $[a, b]$. Clearly, S is order bounded if and only if $S \subseteq [a, b]$ for some elements a and b of X.

In the classical theory of partially ordered sets, the supremum is defined as the least upper bound. For real numbers we have a stronger notion: an element s of $\mathbf{R}$ is the supremum of a nonempty subset S if it is an upper bound for S and if for each $x < b$ there exists $a \in S$ with $x < a$. Classically, the two definitions are equivalent but this does not hold constructively [50]. This non-equivalence is due to the fact that the stronger definition is based on the affirmative concept of strict order, whereas the other one uses the weak relation "less than or equal to". To obtain a generalization of the stronger supremum for partially ordered sets, one needs the positive notion of an excess relation rather than the negative one of a partial order.

The definition of join of two elements of a lattice [55] can be easily extended to a general definition of the supremum. Consider an excess relation $\nleq$ on X, a nonempty subset S of X, and $s \in X$, an upper bound for S. We say that s is a

- **supremum** of S if $(x \in X \wedge s \nleq x) \Rightarrow \exists a \in S\ (a \nleq x)$;
- **weak supremum** of S if $(\ \forall a \in S\ (a \leq b)) \Rightarrow s \leq b$.

If S has a (weak) supremum, then that (weak) supremum is unique. We denote by $\sup S$ and w-sup S the supremum and the weak supremum of S, respectively, when they exist. The **infimum** $\inf S$ and the **weak infimum** w-inf S are defined similarly, as expected. A lower bound m for S is called the

- **infimum** of S if $(x \in X \wedge x \nleq m) \Rightarrow \exists a \in S\ (x \nleq a)$;
- **weak infimum** of S if $(\ \forall a \in S\ (b \leq a)) \Rightarrow b \leq m$.

Since each (weak) infimum with respect to the excess relation $\nleq$ is a (weak) supremum with respect to the dual relation $\ngeq$, we will obtain dual properties for (weak) supremum and (weak) infimum. Most of the results will be given for the suprema, without mentioning the corresponding counterparts for infima.

The next result is a necessary condition for the existence of the (weak) supremum.

Proposition 2.4.1. *Let S be a subset of a partially ordered set X and $x, y \in X$ such that y exceeds x.*

(i) *If* $\sup S$ *exists, then either y exceeds each element of S or else there exists a in S with $a \nleq x$.*

(ii) *If the weak supremum of S exists, then either y exceeds each element of S or it is contradictory for x to be an upper bound of S.*

Proof. If $s = \sup S$, then either y exceeds s or else s exceeds x. In the former case, $y \nleq a$ for all a in S. In the latter one, according to the definition of supremum, there exists an element a of S with $a \nleq x$. The weak supremum is handled similarly. □

Proposition 2.4.2. *For an upper bound s of S, the following conditions are equivalent.*

(1) $s = \text{w-sup}\, S$.

(2) $\neg(s \leq x) \Rightarrow \neg\, \forall a \in S\ (a \leq x)$.

(3) $s \nleq x \Rightarrow \neg\, \forall a \in S\ (a \leq x)$.

Proof. From the definition of w-sup S it follows that (1) implies (2). Since $s \nleq x$ entails $\neg(s \leq x)$, (2) implies (3). To prove that (1) is a consequence of (3), take an upper bound b of S and suppose that $s \nleq b$. Then, according to (3), b is not an upper bound for S, a contradiction. Therefore $\neg(s \nleq b)$, that is, $s \leq b$. □

For real numbers, the natural excess relation is given by the strict order relation: x exceeds y if x is greater than y. In this case the equivalence between (1) and (3) was proved in [50] (Proposition 4.7). By applying the general definition of supremum, we obtain the usual constructive definition [8] for the supremum of a subset of $\mathbf{R}$. As shown by Mandelkern (Proposition 4.13 in [50]), a subset S of $\mathbf{R}$ has a weak supremum s not only when $s = \sup S$ but also when s is the supremum of the set

$$\neg\neg S = \{a \in X : \neg\neg(a \in S)\}.$$

We will extend these results to the general case.

Proposition 2.4.3. *Let X be a partially ordered set, S a subset of X, and s an element of X. Then*

$$s = \sup S \Rightarrow s = \sup(\neg\neg S) \Rightarrow s = \text{w-}\sup(\neg\neg S) \Leftrightarrow s = \text{w-}\sup S.$$

Proof. To prove the leftmost implication, it suffices to prove that each upper bound of S is an upper bound for $\neg\neg S$ too. Let b be an upper bound for S, a an arbitrary element of $\neg\neg S$, and assume that $a \not\leq b$. If $a \in S$, then $a \leq b$, contradictory to $a \not\leq b$. Therefore $\neg(a \in S)$, but this is contradictory to $a \in \neg\neg S$.

It follows from Proposition 2.4.2 and the definition of supremum that each supremum is also a weak supremum, so that the second implication is proved. Since each upper bound of $\neg\neg S$ is an upper bound for S and vice versa, s is the least upper bound of $\neg\neg S$ if and only if it is the least upper bound of S. $\square$

We cannot expect to prove constructively that the existence of $\sup(\neg\neg S)$ entails the existence of $\sup S$. If the supremum of each subset of $\mathbf{R}$ exists whenever $\sup(\neg\neg S)$ exists, then LPO holds [50]. We will give a Brouwerian example, that is more direct than the one given in [50]. Let (a_n) be an arbitrary binary sequence and consider the set $S = \{a_n + 1 : n \in \mathbf{N}\} \cup \{x \in \mathbf{R} : x = 2 \text{ if } \forall n(a_n = 0)\}$. Assuming that $2 \notin S$ we see that $a_n = 0$ for all n, a contradiction. It follows that $\neg\neg(2 \in S)$; that is, $2 \in (\neg\neg S)$ and therefore $2 = \sup(\neg\neg S)$. If $\sup S$ exists, then, according to Proposition 2.4.3, $\sup S$=2. We can observe that in this case $2 \in S$ and either a_n+1=2 for some n, or $a_n = 0$ for all n.

An open problem raised in [50] requires a Brouwerian example for the implication s=w-sup $S \Rightarrow s = \sup(\neg\neg S)$ in the real case. This problem is still unsolved. However, we can show that for arbitrary partially ordered sets, this implication entails a nonconstructive principle.

Proposition 2.4.4. *If for each partially ordered set X and each subset S of X, the supremum of $\neg\neg S$ exists whenever the weak supremum of S exists, then WLPO holds.*

Proof. Consider X=$\mathbf{R}^2$. Let (a_n) be an arbitrary binary sequence and $S = \{(0, 2)\}\cup$

$\{x \in \mathbf{R}^2 : x = (2,0) \text{ if } \exists n(a_n = 1) \wedge x = (2,1) \text{ if } \forall n(a_n = 0)\}$. It is easily to prove that w-sup $S = (2,2)$. If we assume that $(2,2) = \sup(\neg\neg S)$, then there exists $x = (x_1, x_2) \in (\neg\neg S)$ such that $(x_1, x_2) \not\leq (1,2)$, therefore $1 < x_1$. If $x_1 \neq 2$, then $\neg(x \in S)$, a contradiction. It follows that $x_1 = 2$ and, similarly, we can prove that x_2 equals 0 or 1, so that either $(2,0) \in \neg\neg S$ or $(2,1) \in \neg\neg S$. Clearly, this entails WLPO. $\square$

In the remainder of this section we will assume that X is an ordered vector space. The classical identities regarding $\sup(A+B)$ and $\sup(\alpha A)$[3] can be proved constructively.

Proposition 2.4.5. *Let A and B be subsets of the ordered space X. The following statements hold.*

(i) *If* $\sup A$ *and* $\sup B$ *exist, then the supremum of the set $A+B$ exists and* $\sup(A+B) = \sup A + \sup B$.

(ii) *If* $\sup A$ *exists and $\alpha \geq 0$, then* $\sup \alpha A$ *exists and* $\sup \alpha A = \alpha \sup A$.

(iii) *If* $\sup A$ *exists and $\alpha \leq 0$, then* $\inf \alpha A$ *exists and* $\inf \alpha A = \alpha \sup A$.

Proof. (i) The proof is similar to the classical one.[4] Let s_1 and s_2 be the suprema of A and B, respectively, and x in X, such that $s_1 + s_2 \not\leq x$. Clearly, $s_1 + s_2$ is an upper bound of $A + B$ and there exists a in A such that $a \not\leq x - s_2$. Since $s_2 = \sup B$ and $s_2 \not\leq x - a$, it follows that there exists b in B such that $b \not\leq x - a$, that is, $a + b \not\leq x$. Consequently, $s_1 + s_2$ is the supremum of $A + B$.

(ii) Let s be the supremum of A, α a nonnegative number, and $x \in X$ with $\alpha s \not\leq x$. Clearly, αs is an upper bound for αA and we have to prove that $\alpha a \not\leq x$ for some $a \in A$. By applying the axioms of an ordered vector space, we obtain $(\alpha+1)s \not\leq s+x$ hence $s \not\leq (\alpha+1)^{-1}(s+x)$. Since $s = \sup A$, we can find $a \in A$ such that $a \not\leq (\alpha+1)^{-1}(s+x)$ and, equivalently, $(\alpha+1)a \not\leq s+x$; that is, $\alpha a \not\leq s - a + x$.

[3] The sets $\{a+b : a \in A,\ b \in B\}$ and $\{\alpha a : a \in A\}$ are denoted, as usual, by $A+B$ and αA.
[4] The classical properties of sup and inf and their proofs can be found, for example, in [29].

As a consequence, either $\alpha a \not\leq x$ or $x \not\leq s - a + x$. The latter is equivalent to $a \not\leq s$, contradictory to $s = \sup A$.

(iii) If $s = \sup A$ and $\alpha \leq 0$, then αs is a lower bound for αA and, according to (ii), $-\alpha s$ is the supremum of $-\alpha A$. Let x be an element of X with $x \not\leq \alpha s$. Then $-\alpha s$ exceeds $-x$ and, since $-\alpha s = \sup(-\alpha A)$, there exists a in A such that $-\alpha a$ exceeds $-x$; that is, $x \not\leq \alpha a$. Therefore αs is the infimum of αA. $\square$

Taking into account that for all x in X, $\sup\{x\} = x$, we see from (i) that $\sup(x + A) = x + \sup A$ whenever $\sup A$ exists. The corresponding results for the weak supremum and weak infimum can be proved in a similar way.

For an element x of an ordered vector space, the **positive part** of x is defined by

$$x^+ = \sup\{x, 0\},$$

the **negative part** of x is the vector

$$x^- = \sup\{-x, 0\},$$

and the **modulus** of x is

$$|x| = \sup\{x, -x\},$$

provided these suprema exists.

It follows from Proposition 2.4.5 that to guarantee the existence of x^+, x^-, and $|x|$ it suffices to prove that one of them exists. Moreover, in this case

$$x = x^+ - x^-,$$

$$|x| = x^+ + x^-,$$

and

$$\inf\{x^+, x^-\} = 0.$$

The classical proofs (Theorem 11.7 of [48]) are constructively valid.

Proposition 2.4.6. *Let x be an element of the ordered vector space X such that $|x|$ exists.*

(i) *The conditions* $x \neq 0$ *and* $|x| \neq 0$ *are equivalent.*

(ii) *For all real numbers* α, $|\alpha x|$ *exists and* $|\alpha x| = |\alpha||x|$.

Proof. (i) If $x \neq 0$, then $x^+ - x^- \neq 0$ and therefore $x^+ \neq 0$ or $x^- \neq 0$. In the former case, $0 < x^+ \leq x^+ + x^- = |x|$ and in the latter one $0 < x^- \leq |x|$. Conversely, if $|x| \neq 0$, then $|x| \not\leq 0$. Therefore either x or $-x$ exceeds 0.

(ii) As a consequence of Proposition 2.4.5, for all $\alpha \neq 0, |\alpha x|$ exists and equals $|\alpha||x|$. Consider now an arbitrary real number α and assume that $\alpha x \not\leq |\alpha||x|$. If $\alpha \neq 0$, then $|\alpha||x| = \sup\{\alpha x, -\alpha x\}$; whence $\alpha x \leq |\alpha||x|$, a contradiction. Therefore $\alpha = 0$, and both αx and $|\alpha||x|$ equal 0, in contrast with the condition $\alpha x \not\leq |\alpha||x|$. Consequently, $\alpha x \leq |\alpha||x|$ and, similarly, $-\alpha x \leq |\alpha||x|$; in other words, $|\alpha||x|$ is an upper bound of the set $\{\alpha x, -\alpha x\}$.

Let y be an element of X such that $|\alpha||x| \not\leq y$. We have to prove that either αx or $-\alpha x$ exceeds y. If $|\alpha||x| \not\leq y$, then either $\alpha x \not\leq y$ or $|\alpha||x| \not\leq \alpha x$. In the latter case, $|\alpha||x| \not\leq 0$ or $0 \not\leq \alpha x$ and each condition entails $\alpha \neq 0$. It follows that $|\alpha||x|$ is the supremum of the set $\{\alpha x, -\alpha x\}$. In consequence, either αx or $-\alpha x$ exceeds y. □

As pointed out in [44], $s \in X$ is the least upper bound of A if and only if

$$s + X^+ = \bigcap_{a \in A} (a + X^+).$$

To obtain a characterization for the supremum, instead of X^+ we will use the set $X^e = \{x \in X : 0 \not\leq x\}$.

Proposition 2.4.7. *Let* A *be a nonempty subset of the ordered vector space* X, *and* s *an element of* X.

(i) *The vector* s *is an upper bound of* A *if and only if* $s \in \sim (A + X^e)$.

(ii) *Provided that (i) holds,* $s = \sup A$ *if and only if* $s + X^e \subseteq A + X^e$.

Proof. (i) If a belongs to A and x to X^e, then $a \nleq a+x$. If, in addition, s is an upper bound of A, the condition $a \nleq s$ is contradictory, so that $s \nleq a+x$. Conversely, let a be an arbitrary element of A and assume that a exceeds s. Then $s-a$ belongs to X^e and $s = a + s - a$, which contradicts the hypothesis $s \in\sim (A + X^e)$. Therefore $\neg(a \nleq s)$; that is, $a \leq s$.

(ii) If $s = \sup A$ and x belongs to X^e, then s exceeds $s+x$ and, from the definition of supremum, it follows that there exists an element a of A with $a \nleq s+x$. Consequently, $s+x-a$ belongs to X^e and $s+x = a+s+x-a \in A+X^e$.

Conversely, let x be an element of X such that s exceeds x. Then $x-s$ is an element of X^e and x belongs to $s+X^e$. Therefore $x \in A+X^e$; that is, there exists a in A with $x-a \in X^e$. The last condition is equivalent to $a \nleq x$. Hence $s = \sup A$. □

2.5 Lattices

Linear order in lattices was investigated constructively in [33] and [55]. The general case, when the lattice operations are compatible with a partial order relation was investigated by von Plato [55]. The following definition is the positive one introduced in [55]. Let L be a nonempty set endowed with an excess relation $\nleq$ and two binary operations, **meet** and **join**, denoted by $\wedge$ and $\vee$. It is said that L is a **lattice** if the following axioms are satisfied for all a, b, c in L:

M1 $a \wedge b \leq a$ and $a \wedge b \leq b$;

M2 $c \nleq a \wedge b \Rightarrow (c \nleq a$ or $c \nleq b)$;

J1 $a \leq a \vee b$ and $b \leq a \vee b$;

J2 $a \vee b \nleq c \Rightarrow (a \nleq c$ or $b \nleq c)$.

In other words, taking into account the definition of supremum and infimum, a partially ordered set L is a lattice if for all a and b in L, $a \vee b = \sup\{a, b\}$ and

$a \wedge b = \inf\{a, b\}$ exist. As a consequence, for each pair x, y of elements of a lattice, we may also write $a \vee b$ and $a \wedge b$ for $\sup\{a, b\}$ and $\inf\{a, b\}$, respectively.

In a lattice the conditions $a \nleq b$, $a \wedge b \neq a$ and $b \neq a \vee b$ are equivalent. Moreover, as shown in [55], the lattice operations meet and join satisfy the following properties:

L1 $a \wedge b \neq a \wedge c \Rightarrow b \neq c$ and $a \vee b \neq a \vee c \Rightarrow b \neq c$;

L2 $a \wedge a = a$ and $a \vee a = a$;

L3 $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;

L4 $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;

L5 $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

Conversely, let us consider a nonempty set L with an apartness relation $\neq$ and two binary operations $\wedge$ and $\vee$ satisfying the conditions L1-L5. Define a relation $\nleq$ on L by

$$a \nleq b \Leftrightarrow a \wedge b \neq a$$

or, equivalently, by

$$a \nleq b \Leftrightarrow a \vee b \neq b.$$

Then, according to [55], $\nleq$ is an excess relation that satisfies the axioms M1, M2, J1, and J2. More precisely, one can prove the following results.

Proposition 2.5.1. *Let L be a nonempty set endowed with an apartness relation $\neq$ and two binary operations $\wedge$ and $\vee$. Let $\nleq_m$ and $\nleq_j$ be two binary relations on L, defined on L by:*

$$a \nleq_m b \Leftrightarrow a \wedge b \neq a,$$

$$a \nleq_j b \Leftrightarrow a \vee b \neq b.$$

(i) *If the operation $\wedge$ satisfies the left-hand side conditions L1-L4, then $\nleq_m$ is an excess relation, the relation $\neq$ is the apartness relation induced by the symmetrization of $\nleq_m$, and the conditions M1 and M2 hold.*

(ii) *If $\vee$ satisfies the right-hand side conditions L1-L4, then $\not\leq_j$ is an excess relation whose corresponding apartness relation is $\neq$, and the conditions J1 and J2 hold.*

(iii) *Assume that $\wedge$ and $\vee$ satisfy the conditions L1-L4. In this case, the excess relations $\not\leq_m$ and $\not\leq_j$ coincide if and only if both conditions L5 are satisfied.*

Proof. To prove that $\not\leq_m$ leads to the apartness relation $\neq$, we have to show that for all a and b in L, $a \neq b$ if and only if $a \not\leq_m b$ or $b \not\leq_m a$. If $a \neq b$, then either $a \neq a \wedge b$ or $a \wedge b \neq b$, that is, either $a \not\leq_m b$ or $b \not\leq_m a$. The converse implication follows from the strong extensionality (L1) of meet. In a similar way, under the hypotheses of (ii), one can prove that $a \neq b$ if and only if $a \not\leq_j b$ or $b \not\leq_j a$.

The remainder of the statements (i) and (ii) follows from the proofs of Theorem 7.1 and Theorem 7.2 of [55]. It suffices to separate the relations $\not\leq_m$ and $\not\leq_j$ and to observe that L5 is not necessary.

To prove (iii), assume that L1-L4 hold. If, in addition, $\wedge$ and $\vee$ satisfy L5, the condition $a \not\leq_m b$ is equivalent to $a \wedge b \neq a \wedge (a \vee b)$. It follows from L1 that $b \neq a \vee b$, that is, $a \not\leq_j b$. Similarly, $a \not\leq_j b$ entails $a \not\leq_m b$. Conversely, it is straightforward to see that L5 is a consequence of the equality between $\not\leq_m$ and $\not\leq_j$. $\square$

We end this section with equivalent conditions for supremum and weak supremum.

Proposition 2.5.2. *Let A be a nonempty subset of a lattice L and s an upper bound of A.*

(i) *The element s is the supremum of A if and only if for all x in L with $x < s$ there exists a in A with $a \not\leq x$.*

(ii) *The following conditions are equivalent.*

(1) $s = \text{w-sup}\, A$.

(2) $x \in L \wedge \neg\neg(x < s) \Rightarrow \neg(\forall a \in A\ (a \leq x))$.

(3) $x \in L \wedge x < s \Rightarrow \neg(\forall a \in A\ (a \leq x))$.

Proof. (i) Assume that $s = \sup A$. Since $s \not\leq x$ whenever $x < s$, the existence of a in A with $a \not\leq x$ is guaranteed by the definition of supremum. To prove the converse implication, let x be an element of L such that $s \not\leq x$. Therefore $s \wedge x \neq s$; that is, $s \wedge x < s$ and, according to the hypothesis, there exists an element a of A that exceeds $s \wedge x$. The last condition is equivalent to $a \wedge (s \wedge x) < a$ and, as $a \wedge s = a$, to $a \wedge x < a$. Consequently, there exists an element a of A such that $a \not\leq x$; whence $s = \sup A$.

(ii) It follows from Proposition 2.4.2 and the implication $\neg\neg(x < s) \Rightarrow \neg\,(s \leq x)$ that (1) entails (2). Clearly, (3) is a consequence of (2). To prove that (3) implies (1), assume that b is an upper bound of A and s exceeds b. Then $s \wedge b < s$ and, as a consequence, it is contradictory for $s \wedge b$ to be an upper bound of A. If a is an arbitrary element of A, then $a \leq s$ and $a \leq b$, therefore $a \leq s \wedge b$, a contradiction. Consequently, if b is an upper bound of A, then $\neg(s \not\leq b)$, that is, $s \leq b$. In other words, s is the weak supremum of A. □

2.6 Riesz spaces

An ordered vector space X is said to be a **Riesz space** or a **vector lattice** if each element of X has a positive part. In this case, according to Proposition 2.4.5, for all elements x and y in X, $\sup\{x, y\}$ exists and

$$\sup\{x, y\} = y + \sup\{x - y, 0\}$$

—that is,

$$\sup\{x, y\} = y + (x - y)^+.$$

Furthermore, the infimum of $\{x, y\}$ exists and

$$\inf\{x, y\} = -\sup\{-x, -y\}.$$

Therefore a Riesz space is an ordered vector space that is also a lattice.

Corollary 2.6.1. *Let X be an ordered vector space and $X^e = \{x \in X : 0 \not\leq x\}$. The following assertions are equivalent:*

(1) *The space X is a vector lattice.*

(2) *For each pair x, y of elements of X, there is an element z of X such that $z \in\sim (x + X^e)$, $z \in\sim (y + X^e)$, and $z + X^e \subseteq (x + X^e) \cup (y + X^e)$.*

(3) *For each element x of X, there exists an element z of X with the properties $z \in\sim (X^e \cup (x + X^e))$ and $z + X^e \subseteq X^e \cup (x + X^e)$.*

Proof. We can apply Proposition 2.4.7 for $A = \{x, y\}$ and then for $A = \{x, 0\}$. □

We have already seen (Proposition 2.3.5) that for any ordered vector space, $X^+ = \neg X^e$ and $X^e \subseteq\sim X^+$. If, in addition, X is a Riesz space, then $X^e =\sim X^+$. To prove that $\sim X^+ \subseteq X^e$, let x be an arbitrary vector in $\sim X^+$. It follows that $x \neq x \vee 0$ and, equivalently, $0 \not\leq x$. Therefore $x \in X^e$ whenever $x \in\sim X^+$.

As for lattices, an apartness relation can be used as a primary relation to define a Riesz space.

Proposition 2.6.2. *Let X be a real vector space with a tight apartness relation $\neq$. Assume that $\vee$ is a binary operation on X satisfying the following conditions for all x, y, and z in X:*

R1 $x \vee y \neq x \vee z \Rightarrow y \neq z$;

R2 $x \vee x = x$;

R3 $x \vee y = y \vee x$;

R4 $(x \vee y) \vee z = x \vee (y \vee z)$;

R5 $x \vee y \neq y \Rightarrow (x + z) \vee (y + z) \neq (y + z)$;

R6 $(\alpha > 0$ *and* $x \vee 0 \neq 0) \Rightarrow \alpha x \vee 0 \neq 0$;

R7 $\alpha x \neq 0 \Rightarrow \alpha \neq 0$.

Then X is a Riesz space with respect to the excess relation $\not\leq$ defined by

$$x \not\leq y \Leftrightarrow x \vee y \neq y.$$

Proof. It follows from Proposition 2.5.1 that the conditions R1-R4 ensure that $\nleq$ is an excess relation on X and that for each pair x, y of elements of X, $\sup\{x, y\} = x \vee y$. The vector space X is an ordered vector space if and only if $\nleq$ satisfies the conditions 02, 03, and 05 (Section 2.3). Taking into account the definition of $\nleq$, we see that these conditions are none other than the hypotheses R5-R7. $\square$

In this case, we can define a binary operation $\wedge$ on X by

$$x \wedge y = -(-x) \vee (-y).$$

Then

$$x \wedge y = \inf\{x, y\}.$$

Equivalently, we can start with an operation $\wedge$ that satisfies the conditions R1-R4, R7, and the following two conditions:

R8 $x \wedge y \neq x \Rightarrow (x + z) \wedge (y + z) \neq (x + z)$;

R9 $(\alpha > 0 \text{ and } x \wedge 0 \neq x) \Rightarrow \alpha x \wedge 0 \neq \alpha x$.

A straightforward example is the set $\mathbf{R}$ of the real numbers. For x and y in $\mathbf{R}$ we will denote $\max(x, y)$ and $\min(x, y)$[5] by $x \vee y$, respectively $x \wedge y$. Clearly, $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$; so the notation is consistent with the one given in Section 2.5.

Following Palmgren [53], we say that a nonempty subset S of $\mathbf{R}$ is **upper located** if for all real numbers α, β with $\alpha < \beta$, either β is an upper bound of S or else there exists $a \in S$ with $a > \alpha$. A **lower located** subset is defined correspondingly, as expected. The definitions can be extended without any modification from $\mathbf{R}$ to any linearly ordered set with a dense order. If S is a nonempty subset of $\mathbf{R}$, then $\sup S$ exists if and only if S is bounded above and upper located (Proposition 4.3 in Chapter 3 of [11]).[6]

[5] By $\max(x, y)$ and $\min(x, y)$ we mean the maximum and minimum of the real numbers x, y, as defined constructively in [8].

[6] In an axiomatic description of $\mathbf{R}$ [17], one of the axioms states that $\sup S$ exists whenever S is bounded above and upper located. As a consequence, $\sup\{x, y\}$ exists for each pair x, y of real numbers.

Other ordered vector spaces have a lattice structure induced in a natural way by the lattice operations on $\mathbf{R}$. Consider, for instance, the space $C[0,1]$. For each pair f, g of elements define the function $f \vee g$ by

$$f \vee g(x) = \sup\{f(x), g(x)\} \quad (x \in [0,1]).$$

Then $f \vee g$ is an element of $C[0,1]$ [8]. Moreover, it can be easily verified that $f \vee g$ is the supremum of the set $\{f, g\}$. Taking into account that $C[0,1]$ is an ordered vector space (Section 2.3), we conclude that this space is also a vector lattice.

We now give a counterexample: an ordered vector space that is not a vector lattice. We have already seen (Section 2.3) that $\mathbf{R}^2$ is an ordered vector space with respect to the excess relation $\not\leq$ defined by

$$(x_1, x_2) \not\leq (y_1, y_2) \Leftrightarrow (x_1 \neq y_1 \vee x_2 > y_2).$$

Since $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 = y_1$ and $x_2 \leq y_2$, it follows that no upper bound, let alone the supremum, of the set $\{(0,0), (1,0)\}$ exists.

Chapter 3

Dedekind complete sets

Classically, a partially ordered set X is said to be Dedekind complete if each nonempty subset of X that is bounded above has a supremum. In this case, each nonempty subset that is bounded below has an infimum. Dedekind completeness plays a crucial role in the classical theory of ordered vector spaces. The most extensive part of the classical theory deals with Dedekind complete Riesz spaces and, furthermore, several important classical results are based on the Dedekind completeness of $\mathbf{R}$.

If we are working constructively, the first problem is to obtain a good substitute for the classical definition. On the one hand, we should be able to prove at least the (constructive) Dedekind completeness of $\mathbf{R}$. On the other hand, $\mathbf{R}$ does not satisfy the classical definition. Indeed, as we have already seen (Section 1.4), if each nonempty subset of $\mathbf{R}$ that is bounded above has a supremum (respectively, a weak supremum), then LPO (respectively WLPO) holds. However, we have a constructive counterpart of the least–upper–bound principle: a nonempty subset of $\mathbf{R}$ that is bounded above has a supremum if and only if it is upper located. As pointed out by Ishihara and Schuster [43], this equivalence expresses constructively the order completeness of the real number line. Furthermore, the definitions of upper and lower locatedness were extended by Palmgren [53] to the case of a dense linear order. According to [53], a set X endowed with a dense linear order is order complete if each nonempty subset of X that is bounded above and upper located has a weak supremum. It can be proved that upper locatedness and the existence of the weak supremum are sufficient conditions for the existence of the supremum and, as a consequence, that the two definitions of order completeness for dense linear orders are equivalent.

In [5] we introduced generalizations for arbitrary partially ordered sets of the definitions of upper and lower locatedness, and we used them to obtain a general constructive definition of Dedekind completeness. In accordance with classical mathematics (see also Theorem 3.10 of [53] for the constructive linear case), we can prove (Section 3.2) the equivalence between the description of Dedekind completeness with upper locatedness and suprema and the one with lower locatedness and infima.

3.1 Order locatedness

We will present a general definition of upper locatedness. As a main result of this section, we will prove that for an arbitrary subset S of a partially ordered set, $\sup S$ can be computed if and only if S is upper located and has a weak supremum.

Lemma 3.1.1. *Let S be a nonempty subset of the partially ordered set X and consider the following conditions:*

(1) *For each pair x, y of elements of X such that $y \not\leq x$, either there exists an element a of S with $a \not\leq x$ or else there exists an upper bound b of S with $y \not\leq b$.*

(2) *For all elements x and y of X such that y exceeds x, either there exists $a \in S$ with $a \not\leq x$ or else $y \not\leq a$ for all elements $a \in S$.*

Then the former condition entails the latter. If, in addition, the excess relation on X is a dense linear order, then each condition is equivalent to the upper locatedness of S.

Proof. Assuming (1), let us consider x and y such that y exceeds x. We may assume that there exists an upper bound b of S with $y \not\leq b$. Let a be an arbitrary element of S. Since the condition $a \not\leq b$ is contradictory, it follows that $y \not\leq a$. Therefore condition (2) holds.

Now let us assume that X has a dense linear order. It follows that y exceeds x if and only if $x < y$. According to the definition (Section 2.6), S is upper located whenever (2) holds. We will prove that upper locatedness is a sufficient condition for (1). Let x, y be a pair of elements of X with $x < y$. Then there exists z in X with $x < z < y$. Either $x < a$ for some $a \in S$ or z is an upper bound of S such that y exceeds z. Consequently, S satisfies the condition (1). □

According to Lemma 3.1.1, both conditions (1) and (2) are generalizations of upper locatedness. We say that a nonempty subset S is **upper located** if it satisfies condition (1). The subset S is said to be **weakly upper located** if for all x, y in X such that y exceeds x, either it is contradictory that x be an upper bound of S or else there exists an upper bound b of S with $y \not\leq b$. **Lower located** and **weakly lower located** sets are defined correspondingly.

Let us consider now several examples. The set X and the subsets $\{a\}$, $a \in X$ are both upper located and lower located. The subset S will be called **unbounded above** if for each $x \in X$ there is an element a in S that exceeds x. Similarly, S is said to be **unbounded below** if for each $x \in X$ there exists $a \in S$ such that $x \not\leq a$. Each subset of X that is unbounded above is upper located and, needless to say, each subset that is unbounded below is lower located.

Proposition 3.1.2. *Let S be a nonempty subset of the partially ordered set X. Then S has a supremum if and only if it is upper located and its weak supremum exists.*

Proof. Let s be the supremum of S and x, y, a pair of elements of X such that y exceeds x. Then either $y \not\leq s$ or $s \not\leq x$. In the former case, y exceeds an upper bound of S, namely, s and in the latter one, there exists an element of S that exceeds x.

Conversely, assume that S is upper located and let w be the weak supremum of S. We will prove that $w = \sup S$. To this end, let x be an element of X such that $w \not\leq x$. If b is an upper bound of S, then the condition $w \not\leq b$ is contradictory to the definition of weak supremum. Since S is upper located, it follows that there exists a in S that exceeds x. By the definition of supremum, it follows that $w = \sup S$. $\square$

As a consequence, to define the Dedekind completeness of $\mathbf{R}$ we can use either suprema, as in [43] or, equivalently, weak suprema [53]. In the next section we will extend the definition of Dedekind completeness to the general case of an arbitrary partially ordered set.

Proposition 3.1.2 shows that the existence of $\sup S$ is a sufficient condition for the upper locatedness of S. Similarly, the existence of the weak supremum entails weakly upper locatedness.

Proposition 3.1.3. *If S has a weak supremum, then S is weakly upper located.*

Proof. Let x, y be elements of X such that $y \nleq x$. If y exceeds the weak supremum w, we have nothing to prove. If $w \nleq x$, suppose that x is an upper bound of S. Since w is the weak supremum of S, it follows that $w \leq x$, contradictory to the condition $w \nleq x$. $\square$

3.2 Dedekind completeness

The partially ordered set X is said to be **Dedekind complete** if each nonempty subset of X that is upper located and bounded above has a weak supremum. In this case the weak supremum is actually a supremum (Proposition 3.1.2). Proposition 4.3 in Chapter 2 of [11] guarantees the order completeness of $\mathbf{R}$. We will prove in Section 3.4 that for each n, $\mathbf{R}^n$ is Dedekind complete.

Since each subset of X is classically upper located, this definition of Dedekind completeness is classically equivalent to the traditional one. As in the classical case, we can use lower locatedness, instead of upper locatedness, to define Dedekind completeness. For a dense linear order this was proved by Palmgren. Our next result is the generalization of Theorem 3.10 of [53].

Proposition 3.2.1. *The partially ordered set X is Dedekind complete if and only if each nonempty subset of X that is lower located and bounded below has a weak infimum.*

Proof. Let us assume that X is Dedekind complete and consider a nonempty subset S that is lower located and bounded below. We prove that $\inf S$ exists. As in the classical proof, we consider the nonempty set B of all lower bounds of S. To prove that B is upper located, let x and y be elements of X with $y \nleq x$. Since S is lower located, it follows that either there exists $a \in S$ with $y \nleq a$ or else there exists a lower bound b of S with $b \nleq x$. Therefore either y exceeds the upper bound a of B or there exists an element of B that exceeds x; in which case, B is upper located.

Let s be the supremum of B. We prove that s is the infimum of S. If $s \nleq a$ for some a in S, then in view of the definition of supremum, there exists $b \in B$ with $b \nleq a$, a contradiction. Therefore $s \leq a$ for all a in S. Let us consider now an element z in X with $z \nleq s$. Since S is lower located, either $z \nleq a$ for some a in S or there exists an element of B that exceeds s. The latter condition is contradictory, so $s = \inf S$. The converse implication can be proved in a similar way. □

We will say that X is **Dedekind incomplete** if there exists a subset S of X that is nonempty, upper located and bounded above, but does not have a supremum. Clearly, this is classically equivalent to the negation of Dedekind completeness. However, to prove constructively that a partially ordered set is Dedekind incomplete, it is not sufficient to show that its Dedekind completeness is contradictory.

In classical functional analysis, the vector space $C[0,1]$ is the standard example of an Archimedean Riesz space that is not Dedekind complete. To prove that $C[0,1]$ is Dedekind incomplete let us consider, as in the classical proof (Example 7.5 of [1]), the sequence $(f_n)_{n\geq 3}$ of continuous functions f_n on $[0,1]$ that satisfy:

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{n}, \\ \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

and f_n is linear on $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$. Clearly, the set $S = \{f_n : n \geq 3\}$ is bounded above by the function e defined by $e(x) = 1$ for all x. Since S does not have a weak supremum, let alone a supremum, it is sufficient to prove that S is upper located.

Let f and g be two elements of $C[0,1]$ such that $g \not\leq f$, that is, $f(x_0) < g(x_0)$ for some x_0. It follows that there exist x_1 and x_2 in $[0,1]$ such that $x_1 < x_2$ and $f(x) < g(x)$ whenever $x_1 \leq x \leq x_2$. Either $x_1 < 1/2$ or $1/2 < x_2$. In the former case, either $f(x_1) < 1$ or $g(x_1) > 1$. If $f(x_1) < 1$, then there exists n with $f_n(x_1) = 1 > f(x_1)$; hence $f_n \not\leq f$. If $g(x_1) > 1$, then $g \not\leq e$. Consider now the case $x_2 > 1/2$. If $f(x_2) < 0$, then $f_n \not\leq f$ for all n. If $g(x_2) > 0$, then we pick an upper bound h of S with $h(x_2) = 0$. Consequently, if g exceeds f, then either $f_n \not\leq f$ for some n or there exists an upper bound u of S (namely, e or h) such that $g \not\leq u$. This ensures that S is upper located.

3.3 The product order

A Cartesian product of partially ordered sets can be ordered in a natural way. Let $X = X_1 \times X_2 \times \cdots \times X_n$ be the Cartesian product of the sets $X_1, X_2, \ldots, X_n$ and for each $i \in \{1, 2, \ldots, n\}$ consider an excess relation $\not\leq_i$ on X_i. Define the relation $\not\leq$ on X by

$$(x_1, x_2, \ldots, x_n) \not\leq (y_1, y_2, \ldots, y_n) \Leftrightarrow \exists i \in \{1, 2, \ldots, n\} \ (x_i \not\leq_i y_i).$$

Since all the relations $\not\leq_i$ are excess relations, it is straightforward to see that this relation $\not\leq$ on X also satisfies the axioms of an excess relation. The general method described in Section 2.2 leads to the following definitions of apartness, equality, partial order and strict partial order on the Cartesian product, as in the classical case:

$$(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n) \Leftrightarrow \exists i \in \{1, 2, \ldots, n\} \ (x_i \neq_i y_i);$$

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \Leftrightarrow \forall i \in \{1, 2, \ldots, n\} \ (x_i =_i y_i);$$

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \Leftrightarrow \forall i \in \{1, 2, \ldots, n\} \ (x_i \leq_i y_i);$$

$$(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n) \Leftrightarrow \forall i \in \{1, 2, \ldots, n\} \ (x_i \leq_i y_i) \wedge \\ \exists j \in \{1, 2, \ldots, n\} \ (x_j <_j y_j).$$

The notation of the relations is self–explanatory. From now on, unless otherwise stated, the Cartesian product of the partially ordered sets X_1, X_2, ..., X_n will be considered ordered by an excess relation, as above.

Proposition 3.3.1. *If X_1, X_2, ..., X_n are ordered vector spaces, then $X_1 \times \cdots \times X_n$ is an ordered vector space.*

Proof. This easily follows from the definition of an ordered vector space and the definition of the excess relation on the Cartesian product. □

For each i, $1 \leq i \leq n$, let us consider the projection π_i of $X = X_1 \times X_2 \times \cdots \times X_n$ onto X_i, defined by

$$\pi_i(x_1, x_2, \ldots, x_n) = x_i.$$

The next result enables us to calculate the (weak) supremum of a subset S of X by computing the (weak) suprema of the projections $\pi_i(S)$, and vice versa.

Proposition 3.3.2. *Let $X_1, X_2, \ldots, X_n$ be partially ordered sets, let S be a nonempty subset of $X = X_1 \times X_2 \times \cdots \times X_n$ that is bounded above, and let $s = (s_1, s_2, \ldots, s_n)$ be an element of X. Then, the following statements hold.*

(i) $s = \sup S \Leftrightarrow \forall i \in \{1, 2, \ldots, n\} \ (s_i = \sup \pi_i(S))$.

(ii) $s = \text{w–sup}\, S \Leftrightarrow \forall i \in \{1, 2, \ldots, n\} \ (s_i = \text{w–sup}\, \pi_i(S))$.

Proof. (i) Clearly, s is an upper bound for S if and only if for each i, s_i is an upper bound of $\pi_i(S)$. Assuming that $s = \sup S$, we prove that $s_1 = \sup \pi_1(S)$. For each $x_1 \in X_1$ with $s_1 \not\leq_1 x_1$ we have to find an element $a_1 \in \pi_1(S)$ such that $a_1 \not\leq_1 x_1$. If $s_1 \not\leq_1 x_1$, then $s \not\leq (x_1, s_2, \ldots, s_n)$, so there exists $a = (a_1, a_2, \ldots, a_n) \in S$ with $a \not\leq (x_1, s_2, \ldots, s_n)$. It follows that either $a_1 \not\leq_1 x_1$ or else $a_j \not\leq_j s_j$ for some $j \geq 2$. Since s is an upper bound for S, the latter case is contradictory, so $a_1 \not\leq_1 x_1$ and $s_1 = \sup \pi_1(S)$. Similarly, $s_i = \sup \pi_i(S)$ for each $i \geq 2$.

To prove the converse implication, let us assume that for all i, $s_i = \sup \pi_i(S)$. Consider $x = (x_1, x_2, \ldots, x_n) \in S$ with $s \not\leq x$—that is, $s_j \not\leq_j x_j$ for some j. Since $s_j = \sup \pi_j(S)$, there exists $a_j \in \pi_j(S)$ such that $a_j \not\leq_j x_j$. If a is an element of S with $\pi_j(a) = a_j$, then $a \not\leq x$. Consequently, $s = \sup S$.

(ii) This can be proved in a similar way. □

Clearly, the corresponding properties for the infimum are also valid. As a consequence, we can define lattice operations on a Cartesian product of lattices in a natural way.

Corollary 3.3.3. *The Cartesian product of the lattices $L_1, L_2, \ldots, L_n$ is a lattice with respect to the operations $\vee$ and $\wedge$ defined by*

$$(x_1, x_2, \ldots, x_n) \vee (y_1, y_2, \ldots, y_n) = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n),$$
$$(x_1, x_2, \ldots, x_n) \wedge (y_1, y_2, \ldots, y_n) = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n).$$

Moreover, if $L_1, L_2, \ldots, L_n$ are Riesz spaces, then $L_1 \times L_2 \times \cdots \times L_n$ is a Riesz space.

Proof. The first statement is a direct consequence of Proposition 3.3.2. To complete the proof, we observe that the Cartesian product is both an ordered vector space (Proposition 3.3.1) and a lattice. □

Lemma 3.3.4. *Let S be a nonempty subset of $X = X_1 \times\ X_2 \times \cdots \times X_n$ that is bounded above. Then S is upper located if and only if each projection $\pi_i(S)$ is upper located.*

Proof. Assuming that S is upper located, we prove, for example, that $\pi_1(S)$ is upper located. Consider an element $a = (a_1, \ldots, a_n)$ of S and an upper bound $b = (b_1, \ldots, b_n)$ of S. If x_1 and y_1 are elements of X_1 such that $y_1 \not\leq_1 x_1$, then $(y_1, a_2, \ldots, a_n) \not\leq (x_1, b_2, \ldots, b_n)$. It follows that either there exists an upper bound $b' = (b'_1, \ldots, b'_n)$ of S with $(y_1, a_2, \ldots, a_n) \not\leq b'$ or else there exists an element $a' = (a'_1, \ldots, a'_n)$ of S that exceeds $(x_1, b_2, \ldots, b_n)$. In the former case, b'_1 is an upper bound of $\pi_1(S)$ and, as $a_i \leq b'_i$ for all $i \geq 2$, y_1 exceeds b'_1. In the latter a'_1 is an element of $\pi_1(S)$ that exceeds x_1. This proves the upper locatedness of $\pi_1(S)$; the other projections are proved to be upper located in a similar way.

Conversely, assume that each projection of S is upper located, and let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be elements of S such that y exceeds x. It follows that $y_i \not\leq_i x_i$ for some i. Since $\pi_i(S)$ is upper located, either there exists an upper bound b_i of $\pi_i(S)$ with $y_i \not\leq_i b_i$ or there exists an element $a = (a_1, \ldots a_n)$ in X such that $a_i \not\leq_i x_i$. In the former case, taking into account that S is bounded above, we can easily construct an upper bound b of S such that y exceeds b. In the latter, a exceeds x, which ensures that S is upper located. □

Note that a similar result can be obtained for weakly upper located sets.

Proposition 3.3.5. *The partially ordered set $X = X_1 \times \cdots \times X_n$ is Dedekind complete if and only if for each i $(1 \leq i \leq n)$, X_i is Dedekind complete.*

Proof. Suppose first that X is Dedekind complete, and let S_1 be a nonempty subset of X_1 that is upper located and bounded above. For each i $(2 \leq i \leq n)$, pick an element $a_i \in X_i$. Then, according to Lemma 3.3.4, the set $S_1 \times \{a_2\} \times \cdots \times \{a_n\}$ is upper located. This subset of X is also bounded above; hence its supremum exists. By Lemma 3.3.2, the supremum of S_1 exists, which guarantees the Dedekind completeness of X_1.

The converse implication is a direct consequence of Lemmas 3.3.2 and 3.3.4. □

3.4 An example: $\mathbf{R}^{\mathrm{n}}$

We investigate a specific example: the Cartesian product $\mathbf{R}^n$ of n copies of $\mathbf{R}$. Since $\mathbf{R}$ is a Dedekind complete Riesz space, the following result is a direct consequence of Proposition 3.3.5.

Corollary 3.4.1. *For each positive integer n, $\mathbf{R}^n$ is a Dedekind complete Riesz space with respect to the standard product order.*

We will prove equivalent conditions for the existence of the supremum of a subset of $\mathbf{R}^n$.

Proposition 3.4.2. *If S is a nonempty subset of $\mathbf{R}^n$, then the following conditions are equivalent.*

(1) *The supremum of S exists.*

(2) *There exists an element $s \in \mathbf{R}^n$ such that s is an upper bound of S and for each $x \in \mathbf{R}^n$ with $x < s$, at least an element a of S exceeds x.*

(3) *The set S is bounded above and upper located.*

(4) *The set S is bounded above, and for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$ with $x_i < y_i$ for each $i \in \{1, \ldots, n\}$, either y is an upper bound of S or there exists a in S such that $a \nleq x$.*

(5) *The projections $\pi_i(S)$ are bounded above and upper located.*

Proof. To avoid cumbersome notation, we will assume that $n = 2$. First we prove that (3) entails (4). Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $\mathbf{R}^2$ such that $x_1 < y_1$ and $x_2 < y_2$. Pick an element $a = (a_1, a_2)$ of S, and consider the elements $z = (y_1, a_2)$ and $w = (a_1, y_2)$. Both z and w exceed x; whence either there exists an element of S that exceeds x or else we can construct upper bounds (b_1, b_2) and (b_1', b_2') of S with $z \nleq (b_1, b_2)$ and $w \nleq (b_1', b_2')$. In the latter case, $b_1 < y_1$ and $b_2' < y_2$, so y is an upper bound of S.

To prove that (4) entails (5), consider an upper bound (b_1, b_2) of S. If α and β are two real numbers with $\alpha < \beta$, set $x = (\alpha, b_2)$ and $y = (\beta, b_2 + 1)$. Then either y is an upper bound of S or there exists $a = (a_1, a_2)$ in S with $a \nleq x$. In the former case, β is an upper bound of $\pi_1(S)$; in the latter, $\alpha < a_1$. Consequently, $\pi_1(S)$ is upper located; $\pi_2(S)$ is proved to be upper located in a similar way.

The Dedekind completeness of $\mathbf{R}$, together with Lemma 3.3.2 guarantee the equivalence of (5) and (1). According to Corollary 3.4.1, (1) and (3) are equivalent. Since $\mathbf{R}^n$ is a Riesz space, the equivalence of (1) and (2) follows from Proposition 2.5.2(i). □

A subset S of $\mathbf{R}^n$ is said to be **totally bounded** if for each $\varepsilon > 0$ there exists a set $\{a_1, \ldots, a_k\}$ of points of S such that for each x in S at least one of the numbers $||x - a_1||, \ldots, ||x - a_k||$ is less than ε.

Corollary 3.4.3. *If S is a totally bounded subset of $\mathbf{R}^n$, then S has a supremum and an infimum.*

Proof. The projections $\pi_i(S)$, $1 \leq i \leq n$, are totally bounded subsets of $\mathbf{R}$ and, according to Theorem 3 in Chapter 2 of [8], their suprema and infima exist. Consequently, $\sup S$ and $\inf S$ exist. □

Note that for $n \geq 2$ the condition in the left–hand side of (4) (Proposition 3.4.2) cannot be replaced by the weaker condition $x < y$.

Proposition 3.4.4. *Let $n \geq 2$ be an integer, and S a nonempty subset of $\mathbf{R}^n$ that is bounded above. If, for all x and y in $\mathbf{R}^n$ with $x < y$, either y is an upper bound of S or else there exists a in S such that $a \not\leq x$, then LPO holds.*

Proof. If S satisfies the hypothesis, then $\sup S$ exists. Let $s = (s_1, \ldots, s_n)$ be the supremum of S and take an arbitrary real number α. If $x = (\alpha, s_2, \ldots, s_n)$ and $y = (\alpha, s_2 + 1, \ldots, s_n + 1)$, then $x < y$, and either y is an upper bound of S or else we can find an element $a = (a_1, \ldots, a_n)$ in S that exceeds x. In the former case, α is an upper bound of $\pi_1(S)$; whence $s_1 \leq \alpha$. In the latter case, either $\alpha < a_1$ or else $s_j < a_j$ for some $j \geq 2$. Since $s = \sup S$, the latter condition is contradictory. Consequently, for each real number α, either $\alpha \geq s_1$ or $\alpha < s_1$. This property entails LPO. □

We have corresponding results for the weak supremum. The proofs are similar and hence omitted.

Proposition 3.4.5. *For a nonempty subset S of $\mathbf{R}^n$, the following conditions are equivalent.*

(1) *The weak supremum of S exists.*

(2) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$s \nleq x \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(3) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$\neg(s \leq x) \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(4) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$\neg\neg(\forall a \in S\ (a \leq x)) \Rightarrow (s \leq x).$$

(5) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$x < s \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(6) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$\neg\neg(x < s) \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(7) *There exists* $s \in \mathbf{R}^n$ *such that* s *is an upper bound of* S *and*

$$\neg\neg(\forall a \in S\ (a \leq x)) \Rightarrow \neg(x < s).$$

(8) *The set* S *is bounded above and weakly upper located.*

(9) *The set* S *is bounded above, and for all* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *in* $\mathbf{R}^n$ *with* $x_i < y_i$ *for each* $i \in \{1, \ldots, n\}$*, either* y *is an upper bound of* S *or else it is contradictory that* x *be an upper bound of* S*.*

(10) *The projections* $\pi_i(S)$ *are bounded above and weakly upper located.*

Proposition 3.4.6. *Let* $n \geq 2$ *be an integer, and* S *a nonempty subset of* $\mathbf{R}^n$ *that is bounded above. If, for all* x *and* y *in* $\mathbf{R}^n$ *with* $x < y$*, either* y *is an upper bound of* S *or else it is contradictory that* x *be an upper bound of* S*, then WLPO holds.*

We end with a condition equivalent to the existence of the (weak) supremum of an order bounded subset of $\mathbf{R}^n$.

Proposition 3.4.7. *Let S be a nonempty subset of $\mathbf{R}^n$ that is order bounded.*

(i) *The supremum of S exists if and only if, for all x and y in $\mathbf{R}^n$ with $y \nleq x$, either $y \nleq a$ for all a in S or else there exists a in S such that $a \nleq x$.*

(ii) *The weak supremum of S exists if and only if, for all x and y in $\mathbf{R}^n$ with $y \nleq x$, either $y \nleq a$ for all a in S or it is contradictory that x be an upper bound of S.*

Proof. We prove only (i), the proof of (ii) being similar. If the supremum of S exists, then S is upper located and, according to Lemma 3.1.1, the condition in the right–hand side holds.

Conversely, let $b = (b_1, \ldots, b_n)$ an upper bound of S, and let $m = (m_1, \ldots, m_n)$ be a lower bound. If α and β are real numbers with $\alpha < \beta$, then $(\beta, m_2, \ldots, m_n) \nleq (\alpha, b_2, \ldots, b_n)$. It follows that either $(\beta, m_2, \ldots, m_n) \nleq a$ for all a in S or else there exists an element $a = (a_1, \ldots a_n)$ in S such that $(a_1, \ldots, a_n) \nleq (\alpha, b_2, \ldots, b_n)$. In the former case, β is an upper bound of $\pi_1(S)$; in the latter, there exists a_1 in $\pi_1(S)$ with $\alpha < a_1$. Consequently, we see that $\sup \pi_1(S)$ exists. Similarly, we prove that $\sup \pi_i(S)$ exists for each i. This proves the existence of $\sup S$. $\square$

Chapter 4

Order units

Ordered vector spaces with certain additional properties are examined in this chapter. We first deal with Archimedean spaces—that is, ordered vector spaces that satisfy a generalization of the Axiom of Archimedes (Section 4.1). Many Riesz spaces have norms that are increasing on the positive cone. Classically, all these Riesz spaces are Archimedean. A constructive counterpart of this result is presented in Section 4.2

Order units are considered in Sections 4.3 and 4.4. A positive vector e is an order unit of the ordered vector space X if the union of all order intervals $[-\alpha e, \alpha e]$ $(\alpha > 0)$ covers X. In the classical theory, if X is an Archimedean space with an order unit e, then the Minkowski functional of the order interval $[-e, e]$ is a norm and $[-e, e]$ is the closed unit ball. We will use Ishihara's results on the constructive existence of Minkowski functionals [37] to obtain similar results constructively.

4.1 Archimedean spaces

The ordered vector space X is called **(weakly) Archimedean** if for each vector x of X^+, the (weak) infimum of the set $\{n^{-1}x : n = 1, 2, \dots\}$ exists. If X is weakly Archimedean, then the weak infimum is necessarily the null vector 0. Indeed, let S

be the set $\{n^{-1}x : n = 1, 2, \dots\}$ and let z be the weak infimum of S. Then $z \leq \frac{1}{2n}x$ for all positive integers n; whence $2z$ is a lower bound of S. Therefore $2z \leq z$ and, equivalently, $z \leq 0$. On the other hand, 0 is a lower bound of S, which ensures that $z \geq 0$. Consequently, $z = 0$.

As in the classical case, one can easily prove that X is (weakly) Archimedean if and only if for each $x \in X^+$, and for each sequence (α_n) of nonnegative numbers such that (α_n) converges to zero, the (weak) infimum of the set $\{\alpha_n x : n \in \mathrm{N}\}$ exists. A similar result is obtained when the condition (α_n) converges to zero is replaced by the condition $\inf\{\alpha_n : n \in \mathrm{N}\} = 0$. As a consequence, X is (weakly) Archimedean if and only if for each sequence (α_n) of real numbers with $\sup\{\alpha_n : n \in \mathrm{N}\} = \alpha$, and for each vector $x \geq 0$, the (weak) supremum of the set $\{\alpha_n x : n \in \mathrm{N}\}$ exists and equals αx.

Suppose now that for all $x \in X^+$, the sets $\{n^{-1}x : n = 1, 2, \dots\}$, are lower located. If X is either weakly Archimedean or Dedekind complete, then X is Archimedean. In the former case, it follows from Proposition 3.1.2 that X is Archimedean. If X is Dedekind complete, then the lower locatedness of the sets $\{n^{-1}x : n = 1, 2, \dots\}$ guarantees the existence of their infima.

Clearly, X is weakly Archimedean if and only if for each $y \in X$ and $x \in X^+$, the condition $ny \leq x$ for all positive integers n entails $y \leq 0$. In this case, for each strictly positive y, the set $\{ny : n \in \mathrm{N}\}$ cannot be bounded above.

A partially ordered set is said to be **directed upwards** if for each pair x, y of elements of X, there is an element z such that $x \leq z$ and $y \leq z$. A **directed downwards** set is defined correspondingly, as expected. For example, all lattices are directed both upwards and downwards. An ordered vector space is directed upwards if and only if it is directed downwards and, in this case, it will be referred simply, as a **directed vector space**. It is easy to prove that a directed vector space is weakly Archimedean if and only if for each vector y that exceeds 0, the set $\{ny : n \in \mathrm{N}\}$ is not bounded above. Equivalently, $y \leq 0$ whenever the set $\{ny : n \in \mathrm{N}\}$ is bounded above. The corresponding result for Archimedean spaces is given in the next proposition.

Proposition 4.1.1. *Let X be a directed vector space. Then X is Archimedean if and only if for each element y of X that exceeds 0, the set $\{ny : n \in \mathbf{N}\}$ is unbounded above.*

Proof. Assume that X is Archimedean, and that y is an element of X that exceeds 0. Let x be an arbitrary element of X. Since X is directed, there is an element $z \in X^+$ such that $x \leq z$. Then $\inf\{n^{-1}z : n = 1, 2, \dots\} = 0$ and, as y exceeds 0, it follows that $y \not\leq n^{-1}z$ for some positive integer n. Therefore ny exceeds z and, as a consequence, $ny \not\leq x$. The converse implication is straightforward. □

Assume now that X is a linearly ordered vector space. As a direct consequence of Proposition 4.1.1, X is Archimedean if and only if it satisfies the **Axiom of Archimedes**:

$$\forall x, y \in X \ (y > 0 \Rightarrow \exists n \in \mathbf{N} \ (ny > x)).$$

Not every Archimedean space is Dedekind complete. It is straightforward to verify that $C[0,1]$ is Archimedean; but, as we have already seen (Section 3.2), this Riesz space is Dedekind incomplete. However, as a consequence of the next proposition, a nontrivial[1] Archimedean space with a linear order is necessarily Dedekind complete.

Proposition 4.1.2. *Let X be a nontrivial Archimedean space. Then X is a linearly ordered vector space if and only if $X = \mathbf{R}u$ for some vector $u \neq 0$.*

Proof. If the space X is nontrivial and has a linear order, then we may assume, without loss of generality, that there exists a strictly positive vector $u > 0$ in X. Since X is Archimedean, for each x in X we can find the positive integers m and n such that $mu > x$ and $nu > -x$. As a consequence, the set $S_x = \{\lambda \in \mathbf{R} : x \leq \lambda u\}$ is nonempty, and $-n$ is a lower bound of S_x. We show that the infimum of S_x exists. Given two real numbers α and β with $\alpha < \beta$, we have to prove that either α is a lower bound of S_x or $\beta > \lambda$ for some $\lambda \in S_x$. Since $\alpha u < \beta u$, either $\alpha u < x$ or

[1] An ordered vector space X is said to be nontrivial if there exists $x \in X$ with $x \neq 0$.

$x < \beta u$. In the former case $\alpha \leq \lambda$ for all $\lambda \in S_x$, and in the latter one there exists a positive integer n such that $n^{-1}u < \beta u - x$. It follows that $\beta - n^{-1} \in S_x$ and, consequently, S_x has an infimum.

For each vector x of X, denote the infimum of S_x by $\varphi(x)$ and assume that $\varphi(x)u \neq x$. It follows that either $\varphi(x)u < x$ or $x < \varphi(x)u$. Therefore there exists a positive integer n such that $(\varphi(x) + 1/n)u < x$ or $x < (\varphi(x) - 1/n)u$. Since both conditions are contradictory to the definition of $\varphi(x)$, we obtain $\varphi(x)u = x$. Therefore $X = \mathbf{R}u$.

Conversely, if $X = \mathbf{R}u$ with $u \neq 0$, then we may assume that $u > 0$. If $\alpha u \not\leq \beta u$, then it follows from the definition of an ordered vector space that $\alpha > \beta$ and therefore $\alpha u > \beta u$. Consequently, the excess relation on $\mathbf{R}u$ is a linear order. $\square$

As a consequence, under the above hypotheses $X = \mathbf{R}u$ is a unidimensional normed space, with the norm defined by $\|x\| = |\varphi(x)|$ for all x. Since $\mathbf{R}$ is Dedekind complete, every Archimedean space $\mathbf{R}u$ is Dedekind complete.

4.2 Lattice norms

This section deals with ordered vector spaces endowed with norms that are somewhat related to the order relations. We will discuss two types of connection between norm and order. First, let us consider the case when the apartness defined by the norm coincides with the one given by the excess relation. A vector space with this property will be called a **normed ordered vector space**. In other words, X is a normed ordered vector space with respect to the excess relation $\not\leq$ and the norm $\|\ \|$ if for each vector x,

$$\|x\| \neq 0 \Leftrightarrow (x \not\leq 0 \vee 0 \not\leq x).$$

This condition is automatically satisfied in classical mathematics. However, if each norm on an ordered vector space satisfies the above condition, then Markov's Principle holds. To prove this, let us observe that for each real number a with $\neg(a \leq 0)$, the mapping $x \mapsto a|x|$ from $\mathbf{R}$ to $\mathbf{R}$ is a norm. If $\mathbf{R}$ is a normed ordered vector space with respect to this norm, then $a > 0$. Conversely, Markov's Principle implies that each ordered vector space with a norm is a normed ordered vector space. This is a straightforward consequence of the equivalence

$$x = 0 \Leftrightarrow \|x\| = 0.$$

Now let us examine another situation which occurs frequently in functional analysis: when a Riesz space has a norm that satisfies the condition

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|.$$

We say that this norm is a **weak lattice norm**. The classical name **lattice norm** is reserved for a norm that satisfies the classically equivalent condition:

$$\|x\| > \|y\| \Rightarrow |x| \nleq |y|.$$

Every lattice norm satisfies the condition

$$\|x\| > 0 \Rightarrow x \neq 0.$$

Indeed, if $\|x\| > 0 = \|0\|$, then $|x| > 0$ or, equivalently, $x \neq 0$.

If X is a Riesz space endowed with a weak lattice norm, then the mappings defined by:

$$(x, y) \mapsto x \vee y \quad ((x, y) \in X \times X),$$

$$(x, y) \mapsto x \wedge y \quad ((x, y) \in X \times X),$$

$$x \mapsto x^+ \quad (x \in X),$$

$$x \mapsto x^- \quad (x \in X),$$

$$x \mapsto |x| \quad (x \in X),$$

are uniformly continuous. The classical proof (see for example Proposition 5.2 of [65]) is constructively valid. Indeed, it is sufficient to observe that the inequalities

$$|x \vee y - x' \vee y'| \leq |x - x'| + |y - y'|,$$

$$|x \wedge y - x' \wedge y'| \leq |x - x'| + |y - y'|$$

hold constructively in each Riesz space. Since the mapping $x \mapsto x^-$ is continuous, the positive cone $X^+ = \{x \in X : x^- = 0\}$ is closed. In this case, if (x_n) is a decreasing sequence of elements of X that converges to x, then x is the weak infimum of the set $\{x_n : n \in \mathbf{N}\}$. In turn, this condition shows that X is weakly Archimedean. The classical proofs of these results, as given in [1], [65], or [73], do not require any modification.

Let us consider now a Riesz space X with a lattice norm. The space X is called a **normed Riesz space** if it is a normed ordered vector space with respect to the lattice norm. The Brouwerian example from the beginning of this section shows that if each Riesz space with a lattice norm is a normed Riesz space, then Markov's Principle holds.

Lemma 4.2.1. *Let X be a normed Riesz space and (x_n) a decreasing sequence of elements of X. If (x_n) converges to x, then $x = \inf\{x_n : n \in \mathbf{N}\}$.*

Proof. We have already seen that x is the weak infimum of the set $\{x_n : n \in \mathbb{N}\}$. To end the proof let us consider an element y with $x < y$. Then $\|y - x\| > 0$ and there exists n_0 such that $\|x_n - x\| < \|y - x\|$ for all $n > n_0$. For such an n, $|y - x| \nleq |x_n - x|$; whence $y - x \nleq x_n - x$, or, equivalently, $y \nleq x_n$. □

The following result is a direct consequence of this lemma.

Proposition 4.2.2. *Each normed Riesz space is Archimedean.*

4.3 The Minkowski functional associated with an order unit

Under certain hypotheses, one can define a weak lattice norm on the Archimedean space X such that the closed unit ball is a closed order interval. It turns out that this norm is the Minkowski functional of that order interval.

Let X be an ordered vector space. A vector e is said to be an **order unit** if for each $x \in X$ there is a positive number α such that $x \leq \alpha e$. For example, any

$x = (x_1, \ldots, x_n)$ is an order unit in $\mathbf{R}^n$ whenever $x_i > 0$ for each i. Clearly, an ordered vector space with an order unit is necessarily directed. If e is an order unit, then $e \geq 0$. Furthermore, we prove that $e > 0$ whenever X is nontrivial.

Lemma 4.3.1. *Let X be a nontrivial ordered vector space with an order unit e. Then e is strictly positive.*

Proof. Let x be a vector of X such that $x \neq 0$. Then there exists a vector y (namely, x or $-x$) that exceeds 0. Let α be a positive number with $y \leq \alpha e$. Since y exceeds 0 and y does not exceed αe, it follows that $\alpha e \not\leq 0$. The vector αe is positive and exceeds 0; whence it is strictly positive. Therefore e is strictly positive. □

A subset A of a real vector space X is said to be **balanced** if $\lambda A \subseteq A$ for all real numbers λ such that $|\lambda| \leq 1$. For each positive element x of an ordered vector space, the order interval $[-x, x]$ is balanced and $\lambda[-x, x] = [-\lambda x, \lambda x]$ for all $\lambda > 0$. A convex subset A is said to be **absorbing** if every x in X belongs to one of the sets λA with $\lambda > 0$. If X is an ordered vector space and e is an order unit, then the order interval $[-e, e]$ is a convex absorbing set. Classically, for a convex absorbing subset C of a vector space X, the **Minkowski functional** μ of C is defined by

$$\mu(x) = \inf\{\lambda > 0 : x \in \lambda C\} \quad (x \in X).$$

However, as pointed out by Ishihara [37], we cannot expect to prove constructively that each convex absorbing set has a Minkowski functional. Furthermore, if for each Archimedean Riesz space with an order unit e, the set $[-e, e]$ has a Minkowski functional, then WLPO holds. To prove this, let us consider the space ℓ^∞ of all bounded sequences of real numbers with the usual excess relation defined by

$$(a_n) \not\leq (b_n) \Leftrightarrow \exists n \, (a_n > b_n)$$

We omit the simple proof that ℓ^∞ is an Archimedean Riesz space and $e = (1, 1, 1, \ldots)$ is an order unit. Let (a_n) be an arbitrary binary sequence and assume that the infimum μ of the set $\{\lambda > 0 : (a_n) \leq \lambda e\}$ exists. Either $\mu < 1$ or $\mu > 0$ and, as a consequence, either $(a_n) = 0$ or $\neg((a_n) = 0)$.

The following lemma of Ishihara [37] provides an equivalent condition for the existence of the Minkowski functional.

Lemma 4.3.2. *Let C be a convex absorbing subset of X. Then C has a Minkowski functional if and only if for all x in X and all positive real numbers s, t with $s < t$, either $x \notin sC$ or $x \in tC$.*

An equivalent condition for the existence of infima can be obtained in a slightly more general case. Furthermore, similar conditions hold for weak infima.

Lemma 4.3.3. *Let S be a nonempty set, and let $(C_\alpha)_{\alpha \geq 0}$ be a family of subsets of X such that $\bigcup_{\alpha \geq 0} C_\alpha = S$ and $C_\alpha \subseteq C_\beta$ whenever $0 \leq \alpha < \beta$. Then each element $x \in S$ satisfies the following conditions:*

(i) *The infimum of the set $\{\lambda > 0 : x \in C_\lambda\}$ exists if and only if $0 < \alpha < \beta$ entails $x \notin C_\alpha$ or $x \in C_\beta$.*

(ii) *The weak infimum of the set $\{\lambda > 0 : x \in C_\lambda\}$ exists if and only if $0 < \alpha < \beta$ entails $x \notin C_\alpha$ or $\neg\neg(x \in C_\beta)$.*

Proof. (i) We have to prove that the set $\{\lambda > 0 : x \in C_\lambda\}$ is lower located if and only if for all α, β with $0 < \alpha < \beta$, either $x \notin C_\alpha$ or $x \in C_\beta$. The proof of Lemma 4.3.2 can be easily adapted for the general case.

(ii) The proof is similar to that of (i). Let x be an arbitrary element of S and let $S_x = \{\lambda > 0 : x \in C_\lambda\}$. If μ is the weak infimum of S_x and $0 < \alpha < \beta$, then either $\alpha < \mu$ or $\mu < \beta$. In the former case, if $x \in C_\alpha$, then $\mu \leq \alpha$, which is contradictory. In the latter case, assume that $x \notin C_\beta$ and let $\lambda > 0$ such that $x \in C_\lambda$. If $\lambda < \beta$, then $x \in C_\beta$, again a contradiction. Therefore $\lambda \geq \beta$, that is, β is a lower bound of S_x; as a consequence, $\beta \leq \mu$, which is contradictory to $\mu < \beta$. It follows that $\neg\neg(x \in C_\beta)$, as desired.

To obtain the converse implication we have to prove that the set S_x is weakly lower located. To this end, let us consider numbers α and β with $\alpha < \beta$. Either $\alpha < 0$, and so α is a lower bound of S_x, or else $0 < \beta$. In the latter case, let α' and β' be such that $\max(0, \alpha) < \alpha' < \beta' < \beta$. Either $\neg(x \in C_{\alpha'})$ or $\neg\neg(x \in C_{\beta'})$. If $\neg(x \in C_{\alpha'})$, then α is a lower bound of S_x. In the latter case, β is not a lower bound of S_x. Therefore S_x is weakly lower located. $\square$

Corollary 4.3.4. *Let X be an ordered vector space with an order unit e. Then for each vector x of X, the following conditions are equivalent.*

(1) *The set $\{\lambda > 0 : x \in [-\lambda e, \lambda e]\}$ has an infimum.*

(2) *The set $\{\lambda > 0 : x \in [-\lambda e, \lambda e]\}$ has a weak infimum.*

(3) *If $0 < \alpha < \beta$, then either $x \in [-\beta e, \beta e]$ or $\neg(x \in [-\alpha e, \alpha e])$.*

Proof. We apply Lemma 4.3.3 to the sets $C_\lambda = [-\lambda e, \lambda e]$, $\lambda \in [0, \infty)$, and we take into account that $\neg\neg(x \in [-\beta e, \beta e])$ if and only if $x \in [-\beta e, \beta e]$. Indeed, since for any statements P and Q, $\neg\neg(P \wedge Q)$ and $(\neg\neg P) \wedge (\neg\neg Q)$ are equivalent, it follows that the former condition is equivalent to $\neg\neg(x \leq \beta e) \wedge \neg\neg(-x \leq \beta e)$. It is sufficient now to observe that for each pair y, z of vectors, $\neg\neg(y \leq z)$ means $\neg\neg\neg(y \not\leq z)$, which is equivalent to $\neg(y \not\leq z)$—that is, $y \leq z$.[2] $\square$

Since $[-e, e]$ is a convex absorbing subset of X, the equivalence of (1) and (3) is also a direct consequence of Lemma 4.3.2. Let us assume now that for each x in X, the (weak) infimum $\mu(x)$ of the set $\{\lambda > 0 : x \in [-\lambda e, \lambda e]\}$ exists. Taking into account that $[-e, e]$ is also a balanced set, we see that the Minkowski functional μ is a seminorm (Chapter 2 of [23]). If in addition X is weakly Archimedean, we can prove, as in the classical case [71], that the Minkowski functional is a norm.

Proposition 4.3.5. *Let X be an ordered vector space with an order unit e, and assume that for all real numbers α and β with $0 < \alpha < \beta$, either $x \in [-\beta e, \beta e]$ or else $\neg(x \in [-\alpha e, \alpha e])$. Then the following statements hold.*

(i) *If X is weakly Archimedean, then the Minkowski functional μ of $[-e, e]$ is a norm, $[-e, e]$ is the closed unit ball with respect to this norm, and for all $x \in X$, $x \in [-\mu(x)e, \mu(x)e]$.*

(ii) *If X is Archimedean, then $\mu(x) > 0$ whenever $x \neq 0$. Moreover, X is a normed ordered vector space with respect to the norm μ if and only if each vector x of*

[2] Although the classical principle $\neg\neg P \Rightarrow P$ is rejected in constructive mathematics, one can easily prove the implication $\neg\neg\neg P \Rightarrow \neg P$. This was first observed by Brouwer.

X satisfies the following condition:

$$\forall\lambda \in \mathbf{R}\ (0 < \lambda \Rightarrow x \in [-\lambda e, \lambda e] \vee x \neq 0). \tag{4.1}$$

Proof. (i) To prove that μ is a norm, we need only show that $\mu(x) = 0$ entails $x = 0$. If $\mu(x) = 0$, then for all positive integers n, $x \in [-n^{-1}e, n^{-1}e]$. Since X is weakly Archimedean and both x and $-x$ are lower bounds of the set $\{n^{-1}e : n \in \mathbf{N}\}$, it follows that $x \leq 0$ and $-x \leq 0$ hence $x = 0$. Clearly, every vector x belongs to all the order intervals $[(-\mu(x)-n^{-1})e, (\mu(x)+n^{-1})e]$ $(n \in \mathbf{N})$. It follows that for all positive integers n, $x - \mu(x)e \leq n^{-1}e$ and $-x - \mu(x)e \leq n^{-1}e$. Therefore $x - \mu(x)e \leq 0$ and $-x - \mu(x)e \leq 0$; whence $-\mu(x)e \leq x \leq \mu(x)e$. It is now straightforward to observe that $x \in [-e, e]$ if and only if $\mu(x) \leq 1$—that is, $[-e, e]$ is the closed unit ball.

(ii) Let x be a vector of X with $x \neq 0$. Without loss of generality we may assume that $x \nleq 0$. Since X is Archimedean, there exists a positive integer n such that $x \nleq n^{-1}e$ and therefore $\mu(x) \geq 1/n$. It follows that X is a normed ordered vector space with respect to μ if and only if

$$\mu(x) > 0 \Rightarrow x \neq 0. \tag{4.2}$$

To end the proof, it is sufficient to show that for all vectors x the conditions (4.1) and (4.2) are equivalent. Suppose that x satisfies (4.1) and $\mu(x) > 0$. If $0 < \lambda < \mu(x)$, then either $x \neq 0$ or $x \in [-\lambda e, \lambda e]$. The latter case is contradictory, so $x \neq 0$. Conversely, if λ is a strictly positive number, then either $\mu(x) > 0$ or $\mu(x) < \lambda$. In the former case $x \neq 0$ and in the latter one, $x \in [-\lambda e, \lambda e]$. □

Assume now that, in addition, X is a Riesz space. Classically, X is Archimedean if and only if it is a normed Riesz space with respect to the Minkowski functional of $[-e, e]$. The following two propositions are constructive counterparts of this result; both are classically equivalent to the classical theorem.

Proposition 4.3.6. *Let X be a Riesz space with an order unit e, and assume that the Minkowski functional μ of $[-e, e]$ exists. Then X is weakly Archimedean if and only if μ is a norm. In this case, μ is a weak lattice norm.*

Proof. Assume that μ is a norm. To prove that X is weakly Archimedean, let x, y be a pair of vectors of X such that $x \geq 0$ and $y > 0$. We have to prove that y cannot be a lower bound of the set $\{n^{-1}x : n \in \mathbf{N}\}$. To this end, assume that y is a lower bound and let ε be a strictly positive number. Since $y \leq n^{-1}(\mu(x) + \varepsilon)e$ for all n, it follows that $\mu(y) \leq n^{-1}(\mu(x)+\varepsilon)$ for all n or, equivalently, that $\mu(y) = 0$. Since μ is a norm, it follows that $y = 0$, which is a contradiction. The converse implication has been already proved (Proposition 4.3.5(i)). To end the proof we need only observe that $|x| \leq |y|$ entails $\mu(x) \leq \mu(y)$, so μ is a weak lattice norm. □

Proposition 4.3.7. *Let X be a Riesz space with an order unit e such that every vector x satisfies the condition*

$$0 < \alpha < \beta \Rightarrow (|x| \leq \beta e \vee |x| \not\leq \alpha e).$$

Then the Minkowski functional μ of $[-e, e]$ exists. Furthermore, X is Archimedean if and only if it is a normed ordered vector space with respect to μ. In this case μ is a lattice norm.

Proof. Clearly, the hypothesis is sufficient for the existence of μ. We prove that if μ is a norm, then is necessarily a lattice norm. To this end, let x, y be a pair of vectors of X with $\mu(x) > \mu(y)$. Then there exist $\alpha > \mu(y)$ and β such that $0 < \alpha < \beta < \mu(x)$; whence either $|x| \leq \beta e$ or else $|x| \not\leq \alpha e$. The former condition is contradictory, so the latter is the case. Since $|y| \leq \mu(y)e \leq \alpha e$ and $|x| \not\leq \alpha e$, it follows that $|x| \not\leq |y|$; whence μ is a lattice norm.

If X is Archimedean, then μ is a norm and $x \neq 0$ entails $\mu(x) > 0$ (Proposition 4.3.5). Since μ is a lattice norm, $\mu(x) > 0$ entails $x \neq 0$. Consequently, X is a normed ordered vector space. Conversely, let us assume that X is a normed ordered vector space with respect to μ. Having a lattice norm, X is a normed Riesz space, and according to Proposition 4.2.2, X is Archimedean. □

Corollary 4.3.8. *Let X be a Riesz space with an order unit e. Then the following statements are equivalent.*

(1) *The Minkowski functional μ of $[-e, e]$ exists and X is a normed Riesz space with respect to μ.*

(2) *For each vector $y > 0$ of X, 0 is bounded away*[3] *from the set $\{\lambda > 0 : |y| \leq \lambda e\}$ and every vector x satisfies the condition*

$$0 < \alpha < \beta \Rightarrow (|x| \leq \beta e \vee |x| \nleq \alpha e).$$

Proof. If X is a normed Riesz space, then X is Archimedean. Therefore $y > 0$ entails $\mu(y) > 0$. Since $\mu(y)$ is the infimum of the set $\{\lambda > 0 : |y| \leq \lambda e\}$, it follows that 0 is bounded away from this set. Consider now an arbitrary vector x. If $0 < \alpha < \beta$, then either $\mu(x) < \beta$ or $\mu(x) > \alpha$. In the former case, $|x| \leq \beta e$, and in the latter, $\mu(x) > \mu(\alpha e)$. Since μ is a lattice norm, it follows that $|x| \nleq \alpha e$.

Let us prove now that (2) implies (1). Clearly, the hypotheses guarantee that μ exists and $\mu(x) > 0$ for all $x > 0$. If $x \neq 0$, then $|x| > 0$ and $0 < \mu(|x|) = \mu(x)$. Therefore μ is a norm. As we have shown in the proof of the preceding proposition, μ is a lattice norm and, in addition, X is a normed ordered vector space. Consequently, X is a normed Riesz space. $\square$

4.4 Krein spaces

Let us consider a normed ordered vector space X and suppose that the positive cone X^+ has a nonempty interior. Following Vulikh [70], we say that X is a **Krein space** and we write $x \gg y$ whenever $x - y$ belongs to the interior of X^+. For the classical theory of Krein spaces the reader is referred to [47] or to Chapter XIII of [70].

Proposition 4.4.1. *Let x be a vector of the normed ordered vector space X. If X is nontrivial and $x \gg 0$, then $x > 0$.*

Proof. First, let us observe that $x \gg 0$ entails $\neg(x = 0)$ [70]. Consider now $r > 0$ such that $y \geq 0$ whenever $\|x - y\| < r$. If $\|x\| < r$, then we can find a positive number α such that $(1 + \alpha)\|x\| < r$. Therefore $-\alpha x \geq 0$ so $x \leq 0$; since $x \in X^+$, it follows that $x = 0$. This contradiction ensures that $\|x\| \geq r > 0$. Taking into

[3] A real number α is bounded away from the subset S of $\mathbf{R}$ if there exists $r > 0$ such that $|\lambda - \alpha| \geq r$ for all $\lambda \in S$.

account that X is a normed ordered vector space, we obtain $x \neq 0$. Thus $x \geq 0$ and $x \neq 0$; therefore $x > 0$. □

As in the classical case, an element e of a Krein space is an order unit if and only if $e \gg 0$. Therefore an element e of a Krein space is an order unit if and only if the order interval $[-e, e]$ has a nonempty interior. Furthermore, in this case 0 is an interior point of $[-e.e]$.

As shown by Ishihara [37], the existence of the Minkowski functional of $[-e, e]$ is intimately connected with the locatedness of this order interval. A subset S of a metric space X is said to be **located** in X if

$$d(x, S) = \inf\{d(x, y) : y \in S\}$$

can be computed for each x in X. We say that S is **weakly located** if the weak infimum $wd(x, S)$ of the set $\{d(x, y) : y \in S\}$ exists for each x. Note that if each weakly located subset of $\mathbf{R}$ is located, then LPO holds.

If C is a located convex absorbing subset of a normed linear space and the interior of C is nonempty, then the Minkowski functional of C exists (Proposition 1 of [37]). Therefore if X is both a Krein space and a normed space, then the Minkowski functional of $[-e, e]$ exists provided that $e \gg 0$ and $[-e, e]$ is located. Moreover, in this case Ishihara's proof can be adapted in order to prove that even the weak locatedness of $[-e, e]$ is sufficient.

Proposition 4.4.2. *Let $e \gg 0$ be an element of the Krein space X. If X is a normed space and the order interval $[-e, e]$ is weakly located, then its Minkowski functional exists.*

Proof. Let $\delta > 0$ such that the open ball[4] $B(0, \delta)$ is a subset of $[-e, e]$. Let α, β be real numbers with $0 < \alpha < \beta$, and set $\varepsilon = \delta(\beta - \alpha)$. For each vector x, either $wd(x, [-\alpha e, \alpha e]) > 0$ or $wd(x, [-\alpha e, \alpha e]) < \varepsilon$. In the former case, $x \notin [-\alpha e, \alpha e]$. In the latter, assume that $x \notin [-\beta e, \beta e]$, and let y be a vector in $[-\alpha e, \alpha e]$. Then $x - y$

[4] The open ball of center a and radius ρ : $\{x \in X : \|x - a\| < \rho\}$, will be denoted, as usual, by $B(a, \rho)$.

does not belong to $[(\alpha-\beta)e, (\beta-\alpha)e]$. If $\|x-y\| < \epsilon$, then $\|x-y\|/(\beta-\alpha) < \delta$; whence $x-y \in [(\alpha-\beta)e, (\beta-\alpha)e]$, which is contradictory. Therefore $\|x-y\| \geq \epsilon$ for all $y \in [-\alpha e, \alpha e]$. This is not possible, because $wd(x, [-\alpha e, \alpha e]) < \varepsilon$. Consequently, $\neg\neg(x \in [-\beta e, \beta e])$ and, in view of Lemma 4.3.3 and Corollary 4.3.4, the Minkowski functional of $[-e, e]$ exists. □

We end this chapter with the finite–dimensional case. First, let us consider the space $\mathbf{R}^n$. An element $e = (e_1, \ldots, e_n)$ is an order unit if and only if $e_i > 0$ for all $i \in \{1, \ldots, n\}$. Then the interval

$$[-e, e] = [-e_1, e_1] \times \cdots \times [-e_n, e_n]$$

is totally bounded and therefore located (Proposition 4.6 in Chapter 4 of [11]). Consequently, the Minkowski functional of $[-e, e]$ exists. It remains an open problem to decide whether this result can be extended to an arbitrary finite–dimensional space. Nevertheless, the following result holds.

Corollary 4.4.3. *Let X be a finite–dimensional normed ordered vector space with an order unit e. Then the following conditions are equivalent.*

(1) *The Minkowski functional of $[-e, e]$ exists.*

(2) *The order interval $[-e, e]$ is located.*

(3) *The order interval $[-e, e]$ is weakly located.*

Proof. The interior of $[-e, e]$ contains the null vector 0 (Lemma 3 of [19]), so X is a Krein space and $e \gg 0$. It follows from Theorem 2 in [37] that in the finite–dimensional case (1) and (2) are equivalent. Every located set is weakly located, so (2) entails (3). To end the proof, we need only observe that the implication (3) $\Rightarrow$ (1) is given by Proposition 4.4.2. □

Chapter 5

Positive operators

The study of positive operators—that is, linear mappings that preserve the order structures—was originated in the 1930s. The set of positive operators between two ordered vector spaces X and Y is a positive cone in the vector space of all order bounded linear mappings of X into Y, so this space is partially ordered in a natural way. Considering an appropriate excess relation, we prove a similar result constructively.

Having obtained an order structure on this space of operators, we next seek to define lattice operations. Classically, this is possible whenever X is a Riesz space and Y is a Dedekind complete Riesz space. (For the classical theory, the reader is referred to the book [3].) As a consequence, every order bounded linear functional on a Riesz space has a modulus. Unlike in the classical case, we cannot expect to prove this result constructively. The main reason is, as one could expect, that certain suprema cannot be always computed. We discuss this matter in Section 5.3. Nevertheless, when X is a Riesz space with an order unit, the modulus of a linear functional on X exists, whenever a certain supremum can be computed (Section 5.4).

5.1 Monotonicity

Since a linear operator is positive if and only if it is increasing, we should clarify the constructive definitions of monotone functions. Apart from the classical distinction between increasing and strictly increasing functions, other relevant distinctions should be made in the constructive theory. Let X and Y be two partially ordered sets. A mapping $f : X \to Y$ is said to be

- **increasing** if $x < y$ implies $f(x) \leq f(y)$;
- **strongly increasing** if $f(x) \nleq f(y)$ implies $x \nleq y$;
- **strictly increasing** if $x < y$ implies $f(x) < f(y)$;
- **almost strictly increasing** if $x < y$ implies $\neg\neg(f(x) < f(y))$.

Real-valued functions on a subset of $\mathbf{R}$ that satisfy such properties of monotonicity were investigated in detail by Mandelkern [49, 50].[1] Proposition 12.5 of [50] can be easily adapted for the general case. We need only replace the strict order relation $<$ on $\mathbf{R}$ by the excess relations on X, respectively Y. We can also obtain the next proposition as a corollary of Proposition 6.5.1.

Proposition 5.1.1. *The following are equivalent conditions for a function f from X to Y.*

(1) *The function f is increasing.*

(2) $x \leq y \Rightarrow f(x) \leq f(y)$;

(3) $f(x) \nleq f(y) \Rightarrow \neg(x \leq y)$;

(4) $f(x) \nleq f(y) \Rightarrow \neg(x < y)$;

(5) $\neg(f(x) \leq f(y)) \Rightarrow \neg(x \leq y)$;

[1] We prefer to use the terms "strictly increasing" and "increasing", as in the classical literature, rather than "increasing", respectively "nondecreasing", as in Mandelkern's terminology.

(6) $\neg(f(x) \leq f(y)) \Rightarrow \neg(x < y)$.

In a similar way we can adapt Proposition 12.6 of [50] to obtain equivalent conditions for almost strict monotonicity.

Proposition 5.1.2. *Let f be a function from X to Y. Then the following conditions are equivalent.*

(1) *The function f is almost strictly increasing.*

(2) $\neg\neg(x < y) \Rightarrow \neg\neg(f(x) < f(y))$;

(3) $\neg(f(x) < f(y)) \Rightarrow \neg(x < y)$.

Every strictly increasing function is almost strictly increasing and every almost strictly increasing function is increasing. However, if every almost strictly increasing linear function is strictly increasing, then Markov's principle holds. To prove this, let us consider the function $f : \mathbf{R} \to \mathbf{R}$, defined by $f(x) = ax$, where a is a real number such that $\neg(a \leq 0)$. Then f is almost strictly increasing; furthermore, f is strictly increasing if and only if $a > 0$.

We now examine the strong monotonicity. First, note that every strictly increasing function between subsets of $\mathbf{R}$ is strongly increasing if and only if Markov's principle holds. To prove this, let a be a number such that $\neg(a \leq 0)$ and let $f : \{0, a\} \to \mathbf{R}$ be defined by $f(0) = 0$ and $f(a) = 1$. Then f is strictly increasing. If f is strongly increasing, then $a \neq 0$. The converse implication is straightforward.

Even if we consider a compact subset X of the real number line, we cannot guarantee that every strictly increasing real–valued function on X is strongly increasing. Let f be defined as above and assume that a is **pseudopositive**; that is, it satisfies the property

$$\forall x \in \mathbf{R}(\neg\neg(0 < x) \vee \neg\neg(x < a)).$$

Then (see Theorem 2 of [36]) f is strongly increasing if and only if $a > 0$. Therefore every strictly increasing function on a compact subset of $\mathbf{R}$ is strongly increasing if and only if **weak Markov principle (WMP)**:

every pseudopositive real number is positive

holds.

Let X and Y be two sets with apartness relations. A mapping f from X to Y is said to be **strongly extensional** if for all x, y in X,

$$f(x) \neq f(y) \Rightarrow x \neq y.$$

The following result shows that the stronger monotonicity is closely related to strong extensionality.[2] We will prove it in a more general setting in Section 6.5 (Proposition 6.5.6).

Proposition 5.1.3. *Let X and Y be two partially ordered sets. If f is a strongly increasing mapping of X into Y, then f is increasing and strongly extensional. The converse implication is valid whenever X is a lattice.*

5.2 Positivity

A linear mapping T between two ordered vector spaces X and Y is said to be

- **positive** if $\forall x \in X \; (x > 0 \Rightarrow Tx \geq 0)$;
- **strongly positive** if $\forall x \in X \; (0 \not\leq Tx \Rightarrow 0 \not\leq x)$;
- **strictly positive** if $\forall x \in X \; (x > 0 \Rightarrow Tx > 0)$;
- **almost strictly positive** if $\forall x \in X \; (x > 0 \Rightarrow \neg\neg(Tx > 0))$.

A positive linear mapping from X to $\mathbf{R}$ is called a **positive functional**. A **strongly positive functional**, a **strictly positive functional**, and an **almost strictly positive functional** are defined correspondingly. Clearly, each variant of positivity

[2] A strongly increasing function is also called "antidecreasing" [49, 50]. To emphasize the relation between strong extensionality and this type of monotonicity, we use the former term.

is related to a certain type of monotonicity. For instance, a positive operator is an increasing linear function between two ordered vector spaces.

The following two results illustrate the connection between strong positivity and strong extensionality.

Corollary 5.2.1. *A linear mapping T of the Riesz space X into the ordered vector space Y is strongly positive if and only if T is positive and for all x in X^+,*

$$Tx > 0 \Rightarrow x \neq 0.$$

Proof. Clearly, T is strongly positive if and only if it is positive and strongly extensional (Proposition 5.1.3). We need only show that the positivity of T and the displayed implication guarantee the strong extensionality of T. To prove this, let x be a vector of X with $Tx \neq 0$. Then $Tx^+ \neq Tx^-$, so either $Tx^+ > 0$ or else $Tx^- > 0$. It follows that either $x^+ > 0$ or $x^- > 0$, and therefore that $x \neq 0$. □

Corollary 5.2.2. *If X and Y are normed ordered vector spaces and, in addition, X is a Banach space and a Riesz space, then every positive operator from X to Y is strongly positive.*

Proof. Every linear mapping of a Banach space into a normed space is strongly extensional (Corollary 2 of [20]). The strong positivity follows now from Proposition 5.1.3. □

In the remainder of this section we consider only linear functionals.

Proposition 5.2.3. *A positive functional on X maps every order interval $[a, b]$ of X onto a totally bounded subset of* **R**.

Proof. For a positive functional φ on X and a positive element x of X, we have $\varphi[0, x] \subseteq [0, \varphi(x)]$. If $\varepsilon > 0$ then either $\varphi(x) < \varepsilon$ or $\varphi(x) > 0$. In the first case, $\{0\}$ is an ε–approximation to $\varphi[0, x]$. In the second case, if $\alpha \in [0, \varphi(x)]$, then

$$y = \alpha(\varphi(x))^{-1}x \in [0, x]$$

and $\varphi(y) = \alpha$. It follows that $\varphi[0,x] = [0,\varphi(x)]$ and therefore that we can find a finite ε–approximation to $[0,\varphi(x)]$.

Now consider an arbitrary order interval $[a,b]$ of X. Since φ is linear,

$$\varphi[a,b] = \varphi(a) + \varphi[0, b-a],$$

which, being a translate of a totally bounded set, is totally bounded. □

Following [44], we say that a nonempty subset S of a partially ordered set X is **order convex** if for all a, b in S and all x in X, $a \leq x \leq b$ implies that $x \in S$.[3] In other words, S is order convex if $[a,b] \subseteq S$ whenever $a, b \in S$ and $a \leq b$. The positivity of linear functionals is related to the order convexity of their kernels. As in the classical case, we can easily prove that if either φ or $-\varphi$ is a positive functional, then ker φ is order convex and, equivalently, ker $\varphi \cap X^+$ is order convex. The converse implication is classically valid for functionals on a Riesz space, but not constructively. However, the following results hold.

Lemma 5.2.4. *Let X be a Riesz space.*

(i) *If φ is a nonzero linear functional on X and* ker φ *is order convex, then either φ or $-\varphi$ is positive.*

(ii) *If for every linear functional φ on X with* ker φ *order convex, either φ or $-\varphi$ is positive, then LLPO holds.*

Proof. (i) Since φ is nonzero and X is a Riesz space, there exists an element $x \in X^+$ with $\varphi(x) \neq 0$. Without loss of generality we may assume that $\varphi(x) = 1$. We show that φ is positive. To this end, let $y > 0$ be an element of X and assume that $\varphi(y) < 0$. From now on we can follow the classical proof (Proposition 1.5.5 of [44]). Indeed, there exists $z > 0$ with $\varphi(z) = -1$; whence $x + z$ is an element of ker φ, while x is not, contradictory to the order–convexity of ker φ. Therefore $\neg(\varphi(y) < 0)$; that is, $\varphi(y) \geq 0$.

[3] An order convex subset is also called a **full** subset [54].

(ii) For each real number a, let φ_a be the functional on $\mathbf{R}$ defined by $\varphi_a(x) = ax$. To prove that ker φ_a is order convex, let $x \in \ker\ \varphi_a \cap X^+$ and let $0 \leq y \leq x$. Then

$$0 \leq |a|y \leq |a|x = |ax| = 0.$$

It follows that $|a|y = 0$ and therefore $ay = 0$. Clearly, $a \geq 0$ whenever φ_a is positive; and $a \leq 0$ whenever $-\varphi_a$ is positive. Consequently, for every real number a either $a \geq 0$ or $a \leq 0$; this is equivalent to LLPO. □

Whereas the positivity of a functional φ is related to the order convexity of ker φ, almost strictly positivity is related to the property $\ker\ \varphi \cap X^+ = \{0\}$. It is straightforward to see that φ satisfies this condition whenever φ or $-\varphi$ is almost strictly positive. Let us examine the converse implication, which holds classically for functionals on a Riesz space (Proposition 1.9.5 of [44]).

Proposition 5.2.5. *Let X be a Riesz space.*

(i) *If φ is a nonzero linear functional on X and* $\ker\ \varphi \cap X^+ = \{0\}$, *then either φ or $-\varphi$ is almost strictly positive.*

(ii) *If for every linear functional φ on X with* $\ker\ \varphi \cap X^+ = \{0\}$, *either φ or $-\varphi$ is almost strictly positive, then the following statement holds.*

$$\forall a \in \mathbf{R}\ (\neg\neg(a > 0 \vee a < 0) \Rightarrow (\neg\neg(a > 0) \vee \neg\neg(a < 0))).$$[4]

Proof. (i) We may assume that $\varphi(x) > 0$ for some $x \in X^+$. Let y be a strictly positive element of X and assume that $\varphi(y) \leq 0$. On the one hand, ker φ is order convex and, according to Lemma 5.2.4, φ is positive. On the other hand, $y > 0$ and $\varphi(y) \leq 0$. Therefore $\varphi(y) = 0$ and, as a consequence of the hypothesis, $y = 0$. This contradiction ensures that $\neg(\varphi(y) \leq 0)$; hence φ is almost strictly positive.

(ii) For each real number a with $\neg\neg(a > 0 \vee a < 0)$—that is, $\neg(a = 0)$—consider the functional φ_a as in Lemma 5.2.4(ii). Let x be an element of $\ker\ \varphi_a \cap X^+$, and assume that $x \neq 0$. Then $a = 0$, a contradiction. Therefore $x = 0$, and φ_a satisfies

[4] This statement, weaker than Markov's principle and than LLPO, is called the **disjunctive version of Markov's principle** [39].

the hypothesis. If either φ_a or $-\varphi_a$ is almost strictly positive, then either $\neg(a \leq 0)$ or else $\neg(a \geq 0)$. $\square$

To deal with strict positivity instead of almost strict positivity, we should replace the condition ker $\varphi \cap X^+ = \{0\}$ by the classically equivalent condition

$$x > 0 \Rightarrow \varphi(x) \neq 0.$$

Corollary 5.2.6. *Let φ be a nonzero functional on the Riesz space X. Then φ satisfies the condition*

$$x > 0 \Rightarrow \varphi(x) \neq 0$$

if and only if either φ or $-\varphi$ is strictly positive.

Proof. If φ or $-\varphi$ is strictly positive, then $\varphi(x) \neq 0$ whenever $x > 0$. To prove the converse implication, let us observe that the latter condition ensures that the null vector is the only element of ker $\varphi \cap X^+$. Since φ is nonzero, it follows from Proposition 5.2.5 that either φ or $-\varphi$ is almost strictly positive. In the former case, $\neg\neg(\varphi(x) > 0)$ and $\varphi(x) \neq 0$ whenever $x > 0$; that is, $\varphi(x) > 0$ for all $x > 0$. In the latter case, it follows in a similar way that $-\varphi$ is strictly positive. $\square$

5.3 Order bounded operators

A function between two partially ordered sets X and Y is called **order bounded** if it maps order subsets of X onto order bounded subsets of Y. Clearly, a linear mapping between the ordered vector spaces X and Y is order bounded if and only it maps every interval $[0, x]$ of X onto an order bounded subset of Y. As a consequence, every positive operator is order bounded. Denote by $\mathcal{L}_b(X, Y)$ the set of all order bounded linear operators from X into Y. Then $\mathcal{L}_b(X, Y)$ is a vector space with respect to the usual operations of addition and multiplication by scalars.

Consider now two Riesz spaces X and Y. The canonical excess relation on $\mathcal{L}_b(X, Y)$ is defined by

$$S \not\leq T \Leftrightarrow \exists x \in X^+ \; (Sx \not\leq Tx).$$

We omit the simple proof that this does define an excess relation, with respect to which $\mathcal{L}_b(X,Y)$ is an ordered vector space whose positive cone is the set of positive operators between X and Y. Furthermore, $\mathcal{L}_b(X,Y)$ is an Archimedean space, respectively a weakly Archimedean one, whenever Y has the same property. In Section 3 of [4] we proved several results about the space $\mathcal{L}_b(X,Y)$ in the particular case $Y = \mathbf{R}$. In the remainder of this section we present these results in the general case.

The partial order $\leq$ and the apartness relation $\neq$ corresponding to the excess relation $\nleq$ are given by

$$S \leq T \Leftrightarrow \forall x \in X^+ \ (Sx \leq Tx),$$

$$S \neq T \Leftrightarrow \exists x \in X^+ \ (Sx \neq Tx).$$

The relation $\neq$ is the standard apartness on $\mathcal{L}_b(X,Y)$ given by

$$S \neq T \Leftrightarrow \exists y \in X \ (Sy \neq Ty).$$

Indeed, if $Sy \neq Ty$, then either $Sy^+ - Ty^+ \neq 0$ or else $Sy^+ - Ty^+ \neq Sy - Ty$. In the former case, $Sy^+ \neq Ty^+$, and in the latter, $Sy^- \neq Ty^-$. Consequently, there exists an element $x \in X^+$ such that $Sx \neq Tx$.

Classically, if $T : X^+ \to Y^+$ is additive; that is, $T(x+y) = Tx + Ty$ holds for all $x, y \in X^+$, then the mapping $S : X \to Y$ given by

$$Sx = Tx^+ - Tx^-$$

is additive and extends T. Indeed, it is straightforward to see that $Sx = Tx$ for each $x \in X^+$; and, as in the classical proof (see [3]), we can show that Sx is independent of the particular representation of x as a difference of positive elements. It follows that for each pair y, z of vectors of X, we have

$$S(y+z) = T(y^+ + z^+) - T(y^- + z^-) = Ty^+ + Tz^+ - Ty^- - Tz^- = Sy + Sz.$$

Lemma 5.3.1. *Let $T : X^+ \to Y^+$ be an additive mapping such that $T(\alpha x) = \alpha Tx$ for all $\alpha \geq 0$ and all $x \in X^+$. Then the mapping S defined as above is a positive operator.*

Proof. We need only prove the homogenity of S. To this end, let α be a real number and let x be an arbitrary vector in X. Then

$$S(\alpha x) = S(\alpha^+ x^+ - \alpha^- x^+) + S(\alpha^- x^- - \alpha^+ x^-)$$

and we can apply the definition of S together with the positive homogenity of T. $\square$

When Y is weakly Archimedean, the positivity homogenity of T is not necessary. It was proved by Kantorovich [46] that every additive mapping $T : X^+ \to Y^+$ extends uniquely to a positive operator between X and Y, provided that Y is weakly Archimedean. The classical proof (see also Theorem 1.7 of [3]) is constructively valid.

Proposition 5.3.2. *Let T be a linear operator between the Riesz spaces X and Y such that* $\sup\{Ty : 0 \le y \le x\}$ *exists for each $x \in X^+$. Then the function $T^+ : X \to Y$ given by*

$$T^+ x = \sup\{Ty : 0 \le y \le x^+\} - \sup\{Ty : 0 \le y \le x^-\},$$

is a positive operator. Moreover, T^+ is the supremum of the set $\{T, 0\}$ in $\mathcal{L}_b(X, Y)$.

Proof. By applying the Riesz decomposition property (Theorem 6.4 in [73]) and the linearity of T, we find, as in the classical case, that

$$\{Ty : 0 \le y \le x_1 + x_2\} = \{Ty : 0 \le y \le x_1\} + \{Ty : 0 \le y \le x_2\}$$

for all x_1, x_2 in X^+. It follows from Proposition 2.4.5(i) that T^+ is additive on X^+.

Given $\alpha \ge 0$ and $x \in X^+$, assume that $\alpha T^+ x \ne T^+(\alpha x)$. If $\alpha > 0$, then it is easy to check that $\alpha T^+ x = T^+(\alpha x)$, which is contradictory. Therefore $\alpha = 0$; hence $\alpha T x^+ = 0 = T^+(\alpha x)$.

Now, Lemma 5.3.1 ensures that T^+ is a positive operator. For $x \in X^+$,

$$Tx \le \sup\{Ty : 0 \le y \le x\} = T^+ x$$

and $T^+ x \ge 0$; whence T^+ is an upper bound of the set $\{T, 0\}$. Let $S \in \mathcal{L}_b(X, Y)$ with $T^+ \not\le S$; that is, $T^+ x \not\le Sx$ for some x in X^+. Then there exists $y \in [0, x]$ such that $Ty \not\le Sx$. Hence either $Ty \not\le Sy$ or else $Sy \not\le Sx$. In the former case, $T \not\le S$; and in the latter, $0 \le x - y \le x$ and $0 \not\le S(x - y)$, so $0 \not\le S$. Consequently, $T^+ = \sup\{T, 0\}$. $\square$

Corollary 5.3.3. *For any positive operator $T : X \to Y$, the positive part T^+ exists and equals T.*

Proof. If $x \in X^+$, then $\sup\{Ty : 0 \leq y \leq x\}$ exists and equals Tx. $\square$

Let $T \in \mathcal{L}_b(X, Y)$. If T^+ exists, then the negative part $T^- = \sup\{-T, 0\}$ and the modulus $|T| = \sup\{-T, T\}$ exist (Section 2.4). As a consequence, T is **regular** in the sense that it can be expressed as a difference of two positive operators (namely, T^+ and T^-). Taking into account that $T^- = T^+ - T$ and $|T| = T^+ + T^-$, it is easy to check that for all $x \in X^+$, the following identities hold:

$$T^-x = -\inf\{Tz; 0 \leq z \leq x\},$$
$$|T|x = \sup\{|Ty| : |y| \leq x\}.$$

If $\sup\{Tx : 0 \leq y \leq x\}$ exists for all $T \in \mathcal{L}_b(X, Y)$ and for all $x \in X^+$, then the ordered vector space $\mathcal{L}_b(X, Y)$ has the structure of a Riesz space. In this case, as we have already seen in Section 2.6, for all S and T we have $S \vee T = T + (S - T)^+$ and $S \wedge T = -(-S) \vee (-T)$. As in the classical case (see Theorem 1.13 of [3], it follows that $S \vee T$ and $S \wedge T$ satisfy the following conditions for all $x \in X^+$:

$$(S \vee T)x = \sup\{Sy + Tz : y, z \in X^+,\ y + z = x\},$$

$$(S \wedge T)x = \inf\{Sy + Tz : y, z \in X^+,\ y + z = x\},$$

Corollary 5.3.4. *For all positive integers m and n, $\mathcal{L}_b(\mathbf{R}^n, \mathbf{R}^m)$ is a Riesz space.*

Proof. Every linear function T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is uniformly continuous. For any positive vector $x = (x_1, \ldots x_n)$ in $\mathbf{R}^n$, the order interval

$$[0, x] = [0, x_1] \times \cdots \times [0, x_n]$$

is compact (Proposition 6, Chapter 4 in [8]), and therefore $\sup\{Ty : y \in [0, x]\}$ exists. $\square$

The last result of this section shows that the strong extensionality of T is related, as expected, to the strong extensionality of T^+, T^-, and $|T|$.

Proposition 5.3.5. *Let $T \in \mathcal{L}_b(X, Y)$ be such that T^+ exists. Then the following properties are equivalent.*

(1) *The function T is strongly extensional.*

(2) *The modulus $|T|$ is strongly positive.*

(3) *Both T^+ and T^- are strongly positive.*

Proof. Assume that T is strongly extensional and let x be an element of X^+ with $|T|x > 0$. Then there exists $y \in X$ with $|y| \leq x$ such that $Ty \not\leq 0$. It follows that $y \neq 0$; whence $|y| > 0$. Therefore $x > 0$ and, according to Corollary 5.2.1, $|T|$ is strongly positive.

Now we prove that (2) entails (3). To this end, let x be an element of X^+ with $T^+x > 0$. Then $|Tx| \geq T^+x > 0$ and, since $|T|$ is strongly extensional, it follows that $x \neq 0$. Therefore T^+ is strongly positive. The positive operator T^- is strongly extensional, being the sum of the strong extensional mappings T^+ and $-|T|$. Furthermore, T^- is strongly positive (Proposition 5.1.3).

The implication (3) $\Rightarrow$ (1) is straightforward. If T^+ and T^- are strongly positive, then they are strongly extensional, and so is their difference T. $\square$

5.4 The order dual

Classically, when Y is Dedekind complete, the space $\mathcal{L}_b(X, Y)$ is a Dedekind complete Riesz space. This was proved by F. Riesz [63] for the case $Y = \mathbf{R}$ and by L.V. Kantorovich [46] in the general setting. However, we cannot prove this result constructively.

Let X be a Riesz space. The space $\mathcal{L}_b(X, \mathbf{R})$ is called the **order dual** of X, and is usually denoted by $X^\sim$. Since $\mathbf{R}$ is Archimedean, the ordered vector space $X^\sim$ is Archimedean. As we have already seen in the preceding section, to guarantee that $X^\sim$ is a Riesz space we need to construct φ^+ for all $\varphi \in X^\sim$. We shall prove that whenever X has an order unit e and $\varphi \in X^\sim$, the computability of $\varphi^+(e)$ ensures the existence of $\varphi^+(x)$ for any x.

Lemma 5.4.1. *Let X be a Riesz space, φ an element of $X^\sim$, ε a positive number, and x, y elements of X such that $0 \leq y \leq x$. Then*

$$\forall z \in [0, x] \quad (\varphi(z) \leq \varphi(y) + \varepsilon)$$

if and only if

$$\forall z_1 \in [0, x - y]\, \forall z_2 \in [0, y] \quad (\varphi(z_1 - z_2) \leq \varepsilon).$$

Proof. In [8] (Lemma 1, Chapter 8), the space of all measures on a locally compact space is considered. The proof remains valid when this space is replaced by the order dual $X^\sim$ of an arbitrary Riesz space X. □

In [4] we proved the following analogue of Theorem 1, Chapter 8 in [8].

Proposition 5.4.2. *Let X be a Riesz space with an order unit e. If φ is a linear functional on X such that* $\sup\{\varphi(z) : 0 \leq z \leq e\}$ *exists, then $\varphi \in X^\sim$ and φ^+ exists.*

Proof. It suffices to prove that $\sup\{\varphi(z) : 0 \leq z \leq x\}$ exists for any $x \in X^+$. In this case, φ^+ exists and, furthermore, φ is order bounded—that is, $\varphi \in X^\sim$. Indeed, for each pair a, b of vectors with $a \leq b$, $\varphi[a, b] = \varphi(a) + \varphi[0, b - a]$ and this set is bounded above because it has a supremum. The linearity of φ ensures that the set $\varphi[a, b]$ is also bounded below.

Let us prove now the existence of φ^+. If $\varepsilon > 0$ then we can find y such that $0 \leq y \leq e$ and

$$\sup\{\varphi(z) : 0 \leq z \leq e\} - \varepsilon < \varphi(y).$$

Therefore $\varphi(z) < \varphi(y) + \varepsilon$ for any z with $0 \leq z \leq e$; whence $\varphi(z_1 - z_2) \leq \varepsilon$ whenever $0 \leq z_1 \leq e - y$ and $0 \leq z_2 \leq y$.

Since $0 \leq y \leq e$ for any $x \in X$ with $0 \leq x \leq e$, we have $0 \leq x \wedge y$ and $x \vee y \leq e$. From the identity $x + y = x \vee y + x \wedge y$, it follows that $x + y \leq e + x \wedge y$. Consequently, $x - x \wedge y \leq e - y$. Hence $\varphi(z_1 - z_2) \leq \varepsilon$ whenever $0 \leq z_1 \leq x - x \wedge y$ and $0 \leq z_2 \leq x \wedge y$. According to Lemma 1, $\varphi(z) \leq \varphi(x \wedge y) + \varepsilon$ for any z with $0 \leq z \leq x$.

Given $a, b \in \mathbf{R}$ with $a < b$, set $\varepsilon = \frac{1}{2}(b - a)$. Then either $\varphi(x \wedge y) < a + \varepsilon$ or $a < \varphi(x \wedge y)$. In the first case,

$$\varphi(x \wedge y) + \varepsilon < a + 2\varepsilon = b,$$

and so b is an upper bound for the set $\{\varphi(z) : 0 \leq z \leq x\}$. In the second case, $0 \leq x \wedge y \leq x$ and $\varphi(x \wedge y) > a$. Therefore the set $\{\varphi(z) : 0 \leq z \leq x\}$ has a supremum.

If x is an arbitrary positive element of X^+, then there exists $\lambda > 0$ such that $\frac{1}{\lambda}x \leq e$. Hence

$$s = \sup\left\{\varphi(z) : 0 \leq z \leq \frac{1}{\lambda}x\right\}$$

exists, as therefore does

$$\lambda s = \sup\{\varphi(z) : 0 \leq z \leq x\}.$$

$\square$

Chapter 6

Applications in mathematical economics

6.1 Preference and utility: an introduction

Microeconomics aims to model economic activity as an interaction of individual agents. Each agent is supposed to have a preference relation over the set of possible choices. An important problem is the possibility of measuring numerically the preferences by assigning a number (utility) to each possible choice such that one alternative is preferred to the other if and only if the utility of the former is greater than the one of the latter. A basic assumption made by pioneers of classical microeconomics such as Edgeworth and Pareto was that the preferences could always be measured in this way. However, this assumption was challenged by economists such as Wold [72], who saw the need of specifying conditions under which preferences could be represented numerically. From a mathematical point of view, the abstract problem is to find for a certain ordering of the set X, an order–preserving mapping u of X into $\mathbf{R}$. Furthermore, when X has a topological structure, the function u is required to be continuous.[1]

[1] A variety of approaches to the problem of representation are discussed in [21].

6.2 Basic definitions

We present the constructive notions of preference and weak preference, as given in [12, 13]. Let X be a nonempty set, and $\succ$ a binary relation on X. Denote by $\succeq$ the binary relation on X defined by

$$x \succeq y \Leftrightarrow \neg(y \succ x).$$

We say that $\succ$ is a **preference relation** if it satisfies the following axioms.

P1 $x \succ y \Rightarrow \neg(y \succ x)$;

P2 $x \succ y \Rightarrow \forall z\ (x \succ z \vee z \succ y)$.

If the property P2 is replaced by

P3 $((x \succ y \wedge y \succeq z) \vee (x \succeq y \wedge y \succ z)) \Rightarrow x \succ z$,

then the relation $\succ$ is called a **weak preference**. Clearly, any preference relation is a weak preference but, as shown by Bridges [13], we cannot expect to prove the converse implication constructively.

The relation $\succeq$ associated with a weak preference $\succ$ is called a **preference–indifference relation**. The corresponding **indifference relation** is the relation $\sim$ defined by

$$x \sim y \Leftrightarrow (x \succeq y \wedge y \succeq x).$$

Important types of set associated with a weak preference $\succ$ are

the **upper contour set** at a :	$[a, \rightarrow)$	$=$	$\{x \in X : x \succeq a\}$,
the **strict upper contour set** at a :	$(a, \rightarrow)$	$=$	$\{x \in X : x \succ a\}$,
the **lower contour set** at a :	$(\leftarrow, a]$	$=$	$\{x \in X : a \succeq x\}$,
the **strict lower contour set** at a :	$(\leftarrow, a)$	$=$	$\{x \in X : a \succ x\}$.

Let $\succ$ be a weak preference on X. A **utility function** for $\succ$ is a mapping $u : X \rightarrow \mathbf{R}$ such that

$$x \succ y \Leftrightarrow u(x) > u(y).$$

Such a map is said to **represent** $\succ$. A weak preference that is represented by a utility function is necessarily a preference.

6.3 Strong extensionality

Strong extensionality and hyperextensionality are two important notions in the constructive study of continuous functions. We introduce a corresponding definition of hyperextensionality for weak preferences and we show that, as in the case of functions, a hyperextensional weak preference is necessarily strongly extensional. We will use these notions in Section 6.4 for a constructive examination of continuous weak preferences. It turns out that the preference $\succ$ can be represented by a continuous utility function only when $\succ$ is strongly extensional.

Let X be a nonempty set with an apartness relation $\neq$. An irreflexive binary relation R on X is said to be **strongly extensional** if $x \neq y$ whenever xRy. If X and Y are sets endowed with apartness relations, f is a mapping of X into Y, and R_f is the binary relation on X defined by

$$xR_f y \Leftrightarrow f(x) \neq f(y),$$

then the strong extensionality of R_f is nothing else than the usual strong extensionality of the mapping f. A general treatment of strong extensionality can be found in [69] (Vol.2, Chapter 8).

Throughout this section we consider only binary relations on metric spaces. The standard apartness relation on a metric space X is

$$x \neq y \Leftrightarrow d(x, y) > 0.$$

Ishihara's Lemmas 3 and 4 in [36] can be easily adapted to a more general case, as follows.

Lemma 6.3.1. *Let $\not\leq$ be an excess relation on the metric space X, and let x, y be points of X such that $x \not\leq y$. Then the following properties hold.*

(i) *The subset $\{x, y\}$ is closed.*

(ii) *If X is complete, then $d(x, y)$ is pseudopositive.*

Proof. (i) Let (z_n) be a sequence of $\{x, y\}$ converging to a limit z in X. Then either $x \not\leq z$ or else $z \not\leq y$. In the former case, suppose that $z \neq y$. Then $z_n \neq y$ hence $z_n = x$, for all sufficiently large n. It follows that $z = x$, in contradiction to $x \not\leq z$. Consequently, $z = y$ and, in a similar way, we prove that $z = x$ in the latter case.

(ii) As in the proof of Lemma 4 of [36], let $t \in \mathbf{R}$ and define an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 \ &\Rightarrow \ d(x, y) < 1/(n+1), \\ \lambda_n = 1 \ &\Rightarrow \ d(x, y) > 0. \end{aligned}$$

We may assume that $\lambda_1 = 0$. Construct the sequence (z_n) in $\{x, y\}$ as follows: if $\lambda_n = 0$, then set $z_n = x$; if $\lambda_n = 1 - \lambda_{n-1}$, set $z_k = y$ when $0 < t$, and set $z_k = x$ for all $k \geq n$ when $t < d(x, y)$. The sequence (z_n) is Cauchy hence convergent to a limit z. It follows from (i) that either $z = x$ or else $z = y$. Taking into account that $x = y$ is contradictory, we obtain as in Ishihara's proof that $\neg\neg(t < d(x, y))$ when $z = x$ and $\neg\neg(0 < t)$ when $z = y$. Consequently, $d(x, y)$ is pseudopositive. □

Considering the excess relation R_f defined in the beginning of this section, we obtain as a corollary Ishihara's lemmas. Lemma 6.3.1 can also be applied for preferences on a metric space. As a consequence, we obtain the next result.

Lemma 6.3.2. *The following are equivalent.*

(1) *Every preference relation on a complete metric space is strongly extensional.*

(2) *The weak Markov principle (WMP).*

Proof. It follows from the second part of Lemma 6.3.1 that (2) implies (1). Now let a be a pseudopositive number; then the set $X = \{0, a\}$ is a complete metric space [51]. If $\succ$ is the subset $\{(a, 0)\}$ of X^2, then $\succ$ is a preference on X. Its strong extensionality entails the strict positivity of a. Therefore WMP holds. □

If the completeness of the metric space is not required, we obtain a similar result with WMP replaced by Markov's principle. Moreover, in this case we may consider weak preferences, or even arbitrary irreflexive binary relations, instead of preferences.

A mapping between metric spaces is called **hyperextensional** if for each pair α, β of real numbers with $0 < \alpha < \beta$ and for each sequence (x_n) that converges to a limit x, either $d(f(x_n), f(x)) < \beta$ for all n or $d(f(x_n), f(x)) > \alpha$ for some n. If f is hyperextensional, then f is strongly extensional (Lemma 2 of [42]). We introduce a similar notion of hyperextensionality and we prove a similar result for weak preference relations.

Let $\succ$ be a weak preference relation on the metric space X. We say that $\succ$ is **hyperextensional** if for each convergent sequence (x_n) of elements of X and for all a, b in X with $b \succ a$, the following conditions are satisfied:

H1 $\forall n\, (b \succ x_n) \vee \exists n\, (x_n \succ a)$;

H2 $\forall n\, (x_n \succ a) \vee \exists n\, (b \succ x_n)$.

Proposition 6.3.3. *If a weak preference satisfies one of the conditions of hyperextensionality, then it is strongly extensional.*

Proof. We prove that H1 implies strong extensionality. Suppose that $b \succ a$ and construct an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad d(a,b) < 1/n, \\ \lambda_n = 1 \quad &\Rightarrow \quad d(a,b) > 0. \end{aligned}$$

We may assume that $\lambda_1 = 0$. Define, as in the proof of the corresponding theorem for mappings [42], a sequence z_n as follows: if $\lambda_n = 1 - \lambda_{n-1}$, set $z_n = b$; otherwise set $z_n = a$. Then (z_n) is convergent; therefore either $b \succ z_n$ for all n or else $z_n \succ a$ for some n. In the former case, if $\lambda_n = 1 - \lambda_{n-1}$ for some n, then $b \succ b$, a contradiction. Therefore $\lambda_n = 0$ for all n hence $a = b$, contradictory to $b \succ a$. It follows that there exists N with $z_N \succ a$. If $\lambda_N = 0$, then $a \succ a$, a contradiction. This ensures that $\lambda_N = 1$; hence $d(a,b) > 0$. In a similar way we can prove that the weak preference $\succ$ is strongly extensional whenever it satisfies condition H2. $\square$

6.4 Continuity

A weak preference on the metric space X is continuous if for each x in X the strict contour sets $(x, \rightarrow)$ and $(\leftarrow, x)$ are open in X; in which case the contour sets $[x, \rightarrow)$ and $(\leftarrow, x]$ are closed. A stronger notion of continuity, called uniform continuity, was introduced by Bridges [12, 14]. We introduce two other variants of continuity and examine their relations to strong extensionality and hyperextensionality. It turns out that each of these five variants of continuity corresponds to a certain type of continuity of functions.

Let $f : X \to Y$ be a function between two metric spaces. The various types of continuity that we will use in this section are given in the following definition. The function f is

- **uniformly continuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$;
- **pointwise continuous** at x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$;
- **sequentially continuous** at x if $x_n \to x$ implies $f(x_n) \to f(x)$;
- **nearly continuous** at x if $x \in \overline{S}$ implies $f(x) \in \overline{f(S)}$;[2]
- **nondiscontinuous** at x if $x_n \to x$ and $\forall n(d(f(x_n), f(x)) \geq \delta)$ imply $\delta \leq 0$.

Each type of continuity implies the next [24, 36, 38, 40, 42]. Similar results can be obtained for weak preferences. Let $\succ$ be a weak preference on the metric space X. We say that $\succ$ is

- **uniformly continuous** if for each pair a, b of elements of X with $a \succ b$, there exists $r > 0$ such that $d(x, y) < r$ entails either $a \succ x$ or $y \succ b$.
- **(pointwise) continuous** at $a \in X$ if both sets $(a, \rightarrow)$ and $(\leftarrow, a)$ are open.

[2] The closure of S is denoted, as usual, by $\overline{S}$.

- **sequentially continuous** at a if $x_n \to x$ implies that there exists a positive integer N such that

$$x \succ a \Rightarrow \forall n \geq N \ (x_n \succ a),$$

$$a \succ x \Rightarrow \forall n \geq N \ (a \succ x_n).$$

- **nearly continuous** at a if for each subset S of X and for each $x \in \overline{S}$ we have:

$$x \succ a \Rightarrow \exists s \in S \ (s \succ a),$$

$$a \succ x \Rightarrow \exists s \in S \ (a \succ s).$$

- **nondiscontinuous** at a if the sets $[a, \rightarrow)$ and $(\leftarrow, a]$ are closed.

We say that a weak preference or a function is **pointwise continuous** whenever it is pointwise continuous at every point of X. We apply the same convention for the other variants of continuity. Every uniformly continuous weak preference is a preference [12]; but we cannot expect to prove constructively a similar result for pointwise continuity [13].

Each uniformly continuous preference is necessarily pointwise continuous. For preferences on a compact metric space, these notions are classically equivalent. However, a recursive counterexample, due to Bridges [12], shows that we cannot prove constructively that every continuous preference on a compact space is uniformly continuous.

Proposition 6.4.1. *Every weak preference that is pointwise continuous at a is also sequentially continuous at that point. The latter implies near continuity at a which, in turn, implies nondiscontinuity at a.*

Proof. Let $\succ$ be a weak preference on the metric space X and assume that $\succ$ is pointwise continuous at $a \in X$. If $x \succ a$ and $(a, \rightarrow)$ is open, then there exists $r > 0$ such that $y \succ a$ whenever $d(x, y) < r$. It follows that for every sequence (x_n) of elements of X with $x_n \to x$, $x_n \succ a$ for all sufficiently large n. The case $a \succ x$ is similar.

Suppose now that $\succ$ is sequentially continuous at a. To prove that $\succ$ is nearly continuous at a, let S be a subset of X and let $x \in \overline{S}$ with $x \succ a$. Then there exists a sequence (x_n) of elements of S with $x_n \to x$. Since $x \succ a$, it follows that $x_n \succ a$ for all sufficiently large n. The case $a \succ x$ is handled similarly.

We prove now that near continuity at a implies nondiscontinuity at that point. We have to prove that the contour sets $[a, \to)$ and $(\leftarrow, a]$ are closed. To this end, consider a sequence (x_n) in X such that $x_n \to x$ and $x_n \succeq a$ for all n. Assume that $a \succ x$. Since $x_n \to x$ and $x_n \succeq a$ for all n, it follows that x belongs to the closure of $[a, \to)$. Since $\succ$ is nearly continuous at a, it follows that $a \succ s$ for some $s \in [a, \to)$, a contradiction. Consequently, $\neg(a \succ x)$; whence $x \in [a, \to)$. In a similar manner we prove that the lower contour set at a is closed; so the weak preference $\succ$ is nondiscontinuous at a. □

We now examine the relations between continuity and strong extensionality (respectively, hyperextensionality). The example in Lemma 6.3.2 shows that nondiscontinuity does not imply strong extensionality. Indeed, if $\neg\neg(a > 0)$ and the preference $\succ$ is defined by

$$x \succ y \Leftrightarrow (x = a \wedge y = 0),$$

then the contour sets $(\leftarrow, 0] = \{0\}$, $(\leftarrow, a] = \{0, a\} = [0, \to)$, and $[a, \to) = \{a\}$ are closed in $\{0, a\}$; whence $\succ$ is nondiscontinuous. On the other hand, even if a is pseudopositive, this preference is not strongly extensional. Consequently, in view of Lemma 6.3.2, if every nondscontinuous preference on a complete metric space is strongly extensional, then WMP holds, and vice versa. Similarly, if every preference on a metric space is strongly extensional, then Markov's principle holds; the converse is also valid.

The next proposition shows that, as in the case of functions, strong extensionality is a necessary condition for near continuity.

Proposition 6.4.2. *Every nearly continuous weak preference is strongly extensional.*

Proof. Let $\succ$ be a nearly continuous weak preference and let $a \succ b$. To prove that

$a \neq b$, we use the standard constructive technique: construct an increasing binary sequence (λ_n) such that

$$\lambda_n = 0 \;\Rightarrow\; d(a,b) < 1/n,$$
$$\lambda_n = 1 \;\Rightarrow\; d(a,b) > 0,$$

and prove that $\lambda_n = 1$ for some n. We may assume that $\lambda_1 = 0$. Define, as in the proof of Proposition 1 of [24], the sets S_n as follows. If $\lambda_n = 0$, set $S_n = \{a\}$; if $\lambda_n = 1$, set $S_n = \{b\}$, and let $S = \bigcup_{n=1}^{\infty} S_n$. Then $b \in \overline{S}$ so, since $\succ$ is nearly continuous, it follows that there exists $s \in S$ with $a \succ s$. Therefore $s \in S_N$ for some N. If $\lambda_N = 0$, then $s = a$, a contradiction. This ensures that $\lambda_N = 1$. □

As a consequence, we cannot expect to prove that every nondiscontinuous preference is nearly continuous.

A weak preference relation on X is said to be **dense** if for all x, y in X such that $x \succ y$, there exists z with $x \succ z \succ y$. A function between two metric spaces is sequentially continuous if and only if is nondiscontinuous and strongly extensional [42]. For a preference, sequential continuity guarantees the other two properties. In addition, if the preference is dense, then sequential continuity is also necessary.

Proposition 6.4.3. *Every sequentially continuous preference is hyperextensional.*

Proof. Let (x_n) be a sequence in X that converges to a limit x. If a and b are elements of X with $a \succ b$, then either $a \succ x$ or else $x \succ b$. In the former case, there exists N such that $a \succ x_n$ for all $n > N$. Therefore either $a \succ x_n$ for all n or else $x_n \succ b$ for some $n \in \{1, \ldots, N\}$. In the latter case, $x_n \succ b$ for all sufficiently large n. Consequently, the preference $\succ$ satisfies the condition H1. The property H2 can be proved in a similar manner. □

Proposition 6.4.4. *Let $\succ$ be a dense preference on the metric space X. Then $\succ$ is sequentially continuous if and only if it is nondiscontinuous and hyperextensional.*

Proof. Every sequentially continuous preference is nondiscontinuous (Proposition 6.4.1) and hyperextensional, so we need only prove the converse implication. The technique used in the corresponding proof for functions (Theorem 1 of [42]) can be applied here. Let x_n be a sequence in X that converges to a limit x. We prove that $a \succ x_n$ for all sufficiently large n, provided $a \succ x$. To this end, let y be an element of X with $a \succ y \succ x$, and construct an increasing binary sequence (λ_n) such that

$$\lambda_n = 0 \quad \Rightarrow \quad x_k \succ y \text{ for some } k \geq n,$$
$$\lambda_n = 1 \quad \Rightarrow \quad a \succ x_k \text{ for all } k \geq n.$$

We may assume that $\lambda_1 = 0$. Define a sequence y_n in X as follows: if $\lambda_n = 1 - \lambda_{n-1}$, choose $k \geq n-1$ such that $x_k \succ y$ and set $y_n = x_k$; otherwise set $y_n = x$. Then $y_n \to x$ and, according to H1, either $y \succ y_n$ for all n or else $y_n \succ x$ for some n. In the former case, suppose that $\lambda_n = 1 - \lambda_{n-1}$. Then there exists $k \geq n-1$ such that $x_k \succ y$ and $y_n = x_k$, which is contradictory. Therefore $\lambda_n = 0$ for all n, in which case there exists a subsequence (x_{k_n}) such that $x_{k_n} \succ y$ for all n. Since the preference $\succ$ is nondiscontinuous, it follows that $x \succeq y$, contradictory to $y \succ x$. Consequently, the latter is the case, so $y_n \succ x$ for some n. If $\lambda_n = 0$, then $y_n = x$, a contradiction. This ensures that $\lambda_n = 1$, and hence that $a \succ x_k$ for all sufficiently large k.

In a similar way we can prove that the condition H2 together with the nondiscontinuity of the preference $\succ$ guarantee that if $x \succ b$, then $x_n \succ b$ for all sufficiently large n. $\square$

In the remainder of this section we show that the continuity of a representation u entails the corresponding type of continuity for the preference represented by u. For uniform continuity and pointwise continuity this was proved in [12]. The next proposition deals with the other three variants of continuity.

Proposition 6.4.5. *Let $\succ$ be a preference represented by the utility function u.*

(i) *If u is sequentially continuous, then $\succ$ is sequentially continuous.*

(ii) *If u is nearly continuous, then $\succ$ is nearly continuous.*

(iii) *If u is nondiscontinuous, then $\succ$ is nondiscontinuous.*

Proof. (i) If $x_n \to x$ and $x \succ a$, then $u(x_n) \to u(x)$ and $u(x) > u(a)$. Then there exists N such that $u(x_n) > u(a)$ for all $n \geq N$. Therefore $x_n \succ a$ for all $n \geq N$. The case $a \succ x$ is handled similarly.

(ii) If $x \in \overline{S}$ and $x \succ a$, then $u(x) \in \overline{u(S)}$ and $u(x) > u(a)$. Since $u(x) \in \overline{u(S)}$, it follows that $|u(x) - u(s)| < u(x) - u(a)$ for some $s \in S$. Then $u(s) > u(a)$; whence $s \succ a$. The case $a \succ x$ is similar.

(iii) We prove that for each a, the upper contour set at a is closed. To this end, let (x_n) be a sequence in $[a, \to)$ with $x_n \to x$, and assume that $a \succ x$. Then $u(x_n) \geq u(a) > u(x)$ for all n; whence

$$|u(x_n) - u(x)| = u(x_n) - u(x) \geq u(a) - u(x)$$

for all n. The nondiscontinuity of u ensures that $u(a) - u(x) \leq 0$, a contradiction. Therefore $x \succeq a$. The set $(\leftarrow, a]$ is proved to be closed analogously. □

As a consequence, if the preference relation $\succ$ is represented by a nearly continuous utility function, then $\succ$ is necessarily strongly extensional. (See Proposition 6.4.2.)

6.5 Monotone weak excess relations

In this section we deal with binary relations defined on partially ordered sets. The ordering of a set X enables one to consider various properties of monotonicity. Classically, there are two main notions of monotonicity, but more distinctions occur under a constructive scrutiny.

We introduce the notion of weak cotransitivity of a binary relation, a generalization of cotransitivity. We also define the weak excess relation, which is more general than the excess relation. Several notions of monotonicity of a weak excess relation on a partially ordered set are introduced. In particular, we obtain various types of monotonicity of functions, as defined in Section 5.1. Then we examine a specific case: when the weak excess relation is a weak preference.

Let X be a nonempty set. A binary relation R on X is said to be **weakly cotransitive** if each pair x, y of elements of X satisfies the condition

$$\forall z \in X \ ((xRy \wedge \neg(zRy)) \vee (\neg(yRx) \wedge (yRz)) \Rightarrow xRz.$$

Clearly, every cotransitive relation is weakly cotransitive. We say that R is a **weak excess relation** if it is both irreflexive and weakly cotransitive. For instance, every weak preference is a weak excess relation. Every excess relation is also a weak excess relation. The converse implication does not hold constructively. Indeed, as we have already seen, we cannot prove constructively that every weak preference relation is a preference.

Let S be a nonempty subset of the partially ordered set X and R a weak excess relation on X. We say that R is

- **strictly increasing** on S if for all x, y in S,

$$x > y \Rightarrow (xRy \wedge \neg(yRx));$$

- **almost strictly increasing** on S if for all x, y in S,

$$x > y \Rightarrow (\neg\neg(xRy) \wedge \neg(yRx));$$

- **uniformly increasing** on S if for all x, y in S and for all z in X,

$$x > y \Rightarrow (zRy \vee \neg(zRx));$$

- **weakly uniformly increasing** on S if for all x, y in S and for all z in X,

$$x > y \Rightarrow (\neg\neg(zRy) \vee \neg(zRx));$$

- **increasing** on S if for all x, y in S,

$$x > y \Rightarrow \neg(yRx);$$

- **strongly increasing** on S if for all x, y in S,

$$xRy \Rightarrow x \nleq y.$$

Clearly, the first two conditions are classically equivalent, as are the remaining four definitions. To give an example, let us consider two partially ordered sets X and Y and a function f from X to Y. Then the binary relation f on X defined by

$$xfy \Leftrightarrow f(x) \not\leq f(y)$$

is an excess relation on X. In this case, the above definitions of monotonicity of f are consistent with the definitions given in Section 5.1. To verify this, we need only observe that

$$\begin{aligned} \neg(yfx) &\Leftrightarrow f(y) \leq f(x), \\ ((xfy) \wedge \neg(yfx)) &\Leftrightarrow f(x) > f(y), \end{aligned}$$

and

$$(\neg\neg(xfy) \wedge \neg(yfx)) \Leftrightarrow (\neg\neg(xfy) \wedge \neg\neg\neg(yfx)) \Leftrightarrow \neg\neg(xfy \wedge \neg(yfx)).$$

If the relation R is a weak preference $\succ$ on X, then the following conditions are satisfied for each pair x, y of elements of X.

$$\begin{aligned} \neg(yRx) &\Leftrightarrow x \succeq y, \\ (xRy \wedge \neg(yRx)) &\Leftrightarrow (x \succ y \wedge x \succeq y) \Leftrightarrow x \succ y, \\ (\neg\neg(xRy) \wedge \neg(yRx)) &\Leftrightarrow (\neg(y \succeq x) \wedge x \succeq y) \Leftrightarrow \neg(y \succeq x). \end{aligned}$$

As a consequence of the above properties, the weak preference $\succ$ is

- **strictly increasing** on S if for all x, y in S,

$$x > y \Rightarrow x \succ y;$$

- **almost strictly increasing** on S if for all x, y in S,

$$x > y \Rightarrow \neg(y \succeq x);$$

- **uniformly increasing** on S if for all x, y in S and for all z in X,

$$x > y \Rightarrow (x \succeq z \vee z \succ y);$$

- **weakly uniformly increasing** on S if for all x, y in S and for all z in X,

$$x > y \Rightarrow (x \succeq z \vee \neg(y \succeq z));$$

- **increasing** on S if for all x, y in S,

$$x > y \Rightarrow x \succeq y;$$

- **strongly increasing** on S if for all x, y in S,

$$x \succ y \Rightarrow x \not\leq y.$$

Proposition 6.5.1. *Let R be a weak excess relation on the partially ordered set X, and let S be a nonempty subset of S. Then the following conditions are equivalent.*

(1) *The relation R is increasing on S.*

(2) *For all x, y in S, $x \geq y \Rightarrow \neg(yRx)$.*

(3) *For all x, y in S, $yRx \Rightarrow \neg(x \geq y)$.*

(4) *For all x, y in S, $yRx \Rightarrow \neg(x > y)$.*

(5) *For all x, y in S, $\neg\neg(yRx) \Rightarrow \neg(x \geq y)$.*

(6) *For all x, y in S, $\neg\neg(yRx) \Rightarrow \neg(x > y)$.*

Proof. To prove that (1) entails (2), let x, y be a pair of elements of S such that $x \geq y$, and assume that yRx. If $x \neq y$, then $x > y$, so the condition yRx is contradictory. Therefore $x = y$, so $\neg(yRx)$. Consequently, the supposition yRx leads to a contradiction.

The implications (2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) are straightforward. We show that (4) implies (6). To this end, let x, y be elements of S with $\neg\neg(yRx)$ and assume that $x > y$. If yRx, then $\neg(x > y)$, a contradiction. Therefore $\neg(yRx)$, which is contradictory to $\neg\neg(yRx)$. It follows that the condition $x > y$ is contradictory.

To show that (6) entails (5), let $\neg\neg(yRx)$, and assume that $x \geq y$. It follows from (6) that $\neg(x > y)$, so $x = y$. In this case $\neg(yRx)$, a contradiction. It remains to prove (5)$\Rightarrow$(1). The contrapositive of (5) is $\neg\neg(x \geq y) \Rightarrow \neg\neg\neg(yRx)$, which is equivalent to (2); and this, in turn, entails (1). $\square$

If the relation R is a mapping f of X into a partially ordered set Y, and $S = X$, we obtain the equivalent conditions given in Proposition 5.1.1. For a weak preference $\succeq$, we obtain the following conditions.

Corollary 6.5.2. *Let $\succ$ be a weak preference on the partially ordered set X, and let S be a nonempty subset of X. Then the following conditions are equivalent.*

(1) *The relation $\succ$ is increasing on S.*

(2) *For all x, y in S, $x \geq y \Rightarrow\ x \succ y$.*

(3) *For all x, y in S, $y \succ x \Rightarrow \neg(x \geq y)$.*

(4) *For all x, y in S, $y \succ x \Rightarrow \neg(x > y)$.*

(5) *For all x, y in S, $\neg(x \succeq y) \Rightarrow \neg(x \geq y)$.*

(6) *For all x, y in S, $\neg(x \succeq y) \Rightarrow \neg(x > y)$.*

The next proposition is the generalization of Proposition 5.1.2.

Proposition 6.5.3. *Let S be a nonempty subset of the partially ordered set X. Let R be a weak excess relation on X and ρ the binary relation on X defined by*

$$x\rho y \Leftrightarrow (xRy \wedge \neg(yRx)).$$

Then the following conditions are equivalent.

(1) *The relation R is almost strictly increasing on S.*

(2) *For all x, y in S, $x > y \Rightarrow \neg\neg(x\rho y)$.*

(3) *For all x, y in S, $\neg\neg(x > y) \Rightarrow \neg\neg(x\rho y)$.*

(4) *For all x, y in S, $\neg(x\rho y) \Rightarrow \neg(x > y)$.*

Proof. For all x, y in X,

$$\neg\neg(x\rho y) \Leftrightarrow (\neg\neg(xRy) \wedge \neg\neg\neg(yRx)) \Leftrightarrow (\neg\neg(xRy) \wedge \neg(yRx)).$$

It follows that the conditions (1) and (2) are equivalent. Clearly, (3) implies (2). Taking into account that $\neg\neg\neg P \Leftrightarrow \neg P$, we observe that (4) is the contrapositive of (2) so is entailed by (2). Finally, (3) is the contrapositive of (4). □

For a weak preference $\succ$ we obtain the following equivalent conditions.

Corollary 6.5.4. *Let $\succ$ be a weak preference relation on the partially ordered set X, and let S be a nonempty subset of X. Then the following conditions are equivalent.*

(1) *The weak preference $\succ$ is almost strictly increasing on S.*

(2) *For all x, y in S, $\neg\neg(x > y) \Rightarrow \neg(y \succeq x)$.*

(3) *For all x, y in S, $y \succeq x \Rightarrow \neg(x > y)$.*

Clearly, any strictly increasing weak excess relation is almost strictly increasing. It is also straightforward to see that every almost strictly increasing weak excess relation is increasing, and that every uniformly increasing one is weakly uniformly increasing.

For a preference, the strict monotonicity entails the uniform one.

Proposition 6.5.5. *Let S be a nonempty subset of the partially ordered set X, and let R be a weak excess relation on X. Then the following implications hold.*

(i) *If R is weakly uniformly increasing on S, then R is increasing.*

(ii) *If R is a preference that is strictly increasing on S, then R is uniformly increasing on S.*

(iii) *If every increasing preference is uniformly increasing, then LPO holds. Furthermore, if every increasing preference is weakly uniformly increasing, then WLPO holds.*

Proof. (i) Let x, y be elements of S with $x > y$. Since R is weakly uniformly increasing, it follows that either $\neg(yRx)$ or $\neg\neg(yRy)$. The latter contradicts the irreflexivity of R, so the former is the case.

(ii) Suppose that x, y are elements of S with $x > y$, and let z be an arbitrary element of S. Then xRy and, taking into account that R is cotransitive, it follows that either xRz; in which case, $\neg(zRx)$ or zRy.

(iii) Let $\succ$ be the binary relation on $\mathbf{R}^2$ defined by

$$(x_1, x_2) \succ (y_1, y_2) \Leftrightarrow x_1 > y_1.$$

Then the relation $\succ$ is a preference and is increasing on $\mathbf{R}^2$. If $\succ$ is uniformly increasing on $\mathbf{R}^2$, then for all real numbers a either $(0,1) \succeq (a, 0)$ or $(a, 0) \succ (0, 0)$. Therefore for all a, we have either $0 \geq a$ or $a > 0$; this implies LPO. If the relation $\succ$ is weakly uniformly increasing, then for all $a \in \mathbf{R}$, either $(0,1) \succeq (a, 0)$ or $\neg(0,0) \succ (a, 0)$. It follows that for all a, either $0 \geq a$ or else $\neg(0 \geq a)$. As a consequence, we obtain WLPO. □

The last proposition of this section deals with strongly increasing weak excess relations. As a consequence, we obtain Proposition 5.1.3.

Proposition 6.5.6. *Let R be a weak excess relation on the partially ordered set X, and let S be a subset of X.*

(i) *If R is strongly increasing on S, then R is increasing and strongly extensional on S.*

(ii) *The converse implication is valid provided S is a lattice.*

Proof. Let R be strongly increasing, and let x, y be a pair of elements of X such that $x > y$. If yRx, then $y \not\leq x$, a contradiction. Therefore $\neg(yRx)$ and hence R is increasing. If xRy, then $x \not\leq y$; this ensures that $x \neq y$. Consequently, R is strongly extensional.

Suppose now that S is a lattice and that R is increasing and strongly extensional. Let x, y be elements of S such that xRy. Since $x \vee y \geq x$ and R is increasing, it follows that $\neg(xR(x \vee y))$. The conditions xRy and $\neg(xR(x \vee y))$, together with the weak cotransitivity of R, imply that $(x \vee y)Ry$. The strong extensionality of R ensures now that $x \vee y \neq y$ or, equivalently, $x \not\leq y$. Consequently, R is strongly increasing. $\square$

6.6 Unit elements

From now on we consider weak preference relations defined on the positive cone of an ordered vector space X. Let $\succ$ be a weak preference relation on the positive cone X^+. An element e of X^+ is said to be a **unit element** if it satisfies the following two conditions:

U1 $\forall x \in X^+ \exists \lambda \, (\lambda > 0 \wedge \lambda e \succeq x)$;

U2 $(0 \leq \alpha < \beta \wedge \alpha e \succeq x) \Rightarrow \beta e \succeq x$.

Clearly, a weak preference relation satisfies the condition U2 if and only it is increasing on the set

$$\mathbf{R}^+ e = \{\lambda e : \lambda \geq 0\}.$$

Throughout this section for every $x \in X^+$, $u(x)$ represents the number

$$u(x) = \inf\{\lambda > 0 : \lambda e \succeq x\},$$

provided this infimum exists. When $u(x)$ exists for all $x \in X^+$, it is natural to examine whether or not the relation $\succ$ can be represented by u. For example, in order that $\succ$ be represented by u, it is necessary that $x \succeq 0$ for all x. Indeed, $0 \succ x$ entails $u(0) > u(x)$, in contradiction to $u(x) \geq 0 = u(0)$.

Lemma 6.6.1. *Let e be a unit element for the weak preference $\succ$ and $x \in X^+$ such that $u(x)$ exists.*

(i) *If $[x, \rightarrow)$ is closed, then $u(x)e \succeq x$.*

(ii) *If $(\leftarrow, x]$ is closed and $x \succeq 0$, then $x \succeq u(x)e$.*

(iii) *If* $x \succeq u(x)e$, *then* $x \succeq 0$.

Proof. (i) If $\alpha > u(x)$, then, by definition of infimum, there exists β such that $\alpha > \beta > u(x)$ and $\beta e \succeq x$, hence $\alpha e \succeq x$. Since $[x, \rightarrow)$ is closed, it follows that $u(x)e \in [x, \rightarrow)$; that is, $u(x)e \succeq x$.

(ii) Suppose that $u(x)e \succ x$. If $u(x) > 0$, then for any ε with $0 < \varepsilon < u(x)$ we have $\neg((u(x) - \varepsilon)e \succeq x)$ hence $x \succeq (u(x) - \varepsilon)e$. Since $(\leftarrow, x]$ is closed, it follows that $x \succeq u(x)e$, a contradiction. Therefore $u(x) = 0$ and so $0e \succ x$, which is contradictory to $x \succeq 0$. Consequently, $x \succeq u(x)e$.

(iii) Assume that $0 \succ x$. Since $u(x) \geq 0$, we have $u(x)e \succeq 0$; whence

$$u(x)e \succ x \succeq u(x)e,$$

which is a contradiction. □

Proposition 6.6.2. *Let* $e \in X^+$ *be a unit element for the weak preference* $\succ$. *Then the following conditions are equivalent.*

(1) *For each* $x \in X^+$, $\inf\{\lambda > 0 : \lambda e \succ x\}$ *exists;*

(2) *For each* $x \in X^+$, $\text{w–inf}\{\lambda > 0 : \lambda e \succ x\}$ *exists;*

(3) *The weak preference* $\succ$ *is weakly uniformly increasing on the set* $\{\lambda e : \lambda > 0\}$.

(4) *The weak preference* $\succ$ *is weakly uniformly increasing on* $\mathbf{R}^+e$.

Proof. For every $\lambda \geq 0$ let $C_\lambda = (\leftarrow, \lambda e]$. Then the union of the sets C_λ $(\lambda \geq 0)$ covers X^+, and $C_\alpha \subseteq C_\beta$ whenever $0 < \alpha < \beta$, so we can apply Lemma 4.3.3. It follows that each of the conditions (1) and (2) is equivalent to the property

$$0 < \alpha < \beta \Rightarrow (\beta e \succeq x) \vee \neg(\alpha e \succeq x)).$$

In turn, this is equivalent to the condition

$$0 \leq \alpha < \beta \Rightarrow (\beta e \succeq x) \vee \neg(\alpha e \succeq x)).$$

□

Corollary 6.6.3. *Let $\succ$ be a nondiscontinuous weak preference with a unit element e. If $\succ$ is weakly uniformly increasing on $\mathbf{R}^+e$ and $x \succeq 0$ for all $x \in X^+$, then the following conditions are obtained.*

(i) *If $\succ$ is strictly increasing on $\mathbf{R}^+e$, then*

$$u(x) > u(y) \Rightarrow x \succ y \text{ and}$$
$$u(x) \geq u(y) \Leftrightarrow x \succeq y.$$

(ii) *If $\succ$ is strongly increasing on $\mathbf{R}^+e$, then $x \succ y \Rightarrow u(x) > u(y)$.*

(iii) *If $\succ$ is both strictly increasing and strongly increasing on $\mathbf{R}^+e$, then $\succ$ is a preference relation.*

Proof. To prove (i) and (ii), we take into account that, in view of Lemma 6.6.1, $u(x)e \sim x$ for all x. If $\succ$ is strictly increasing and strongly increasing on $\mathbf{R}^+e$, then u is a utility function that represents $\succ$, therefore $\succ$ is a preference relation. □

Under certain additional hypotheses we can guarantee that the representation u is uniformly continuous on compact sets.[3] As expected, this property of u is related to a certain property of continuity of the preference. We say that the weak preference $\succ$ is **strongly continuous on compact sets** with respect to the unit element e if it satisfies the following condition.

SC *For each compact $K \subset X^+$ and each $\varepsilon > 0$, there exists $r > 0$ such that for all $\lambda > 0$ and all x, y in K with $\|x-y\| < r$, either $(\lambda+\varepsilon)e \succ x$ or $y \succ \lambda e$.*

Proposition 6.6.4. *Let $\succ$ be a weak preference on the positive cone of an ordered vector space X and let e be a unit element. Suppose that $\succ$ is nondiscontinuous, strongly extensional on $\mathbf{R}^+e$, and satisfies the condition $x \succeq 0$ for all $x \in X^+$. Then the following properties are equivalent.*

[3] The single word "continuous" is usually applied for functions which are uniformly continuous on each compact subset of their domain. We cannot expect to prove constructively that any pointwise continuous function on a compact set is uniformly continuous.

(1) *The weak preference relation $\succ$ is strongly continuous on compact sets with respect to e.*

(2) *The relation $\succ$ is a strictly increasing preference on $\mathbf{R}^+e$, $u(x)$ exists for every x, u is a utility function that represents $\succ$, and u is uniformly continuous on compact sets.*

Proof. (1) $\Rightarrow$ (2). Assume that $\succ$ satisfies SC and let α and β such that $0 \leq \alpha < \beta$. Set $\varepsilon = \frac{1}{2}(\beta - \alpha)$ and $\lambda = \alpha + \varepsilon$. Since $\alpha + \varepsilon > 0$ and the set $\{\alpha e\}$ is compact, we have either $(\alpha + 2\varepsilon)e \succ \alpha e$ or $\alpha e \succ (\alpha + \varepsilon)e$. The latter is contradictory, hence $(\alpha + 2\varepsilon)e \succ \alpha e$, that is, $\beta e \succ \alpha e$. Therefore $\succ$ is strictly increasing on $\mathbf{R}^+e$. From Proposition 6.5.5, it follows that $\succ$ is uniformly increasing on $\mathbf{R}^+e$, so $u(x)$ exists for every x.

Since $\mathbf{R}^+e$ is a lattice and $\succ$ is strongly extensional on $\mathbf{R}^+e$, it follows from Proposition 6.5.6 that $\succ$ is strongly increasing on $\mathbf{R}^+e$. By Corollary 6.6.3, $\succ$ is a preference on X^+, and u is a representation for $\succ$.

Consider now a compact set K, $\varepsilon > 0$, and $r > 0$ as in the definition of strong continuity. For x, y in K with $\|x - y\| < r$, we have either $(u(y) + 2\varepsilon)e \succ x$ or $y \succ (u(y)+\varepsilon)e$. Since the latter is contradictory, it follows that $(u(y)+2\varepsilon)e \succ u(x)e$ and therefore $u(y)+2\varepsilon \geq u(x)$. Similarly, $u(x)+2\varepsilon \geq u(y)$. Therefore $|u(x)-u(y)| \leq 2\varepsilon$ whenever $x, y \in K$ and $\|x - y\| < r$, that is, u is uniformly continuous on K. Consequently, (1) entails (2).

(2) $\Rightarrow$ (1). Let K be a compact and let $\varepsilon > 0$. Since u is uniformly continuous on K, there exists $\delta > 0$ such that $|u(x) - u(y)| < \varepsilon/3$ for all x, y in K with $\|x - y\| < \delta$. Since $\succ$ is a preference, either $(\lambda + \varepsilon)e \succ x$ or $x \succ (\lambda + 2\varepsilon/3)e$. In the latter case, $u(x) > \lambda + 2\varepsilon/3$, which entails $u(y) \geq \lambda + \varepsilon/3$. Therefore $u(y) > \lambda$, so $y \succeq u(y)e \succ \lambda e$. □

Proposition 6.6.5. *Let $\succ$ be a weak preference on the positive cone X^+ and let e be a unit element. Suppose that $\succ$ is nearly continuous, satisfies SC, and $x \succeq 0$ for all $x \in X^+$. Then $u(x)$ exists for every x, u is uniformly continuous on compact sets, and represents $\succ$.*

Proof. In view of Proposition 6.4.2, the weak preference $\succ$ is strongly extensional. The conclusion now follows as a direct consequence of the preceding proposition. $\square$

If in the hypotheses of Proposition 6.6.4 we consider a weak preference that is weakly uniformly increasing on $\mathbf{R}^+e$ rather than a strongly extensional one, then the existence of u is still guaranteed. However, when $\succ$ is a weak preference that is not strongly extensional on $\mathbf{R}^+e$, the best we can obtain is a "weak representation".

Proposition 6.6.6. *Let $\succ$ be a weak preference on the positive cone X^+ of X such that $x \succeq 0$ for all x. Let e be a unit element and suppose that $\succ$ is nondiscontinuous, weakly uniformly increasing on $\mathbf{R}^+e$, and satisfies the property SC with respect to e. Then for each $x \in X^+$, $u(x)$ exists. Furthermore, u is uniformly continuous on compact sets and satisfies the property*

$$u(x) \geq u(y) \Leftrightarrow x \succeq y.$$

Proof. It follows, as in the proof of Proposition 6.6.4, that $\succ$ is strictly increasing on $\mathbf{R}^+e$, and u is uniformly continuous on compact sets. By Corollary 6.6.3 (i), $u(x) \geq u(y) \Leftrightarrow x \succeq y$. $\square$

If every weak preference defined on the positive cone of a complete Riesz space and satisfying the above conditions is represented by u, then Markov's principle holds. To prove this, consider the complete Riesz space $\mathbf{R}$ and define the relation $\succ$ on $\mathbf{R}^+ = [0, \infty)$ by $x \succ y$ if $\neg\neg(x > y)$. Then $\succ$ is a nondiscontinuous weak preference, 1 is a unit element, and $u(x) = x$ for all x. Moreover, $\succ$ satisfies SC with respect to $e = 1$. However, if u represents $\succ$, then $\neg\neg(x > y)$ entails $u(x) > u(y)$, that is $x > y$. Therefore Markov's principle holds.

It remains an open problem to prove or disprove that the preference $\succ$ can be represented by u without assuming the strong extensionality on $\mathbf{R}^+e$. More precisely, we have to prove or disprove the following statement:

> *If $\succ$ is a nondiscontinuous preference on X^+ that satisfies SC with respect to the unit element e, and $x \succeq 0$ for all x, then $\succ$ is strongly extensional on $\mathbf{R}^+e$.*

If this was true, then every strictly increasing preference on $\mathbf{R}^+$ would be strongly extensional.[4] In particular, every strictly increasing mapping of $\mathbf{R}^+$ into $\mathbf{R}$ would be strongly extensional.

6.7 Uniformly proper preferences

Following [2], we say that an element $e \in X^+$ is said to be an **extremely desirable bundle** for the weak preference $\succ$ if $x + \alpha e \succ x$ for all $x \in X^+$ and all $\alpha > 0$. For example, if $\succ$ is strictly monotone, each element $e > 0$ is an extremely desirable bundle.

Let X be a normed ordered vector space. A vector e is said to be a **vector of uniform properness** [52] for the preference $\succ$ if there exists $r > 0$ such that

$$(\alpha > 0 \wedge x - \alpha e + z \in X^+ \wedge \|z\| < \alpha r) \Rightarrow x \succ x - \alpha e + z.$$

Classically, every nondiscontinuous preference $\succ$ that has a vector of uniform properness e and satisfies the condition $x \succeq 0$ for all x, can be represented by the continuous utility function u defined by

$$u(x) = \inf\{\lambda > 0 : \lambda v \succeq x\}.$$

The following result is a constructive counterpart of this theorem.[5]

Proposition 6.7.1. *Let $\succ$ be a nondiscontinuous weak preference on X^+ such that $x \succeq 0$ for any $x \in X^+$, and let e be a vector of uniform properness. If $\succ$ is strongly extensional on $\mathbf{R}^+ e$, then the mapping u associated, as above, with e is defined for all x and represents $\succ$.*

If, in addition, $\succ$ is strongly continuous on compact sets with respect to e, then u is continuous.

Proof. If $z \in X^+$ and $\alpha > \|z\|/r$, then $\alpha e \succ \alpha e - \alpha e + z$ and, as a consequence, e is a unit element.

[4] Note that, according to Lemma 6.3.1(ii), the latter statement is a consequence of WMP.

[5] For the classical result, see Theorem 1 of [56].

If $x \in X^+$ and $\alpha > 0$, then

$$x + \alpha e \succ x + \alpha e - \alpha e + 0,$$

so e is an extremely desirable bundle. Therefore $\succ$ is strictly increasing on $\mathbf{R}^+ e$. It follows that $\succ$ is uniformly increasing on $\mathbf{R}^+ e$, and therefore $u(x)$ exists for all x. In view of Corollary 6.6.3, the relation $\succ$ is a preference that is represented by u. According to Proposition 6.6.4, u is uniformly continuous on compact sets, provided the preference $\succ$ satisfies SC. $\square$

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Symbols

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