Linear and Nonlinear Integral Equation Population Models

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Abstract

This thesis is concerned with population models by using integral equations. These equations are formulated by using concepts of the continuous time delay (bounded and unbounded), concepts of population history, concepts of birth of new offsprings and concepts of survival of each individuals. The thesis consists of four parts. The first part deals with formulation of the integral equations for population dynamics. This part starts with the introduction of the integral equation models for population dynamics. Then, the rest of the first part will discuss the issues of time delay, formulation of the birth rate, the issues of population history and the formulation of survival rate. The second part will cover the topics about the linear equations. This part will deal with the assumptions of the linear equations, the relationships with Volterra integral equations, the reducibility of the integral equations (unbounded delay) to systems of ordinary differential equations and the asymptotic stability of zero solutions of the linear integral equations. The third part concerns only global stability of zero solutions of the special forms of the integral equations by using Lyapunov functionals. Finally, the fourth part is devoted to the analysis of the nonlinear integral equations I formulated in the beginning of this thesis. This part is concerned with the assumptions of the equations, the steady states and local stability of the steady states. This part will also consider the analysis of an example, its steady states, the local stability of the steady states, the construction of the periodic solutions and the numerical solutions.

Preface

The material in this thesis is intended to present the subject of population models by using integral equations governing the growth dynamics of single species. The integral equations are formulated by using the concepts of population history, concepts of birth of new offsprings and concepts of survival of each individuals. Roughly speaking, I am dealing with two kinds of integral equations - equations with bounded (finite) distributed delay and equations with unbounded (infinite) distributed delay.

Chapter 1 deals with formulation of the integral equations for population dynamics, and it starts with the introduction of the integral equation models for population dynamics. In Section 1.1, I will discuss the issues of unbounded distributed delay and bounded distributed delay. Section 1.2 deals with formulation of the birth rate. Section 1.3 deals with formulation of the survival rate of each individuals and the population history. Finally, I will discuss the formulation of integral equation population models in Section 1.4.

Chapter 2 deals with some issues of linear integral equations, and it starts with the discussions of the assumptions of the linear equations and of the relationships with Volterra integral equation (Section 2.1). In Section 2.2, I will discuss some of the exact solutions of the linear equations under the assumptions I made in Section 2.1. Then, most of Section 2.4 is devoted to the discussion of the reducibility of the integral equations with infinite delays (linear and nonlinear) to the systems of ordinary differential equations. Finally, the rest of the parts of Chapter 2 (Section 2.5, 2.6 and 2.7) is dominated by analysis of the asymptotic stability of zero solutions of the linear integral equations by using Lyapunov functionals.

Chapter 3 is given to the analysis of the global stability of zero solutions of the special forms of the integral equations by using Lyapunov functionals, which I formulated in Chapter 1.

Chapter 4 is devoted to the analysis of the integral equations (nonlinear) I formulated in Chapter 1 and Chapter 4 starts with making the assumptions of the integral equations (Section 4.1). In Section 4.2, I will discuss the steady states of the integral equations. In Section 4.3, I will introduce the linearized equations which are obtained from the integral equations which I formulated in Chapter 1 by using first order Taylor series and I will discuss the local stability of the steady states I obtained in Section 4.2 by applying the results obtained in Section 2.5, 2.6 and 2.7. Then, the rest of Chapter 4 is dominated by the analysis of an example of the integral equation with infinite delay. I will discuss the steady states in Section 4.4 (Section 4.4.1), the local stability of the steady states in Section 4.4 (Section 4.4.2), the construction of the periodic solutions by using Poincaré-Lindsted method in Section 4.5 and the numerical solution in Section 4.7. In addition, Section 4.7.5 will provide some issues about the numerical method to the integral equations I formulated in Chapter 1.

I owe my deepest appreciation and acknowledge support from my family. I thank Hisashi Yamamoto (my father), Kikuko Yamamoto (my mother) and Katsuya Yamamoto (my brother) for their support and patience. I would like to thank the members of Instituto de Ingeniería de Gerencia Moderna, S.A. for their support. I also would like to thank Professor D. Wall in University of Canterbury for many constructive comments on this research and Professor G. Wake in University of Canterbury. Finally, I would like to thank Professor T. A. Burton in Southern Illinois University and Dr. T. He in University of Technology in Sydney for some useful comments.

Contents

Chapter 1

The formulation of integral equation models

In this chapter, I will deal with the topics as follows:

1. I will introduce 2 equations below as the background of population models.

$$\begin{aligned} n(t) &= n_0 f(t) + k \int_0^t f(t-\tau) n(\tau)\, d\tau \\ N_{st}(t) &= \int_0^\infty F_{st}(\tau, t : N_{st}) B_{st}(N_{st}(t-\tau)) N_{st}(t-\tau)\, d\tau \end{aligned}$$

   (I call the above (1.1) and (1.2) respectively.)

   Note : (1.1) and (1.2) are introduced to show some backgrounds of integral equation population models. They are not related each other.

2. I will introduce the equation as follows

$$\begin{aligned} N(t) &= \int_0^\infty FB(N(t-\tau)) N(t-\tau)\, d\tau \\ N(t) &= \int_0^M FB(N(t-\tau)) N(t-\tau)\, d\tau, \; M \in \Re^+ \end{aligned}$$

   (They are numbered (1.3) and (1.4), respectively) and I will explain how to formulate them. (The main purpose of this thesis is to analyze the above 2 types of equations (1.3) and (1.4).)

3. I will discuss the issues of unbounded distributed delay and bounded distributed delay.

4. I will discuss the concepts of birth rate, of survival rate and of population history, which are related to the formulation of (1.3) and (1.4).

1.0.1 The equation (1.1)

Jerri [65] (on the pages 43 and 44) explains how to formulate a human population model by using an integral equation model, namely,

$$n(t) \quad = \quad n_0 f(t) + k \int_0^t f(t-\tau) n(\tau)\, d\tau \tag{1.1}$$

where

- k is a positive constant.
- n_0 is an initial number of the population.
- $f(t)$ is a continuous function and represents the survival rate of the population ($f(0) = 1$).

(1.1) is a linear Volterra integral equation of the second kind in $n(t)$ which represents the number of population at time t, and it is a population model which explains the population growth or decay by using a continuous addition or subtraction to the population through the new births.

Note : About the formulation, it is similar to those of (1.3) and (1.4) and so, see later in this chapter. (The formulation is also explained in Jerri [65]).

1.0.2 The equation (1.2)

In Roberts [81], the following integral equation is introduced

$$N_{st}(t) \quad = \quad \int_0^\infty F_{st}(\tau, t : N_{st}) B_{st}(N_{st}(t-\tau)) N_{st}(t-\tau)\, d\tau \tag{1.2}$$

where

- $N_{st}(t)$ is the number of population at time t.
- F_{st} is the survival rate which an individual animal that was alive at time $t - \tau$ survives until time t, given the history of the population density N_{st} between times $t - \tau$ and t.
- F_{st} depends on t only by virtue of being dependent on the history of the population dynamics between $t - \tau$ and t.
- B_{st} represents the birth rate which depends only on N_{st}.

Note : $F_{st}(\tau, t : N_{st})$ means that F_{st} depends on t, τ and N_{st} such that $\{N_{st}(s) : t - \tau \leq s \leq t\}$.

1.0.3 The equations (1.3) and (1.4)

This equation (1.2) is very close to the equations below (especially (1.3)) which I am formulating in this chapter and analyzing in the later chapters, namely,

$$N(t) = \int_0^\infty FB(N(t-\tau))N(t-\tau)\,d\tau \tag{1.3}$$

$$N(t) = \int_0^M FB(N(t-\tau))N(t-\tau)\,d\tau,\ M \in \Re^+ \tag{1.4}$$

where

$$F = F(\tau, N(t-\tau)), \tag{1.5}$$

$$F = F\left(\tau, N(t-\tau), N\left(t-\frac{1}{2}\tau\right), N(t)\right) \text{ or} \tag{1.6}$$

$$F = F\left(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1\right) \tag{1.7}$$

and where

- $N(t)$ represents the number of population at time t.
- F represents the survival rate which individuals that were alive at time $t-\tau$ survive until time t, provided that the history of the number of population N between times $t-\tau$ ($\tau > 0$) and t. (There are three types of F introduced above. All of them have already included the concepts of the history of population.)
- B represents the birth rate at time $t-\tau$ which depends on the number of the population at time $t-\tau$ ($N(t-\tau)$).
- I mainly regard τ as the age of individuals since it is the best way to describe these equations in the context of the real situation. I consider the time between $t-\tau$ and t. So, it is possible to calculate $N(t)$ by considering the time interval between $t-\tau$ and t. That is, I can calculate $N(t)$ by considering the time between when the individuals born τ time ago and t is now the present time.

Moreover, I also have to state:

1. (1.3) and (1.4) are also population models which explain the population growth or decay by using a continuous addition or subtraction to the population through new births.
2. In principal, (1.3) and (1.4) do not have initial values. This is probably the biggest difference from (1.1). I will discuss this issue in Section 2.2, Section 2.4 and Section 4.7.
3. I will consider both of the cases when the integration is bounded (finite) (1.4) and when that is unbounded (infinite) (1.3). (1.4) is more practically useful than (1.3) since in the biological world, infinite time is not used. I will discuss this issue later in this chapter (Section 1.1).

1.0.4 Comments and a remark

Then, I have some comments as follows:

Comment 1.0.4.1

1. If you are interested in age-structured population model, Gurney and Nisbet [56] introduces an age-structured population model by using an integral equation (from page 62 to page 85).

2. (1.2) has some assumptions which not always true in the real situations. For example, it is assumed that $\frac{\partial B_{st}}{\partial N_{st}} \leq 0$ and $\frac{\partial F_{st}}{\partial N_{st}} \leq 0$. These mean that birth rate and survival rate never increase as the population density increases. This is not always true in the real situation (e.g. human population growth or decay). For the detail, see Roberts [81].

3. I made also assumptions for the birth rate B and for survival rate F to formulate (1.3) and (1.4). They will appear later in this chapter (Assumption 1.2.2.1 in Section 1.2.2 and Assumption 1.3.3.1 in Section 1.3.3).

4. The assumptions of (1.3) and (1.4) will be introduced in the beginning of Chapter 4 (Assumption 4.1.0.6) and also in Section 2.1, which only deals with the linear equations (Assumption 2.1.1.1).

5. I will explain the concepts of B in Section 1.2.1, and I will explain the concepts of F in Section 1.3.1 and the population history in Section 1.3.2.

Remark:

- $\int_0^\infty G(t,\tau,N(t-\tau))\,d\tau$ and $\int_0^M G(t,\tau,N(t-\tau))\,d\tau$ are called distributed delay terms (nonlinear) and so, (1.3) and (1.4) when $F = F(\tau, N(t-\tau))$ can be regarded as a special case of them.

- There are many authors who have already studied functional differential equations or integral equations with nonlinear distributed delay terms. For example, Huang and Bo [63] studied Liapunov functionals and the global stability of the zero solution of the system of functional differential equations which include $\int_{-\infty}^t G(t,\tau,N(\tau))\,d\tau$. Burton [10] studied the periodic solution of the functional differential equation which also includes $\int_{-\infty}^t G(t,\tau,N(\tau))\,d\tau$. Burton and Hatvani [14] studied the stability theorems of the zero solution of functional differential equations which include the term $\int_{t-M}^t G(t,\tau,N(\tau))\,d\tau$. Burton [11] studied the periodic solution and Liapunov functional to obtain the stability of the zero solutions of the integral equation which includes $\int_{-\infty}^t G_1(t,\tau)G_2(\tau,N(\tau))\,d\tau$. Burton [12] studied the Liapunov functionals to obtain the stability of the zero solutions of the integral equations and the functional differential equations which include $\int_{t-M}^{t} G_1(t,\tau)G_2(N(\tau))\,d\tau$. Finally, He and Ruan [59], and He, Ruan and Xia [60] studied the global stability of the positive steady state of the system of the functional differential equation which includes the term like $\int_0^\infty G_3(\tau)G_4(N(t-\tau))\,d\tau$

- By letting $\tau \to t-\tau$ or the vice versa, it is possible to have

$$\begin{aligned}\int_0^\infty G(t,\tau,N(t-\tau))\,d\tau &\equiv \int_{-\infty}^{t} G(t,\tau,N(\tau))\,d\tau \text{ and}\\ \int_0^M G(t,\tau,N(t-\tau))\,d\tau &\equiv \int_{t-M}^{t} G(t,\tau,N(\tau))\,d\tau\end{aligned}$$

  with alternative redefinition of G. I will explain this in Section 2.2.

- I stated some results of Burton [12] which are very important ones for the Chapter 3. For the detail, refer the chapter.

1.1 Unbounded and bounded distributed delay

The right hand side of (1.3) is $\int_0^\infty FBN(t-\tau)\,d\tau$. This is integrated from 0 to ∞, which has no bound in the region of the integration. So, this is called unbounded distributed delay. On the other hand, the right hand side of (1.4) is $\int_0^M FBN(t-\tau)\,d\tau$. This is integrated from 0 to M, which has a bound in the region of the integration. So, this is called bounded distributed delay.

Remark:

- In practical sense, $\int_0^M FBN(t-\tau)\,d\tau$ is more useful than $\int_0^\infty FBN(t-\tau)\,d\tau$. Since every organism has the length of life span which will be estimated in the finite values and time infinity (∞) is not clear in the sense of biological science. So, strictly speaking, $\int_0^\infty FBN(t-\tau)\,d\tau$ has no use to estimate the number of the population.

- There are some mathematical reasons why $\int_0^\infty FBN(t-\tau)\,d\tau$ should be analyzed. Firstly, the analysis to (1.3) is less complicated than that to (1.4). (See Section 2.5.1, 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2, 2.7.1 and 2.7.2).

- Secondly, some of the types of (1.3) is reducible to systems of ordinary differential equations which are probably simplest to analyze in mathematical sense. (See Section 2.4)

- Thirdly, for the survival rate, the families of exponential functions like $e^{-\tau}$ or the families of fractional function of τ like $1/(1+\tau)$ are often used. As $\tau \to \infty$, they will become zero. This is theoretically true, since I regard τ as the age of the individuals. There are no individuals which are alive forever. Therefore, it is appropriate that the survival rate becomes zero as $\tau \to \infty$.

1.2 The birth rate

1.2.1 The concept of mass action to the birth rate

As I stated in Section 1.0.3, B represents the birth rate which depends on $N(t)$ and I am using the concept of mass action to calculate the number of new born at time t, namely, I calculate the number of new born by multiplying the number of the population at time t ($N(t)$) with the birth rate at time t, ($B(N(t))$). The Figure 1.1 below explains about the concept of mass action of the birth rate schematically.

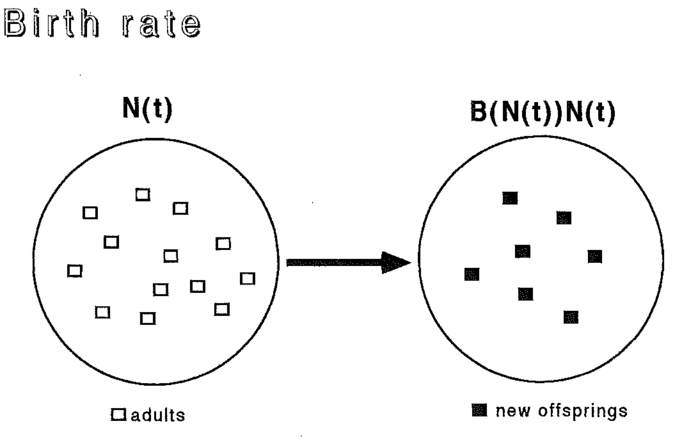

Figure 1.1: (This shows that at time t, 7 new offsprings are made from 13 adults, that is, 7 new individuals are added to the total number of the population. Therefore, the total number of the population becomes 20 at time t.)

1.2.2 Assumptions of the birth rate and the comment on the birth rate

I have made the assumptions of B as follows:

Assumption 1.2.2.1

* As I stated in Comment 1.0.4.1, I do not assume that $\frac{\partial B}{\partial N} \leq 0$ as in Roberts [81]. However, it is assumed that $\frac{\partial B}{\partial N} \leq 0$, as $N \longrightarrow \infty$. This means that the slope of a function B will eventually be negative if the number of population increases indefinitely.

Comment 1.2.2.1

1. I made Assumption 1.2.2.1 since I considered the positive factors and negative factors of the (environmental) conditions of giving births. If the size of the population is ideal (not too large number or not too small number of the population size) and other environmental conditions (temperature, climate, food source and so on) are also relatively ideal (mild climate, proper amount of rainfall and so on), the birth rate gets higher. This is because if the conditions

are nice to individuals, it is easier for the individuals to reproduce. (Indeed, I assume that if $\frac{\partial B}{\partial N} > 0$, the environmental conditions and the size of population are appropriate to make the birth rate higher.) On the other hand, if the size of the population is too small or too large, the birth rate gets lower because if the size gets smaller, the individuals have less chance to meet and because if the size gets too big, some of environmental factors (food source, space of living and so on) will be less for all individuals. These cause stress to individuals and so the birth rate gets lower. The assumption above considers these concepts of the growth of the population. In particular, the assumption ($\frac{\partial B}{\partial N} \leq 0$, as $N \longrightarrow \infty$) represents the case that the size of the population is getting too large. (Indeed, I assume that if $\frac{\partial B}{\partial N} < 0$, the poor environmental conditions and the too small or too large size of population cause the birth rate to decrease.)

2. B just depends on N, that is, the birth rate is determined just by the total number of population. So, you can say that B should include more factors. As I discussed in the above, the living conditions like-temperature, climate, food source and so on are significant factors to determine the chance to give births. There are chances to obtain better birth rate by including more natural concepts of the conditions like the above. However, there are too many factors that affect living organisms in this planet, so, it is impossible to express all of them mathematically. Moreover, it is not always true that the more concepts included, the more accurate birth rate is obtained. Mathematical concepts come from human imagination, on the other hand, the birth rate in principal, must be produced from the biological situation. Mathematical concepts are not basically made to explain the biological concepts. Therefore, there have been no exact birth rate created by mathematicians. The best I can do is to pretend that the birth rate is right and then to obtain the some data of the growth or decay of the population in number by calculating (1.3) and (1.4). However, there is nothing I can do to know how good this birth rate B is fit to the real situations. This requires some experiments. As I said in the above, the total number of the population can be a good indicator about how the environmental condition is. If the number increases, the condition is relatively nice, if it decreases, that is relatively worse and if the number is too big, it decreases because too many number of population will create so many negative factors to the each individuals. So, I just use $B(N)$ as the birth rate.

3. I insist that the similar argument will apply to the survival rate F. (See Comment 1.3.3.1 of Section 1.3.3.)

1.3 The survival rate

As I stated in Section 1.0.3, F represents the survival rate for individuals that were alive at time $t - \tau$ and that survive until time t, provided that the history of the number of population N between times $t - \tau$ and t. At the beginning, I will explain about the most basic concept of the survival rate and Figure 1.2 below explains about the concepts of the survival rate schematically.

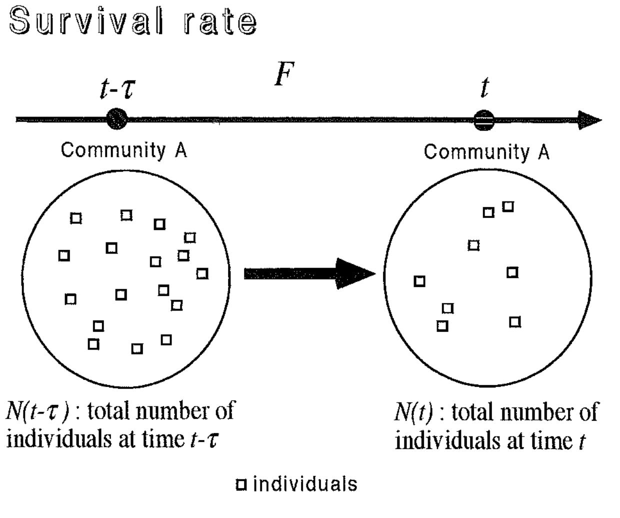

Figure 1.2: (At time $t - \tau$, there were 17 individuals but after τ time past, there are 8 individuals. That is, 8 of them just survived between time $t - \tau$ and t. Hence, the survival rate is 8/17.)

1.3.1 The basic concept of the survival rate

Every living organism in this world has a very common destiny which is death. If new individuals have been given their lives at certain time, they must die sooner or later. Survival rate F is a function which express the possibility that the number of individuals born at time $t - \tau$ survives until time t. I supposed to use the functions of the survival rate based on the functions which depend on τ and the total number of the population N. Moreover, I mainly consider τ as the ages of the individuals. So, I can regard F as a probability function which express the possibility that the number of individuals that have been born τ age (time) ago survives until time t. So, F tells that the possibility of the survival of the individuals since they have been born.

1.3.2 The history of the population

F is the survival rate in the period of from $t - \tau$ to t. Between the period, what might happen to the number of population? As I discussed in Comment 1.2.2.1 in Section 1.2, if the size of the population is ideal and other environmental conditions (temperature, climate, food source and so on) are also relatively ideal (mild climate, proper amount of rainfall and so on), each individuals will have more chance to survive. On the other hand, if the size of the population is too small or too large, or if the environmental conditions are not nice to organisms (too low temperature, too small amount of rainfall and so on), each individuals will have less chance to survive. This will depend on how long the period is. For (1.4), the length of the period depends on the value of M. As M gets larger, there will be more things happening which affect the number of population or which affect the chance of the survival of each individuals. For (1.3), it is possible to assume that the length of the period is very very long. So, there have been many things happened to affect the chance of the survivals. Hence, it might be better to include the values of the total number of the population N between $t-\tau$ and t and some functions which depend on τ, to estimate F. I have already introduced three types of F in Section 1.0.3 ((1.5) which is $F = F(\tau, N(t-\tau))$, (1.6) which is $F = F(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$ and (1.7) which is $F = F(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1)$). Now, I will explain about each F.

The survival rate (1.5) Firstly, I consider the case when $F = F(\tau, N(t-\tau))$. In this case, F is a function which just depends on τ and $N(t-\tau)$. This means that F will be calculated by using $N(t-\tau)$ (the total number of the population at $t-\tau$ (τ time ago)) and τ. (1.5) could be too simple as a survival rate which includes the concepts of the population history since it just depends on $N(t-\tau)$ and τ. However, as I discussed in Comment 1.2.2.1 in Section 1.2.2, it is often true that simpler equation is better than the more complicated equation in applications. So, it is worthwhile analyzing (1.5).

The survival rate (1.6) Secondly, I consider the case when $F = F(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$. F is a function which depends on τ, $N(t-\tau)$, $N(t-\tau/2)$ and $N(t)$. That is, the value of F will be calculated by using the total number of population at $t-\tau$, $t-\tau/2$ and t ($N(t-\tau)$, $N(t-\tau/2)$ and $N(t)$) and τ. F is the survival rate during the period of from $t-\tau$ to t. So, it could be better to use more values like $N(t-\tau)$, $N(t-\tau/2)$ and $N(t)$ than to use one value $N(t-\tau)$ in (1.5). To obtain the survival rate, the most affective way is to derive from the actual data which is taken by survey and so on. The data will not be continuous. So, it is worthwhile expressing the population history by using discrete number of N's like $N(t-\tau)$, $N(t-\tau/2)$ and $N(t)$.

The survival rate (1.7) Finally, I consider the case $F = F(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1)$. This is the case when F depends on τ and $K(t,\tau) \equiv \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1$. This is useful if the history of the total number of the population is defined continuously from $t-\tau$ and t. Since $K(t,\tau)$ is integrated from time $t-\tau$ to time t, $K(t,\tau)$ uses the information of the total number of the population from $t-\tau$ to t continuously. I also consider k_2 which is a function of the environmental fluctuation which affects the population history from 0 to τ (from $t-t_1 = t$ to $t-t_1 = t-\tau$).

Since I am evaluating $N(t)$, it is important to include the concepts which explain the nature, environment and so on from 0 to τ, which is k_2. I name $\int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1$ as population history which include concepts of the environmental impacts in the past.

Note :

1. The integration $\int_{t-\tau}^{t} N(t_1)\,dt_1$ is the total of N from $t-\tau$ to t and it is possible to use it as one of the parameter of the population history to the survival rate. However, $\int_{t-\tau}^{t} N(t_1)\,dt_1$ will be too simple to apply as one of the parameter of the population history, since I question whether the population history will be explained by simply using $\int_{t-\tau}^{t} N(t_1)\,dt_1$.

2. $\int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1$ is more appropriate than $\int_{t-\tau}^{t} N(t_1)\,dt_1$, since the function k_2 can answer my question. $\int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1$ is not just total of the total number of the population but includes the environmental impacts from $t-\tau$ to t.

Therefore, it is possible to assume that there is a continuous accumulation of the impacts to the survival rate from $t-\tau$ to t, which is calculated by using the total number of the population and environmental effects at the time between $t-\tau$ to t. $N(t_1)$ is the total number of the population at time t_1 and then, in a particular time interval $\Delta_j t_1$ about time t_{1_j} for $j = 1, 2, 3, \dots$ ($t_{1_0} = t - \tau$), $N(t_{1_j})\Delta_j t_1$ is the total of N of time $t - t_{1_j}$. However, I also consider the environmental fluctuation of the environment to each individuals, k_2, namely, only a fraction of $k_2(t-t_{1_j})$ of the total $(N(t_{1_j})\Delta_j t_1)$ will be taken as the value of population history. Hence, $k_2(t-t_{1_j})N(t_{1_j})\Delta_j t_1$ is the parameter for the population history of (1.7). If I just consider (1.4) and if this process is repeated for all the m subintervals of the time interval $(t-\tau, t)$ ($t_{1_0} = t-\tau$ and $t_{1_m} = t$), I get the partial sum

$$K_m(t,\tau) = \sum_{j=1}^{m} k_2(t-t_{1_j})N(t_{1_j})\Delta_j t_1$$

if this is passed to the limit, this will become the integral

$$K(t,\tau) \equiv \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1$$

Then, by letting $M \to \infty$, I obtain the partial sum for (1.3).

$$K_{m_1}(t,\tau) = \lim_{M\to\infty} K_m(\tau)$$

hence, if this is passed to the limit, again this will become the integral

$$K(t,\tau) \equiv \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1$$

1.3.3 Assumptions of the survival rate

I have made the assumptions to F as follows:

Assumption 1.3.3.1

1. As I stated in Comment 1.0.4.1, I do not assume that $\frac{\partial F}{\partial N} \leq 0$ as in Roberts [81] and there are 3 types of Fs ((1.5), (1.6) and (1.7)) and so, I make assumptions for each F.

2. For (1.5), it is assumed that $\frac{\partial F}{\partial (N(t-\tau))} \leq 0$, as $N(t-\tau) \longrightarrow \infty$. This means that the slope of a function F will eventually be negative if the number of population at $t-\tau$ keeps increasing.

3. For (1.6), it is assumed that $\frac{\partial F}{\partial N(\cdot)} \leq 0$, as $N(\cdot) \longrightarrow \infty$, where $N(\cdot) = N(t-\tau)$, $N(t-(1/2)\tau)$ or $N(t)$. This means that the slope of a function F will eventually be negative if the number of population at each time ($t-\tau$, $t-(1/2)\tau$ and t) keeps increasing.

4. For (1.7), it is possible to have several assumptions since $\int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1$ also includes the concept of the environmental impacts. For example, suppose that $\frac{\partial F}{\partial(\int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1)} \leq 0$, as $\int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1 \longrightarrow \infty$. This means that the slope of a function F will eventually be negative if the impacts get larger.

5. $\frac{\partial F(\tau,\omega)}{\partial \tau} \leq 0$, as $\tau \longrightarrow \infty$ where ω is a positive constant. Since I regard τ as the age of the individuals, it is reasonable to think that the survival rate will decline if the age of the individuals are high.

Comment 1.3.3.1

- As in Comment 1.2.2.1, I also considered the positive factors and negative factors of the conditions to the survival. If the environmental conditions are nicer for the individuals, the survival rate is higher. On the other hand, if the environmental conditions are worse to the individuals, the survival rate is lower. The assumptions for (1.5) and (1.6) above also consider concepts of the growth of the population. In particular, the assumption ($\frac{\partial F}{\partial N(\cdot)} \leq 0$, as $N(\cdot) \longrightarrow \infty$, where $N(\cdot) = N(t-\tau)$, $N(t-(1/2)\tau)$ and $N(t)$) is representing the case that the size of the population is getting too large. (too large size of the population causes the possibility of the survival to decline.)

- The assumption for (1.6) considers N at every time considered in (1.6) ($t-\tau$, $t-(1/2)\tau$ and t). This is because I consider a lot about the fluctuation of the population growth. However, if I use the history of the population like (1.5) or (1.7), it is impossible to consider the fluctuation concepts. Since (1.5) just depends on $N(t-\tau)$ and τ and (1.7) depends on $\int_{t-\tau}^{t} k_2(t-t_2)N(t_1)\,dt_1$ and τ, it is not easy to express the fluctuation like (1.6).

- As in Comment 1.2.2.1, F can be improved by including more biological concepts or by using different kind of concepts in theory. However, in practice, there is no guarantee of the improvement.

1.4 Formulation of equations (1.3) and (1.4)

Now, I am ready to formulate (1.3) and (1.4) for each type of F ($F(\tau, N(t-\tau))$, $F(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$ and $F(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1)$). Under normal circumstances, there is a continuous addition to the number of the population through new births. If new individuals are born at the rate $B(N(t-\tau))$ at the time $t-\tau$, then in a particular time interval $\Delta_i\tau$ about the time $t-\tau_i$ for $i=1,2,\ldots,m$, there are $B(N(t-\tau_i))N(t-\tau_i)\Delta_i\tau$ numbers of new born added, if they survive, who will be of age τ_i at time t. However, strictly speaking, all of the new born will not survive during $t-\tau$ and t. I assume that only the fraction F of the new born will survive to age τ_i. As I mentioned repeatedly, I have 3 types of F to consider. They are:

$$F = F(\tau_i, N(t-\tau_i)), \tag{1.8}$$

$$F = F\left(\tau_i, N(t-\tau_i), N\left(t-\frac{1}{2}\tau_i\right), N(t)\right) \text{ or} \tag{1.9}$$

$$F = F\left(\tau_i, \int_{t-\tau_i}^{t} k_2(t-t_1)N(t_1)\,dt_1\right) \tag{1.10}$$

The case (1.8) Firstly, I consider the case when $F = F(\tau_i, N(t-\tau_i))$. The final addition to the population at time t, from the new offsprings born in the interval $\Delta_i\tau$ about time τ_i is

$$F(\tau_i, N(t-\tau_i))B(N(t-\tau_i))N(t-\tau_i)\Delta_i\tau$$

Now if this process is repeated for all the m subintervals of the time interval $(0, M)$ like Figure 1.3, I get the partial sum

$$N_m(t) = \sum_{i=1}^{m} F(\tau_i, N(t-\tau_i))B(N(t-\tau_i))N(t-\tau_i)\Delta_i\tau \tag{1.11}$$

as the number of individuals added through new births which, if passed to the limit, becomes the integral and I get (1.4) when $F = F(\tau, N(t-\tau))$. Then, as $M \longrightarrow \infty$, I get (1.3) when $F = F(\tau, N(t-\tau))$.

The case (1.9) Secondly, I consider the case when $F = F(\tau_i, N(t-\tau_i), N(t-\frac{1}{2}\tau_i), N(t))$. The final addition to the population at time t, from the new offsprings born in the interval $\Delta_i\tau$ about time τ_i is

$$F\left(\tau_i, N(t-\tau_i), N\left(t-\frac{1}{2}\tau_i\right), N(t)\right)B(N(t-\tau_i))N(t-\tau_i)\Delta_i\tau$$

Again, if this process is repeated for all the m subintervals of the time interval $(0, M)$ like Figure 1.3 below, I get the partial sum

$$N_m(t) = \sum_{i=1}^{m} F\left(\tau_i, N(t-\tau_i), N\left(t-\frac{1}{2}\tau_i\right), N(t)\right) B(N(t-\tau_i))N(t-\tau_i)\Delta_i\tau \qquad (1.12)$$

as the number of individuals added through new births which, if passed to the limit, becomes the integral and I get (1.4) when $F = F(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$. Then, as $M \longrightarrow \infty$, I get (1.3) when $F = F(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$.

The case (1.10) Thirdly, I consider the case when $F = (\tau_i, \int_{t-\tau_i}^{t} k_2(t-t_1)N(t_1)\,dt_1)$. The final addition to the population at time t, from the new offsprings born in the interval $\Delta_i\tau$ about time τ_i is

$$F\left(\tau_i, \int_{t-\tau_i}^{t} k_2(t-t_1)N(t_1)\,dt_1\right) B(N(t-\tau_i))N(t-\tau_i)\Delta_i\tau$$

Again, if this process is repeated for all the m subintervals of the time interval $(0, M)$ like Figure 1.3 below, I get the partial sum

$$N_m(t) = \sum_{i=1}^{m} F\left(\tau_i, \int_{t-\tau_i}^{t} k_2(t-t_1)N(t_1)\,dt_1\right) B(N(t-\tau_i))N(t-\tau_i)\Delta_i\tau \qquad (1.13)$$

as the number of individuals added through new births which, if passed to the limit, becomes the integral and I get (1.4) when $F = F(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1)$. Then, as $M \longrightarrow \infty$, I get (1.3) when $F = F(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1)$.

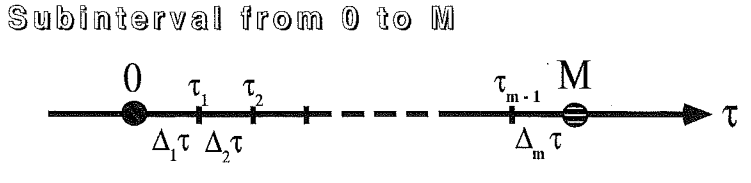

Figure 1.3: ($\tau_0 = 0$, $\tau_m = M$ and $\tau_i < \tau_{i+1}$ for $i = 1, 2, \ldots, m$. $\Delta_i\tau$ is an interval between τ_i and τ_{i+1}.)

1.5 Some examples

I show some examples of (1.3) and (1.4). (Of course, I consider Assumption 1.2.2.1 and Assumption 1.3.3.1). When $B = re^{-qq_1N(t-\tau)}$ and $F = e^{-q(\tau+q_2N(t-\tau))}r(1-be^{-p\tau}N(t-\tau))$ (p, q_1, q_2, b, c and r are all positive constants), it is possible to obtain

$$N(t) = \int_0^{\infty} e^{-q(\tau+cN(t-\tau))}r(1-be^{-p\tau}N(t-\tau))N(t-\tau)\,d\tau$$

where $c = q_1 + q_2$. I will analyze the equation in Section 2.4.2, 4.4, 4.5, 4.6 and 4.7. When $B = (1-b_2N(t-\tau))$ and $F = (b_4\tau + N(t-\tau))e^{-b_3(\tau+N(t-\tau))^2}$ (b_2, b_3 and b_4 are all positive constants), I get

$$\begin{aligned} N(t) &= \int_0^{\infty} (b_4\tau + N(t-\tau))e^{-b_3(\tau+N(t-\tau))^2}(1-b_2N(t-\tau))N(t-\tau)\,d\tau \text{ or} \\ N(t) &= \int_0^{M} (b_4\tau + N(t-\tau))e^{-b_3(\tau+N(t-\tau))^2}(1-b_2N(t-\tau))N(t-\tau)\,d\tau \end{aligned}$$

When $B = \frac{b_5 N(t-\tau)^2}{1+b_6N(t-\tau)}$ and $F = e^{-(b_7\tau + \int_{t-\tau}^{t} b_8(t-t_1)e^{-(t-t_1)}N(t_1)\,dt_1)}$ (b_5, b_6, b_7 and b_8 are all positive constants), I get

$$\begin{aligned} N(t) &= \int_0^{\infty} e^{-(b_7\tau + \int_{t-\tau}^{t} b_8(t-t_1)e^{-(t-t_1)}N(t_1)\,dt_1)} \frac{b_5N(t-\tau)^2}{1+b_6N(t-\tau)}\,d\tau \text{ or} \\ N(t) &= \int_0^{M} e^{-(b_7\tau + \int_{t-\tau}^{t} b_8(t-t_1)e^{-(t-t_1)}N(t_1)\,dt_1)} \frac{b_5N(t-\tau)^2}{1+b_6N(t-\tau)}\,d\tau \end{aligned}$$

Chapter 2

Linear equations with distributed delay

In this chapter, I will deal with the topics as follows:

1. I will introduce the linear equations (2.1) and (2.2) and the assumptions.

2. I will discuss the solutions of some of the special cases of (2.1) and (2.2).

3. I will discuss the reducibility of some of the types of (2.1) to systems of ordinary differential equations. (Ordinary differential equations are not as complicated as integral equations to study and so, by reducing the integral equations into the systems of differential equations, the analysis will become easier.)

4. I will discuss the asymptotic stability of the zero solution of (2.1) and (2.2). (This result is also important to obtain the local stability of (1.3) and (1.4) in Section 4.3.1.)

Note : In Section 2.1.2, I will introduce equation below, namely,

$$y(t) \;=\; \int_0^\infty k(t-s)y(s)\,ds \text{ for } t \in \Re_+$$

In Section 2.1.3, I will introduce

$$x(t) \;=\; \int_{-\infty}^{t} l(t,s)x(s)\,ds,\ t \in (-\infty, 0]$$

However, the purpose of these 2 sections is to show some background studies about integral equations. The results appeared in these sections are not related to the main results of this thesis.

2.1 Assumptions of the linear equations and relationships with Volterra integral equation

In this section, I introduce equations like (2.1) and (2.2) below, give the assumptions required and talk about some the relationships with Volterra equations.

2.1.1 Introduction of the linear equations (2.1) and (2.2)

$$Y(t) = \int_0^\infty k(\cdot)Y(\cdot)\,d\tau \tag{2.1}$$

$$Y(t) = \int_0^M k(\cdot)Y(\cdot)\,d\tau,\ M \in \Re^+ \tag{2.2}$$

where

$$\begin{aligned}
k(\cdot)Y(\cdot) &= k(\tau)Y(t-\tau),\\
k(\cdot)Y(\cdot) &= k(\tau)Y(t-\tau) + k_1(\tau)Y\left(t-\frac{1}{2}\tau\right) + k_2(\tau)Y(t),\\
k(\cdot)Y(\cdot) &= k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1 \text{ or}\\
k(\cdot)Y(\cdot) &= k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1
\end{aligned}$$

Note : $k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1$ can be regarded as a special case of $k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1$ (when $k_2(t-t_1) = 1$). However, since

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau &= \int_0^\infty k_1(\tau)[Y(t) - Y(t-\tau)]\,d\tau\\
\frac{d}{dt}\int_0^M k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau &= \int_0^M k_1(\tau)[Y(t) - Y(t-\tau)]\,d\tau
\end{aligned}$$

I am able to give two statements as follows:

1. When k and k_i, $i = 1, 2$ have some special forms, I can convert (2.1) to systems of ordinary differential equation when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1$ more simply than I can do when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1$.

2. I also can construct Lyapunov functionals like Functional 2.7.1.1 and Functional 2.7.2.1 to obtain the asymptotic stability, which I cannot construct when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1$.

To the detail, see Section 2.4.3 and Section 2.7.

Remark:

- $\int_0^\infty k(\tau)Y(t-\tau)\,d\tau$ and $\int_0^M k(\tau)Y(t-\tau)\,d\tau$ are called distributed delay terms (linear). Integrodifferential equations which include them are already studied by many authors - for example, Ahlip and King [1], Beretta and Solimano [3], Cushing [25], Gopalsamy [41], [42], [43], [45] and [47], Gomatam and Macdonald [39], He and Ruan [59], He, Ruan and Xia [60], He [57], Bellen, Kolmanovskii, Torelli and Vermiglio [5], Landman [70], Macdonald [74] and [75] and Morris [78] studied the integrodifferential equations which included $\int_0^\infty k(\tau)Y(t-\tau)\,d\tau$. Drozdov, Kolmanovskii and Trigiante [31], Chen, L., Chen, X. D. and Zhang [17] and Gopalsamy [40] studied a system of integrodifferential equations which included $\int_0^M k(\tau)Y(t-\tau)\,d\tau$. Gopalsamy and He [52], Gripenberg [55], He [58], Kuang and Smith [68], Kuang [67] and He [58] studied a system of integrodifferential equations which included both of $\int_0^\infty k(\tau)Y(t-\tau)\,d\tau$ and $\int_0^M k(\tau)Y(t-\tau)\,d\tau$.

- In some of the above publications, they used $\int_{-\infty}^0 k(\tau)Y(t+\tau)\,d\tau$ and $\int_{-M}^0 k(\tau)Y(t+\tau)\,d\tau$ instead of using $\int_0^\infty k(\tau)Y(t-\tau)\,d\tau$ and $\int_0^M k(\tau)Y(t-\tau)\,d\tau$. However, it is obvious that by letting $\tau \to -\tau$,

$$\begin{aligned}\int_{-\infty}^0 k(\tau)Y(t+\tau)\,d\tau &\equiv \int_0^\infty k(-\tau)Y(t-\tau)\,d\tau \text{ and}\\ \int_{-M}^0 k(\tau)Y(t+\tau)\,d\tau &\equiv \int_0^M k(-\tau)Y(t-\tau)\,d\tau\end{aligned}$$

Moreover, some of them also used $\int_{-\infty}^t k(t-\tau)Y(\tau)\,d\tau$ and $\int_{t-M}^t k(t-\tau)Y(\tau)\,d\tau$ instead of using $\int_0^\infty k(\tau)Y(t-\tau)\,d\tau$ and $\int_0^M k(\tau)Y(t-\tau)\,d\tau$. As I mentioned in Section 1.0.4, by letting $\tau \to t-\tau$ or the vice versa, it is possible to have

$$\begin{aligned}\int_{-\infty}^t k(t-\tau)Y(\tau)\,d\tau &\equiv \int_0^\infty k(\tau)Y(t-\tau)\,d\tau \text{ and}\\ \int_{t-M}^t k(t-\tau)Y(\tau)\,d\tau &\equiv \int_0^M k(\tau)Y(t-\tau)\,d\tau\end{aligned}$$

with alternative redefinition of k.

- Gripenberg [55] studied an integral equation like

$$x(t) = \int_{-\infty}^t l(t,s)x(s)\,ds,\ t\in(-\infty,0] \tag{2.3}$$

He proved Theorem 2.1.3.1 which is about the existence of the positive solution of (2.3). I discuss this in Section 2.1.3 below. In addition, if $l(t,s)=l(t-s)$, (2.3) is related to (2.1) when

$$k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau)$$

- In Section 2.4, I am discussing about the reducibility of (2.1) to the systems of ordinary differential equations, the original idea comes from Macdonald [74] and [75].

- In Section 2.5.1, 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2, 2.7.1 and 2.7.2, I will construct the Lyapunov functionals to obtain the sufficient conditions of the zero solutions of all the types of (2.1) and (2.2). Some of the ideas to construct the functionals are quite similar to those of the above publications, especially, of Gopalsamy [41] and [47]. (It is not true that all of the authors of the above studied about Lyapunov functionals). I, particularly, use the results of Gopalsamy [47]. To the detail, see Section 2.5 and 2.5.1.

- In Section 4.5 (Section 4.5.1, 4.5.3, 4.5.4, 4.5.5), I construct a 2π periodic solution of an example of (2.1). When doing the analysis, I use the results of Landman [70] and Morris [78].

- Indeed, Volterra had already formulated functional differential equation predator-prey models which include $\int_0^\infty k(\tau)Y(t-\tau)\,d\tau$ or $\int_0^M k(\tau)Y(t-\tau)\,d\tau$. He had already realized that distributed delay is the one way to explain the time delay which continuously affects the solution for a certain period of earlier time. Again, they are all about integrodifferential equations and moreover, he did not study them so deeply.

The assumptions of (2.1) and (2.2) As the first step of analysis of (2.1) and (2.2), it is assumed as follows

Assumption 2.1.1.1

1. $k : (-\infty,\infty) \to (-\infty,\infty)$ and $k_i : (-\infty,\infty) \to (-\infty,\infty)$ for $i = 1, 2$.

2. $\dot{k} = \frac{dk}{dt}$ and $\dot{k_i} = \frac{dk_i}{dt}$ and $k,\ k_i \in C^1$.

3. $|\int_0^\infty k(\tau)\,d\tau| < \infty$, $|\int_0^\infty k_i(\tau)\,d\tau| < \infty$ and $|\int_0^M k(\tau)\,d\tau| < \infty$, $|\int_0^M k_i(\tau)\,d\tau| < \infty$.

4. $|k(M)| < \infty$ and $|k_i(M)| < \infty$.

5. $Y(t)$ is continuous and differentiable for $-\infty < t < \infty$.

6. Without loss of generality $t = 0$ can be fixed as a reference time and it is possible to suppose that it is given the function $Y_0(t)$, $-\infty < t \leq 0$, such that $Y(t) = Y_0(t)$ for $-\infty < t \leq 0$.

7. If only equations (2.2) is considered, it should be possible to be given the function $Y_0(t)$, $-M < t \leq 0$, such that $Y(t) = Y_0(t)$ for $-M < t \leq 0$.

2.1.2 Weiner-Hopf equation

Corduneanu [22](page 158) studied the equation

$$y(t) = h(t) + \int_0^\infty k(t-s)y(s)\,ds \text{ for } t \in \Re_+ \tag{2.4}$$

This is called Weiner-Hopf equation. Then, he established the basic property of the equation

$$y(t) = \int_0^\infty k(t-s)y(s)\,ds \text{ for } t \in \Re_+ \qquad (2.5)$$

under an assumption:

Assumption 2.1.2.1

$v = -ind[1 - \tilde{k}(s)] \leq 0$, where $\tilde{k}(s)$ is the Fourier transform of $k(s)$ and $ind[1 - \tilde{k}(s)] = (2\pi)^{-1}[arg(1 - \tilde{k}(s))]_{s=-\infty}^{s=\infty}$. (where the *ind* denotes the index.)

Then, he stated the theorem below which is under the assumption above,

Theorem 2.1.2.1

(2.5) has only solution $y(t) \equiv 0$.

2.1.3 Gripenberg's result

However, as I mentioned in Section 2.1.1 above, Gripenberg [55] studied the integral equation (2.3), namely,

$$x(t) = \int_{-\infty}^t l(t,s)x(s)\,ds,\ t \in (-\infty, 0]$$

Then, there are the assumptions as follows:

Assumption 2.1.3.1

1. $l(t,s)$ is measurable and nonnegative on the set $\{(t,s)| -\infty < s \leq t \leq 0\}$,
2. $l(t,t)$ is locally integrable on $\Re^-$,
3. there exist a measurable function a: $\Re^- \to (0,\infty)$ such that for a.e. $t \in \Re^-$, every $s \in (-\infty, t]$ and $v \in (-\infty, s]$, $l(t,v)/a(v) \leq l(t,s)/a(s) \leq l(t,t)/a(t)$,
4. there exist a number $T_0 \in \Re^-$ and a nonnegative, nonzero function z such that $a(t)z(t)$ is integrable on $(-\infty, T_0]$ and

$$z(t) \leq \int_{-\infty}^t l(t,s)z(s)\,ds \quad \text{a.e. } t \in (-\infty, T_0]$$

Then, he proved the result as follows.

Theorem 2.1.3.1

There exists a unique solution x of the equation above such that $a(t)x(t)$ is locally integrable on $\Re^-$ and $\lim_{T\to-\infty}\int_T^0 a(t)x(t)\,dt = 1$. Moreover, this solution x is nonnegative and can be found with the aide of an iteration procedure.

The above 2 results seem to be countering each other, since Theorem 2.1.2.1 says that there is no solution except for $y=0$, on the other hand, Theorem 2.1.3.1 says that there are positive solution of (2.3). However, they have made different assumption (Assumption 2.1.3.1 and 2.1.2.1). This causes the different results. I have also different assumption (Assumption 2.1.1.1) to (2.6) below

$$Y(t) = \int_0^\infty k(\tau)Y(t-\tau)\,d\tau \tag{2.6}$$

so, of course, I will show you the different kinds of results in the sections and chapters as follows from their results, especially from the result in Corduneanu [22].

2.2 Some special cases and the solutions

In this section, under the Assumption 2.1.1.1, I will show some solution of the special cases of (2.6) above and (2.7) below.

$$Y(t) = \int_0^M k(\tau)Y(-\tau)\,d\tau \tag{2.7}$$

Delay is finite (2.7) By letting $(\tau \to t-\tau)$, (2.7) is convertible to

$$\begin{aligned} Y(t) &= \int_{t-M}^t k(t-\tau)Y(\tau)\,d\tau \\ &= \int_0^t k(t-\tau)Y(\tau)\,d\tau - \int_0^{t-M} k(t-\tau)Y(\tau)\,d\tau \\ &= \int_0^t k(t-\tau)Y(\tau)\,d\tau + f(t) \qquad (2.8) \\ \text{where } f(t) &= -\int_0^{t-M} k(t-\tau)Y(\tau)\,d\tau \end{aligned}$$

with alternative redefinition of k. Since $t \in [0,\infty)$,

$$Y(0) = \int_{-M}^0 k(-\tau)Y(\tau)\,d\tau$$

can be regarded as the initial condition of (2.8). So, (2.8) (or (2.7)) is like a non-homogeneous Volterra integral equation. Then, when $k(\tau) = (wbe^{-\tau}+1-w)e^{-\tau}$ and when

$$w = \frac{3\left(e^M\right)^3 + (-3\sin(M)-\cos(M))\left(e^M\right)^2 + (-\sin(M)-3\cos(M))\,e^M + 1}{\left(e^M\right)^3 + (-3\sin(M)-\cos(M))\left(e^M\right)^2 + (3\sin(M)-\cos(M))\,e^M + 1}$$

$$b = 5\,\frac{\left(-\sin(M)-\cos(M)+e^M\right)\left(e^M\right)^2}{3\left(e^M\right)^3 + (-3\sin(M)-\cos(M))\left(e^M\right)^2 + (-\sin(M)-3\cos(M))\,e^M + 1}$$

(2.7) has a general solution of

$$Y(t) = K_1\cos(t) + K_2\sin(t) \text{ where } K_1,\, K_2 \in \Re$$

Delay is infinite (2.6) It is obvious that it is possible to proceed the same transformation to the equation (2.6) as that of (2.7):

$$\begin{aligned} Y(t) &= \int_{-\infty}^{t} k(t-\tau)Y(\tau)\,d\tau \\ &= \int_{0}^{t} k(t-\tau)Y(\tau)\,d\tau - \int_{0}^{-\infty} k(t-\tau)Y(\tau)\,d\tau \\ &= \int_{0}^{t} k(t-\tau)Y(\tau)\,d\tau + f_1(t) \qquad (2.9) \\ \text{where } f_1(t) &= -\int_{0}^{-\infty} k(t-\tau)Y(\tau)\,d\tau \end{aligned}$$

with alternative redefinition of k. Since $t \in [0, \infty)$,

$$Y(0) = \int_{-\infty}^{0} k(-\tau)Y(\tau)\,d\tau$$

can be regarded as the initial condition of (2.9). So, (2.9) (or (2.6)) is also like a non-homogeneous Volterra integral equation. Again, when $k(\tau) = (wbe^{-\tau} + 1 - w)e^{-\tau}$ and when $b = 5/3$, $w = 3$, (2.6) has a general solution of

$$Y(t) = K_1 \cos(t) + K_2 \sin(t) \text{ where } K_1,\ K_2 \in \Re$$

When $k(\tau) = (wbe^{-\tau} + 1 - w)e^{-\tau}$ and when $b = 13/9$, $w = 9/4$, (2.6) has a general solution of

$$Y(t) = K_3 e^{-\frac{1}{2}(t)} \cos(t) + K_4 e^{-\frac{1}{2}(t)} \sin(t) \text{ where } K_3,\ K_4 \in \Re$$

Comment 2.2.0.1

- The non homogeneous Volterra integral equations like

$$Y(t) = \int_{0}^{t} k(t-\tau)Y(\tau)\,d\tau + f_2(t) \qquad (2.10)$$

  do not have the general solution like $K_1 \cos(t) + K_2 \sin(t)$ or $K_3 e^{-\frac{1}{2}(t)} \cos(t) + K_4 e^{-\frac{1}{2}(t)} \sin(t)$ (2.9) and (2.8) have.

- I take

$$Y(t) = \int_{0}^{t} (wbe^{-(t-\tau)} + 1 - w)e^{-(t-\tau)} Y(\tau)\,d\tau + f_2(t) \qquad (2.11)$$

  as an example. When $b = 5/3$, $w = 3$, and if $f_2(t) = e^{-2t}$, (2.11) has only a particular solution of $Y(t) = \cos(t) + \sin(t)$, and if $f_2(t) = te^{-2t}$, (2.11) has only a particular solution of $Y(t) = (1/5)\cos(t) + (3/5)\sin(t) - (1/5)e^{-2t}$ or when $b = 13/9$ and $w = 9/4$ and if $f_2(t) = e^{-2t}$, (2.11) has only a particular solution of $Y(t) = e^{-\frac{1}{2}(t)} \cos(t) + \frac{1}{2} e^{-\frac{1}{2}(t)} \sin(t)$, and if $f_2(t) = te^{-2t}$, (2.11) has only a particular solution of $Y(t) = -\frac{4}{13}e^{-2(t)} + \frac{4}{13}e^{-\frac{1}{2}(t)} \cos(t) + \frac{7}{13}e^{-\frac{1}{2}(t)} \sin(t)$.

- (2.6) and (2.7) (or (2.8) and (2.9)) have $f_1(t) = -\int_0^{-\infty} k(t-\tau)Y(\tau)\,d\tau$ and $f(t) = -\int_0^{t-M} k(t-\tau)Y(\tau)\,d\tau$ respectively. This is why (2.6) and (2.7) can have general solutions rather than particular solutions.

- By using the same procedures to (2.6) and (2.7), when $F = F(\tau, N(t-\tau))$, (1.3) and (1.4) can be transformed to

$$\begin{aligned} N(t) &= \int_0^t F(t-\tau, N(\tau))B(N(\tau))N(\tau)\,d\tau + g_1(t) & (2.12) \\ \text{where } g_1(t) &= -\int_0^{-\infty} F(t-\tau, N(\tau))B(N(\tau))N(\tau)\,d\tau \\ N(t) &= \int_0^t F(t-\tau, N(\tau))B(N(\tau))N(\tau)\,d\tau + g_2(t) & (2.13) \\ \text{where } g_1(t) &= -\int_0^{t-M} F(t-\tau, N(\tau))B(N(\tau))N(\tau)\,d\tau \end{aligned}$$

respectively, with alternative redefinition of F and B.

- I will discuss this again in Section 4.7.5.

2.3 Differentiability and the differentiated forms

It is already assumed that $Y(t)$ is continuous and differentiable for $0 < t < \infty$, This is an important assumption to explain the relationship of (2.1) with ordinary differential equations and to construct Lyapunov functionals to obtain the condition of stability of zero solution of (2.1) and (2.2) since all the Lyapunov functionals will be constructed after all the forms of (2.1) and (2.2) are converted to functional differential equations (see Section 2.5.1, 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2, 2.7.1 and 2.7.2). Because of this assumption, I can show that (2.6) and (2.7) are equivalent to (2.14) and (2.15) respectively. Again, let $(\tau \to t-\tau)$, then, (2.6) is convertible to

$$Y(t) = \int_{-\infty}^{t} k(t-\tau)Y(\tau)\,d\tau$$

and by differentiating the above with respect to t,

$$\dot{Y}(t) = k(0)Y(t) + \int_{-\infty}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau$$

So, (2.6) will eventually be equivalent to

$$\dot{Y}(t) = k(0)Y(t) + \int_0^{\infty} \dot{k}(\tau)Y(t-\tau)\,d\tau \tag{2.14}$$

Similarly, you will find (2.7) is equivalent to

$$\dot{Y}(t) = k(0)Y(t) - k(M)Y(t-M) + \int_0^{M} \dot{k}(\tau)Y(t-\tau)\,d\tau \tag{2.15}$$

Generally speaking, all kinds of (2.1) and (2.2) have the equivalent functional differential equation forms like (2.14) and (2.15) respectively by using very similar procedures to obtain both of them. The equivalent forms will appear in later sections (Section 2.5.1, 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2 , 2.7.1 and 2.7.2).

2.4 Convertibility to ordinary differential equations and the initial conditions

I recall that it is assumed that $Y(t)$ is continuous and differentiable for $-\infty < t < \infty$ in the Assumption 2.1.1.1. Gomatam and Macdonald [39] studied some functional equations with infinite delays (integrodifferential equations) for competing species. Macdonald [74] and [75] also studied some functional equations with infinite delays (integrodifferential equations) for prey-predator models. They showed that for the certain forms of k (they use $k(z) = a^p z^{p-1} e^{-az}(p!)^{-1}$ where $a \in (0, \infty)$ and $p \in I$), then system like

$$\begin{aligned} \dot{N}(t) &= f_1(N, P) \\ \dot{P}(t) &= f_2(Q, P) \text{ where } Q(t) = \int_{-\infty}^{t} k(t-\tau)N(\tau)\, d\tau \end{aligned}$$

will be reducible to

$$\begin{aligned} \dot{N}(t) &= f_1(N, P) \\ \dot{P}(t) &= f_2(Q, P) \\ \dot{Q}(t) &= a(N - Q) \end{aligned}$$

when $p = 1$. You can study about the reducibility to the systems of ordinary differential equations by also referring Macdonald [76], Cushing [26] and Lenhart and Travis [71]. Then, let me now study the equation

$$Y(t) = \int_0^{\infty} ae^{-a\tau} Y(t-\tau)\, d\tau = \int_{-\infty}^{t} ae^{-a(t-\tau)} Y(\tau)\, d\tau \text{ for } a > 0 \tag{2.16}$$

(2.16) has only constant solution K, since by differentiating (2.16) with respect to t, I obtain

$$\dot{Y}(t) = -a \int_{-\infty}^{t} ae^{-a(t-\tau)} Y(\tau)\, d\tau + aY(t) = -aY(t) + aY(t) = 0$$

Now, (2.17) below has also a constant solution K,

$$Y(t) = \int_0^{\infty} \Big((4w-2)e^{-\tau} - 2w + 2\Big) e^{-\tau} Y(t-\tau)\, d\tau \tag{2.17}$$

then, similarly, I can convert (2.17) into:

$$\begin{aligned} \dot{Y}(t) &= 2wY(t) - 2Y_1(t) - Y_2(t) \\ \dot{Y_1}(t) &= (-2+4w)Y(t) - 2Y_1(t) \\ \dot{Y_2}(t) &= (2-2w)Y(t) - Y_2(t) \text{ where } Y(t) = Y_1(t) + Y_2(t) \end{aligned} \tag{2.18}$$

However, as you can see, the conversion does not give any kind of information about the initial conditions or boundary conditions. So, even if you can see the general solutions of (2.17), you cannot have particular solutions of it. However, if I solve (2.18) by using the initial conditions $Y(0) = A$, $Y_1(0) = A/2$ and $Y_1(0) = A/2$ ($Y(0) = Y_1(0) + Y_2(0)$), I get a particular solution of

$$Y(t) = \frac{A}{2(2w-3)}\Big((4w-3)e^{(2w-3)t} - 3\Big)$$

and if I solve the above by using the initial conditions $Y(-10) = A$, $Y_1(-10) = A/2$ and $Y_1(-10) = A/2$ ($Y(-10) = Y_1(-10) + Y_2(-10)$), I get a particular solution of

$$Y(t) = \frac{A}{2(2w-3)}\Big((4w-3)e^{(2w-3)(t+10)} - 3\Big)$$

Then, both of the above solution satisfies (2.17) by assuming the integral exists. That is, $Y(t) = K$ is a solution of (2.17) but this is not the only solution. As far as the initial condition of (2.18) satisfies the condition ($Y(C) = Y_1(C) + Y_2(C)$), the solution will satisfy (2.17). This will not solve the issues about starting problems of this kind of functional equations but if they are convertible like the previous case, you can set the initial conditions as I used to the case of (2.17).

Remark: If I have a integral equation like

$$\begin{aligned} Y(t) &= \int_{-\infty}^{t} k(t-\tau, y(\tau))\,d\tau \ ,0<t<\infty \\ &= \int_{-\infty}^{0} k(t-\tau, y(\tau))\,d\tau + \int_{0}^{t} k(t-\tau, y(\tau))\,d\tau \ ,0<t<\infty \\ &= f(t) + \int_{0}^{t} k(t-\tau, y(\tau))\,d\tau \ ,0<t<\infty \end{aligned} \tag{2.19}$$

in some special cases, it might be possible to convert it to ode systems like

$$\begin{aligned} \dot{Y}(t) &= k_1(Y(t), Y_1(t), \ldots) \\ \dot{Y}_1(t) &= k_2(Y(t), Y_1(t), \ldots) \end{aligned}$$

Then, by specifying the initial condition of the above properly like $\dot{Y}(0) = Y_0$, etc, the solution of the above will satisfy (2.19) with $f(0) = Y_0$ and $f(t) = 0$ for $t > 0$. This generally means that I do not obtain the solution of the integral equation (2.19) by solving differential equation systems which are reduced from it, unless $f(t)$ is consistent with the initial conditions. (I will discuss this issue in Section 4.7, again.)

2.4.1 Reducibility of the equations which include periodic functions to systems of ordinary differential equations

I will show the reducibility of (2.20) to the ordinary differential equation system. Even if it includes a period forcing function $\sin b(t-\tau)$, it does not matter to do the same procedures as I mentioned in the above. These are not mentioned in Macdonald [74], Macdonald [75], Macdonald [76], Cushing [26] and Lenhart and Travis [71].

$$Y(t) = \int_{-\infty}^{t} a e^{-a(t-\tau)} \sin b(t-\tau)\, Y(\tau)\,d\tau \tag{2.20}$$

Firstly, differentiate (2.20) with respect to t. Then,

$$\dot{Y} = -a\int_{-\infty}^{t} ae^{-a(t-\tau)}\sin b(t-\tau)\,Y(\tau)\,d\tau + \int_{-\infty}^{t} abe^{-a(t-\tau)}\cos b(t-\tau)\,Y(\tau)\,d\tau$$

Let

$$Y_1(t) = \int_{-\infty}^{t} abe^{-a(t-\tau)}\cos b(t-\tau)\,Y(\tau)\,d\tau$$

So,

$$\dot{Y_1} = -a\int_{-\infty}^{t} abe^{-a(t-\tau)}\cos b(t-\tau)\,Y(\tau)\,d\tau - b^2\int_{-\infty}^{t} ae^{-a(t-\tau)}\sin b(t-\tau)\,Y(\tau)\,d\tau + abY(t)$$

then,

$$\begin{aligned}\dot{Y} &= -aY(t) + Y_1(t)\\ \dot{Y_1} &= -aY_1(t) - bY(t) + abY(t)\end{aligned}$$

2.4.2 Reducibility of the nonlinear integral equation to a system of ordinary differential equations

Now, I introduce a nonlinear equation, which is reducible to a system of ordinary differential equations, namely,

$$N(t) = \int_{-\infty}^{t} e^{-q(t-\tau+cN(\tau))}r\Big(1 - be^{-p(t-\tau)}N(\tau)\Big)N(\tau)\,d\tau \tag{2.21}$$

Since

$$\begin{aligned}\dot{N}(t) &= -q\int_{-\infty}^{t} e^{-q(t-\tau+cN(\tau))}r\Big(1 - be^{-p(t-\tau)}N(\tau)\Big)N(\tau)\,d\tau\\ &\quad + N_1(t) + re^{-qcN(t)}(1-bN(t))N(t)\\ &= -qN(t) + N_1(t) + re^{-qcN(t)}(1-bN(t))N(t)\\ \text{where } N_1(t) &= \int_{-\infty}^{t} rbpe^{-q(t-\tau+cN(\tau))}e^{-p(t-\tau)}N(\tau)^2\,d\tau\end{aligned}$$

and since

$$\begin{aligned}\dot{N_1}(t) &= -(p+q)\int_{-\infty}^{t} rbpe^{-q(t-\tau+cN(\tau))}e^{-p(t-\tau)}N(\tau)^2\,d\tau + rbpe^{-qcN(t)}N(t)^2\\ &= -(p+q)N_1(t) + rbpe^{-qcN(t)}N(t)^2\end{aligned}$$

(2.21) is equivalent to

$$\begin{aligned}\dot{N}(t) &= -qN(t) + N_1(t) + re^{-qcN(t)}(1-bN(t))N(t)\\ \dot{N_1}(t) &= -(p+q)N_1(t) + rbpe^{-qcN(t)}N(t)^2\end{aligned} \tag{2.22}$$

This equation is one of the special case of (1.3) when when $F = F(\tau, N(t-\tau))$, which will be analyzed in Section 4.4, 4.5, 4.6 and 4.7.

2.4.3 Reducibility of the other types of equations belonging to (2.1) to systems of ordinary differential equations

Next, I will explain the reducibility to all of kinds of (2.1). Firstly, I consider

$$k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1$$

Then, by converting this to

$$\dot{Y}(t) = k(0)Y(t)\int_{-\infty}^{t}\Big(\dot{k}(t-\tau) - k_1(t-\tau)\Big)Y(\tau)\,d\tau + \int_0^{\infty} k_1(\tau)\,d\tau Y(t)$$

it is possible to apply the similar procedure as in the above if k and k_1 have the special forms (for example, see k in (2.16) or (2.20)).

Remark:

- As I mentioned in Section 2.1, (2.1) is reducible to a functional differential equation like the above only when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau)+k_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1$, it is not possible to apply the same for the more general case when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau)+k_1(\tau)\int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1$. However, there are other ways to convert this. This will appear later.

Next, I consider the case when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)Y(t-\frac{1}{2}\tau) + k_2(\tau)Y(t-\tau)$. As I told in Section 2.2, by letting $\tau \Rightarrow t-\tau$, I can have

$$\int_0^{\infty} k(\tau)Y(t-\tau)\,d\tau = \int_{-\infty}^{t} k(t-\tau)Y(\tau)\,d\tau$$

then, by differentiating the above with respect to t,

$$\frac{d}{dt}\int_{-\infty}^{t} k(t-\tau)Y(\tau)\,d\tau = k(0)Y(t) + \int_{-\infty}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau$$

and this time by letting $\tau \Rightarrow t-(1/2)\tau$, I can have

$$\int_0^{\infty} k_1(\tau)Y\left(t-\frac{1}{2}\tau\right)d\tau = 2\int_{-\infty}^{t} k_1(2(t-\tau))Y(\tau)\,d\tau$$

then, by differentiating the above with respect to t,

$$\frac{d}{dt}\int_{-\infty}^{t} k_1(2(t-\tau))Y(\tau)\,d\tau = 2k_1(0)Y(t) + 4\int_{-\infty}^{t} \dot{k}_1(2(t-\tau))Y(\tau)\,d\tau$$

So, I eventually have

$$\begin{aligned}\dot{Y}(t) &= W\Big[(k(0)+2k_1(0))Y(t) \\ &+ \int_{-\infty}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau + 4\int_{-\infty}^{t} \dot{k}_1(2(t-\tau))Y(\tau)\,d\tau\Big] \qquad (2.23)\\ \text{where } W &= \frac{1}{1-\int_0^{\infty} k_2(\tau)\,d\tau},\ \int_0^{\infty} k_2(\tau)\,d\tau \neq 1\end{aligned}$$

I take an example like

$$Y(t) = \int_0^\infty \frac{1}{4} bae^{-a\tau} Y\left(t - \frac{1}{2}\tau\right) d\tau = \int_{-\infty}^t \frac{1}{2} bae^{-2a(t-\tau)} Y(\tau)\, d\tau \text{ for } a \text{ and } b \text{ constants.}$$

So, this is equivalent to

$$\dot{Y}(t) = -2a \int_{-\infty}^t \frac{1}{2} bae^{-a(t-\tau)} Y(\tau)\, d\tau + \frac{1}{2} baY(t) = -2aY(t) + \frac{1}{2} baY(t)$$

Next, when

$$k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau) \int_{t-\tau}^t k_2(t-t_1)Y(t_1)\, dt_1$$

by letting $\tau \Rightarrow t - \tau$, I can have

$$Y(t) = \int_{-\infty}^t k(t-\tau)Y(\tau)\, d\tau + \int_{-\infty}^t k_1(t-\tau) \int_\tau^t k_2(t-t_1)Y(t_1)\, dt_1\, d\tau \tag{2.24}$$

and by differentiating this with respect to t, it is possible to have:

$$\begin{aligned} \dot{Y}(t) &= k(0)Y(t) + k_1(0) \int_0^\infty k_2(\tau)\, d\tau Y(t) + \int_{-\infty}^t \dot{k}_1(t-\tau) \int_\tau^t k_2(t-t_1)Y(t_1)\, dt_1\, d\tau \\ &\quad + \int_{-\infty}^t \dot{k}(t-\tau)Y(\tau)\, d\tau + \int_{-\infty}^t k_1(t-\tau) \int_\tau^t \dot{k}_2(t-t_1)Y(t_1)\, dt_1\, d\tau \end{aligned} \tag{2.25}$$

and so, some of the special cases of the above are still possible to be reduced to a system of ordinary differential equations. For example,

$$Y(t) = \int_0^\infty ce^{-\tau} \int_{t-\tau}^t e^{-(t-t_1)} Y(t_1)\, dt_1\, d\tau = \int_{-\infty}^t ce^{-(t-\tau)} \int_\tau^t e^{-(t-t_1)} Y(t_1)\, dt_1\, d\tau$$

Then, by differentiating the above with respect to t, I get

$$\dot{Y}(t) = -2 \int_{-\infty}^t ce^{-(t-\tau)} \int_\tau^t e^{-(t-t_1)} Y(t_1)\, dt_1\, d\tau + cY(t) = -2Y(t) + cY(t)$$

which is one of the simplest cases of the conversion to the ordinary differential equation from (2.24) or equivalently, (2.1) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau) \int_{t-\tau}^t k_2(t-t_1)Y(t_1)\, dt_1$. So, next, let me try

$$\begin{aligned} Y(t) &= \int_0^\infty ce^{-\tau} \int_{t-\tau}^t e^{-(t-t_1)} \Big[(b+a)e^{-(t-t_1)} - b(t-t_1)\Big] Y(t_1)\, dt_1\, d\tau \\ &= \int_{-\infty}^t ce^{-(t-\tau)} \int_\tau^t e^{-(t-t_1)} \Big[(b+a)e^{-(t-t_1)} - b(t-t_1)\Big] Y(t_1)\, dt_1\, d\tau \end{aligned} \tag{2.26}$$

When $a = 5/3c$ and $b = 25/9c$, this has a solution of $Y(t) = K_1 \cos(t) + K_2 \sin(t)$ where (K_1, $K_2 \in \Re$). Now, let

$$\begin{aligned} Y_1(t) &= \int_{-\infty}^t ce^{-(t-\tau)} \int_\tau^t (b+a)e^{-2(t-t_1)} Y(t_1)\, dt_1\, d\tau \\ Y_2(t) &= -\int_{-\infty}^t ce^{-(t-\tau)} \int_\tau^t b(t-t_1)e^{-(t-t_1)} Y(t_1)\, dt_1\, d\tau \\ \Big(Y(t) &= Y_1(t) + Y_2(t)\Big) \\ Y_3(t) &= -\int_{-\infty}^t ce^{-(t-\tau)} \int_\tau^t be^{-(t-t_1)} Y(t_1)\, dt_1\, d\tau \end{aligned}$$

Then, by differentiating $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ in the above with respect to t, I obtain:

$$\begin{aligned}
\dot{Y_1}(t) &= -3\int_{-\infty}^{t} ce^{-(t-\tau)}\int_{\tau}^{t}(b+a)e^{-2(t-t_1)}Y(t_1)\,dt_1\,d\tau + c(b+a)Y(t) \\
&= -3Y_1(t) + c(b+a)Y(t) \\
\dot{Y_2}(t) &= 2\int_{-\infty}^{t} ce^{-(t-\tau)}\int_{\tau}^{t} b(t-t_1)e^{-(t-t_1)}Y(t_1)\,dt_1\,d\tau \\
&\quad -\int_{-\infty}^{t} ce^{-(t-\tau)}\int_{\tau}^{t} be^{-(t-t_1)}Y(t_1)\,dt_1\,d\tau \\
&= -2Y_2(t) + Y_3(t) \\
\dot{Y_3}(t) &= 2\int_{-\infty}^{t} ce^{-(t-\tau)}\int_{\tau}^{t} be^{-(t-t_1)}Y(t_1)\,dt_1\,d\tau - cbY(t) \\
&= -2Y_3(t) - cbY(t)
\end{aligned}$$

Since $Y(t) = Y_1(t) + Y_2(t)$, I eventually have a system of ordinary differential equations as follows:

$$\begin{aligned}
\dot{Y}(t) &= \dot{Y_1}(t) + \dot{Y_2}(t) \\
&= -3Y_1(t) + c(b+a)Y(t) - 2Y_2(t) + Y_3(t) \\
\dot{Y_1}(t) &= -3Y_1(t) + c(b+a)Y(t) \\
\dot{Y_2}(t) &= -2Y_2(t) + Y_3(t) \\
\dot{Y_3}(t) &= -2Y_3(t) - cbY(t)
\end{aligned}$$

2.5 Asymptotic stability of linear equations

In this section, I deal with the asymptotic stability of the zero solutions of (2.1) and (2.2). They are linear equations. But the asymptotic stability of linear equations is important to decide the local stability of nonlinear equations (which will be studied later), since the standard way to determine the local stability is to use the characteristic equations obtained by linearising the nonlinear equations. Hence, (2.1) and (2.2) will be required to see the stability for non-linear equations. Gopalsamy [47] (from page 3 to page 9) introduced several functionals to the linear functional differential equations. For example, he introduced the Functional 2.5.0.1 as follows on the integrodifferential equation (2.27)

$$\dot{x}(t) = -\int_{0}^{\infty} K(s)x(t-s)\,ds \tag{2.27}$$

with $K : [0,\infty) \mapsto [0,\infty)$ and K is continuous.

Functional 2.5.0.1

$$\begin{aligned}
v(x)(t) &= v_1(x)(t) + v_2(x)(t) \\
\textit{where } v_1(x)(t) &= \left(x(t) - \int_{0}^{\infty} K(s)\int_{t-s}^{t} x(u)\,du\,ds\right)^2 \\
\textit{and } v_2(x)(t) &= \int_{0}^{\infty} K(s)\,ds\int_{0}^{\infty} K(s)\int_{t-s}^{t}\int_{u}^{t} x(w)^2\,dw\,du\,ds
\end{aligned}$$

Then, under the Assumption 2.5.0.1, below

Assumption 2.5.0.1

$$\int_0^\infty K(s)\,d\,s < \infty \text{ and } \int_0^\infty K(s)s\,d\,s < 1$$

he proved that every nonzero solution of (2.27) satisfies $\lim_{t\to\infty} x(t) = 0$. He also introduced the Functional 2.5.0.2 as follows on the functional differential equation (2.28), namely,

$$\dot{x}(t) = a_1 x(t-\tau_1) + a_2 x(t-\tau_2), \quad \text{for } t > 0 \tag{2.28}$$

where a_1, a_2, τ_1, $\tau_2 \in (0, \infty)$.

Functional 2.5.0.2

$$\begin{aligned} v(x)(t) &= v_1(x)(t) + v_2(x)(t) \\ \textit{where } v_1(x)(t) &= \left(x(t) - a_1 \int_{t-\tau_1}^{t} x(s)\,d\,s - a_2 \int_{t-\tau_2}^{t} x(s)\,d\,s\right)^2 \\ \textit{and } v_2(x)(t) &= (a_1 + a_2)\left[a_1 \int_{t-\tau_1}^{t} \int_{s}^{t} x(u)^2\,d\,u\,d\,s + a_2 \int_{t-\tau_2}^{t} \int_{s}^{t} x(u)^2\,d\,u\,d\,s\right] \end{aligned}$$

Then, under the Assumption 2.5.0.2 below,

Assumption 2.5.0.2

$$a_1\tau_1 + a_2\tau_2 < 1$$

he proved that every nonzero solution of (2.28) satisfies $\lim_{t\to\infty} x(t) = 0$.

2.5.1 The functional to obtain of the condition of the stability of (2.6)

Now, let me start analyzing the stability of the zero solution of (2.1) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau)$. I assumed that $Y(t)$ is continuous and differentiable for $-\infty < t < \infty$ and I have shown that (2.6) is equivalent to (2.14) in section 2.3. Then, (2.14) can be modified like:

$$\begin{aligned} \dot{Y}(t) &= k(0)Y(t) + \int_0^\infty \dot{k}(\tau)Y(t-\tau)\,d\tau \\ &= \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))Y(t-\tau)\,d\tau \qquad (2.29) \\ \text{since } Y(t) &= \int_0^\infty k(\tau)Y(t-\tau)\,d\tau \end{aligned}$$

Then, let me make assumptions as follows:

Assumption 2.5.1.1

$$\int_0^\infty \tau|(k(0)k(\tau) + \dot{k}(\tau))|\,d\tau < \infty \text{ and } \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau < 0$$

and I am introducing the Lyapunov Functional 2.5.1.1 on (2.29)

Functional 2.5.1.1

$$\begin{aligned} v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\ \textit{where } v_1(Y)(t) &= \left(Y(t) + \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau)) \int_{t-\tau}^t Y(u)\,du\,d\tau\right)^2 \\ \textit{and } v_2(Y)(t) &= -\int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau \int_0^\infty |k(0)k(\tau) + \dot{k}(\tau)| \int_{t-\tau}^t \int_u^t Y(w)^2\,dw\,du\,d\tau \end{aligned}$$

Then, since

$$\begin{aligned} \dot{v_1} &= 2\left(Y(t) + \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau)) \int_{t-\tau}^t Y(u)\,du\,d\tau\right)\Big(\dot{Y}(t) \\ &\quad + \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))(Y(t) - Y(t-\tau))\,d\tau\Big) \\ &= 2\int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau\left(Y(t) + \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau)) \int_{t-\tau}^t Y(u)\,du\,d\tau\right)Y(t) \\ &\leq 2\int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau\left(Y(t)^2 - \int_0^\infty |k(0)k(\tau) + \dot{k}(\tau)| \int_{t-\tau}^t |Y(u)||Y(t)|\,du\,d\tau\right) \\ &\leq \int_0^\infty \Big(k(0)k(\tau) + \dot{k}(\tau)\Big)\,d\tau\Big(2Y(t)^2 - \int_0^\infty |k(0)k(\tau) + \dot{k}(\tau)| \int_{t-\tau}^t Y(u)^2\,du\,d\tau \\ &\quad - Y(t)^2 \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\,d\tau\Big) \\ \dot{v_2} &= \int_0^\infty \Big(k(0)k(\tau) + \dot{k}(\tau)\Big)\,d\tau \int_0^\infty |k(0)k(\tau) + \dot{k}(\tau)|\left(\int_{t-\tau}^t Y(u)^2\,du - \tau Y(t)^2\right)d\tau \end{aligned}$$

I obtain

$$\dot{v} \leq 2Y(t)^2 \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau\left(1 - \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\,d\tau\right) \tag{2.30}$$

So, if

$$1 - \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\,d\tau > 0 \tag{2.31}$$

It is found by (2.30) and (2.31) that for any nonzero solution of $Y(t)$ of (2.6), $v(t)$ is non increasing in t. This implies that $0 \leq v(Y)(t) \leq v(Y)(0)$. However, this implies that

$$Y(t) \leq (v(Y)(0))^{1/2} + \int_0^\infty \int_{t-\tau}^t |(k(0)k(\tau) + \dot{k}(\tau))||Y(u)|\,du\,d\tau \tag{2.32}$$

Now, from (2.32) and by putting $m(t) = \sup_{u\in[-\tau,t]} |Y(u)|$

$$m(t)\left(1 - \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\,d\tau\right) \leq (v(Y)(0))^{1/2} \tag{2.33}$$

Using (2.31) and (2.33), any arbitrary nonzero solution of (2.6) is uniformly bounded on $[0,\infty)$ if (2.31) holds, which follows immediately that $\dot{Y}(t)$ is also uniformly bounded on $[0,\infty)$. This implies that Y is uniformly continuous on $[0,\infty)$. So, by integrating both sides of (2.30), I have

$$v(Y)(0) \geq v(Y)(t) - 2\int_0^t Y(s)^2\,ds \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau\left(1 - \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\,d\tau\right)$$

Hence, $Y^2 \in L^1[0,\infty)$. Then, by using the lemma below which is proved by Barbalat in 1959 (You can see the proof in Gopalsamy [47] as the proof of Lemma 1.2.2), the steady state of zero solution of (2.6) is asymptotically stable.

Theorem 2.5.1.1 *[Lemma in Gopalsamy [47]]*

◇ *Let f be a nonnegative function defined on $[0,\infty)$ such that f is integrable on $[0,\infty)$ and uniformly continuous on $[0,\infty)$. Then, $\lim_{t\to\infty} f(t) = 0$*

Then, I can have

Conclusion 2.5.1.1

◇ *In conclusion, if it satisfies the condition that*

$$1 - \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\,d\tau > 0 \text{ and } \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau < 0$$

the zero solution of (2.29) (or equivalently (2.6)) is asymptotically stable.

Now, let me try the example as follows (Example 2.5.1.1).

Example 2.5.1.1

◇ I try $k(\tau) = e^{-a\tau}$ and $a > 1$,then,

$$\int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\,d\tau = \frac{1}{a} - 1 < 0 \text{ and } \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\,d\tau = \frac{-1+a}{a^2} < 1$$

Hence, if $a > 1$, the zero solution of (2.6) when $k(\tau) = e^{-a\tau}$ is asymptotically stable.

Note: From now on, I will explain how to obtain the conditions of the asymptotic stability of the zero solution of all the types of (2.1) and (2.2). That is, I will tell how to obtain the conditions that the derivative of the functional becomes negative ($\dot{v} \leq 0$) like (2.30) and (2.31). It is possible to explain like the above argument by using Lemma in Gopalsamy [47] but it is so similar, so, I omit them in the later sections (Section 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2, 2.7.1 and 2.7.2).

2.5.2 The functional to obtain of the condition of the stability of (2.7)

Now, I have already shown that (2.7) is equivalent to (2.15) in the Section 2.3, namely,

$$\dot{Y}(t) = k(0)Y(t) - k(M)Y(t-M) + \int_0^M \dot{k}(\tau)Y(t-\tau)\,d\tau$$

Moreover, (2.7) and (2.15) are equivalent to (2.34) below, namely,

$$\dot{Y}(t) = \int_0^M (k(0)k(\tau) + \dot{k}(\tau))Y(t-\tau)\,d\tau - k(M)\int_0^M k(\tau)Y(t-\tau-M)\,d\tau \tag{2.34}$$

$$\text{since } Y(t) = \int_0^M k(\tau)Y(t-\tau)\,d\tau \text{ and } Y(t-M) = \int_0^M k(\tau)Y(t-\tau-M)\,d\tau$$

Then, I make assumptions as follows.

Assumption 2.5.2.1

$$\int_0^M \tau|(k(0)k(\tau)+\dot{k}(\tau))|\,d\tau<\infty,\quad \int_0^M (M+\tau)|k(M)k(\tau)|\,d\tau<\infty$$

$$\text{and } L=\int_0^M (k(0)k(\tau)+\dot{k}(\tau)-k(M)k(\tau))\,d\tau<0$$

and let me apply the Functional (2.5.2.1) on (2.34) in order to obtain the conditions for the asymptotic stability of the zero solution of (2.34).

Functional 2.5.2.1

$$\begin{aligned}
v(Y)(t) &= v_1(Y)(t)+v_2(Y)(t)\\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t)+\int_0^M (k(0)k(\tau)+\dot{k}(\tau))\int_{t-\tau}^t Y(u)\,du\,d\tau\\
&\quad -\int_0^M k(M)k(\tau)\int_{t-\tau-M}^t Y(u)\,du\,d\tau\Bigg)^2\\
\textit{and } v_2(Y)(t) &= -L\int_0^M |k(0)k(\tau)+\dot{k}(\tau)|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau\\
&\quad -L\int_0^M |k(M)k(\tau)|\int_{t-\tau-M}^t\int_u^t Y(w)^2\,dw\,du\,d\tau
\end{aligned}$$

Then, I have

$$\dot{v}(Y)(t)\ \le\ 2LY(t)^2\left(1-\int_0^M (\tau|k(0)k(\tau)+\dot{k}(\tau)|+(M+\tau)|k(M)k(\tau)|)\,d\tau\right)$$

Therefore, I can have

Conclusion 2.5.2.1

◇ *If the conditions as follows are satisfied,*

$$\begin{aligned}
0 &> \int_0^M (k(0)k(\tau)+\dot{k}(\tau)-k(M)k(\tau))\,d\tau\\
0 &< 1-\int_0^M (\tau|k(0)k(\tau)+\dot{k}(\tau)|+(M+\tau)|k(M)k(\tau)|)\,d\tau
\end{aligned}$$

the zero solution of (2.34) (or equivalently (2.7)) is asymptotically stable.

Then, let me try the example below which uses the same k as in Example (2.5.1.1).

Example 2.5.2.1

1. $k(\tau)=e^{-a\tau}$ and $a>1$, then,

$$L=\frac{(e^{-aM}-1)(e^{-aM}-1+a)}{a}<0 \text{ for } a>1 \text{ and } M>0$$

2. Moreover,

$$\begin{aligned}
\int_0^M (\tau|k(0)k(\tau)+\dot{k}(\tau)|+(M+\tau)|k(M)k(\tau)|)\,d\tau &= -\frac{(2aM+1)e^{-2aM}}{a^2}\\
&\quad -\frac{(-2+a)(aM+1)e^{-aM}}{a^2}+\frac{-1+a}{a^2}
\end{aligned}$$

   This is always less than 1, if $a>1$ and $M>0$.

Hence, if $a>1$ and $M>0$, the zero solution is stable.

2.5.3 The functional on (2.39)

In this section, I will introduce Functional 2.5.3.1 on (2.39) to obtain the conditions of the asymptotic stability of the zero solution of (2.1) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1$, namely,

$$Y(t) = \int_0^\infty k(\tau)Y(t-\tau)\,d\tau + \int_0^\infty k_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1\,d\tau \tag{2.35}$$

Firstly, let me show the reducibility to the functional differential equation (2.39) from the integral equation (2.35). Again, by letting $\tau \Rightarrow t-\tau$ I get (2.36), that is,

$$Y(t) = \int_{-\infty}^{t} k(t-\tau)Y(\tau)\,d\tau + \int_{-\infty}^{t} k_1(t-\tau)\int_{\tau}^{t} Y(t_1)\,dt_1\,d\tau \tag{2.36}$$

and by differentiating (2.36) above with respect to t and then, by letting $t-\tau \Rightarrow \tau$, I get:

$$\begin{aligned} \dot{Y}(t) &= k(0)Y(t) + \int_0^\infty k_1(\tau)\,d\tau Y(t) \\ &\quad + \int_0^\infty \dot{k}(\tau)Y(t-\tau)\,d\tau + \int_0^\infty \dot{k}_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1\,d\tau \end{aligned} \tag{2.37}$$

Then, (2.37) will be equivalent to

$$\begin{aligned} \dot{Y}(t) &= \int_0^\infty \left(k(0)k(\tau) + \dot{k}(\tau) + k(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)Y(t-\tau)\,d\tau \\ &\quad + \int_0^\infty \left(k(0)k_1(\tau) + \dot{k}_1(\tau) + k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)\int_{t-\tau}^{t} Y(t_1)\,dt_1\,d\tau \\ \text{since } Y(t) &= \int_0^\infty k(\tau)Y(t-\tau)\,d\tau + \int_0^\infty k_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1\,d\tau \end{aligned} \tag{2.38}$$

Then, by letting $t_2 = t - t_1$, I have

$$dt_2 = -dt_1,\ t_1 = t-\tau \Rightarrow t_2 = \tau \text{ and } t_1 = t \Rightarrow t_2 = 0$$

(2.38) will be equivalent to

$$\begin{aligned} \dot{Y}(t) &= \int_0^\infty \left(k(0)k(\tau) + \dot{k}(\tau) + k(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)Y(t-\tau)\,d\tau \\ &\quad + \int_0^\infty \left(k(0)k_1(\tau) + \dot{k}_1(\tau) + k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)\int_0^{\tau} Y(t-t_2)\,dt_2\,d\tau \end{aligned} \tag{2.39}$$

Secondly, let me state the assumptions as follows.

Assumption 2.5.3.1

$$\begin{aligned} \infty &> \int_0^\infty \tau\left|k(0)k(\tau) + \dot{k}(\tau) + k(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right|d\tau \\ \infty &> \int_0^\infty \frac{1}{2}\tau^2\left|k(0)k_1(\tau) + \dot{k}_1(\tau) + k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right|d\tau \\ L &\equiv \int_0^\infty \left(k(0)k(\tau) + \dot{k}(\tau) + k(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)d\tau \\ &\quad + \int_0^\infty \tau\left(k(0)k_1(\tau) + \dot{k}_1(\tau) + k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)d\tau \\ &< 0 \end{aligned}$$

Thirdly, I will apply the Functional 2.5.3.1 on (2.39) as follows to obtain the conditions of the stability of the zero solution of (2.39).

Functional 2.5.3.1

$$\begin{aligned}
v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \Bigg[Y(t) + \int_0^\infty \Big(\dot{k}(\tau) + k(\tau)\Big(k(0) + \int_0^\infty k_1(\tau_1)\,d\tau_1\Big)\Big)\int_{t-\tau}^t Y(u)\,du\,d\tau \\
&\quad + \int_0^\infty \Big(\dot{k}_1(\tau) + k_1(\tau)\Big(k(0) + \int_0^\infty k_1(\tau_1)\,d\tau_1\Big)\Big)\int_0^\tau \int_{t-t_2}^t Y(u)\,du\,dt_2\,d\tau\Bigg]^2 \\
\textit{and } v_2(Y)(t) &= -L\int_0^\infty \Bigg|\dot{k}_1(\tau) \\
&\quad + k_1(\tau)\Big(k(0) + \int_0^\infty k_1(\tau_1)\,d\tau_1\Big)\Bigg|\int_0^\tau \int_{t-t_2}^t \int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau \\
&\quad - L\int_0^\infty \Bigg|\dot{k}(\tau) + k(\tau)\Big(k(0) + \int_0^\infty k_1(\tau_1)\,d\tau_1\Big)\Bigg|\int_{t-\tau}^t \int_u^t Y(w)^2\,dw\,du\,d\tau
\end{aligned}$$

After some calculation, I get

$$\begin{aligned}
\dot{v} &\leq 2L_1LWY(t)^2 \qquad (2.40) \\
\text{where } L_1 &= \Bigg(1 - \int_0^\infty \tau\Bigg|k(0)k(\tau) + \dot{k}(\tau) + k(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\Bigg|\,d\tau \\
&\quad - \int_0^\infty \frac{1}{2}\tau^2\Bigg|k(0)k_1(\tau) + \dot{k}_1(\tau) + k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\Bigg|\,d\tau\Bigg)
\end{aligned}$$

Finally, I state the result as follows.

Conclusion 2.5.3.1

If $L_1 > 0$ and $L < 0$, I can conclude that $\dot{v} < 0$. So, the zero solution of (2.39) (or equivalently, (2.35)) is asymptotically stable.

2.5.4 The functional on (2.44)

In this section, I will introduce Functional 2.5.4.1 on (2.44) to obtain the conditions of the asymptotic stability of the zero solution of (2.2) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1$, namely,

$$Y(t) = \int_0^M k(\tau)Y(t-\tau)\,d\tau + \int_0^M k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau \qquad (2.41)$$

Firstly, let me show the reducibility to the functional differential equation (2.44) from the above type of the integral equation (2.41). Again, by letting $\tau \Rightarrow t-\tau$, I get (2.42), namely,

$$Y(t) = \int_{t-M}^t k(t-\tau)Y(\tau)\,d\tau + \int_{t-M}^t k_1(t-\tau)\int_\tau^t Y(t_1)\,dt_1\,d\tau \qquad (2.42)$$

and by differentiating (2.42) with respect to t, and then, by letting $t-\tau \Rightarrow \tau$, I get:

$$
\begin{aligned}
\dot{Y}(t) &= k(0)Y(t)+\int_0^M k_1(\tau)\,d\tau Y(t)+\int_0^M \dot{k}(\tau)Y(t-\tau)\,d\tau - k(M)Y(t-M) \\
&\quad +\int_0^M \dot{k}_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1\,d\tau - k_1(M)\int_0^M Y(t-t_1)\,dt_1 \qquad (2.43)
\end{aligned}
$$

Then, again as in Section 2.5.3, by letting $t_2 = t-t_1$, I have

$$
dt_2 = -dt_1,\ t_1 = t-\tau \Rightarrow t_2 = \tau \text{ and } t_1 = t \Rightarrow t_2 = 0
$$

and from (2.41), I have

$$
\begin{aligned}
Y(t) &= \int_0^M k(\tau)Y(t-\tau)\,d\tau + \int_0^M k_1(\tau)\int_0^{\tau} Y(t-t_2)\,dt_2\,d\tau \\
Y(t-M) &= \int_0^M k(\tau)Y(t-\tau-M)\,d\tau + \int_0^M k_1(\tau)\int_0^{\tau} Y(t-t_2-M)\,dt_2\,d\tau
\end{aligned}
$$

Then, after some calculations, I have

$$
\begin{aligned}
\dot{Y}(t) &= \int_0^M \left(k(0)k(\tau)+\dot{k}(\tau)+k(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)Y(t-\tau)\,d\tau \\
&\quad +\int_0^M \left(k(0)k_1(\tau)+\dot{k}_1(\tau)+k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)\int_0^{\tau} Y(t-t_2)\,dt_2\,d\tau \\
&\quad -k_1(M)\int_0^M Y(t-t_1)\,dt_1\,d\tau - k(M)\int_0^M k(\tau)Y(t-\tau-M)\,d\tau \\
&\quad -k(M)\int_0^M k_1(\tau)\int_0^{\tau} Y(t-t_2-M)\,dt_2\,d\tau \qquad (2.44)
\end{aligned}
$$

Secondly, let me state the assumptions as follows.

Assumption 2.5.4.1

$$
\int_0^M \tau\left|k(0)k(\tau)+\dot{k}(\tau)+k(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right|d\tau < \infty,
$$

$$
\int_0^M \frac{1}{2}\tau^2\left|k(0)k_1(\tau)+\dot{k}_1(\tau)+k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right|d\tau < \infty,
$$

$$
\int_0^M |k(M)k_1(\tau)|\left(\frac{1}{2}\tau^2+M\tau\right)d\tau < \infty,\ \int_0^M |k(M)k(\tau)|\left(M+\tau\right)d\tau < \infty
$$

$$
\begin{aligned}
\text{and } L &\equiv \int_0^M \left(k(0)k(\tau)+\dot{k}(\tau)+k(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)d\tau - k(M)\int_0^M \left(k_1(\tau)\tau+k(\tau)\right)d\tau \\
&\quad +\int_0^M \left(k(0)k_1(\tau)+\dot{k}_1(\tau)+k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)\tau\,d\tau - k_1(M)M \\
&< 0
\end{aligned}
$$

Thirdly, let me apply the Functional 2.5.4.1 on (2.44) as follows to obtain the conditions of the stability of the zero solution of (2.44).

Functional 2.5.4.1

$$
\begin{aligned}
v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t) + \int_0^M \Big(\dot{k}(\tau) + k(\tau)\Big(k(0) + \int_0^M k_1(\tau_1)\,d\tau_1\Big)\Big) \int_{t-\tau}^{t} Y(u)\,du\,d\tau \\
&\quad + \int_0^M \Big(\dot{k}_1(\tau) + k_1(\tau)\Big(k(0) + \int_0^M k_1(\tau_1)\,d\tau_1\Big)\Big) \int_0^{\tau}\int_{t-t_2}^{t} Y(u)\,du\,dt_2\,d\tau \\
&\quad -k(M)\int_0^M k_1(\tau)\int_0^{\tau}\int_{t-t_2-M}^{t} Y(u)\,du\,dt_2\,d\tau \\
&\quad -k_1(M)\int_0^M\int_{t-t_1}^{t} Y(u)\,du\,dt_1 - k(M)\int_0^M k(\tau)\int_{t-\tau-M}^{t} Y(u)\,du\,d\tau\Bigg)^2 \\
\textit{and } v_2(Y)(t) &= -L\int_0^M \Big|\dot{k}_1(\tau) \\
&\quad + k_1(\tau)\Big(k(0) + \int_0^M k_1(\tau_1)\,d\tau_1\Big)\Big| \int_0^{\tau}\int_{t-t_2}^{t}\int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau \\
&\quad -L\int_0^M \Big|\dot{k}(\tau) + k(\tau)\Big(k(0) + \int_0^M k_1(\tau_1)\,d\tau_1\Big)\Big| \int_{t-\tau}^{t}\int_u^t Y(w)^2\,dw\,du\,d\tau \\
&\quad -L\int_0^M \Big|k(M)k_1(\tau)\Big| \int_0^{\tau}\int_{t-t_2-M}^{t}\int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau \\
&\quad -L\Big|k_1(M)\Big| \int_0^M\int_{t-t_1}^{t}\int_u^t Y(w)^2\,dw\,du\,dt_1 \\
&\quad -L\int_0^M \Big|k(M)k(\tau)\Big| \int_{t-\tau-M}^{t}\int_u^t Y(w)^2\,dw\,du\,d\tau
\end{aligned}
$$

After some calculation, I get

$$
\begin{aligned}
\dot{v} &\leq 2LL_1Y(t)^2 \qquad (2.45)\\
\text{where } L_1 &= \Bigg(1 - \int_0^M \tau\Big|k(0)k(\tau) + \dot{k}(\tau) + k(\tau)\int_0^M k_1(\tau)\,d\tau\Big|\,d\tau \\
&\quad - \int_0^M \frac{1}{2}\tau^2\Big|k(0)k_1(\tau) + \dot{k}_1(\tau) + k_1(\tau)\int_0^M k_1(\tau)\,d\tau\Big|\,d\tau - \frac{1}{2}|k_1(M)|M^2 \\
&\quad - \int_0^M |k(M)k_1(\tau)|\Big(\frac{1}{2}\tau^2 + M\tau\Big)\,d\tau - \int_0^M |k(M)k(\tau)|\Big(M+\tau\Big)\,d\tau\Bigg)
\end{aligned}
$$

Finally, I state the result as follows.

Conclusion 2.5.4.1

If $L_1 > 0$ and $L < 0$, I can conclude that $\dot{v} < 0$. So, the zero solution of (2.44) (or equivalently (2.41)) is asymptotically stable.

2.5.5 The functional on (2.47)

In this section, I will introduce Functional 2.5.5.1 and 2.5.5.2 on (2.47) to obtain the conditions of the asymptotic stability of the zero solution of (2.1) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)Y(t-$

$\frac{\tau}{2}) + k_2(\tau)Y(t)$, namely,

$$Y(t) \;=\; \int_0^\infty k(\tau)Y(t-\tau) + k_1(\tau)Y\left(t - \frac{\tau}{2}\right) + k_2(\tau)Y(t)\,d\tau \tag{2.46}$$

I explained in Section 2.4.3 how to convert the integral equation (2.46) above into an integro-differential equation (2.23). Then, again, by letting $t - \tau \Rightarrow \tau$, I have

$$\int_{-\infty}^t \dot{k}(t-\tau)Y(\tau)\,d\tau \;=\; \int_0^\infty \dot{k}(\tau)Y(t-\tau)\,d\tau$$

and by letting $\tau \Rightarrow t - (1/2)\tau$, I have

$$4\int_{-\infty}^t \dot{k}_1(2(t-\tau))Y(\tau)\,d\tau \;=\; 2\int_0^\infty \dot{k}_1(\tau)Y\left(t - \frac{1}{2}\tau\right)d\tau$$

Hence, (2.23) is equivalent to the below:

$$\begin{aligned}\dot{Y}(t) \;&=\; W(k(0) + 2k_1(0))Y(t) + W\int_0^\infty \dot{k}(\tau)Y(t-\tau)\,d\tau + 2W\int_0^\infty \dot{k}_1(\tau)Y\left(t - \frac{1}{2}\tau\right)d\tau \\ &\text{where } W = \frac{1}{1 - \int_0^\infty k_2(\tau)\,d\tau}, \ \int_0^\infty k_2(\tau)\,d\tau \neq 1\end{aligned}$$

Then, this will become:

$$\begin{aligned}\dot{Y}(t) \;&=\; W\int_0^\infty [(k(0) + 2k_1(0))k(\tau) + \dot{k}(\tau)]Y(t-\tau)\,d\tau \\ &\quad + W(k(0) + 2k_1(0))\int_0^\infty k_2(\tau)\,d\tau Y(t) \\ &\quad + W\int_0^\infty [(k(0) + 2k_1(0))k_1(\tau) + 2\dot{k}_1(\tau)]Y\left(t - \frac{\tau}{2}\right)d\tau \qquad (2.47) \\ \text{due to } Y(t) \;&=\; \int_0^\infty k(\tau)Y(t-\tau) + k_1(\tau)Y\left(t - \frac{\tau}{2}\right) + k_2(\tau)Y(t)\,d\tau\end{aligned}$$

Since I must have $WL < 0$ to obtain $\dot{v} < 0$ as in the below (see (2.48) and (2.49)), where

$$L \equiv \int_0^\infty \left[\Big(k(0) + 2k_1(0)\Big)\Big(k(\tau) + k_1(\tau) + k_2(\tau)\Big) + \dot{k}(\tau) + 2\dot{k}_1(\tau)\right]d\tau$$

I must separate 2 cases ($W > 0$ and $L < 0$ or $W < 0$ and $L > 0$) to proceed the analysis.

$\mathbf{W > 0}$ and $\mathbf{L < 0}$

Firstly, I will proceed for the case when $W > 0$ and $L < 0$. Then, I need the assumption as follows:

Assumption 2.5.5.1

$$\int_0^\infty \tau\left|(k(0) + 2k_1(0))k(\tau) + \dot{k}(\tau)\right|d\tau < \infty, \ \int_0^\infty \tau\left|(k(0) + 2k_1(0))k_1(\tau) + 2\dot{k}_1(\tau)\right|d\tau < \infty$$

and $W > 0$ and $L < 0$

Then, I apply the Functional 2.5.5.1 on (2.47) as follows to obtain the conditions of the stability of the zero solution of (2.46).

Functional 2.5.5.1

$$\begin{aligned}
v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t) + W\int_0^\infty \Big((k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big)\int_{t-\tau}^{t} Y(u)\,du\,d\tau \\
&\quad + W\int_0^\infty (k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\int_{t-\frac{\tau}{2}}^{t} Y(u)\,du\,d\tau\Bigg)^2 \\
\textit{and } v_2(Y)(t) &= -LW^2\Bigg(\int_0^\infty \Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\int_{t-\tau}^{t}\int_u^t Y(w)^2\,dw\,du\,d\tau \\
&\quad + \int_0^\infty \Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\int_{t-\frac{\tau}{2}}^{t}\int_u^t Y(w)^2\,dw\,du\,d\tau\Bigg)
\end{aligned}$$

Then, I get

$$\begin{aligned}
\dot{v} &\leq 2LL_1WY(t)^2 \qquad (2.48)\\
\text{where } L_1 &= \Bigg(1 - W\int_0^\infty \tau\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\,d\tau \\
&\quad -\frac{W}{2}\int_0^\infty \tau\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\,d\tau\Bigg)
\end{aligned}$$

Now, I state the result as follows.

Conclusion 2.5.5.1

If $L_1 > 0$, $W > 0$ and $L < 0$, I can conclude that $\dot{v} < 0$. So, the zero solution of (2.46) (or equivalently (2.47)) is asymptotically stable.

$\mathbf{W < 0}$ and $\mathbf{L > 0}$

Secondly, I proceed for the case when $W < 0$ and $L > 0$. Similarly to the case when $W > 0$ and $L < 0$, I must have the assumptions as follows:

Assumption 2.5.5.2

$$\int_0^\infty \tau\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\,d\tau < \infty, \quad \int_0^\infty \tau\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\,d\tau < \infty$$

and $W < 0$ and $L > 0$

by using the Functional 2.5.5.2 on (2.47) as follows

Functional 2.5.5.2

$$\begin{aligned}
v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t) + W\int_0^\infty \Big((k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big)\int_{t-\tau}^{t} Y(u)\,du\,d\tau \\
&\quad + W\int_0^\infty (k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\int_{t-\frac{\tau}{2}}^{t} Y(u)\,du\,d\tau\Bigg)^2 \\
\textit{and } v_2(Y)(t) &= LW^2\Bigg(\int_0^\infty \Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\int_{t-\tau}^{t}\int_u^t Y(w)^2\,dw\,du\,d\tau \\
&\quad + \int_0^\infty \Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\int_{t-\frac{\tau}{2}}^{t}\int_u^t Y(w)^2\,dw\,du\,d\tau\Bigg)
\end{aligned}$$

I get

$$
\begin{aligned}
\dot{v} &\leq 2LL_2WY(t)^2 \\
\text{where } L_2 &= \Bigg(1+W\int_0^\infty \tau\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\,d\tau \\
&\quad +\frac{W}{2}\int_0^\infty \tau\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k}_1(\tau)\Big|\,d\tau\Bigg)
\end{aligned} \tag{2.49}
$$

Finally, I state the result as follows.

Conclusion 2.5.5.2

If $L_2 > 0$, $W < 0$ and $L > 0$, I can conclude that $\dot{v} < 0$. So, the zero solution of (2.46) (or equivalently (2.47)) is asymptotically stable.

2.5.6 The functional on (2.51)

In this section, I will introduce Functional 2.5.6.1 and 2.5.6.2 on (2.51) to obtain the conditions of the asymptotic stability of the zero solution of (2.2) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)Y(t-\frac{\tau}{2}) + k_2(\tau)Y(t)$, namely,

$$
Y(t) = \int_0^M k(\tau)Y(t-\tau) + k_1(\tau)Y\left(t-\frac{\tau}{2}\right) + k_2(\tau)Y(t)\,d\tau \tag{2.50}
$$

Then, by using a similar process as in the section above (Section 2.5.5), it is possible to obtain the below, that is,

$$
\begin{aligned}
\dot{Y}(t) &= W(k(0)+2k_1(0))Y(t) - Wk(M)Y(t-M) - 2Wk_1(M)Y\left(t-\frac{M}{2}\right) \\
&\quad +W\int_0^M \dot{k}(\tau)Y(t-\tau)\,d\tau + 2W\int_0^M \dot{k}_1(\tau)Y\left(t-\frac{\tau}{2}\right)d\tau \\
&\quad \text{where } W = \frac{1}{1-\int_0^M k_2(\tau)\,d\tau},\ 1-\int_0^M k_2(\tau)\,d\tau \neq 0
\end{aligned}
$$

Then, it is possible to obtain:

$$
\begin{aligned}
\dot{Y}(t) &= W\int_0^M \Big[(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big]Y(t-\tau)\,d\tau - Wk(M)Y(t-M) \\
&\quad +W\int_0^M \Big[(k(0)+2k_1(0))k_1(\tau)+2\dot{k}_1(\tau)\Big]Y\left(t-\frac{1}{2}\tau\right)d\tau \\
&\quad +W(k(0)+2k_1(0))\int_0^M k_2(\tau)\,d\tau Y(t) - 2Wk_1(M)Y\left(t-\frac{M}{2}\right) \qquad (2.51) \\
\text{since } Y(t) &= \int_0^M k(\tau)Y(t-\tau)+k_1(\tau)Y\left(t-\frac{\tau}{2}\right)+k_2(\tau)Y(t)\,d\tau
\end{aligned}
$$

Again, $WL < 0$ is required to obtain $\dot{v} < 0$ as in the below (see (2.52) and (2.53)), where

$$
L \equiv \int_0^M \Big[\Big(k(0)+2k_1(0)\Big)\Big(k(\tau)+k_1(\tau)+k_2(\tau)\Big)\Big]\,d\tau - k(0) - 2k_1(0)
$$

So, I must separate 2 cases ($W > 0$ and $L < 0$ or $W < 0$ and $L > 0$) to proceed the analysis.

W > 0 and L < 0

Firstly, let me proceed for the case ($W > 0$ and $L < 0$). Then, the assumptions required is as follows.

Assumption 2.5.6.1

$$\int_0^M \tau\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\,d\tau<\infty,\;\int_0^M \tau\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\,d\tau<\infty$$

and $W > 0$ and $L < 0$

Then, by using the Functional 2.5.6.1 on (2.51) as follows

Functional 2.5.6.1

$$\begin{aligned}
v(Y)(t) &= v_1(Y)(t)+v_2(Y)(t)\\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t)+W\int_0^M \Big((k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big)\int_{t-\tau}^t Y(u)\,du\,d\tau\\
&\quad +W\int_0^M (k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\int_{t-\frac{\tau}{2}}^t Y(u)\,du\,d\tau\\
&\quad -Wk(M)\int_{t-M}^t Y(s)\,ds-2Wk_1(M)\int_{t-\frac{M}{2}}^t Y(s)\,ds\Bigg)^2\\
\textit{and } v_2(Y)(t) &= -LW^2\Bigg(\int_0^M \Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau\\
&\quad +\int_0^M \Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\int_{t-\frac{\tau}{2}}^t\int_u^t Y(w)^2\,dw\,du\,d\tau\\
&\quad +|k(M)|\int_{t-M}^t\int_s^t Y(u)^2\,du\,ds+2|k_1(M)|\int_{t-\frac{M}{2}}^t\int_s^t Y(u)^2\,du\,ds\Bigg)
\end{aligned}$$

it is possible to have

$$\begin{aligned}
\dot{v} &\le 2L_1LWY(t)^2 \qquad (2.52)\\
\text{where } L_1 &= 1-W\int_0^M \tau\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\,d\tau\\
&\quad -\frac{W}{2}\int_0^M \tau\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\,d\tau-WM|k(M)|-WM|k_1(M)|
\end{aligned}$$

Hence, I can state the conclusion as follows

Conclusion 2.5.6.1 *If $L_1 > 0$, and $W > 0$ and $L < 0$, I can conclude that $\dot{v} < 0$. So, the zero solution of (2.50) (or equivalently (2.51)) is asymptotically stable.*

W < 0 and L > 0

Secondly, let me proceed for the case when $W < 0$ and $L > 0$. Similarly to the case when $W > 0$ and $L < 0$, I can have the assumptions as follows:

Assumption 2.5.6.2

$$\int_0^M \tau\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\,d\tau<\infty,\ \int_0^M \tau\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\,d\tau<\infty$$

and $W<0$ and $L>0$

Then, this time by using the Functional 2.5.6.2 on (2.51) as follows

Functional 2.5.6.2

$$\begin{aligned}
v(Y)(t) &= v_1(Y)(t)+v_2(Y)(t)\\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t)+W\int_0^M\Big((k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big)\int_{t-\tau}^t Y(u)\,du\,d\tau\\
&\quad +W\int_0^M (k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\int_{t-\frac{\tau}{2}}^t Y(u)\,du\,d\tau\\
&\quad -Wk(M)\int_{t-M}^t Y(s)\,ds-2Wk_1(M)\int_{t-\frac{M}{2}}^t Y(s)\,ds\Bigg)^2\\
\textit{and } v_2(Y)(t) &= LW^2\Bigg(\int_0^M\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau\\
&\quad +\int_0^M\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\int_{t-\frac{\tau}{2}}^t\int_u^t Y(w)^2\,dw\,du\,d\tau\\
&\quad +|k(M)|\int_{t-M}^t\int_s^t Y(u)^2\,du\,ds+2|k_1(M)|\int_{t-\frac{M}{2}}^t\int_s^t Y(u)^2\,du\,ds
\end{aligned}$$

I get

$$\begin{aligned}
\dot{v} &\le 2LL_2WY(t)^2 \qquad (2.53)\\
\text{where } L_2 &= 1+W\int_0^M \tau\Big|(k(0)+2k_1(0))k(\tau)+\dot{k}(\tau)\Big|\,d\tau\\
&\quad +\frac{W}{2}\int_0^M \tau\Big|(k(0)+2k_1(0))k_1(\tau)+2\dot{k_1}(\tau)\Big|\,d\tau+WM|k(M)|+WM|k_1(M)|
\end{aligned}$$

Hence, I can conclude the result as follows.

Conclusion 2.5.6.2

If $L_2>0$, $W<0$ and $L>0$, I can conclude that $\dot{v}<0$. So, the zero solution of (2.50) (or equivalently (2.51)) is asymptotically stable.

2.5.7 The functional on (2.56)

In this section, I will introduce Functional 2.5.7.1 on (2.56) to obtain the conditions of the asymptotic stability of the zero solution of (2.1) when $k(\cdot)Y(\cdot)=k(\cdot)Y(\cdot)=k(\tau)Y(t-\tau)+k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1$, namely,

$$Y(t) = \int_0^\infty k(\tau)Y(t-\tau)\,d\tau+\int_0^\infty k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1\,d\tau \qquad (2.54)$$

Firstly, let me show the reducibility to the functional differential equation (2.56) from the integral equation (2.54). As I explained in Section 2.4.3, by letting $\tau \Rightarrow t-\tau$, I get (2.24), that is,

$$Y(t) = \int_{-\infty}^{t} k(t-\tau)Y(\tau)\,d\tau + \int_{-\infty}^{t} k_1(t-\tau)\int_{\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau$$

and again as in Section 2.4.3, by differentiating (2.24) with respect to t, I get (2.25), namely,

$$\begin{aligned}\dot{Y}(t) &= k(0)Y(t) + k_2(0)\int_{-\infty}^{t} k_1(t-\tau)\,d\tau Y(t) + \int_{-\infty}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau \\ &\quad + \int_{-\infty}^{t} k_1(t-\tau)\int_{\tau}^{t} \dot{k}_2(t-t_1)Y(t_1)\,dt_1\,d\tau + \int_{-\infty}^{t} \dot{k}_1(t-\tau)\int_{\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau\end{aligned}$$

Then, by letting $t-\tau \Rightarrow \tau$,

$$\begin{aligned}\dot{Y}(t) &= k(0)Y(t) + k_2(0)\int_{0}^{\infty} k_1(\tau)\,d\tau Y(t) + \int_{0}^{\infty} k_1(\tau)\int_{t-\tau}^{t} \dot{k}_2(t-t_1)Y(t_1)\,dt_1\,d\tau \\ &\quad + \int_{0}^{\infty} \dot{k}(\tau)Y(t-\tau)\,d\tau + \int_{0}^{\infty} \dot{k}_1(\tau)\int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau \qquad (2.55)\end{aligned}$$

Then, (2.55) will be equivalent to

$$\begin{aligned}\dot{Y}(t) &= \int_{0}^{\infty}\left[\dot{k}(\tau) + \left(k(0) + k_2(0)\int_{0}^{\infty} k_1(\tau_1)\,d\tau_1\right)k(\tau)\right]Y(t-\tau)\,d\tau \\ &\quad + \int_{0}^{\infty}\int_{t-\tau}^{t}\Bigg[\Bigg(k(0)k_1(\tau) + \dot{k}_1(\tau) \\ &\quad + k_2(0)k_1(\tau)\int_{0}^{\infty} k_1(\tau_1)\,d\tau_1\Bigg)k_2(t-t_1) + k_1(\tau)\dot{k}_2(t-t_1)\Bigg]Y(t_1)\,dt_1\,d\tau \\ \text{since } Y(t) &= \int_{0}^{\infty} k(\tau)Y(t-\tau)\,d\tau + \int_{0}^{\infty} k_1(\tau)\int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau\end{aligned}$$

As I mentioned in Section 2.5.3 and 2.5.4, by letting $t_2 = t - t_1$, I have

$$dt_2 = -dt_1,\ t_1 = t-\tau \Rightarrow t_2 = \tau \text{ and } t_1 = t \Rightarrow t_2 = 0$$

So, the above equation will be equivalent to

$$\begin{aligned}\dot{Y}(t) &= \int_{0}^{\infty}\left[\dot{k}(\tau) + \left(k(0) + k_2(0)\int_{0}^{\infty} k_1(\tau_1)\,d\tau_1\right)k(\tau)\right]Y(t-\tau)\,d\tau \\ &\quad + \int_{0}^{\infty}\int_{0}^{\tau}\Bigg[\Bigg(k(0)k_1(\tau) + \dot{k}_1(\tau) \\ &\quad + k_2(0)k_1(\tau)\int_{0}^{\infty} k_1(\tau_1)\,d\tau_1\Bigg)k_2(t_2) + k_1(\tau)\dot{k}_2(t_2)\Bigg]Y(t-t_2)\,dt_2\,d\tau \qquad (2.56)\end{aligned}$$

Secondly, it is assumed as follows.

Assumption 2.5.7.1

$$\begin{aligned}\infty &> \int_{0}^{\infty} \tau\left|\dot{k}(\tau) + \left(k(0) + k_2(0)\int_{0}^{\infty} k_1(\tau_1)\,d\tau_1\right)k(\tau)\right|d\tau \\ \infty &> \int_{0}^{\infty}\int_{0}^{\tau} t_2\left|\left(k(0)k_1(\tau) + \dot{k}_1(\tau) + k_2(0)k_1(\tau)\int_{0}^{\infty} k_1(\tau_1)\,d\tau_1\right)k_2(t_2) + k_1(\tau)\dot{k}_2(t_2)\right|dt_2\,d\tau\end{aligned}$$

$$
\begin{aligned}
L &\equiv \int_0^\infty \left[\dot{k}(\tau) + \left(k(0) + k_2(0)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k(\tau)\right]d\tau \\
&+ \int_0^\infty \int_0^\tau \left[\left(k(0)k_1(\tau) + \dot{k}_1(\tau)\right.\right. \\
&\left.\left.+ k_2(0)k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k_2(t_2) + k_1(\tau)\dot{k}_2(t_2)\right]dt_2\,d\tau \\
&< 0
\end{aligned}
$$

Thirdly, I apply the Functional 2.5.7.1 on (2.56) as follows to obtain the conditions of the stability of the zero solution of (2.56).

Functional 2.5.7.1

$$
\begin{aligned}
v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \left(Y(t)\right. \\
&+ \int_0^\infty \left[\dot{k}(\tau) + \left(k(0) + k_2(0)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k(\tau)\right]\int_{t-\tau}^t Y(u)\,du\,d\tau \\
&+ \int_0^\infty \int_0^\tau \left[\left(k(0)k_1(\tau) + \dot{k}_1(\tau) + k_2(0)k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k_2(t_2)\right. \\
&\left.\left. + k_1(\tau)\dot{k}_2(t_2)\right]\int_{t-t_2}^t Y(u)\,du\,dt_2\,d\tau\right)^2 \\
\textit{and } v_2(Y)(t) &= -L\int_0^\infty \left|\dot{k}(\tau) + \left(k(0) + k_2(0)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k(\tau)\right|\int_{t-\tau}^t \int_u^t Y(w)^2\,dw\,du\,d\tau \\
&-L\int_0^\infty \int_0^\tau \left|\left(k(0)k_1(\tau) + \dot{k}_1(\tau) + k_2(0)k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k_2(t_2)\right. \\
&\left. + k_1(\tau)\dot{k}_2(t_2)\right|\int_{t-t_2}^t \int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau
\end{aligned}
$$

Then, after some simplification,

$$
\begin{aligned}
\dot{v}(t) &\leq 2LL_1Y(t)^2 \qquad (2.57) \\
\text{where } L_1 &= 1 - \int_0^\infty \tau\left|\dot{k}(\tau) + \left(k(0) + k_2(0)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k(\tau)\right|d\tau \\
&- \int_0^\infty \int_0^\tau t_2\left|\left(k(0)k_1(\tau) + \dot{k}_1(\tau)\right.\right. \\
&\left.\left.+ k_2(0)k_1(\tau)\int_0^\infty k_1(\tau_1)\,d\tau_1\right)k_2(t_2) + k_1(\tau)\dot{k}_2(t_2)\right|dt_2\,d\tau
\end{aligned}
$$

Finally, I can state the result as follows.

Conclusion 2.5.7.1

If $L_1 > 0$ and $L < 0$, $\dot{v} < 0$. So, the zero solution of (2.56) or (equivalently, (2.54)) is asymptotically stable.

2.5.8 The functional on (2.61)

In this section, I will introduce Functional 2.5.8.1 on (2.61) to obtain the conditions of the asymptotic stability of the zero solution of (2.2) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1$, namely,

$$Y(t) = \int_0^M k(\tau)Y(t-\tau)\,d\tau + \int_0^M k_1(\tau)\int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau \tag{2.58}$$

Firstly, I have to show the reducibility into the functional differential equation (2.61) below from the above type of the integral equation (2.58) by using the similar procedures as in Section 2.5.3, 2.5.4 and 2.5.7. Again, by letting $\tau \Rightarrow t-\tau$, (2.58) will be like:

$$Y(t) = \int_{t-M}^{t} k(t-\tau)Y(\tau)\,d\tau + \int_{t-M}^{t} k_1(t-\tau)\int_{\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau \tag{2.59}$$

and by differentiating (2.59) with respect to t, I get:

$$\begin{aligned}
\dot{Y}(t) &= k(0)Y(t) + k_2(0)\int_{t-M}^{t} k_1(t-\tau)\,d\tau Y(t) + \int_{t-M}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau \\
&\quad + \int_{t-M}^{t} k_1(t-\tau)\int_{\tau}^{t} \dot{k}_2(t-t_1)Y(t_1)\,dt_1\,d\tau + \int_{t-M}^{t} \dot{k}_1(t-\tau)\int_{\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau \\
&\quad - k(M)Y(t-M) - k_1(M)\int_{t-M}^{t} k_2(t-t_1)Y(t_1)\,dt_1
\end{aligned}$$

Then, this time, by letting $t-\tau \Rightarrow \tau$,

$$\begin{aligned}
\dot{Y}(t) &= k(0)Y(t) + k_2(0)\int_0^M k_1(\tau)\,d\tau Y(t) + \int_0^M \dot{k}(\tau)Y(t-\tau)\,d\tau \\
&\quad + \int_0^M k_1(\tau)\int_{t-\tau}^{t} \dot{k}_2(t-t_1)Y(t_1)\,dt_1\,d\tau + \int_0^M \dot{k}_1(\tau)\int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau \\
&\quad - k(M)Y(t-M) - k_1(M)\int_0^M k_2(t_1)Y(t-t_1)\,dt_1
\end{aligned} \tag{2.60}$$

Then, again, as I mentioned in Section 2.5.3, 2.5.4 and 2.5.7 by letting $t_2 = t - t_1$, I have

$$dt_2 = -dt_1,\ t_1 = t-\tau \Rightarrow t_2 = \tau \text{ and } t_1 = t \Rightarrow t_2 = 0$$

So, I can have

$$\begin{aligned}
Y(t) &= \int_0^M k(\tau)Y(t-\tau)\,d\tau + \int_0^M k_1(\tau)\int_0^{\tau} k_2(t_2)Y(t-t_2)\,dt_2\,d\tau \\
Y(t-M) &= \int_0^M k(\tau)Y(t-\tau-M)\,d\tau + \int_0^M k_1(\tau)\int_0^{\tau} k_2(t_2)Y(t-t_2-M)\,dt_2\,d\tau
\end{aligned}$$

Then, (2.60) will be eventually equivalent to :

$$\begin{aligned}
\dot{Y}(t) &= \int_0^M \left[\dot{k}(\tau) + \left(k(0) + k_2(0)\int_0^M k_1(\tau_1)\,d\tau_1\right)k(\tau)\right]Y(t-\tau)\,d\tau \\
&\quad + \int_0^M \int_0^{\tau} \left[\left(k(0)k_1(\tau) + \dot{k}_1(\tau) + k_2(0)k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)k_2(t_2)\right.
\end{aligned}$$

$$
\begin{aligned}
&+k_1(\tau)\dot{k_2}(t_2)\Big] Y(t-t_2)\,dt_2\,d\tau \\
&-k_1(M)\int_0^M k_2(t_1)Y(t-t_1)\,dt_1 - k(M)\int_0^M k(\tau)Y(t-\tau-M)\,d\tau \\
&-k(M)\int_0^M k_1(\tau)\int_0^\tau k_2(t_2)Y(t-t_2-M)\,dt_2\,d\tau
\end{aligned}
\tag{2.61}
$$

Secondly, let me state the assumptions as follows.

Assumption 2.5.8.1

$$
\int_0^M \int_0^\tau t_2\Big|L_1(\tau,t_2)\Big|\,dt_2\,d\tau < \infty,\ \int_0^M \tau\Big|L_2(\tau)\Big|\,d\tau < \infty
$$

$$
\begin{aligned}
\textit{where } L_1 &= \left(k(0)k_1(\tau)+\dot{k_1}(\tau)+k_2(0)k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)k_2(t_2)+k_1(\tau)\dot{k_2}(t_2) \\
L_2 &= \dot{k}(\tau)+\left(k(0)+k_2(0)\int_0^M k_1(\tau_1)\,d\tau_1\right)k(\tau)
\end{aligned}
$$

$$
\int_0^M |k(M)k(\tau)|(M+\tau)\,d\tau < \infty,\ \int_0^M t_1|k_1(M)k_2(t_1)|\,dt_1 < \infty
$$

$$
\int_0^M \int_0^\tau \Big|k(M)k_1(\tau)k_2(t_2)\Big|(t_2+M)\,dt_2\,d\tau < \infty
$$

$$
\begin{aligned}
\textit{and } L \equiv\ & \int_0^M \left[\dot{k}(\tau)+\left(k(0)+k_2(0)\int_0^M k_1(\tau_1)\,d\tau_1\right)k(\tau)\right]d\tau \\
&+\int_0^M\int_0^\tau \left[\left(k(0)k_1(\tau)+\dot{k_1}(\tau)+k_2(0)k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)k_2(t_2)+k_1(\tau)\dot{k_2}(t_2)\right]dt_2\,d\tau \\
&-\int_0^M k(M)k(\tau)\,d\tau-\int_0^M k_1(M)k_2(t_1)\,dt_1-\int_0^M\int_0^\tau k(M)k_1(\tau)k_2(t_2)\,dt_2\,d\tau \\
<\ & 0
\end{aligned}
$$

Thirdly, let me apply the Functional 2.5.8.1 on (2.61) as follows

Functional 2.5.8.1

$$
\begin{aligned}
v(Y)(t) &= v_1(Y)(t)+v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t)+\int_0^M L_2(\tau)\int_{t-\tau}^t Y(u)\,du\,d\tau+\int_0^M\int_0^\tau L_1(\tau,t_2)\int_{t-t_2}^t Y(u)\,du\,dt_2\,d\tau \\
&\quad -k(M)\int_0^M k_1(\tau)\int_0^\tau k_2(t_2)\int_{t-t_2-M}^t Y(u)\,du\,dt_2\,d\tau \\
&\quad -k_1(M)\int_0^M k_2(t_1)\int_{t-t_1}^t Y(u)\,du\,dt_1-k(M)\int_0^M k(\tau)\int_{t-\tau-M}^t Y(u)\,du\,d\tau\Bigg)^2 \\
\textit{and } v_2(Y)(t) &= -L\int_0^M \Big|L_2(\tau)\Big|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau \\
&\quad -L\int_0^M\int_0^\tau \Big|L_1(\tau,t_2)\Big|\int_{t-t_2}^t\int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau \\
&\quad -L\int_0^M\int_0^\tau \Big|k(M)k_1(\tau)k_2(t_2)\Big|\int_{t-t_2-M}^t\int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau
\end{aligned}
$$

$$-L\int_0^M \Big|k(M)k(\tau)\Big| \int_{t-\tau-M}^t \int_u^t Y(w)^2\,dw\,du\,d\tau$$
$$-L\int_0^M \Big|k_1(M)k_2(t_1)\Big| \int_{t-t_1}^t \int_u^t Y(w)^2\,dw\,du\,dt_1$$

After some calculations,

$$\begin{aligned}
\dot{v}(t) &\leq 2LL_3Y(t)^2 \qquad (2.62)\\
\text{where } L_3 &= 1-\int_0^M \tau\Big|L_2(\tau)\Big|\,d\tau - \int_0^M\int_0^\tau t_2\Big|L_1(\tau,t_2)\Big|\,dt_2\,d\tau - \int_0^M t_1|k_1(M)k_2(t_1)|\,dt_1\\
&\quad -\int_0^M |k(M)k(\tau)|(M+\tau)\,d\tau - \int_0^M\int_0^\tau \Big|k(M)k_1(\tau)k_2(t_2)\Big|(t_2+M)\,dt_2\,d\tau
\end{aligned}$$

Finally, I can state the result as follows.

Conclusion 2.5.8.1

If $L_3 > 0$ and $L < 0$, $\dot{v} < 0$. So, the zero solution of (2.61) (or equivalently (2.58)) is asymptotically stable.

2.6 On the Lyapunov functionals I

In this section, I will discuss the other two kinds of functionals to obtain the conditions for the stability of the zero solutions of (2.6) and (2.7). One is applied when $k(\tau) = k_1(\tau) + k_2(\tau)$, the other is applied when differentiate Y with respect to t twice. They will give the different conditions from those of Functional 2.5.1.1 and 2.5.2.1 and it is true that they are sometime more useful than Functional 2.5.1.1 and 2.5.2.1 or sometime are not as useful as Functional 2.5.1.1 and 2.5.2.1.

2.6.1 Lyapunov functionals for a special type of function k

In Section 2.5.2, I have shown that by using the Functional 2.5.2.1, the zero solution of (2.7) is asymptotically stable if the following conditions are satisfied.

$$\begin{aligned}
L \equiv \int_0^M \Big(k(0)k(\tau) + \dot{k}(\tau) - k(M)k(\tau)\Big)\,d\tau &< 0\\
1-\int_0^M \Big(\tau|k(0)k(\tau)+\dot{k}(\tau)| + (M+\tau)|k(M)k(\tau)|\Big)\,d\tau &> 0
\end{aligned}$$

However, in general, the Lyapunov functionals give only sufficient conditions of the stability of the steady states, so, it is possible that even if at least one of the above conditions are not satisfied, the zero solution of (2.7) is asymptotically stable. For example,

Example 2.6.1.1

If $k(\tau) = -2e^{-(3/2)\tau} + e^{-2\tau}$ and $M = 20$,

$$\int_0^M \Big(k(0)k(\tau) + \dot{k}(\tau) - k(M)k(\tau)\Big)\,d\tau \approx 1.83 > 0$$

So, the condition in Section 2.5.2 (and elsewhere) does not apply. However, if $k(\tau)$ is a function such that $k(\tau) = k_1(\tau) + k_2(\tau) \Rightarrow \dot{k}(\tau) = \dot{k}_1(\tau) + \dot{k}_2(\tau)$, where $k_i : (-\infty, \infty) \to (-\infty, \infty)$, $i = 1, 2$, $k_i \in C^1$ and it is assumed that $|\int_0^\infty k_i(\tau)\, d\tau| < \infty$ and $|\int_0^M k_i(\tau)\, d\tau| < \infty$ and it assumed that $|k_i(M)| < \infty$. It is sometimes possible to obtain better conditions by using a different functional (Functional 2.6.1.1) from Functional 2.5.2.1. Now, let me show the reducibility to the functional differential equation (2.63) from the (2.34) in Section 2.5.2.

$$
\begin{aligned}
\dot{Y}(t) &= (k_1(0) + k_2(0))Y(t) - (k_1(M) + k_2(M)))Y(t-M) \\
&\quad + \int_0^M \left[\dot{k}_1(\tau) + \dot{k}_2(\tau)\right] Y(t-\tau)\, d\tau \\
&= k_1(0)Y(t) + \int_0^M \left[k_2(0)k(\tau) + \dot{k}(\tau)\right] Y(t-\tau)\, d\tau \\
&\quad -k_2(M)\int_0^M k(\tau)Y(t-\tau-M)\, d\tau - k_1(M)Y(t-M) \qquad (2.63) \\
\text{since } Y(t) &= \int_0^M (k_1(\tau) + k_2(\tau))Y(t-\tau)\, d\tau \\
\text{and } Y(t-M) &= \int_0^M (k_1(\tau) + k_2(\tau))Y(t-\tau-M)\, d\tau
\end{aligned}
$$

Then, it is assumed that

Assumption 2.6.1.1

$$
\int_0^M \tau|k_2(0)k(\tau) + \dot{k}_1(\tau) + \dot{k}_2(\tau)|\, d\tau < \infty, \quad \int_0^M (M+\tau)|k_2(M)k(\tau)|\, d\tau < \infty
$$
$$
\textit{and } L_1 \equiv \int_0^M ((k_2(0) - k_2(M))k(\tau) + \dot{k}_2(\tau))\, d\tau < 0
$$

and let me apply the Functional 2.6.1.1 on (2.63) as follows to obtain the conditions of the stability of the zero solution.

Functional 2.6.1.1

$$
\begin{aligned}
v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \Bigg(Y(t) + \int_0^M \left[k_2(0)(k_1(\tau) + k_2(\tau)) + \dot{k}_1(\tau) + \dot{k}_2(\tau)\right] \int_{t-\tau}^t Y(u)\, du\, d\tau \\
&\quad - \int_0^M k_2(M)k(\tau) \int_{t-\tau-M}^t Y(u)\, du\, d\tau - k_1(M)\int_{t-M}^t Y(u)\, du\Bigg)^2 \\
\textit{and } v_2(Y)(t) &= -L_1 \int_0^M \left|k_2(0)(k_1(\tau) + k_2(\tau)) + \dot{k}_1(\tau) + \dot{k}_2(\tau)\right| \int_{t-\tau}^t \int_u^t Y(w)^2\, dw\, du\, d\tau \\
&\quad -L_1 \int_0^M |k_2(M)k(\tau)| \int_{t-\tau-M}^t \int_u^t Y(w)^2\, dw\, du\, d\tau \\
&\quad -L_1|k_1(M)| \int_{t-M}^t \int_u^t Y(w)^2\, dw\, du
\end{aligned}
$$

after doing some calculations,

$$
\begin{aligned}
\dot{v} &\leq 2Y(t)^2 L_1 L_2 \qquad (2.64) \\
\text{where } L_2 &= 1 - \int_0^M \tau|k_2(0)k(\tau) + \dot{k}(\tau)|\, d\tau - \int_0^M |k_2(M)k(\tau)|(M+\tau)\, d\tau - |k_1(M)|M
\end{aligned}
$$

Therefore, I can have the conclusion like:

Conclusion 2.6.1.1

If $L_1 < 0$ and $L_2 > 0$, the zero solution of (2.7) is asymptotically stable.

Now, again, let me try the same k as in Example 2.6.1.1.

Example 2.6.1.2

When $k(\tau) = -2e^{-(3/2)\tau} + e^{-2\tau}$ and $M = 20$, I can have k_1 and k_2 such that $k_1(\tau) = -2e^{-(3/2)\tau}$ and $k_2(\tau) = e^{-2\tau}$. Then, $L_1 \approx -1.83$ and $L_2 \approx 0.8$. Hence, I have shown the asymptotic stability of the zero solution when $k(\tau) = -2e^{-(3/2)\tau} + e^{-2\tau}$ and $M = 20$.

Modified Lyapunov functional on (2.65)

Now, I will construct Functional 2.6.1.2 on (2.65) (or equivalently (2.6)) in order to obtain the conditions for the stability of zero solution of (2.6) when I can split $k(\tau)$ like $k(\tau) = k_1(\tau) + k_2(\tau)$, which are the different conditions from the ones obtained by using Functional 2.5.1.1 in Section 2.5.1. As the first step, let me show the reducibility to the functional differential equation (2.65) from (2.29).

$$\begin{aligned}
\dot{Y}(t) &= \Big(k_1(0) + k_2(0)\Big)Y(t) + \int_0^\infty \Big(\dot{k}_1(\tau) + \dot{k}_2(\tau)\Big)Y(t-\tau)\,d\tau \\
&= k_1(0)Y(t) + \int_0^\infty \Big[k_2(0)k(\tau) + \dot{k}(\tau)\Big]Y(t-\tau)\,d\tau \qquad (2.65)\\
\text{since } Y(t) &= \int_0^\infty (k_1(\tau) + k_2(\tau))Y(t-\tau)\,d\tau
\end{aligned}$$

Now, it is required that

Assumption 2.6.1.2

$$\int_0^\infty \tau\Big|k_2(0)k(\tau) + \dot{k}(\tau)\Big|\,d\tau < \infty \textit{ and } L_1 \equiv \int_0^\infty \Big[k_2(0)k(\tau) + \dot{k}_2(\tau)\Big]\,d\tau < 0$$

Then, I can apply the Functional 2.6.1.2 below on (2.65).

Functional 2.6.1.2

$$\begin{aligned}
v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\
\textit{where } v_1(Y)(t) &= \left(Y(t) + \int_0^\infty \Big[k_2(0)k(\tau) + \dot{k}(\tau)\Big]\int_{t-\tau}^t Y(u)\,du\,d\tau\right)^2 \\
\textit{and } v_2(Y)(t) &= -L_1\int_0^\infty |k_2(0)k(\tau) + \dot{k}(\tau)|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau
\end{aligned}$$

after doing some calculations,

$$\dot{v} \;\le\; 2Y(t)^2L_1L_2 \text{ where } L_2 = 1 - \int_0^\infty \tau|k_2(0)k(\tau) + \dot{k}(\tau)|\,d\tau \qquad (2.66)$$

Hence, this time, I have the conclusion as follow

Conclusion 2.6.1.2

If $L_1 < 0$ and $L_2 > 0$, the zero solution of (2.6) is asymptotically stable.

Again, it is possible to employ $k(\tau) = -2e^{-(3/2)\tau} + e^{-2\tau}$ like Example 2.6.1.1 and 2.6.1.2 as an example which Functional 2.5.1.1 does not apply but Functional 2.6.1.2 applies.

2.6.2 Lyapunov functionals by differentiating Y twice

In this section, I will introduce Functional 2.6.2.1 and 2.6.2.2 on (2.67) and (2.70) respectively to obtain the different conditions of the asymptotic stability of the zero solution of (2.7) and (2.6). Since Y needs to be differentiated twice, it should be assumed that $Y(t)$ is continuous and differentiable at least twice for $0 < t < \infty$ and k, $k_i \in C^2$. Moreover, since (2.7) is differentiated twice, I eventually have a second order functional differential equation((2.67) and (2.70) below). Now, I have shown that after differentiating (2.7) with respect to t once, I have

$$\begin{aligned}
\dot{Y}(t) &= k(0)Y(t) - k(M)Y(t-M) + \int_{t-M}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau \\
&= k(0)Y(t) - k(M)\int_{t-2M}^{t-M} k(t-\tau-M)Y(\tau)\,d\tau + \int_{t-M}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau \\
\text{since } Y(t-M) &= \int_{t-2M}^{t-M} k(t-\tau-M)Y(\tau)\,d\tau
\end{aligned}$$

Then, by differentiating the above with respect to t again,

$$\begin{aligned}
\ddot{Y}(t) &= \dot{k}(0)Y(t) - \dot{k}(M)Y(t-M) + \int_{t-M}^{t} \ddot{k}(t-\tau)Y(\tau)\,d\tau \\
&\quad + k(0)\left(k(0)Y(t) - k(M)Y(t-M) + \int_{t-M}^{t} \dot{k}(t-\tau)Y(\tau)\,d\tau\right) \\
&\quad - k(M)\left(k(0)Y(t-M) - k(M)Y(t-2M) + \int_{t-2M}^{t-M} \dot{k}(t-\tau-M)Y(\tau)\,d\tau\right)
\end{aligned}$$

Then, I get

$$\begin{aligned}
\ddot{Y}(t) &= \int_0^M \left[\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2)k(\tau) + k(0)\dot{k}(\tau)\right]Y(t-\tau)\,d\tau - 2k(M)k(0)Y(t-M) \\
&\quad - \int_0^M \left[\dot{k}(M)k(\tau) + k(M)\dot{k}(\tau)\right]Y(t-\tau-M)\,d\tau + k(M)^2Y(t-2M) \qquad (2.67) \\
\text{since } Y(t) &= \int_0^M k(\tau)Y(t-\tau)\,d\tau
\end{aligned}$$

Since it is more convenient to have first order functional differential equations rather than to have second order functional differential equations in order to construct Lyapunov functionals. So, I need to convert the above into a system of first order functional differential equations. So, by letting $\dot{Y}(t) = Y_1(t)$, I get $\ddot{Y}(t) = \dot{Y}_1(t)$. Hence,

$$\begin{aligned}
\dot{Y}(t) &= -k(M)Y(t-M) + \int_0^M \left[k(0)k(\tau) + \dot{k}(\tau)\right]Y(t-\tau)\,d\tau \\
\dot{Y}_1(t) &= \int_0^M \left[\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2)k(\tau) + k(0)\dot{k}(\tau)\right]Y(t-\tau)\,d\tau - 2k(M)k(0)Y(t-M) \\
&\quad - \int_0^M \left[\dot{k}(M)k(\tau) + k(M)\dot{k}(\tau)\right]Y(t-\tau-M)\,d\tau + k(M)^2Y(t-2M) \qquad (2.68)
\end{aligned}$$

Then, it is assumed that

Assumption 2.6.2.1

$$\int_0^M \tau|\ddot{k}(\tau)+(\dot{k}(0)+k(0)^2+k(0))k(\tau)+(k(0)+1)\dot{k}(\tau)|\,d\tau<\infty,$$

$$\int_0^M (M+\tau)|\dot{k}(M)k(\tau)+k(M)\dot{k}(\tau)|\,d\tau<\infty,\quad \int_0^M |k(0)k(\tau)+\dot{k}(\tau)|\,d\tau<\infty$$

$$L\equiv\int_0^M \Big[\Big(\dot{k}(0)+k(0)+k(0)^2-\dot{k}(M)\Big)k(\tau)+\Big(k(0)+1-k(M)\Big)\dot{k}(\tau)\Big]\,d\tau+k(M)[k(M)-2k(0)-1]<0$$

Then, let me apply the Functional 2.6.2.1 on (2.68) (or (2.67)) as follows.

Functional 2.6.2.1

$$\begin{aligned}
v(Y,Y_1)(t) &= v_1(Y,Y_1)(t)+v_2(Y,Y_1)(t)\\
v_1(Y,Y_1)(t) &= \Bigg(Y(t)+Y_1(t)-k(M)(2k(0)+1)\int_{t-M}^t Y(u)\,du\\
&\quad+\int_0^M\Big[\ddot{k}(\tau)+(\dot{k}(0)+k(0)^2+k(0))k(\tau)+(k(0)+1)\dot{k}(\tau)\Big]\int_{t-\tau}^t Y(u)\,du\,d\tau\\
&\quad-\int_0^M\Big[\dot{k}(M)k(\tau)+k(M)\dot{k}(\tau)\Big]\int_{t-\tau-M}^t Y(u)\,du\,d\tau+k(M)^2\int_{t-2M}^t Y(u)\,du\Bigg)^2\\
v_2(Y,Y_1)(t) &= -L\int_0^M|\dot{k}(M)k(\tau)+k(M)\dot{k}(\tau)|\int_{t-\tau-M}^t\int_u^t Y(w)^2\,dw\,du\,d\tau\\
&\quad-L\int_0^M|\ddot{k}(\tau)+(\dot{k}(0)+k(0)^2+k(0))k(\tau)+(k(0)+1)\dot{k}(\tau)|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau\\
&\quad-L|k(M)(2k(0)+1)|\int_{t-M}^t\int_u^t Y(w)^2\,dw\,du-L|k(M)|\int_{t-M}^t Y(u)^2\,du\\
&\quad-Lk(M)^2\int_{t-2M}^t\int_u^t Y(w)^2\,dw\,du-L\int_0^M\int_{t-\tau}^t|k(0)k(\tau)+\dot{k}(\tau)|Y(u)^2\,du\,d\tau
\end{aligned}$$

Note: I am using the fact below to construct the Functional 2.6.2.1.

$$Y_1(t)=\dot{Y}(t)=-k(M)Y(t-M)+\int_0^M(k(0)k(\tau)+\dot{k}(\tau))Y(t-\tau)\,d\tau$$

Then, after some calculation, I get

$$\begin{aligned}
\dot{v} &\le 2LL_1Y(t)^2 \qquad (2.69)\\
\text{where } L_1 &= 1-\int_0^M(M+\tau)|\dot{k}(M)k(\tau)+k(M)\dot{k}(\tau)|\,d\tau\\
&\quad-\int_0^M\tau|\ddot{k}(\tau)+(\dot{k}(0)+k(0)^2+k(0))k(\tau)+(k(0)+1)\dot{k}(\tau)|\,d\tau\\
&\quad-\int_0^M|k(0)k(\tau)+\dot{k}(\tau)|\,d\tau-|k(M)|-2Mk(M)^2-|k(M)(2k(0)+1)|M.
\end{aligned}$$

Hence, I can state the result as follows.

Conclusion 2.6.2.1

If $L_1>0$ and $L<0$, $\dot{v}<0$. So, the zero solution of (2.67) (or equivalently (2.7)) is asymptotically stable.

Now, let me introduce 2 special cases below (Example 2.6.2.1 and Example 2.6.2.2).

Example 2.6.2.1

When $k(\tau) = \frac{e^{-\tau}(\tau-1)}{(5+\tau e^{-\tau})^2}$ and $M = 30$,

$$\int_0^M (k(0)k(\tau) + \dot{k}(\tau) - k(M)k(\tau))\, d\tau \approx 0.04 > 0$$

So, the conditions required for Conclusion 2.5.2.1 are not satisfied and so Functional 2.5.2.1 is not useful to show the asymptotic stability. However, $L \approx -0.0576 < 0$ and $L_1 \approx 0.912$. So, the conditions required for Conclusion 2.6.2.1 are satisfied. Hence, when $k(\tau) = \frac{e^{-\tau}(\tau-1)}{(5+\tau e^{-\tau})^2}$ and $M = 30$, the zero solution of (2.7) is asymptotically stable.

This is the example showing that Functional 2.6.2.1 superiors to Functional 2.5.2.1. However, the next case (Example 2.6.2.2) shows that Functional 2.6.2.1 inferiors to Functional 2.5.2.1.

Example 2.6.2.2

When $k(\tau) = \frac{e^{-\tau}(1-\tau)}{(\tau e^{-\tau}-5)^2}$ and $M = 30$, $L \approx 0.022 > 0$. So, the conditions required for Conclusion 2.6.2.1 are not satisfied. However, since I have

$$\begin{aligned} \int_0^M (k(0)k(\tau) + \dot{k}(\tau) - k(M)k(\tau))\, d\tau &\approx -0.04 \\ 1 - \int_0^M (\tau|k(0)k(\tau) + \dot{k}(\tau)| + (M+\tau)|k(M)k(\tau)|)\, d\tau &\approx 0.954 \end{aligned}$$

the conditions required for Conclusion 2.5.2.1 are satisfied. So, zero solution of (2.7) is asymptotically stable.

These 2 examples (Example 2.6.2.1 and Example 2.6.2.2) show that it is not possible to say Functional 2.6.2.1 is sharper than Functional 2.5.2.1, conversely it is not possible to say Functional 2.5.2.1 is sharper than Functional 2.6.2.1.

Lyapunov functional on (2.70)

It is obvious that it is possible to construct similar functional (Functional 2.6.2.2) to obtain the conditions of the asymptotic stability of the zero solution of (2.6). Again, after differentiating Y with respect to t once, it is possible to obtain

$$\dot{Y}(t) = k(0)Y(t) + \int_{-\infty}^t \dot{k}(t-\tau)Y(\tau)\, d\tau$$

Then, by differentiating the above with respect to t again, I obtain

$$\begin{aligned} \ddot{Y}(t) &= \dot{k}(0)Y(t) + \int_{-\infty}^t \ddot{k}(t-\tau)Y(\tau)\, d\tau + k(0)\left(k(0)Y(t) + \int_{-\infty}^t \dot{k}(t-\tau)Y(\tau)\, d\tau\right) \\ &= \int_0^\infty (\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2)k(\tau) + k(0)\dot{k}(\tau))Y(t-\tau)\, d\tau \qquad (2.70) \\ \text{due to } Y(t) &= \int_0^\infty k(\tau)Y(t-\tau)\, d\tau \end{aligned}$$

Since it is more convenient to have first order functional differential equations rather than to have second order functional differential equations in order to construct Lyapunov functionals, I need to convert (2.70) into a system of functional differential equations. So, by letting $\dot{Y}(t) = Y_1(t)$, I get $\ddot{Y}(t) = \dot{Y}_1(t)$. Hence,

$$\begin{aligned} \dot{Y}(t) &= \int_0^\infty [k(0)k(\tau) + \dot{k}(\tau)]Y(t-\tau)\,d\tau \\ \dot{Y}_1(t) &= \int_0^\infty [\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2)k(\tau) + k(0)\dot{k}(\tau)]Y(t-\tau)\,d\tau \end{aligned} \tag{2.71}$$

Then, I should have the assumption like:

Assumption 2.6.2.2

$$\begin{aligned} \int_0^\infty |k(0)k(\tau) + \dot{k}(\tau)|\,d\tau &< \infty \\ \int_0^\infty \tau|\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) + (k(0)+1)\dot{k}(\tau)|\,d\tau &< \infty \\ L \equiv \int_0^\infty \Big[\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) + (k(0)+1)\dot{k}(\tau)\Big]\,d\tau &< 0 \end{aligned}$$

Then, I apply the Functional 2.6.2.2 on (2.71) (or (2.70)) as follows.

Functional 2.6.2.2

$$\begin{aligned} &v(Y,Y_1)(t) = v_1(Y,Y_1)(t) + v_2(Y,Y_1)(t) \\ &v_1(Y,Y_1)(t) = \Bigg(Y(t) + Y_1(t) \\ &\quad + \int_0^\infty \Big[\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) + (k(0)+1)\dot{k}(\tau)\Big] \int_{t-\tau}^t Y(u)\,du\,d\tau\Bigg)^2 \\ &v_2(Y,Y_1)(t) = -L\int_0^\infty \int_{t-\tau}^t |k(0)k(\tau) + \dot{k}(\tau)|Y(u)^2\,du\,d\tau \\ &\quad -L\int_0^\infty |\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) + (k(0)+1)\dot{k}(\tau)| \int_{t-\tau}^t \int_u^t Y(w)^2\,dw\,du\,d\tau \end{aligned}$$

Note: I am using the fact below to construct the Functional 2.6.2.2 above.

$$Y_1(t) = \dot{Y}(t) = \int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))Y(t-\tau)\,d\tau$$

then, after some calculation, I get

$$\begin{aligned} \dot{v} &\leq 2LL_1 Y(t)^2 \\ \text{where } L_1 &= 1 - \int_0^\infty |k(0)k(\tau) + \dot{k}(\tau)|\,d\tau \\ &\quad - \int_0^\infty \tau|\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) + (k(0)+1)\dot{k}(\tau)|\,d\tau \end{aligned} \tag{2.72}$$

Hence, I can state the result as follows.

Conclusion 2.6.2.2

If $L_1 > 0$ and $L < 0$, $\dot{v} < 0$. So, the zero solution of (2.6) (or (2.70)) is asymptotically stable.

Again by using $k(\tau) = \frac{e^{-\tau}(\tau-1)}{(5+\tau e^{-\tau})^2}$ or $k(\tau) = \frac{e^{-\tau}(1-\tau)}{(\tau e^{-\tau}-5)^2}$ as in Example 2.6.2.1 and Example 2.6.2.2, it is possible to have the same arguments between Functional 2.5.1.1 and Functional 2.6.2.2 as the ones between Functional 2.5.2.1 and Functional 2.6.2.1 which I discussed in the above. The conclusion is that there are no way to find the superiority of the one to the other. In addition, it is possible to construct the functionals like Functional 2.6.2.1, 2.6.1.1, 2.6.2.2 and 2.6.1.2 to all types of (2.1) and (2.2). Then, by using the functionals, it will be possible to have the different stability conditions of the zero solution of them from the ones obtained in Section 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7 and 2.5.8.

2.7 On the Lyapunov functionals II

In this section, I will discuss Functional 2.7.1.1 on (2.74) to obtain the different conditions of the stability of the zero solution of (2.41), namely,

$$Y(t) = \int_0^M k(\tau)Y(t-\tau)\,d\tau + \int_0^M k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau$$

and I will discuss Functional 2.7.2.1 on (2.77) to obtain the different conditions of the stability of the zero solution of (2.35), namely,

$$Y(t) = \int_0^\infty k(\tau)Y(t-\tau)\,d\tau + \int_0^\infty k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau$$

Note:

1. As I mentioned in Section 2.1, the functionals like Functional 2.7.1.1 and Functional 2.7.2.1 cannot be constructed to (2.1) and (2.2) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1$.

2. I have already introduced Functional 2.5.3.1 on (2.39) in Section 2.5.3 and Functional 2.5.4.1 on (2.44) in Section 2.5.4. However, the functionals I will discuss here need different conditions that the derivative of the functionals become less than zero. Hence, it is possible to obtain the different conditions of the asymptotic stability of the zero solution of (2.1) and (2.2) when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^t k_2(t-t_1)Y(t_1)\,dt_1$, from those made by using Functional 2.5.3.1 and Functional 2.5.4.1.

2.7.1 Functional 2.7.1.1 on (2.74)

In Section 2.5.4, I have shown that by using the Functional 2.5.4.1, the zero solution of (2.44) (or equivalently (2.41)) is asymptotically stable if the following conditions are satisfied.

$$L \equiv \int_0^M \left(k(0)k(\tau) + \dot{k}(\tau) + k(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)d\tau - k(M)\int_0^M \left(k_1(\tau)\tau + k(\tau)\right)d\tau$$

$$
\begin{aligned}
&+\int_0^M \left(k(0)k_1(\tau)+\dot{k}_1(\tau)+k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right)\tau\,d\tau - k_1(M)M \\
< & \ 0 \\
L_1 \ = & \left(1-\int_0^M \tau\left|k(0)k(\tau)+\dot{k}(\tau)+k(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right|d\tau\right. \\
& -\int_0^M \frac{1}{2}\tau^2\left|k(0)k_1(\tau)+\dot{k}_1(\tau)+k_1(\tau)\int_0^M k_1(\tau_1)\,d\tau_1\right|d\tau - \frac{1}{2}|k_1(M)|M^2 \\
& \left.-\int_0^M |k(M)k(\tau)|\Big(M+\tau\Big)\,d\tau - \int_0^M |k(M)k_1(\tau)|\left(\frac{1}{2}\tau^2+M\tau\right)d\tau\right) \\
> & \ 0
\end{aligned}
$$

However, in general, the Lyapunov functionals give only sufficient conditions, so, it is possible that even if at least one of the above conditions is not satisfied, the zero solution of (2.41) is asymptotically stable. So, it is worth constructing a functional which gives different kind of conditions to determine the stability of the zero solution of (2.41). Then, firstly, I have to show how to reduce (2.74) from (2.41). By letting $\tau \Rightarrow t-\tau$, it is possible to have

$$
\int_0^M k(\tau)Y(t-\tau)\,d\tau = \int_{t-M}^t k(t-\tau)Y(\tau)\,d\tau
$$

and as I told in the beginning of this chapter, it is possible to have

$$
\frac{d}{dt}\int_0^M k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau = \int_0^M k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau
$$

So, (2.41) is convertible to a functional differential equation as follows:

$$
\dot{Y}(t) = k(0)Y(t)+\int_{t-M}^t \dot{k}(t-\tau)Y(\tau)\,d\tau - k(M)Y(t-M)+\int_0^M k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau
$$

Then, by letting $t-\tau \Rightarrow \tau$,

$$
\begin{aligned}
\dot{Y}(t) &= k(0)Y(t)+\int_0^M \dot{k}(\tau)Y(t-\tau)\,d\tau - k(M)Y(t-M) \\
&\quad +\int_0^M k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau
\end{aligned} \tag{2.73}
$$

Then, (2.73) will be equivalent to

$$
\begin{aligned}
\dot{Y}(t) &= \int_0^M \Big[\dot{k}(\tau)+k(0)k(\tau)\Big]Y(t-\tau)\,d\tau + \int_0^M k(0)k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau \\
&\quad -k(M)Y(t-M)+\int_0^M k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau \\
\text{since } Y(t) &= \int_0^M k(\tau)Y(t-\tau)\,d\tau + \int_0^M k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau
\end{aligned}
$$

Then, by letting $t_2 = t-t_1$, I have

$$
dt_2 = -dt_1,\ t_1 = t-\tau \Rightarrow t_2 = \tau \text{ and } t_1 = t \Rightarrow t_2 = 0
$$

Hence, the above equation will be equivalent to

$$\begin{aligned}\dot{Y}(t) &= \int_0^M \left[\dot{k}(\tau) + k(0)k(\tau)\right] Y(t-\tau)\,d\tau - k(M)Y(t-M) \\ &\quad + \int_0^M k(0)k_1(\tau)\int_0^\tau Y(t-t_2)\,dt_2\,d\tau + \int_0^M k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau \qquad (2.74)\end{aligned}$$

Next, let me state the assumptions as follows.

Assumption 2.7.1.1

$$\int_0^M \tau\left|k(0)k(\tau) + \dot{k}(\tau) - k_1(\tau)\right| d\tau < \infty, \ \int_0^M \frac{1}{2}\tau^2\left|k(0)k_1(\tau)\right| d\tau < \infty$$

$$\textit{and} \ \ L_2 \ \equiv \ \int_0^M \left[k(0)k(\tau) + \dot{k}(\tau) + \tau k(0)k_1(\tau)\right] d\tau - k(M) < 0$$

Then, I apply the Functional 2.7.1.1 on (2.74) as follows to obtain the conditions of the stability of the zero solution of (2.74).

Functional 2.7.1.1

$$\begin{aligned}v(Y)(t) &= v_1(Y)(t) + v_2(Y)(t) \\ \textit{where } v_1(Y)(t) &= \Bigg(Y(t) + \int_0^M \left[k(0)k(\tau) + \dot{k}(\tau) - k_1(\tau)\right]\int_{t-\tau}^t Y(u)\,du\,d\tau \\ &\quad + \int_0^M k(0)k_1(\tau)\int_0^\tau\int_{t-t_2}^t Y(u)\,du\,dt_2\,d\tau - k(M)\int_{t-M}^t Y(u)\,du\Bigg)^2 \\ \textit{and } v_2(Y)(t) &= -L_2\int_0^M \left|k(0)k_1(\tau)\right|\int_0^\tau\int_{t-t_2}^t\int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau \\ &\quad - L_2\int_0^M \left|k(0)k(\tau) + \dot{k}(\tau) - k_1(\tau)\right|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau \\ &\quad - L_2|k(M)|\int_{t-M}^t\int_u^t Y(w)^2\,dw\,du\end{aligned}$$

After some calculation, I get

$$\begin{aligned}\dot{v} &\leq 2L_2L_3Y(t)^2 \qquad (2.75) \\ \text{where } L_3 &= 1 - \int_0^M \tau\left|k(0)k(\tau) + \dot{k}(\tau) - k_1(\tau)\right| d\tau - \int_0^M \frac{1}{2}\tau^2\left|k(0)k_1(\tau)\right| d\tau - M|k(M)|\end{aligned}$$

Finally, I state the result as follows.

Conclusion 2.7.1.1

If $L_3 > 0$ and $L_2 < 0$, I can conclude that $\dot{v} < 0$. So, the zero solution of (2.41) (or equivalently (2.74)) is asymptotically stable.

Now, let me try Example 2.7.1.1 below, then,

Example 2.7.1.1

When $k(\tau) = \frac{4}{5}e^{-2\tau}$, $k_1(\tau) = -2e^{-2\tau}$ and $M = 40$, $L \approx 0.22 > 0$, $L_2 \approx -0.88 < 0$ and $L_3 \approx 0.54 > 0$.

So, the conditions required for Conclusion 2.7.1.1 are satisfied. Hence, the zero solution of (2.74) (or (2.43)) is asymptotically stable, when $k(\tau) = \frac{4}{5}e^{-2\tau}$, $k_1(\tau) = -2e^{-2\tau}$. Since $L \approx 0.22 > 0$, this is the case Functional 2.5.4.1 does not apply but Functional 2.7.1.1 does. However, if I try Example 2.7.1.2 below

Example 2.7.1.2

When $k(\tau) = \frac{5}{2}e^{-4\tau}$, $k_1(\tau) = \frac{1}{5}(\tau+1)e^{-\frac{7}{5}\tau}$ and $M = 40$, $L \approx -0.3491 < 0$, $L_1 \approx 0.423 > 0$ and $L_3 \approx -0.0548 < 0$.

So, the conditions required for Conclusion 2.5.4.1 are satisfied. Hence, the zero solution of (2.74) (or (2.41)) is asymptotically stable, when $k(\tau) = \frac{5}{2}e^{-4\tau}$ $k_1(\tau) = \frac{1}{5}(\tau+1)e^{-\frac{7}{5}\tau}$. This is the case Functional 2.7.1.1 does not apply but Functional 2.5.4.1 does. Therefore, as I discussed about Functional 2.5.2.1 and Functional 2.6.2.1 in Section 2.6.2, it is not possible to say that one is sharper than the other.

2.7.2 Functional 2.7.2.1 on (2.77)

In this section, I will construct another functional (Functional (2.7.2.1)) on (2.77) or equivalently (2.35) in order to gain another condition for the stability of zero solution of (2.35). Similarly to the above, I reduce (2.77) from (2.35). By letting $\tau \Rightarrow t-\tau$, it is possible to have

$$\int_0^\infty k(\tau)Y(t-\tau)\,d\tau \;=\; \int_{-\infty}^t k(t-\tau)Y(\tau)\,d\tau$$

and it is possible to have

$$\frac{d}{dt}\int_0^\infty k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau \;=\; \int_0^\infty k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau$$

So, (2.35) is convertible to a functional differential equation as follows:

$$\dot{Y}(t) \;=\; k(0)Y(t) + \int_{-\infty}^t \dot{k}(t-\tau)Y(\tau)\,d\tau + \int_0^\infty k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau$$

Then, by letting $t-\tau \Rightarrow \tau$,

$$\dot{Y}(t) \;=\; k(0)Y(t) + \int_0^\infty \dot{k}(\tau)Y(t-\tau)\,d\tau + \int_0^\infty k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau \tag{2.76}$$

Then, (2.76) will be equivalent to

$$\begin{aligned}\dot{Y}(t) \;&=\; \int_0^\infty \Big[\dot{k}(\tau)+k(0)k(\tau)\Big]Y(t-\tau)\,d\tau + \int_0^\infty k(0)k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau \\ &\quad+\int_0^\infty k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau \\ \text{since } Y(t) \;&=\; \int_0^\infty k(\tau)Y(t-\tau)\,d\tau + \int_0^\infty k_1(\tau)\int_{t-\tau}^t Y(t_1)\,dt_1\,d\tau\end{aligned}$$

Then, by letting $t_2 = t - t_1$, I have

$$dt_2 = -dt_1,\; t_1 = t-\tau \Rightarrow t_2 = \tau \text{ and } t_1 = t \Rightarrow t_2 = 0$$

The above equation will be equivalent to

$$\begin{aligned}\dot{Y}(t) &= \int_0^\infty \Big[\dot{k}(\tau)+k(0)k(\tau)\Big]Y(t-\tau)\,d\tau+\int_0^\infty k(0)k_1(\tau)\int_0^\tau Y(t-t_2)\,dt_2\,d\tau \\ &\quad +\int_0^\infty k_1(\tau)[Y(t)-Y(t-\tau)]\,d\tau \end{aligned} \tag{2.77}$$

This time, the assumption below, namely,

Assumption 2.7.2.1

$$\int_0^\infty \tau\Big|k(0)k(\tau)+\dot{k}(\tau)-k_1(\tau)\Big|\,d\tau<\infty,\ \int_0^\infty \frac{1}{2}\tau^2\Big|k(0)k_1(\tau)\Big|\,d\tau<\infty$$

$$\textit{and}\ L_2 \equiv \int_0^\infty \Big[k(0)k(\tau)+\dot{k}(\tau)+\tau k(0)k_1(\tau)\Big]\,d\tau<0$$

is required to apply the Functional 2.7.2.1 on (2.77) as follows to obtain the conditions of the stability of the zero solution of (2.77).

Functional 2.7.2.1

$$\begin{aligned} v(Y)(t) &= v_1(Y)(t)+v_2(Y)(t) \\ \textit{where}\ v_1(Y)(t) &= \Bigg(Y(t)+\int_0^\infty \Big[k(0)k(\tau)+\dot{k}(\tau)-k_1(\tau)\Big]\int_{t-\tau}^t Y(u)\,du\,d\tau \\ &\quad +\int_0^\infty k(0)k_1(\tau)\int_0^\tau\int_{t-t_2}^t Y(u)\,du\,dt_2\,d\tau\Bigg)^2 \\ \textit{and}\ v_2(Y)(t) &= -L_2\int_0^\infty \Big|k(0)k_1(\tau)\Big|\int_0^\tau\int_{t-t_2}^t\int_u^t Y(w)^2\,dw\,du\,dt_2\,d\tau \\ &\quad -L_2\int_0^\infty \Big|k(0)k(\tau)+\dot{k}(\tau)-k_1(\tau)\Big|\int_{t-\tau}^t\int_u^t Y(w)^2\,dw\,du\,d\tau \end{aligned}$$

After some calculation, I get

$$\begin{aligned} \dot{v} &\le 2L_2L_3Y(t)^2 \\ \text{where } L_3 &= 1-\int_0^\infty \tau\Big|k(0)k(\tau)+\dot{k}(\tau)-k_1(\tau)\Big|\,d\tau-\int_0^\infty\frac{1}{2}\tau^2\Big|k(0)k_1(\tau)\Big|\,d\tau \end{aligned} \tag{2.78}$$

Finally, I state the result as follows.

Conclusion 2.7.2.1

If $L_3>0$ and $L_2<0$, I can conclude that $\dot{v}<0$. So, the zero solution of (2.35) (or equivalently (2.77)) is asymptotically stable.

Remark : If I use $k(\tau)=\frac{4}{5}e^{-2\tau}$, $k_1(\tau)=-2e^{-2\tau}$ as in Example 2.7.1.1, I will be able to prove the stability of the zero solution of (2.35) by using Functional 2.7.2.1 but I will not by using Functional 2.5.3.1. If I use $k(\tau)=\frac{5}{2}e^{-4\tau}$, $k_1(\tau)=\frac{1}{5}(\tau+1)e^{-\frac{7}{5}\tau}$ as in Example 2.7.1.2, I will be able to prove the stability of the zero solution of (2.35) by using Functional 2.5.3.1 but I will not by using Functional 2.7.2.1. So, again, I fail to show which functional is sharper.

Note : k, $k_1 \in C^1$ is required to apply Functional 2.5.4.1 and Functional 2.5.3.1 but $k \in C^1$ is only requirement to apply Functional 2.7.1.1 and Functional 2.7.2.1. As you can see in the above, they never depend on $\dot{k}_1$. This could be another advantage of using Functional 2.7.1.1 Functional 2.7.2.1 rather than using Functional 2.5.4.1 Functional 2.5.3.1.

2.8 On the stability analysis

There is another way of obtaining the stability of the steady state of (2.1) and (2.2) when

$$\begin{aligned}
k(\cdot)Y(\cdot) &= k(\tau)Y(t-\tau), \\
k(\cdot)Y(\cdot) &= k(\tau)Y(t-\tau) + k_1(\tau)Y\left(t - \frac{1}{2}\tau\right) + k_2(\tau)Y(t) \text{ or} \\
k(\cdot)Y(\cdot) &= k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^{t} Y(t_1)\,dt_1
\end{aligned}$$

In those cases, you can use the fact the above equations have solutions canonically of the form e^{lt} and we determine the stability by the sign of $Re(l)$. Here is the advantage of using this way rather than using Lyapunov functionals:

- You can see the condition of the instability as well as the stability.

However, there is a disadvantage of using it:

- The characteristic equation of l is not always easy to solve.

If you use the Lyapunov functionals, here is the advantage:

- You can see the condition of the stability relatively easier since all we have to do is to evaluate the integral of the known functions (see Conclusion 2.5.1.1, Conclusion 2.5.2.1...etc, in previous Sections).

However, here is the disadvantage:

- You can find only the sufficient condition of the stability and cannot see anything about the condition for the instability.

Chapter 3

Lyapunov functionals to nonlinear equations

In this chapter, I will obtain the global stability of zero solutions of special forms of (1.3) and (1.4) when $F = F(\tau, N(t-\tau))$ by using the results of Burton [12].

3.1 When the delay is finite

Burton [12] has studied the asymptotic stability of the equations below.

$$x(t) = a(t) - \int_{t-h}^{t} D(t,s)g(x(s))\,ds \tag{3.1}$$

Note:

- When $a(t) = 0$ and $D(t,s) = D(t-s)$, (3.1) will become

  $$x(t) = -\int_{t-h}^{t} D(t-s)g(x(s))\,ds$$

  Then, it is possible to rewrite the above

  $$N(t) = -\int_{t-M}^{t} D(t-\tau)g(N(\tau))\,d\tau$$

  and by letting $t - \tau \longrightarrow \tau$,

  $$N(t) = -\int_{0}^{M} D(\tau)g(N(t-\tau))\,d\tau \tag{3.2}$$

  This can be regarded as a special form of (1.4) when $F = F(\tau, N(t-\tau))$.

Then, he made assumptions as follows:

Assumption 3.1.0.2

$D : \Re \times \Re \longrightarrow \Re$, $D_s(t,s) \geq 0$, $D_{st}(t,s) \leq 0$ and $D(t, t-h) = 0$. It is also supposed that there is a bounded continuous initial function $\phi : [t_0 - h, t_0] \longrightarrow \Re$, where t_0 is the initial value and $a(t) \in L^1[0,\infty)$ is bounded.

Then, the Functional 3.1.0.2 below

Functional 3.1.0.2

$$H(t,x(\cdot)) = \frac{1}{2}\int_{t-h}^{t} D_s(t,s)\left(\int_s^t g(x(v))\,dv\right)^2 ds$$

satisfies

$$2H(t,x(\cdot))\int_{t-h}^{t} D_s(t,s)\,ds \geq (a(t)-x(t))^2$$

and there are positive constants c_1 and M with

$$H'(t,x(\cdot)) \leq -c_1 x(t)g(x(t)) + M|a(t)|$$

Hence,

$$\int_{t_0}^{\infty} x(t)g(x(t))\,dt < \infty$$

and if $\int_{t-h}^{t} D_s(t,s)\,ds$ is bounded then $x(t)$ is bounded. If, in addition, $\int_{t-h}^{t} D_s(t,s)(t-s)\,ds$ is bounded, and if for each $M > 0$ there is a $J > 0$ such that $|x| \leq M$ implies that $|g(x)| \leq J|x|$, then, $x(t) \longrightarrow a(t)$ as $t \longrightarrow \infty$.

Proof given by Burton

$$\begin{aligned} H'(t,x(\cdot)) &= \frac{1}{2}\int_{t-h}^{t} D_{st}(t,s)\left(\int_s^t g(x(v))\,dv\right)^2 ds \\ &\quad -\frac{1}{2}D_s(t,t-h)\left(\int_{t-h}^{t} g(x(v))\,dv\right)^2 \\ &\quad +\int_{t-h}^{t} D_s(t,s)g(x(t))\int_s^t g(x(v))\,dv\,ds \end{aligned}$$

Then,

$$\begin{aligned} &\int_{t-h}^{t} D_s(t,s)g(x(t))\int_s^t g(x(v))\,dv\,ds \\ &= g(x(t))\left[D(t,s)\int_s^t g(x(v))\,dv\Bigg|_{s=t-h}^{s=t} + \int_{t-h}^{t} D(t,s)g(x(s))\,ds\right] \\ &= g(x(t))\int_{t-h}^{t} D(t,s)g(x(s))\,ds \end{aligned}$$

So,

$$\begin{aligned} H'(t,x(\cdot)) &\leq g(x(t))\int_{t-h}^{t} D(t,s)g(x(s))\,ds \\ &= g(x(t))[a(t)-x(t)] \end{aligned} \tag{3.3}$$

By the boundedness of $a(t)$, it is possible to find positive constants c_1 and M such that

$$H'(t,x(\cdot)) \leq -c_1 x(t)g(x(t)) + M|a(t)|$$

To obtain the lower bound on H, by Schwarz inequality,

$$\begin{aligned}
&\left(\int_{t-h}^{t} D_s(t,s)\int_s^t g(x(v))\,dv\,ds\right)^2 \\
&= \left(\int_{t-h}^{t} \sqrt{D_s(t,s)}\sqrt{D_s(t,s)}\int_s^t g(x(v))\,dv\,ds\right)^2 \\
&\leq \int_{t-h}^{t} D_s(t,s)\,ds \int_{t-h}^{t} D_s(t,s)\left(\int_s^t g(x(v))\right)^2 dv\,ds \\
&= 2H(t,x(\cdot))\int_{t-h}^{t} D_s(t,s)\,ds
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\int_{t-h}^{t} D_s(t,s)\int_s^t g(x(v))\,dv\,ds\right)^2 \\
&= \left(D(t,s)\int_s^t g(x(v))\,dv\Big|_{s=t-h}^{s=t} + \int_{t-h}^{t} D(t,s)g(x(s))\,ds\right)^2 \\
&= (a(t)-x(t))^2
\end{aligned}$$

it is possible to have

$$2H(t,x(\cdot))\int_{t-h}^{t} D_s(t,s)\,ds \;\geq\; (a(t)-x(t))^2$$

From (3.3), an integration yields H bounded, which, together with $a(t)$ bounded, yields $x(t)$ bounded by considering the lower bound on H. Thus, $|g(x)| \leq J|x(t)|$ for some $J > 0$. Hence,

$$\int_{t_0}^{\infty} x(t)g(x(t))\,dt < \infty \text{ yields } \int_{t_0}^{\infty} g(x(t))^2\,dt < \infty$$

This means that $\int_{t-h}^{t} g(x(v))^2\,dv \longrightarrow 0$ as $t \longrightarrow \infty$, which he now uses. He has (by Schwarz's inequality)

$$\begin{aligned}
H(t,x(\cdot)) &\leq \int_{t-h}^{t} D_s(t,s)(t-s)\int_s^t g(x(v))^2\,dv\,ds \\
&\leq \int_{t-h}^{t} D_s(t,s)(t-s)\,ds\int_{t-h}^{t} g(x(v))^2\,dv
\end{aligned}$$

which tends to zero since $\int_{t-h}^{t} D_s(t,s)(t-s)\,ds$ is bounded. This yields the conclusion.

Comment 3.1.0.1

- The above results require $D(t,s)$ which has stricted conditions such that $D: \Re \times \Re \longrightarrow \Re$, $D_s(t,s) \geq 0$, $D_{st}(t,s) \leq 0$ and $D(t,t-h) = 0$.

- When $a(t) = 0$, $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$. So, when $a(t) = 0$, it is possible to conclude that the zero solution of (3.1) is globally asymptotically stable.

- As I mentioned in the beginning of this section, when $a(t) = 0$ and $D(t, s) = D(t-s)$, (3.2) can be regarded as a special form of (1.4) when $F = F(\tau, N(t-\tau))$. Hence, I can have the conclusion as follows under Assumption 3.1.0.3 (I just rewrite Assumption 3.1.0.2 to Assumption 3.1.0.3).

Assumption 3.1.0.3

$D : \Re \times \Re \longrightarrow \Re$, $D_\tau(t-\tau) \geq 0$, $D_{\tau t}(t-\tau) \leq 0$ and $D(t, t-M) = 0$. It is also supposed that there is a bounded continuous initial function $\phi : (-\infty, t_0] \longrightarrow \Re$, where t_0 is the initial value.

Conclusion 3.1.0.2

Under Assumption 3.1.0.3, the zero solution of (3.2) is globally stable.

3.2 When the delay is infinite

Next, it is possible to have the same result to the case of infinite delay, which is

$$x(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(x(s))\,ds \tag{3.4}$$

Note :

- Again, when $a(t) = 0$ and $D(t, s) = D(t-s)$, by having the same process as in the above section, (3.4) will become

$$N(t) = -\int_{0}^{\infty} D(\tau)g(N(t-\tau))\,d\tau \tag{3.5}$$

  This can be regarded as a special form of (1.3) when $F = F(\tau, N(t-\tau))$.

Then, it is required slightly different assumptions from the above case, which is:

Assumption 3.2.0.4

$D : \Re \times \Re \longrightarrow \Re$, $D_s(t, s) \geq 0$, $D_{st}(t, s) \leq 0$ and $D(t, -\infty) = 0$. It is also supposed that there is a bounded continuous initial function $\phi : (-\infty, t_0] \longrightarrow \Re$, where t_0 is the initial value and $a(t) \in L^1[0, \infty)$ and is bounded.

Then, the Functional 3.2.0.3 below

Functional 3.2.0.3

$$H(t, x(\cdot)) = \frac{1}{2}\int_{-\infty}^{t} D_s(t,s)\left(\int_{s}^{t} g(x(v))\,dv\right)^2 ds$$

satisfies

$$2H(t, x(\cdot))\int_{-\infty}^{t} D_s(t,s)\,ds \geq (a(t) - x(t))^2$$

and there are positive constants c_1 and M with

$$H'(t, x(\cdot)) \leq -c_1 x(t) g(x(t)) + M|a(t)|$$

Hence,

$$\int_{t_0}^{\infty} x(t) g(x(t))\, dt < \infty$$

and if $\int_{-\infty}^{t} D_s(t, s)\, ds$ is bounded then $x(t)$ is bounded. If, in addition, $\int_{-\infty}^{t} D_s(t, s)(t-s)\, ds$ is bounded, and if for each $M > 0$ there is a $J > 0$ such that $|x| \leq M$ implies that $|g(x)| \leq J|x|$, then, $x(t) \longrightarrow a(t)$ as $t \longrightarrow \infty$. The proof is similar to those of (3.1).

Comment 3.2.0.2

- Again, the above results require $D(t, s)$ which has stricted conditions such that $D : \Re \times \Re \longrightarrow \Re$, $D_s(t, s) \geq 0$, $D_{st}(t, s) \leq 0$ and $D(t, -\infty) = 0$. The one of the case will be $D(t, s) = e^{-a(t-s)}$ for $a > 0$. This is obvious that $D(t, -\infty) = e^{-a(\infty)} = 0$, $D_s(t, s) = ae^{-a(t-s)} \geq 0$ and $D_{st}(t, s) = -a^2 e^{-a(t-s)} \leq 0$.

- When $a(t) = 0$, $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$. So, it is possible to conclude that the zero solution of (3.4) is globally asymptotically stable.

- As I mentioned in the beginning of this section, when $a(t) = 0$ and $D(t, s) = D(t-s)$, (3.5) can be regarded as a special form of (1.3) when $F = F(\tau, N(t-\tau))$. Hence, I can have the conclusion as follows under Assumption 3.2.0.5 (I just rewrite Assumption 3.2.0.4 to Assumption 3.2.0.5).

Assumption 3.2.0.5

$D : \Re \times \Re \longrightarrow \Re$, $D_\tau(t-\tau) \geq 0$, $D_{\tau t}(t-\tau) \leq 0$ and $D(t, -\infty) = 0$. It is also supposed that there is a bounded continuous initial function $\phi : (-\infty, t_0] \longrightarrow \Re$, where t_0 is the initial value.

Conclusion 3.2.0.3

Under Assumption 3.2.0.5, the zero solution of (3.5) is globally stable.

Chapter 4

Nonlinear equations with distributed delay

In this chapter, I will discuss the equations (1.3) and (1.4), namely,

$$N(t) = \int_0^{\infty} FBN(t-\tau)\,d\tau \text{ and } N(t) = \int_0^{M} FBN(t-\tau)\,d\tau, \;\; M \in \Re^+$$

$$\text{where } F = F(\tau, N(t-\tau)),\; F = F\left(\tau, N(t-\tau), N\left(t - \frac{1}{2}\tau\right), N(t)\right) \text{ or}$$

$$F = F\left(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1\right) \text{ and } \; B = B(N(t-\tau))$$

and I will deal with the topics as follows:

1. I make assumptions of (1.3) and (1.4).
2. I will discuss the steady states of (1.3) and (1.4).
3. I will discuss the local stability of the steady states of (1.3) and (1.4).
4. I will discuss an example of (1.3) which is (4.17), namely,

$$N(t) \;\; = \;\; \int_0^{\infty} e^{-q(\tau+cN(t-\tau))} r(1 - be^{-p\tau}N(t-\tau))N(t-\tau)\,d\tau$$

   (I will discuss the local stability of the steady states of (4.17) and the construction of the 2π periodic solution of (4.17)).

5. I will discuss the numerical solution of (4.17).

4.1 The assumptions of (1.3) and (1.4)

I made assumptions of (1.3) and (1.4) as follows.

Assumption 4.1.0.6

1. Without loss of generality $t = 0$ can be fixed as a reference time and it is possible to suppose that given the function $N_0(t)$, $-\infty < t \leq 0$, such that $N(t) = N_0(t)$ for $-\infty < t \leq 0$.

2. $N(t)$ is continuous and differentiable for $-\infty < t < \infty$.

3. If only equations (1.4) is considered, it should be possible to be given the function $N_0(t)$ for $-M < t \leq 0$, such that $N(t) = N_0(t)$ for $-M < t \leq 0$.

4. It is also assumed that

$$B \neq \frac{\alpha_2}{N(t-\tau)} + F_2(N(t-\tau)) \text{ where } \alpha_2 \in \Re.$$

and only when $F = F(\tau, N(t-\tau))$ and $F = F(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$

$$F \neq \frac{\alpha_1}{N(t-\tau)} + F_1(\cdot)$$

where $\alpha_1 \in \Re$ and $F_1(\cdot) = F_1(\tau, N(t-\tau))$ or $F_1(\cdot) = F_1(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$.

4.2 The steady states of (1.3) and (1.4)

Let $N^{(s)}$ be the steady state of (1.3) and (1.4). Then, I can have

$$N^{(s)} = \int_0^\infty F_{N^{(s)}} B_{N^{(s)}} N^{(s)} \, d\tau \tag{4.1}$$

$$N^{(s)} = \int_0^M F_{N^{(s)}} B_{N^{(s)}} N^{(s)} \, d\tau \tag{4.2}$$

where $F_{N^{(s)}} = F(\tau, N^{(s)})$, $F_{N^{(s)}} = F(\tau, N^{(s)}, N^{(s)}, N^{(s)})$ or $F_{N^{(s)}} = F(\tau, N^{(s)} \int_{t-\tau}^t k_2(t-t_1)\, dt_1)$ and $B_{N^{(s)}} = B(N^{(s)})$. Due to the structure of the equations and Assumption 4.1.0.6, $N^{(s)} = 0$ is always a steady state and if $N^{(s)} \neq 0$, I have

$$1 = \int_0^\infty F_{N^{(s)}} B_{N^{(s)}} \, d\tau \text{ and } 1 = \int_0^M F_{N^{(s)}} B_{N^{(s)}} \, d\tau$$

So, if it is assumed that $|\int_0^\infty F_{N^{(s)}} B_{N^{(s)}} \, d\tau| < \infty$ and $|\int_0^M F_{N^{(s)}} B_{N^{(s)}} \, d\tau| < \infty$, it is possible to obtain $N^{(s)}$ such that $N^{(s)} \neq 0$. However, since these equations are made to apply population biology, it is important to find the condition that the above 2 equations have the positive solution. Now,

$$f_1(N^{(s)}) \equiv \int_0^\infty F_{N^{(s)}} B_{N^{(s)}} \, d\tau - 1 = 0 \text{ and } f_2(N^{(s)}) \equiv \int_0^M F_{N^{(s)}} B_{N^{(s)}} \, d\tau - 1 = 0$$

I suppose that $f_1(0) > 0$ and $f_2(0) > 0$. If $f_1(N^{(s)}) \to E < 0$ and $f_2(N^{(s)}) \to E_1 < 0$ as $N^{(s)} \to \infty$, then, it is clear that $f_1(N^{(s)})$ and $f_2(N^{(s)})$ cross the line $N^{(s)} = 0$ at least once but not even number of times. Hence, I can have odd number of positive solutions of $f_1(N^{(s)}) = 0$ and $f_2(N^{(s)}) = 0$. If $f_1(N^{(s)}) \to E > 0$ and $f_2(N^{(s)}) \to E_1 > 0$ as $N^{(s)} \to \infty$ and if there exists P such that $f_1(P) < 0$

and if there exists P_1 such that $f_2(P_1) < 0$ $(P,\ P_1 > 0)$, it is clear that $f_1(N^{(s)})$ and $f_2(N^{(s)})$ cross the line $N^{(s)} = 0$ at least twice but not odd number of times. Hence, I can have even number of positive solutions of $f_1(N^{(s)}) = 0$ and $f_2(N^{(s)}) = 0$. Then, by having an assumption as follows

Assumption 4.2.0.7

- $f_1(0) > 0$ and $f_2(0) > 0$

it is possible to have the conclusion as follows:

Conclusion 4.2.0.4

- *If $f_1(N^{(s)}) \to E < 0$ and if $f_2(N^{(s)}) \to E_1 < 0$ as $N^{(s)} \to \infty$, I can have odd number of positive solutions of $f_1(N^{(s)}) = 0$ and $f_2(N^{(s)}) = 0$.*

- *If $f_1(N^{(s)}) \to E > 0$ and if $f_2(N^{(s)}) \to E_1 > 0$ as $N^{(s)} \to \infty$ and if there exists P such that $f_1(P) < 0$ and if there exists P_1 such that $f_2(P_1) < 0$ $(P,\ P_1 > 0)$, I can have even number of positive solutions of $f_1(N^{(s)}) = 0$ and $f_2(N^{(s)}) = 0$.*

Similarly, by having the assumption as follows

Assumption 4.2.0.8

- $f_1(0) < 0$ and $f_2(0) < 0$

it is possible to have the similar conclusion as follows.

Conclusion 4.2.0.5

- *If $f_1(N^{(s)}) \to E > 0$ and if $f_2(N^{(s)}) \to E_1 > 0$ as $N^{(s)} \to \infty$, I can have odd number of positive solutions of $f_1(N^{(s)}) = 0$ and $f_2(N^{(s)}) = 0$.*

- *If $f_1(N^{(s)}) \to E < 0$ and if $f_2(N^{(s)}) \to E_1 < 0$ as $N^{(s)} \to \infty$ and if there exists P such that $f_1(P) > 0$ and if there exists P_1 such that $f_2(P_1) > 0$ $(P,\ P_1 > 0)$, I can have even number of positive solutions of $f_1(N^{(s)}) = 0$ and $f_2(N^{(s)}) = 0$.*

4.3 Linearization of (1.3) and (1.4)

I proceed with obtaining the Gateaux derivative of (1.3) and (1.4) by letting

$$N(t) = N^{(s)} + \epsilon Y(t) \tag{4.3}$$

$$N(t-\tau) = N^{(s)} + \epsilon Y(t-\tau) \tag{4.4}$$

$$N\left(t - \frac{\tau}{2}\right) = N^{(s)} + \epsilon Y\left(t - \frac{\tau}{2}\right) \tag{4.5}$$

$$N(t_1) = N^{(s)} + \epsilon Y(t_1) \tag{4.6}$$

then, it is possible to approximate F and B by using first order Taylor series like:

$$B(N^{(s)} + \epsilon Y(t-\tau)) = B_{N^{(s)}} + \epsilon D_1 B(N^{(s)})Y(t-\tau) + O(\epsilon^2) \tag{4.7}$$

$$F(\tau, N^{(s)} + \epsilon Y(t-\tau)) = F^{(1)}_{N^{(s)}} + \epsilon D_2 F(\tau, N^{(s)})Y(t-\tau) + O(\epsilon^2) \tag{4.8}$$

where $F^{(1)}_{N^{(s)}} = F(\tau, N^{(s)})$

$$F\left(\tau, N^{(s)} + \epsilon Y(t-\tau), N^{(s)} + \epsilon Y\left(t - \frac{1}{2}\tau\right), N^{(s)} + \epsilon Y(t)\right) \tag{4.9}$$

$$= F^{(2)}_{N^{(s)}} + \epsilon G_{(2)}(\tau, N^{(s)})Y(t-\tau) + \epsilon G_{(3)}(\tau, N^{(s)})Y\left(t - \frac{1}{2}\tau\right) + \epsilon G_{(4)}(\tau, N^{(s)})Y(t) + O(\epsilon^2)$$

where $F^{(2)}_{N^{(s)}} = F(\tau, N^{(s)}, N^{(s)}, N^{(s)})$ and $G_{(i)}(\tau, N^{(s)}) = D_i F(\tau, N^{(s)}, N^{(s)}, N^{(s)})$

$$F\left(\tau, \int_{t-\tau}^{t} k_2(t-t_1)(N^{(s)} + \epsilon Y(t_1))\,dt_1\right)$$

$$= F^{(3)}_{N^{(s)}} + \epsilon D_2 F\left(\tau, N^{(s)} \int_{t-\tau}^{t} k_2(t-t_1)\,dt_1\right) \int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1 + O(\epsilon^2) \tag{4.10}$$

where $F^{(3)}_{N^{(s)}} = F(\tau, N^{(s)} \int_{t-\tau}^{t} k_2(t-t_1)\,dt_1)$

Note:

- D_i represents partial derivative with respect to the i-th place in the functions argument list.

Since the ϵ terms are all required in the linearization, by substituting (4.3), (4.4), (4.7) and (4.8) into (1.3) and (1.4) when $F = F(\tau, N(t-\tau))$, it is possible to obtain

$$Y(t) = \int_0^\infty \left[F^{(1)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(1)}_{N^{(s)}} D_1 B(N^{(s)}) + N^{(s)} B(N^{(s)}) D_2 F(\tau, N^{(s)})\right] Y(t-\tau)\,d\tau \tag{4.11}$$

$$Y(t) = \int_0^M \left[F^{(1)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(1)}_{N^{(s)}} D_1 B(N^{(s)}) + N^{(s)} B(N^{(s)}) D_2 F(\tau, N^{(s)})\right] Y(t-\tau)\,d\tau \tag{4.12}$$

then, by substituting (4.3), (4.4), (4.5), (4.7) and (4.9) into (1.3) and (1.4) when $F = F(\tau, N(t-\tau), N(t-\frac{1}{2}\tau), N(t))$, I obtain

$$Y(t) = \int_0^\infty \left[F^{(2)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(2)}_{N^{(s)}} D_1 B N^{(s)}) + N^{(s)} B(N^{(s)}) G_{(2)}(\tau, N^{(s)})\right] Y(t-\tau)\,d\tau \tag{4.13}$$
$$+ \int_0^\infty N^{(s)} B(N^{(s)}) G_{(3)}(\tau, N^{(s)}) Y\left(t - \frac{1}{2}\tau\right) d\tau + \int_0^\infty N^{(s)} B(N^{(s)}) G_{(4)}(\tau, N^{(s)})\,d\tau Y(t)$$

$$Y(t) = \int_0^M \left[F^{(2)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(2)}_{N^{(s)}} D_1 B N^{(s)}) + N^{(s)} B(N^{(s)}) G_{(2)}(\tau, N^{(s)})\right] Y(t-\tau)\,d\tau \tag{4.14}$$
$$+ \int_0^M N^{(s)} B(N^{(s)}) G_{(3)}(\tau, N^{(s)}) Y\left(t - \frac{1}{2}\tau\right) d\tau + \int_0^M N^{(s)} B(N^{(s)}) G_{(4)}(\tau, N^{(s)})\,d\tau Y(t)$$

then, by substituting (4.3), (4.4), (4.6), (4.7) and (4.10) into (1.3) and (1.4) when $F = F(\tau, \int_{t-\tau}^{t} k_2(t-t_1)N(t_1)\,dt_1)$, I obtain

$$Y(t) = \int_0^\infty \left[F^{(3)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(3)}_{N^{(s)}} D_1 B(N^{(s)})\right] Y(t-\tau) \tag{4.15}$$
$$+ \int_0^\infty N^{(s)} B(N^{(s)}) D_2 F\left(\tau, N^{(s)} \int_{t-\tau}^{t} k_2(t-t_1)\,dt_1\right) \int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1\,d\tau$$

$$Y(t) = \int_0^M \left[F^{(3)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(3)}_{N^{(s)}} D_1 B(N^{(s)})\right] Y(t-\tau) \tag{4.16}$$
$$+ \int_0^M N^{(s)} B(N^{(s)}) D_2 F\left(\tau, N^{(s)} \int_{t-\tau}^{t} k_2(t-t_1)\,dt_1\right) \int_{t-\tau}^{t} k_2(t-t_1) Y(t_1)\,dt_1\,d\tau$$

Then, it is obvious to see the relationships with (2.1) and (2.2) in the previous chapter. Namely, if you consider

$$k(\tau) \;=\; F^{(1)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(1)}_{N^{(s)}} D_1 B(N^{(s)}) + N^{(s)} B(N^{(s)}) D_2 F(\tau, N^{(s)})$$

(4.11) and (4.12) are identical to (2.1) and (2.2) respectively, when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau)$. Similarly, if you consider

$$\begin{aligned} k(\tau) &= F^{(2)}_{N^{(s)}} B(\tau, N^{(s)}) + N^{(s)} F^{(2)}_{N^{(s)}} D_1 B(N^{(s)}) + N^{(s)} B(N^{(s)}) D_2 F(\tau, N^{(s)}, N^{(s)}, N^{(s)}) \\ k_1(\tau) &= N^{(s)} B(N^{(s)}) D_3 F(\tau, N^{(s)}, N^{(s)}, N^{(s)}) \\ k_2(\tau) &= N^{(s)} B(N^{(s)}) D_4 F(\tau, N^{(s)}, N^{(s)}, N^{(s)}) \end{aligned}$$

(4.13) and (4.14) are identical to (2.1) and (2.2) respectively, when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)Y(t-\frac{1}{2}\tau) + k_2(\tau)Y(t)$. Similarly, if you consider ($\int_{t-\tau}^{t} k_2(t-t_1)\,dt_1 = \int_0^{\tau} k_2(t_2)\,dt_2$)

$$\begin{aligned} k(\tau) &= F^{(3)}_{N^{(s)}} B(N^{(s)}) + N^{(s)} F^{(3)}_{N^{(s)}} D_1 B(N^{(s)}) \\ k_1(\tau) &= N^{(s)} B(N^{(s)}) D_2 F\left(\tau, N^{(s)} \int_{t-\tau}^{t} k_2(t-t_1)\,dt_1\right) \end{aligned}$$

(4.15) and (4.16) are identical to (2.1) and (2.2) respectively, when $k(\cdot)Y(\cdot) = k(\tau)Y(t-\tau) + k_1(\tau)\int_{t-\tau}^{t} k_2(t-t_1)Y(t_1)\,dt_1$. Due to Assumption 1.3.3.1 in Section 1.3.3, again, it is assumed that the function k and function k_i, $i = 1, 2$ such that $k : (-\infty, \infty) \to (-\infty, \infty)$ and $k_i : (-\infty, \infty) \to (-\infty, \infty)$ and $\dot{k} = \frac{dk}{dt}$ and $\dot{k}_i = \frac{dk_i}{dt}$ and k, $k_i \in C^1$. It is also assumed that $|\int_0^\infty k(\tau)\,d\tau| < \infty$, $|\int_0^\infty k_i(\tau)\,d\tau| < \infty$ and $|\int_0^M k(\tau)\,d\tau| < \infty$, $|\int_0^M k_i(\tau)\,d\tau| < \infty$. Then, it is possible to use the conditions of stability by using Lyapunov functionals which are constructed in the previous sections (Section 2.5.1, 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2, 2.7.1 and 2.7.2). For the detail, refer them again.

4.3.1 On the linearized equations and the Lyapunov functionals

The case $\mathbf{N^{(s)} = 0}$ From (4.11), (4.12), (4.13), (4.14), (4.15) and (4.16), it is obvious that

$$Y(t) = \int_0^\infty F_{N^{(s)}}\Big|_{N^{(s)}=0} B(0) Y(t-\tau)\,d\tau \text{ and } Y(t) = \int_0^M F_{N^{(s)}}\Big|_{N^{(s)}=0} B(0) Y(t-\tau)\,d\tau$$

are the only equations to consider ($F_{N^{(s)}} = F(\tau, N^{(s)})$, $F_{N^{(s)}} = F(\tau, N^{(s)}, N^{(s)}, N^{(s)})$ or $F_{N^{(s)}} = F(\tau, N^{(s)} \int_{t-\tau}^{t} k_2(t-t_1)\,dt_1)$). So, by considering

$$k(\tau) = F_{N^{(s)}}\Big|_{N^{(s)}=0} B(0)$$

only Functional 2.5.1.1 and 2.5.2.1 are required to obtain sufficient conditions of stability of zero solutions of (4.11), (4.12), (4.13), (4.14), (4.15) and (4.16). Moreover, it can be also useful to use functionals like Functional 2.6.1.1, 2.6.2.1 2.6.1.2 and 2.6.2.2.

Remark:

- To apply Functional 2.6.1.1 and 2.6.1.2 the condition that $k(\tau) = k_1(\tau) + k_2(\tau)$ is required.

Finally, here are the summaries of the conditions of stability of zero solution given by applying Functional 2.5.1.1, 2.5.2.1, 2.6.1.1, 2.6.2.1, 2.6.1.2 and 2.6.2.2. (To the detail, see section 2.5.1, 2.5.2, 2.6.1, 2.6.2)

Conclusion 4.3.1.1

If at least one of the conditions below is satisfied, $N^{(s)} = 0$ of (1.3) is locally stable.

1. *Conditions to apply Functional 2.5.1.1*

$$\int_0^\infty (k(0)k(\tau) + \dot{k}(\tau))\, d\tau < 0 \text{ and } 1 - \int_0^\infty \tau|k(0)k(\tau) + \dot{k}(\tau)|\, d\tau > 0$$

2. *Conditions to apply Functional 2.6.1.2*

$$\int_0^\infty (k_2(0)k(\tau) + \dot{k}_2(\tau))\, d\tau < 0 \text{ and } 1 - \int_0^\infty \tau|k_2(0)k(\tau) + \dot{k}(\tau)|\, d\tau > 0$$

3. *Conditions to apply Functional 2.6.2.2*

$$\begin{aligned} 0 &> \int_0^\infty \left[\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) + (k(0)+1)\dot{k}(\tau)\right] d\tau \\ 0 &< 1 - \int_0^\infty \tau|\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) \\ &\quad +(k(0)+1)\dot{k}(\tau)|\, d\tau - \int_0^\infty |k(0)k(\tau) + \dot{k}(\tau)|\, d\tau \end{aligned}$$

Conclusion 4.3.1.2

If at least one of the conditions below is satisfied, $N^{(s)} = 0$ of (1.4) is locally stable.

1. *Conditions to apply Functional 2.5.2.1*

$$\begin{aligned} 0 &> \int_0^M \left[k(0)k(\tau) + \dot{k}(\tau) - k(M)k(\tau)\right] d\tau \\ 0 &< 1 - \int_0^M \Big(\tau|k(0)k(\tau) + \dot{k}(\tau)| + (M+\tau)|k(M)k(\tau)|\Big)\, d\tau \end{aligned}$$

2. *Conditions to apply Functional 2.6.1.1*

$$\begin{aligned} 0 &> \int_0^M [(k_2(0) - k_2(M))k(\tau) + \dot{k}_2(\tau)]\, d\tau \\ 0 &< 1 - \int_0^M \tau|k_2(0)k(\tau) + \dot{k}(\tau)|\, d\tau - \int_0^M |k_2(M)k(\tau)|(M+\tau)\, d\tau - |k_1(M)|M \end{aligned}$$

3. Conditions to apply Functional 2.6.2.1

$$
\begin{aligned}
0 &> \int_0^M \left[\left(\dot{k}(0) + k(0) + k(0)^2 - \dot{k}(M)\right)k(\tau) + \left(k(0) + 1 - k(M)\right)\dot{k}(\tau)\right] d\tau \\
&\quad + k(M)[k(M) - 2k(0) - 1] \\
0 &< 1 - \int_0^M (M+\tau)|\dot{k}(M)k(\tau) + k(M)\dot{k}(\tau)|\, d\tau \\
&\quad - \int_0^M \tau|\ddot{k}(\tau) + (\dot{k}(0) + k(0)^2 + k(0))k(\tau) + (k(0)+1)\dot{k}(\tau)|\, d\tau \\
&\quad - \int_0^M |k(0)k(\tau) + \dot{k}(\tau)|\, d\tau - |k(M)| - Mk(M)^2 - 2|k(M)k(0)|M
\end{aligned}
$$

The case $\mathbf{N^{(s)} > 0}$ Then, if $N^{(s)} \neq 0$, particularly, $N^{(s)} > 0$, all the Functionals 2.5.1.1, 2.5.2.1, 2.5.5.1, 2.5.5.2, 2.5.6.1, 2.5.6.2, 2.5.7.1, 2.5.8.1, 2.6.1.1, 2.6.2.1, 2.6.1.2 and 2.6.2.2 are useful. If the conditions are satisfied, it is possible to say $Y(t) \longrightarrow 0$ by using Lemma in Gopalsamy [47] (see Theorem 2.5.1.1 in Section 2.5.1) and so, $N(t) \longrightarrow N^{(s)}$. That is, the steady states of (1.3) and (1.4) are locally stable. (To the detail, see Section 2.5.1, 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2, 2.7.1 and 2.7.2). Moreover, when $F = F(\tau, N(t-\tau))$, it is possible to use the conditions of Conclusion 4.3.1.1 and those of Conclusion 4.3.1.2 to obtain the stability of $N^{(s)} > 0$ of (1.3) and of (1.4) respectively. So, I have the conclusion as follows.

Conclusion 4.3.1.3

If at least one of the conditions of Conclusion 4.3.1.1 is satisfied, $N^{(s)} > 0$ of (1.3) when $F = F(\tau, N(t-\tau))$ is locally stable. Then, if at least one of the conditions of Conclusion 4.3.1.2 is satisfied, $N^{(s)} > 0$ of (1.4) when $F = F(\tau, N(t-\tau))$ is also locally stable.

4.4 An example

Now, let me discuss an example, namely

$$
N(t) = \int_0^\infty e^{-q(\tau + cN(t-\tau))} r(1 - be^{-p\tau} N(t-\tau))N(t-\tau)\, d\tau \tag{4.17}
$$

p, q, b, c and r are all positive constants. In the Section 2.4.2, I mentioned that this is convertible to a system of ordinary differential equations (2.22).

4.4.1 Steady States

Note : It is obvious that $N^{(s)} = 0$ is always a steady state of (4.17). However, as I explained in the first chapter, (4.17) is a model to estimate the number of population, so it is important to obtain the value of $N^{(s)} > 0$ if this exists.

Let $N^{(s)}$ represent the steady state of (4.17) and I suppose that $N^{(s)} > 0$. Then, by substituting $N(t) = N^{(s)}$ and $N(t-\tau) = N^{(s)}$ into (4.17), I obtain

$$N^{(s)} = \int_0^\infty e^{-q(\tau + cN^{(s)})} r\Big(1 - be^{-p\tau} N^{(s)}\Big) N^{(s)}\, d\tau$$

$$\Longrightarrow N^{(s)} = \frac{e^{-q(\tau + cN^{(s)})} r\Big(-p - q + bqN^{(s)} e^{-p\tau}\Big) N^{(s)}}{q(p+q)}\Bigg|_{\tau=0}^{\infty}$$

Since $q > 0$ and $p > 0$,

$$N^{(s)} = \frac{e^{-qcN^{(s)}} r\Big(p + q - bqN^{(s)}\Big) N^{(s)}}{q(p+q)} \tag{4.18}$$

which is a nonlinear algebraic equation to obtain $N^{(s)}$. Then, since $N^{(s)} \neq 0$, I obtain

$$1 = \frac{e^{-qcN^{(s)}} r\Big(p + q - bqN^{(s)}\Big)}{q(p+q)} \tag{4.19}$$

Then,

$$f(N^{(s)}) \equiv q(q+p) + \Big[\Big(-1 + bN^{(s)}\Big)q - p\Big] re^{-qcN^{(s)}} = 0$$

Then, $f(0) = (q+p)(q-r)$ and $f(N^{(s)}) = q(q+p) > 0$ as $N^{(s)} \longrightarrow \infty$. So, there are 2 possibilities for (4.18) to have more than 1 positive $N^{(s)}$ according to the Conclusion 4.2.0.4 and 4.2.0.5 in Section 4.2. They are:

1. If $f(0) = (q+p)(q-r) < 0$, that is, if $q < r$, (4.18) has odd number of positive $N^{(s)}$.

2. If $f(0) = (q+p)(q-r) > 0$, that is, if $q > r$, and if there exists $P > 0$ such that $f(P) < 0$, (4.18) has even number of positive $N^{(s)}$.

So, for $q < r$, there is a guarantee that (4.18) has at least one positive $N^{(s)}$. Now, by differentiating $f(N^{(s)})$ with respect to $N^{(s)}$, it is possible to obtain

$$f'(N^{(s)}) = -rqe^{-qcN^{(s)}}\Big(bcqN^{(s)} - b - c(p+q)\Big)$$

Then, by solving $f'(N^{(s)}) = 0$ for $N^{(s)}$,

$$N^{(s)} = \frac{b + c(p+q)}{cbq} \equiv N^{(s^*)}$$

Then, by substituting the $N^{(s^*)}$ above into $f(N^{(s)})$,

$$f(N^{(s^*)}) = q(q+p) + \frac{bre^{-\frac{b+c(p+q)}{b}}}{c} > 0$$

which is the local minima of $f(N^{(s)})$ for $N^{(s)} \in [0, \infty)$ and which is positive. This eliminates the possibility of the existence of the value of P in the above statement and eliminates the possibilities that (4.18) has more than one positive $N^{(s)}$. Hence, I have a conclusion as follows:

Conclusion 4.4.1.1

(4.18) has one positive $N^{(s)}$ if and only if $q < r$.

4.4.2 Linearization of (4.17) and the local stability of the steady state of (4.17)

Now, let me take Gateaux derivative of (4.17) by substituting

$$
\begin{aligned}
N(t) &= N^{(s)} + \epsilon Y(t) && (4.20)\\
N(t-\tau) &= N^{(s)} + \epsilon Y(t-\tau) && (4.21)
\end{aligned}
$$

and linearize $e^{-q(\tau+cN(t-\tau))}$ by

$$
e^{-q(\tau+cN(t-\tau))} = e^{-q(\tau+cN^{(s)})}(1-\epsilon qcY(t-\tau)) + O(\epsilon^2) \qquad (4.22)
$$

which is done by using first order Taylor series. Then, again, since I only require the first order ϵ terms, it is possible to obtain

$$
Y(t) = \int_0^\infty e^{-q(\tau+cN^{(s)})} r\Big[1 + bN^{(s)}e^{-p\tau}\Big(qcN^{(s)}-2\Big) - qcN^{(s)}\Big]Y(t-\tau)\,d\tau \qquad (4.23)
$$

Then, as I discussed in Section 2.3, (4.23) can be modified into a functional differential equation, namely,

$$
\begin{aligned}
\dot{Y}(t) &= r\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b - qcN^{(s)} + 1\Big]e^{-qcN^{(s)}}Y(t) \qquad (4.24)\\
&+ \int_0^\infty r\Big[q\Big(qcN^{(s)}-1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]e^{-q(\tau+cN^{(s)})}Y(t-\tau)\,d\tau
\end{aligned}
$$

and by regarding

$$
\begin{aligned}
k(\tau) &= e^{-q(\tau+cN^{(s)})} r\Big[1 + bN^{(s)}e^{-p\tau}\Big(qcN^{(s)}-2\Big) - qcN^{(s)}\Big]\\
k(0) &= r\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b - qcN^{(s)} + 1\Big]e^{-qcN^{(s)}}\\
\dot{k}(\tau) &= r\Big[q\Big(qcN^{(s)}-1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]e^{-q(\tau+cN^{(s)})}
\end{aligned}
$$

it is possible to use the all the Functional 2.5.1.1, 2.6.1.2 and 2.6.2.2.

Note : I discussed this in Section 4.3.1.

Application of Functional 2.5.1.1 to determine the local stability of (4.17)

It is obvious that the $k(\tau)$ above is differentiable, since there exists $\dot{k}(\tau)$. So, Functional 2.5.1.1 is possible to apply.

Application of Functional 2.6.1.2 to determine the local stability of (4.17)

I need $k(\tau)$ like $k(\tau) = g_1(\tau) + g_2(\tau)$ as well as $k(\tau) \in C^1$ to use Functional 2.6.1.2. However it is possible to split $k(\tau)$ - for instance, $g_1(\tau) = e^{-q(\tau+cN^{(s)})}r[1 + bN^{(s)}e^{-p\tau}(qcN^{(s)}-2)]$ and $g_2(\tau) = -e^{-q(\tau+cN^{(s)})}rqcN^{(s)}$ and there is no doubt about $g_1(\tau), g_2(\tau) \in C^1$ due to the above discussion. So, Functional 2.6.1.2 is one of the candidates to use.

Application of Functional 2.6.2.2 to determine the local stability of (4.17)

There is no doubt about $k(\tau) \in C^2$. So, Functional 2.6.2.2 may apply.

Note : To prove the local stability of (4.17) by using Functional 2.5.1.1, 2.6.1.2 and 2.6.2.2, there are some conditions required. To the detail, see Section 2.5.1, 2.6.1 and 2.6.2 or Section 4.3.1.

Local stability analysis of (4.17) without using Lyapunov functionals

However, in general, the Lyapunov functionals yield only the sufficient conditions of the stability of the steady states and so, it is better to apply the normal way of local stability analysis, which is usually proceeded by using $Y(t) = e^{\lambda t}$ because of the reasons as follows

1. $e^{\lambda t}$ is a solution of (4.23) if

$$re^{-qcN^{(s)}}\left(\frac{1-qcN^{(s)}}{\lambda+q}+\frac{(qcbN^{(s)}-2b)N^{(s)}}{\lambda+p+q}\right)=1$$

   This is similar to what I discussed in section 2.2.

2. $e^{\lambda t}$ is simple to use since if $\lambda > 0$, it will grow, which indicates the instability of the steady state $N^{(s)}$ and if $\lambda < 0$, it will decay, which indicates the stability of the steady state $N^{(s)}$. (I am looking for the behaviour of (4.17) near $N^{(s)}$, so, $e^{\lambda t}$ is a convenient function in order to see whether (4.17) converges to $N^{(s)}$ or not.)

Now, let me proceed perturbation analysis. Firstly, by substituting $Y(t) = e^{\lambda(t)}$ into (4.20) and $Y(t-\tau) = e^{\lambda(t-\tau)}$ into (4.21). So, again, $e^{-q(\tau+cN(t-\tau))}$ is linearized like

$$e^{-q(\tau+cN(t-\tau))} = e^{-q(\tau+cN^{(s)})}(1-\epsilon qce^{\lambda(t-\tau)})+O(\epsilon^2)$$

which is again done by first order Taylor series. It is clear that

$$1 = re^{-qcN^{(s)}}[F_1(\lambda,\tau)+F_2(\lambda,\tau)]\Big|_{\tau=0}^{\infty} \tag{4.25}$$

$$F_1(\lambda,\tau)=\frac{(qcN^{(s)}-1)e^{(-(q+\lambda)\tau)}}{(\lambda+q)} \text{ and } F_2(\lambda,\tau)=\frac{bN^{(s)}(qcN^{(s)}-2)e^{(-(p+q+\lambda)\tau)}}{(\lambda+p+q)}$$

So, $\lambda \neq p+q$ and $\lambda \neq q$ and it is required to have the condition that real part of $\lambda > -q$, otherwise, it is impossible to obtain (4.25) since the integral does not exist. In particular, for λ to have real roots, $\lambda > -q$. Then, (4.25) becomes

$$\begin{aligned} 0 &= \lambda^2 + A\lambda + B \qquad (4.26)\\ \text{where } A &= 2q+p-re^{-qcN^{(s)}}\left[1-qcN^{(s)}-2bN^{(s)}+qcb(N^{(s)})^2\right]\\ \text{and } B &= q(p+q)-re^{-qcN^{(s)}}\left[q+p-cq^2N^{(s)}-cpqN^{(s)}-2bqN^{(s)}+bc(qN^{(s)})^2\right]\end{aligned}$$

Then, let me separate 2 cases to discuss the values of λ, which are $N^{(s)}=0$ and $N^{(s)}>0$.

$\mathbf{N^{(s)} = 0}$ So, if $N^{(s)} = 0$, $\lambda^2 + A\lambda + B = \lambda^2 + (2q+p-r)\lambda - r(p+q) = 0$. Hence, $\lambda = -q-p, -q+r$. However, $\lambda > -q$, so, $\lambda = -q+r$ is the only value λ can take, even if $-q-p < 0$. So, I can have the conclusion as follows.

Conclusion 4.4.2.1

If $-q+r > 0$, $N^{(s)} = 0$ is locally unstable and if $-q+r < 0$, $N^{(s)} = 0$ is locally stable.

Note: I have shown that (4.18) has one positive $N^{(s)}$ if and only if $q < r$ in the previous argument about the steady state of (4.18). So, if there exist a positive steady state ($N^{(s)} > 0$), $N^{(s)} = 0$ is unstable.

$\mathbf{N^{(s)} > 0}$ Now, by using Routh-Hurwitz criterion (the proof is in Anagnost and Desoer [2]), it is possible to obtain the local stability for ($N^{(s)} > 0$) as well. That is, the real parts of the root of (4.26) are negative if and only if $A > 0$, $B > 0$. So, I can conclude as follows:

Conclusion 4.4.2.2

$N^{(s)} > 0$ is locally stable if and only if $A > 0$, $B > 0$, otherwise $N^{(s)} > 0$ is locally unstable.

Note: As I discussed in the above, if λ has restricted region of the roots (real parts of $\lambda > -q$) in (4.26). However if $B > 0$ and $A = 0$ in (4.26), $\lambda = \pm\sqrt{B}i$, that is, there are possibilities of stability switch at these value. In fact, when $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$, or $N^{(s)} = 102.1291175$, $p = 1$, $c = 1$, $r = 7.398532026$ and $q = 0.02$, I have $B \approx 1$ and $A \approx 0$. That is, the solution of (4.26) is purely imaginary and so, (4.17) could have a periodic solution. This is indeed, guaranteed since (4.17) is reducible to a system of ordinary differential equations as I discussed in Section 2.4.2. (For the systems of ordinary differential equations, if the eigenvalues values of the corresponding characteristic equations are purely imaginary, it is possible to conclude that the systems have Hopf bifurcation.) However, the purpose of this thesis is to analyze integral equations, so, I will discuss in the next section without using the fact of reducibility of a system of ordinary differential equations.

4.4.3 A periodic solution of (4.23)

Now, let me substitute $Y(t) = K_1\cos(t) + K_2\sin(t)$ and $Y(t-\tau) = K_1\cos(t-\tau) + K_2\sin(t-\tau)$ (K_1, $K_2 \in \Re$) into (4.23), then, I get

$$\begin{aligned} &K_1\cos(t) + K_2\sin(t) \qquad (4.27)\\ &= \left(\frac{bN^{(s)}(qcN^{(s)}-2)(K_1(p+q)-K_2)}{(q+p)^2+1} - \frac{(qcN^{(s)}-1)(K_1q-K_2)}{q^2+1}\right)re^{-qcN^{(s)}}\cos(t)\\ &\quad+\left(\frac{bN^{(s)}(K_2(p+q)+K_1)(qcN^{(s)}-2)}{(q+p)^2+1} - \frac{(qcN^{(s)}-1)(K_2q+K_1)}{q^2+1}\right)re^{-qcN^{(s)}}\sin(t)\end{aligned}$$

Then, if I substitute the values above ($N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$) into (4.27), it is possible to obtain $K_1\cos(t) + K_2\sin(t) = K_1\cos(t) + K_2\sin(t)$. That is, when $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$, I get

$$\begin{aligned} K_1 &= \left(\frac{bN^{(s)}(qcN^{(s)}-2)(K_1(p+q)-K_2)}{(q+p)^2+1} - \frac{(qcN^{(s)}-1)(K_1q-K_2)}{q^2+1}\right) re^{-qcN^{(s)}} \\ K_2 &= \left(\frac{bN^{(s)}(K_2(p+q)+K_1)(qcN^{(s)}-2)}{(q+p)^2+1} - \frac{(qcN^{(s)}-1)(K_2q+K_1)}{q^2+1}\right) re^{-qcN^{(s)}} \end{aligned}$$

This is as I discussed in Section 2.2 and this is true for all the values of b, $N^{(s)}$, r, p, c and q which make $B = 1$ and $A = 0$ in (4.26). Since $\sqrt{B}$ determines the frequency of the oscillation, $B = 1$ means that I obtain $\lambda = \pm i$ as the roots of $\lambda^2 + A\lambda + B$ in (4.26) and so, $e^{\pm it} = \cos(t) \pm i\sin(t)$ is a solution of (4.25). Hence, I can give the conclusion as follows.

Conclusion 4.4.3.1

If b, $N^{(s)}$, r, p, c and q have values such that $B = 1$ and $A = 0$ in (4.26), $\sin(t)$ and $\cos(t)$ span all the non-zero solutions of (4.23) and the roots of (4.26) are also purely imaginary ($\pm i$).

Remark: The above conclusion is equivalent to the Theorem 2.2 of Landman [70].

4.5 Poincaré-Lindsted method

In this section, I will approximate a periodic solution (2π periodic) of (4.17) when $B = 1$ and $A = 0$ in (4.26) by using Poincaré-Lindsted method. This method is used to construct a periodic solution of the integro-differential equation

$$\dot{y}(t) = \left(1 - \int_{-\infty}^{t} t' y(t') e^{\frac{-t'}{\alpha}}\, dt'\right) y(t) \tag{4.28}$$

in Morris [78], which is called generalised Hutchinson equation. The method is also used in Landman [70], which is very similar to the Poincaré-Lindsted method. She used the method on the equation

$$\dot{N}(t) = \left(1 - \int_{-\infty}^{t} (t-s)e^{-(t-s)} N(s)\, ds\right) \lambda N(t), \quad \lambda,\ c > 0 \tag{4.29}$$

as well as more generalized equation

$$\dot{N}(t) = \left(1 - aN(t) - \int_{-\infty}^{t} k(t-s) F(N(s))\, ds\right) \lambda N(t), \quad t > 0 \tag{4.30}$$

Morris and Landman used this method on integro-differential equations but by differentiating integral equation (4.17) with respect to t, it is possible to obtain the integro-differential equation (4.31) below (Differentiability of $N(t)$ is assumed in Assumption 4.1.0.6 in Section 4.1). So, Poincaré-Lindsted method is able to be applied to (4.17) by using the same procedure as Morris or Landman used. So, as the first step, I need to show (4.31), namely,

$$\dot{N}(t) = r(1-bN(t))N(t)e^{-qcN(t)} + \int_0^\infty r\Big[b(q+p)e^{-p\tau}N(t-\tau)-q\Big]N(t-\tau)e^{-q(\tau+cN(t-\tau))}\,d\tau \tag{4.31}$$

Remark: In Gopalsamy [47](from page 143 to page 148) and in Morris [78], Poincaré-Lindsted method is used to approximate the periodic solution of the functional differential equation like:

$$\dot{u}(t) = ru(t)\left(1-\frac{u(t-h)}{K}\right) \tag{4.32}$$

for $h > 0$ above a certain threshold and where r and K are positive constants and Morris [79] applied Poincaré-Lindsted method to a system of functional differential equations like:

$$\begin{aligned}\dot{H}(t) &= rH(t)\left(1-\frac{H(t-h)}{K}\right)-\alpha H(t)C(t)\\ \dot{C}(t) &= -bc+\gamma H(t)C(t)\end{aligned} \tag{4.33}$$

for $h > 0$ above a certain threshold and where r, α, b, c and K are positive constants. Moreover, in Jordan and Smith [66], the method is introduced and well explained for ordinary differential equations.

4.5.1 Expansion of (4.31) by using Taylor series and rescaling it

I found that the way of Landman [70] is more suitable to this case than that of Morris [78], so, I proceed the expansion by using very similar manner to Landman [70]. Then, before I proceed Poincaré-Lindsted method I have to convert (4.31) to an appropriate form by using Taylor series. However, this time, I am not converting to a linear equation. Again like linearization, let $N(t) = N^{(s)} + Y(t)$, $N(t-\tau) = N^{(s)} + Y(t-\tau)$ and approximate $e^{-q(\tau+N(t-\tau))}$ by

$$e^{-q(\tau+cN^{(s)})} = 1 - qcY(t-\tau) + \frac{(qc)^2}{2}Y(t-\tau)^2 - \frac{(qc)^3}{6}Y(t-\tau)^3 + \cdots$$

which is done by using Taylor series. Now, let me substitute them into (4.31). Then,

$$\begin{aligned}\dot{Y}(t) &= r(1-b(N^{(s)}+Y(t)))(N^{(s)}+Y(t))e^{-qN^{(s)}}\left(1-qcY(t)+\frac{(qc)^2}{2}Y(t)^2-\frac{(qc)^3}{6}Y(t)^3+\cdots\right)\\ &\quad -\int_0^\infty\Bigg[r\Big[b(q+p)e^{-p\tau}(N^{(s)}+Y(t-\tau))-q\Big](N^{(s)}+Y(t-\tau))e^{-q(\tau+cN^{(s)})}\Big[1-qcY(t-\tau)\\ &\quad +\frac{(qc)^2}{2}Y(t-\tau)^2-\frac{(qc)^3}{6}Y(t-\tau)^3+\cdots\Big]\Bigg]\,d\tau\end{aligned} \tag{4.34}$$

Since I look for the solution which has period $2\pi/\omega(\epsilon)$ where $\omega(\epsilon)\to 1$ as $\epsilon \to 0$, it is convenient to re-scale the time variable t as $s = \omega(\epsilon)t$ and to write the solution in this new variable s as

$$y(s) = Y\left(\frac{s}{\omega(\epsilon))}\right) = Y(t)$$

Therefore, the solution $y(s)$ will be 2π periodic in s, that is, I am looking for a solution $y \in P(2\pi)$ in the new variable s satisfies

$$\begin{aligned}
\dot{y}(s) &= r(1 - b(N^{(s)} + y(s)))(N^{(s)} + y(s))e^{-qcN^{(s)}}\left(1 - qcy(s) + \frac{(qc)^2}{2}y(s)^2 + \cdots\right) \\
&- \int_0^\infty \left[r\Big(b(q+p)e^{-p\tau}(N^{(s)} + y(s-\omega\tau)) - q\Big)(N^{(s)} + y(s-\omega\tau))e^{-q(\tau+cN^{(s)})}\Big(1\right. \\
&\left.\left. -qcy(s-\omega\tau) + \frac{(qc)^2}{2}y(s-\omega\tau)^2 + \cdots\right)\right] d\tau \qquad (4.35)
\end{aligned}$$

where $P(p)$ denotes the Banach space of real valued functions which are continuous and p periodic functions of $t \in \Re$.

4.5.2 The expansion of (4.35)

I am looking for a solution in a series

$$y(s) \quad = \quad \epsilon(u_0(s) + \epsilon u_1(s) + \epsilon^2 u_2(s) + \cdots), \quad u_i \in P(2\pi) \qquad (4.36)$$

with

$$\omega = \omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \qquad (4.37)$$

$$r = r(\epsilon) = r_0 + \epsilon r_1 + \epsilon^2 r_2 + \cdots \qquad (4.38)$$

It is now ready to substitute the expansions (4.36), (4.37), (4.38) into (4.35) and equate corresponding coefficients of like powers of ϵ.

4.5.3 The first order

At the first order (order ϵ), I find the homogeneous linear equation

$$\begin{aligned}
\dot{u_0}(s) &= r_0\Big[N^{(s)}\Big(qcN^{(s)} - 2\Big)b - qcN^{(s)} + 1\Big]e^{-qcN^{(s)}}u_0(s) \qquad (4.39) \\
&+ \int_0^\infty r_0\Big[q(qcN^{(s)} - 1) - bN^{(s)}(q+p)\Big(qcN^{(s)} - 2\Big)e^{-p\tau}\Big]e^{-q(\tau+cN^{(s)})}u_0(s-\omega\tau)\, d\tau
\end{aligned}$$

then, as I discussed in Section 4.4.2 by differentiating (4.23) with respect to t, I get (4.24), namely,

$$\begin{aligned}
\dot{Y}(t) &= r\Big[N^{(s)}\Big(qcN^{(s)} - 2\Big)b - qcN^{(s)} + 1\Big]e^{-qcN^{(s)}}Y(t) \\
&+ \int_0^\infty r\Big[q\Big(qcN^{(s)} - 1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)} - 2\Big)e^{-p\tau}\Big]e^{-q(\tau+cN^{(s)})}Y(t-\tau)\, d\tau
\end{aligned}$$

Then, by allowing for the change to the variable s and putting $r = r_0$, (4.24) is identical to (4.39). Furthermore, as I obtained Conclusion 4.4.3.1, by putting $Y(t) = K_1\cos(t) + K_2\sin(t)$ and $Y(t-\tau) = K_1\cos(t-\tau) + K_2\sin(t-\tau)$ (K_1, $K_2 \in \Re$ and K_1, K_2 can be chosen arbitrarily) into

(4.24), I get

$$
\begin{aligned}
&K_2\cos(t) - K_1\sin(t) \\
&= \left(\frac{bN^{(s)}(qcN^{(s)}-2)(K_2(p+q)+K_1)}{1+(p+q)^2} - \frac{(qcN^{(s)}-1)(K_2q+K_1)}{q^2+1}\right) re^{-qcN^{(s)}}\cos(t) \\
&+ \left(\frac{(qcN^{(s)}-1)(K_1q-K_2)}{q^2+1} - \frac{bN^{(s)}(K_1(p+q)-K_2)(qcN^{(s)}-2)}{1+(p+q)^2}\right) re^{-qcN^{(s)}}\sin(t)
\end{aligned} \tag{4.40}
$$

Then, it is obvious that if b, $N^{(s)}$, r and p have the values such that $B > 0$ and $A = 0$ in (4.26). Hence, in (4.40) I have:

$$\left(\frac{bN^{(s)}(qcN^{(s)}-2)(K_2(p+q)+K_1)}{1+(p+q)^2} - \frac{(qcN^{(s)}-1)(K_2q+K_1)}{q^2+1}\right) re^{-qcN^{(s)}} = K_2 \tag{4.41}$$

$$\left(\frac{(qcN^{(s)}-1)(K_1q-K_2)}{q^2+1} - \frac{bN^{(s)}(K_1(p+q)-K_2)(qcN^{(s)}-2)}{1+(p+q)^2}\right) re^{-qcN^{(s)}} = -K_1 \tag{4.42}$$

Hence, $K_2\cos(t) - K_1\sin(t) = K_2\cos(t) - K_1\sin(t)$. However, it is more convenient to choose K_1 and K_2 for definiteness, so, without loss of generality, I can chose $K_1 = 0$ and $K_2 = 1$. Therefore, (4.39) has a unique solution in $P(2\pi)$: $u_0 = \sin(s)$ by considering $r = r_0$.

Note:

- When $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$, it is possible to obtain $B = 1$ and $A = 0$ in (4.26), and when $K_1 = 0$ and $K_2 = 1$, (4.41) and (4.42) will become (4.43) and (4.44) respectively, namely,

$$\left(-\frac{q(qcN^{(s)}-1)}{q^2+1} + \frac{bN^{(s)}(p+q)(qcN^{(s)}-2)}{1+(p+q)^2}\right) re^{-qcN^{(s)}} = 1 \tag{4.43}$$

$$\left(\frac{bN^{(s)}(qcN^{(s)}-2)}{1+(p+q)^2} - \frac{(qcN^{(s)}-1)}{q^2+1}\right) re^{-qcN^{(s)}} = 0 \tag{4.44}$$

- Furthermore, when $B = 1$ and $A = 0$ in (4.26)

$$\left(\frac{bN^{(s)}(qcN^{(s)}-2)}{1+(p+q)^2} - \frac{qcN^{(s)}-1}{q^2+1}\right) = 0 \tag{4.45}$$

  so, $re^{-qcN^{(s)}}$ does not contribute to annihilate (4.44).

4.5.4 The second order

When looking at second order (order ϵ^2), it is required to expand the term $u_0(s-\omega\tau) = \sin(s-\omega\tau)$ in terms of $s-\tau$; this can be expanded as a Taylor series

$$u_0(s-\omega\tau) = \sin(s-\omega\tau) = \sin(s-\tau) + (1-\omega)\tau\cos(s-\tau) + R, \tag{4.46}$$

where the remainder is given by

$$R = -\frac{(1-\omega)^2\tau^2}{2}\sin(s-\xi\tau) \text{ for } \xi \in (\omega, 1) \text{ or } \xi \in (1, \omega)$$

Since I have assumed that $q > 0$, $p > 0$ and $c > 0$,

$$\int_0^\infty \tau e^{-q(\tau)}\, d\tau < \infty, \quad \int_0^\infty \tau^2 e^{-q(\tau)}\, d\tau < \infty,$$

$$\int_0^\infty \tau e^{-qc(\tau)} e^{-p(\tau)}\, d\tau < \infty \text{ and } \int_0^\infty \tau^2 e^{-qc(\tau)} e^{-p(\tau)}\, d\tau < \infty$$

So, it is possible to substitute (4.46) into

$$\int_0^\infty e^{-q(\tau+cN^{(s)})} r_0 \left[-\frac{1}{2} q^2 c(qcN^{(s)} - 2) + \frac{b}{2}(q+p)((qcN^{(s)})^2 - 4qcN^{(s)} + 2)e^{-p\tau} \right] u_0(s-\omega\tau)^2\, d\tau$$

$$\text{and } \int_0^\infty e^{-q(\tau+cN^{(s)})} r_1 [q(qcN^{(s)} - 1) - bN^{(s)}(q+p)(qcN^{(s)} - 2)e^{-p\tau}] u_0(s-\omega\tau)\, d\tau$$

Now, it is possible to look at the equation at $O(\epsilon^2)$:

$$\begin{aligned}
Q_1(s) &= Q_2(s) \qquad\qquad (4.47)\\
Q_1(s) &= \dot{u}_1(s) - r_0\Big[N^{(s)}\Big(qcN^{(s)} - 2\Big)b - qcN^{(s)} + 1\Big]e^{-qcN^{(s)}} u_1(s)\\
&\quad - \int_0^\infty e^{-q(\tau+cN^{(s)})} r_0 \Big[q\Big(qcN^{(s)} - 1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)} - 2\Big)e^{-p\tau}\Big] u_1(s-\tau)\, d\tau\\
Q_2(s) &= r_1 e^{-qcN^{(s)}}\Big[N^{(s)}\Big(qcN^{(s)} - 2\Big)b - qcN^{(s)} + 1\Big]\sin(s) - \omega_1\cos(s)\\
&\quad + r_0 e^{-qcN^{(s)}}\left[\frac{1}{2}qc\Big(qcN^{(s)} - 2\Big) + \left(\frac{-1}{2}\Big(qcN^{(s)}\Big)^2 + 2qcN^{(s)} - 1\right)\right]\sin^2(s)\\
&\quad - \omega_1 \int_0^\infty r_0 e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)} - 1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)} - 2\Big)e^{-p\tau}\Big]\tau\cos(s-\tau)\, d\tau\\
&\quad + r_1 \int_0^\infty e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)} - 1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)} - 2\Big)e^{-p\tau}\Big]\sin(s-\tau)\, d\tau\\
&\quad + r_0 \int_0^\infty \left[\frac{b}{2}(q+p)\Big[\Big(qcN^{(s)}\Big)^2 - 4qcN^{(s)} + 2\Big]e^{-p\tau}\right.\\
&\quad \left. - \frac{1}{2}q^2 c\Big(qcN^{(s)} - 2\Big)\right] e^{-q(\tau+cN^{(s)})}\sin^2(s-\tau)\, d\tau
\end{aligned}$$

$Q_1(s)$ consists of the homogeneous terms in u_1, while $Q_2(s)$ is a known periodic function in $P(2\pi)$. Then, I need the conditions for the solvability of (4.47) to determine ω_1 and λ_1.

Solvability condition of (4.47)

As I have already discussed to obtain conclusion 4.4.3.1, if b, $N^{(s)}$, r, p, c and q have the values such that $B = 1$ and $A = 0$ in (4.26), $\sin(t)$ and $\cos(t)$ span all the non-zero solutions of (4.23))-e.g. $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$. So, I can use the theorem as follows which is mentioned in Landman [70] and is proved by Cushing [25]:

Theorem 4.5.4.1

The non-homogeneous equation (4.47) has a solution in $P(2\pi)$ *if and only if*

$$\int_0^{2\pi} Q_2(s)\sin(s)\, ds = \int_0^{2\pi} Q_2(s)\cos(s)\, ds = 0$$

where $\sin(s)$ *and* $\cos(s)$ *are the two independent solutions of* $Q_1(s) = 0$ *in* $P(2\pi)$.

Now, by applying theorem 4.5.4.1, I have

$$
\begin{aligned}
\int_0^{2\pi} Q_2(s)\sin(s)\,ds &= -\omega_1\int_0^{2\pi}\cos(s)\sin(s)\,ds \\
&+\int_0^{2\pi} r_1 e^{-qcN^{(s)}}\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b-qcN^{(s)}+1\Big]\sin^2(s)\,ds \\
&+\int_0^{2\pi} r_0 e^{-qcN^{(s)}}\left[\frac{1}{2}qc\Big(qcN^{(s)}-2\Big)+\left(2qcN^{(s)}-\frac{1}{2}(qcN^{(s)})^2-1\right)\right]\sin^3(s)\,ds \\
&-\omega_1\int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)(qcN^{(s)}-2)e^{-p\tau}\Big]\tau\sin(s)\cos(s-\tau)\,d\tau\,ds \\
&+r_1\int_0^{2\pi}\int_0^{\infty} e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\sin(s)\sin(s-\tau)\,d\tau\,ds \\
&+\int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\left[\frac{b}{2}(q+p)\Big((qcN^{(s)})^2-4qcN^{(s)}+2\Big)e^{-p\tau}\right. \\
&\left.-\frac{1}{2}q^2c\Big(qcN^{(s)}-2\Big)\right]\sin(s)\sin^2(s-\tau)\,d\tau\,ds
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
\int_0^{2\pi} Q_2(s)\cos(s)\,ds &= -\omega_1\int_0^{2\pi}\cos^2(s)\,ds \\
&+r_1\int_0^{2\pi} e^{-qcN^{(s)}}\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b-qcN^{(s)}+1\Big]\cos(s)\sin(s)\,ds \\
&+\int_0^{2\pi} r_0 e^{-qcN^{(s)}}\left[\frac{1}{2}qc\Big(qcN^{(s)}-2\Big)+\left(2qcN^{(s)}-\frac{1}{2}(qcN^{(s)})^2-1\right)\right]\cos(s)\sin^2(s)\,ds \\
&-\omega_1\int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\tau\cos(s)\cos(s-\tau)\,d\tau\,ds \\
&+r_1\int_0^{2\pi}\int_0^{\infty} e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\cos(s)\sin(s-\tau)\,d\tau\,ds \\
&+\int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\left[\frac{b}{2}(q+p)\Big((qcN^{(s)})^2-4qcN^{(s)}+2\Big)e^{-p\tau}\right. \\
&\left.-\frac{1}{2}q^2c\Big(qcN^{(s)}-2\Big)\right]\cos(s)\sin^2(s-\tau)\,d\tau\,ds
\end{aligned}
\tag{4.49}
$$

Then, by using the expansions $\sin(s-\tau)=\sin(s)\cos(\tau)-\cos(s)\sin(\tau)$ and $\cos(s-\tau)=\sin(s)\sin(\tau)+\cos(s)\cos(\tau)$, and then, by reversing the order of integration, it is possible to get

$$
\begin{aligned}
0 &= r_1\left(\frac{bN^{(s)}(qcN^{(s)}-2)}{1+(p+q)^2}-\frac{qcN^{(s)}-1}{q^2+1}\right)e^{-qcN^{(s)}} \\
&+\omega_1\left(\frac{bN^{(s)}(q+p)^2(qcN^{(s)}-2)}{(1+(p+q)^2)^2}-\frac{q^2(qcN^{(s)}-1)}{(q^2+1)^2}\right)2r_0e^{-qcN^{(s)}} \\
0 &= r_1\left(\frac{bN^{(s)}(q+p)(qcN^{(s)}-2)}{1+(p+q)^2}-\frac{q(qcN^{(s)}-1)}{q^2+1}\right)e^{-qcN^{(s)}} \\
&+\omega_1\left[\left(\frac{bN^{(s)}((q+p)^2-1)(q+p)(qcN^{(s)}-2)}{(1+(p+q)^2)^2}-\frac{q(q^2-1)(qcN^{(s)}-1)}{(q^2+1)^2}\right)r_0e^{-qcN^{(s)}}-1\right]
\end{aligned}
$$

Then, from (4.43) and (4.45), the above equations will be simplified to

$$
\begin{aligned}
0 &= \omega_1\left(\frac{bN^{(s)}(q+p)^2(qcN^{(s)}-2)}{(1+(p+q)^2)^2}-\frac{q^2(qcN^{(s)}-1)}{(q^2+1)^2}\right)2r_0e^{-qcN^{(s)}}\\
0 &= \frac{r_1}{r_0}+\omega_1\Bigg[-1+\left(\frac{bN^{(s)}((q+p)^2-1)(q+p)(qcN^{(s)}-2)}{(1+(p+q)^2)^2}\right.\\
&\left.-\frac{q(q^2-1)(qcN^{(s)}-1)}{(q^2+1)^2}\right)r_0e^{-qcN^{(s)}}\Bigg] \qquad (4.50)
\end{aligned}
$$

These equations have the unique solution by solving for ω_1 and r_1, which is

$$\omega_1 = r_1 = 0 \qquad (4.51)$$

Morris [78] mentioned that this is the usual result of the Poincaré-Lindsted method when applied to an ordinary differential equation (or a functional differential equation). (4.51) gives a necessary and sufficient condition for (4.47) to be uniquely solvable for $u_1 \in P(2\pi)$. Now, Q_2 in (4.47) is just:

$$
\begin{aligned}
Q_2(s) &= r_0e^{-qcN^{(s)}}\left[\frac{1}{2}qc(qcN^{(s)}-2)+\left(\frac{-1}{2}(qcN^{(s)})^2+2qcN^{(s)}-1\right)\right]\sin^2(s)\\
&+r_0\int_0^\infty e^{-q(\tau+cN^{(s)})}\Bigg[\frac{b}{2}(q+p)((qcN^{(s)})^2-4qcN^{(s)}+2)e^{-p\tau}\\
&-\frac{1}{2}q^2c(qcN^{(s)}-2)\Bigg]\sin^2(s-\tau)\,d\tau
\end{aligned}
$$

(4.47) needs to have a non-zero solution in terms of trigonometric polynomials. Such a solution can be found by the method of undetermined coefficients which leads to

$$
\begin{aligned}
u_1 &= \frac{C_2}{C_1}\cos(2s)+\frac{2C_3}{C_1}\sin(2s) \qquad (4.52)\\
C_1 &= \Big[4(N^{(s)})^2(q^2+4)\Big(qcN^{(s)}-2\Big)^2b^2-8N^{(s)}\Big(qcN^{(s)}-2\Big)\Big(qcN^{(s)}-1\Big)(q^2+qp+4)b\\
&+4((q+p)^2+4)\Big(qcN^{(s)}-1\Big)^2\Big]\Big(r_0e^{-q(\tau+cN^{(s)})}\Big)^2\\
&+\Big[-8N^{(s)}(q^2+4)(q+p)\Big(qcN^{(s)}-2\Big)b+8q((q+p)^2+4)\Big(qcN^{(s)}-1\Big)\Big]r_0e^{-q(\tau+cN^{(s)})}\\
&+4(q^2+4)((q+p)^2+4)\\
C_2 &= \Big[-N^{(s)}(q^2+4)\Big(qcN^{(s)}-2\Big)\Big((qcN^{(s)}-2)^2-2\Big)b^2\\
&+\Big[-2+10qcN^{(s)}+2\Big(qcN^{(s)}\Big)^3-9\Big(qcN^{(s)}\Big)^2\Big](q^2+qp+4)b\\
&-qc((q+p)^2+4)\Big(qcN^{(s)}-1\Big)\Big(qcN^{(s)}-2\Big)\Big]\Big(r_0e^{-q(\tau+cN^{(s)})}\Big)^2\\
&+\Big[(q^2+4)\Big((qcN^{(s)}-2)^2-2\Big)(q+p)b-cq^2((q+p)^2+4)\Big(qcN^{(s)}-2\Big)\Big]r_0e^{-q(\tau+cN^{(s)})}\\
C_3 &= \Big[p\Big((qcN^{(s)}-1)^2+1\Big)\Big]b\Big(r_0e^{-q(\tau+cN^{(s)})}\Big)^2\\
&+\Big[(q^2+4)\Big((qcN^{(s)}-2)^2-2\Big)b-qc((q+p)^2+4)\Big(qcN^{(s)}-2\Big)\Big]r_0e^{-q(\tau+cN^{(s)})}
\end{aligned}
$$

Note: It is obvious that $\sin^2(s) = (1/2) - (1/2)\cos(2s)$ and $\sin^2(s-\tau) = (1/2) - (1/2)\cos(2(s-\tau))$, however in order to obtain a uniformly valid expansion, I have to eliminate the constants. So, indeed, C_1, C_2 and C_3 above are obtained by using $Q_2^*(s)$ below instead of using $Q_2(s)$ above.

$$\begin{aligned} Q_2^*(s) &= r_0 e^{-qcN^{(s)}}\Big[-\frac{1}{2}qc(qcN^{(s)}-2) - \Big(\frac{-1}{2}(qcN^{(s)})^2 + 2qcN^{(s)} - 1\Big)\Big]\frac{1}{2}\cos(2s) \\ &+ r_0\int_0^\infty e^{-q(\tau+cN^{(s)})}\Big[-\frac{b}{2}(q+p)((qcN^{(s)})^2 - 4qcN^{(s)} + 2)e^{-p\tau} \\ &+ \frac{1}{2}q^2c(qcN^{(s)}-2)\Big]\frac{1}{2}\cos(2(s-\tau))\,d\tau \end{aligned}$$

Morris [78] mentioned this as well.

4.5.5 The third order

Let me now look at the next equation at (order ϵ^3). Due to the result of the second order, $\omega_1 = r_1 = 0$ and again I use the approximation of $u_0(s-\omega\tau) = \sin(s-\omega\tau)$ of (4.46) such that

$$u_0(s-\omega\tau) = \sin(s-\omega\tau) = \sin(s-\tau) + (1-\omega)\tau\cos(s-\tau) + R,$$

where the remainder is given by $R = -\dfrac{(1-\omega)^2\tau^2}{2}\sin(s-\xi\tau)$ for $\xi \in (\omega, 1)$ or $\xi \in (1,\omega)$. So, I have:

$$\begin{aligned} Q_3(s) &= Q_4(s) \qquad (4.53) \\ Q_3(s) &= \dot{u}_2(s) - r_0\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b - qcN^{(s)} + 1\Big]e^{-qcN^{(s)}}u_2(s) \\ &- \int_0^\infty e^{-q(\tau+cN^{(s)})}r_0\Big[q\Big(qcN^{(s)}-1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]u_2(s-\tau)\,d\tau \\ Q_4(s) &= r_2 e^{-qcN^{(s)}}\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b - qcN^{(s)} + 1\Big]\sin(s) \\ &+ r_0 e^{-qcN^{(s)}}\Big[-\Big((qcN^{(s)}-2)^2 - 2\Big)b + cq\Big(qcN^{(s)}-2\Big)\Big]\sin(s)u_1(s) \\ &+ \frac{r_0}{6}e^{-qcN^{(s)}}\Big[qcb\Big((qcN^{(s)})^2 - 6qcN^{(s)} + 6\Big) - (qc)^2\Big(qcN^{(s)}-3\Big)\Big]\sin^3(s) - \omega_2\cos(s) \\ &- \omega_2\int_0^\infty r_0 e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\tau\cos(s-\tau)\,d\tau \\ &+ r_2\int_0^\infty e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) - bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\sin(s-\tau)\,d\tau \\ &+ \int_0^\infty r_0 e^{-q(\tau+cN^{(s)})}\Big[\Big((qcN^{(s)}-2)^2 - 2\Big)(q+p)be^{-p\tau} \\ &- cq^2\Big(qcN^{(s)}-2\Big)\Big]\sin(s-\tau)u_1(s-\tau)\,d\tau \\ &+ \int_0^\infty \Big[-cqb(q+p)\Big((qcN^{(s)})^2 - 6qcN^{(s)} + 6\Big)e^{-p\tau} \\ &+ q(qc)^2(qcN^{(s)}-3)\Big]\frac{r_0}{6}e^{-q(\tau+cN^{(s)})}\sin^3(s-\tau)\,d\tau \end{aligned}$$

Again, I need the conditions for the solvability of (4.53) to determine ω_2 and λ_2. However as I discussed earlier, in order to obtain ω_1 and λ_1, I can use the theorem as follows which is mentioned in Landman [70] and is proved by Cushing [25]:

Theorem 4.5.5.1

The non-homogeneous equation (4.53) has a solution in $P(2\pi)$ if and only if

$$\int_0^{2\pi} Q_4(s)\sin(s)\,d\,s = \int_0^{2\pi} Q_4(s)\cos(s)\,d\,s = 0$$

where $\sin(s)$ *and* $\cos(s)$ *are the two independent solutions of* $Q_3(s) = 0$ *in* $P(2\pi)$.

$$\begin{aligned}
\int_0^{2\pi} Q_4(s)\sin(s)\,d\,s &= -\omega_2 \int_0^{2\pi} \cos(s)\sin(s)\,d\,s \\
&+r_2 \int_0^{2\pi} e^{-qcN^{(s)}}\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b - qcN^{(s)}+1\Big]\sin^2(s)\,d\,s \\
&+\int_0^{2\pi} r_0 e^{-qcN^{(s)}}\Big[-\Big((qcN^{(s)}-2)^2-2\Big)b + cq\Big(qcN^{(s)}-2\Big)\Big]\sin^2(s)u_1(s)\,d\,s \\
&+\int_0^{2\pi} \frac{r_0}{6}e^{-qcN^{(s)}}\Big[qcb\Big((qcN^{(s)})^2-6qcN^{(s)}+6\Big)-(qc)^2\Big(qcN^{(s)}-3\Big)\Big]\sin^4(s)\,d\,s \\
&-\omega_2 \int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\tau\cos(s-\tau)\sin(s)\,d\,\tau\,d\,s \\
&+r_2 \int_0^{2\pi}\int_0^{\infty} e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\sin(s)\sin(s-\tau)\,d\,\tau\,d\,s \\
&+\int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\Big[\Big((qcN^{(s)}-2)^2-2\Big)(q+p)be^{-p\tau} \\
&-cq^2(qcN^{(s)}-2)\Big]\sin(s)\sin(s-\tau)u_1(s-\tau)\,d\,\tau\,d\,s \\
&+\int_0^{2\pi}\int_0^{\infty} \frac{r_0}{6}\Big[-cqb(q+p)\Big((qcN^{(s)})^2-6qcN^{(s)}+6\Big)e^{-p\tau} \\
&+q(qc)^2(qcN^{(s)}-3)\Big]e^{-q(\tau+cN^{(s)})}\sin(s)\sin^3(s-\tau)\,d\,\tau\,d\,s
\end{aligned} \tag{4.54}$$

$$\begin{aligned}
\int_0^{2\pi} Q_4(s)\cos(s)\,d\,s &= -\omega_2 \int_0^{2\pi} \cos^2(s)\,d\,s \\
&+r_2 \int_0^{2\pi} e^{-qcN^{(s)}}\Big[N^{(s)}\Big(qcN^{(s)}-2\Big)b - qcN^{(s)}+1\Big]\cos(s)\sin(s)\,d\,s \\
&+\int_0^{2\pi} r_0 e^{-qcN^{(s)}}\Big[-\Big((qcN^{(s)}-2)^2-2\Big)b + cq\Big(qcN^{(s)}-2\Big)\Big]\sin(s)\cos(s)u_1(s)\,d\,s \\
&+\int_0^{2\pi} \frac{r_0}{6}e^{-qcN^{(s)}}\Big[qcb\Big((qcN^{(s)})^2-6qcN^{(s)}+6\Big)-(qc)^2\Big(qcN^{(s)}-3\Big)\Big]\cos(s)\sin^3(s)\,d\,s \\
&-\omega_2 \int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\tau\cos(s-\tau)\cos(s)\,d\,\tau\,d\,s \\
&+r_2 \int_0^{2\pi}\int_0^{\infty} e^{-q(\tau+cN^{(s)})}\Big[q\Big(qcN^{(s)}-1\Big) \\
&-bN^{(s)}(q+p)\Big(qcN^{(s)}-2\Big)e^{-p\tau}\Big]\cos(s)\sin(s-\tau)\,d\,\tau\,d\,s \\
&+\int_0^{2\pi}\int_0^{\infty} r_0 e^{-q(\tau+cN^{(s)})}\Big[\Big((qcN^{(s)}-2)^2-2\Big)(q+p)be^{-p\tau} \\
&-cq^2\Big(qcN^{(s)}-2\Big)\Big]\cos(s)\sin(s-\tau)u_1(s-\tau)\,d\,\tau\,d\,s \\
&+\int_0^{2\pi}\int_0^{\infty} \frac{r_0}{6}\Big[-cqb(q+p)\Big((qcN^{(s)})^2-6qcN^{(s)}+6\Big)e^{-p\tau} \\
&+q(qc)^2(qcN^{(s)}-3)\Big]e^{-q(\tau+cN^{(s)})}\cos(s)\sin^3(s-\tau)\,d\,\tau\,d\,s
\end{aligned} \tag{4.55}$$

Then, from (4.52), I have $u_1(s) = \frac{C_2}{C_1}\cos(2s) + \frac{2C_3}{C_1}\sin(2s)$ and $u_1(s-\tau) = \frac{C_2}{C_1}\cos(2s-2\tau) + \frac{2C_3}{C_1}\sin(2s-2\tau)$. Then, by substituting these into (4.54) and (4.55), I obtain

$$\begin{aligned}
0 &= \omega_2\left(\frac{bN^{(s)}(q+p)^2(qcN^{(s)}-2)}{(1+(p+q)^2)^2} - \frac{q^2(qcN^{(s)}-1)}{(q^2+1)^2}\right)2r_0\pi e^{-qcN^{(s)}} \\
&+\left(\frac{qcb((qcN^{(s)})^2-6qcN^{(s)}+6)}{8(1+(p+q)^2)} - \frac{(qc)^2(qcN^{(s)}-3)}{8(q^2+1)}\right)r_0\pi e^{-qcN^{(s)}} \\
&+\left(\frac{b\Big((qcN^{(s)})^2-4qcN^{(s)}+2\Big)}{2(1+(p+q)^2)} - \frac{qc(qcN^{(s)}-2)}{2(q^2+1)}\right)\frac{C_2}{C_1} \\
&+\left(\frac{b(q+p)\Big((qcN^{(s)})^2-4qcN^{(s)}+2\Big)}{2(1+(p+q)^2)} - \frac{q^2c(qcN^{(s)}-2)}{2(q^2+1)}\right)\frac{2C_3}{C_1} \qquad (4.56)\\
0 &= \pi\omega_2\left[\left(\frac{bN^{(s)}((q+p)^2-1)(q+p)(qcN^{(s)}-2)}{(1+(p+q)^2)^2} - \frac{q(q^2-1)(qcN^{(s)}-1)}{(q^2+1)^2}\right)r_0 e^{-qcN^{(s)}} - 1\right] \\
&+r_2\left(\frac{bN^{(s)}(q+p)(qcN^{(s)}-2)}{1+(p+q)^2} - \frac{q(qcN^{(s)}-1)}{q^2+1}\right)\pi e^{-qcN^{(s)}} \\
&+\left(\frac{qcb(q+p)((qcN^{(s)})^2-6qcN^{(s)}+6)}{8(1+(p+q)^2)} - \frac{c^2q^3(qcN^{(s)}-3)}{8(q^2+1)}\right)r_0\pi e^{-qcN^{(s)}} \\
&+\left(\frac{b(q+p)\Big((qcN^{(s)})^2-4qcN^{(s)}+2\Big)}{2(1+(p+q)^2)} - \frac{q^2c(qcN^{(s)}-2)}{2(q^2+1)}\right)\frac{C_2}{C_1} \\
&+\left(\frac{b(q+p)^2\Big((qcN^{(s)})^2-4qcN^{(s)}+2\Big)}{2(1+(p+q)^2)} - \frac{q^3c(qcN^{(s)}-2)}{2(q^2+1)}\right)\frac{2C_3}{C_1} \\
&+\left(-qc-b+2bqcN^{(s)} - \frac{b(qcN^{(s)})^2}{2} + \frac{(qc)^2N^{(s)}}{2}\right)\frac{2C_3}{C_1}r_0\pi e^{-qcN^{(s)}} \qquad (4.57)
\end{aligned}$$

These are the equations which give the values of ω_2 and r_2. Then, again, by using the values $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r_0 = 21.16679874$ and $b = 0.017781473$, I obtain $r_2 = 0.00268149 > 0$. (4.51) tells that the first effects of the nonlinearity on $\omega(\epsilon)$ and $r(\epsilon)$ occur at second order, and from (4.56) and (4.57), it is possible to have the conclusion as follows:

Conclusion 4.5.5.1

The bifurcation of (4.24) is sub-critical if $r_2 < 0$ or supercritical if $r_2 > 0$.

Note: When $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r_0 = 21.16679874$ and $b = 0.017781473$, I obtain $r_2 = 0.00268149 > 0$ from (4.56) and (4.57). When $N^{(s)} = 102.1291175$, $p = 1$, $c = 1$, $r_0 = 7.398532026$ and $q = 0.02$, I obtain $r_2 = 0.03339774 > 0$ from (4.56) and (4.57). So, in both of the cases, I have supercritical periodic solutions.

4.6 The bifurcation diagram

In this section, I will produce the diagrams like the below.

Remark :

1. In Section 4.4.2, I showed that if $q > r$, $N^{(s)} = 0$ of (4.17) is stable or if $q < r$, $N^{(s)} = 0$ of (4.17) is unstable (see Conclusion 4.4.2.1) and I also showed that by using Routh-Hurwitz criterion, if $A > 0$, $B > 0$ in (4.26), $N^{(s)} > 0$ of (4.17) is stable, otherwise unstable (see Conclusion 4.4.2.2).

2. I constructed a 2π periodic solution of (4.17) when $B = 1$ and $A = 0$ in (4.26) by using Poincaré-Lindsted method in Section 4.5 (Section 4.5.1, 4.5.3, 4.5.4, 4.5.5).

3. When $r = q$, there exists only $N^{(s)} = 0$. However, the perturbation analysis in Section 4.4.2 fails since when $N^{(s)} = 0$ and $r = q$, (4.26) has solution $\lambda = -q - p, 0$.

I obtained A and B in Section 4.4.2, such that

$$\begin{aligned} A &= 2q + p - re^{-qcN^{(s)}}\left[1 - qcN^{(s)} - 2bN^{(s)} + qcb(N^{(s)})^2\right] \\ B &= q(p+q) - re^{-qcN^{(s)}}\left[q + p - cq^2N^{(s)} - cpqN^{(s)} - 2bqN^{(s)} + bc(qN^{(s)})^2\right] \end{aligned}$$

Since $p > 0$, $q > 0$, $r > 0$ and $e^{-qcN^{(s)}} > 0$, it is obvious that $q(p+q) > 0$ and the sign of $B_1(b, c, p, q, N^{(s)})$ such that

$$B_1(b, c, p, q, N^{(s)}) = \left((q + p - cq^2N^{(s)} - cpqN^{(s)} - 2bqN^{(s)} + bc(qN^{(s)})^2\right)e^{-qcN^{(s)}}$$

has a big part to determine the sign of the B above. Therefore, I have the conclusion as follows:

Conclusion 4.6.0.2

B is positive if $B_1(b, c, p, q, N^{(s)}) < 0$ or if $0 < B_1(b, c, p, q, N^{(s)}) < \frac{q(p+q)}{r}$.

Similarly, it is obvious that $2q + p > 0$ the sign of $A_1(b, c, p, q, N^{(s)})$ such that

$$A_1(b, c, p, q, N^{(s)}) = \left((1 - qcN^{(s)} - 2bN^{(s)} + qcb(N^{(s)})^2)\right)e^{-qcN^{(s)}}$$

has a big part to determine the sign of the A above. Hence, I have a similar conclusion to A.

Conclusion 4.6.0.3

A is positive if $A_1(b, c, p, q, N^{(s)}) < 0$ or if $0 < A_1(b, c, p, q, N^{(s)}) < \frac{2q+p}{r}$.

Hence, if the conditions of Conclusion 4.6.0.2 and 4.6.0.3 are satisfied, $N^{(s)} > 0$ of (4.17) is stable. This is of course the equivalent condition to that of Conclusion 4.4.2.2 in Section 4.4.2.

4.6.1 Figure 4.1

Since $A_1(b, c, p, q, N^{(s)})$ and $B_1(b, c, p, q, N^{(s)})$ do not depend on $r > 0$ and from Conclusion 4.6.0.2 and 4.6.0.3, I can have another conclusion as follows

Conclusion 4.6.1.1

If $A_1(b,c,p,q,N^{(s)}) < 0$ and $B_1(b,c,p,q,N^{(s)}) < 0$, as far as $r > q > 0$, $A > 0$ and $B > 0$, that is, $N^{(s)} > 0$ of (4.17) is stable.

Comment 4.6.1.1

- As in Conclusion 4.4.1.1 of Section 4.4.1, if $r < q$, $N^{(s)} > 0$ of (4.17) does not exist.
- Moreover, if $N^{(s)} < 0$ and b, p, r, c and q are all positive, $A_1(b,c,p,q,N^{(s)}) \geq 0$ and $B_1(b,c,p,q,N^{(s)}) \geq 0$. (However, I do not consider $N^{(s)} < 0$ as I stated in Section 4.2).

Hence, I can draw the diagram below:

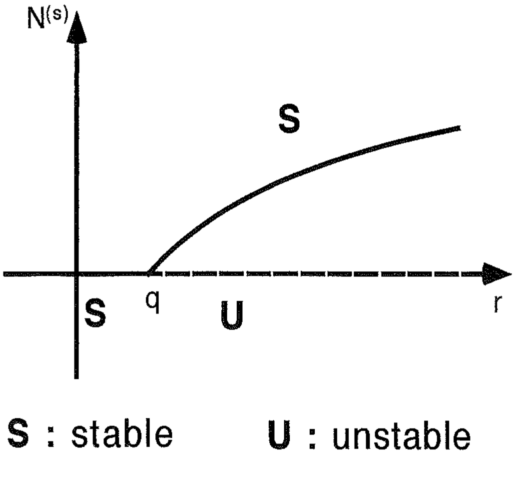

Figure 4.1: $N^{(s)} = 0$ *of (4.17) is stable if $q > r$ and is unstable if $q < r$. $N^{(s)} > 0$ of (4.17) is stable if $A_1(b,c,p,q,N^{(s)}) < 0$, $B_1(b,c,p,q,N^{(s)}) < 0$, $r > q > 0$, $A > 0$ and $B > 0$.*

4.6.2 Figure 4.2

If $A_1(b,c,p,q,N^{(s)}) < 0$, $A = 0$, since b, $N^{(s)}$, p, r, c and q are all positive. Now, let me introduce $g(r)$ such that

$$g(r) \equiv A = 2q + p - rA_1(b,c,p,q,N^{(s)})$$

and suppose that I have values of $N^{(s)}$, p, c, q, r and b such that $A = 0$ and $B > 0$- for example, $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$. It is obvious that $g(r) = 0$. Now, let me fix the values b, $N^{(s)}$, p, c and q, for example, $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$ and $b = 0.017781473$ and change the value of r. If I substitute r^* into $g(r)$ such that $q < r^* < r$, I have $g(r^*) > 0$. Hence, if I also have $B > 0$, $N^{(s)} > 0$ of (4.17) is stable. If I substitute r^{**} into $g(r)$ such that $q < r < r^{**}$, I have $g(r^{**}) < 0$. So, $N^{(s)} > 0$ of (4.17) is unstable. Hence, by also considering Comment 4.6.1.1, it is possible to draw the diagram (Figure 4.2) below:

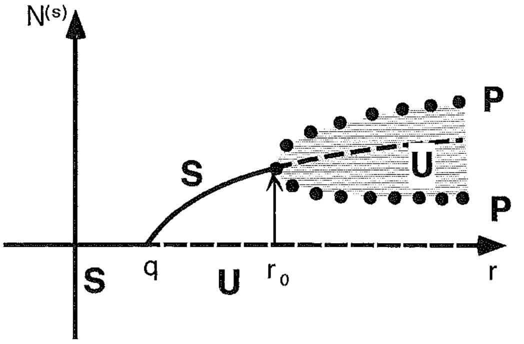

Figure 4.2: $N^{(s)} = 0$ *of (4.17) is stable if* $q > r$ *and is unstable if* $q < r$. *As I discussed in the above* $N^{(s)} > 0$ *of (4.17) is stable if I have* r^* *such that* $q < r^* < r$ *and* $g(r^*) > 0$ *and* $N^{(s)} > 0$ *of (4.17) is unstable if I have* r^{**} *such that* $q < r < r^{**}$ *and* $g(r^{**}) < 0$ *by assuming that I have values of* $N^{(s)}$, p, c, q, r *and* b *such that* $A = 0$ *and* $B > 0$.

Note : The r_0 is regarded as the value when expanding r like (4.38)

$$r = r(\epsilon) = r_0 + \epsilon r_1 + \epsilon^2 r_2 + \cdots$$

in Section 4.5.2. So, when $r_0 = 21.16679874$, the periodicity starts.

4.7 Numerical solution of (4.17)

In this section, I will deal with the numerical solution of (4.17), which is:

$$N(t) = \int_0^\infty e^{-q(\tau+cN(t-\tau))} r(1 - be^{-p\tau} N(t-\tau)) N(t-\tau)\, d\tau$$

Indeed, there is no direct methods which apply the equations (4.17) or more generally (1.3) when $F = F(\tau, N(t-\tau))$. However, it is possible to apply the methods which are used to solve standard Volterra equation of the second kind like

$$f(t) = g(t) + \int_0^t K(t, s, f(s))\, ds \text{ for } 0 \le t \le T \tag{4.58}$$

in order to see the long time behaviour of solutions of them. Of course, I have to slightly modify the methods. Linz [72](page 95 and page 96) and Brunner and Howen [8] introduce some of the numerical methods which are applicable to the equation like (4.58). In particular, Linz made assumptions to the equation (4.58) as follows.

Assumption 4.7.0.1

- $g(t)$ is continuous function in $0 \le t \le T$ for $0 < T$,

- the kernel $K(t, s, y)$ is continuous in $0 \le s \le t \le T$, $-\infty < y < \infty$,
- the kernel satisfies the Lipschitz condition $|K(t, s, y_1) - K(t, s, y_2)| \le L|y_1 - y_2|$ $\forall$ s, t such that $0 \le s \le t \le T$ and $\forall$ y_1, y_2.

Under the one of the assumptions of Assumption 4.1.0.6, which is

- Without loss of generality $t = 0$ can be fixed as a reference time and it is possible to suppose that it is given the function $N_0(t)$, $-\infty < t \le 0$, such that $N(t) = N_0(t)$ for $-\infty < t \le 0$.

I can convert (4.17) like:

$$\begin{aligned} N(t) &= \int_0^\infty e^{-q(\tau + cN(t-\tau))} r(1 - be^{-p\tau} N(t-\tau)) N(t-\tau)\, d\tau \\ &= \int_{-\infty}^t e^{-q((t-\tau) + cN(\tau))} r(1 - be^{-p(t-\tau)} N(\tau)) N(\tau)\, d\tau \\ &= \int_0^t e^{-q((t-\tau) + cN(\tau))} r(1 - be^{-p(t-\tau)} N(\tau)) N(\tau)\, d\tau \\ &\quad + \int_{-\infty}^0 e^{-q((t-\tau) + cN(\tau))} r(1 - be^{-p(t-\tau)} N(\tau)) N(\tau)\, d\tau \end{aligned}$$

Since $t \in [0, \infty)$, it is possible to define $f_4(t)$ such that

$$f_4(t) = \int_{-\infty}^0 e^{-q((t-\tau) + cN(\tau))} r(1 - be^{-p(t-\tau)} N(\tau)) N(\tau)\, d\tau \tag{4.59}$$

then, eventually I have (4.60), which is

$$N(t) = f_4(t) + \int_0^t e^{-q((t-\tau) + cN(\tau))} r(1 - be^{-p(t-\tau)} N(\tau)) N(\tau)\, d\tau \tag{4.60}$$

$$f_4(t) = \int_{-\infty}^0 e^{-q((-\tau) + cN(\tau))} r(1 - be^{-p(-\tau)} N(\tau)) N(\tau)\, d\tau \tag{4.61}$$

where (4.61) can be regarded as the initial condition of (4.60).

Note : I mentioned this conversion in Section 2.2. This is done after letting $\tau \to t - \tau$.

4.7.1 The special case (4.60) and Assumption 4.7.0.1

It is possible to say that (4.60) meets the Assumption 4.7.0.1, due to the statements as follows.

1. $f_4(t)$ in (4.60) is continuous function in $0 \le t \le T$. Hence, assumption of $g(t)$ in (4.58) can apply to $f_4(t)$.
2. the kernel $e^{-q((t-\tau)+cy)} r(1 - be^{-p(t-\tau)} y) y$ is continuous in $0 \le s \le t \le T$, $-\infty < y < \infty$,
3. the kernel satisfies the Lipschitz condition

$$|e^{-q((t-\tau)+cy_1)} r(1 - be^{-p(t-\tau)} y_1) y_1 - e^{-q((t-\tau)+cy_2)} r(1 - be^{-p(t-\tau)} y_2) y_2| \le L|y_1 - y_2|$$

$\forall$ s, t such that $0 \le s \le t \le T$ and $\forall$ y_1, y_2.

Hence, in principal, it is possible to apply the methods of Linz [72] to approximate the long time behaviour of the solution of (4.17).

4.7.2 Procedure 4.62

Linz [72] introduced the procedure as follows to approximate the solution to (4.58)

$$F_n = g(t_n) + h\left(\frac{1}{2}K(t_n, t_0, F_0) + \sum_{i=0}^{n-1} K(t_n, t_i, F_i) + \frac{1}{2}K(t_n, t_n, F_n)\right), \; n = 1, 2, 3, \ldots \tag{4.62}$$

with $F_0 = g(0)$ and F_n denotes the approximate value of $f(t_n)$. Then, he stated about this procedure as follows.

- He supposes that for a given step size $h > 0$ we know the solution at points $t_i = ih$, $i = 0, \ldots, n-1$. An approximation to $f(t_n)$ can then be computed by replacing the integral on the right hand side of (4.58) (or (4.60)) by a numerical integration rule equation for $f(t_n)$. Since $f(t_0) = g(0)$, the approximate solution can be computed in this step by step fashion. The particulars of the algorithm depend primarily on the integration rule we choose.

- The unknown F_n is defined by Procedure 4.62 implicitly, but for the sufficiently small h the equation has a unique solution. In the linear case we can of course solve it directly for F_n; in the nonlinear case we would normally use some iterative technique to solve for F_n to within a desired accuracy.

Note : I chose Procedure 4.62 as the algorithm, which is so called composite trapezoidal rule and I chose Newton's method to solve for F_n.

4.7.3 On the initial condition

I am using Procedure 4.62 to approximate the solution to (4.60). However, there is another issue I have to discuss. That is about initial condition. In section 4.7.1, I stated that assumption of $g(t)$ in (4.58) can apply to $f_4(t)$ in (4.60). However, $g(t)$ is a function whose value is easily determined like e^{-t} or $e^{-t}(\sin t - \cos t)$. This is not like $f_4(t)$ above. Since $f_4(t)$ is like a functional, technically it is not easy to determine the value of $f_4(t)$. In other words, the entire models I have dealt with in this project do not have starting points which are easily evaluated. So, even if the conditions of Assumption 4.7.0.1 is satisfied, the Procedure 4.62 cannot be applied directly because the starting points are not easily determined. However, as I mentioned in the beginning of this section, I am interested in the long time behaviour of (4.17) and as $\tau \longrightarrow -\infty$, $e^{-q((t-\tau)+c\omega)} r(1-be^{-p(t-\tau)}\omega) \longrightarrow 0$ where ω is a positive constant. So, I use the functions which decay as $t \to \infty$, (for instance, e^{-10t}) as initial functions of $f_4(t)$, rather than calculate $\int_{-\infty}^{0} e^{-q((t-\tau)+cN(\tau))} r(1-be^{-p(t-\tau)}N(\tau))N(\tau)\, d\tau$. If I use the Procedure 4.62 under this concept, the approximations of (4.17) near initial values are not reliable but as $t \longrightarrow \infty$, the behaviour is close to the solution of (4.17). There are other things to discuss. For the detail, see later in Section 4.7.5.

4.7.4 The numerical solution of (4.17)

In this section, I will show the numerical solutions of (4.17) by using Procedure 4.62. In general, I used $h = 0.01$ in Procedure 4.62 and $f_4(t) = e^{-10t}$ as an initial function.

Note: The function like $f_4(t) = e^{-10t}$ is appropriate to use as the initial function. This is because e^{-10t} decays to zero as t gets larger and so, it does not affect the long time behaviour of (4.17). Remember that I am just interested in the long time behaviour. The behaviour near the initial conditions ($t = 0$) may not be accurate. (The initial function e^{-10t} is not a true function to calculate the numerical solutions of (4.17).)

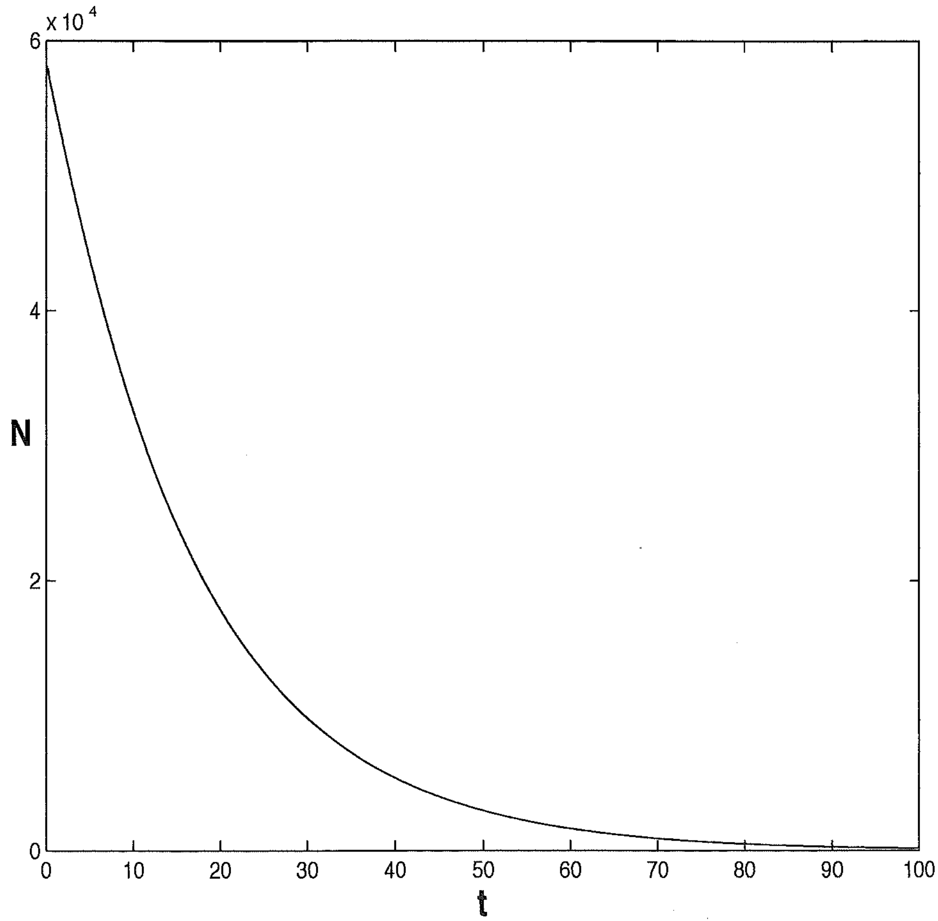

Figure 4.3: ($p = 0.35$, $c = 0.1$, $q = 0.1$, $r = 0.04$ and $b = 0.017781473$)

I use the values $p = 0.35$, $c = 0.1$, $q = 0.1$, $r = 0.04$ and $b = 0.017781473$. It is obvious that $r < q$. This is the only condition required to obtain the local stability of $N^{(s)} = 0$ of (4.17) in Conclusion 4.4.2.1 of Section 4.4.2.

Note : The values $p = 0.35$, $c = 0.1$, $q = 0.1$, $r = 0.04$ and $b = 0.017781473$ can represent when $N^{(s)} = 0$ is locally stable in Figure 4.2 of Section 4.6.2.

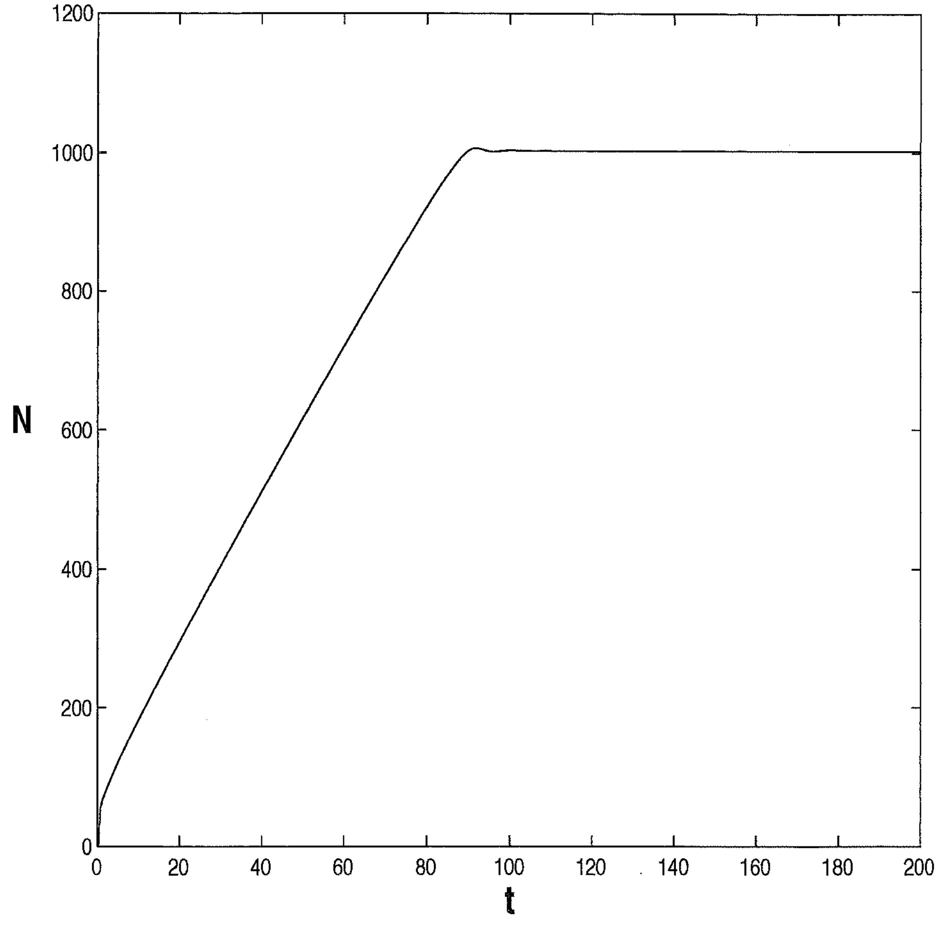

Figure 4.4: ($p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 12$ and $b = 0.017781473$)

I use the values $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 12$ and $b = 0.017781473$ and these are the values which make $A > 0$ and $B > 0$ in (4.26) which is the characteristic equation of (4.17), when (4.17) is linearized to (4.23) (see Section 4.4.2). So, these values satisfy the condition of the local stability of $N^{(s)} > 0$ required for Conclusion 4.4.2.2 in Section 4.4.2.

Note : By substituting the values $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 12$ and $b = 0.017781473$ into (4.19) in Section 4.4.2 and by solving (4.19) for $N^{(s)}$, it is possible to have $N^{(s)} = 1002.838031$. As you can see the Figure 4.4, the solution converges to near 1002.838031. Moreover, the values represent when $N^{(s)} > 0$ is locally stable in Figure 4.2 in Section 4.6.2.

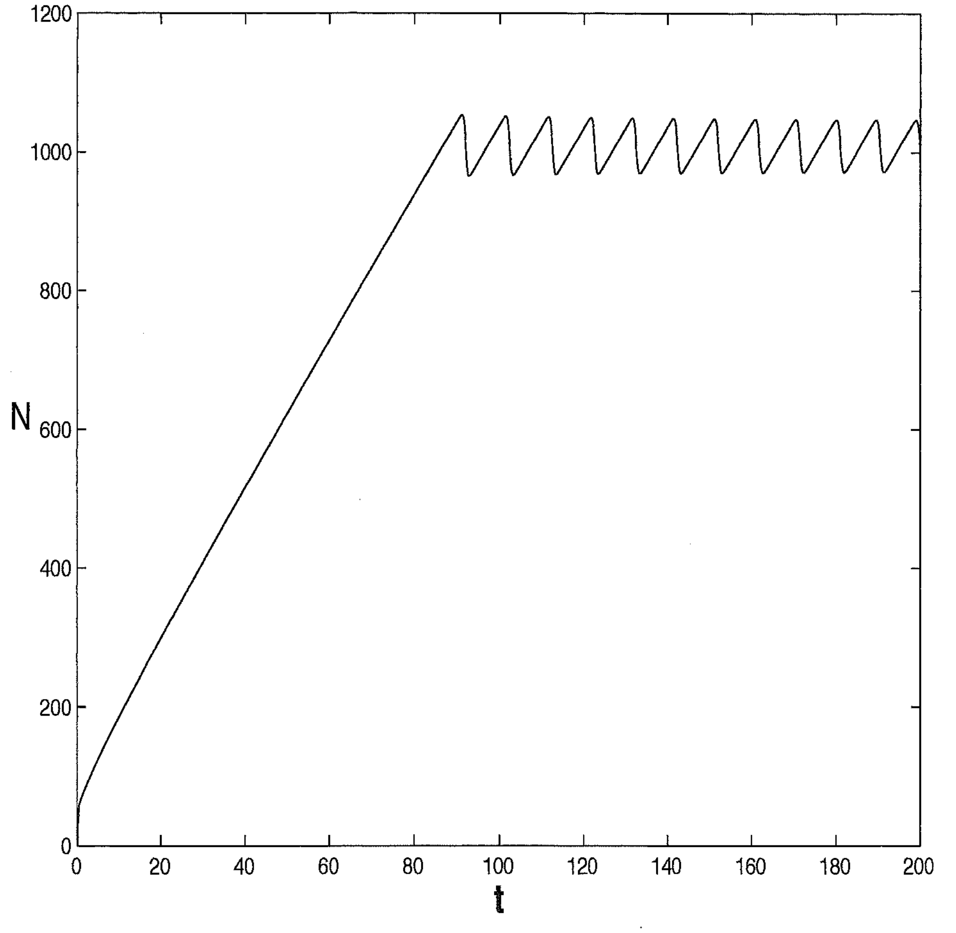

Figure 4.5: ($p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.167$ and $b = 0.017781473$)

I use the values $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.167$ and $b = 0.017781473$. I use slightly bigger value for r than $r = 21.16679874$. Since $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$ are the values in which the periodicity starts and $r = 21.16679874$ can be regarded as r_0 in Section 4.5.2 (see also Section 4.6.2), $r = 21.167$ can give a more clear periodicity than $r = 21.16679874$ can give.

Remark : By substituting $N^{(s)} = 1008.621627$, $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$, it is possible to obtain $A = 0$ and $B = 1 > 0$ in (4.26) which satisfy the condition for the Conclusion 4.4.3.1 in Section 4.4.3 and I also repeatedly used these values as an example when constructing 2π periodic solution of (4.17) by using Poincaré-Lindsted method in Section 4.5 (Section 4.5.1, 4.5.3, 4.5.4, 4.5.5).

4.7.5 Linear equations, Procedure 4.62 and some difficulties to apply Procedure 4.62 to (1.3) and (1.4)

I obtained the numerical solution of (4.17), namely,

$$N(t) = \int_0^\infty e^{-q(\tau+cN(t-\tau))} r(1-be^{-p\tau}N(t-\tau))N(t-\tau)\,d\tau$$

by using Procedure 4.62 and after modifying (4.17) like

$$\begin{aligned} N(t) &= \int_0^t e^{-q((t-\tau)+cN(\tau))} r(1-be^{-p(t-\tau)}N(\tau))N(\tau)\,d\tau \\ &\quad + \int_{-\infty}^0 e^{-q((t-\tau)+cN(\tau))} r(1-be^{-p(t-\tau)}N(\tau))N(\tau)\,d\tau \\ &= f_4(t) + \int_0^t e^{-q((t-\tau)+cN(\tau))} r(1-be^{-p(t-\tau)}N(\tau))N(\tau)\,d\tau \\ \text{where } f_4(t) &= \int_{-\infty}^0 e^{-q((t-\tau)+cN(\tau))} r(1-be^{-p(t-\tau)}N(\tau))N(\tau)\,d\tau \end{aligned}$$

In the Comment 2.2.0.1 of Section 2.2, I showed that

$$Y(t) = \int_{-\infty}^t (wbe^{-(t-\tau)} + 1 - w)e^{-(t-\tau)}Y(\tau)\,d\tau \tag{4.63}$$

has a general solution of $Y(t) = K_1\cos(t) + K_2\sin(t)$ where (K_1, $K_2 \in \Re$) and when $b = 5/3$ and $w = 3$ or it has a general solution of $Y(t) = K_3\cos(t) + K_4\sin(t)$ where (K_3, $K_4 \in \Re$) and when $b = 13/9$ and $w = 9/4$. I also showed that (2.11), namely,

$$Y(t) = \int_0^t (wbe^{-(t-\tau)} + 1 - w)e^{-(t-\tau)}Y(\tau)\,d\tau + f_2(t)$$

When $b = 5/3$, $w = 3$, if $f_2(t) = e^{-2t}$, (2.11) has only a particular solution of $Y(t) = \cos(t) + \sin(t)$ or if $f_2(t) = te^{-2t}$, (2.11) has only a particular solution of $Y(t) = (1/5)\cos(t) + (3/5)\sin(t) - (1/5)e^{-2t}$. When $b = 13/9$ and $w = 9/4$, and if $f_2(t) = e^{-2t}$, (2.11) has only a particular solution of $Y(t) = e^{-\frac{1}{2}(t)}\cos(t) + \frac{1}{2}e^{-\frac{1}{2}(t)}\sin(t)$, and if $f_2(t) = te^{-2t}$, (2.11) has only a particular solution of $Y(t) = -\frac{4}{13}e^{-2(t)} + \frac{4}{13}e^{-\frac{1}{2}(t)}\cos(t) + \frac{7}{13}e^{-\frac{1}{2}(t)}\sin(t)$. This means that by using different $f_2(t)$, (2.11) is converging to different solution when $b = 5/3$, $w = 3$ or (2.11) is converging to zero when $b = 13/9$ and $w = 9/4$. Then, by using the exactly similar process which is used to convert (2.6) into (2.9) in Section 2.2, (4.63) is identical to (2.11). This means that Procedure 4.62 is not the proper procedure to obtain the long time behaviour of (4.63) numerically, when $b = 5/3$, $w = 3$. Since every time changing $f_2(t)$, the long time behaviour is different or generally speaking, Procedure 4.62 is not reliable for the linear equations like (2.6) to obtain the long time behaviour of them. However, Procedure 4.62 is useful for (4.17) even if it has been used the same process above. Since even if I change the $f_4(t)$, they seem to converge to the same values ($N^{(s)} = 0$ or $N^{(s)} > 0$) or the same kind of periodic functions eventually.

The speciality of (4.17)

In Section 4.4, I showed that (4.17) has only 2 steady states in the region $N(t) \geq 0$ ($N^{(s)} = 0$ and $N^{(s)} > 0$) and as I discussed in Section 4.4, if $N^{(s)} = 0$ is stable, $N^{(s)} > 0$ does not exist and if $N^{(s)} = 0$ is unstable, $N^{(s)} > 0$ is stable. So, when I am using the values like $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 12$ and $b = 0.017781473$ (This is the case when $N^{(s)} > 0$ of (4.17) is locally stable and to the detail, see Section 4.7.4), (4.60) converges to $N^{(s)} > 0$ as t gets larger as far as $f_4(t) \to 0$ as $t \to \infty$. Moreover, the periodic solution of (4.17) is unique (This is when $N^{(s)} = 0$ and $N^{(s)} > 0$ are unstable). Hence, when I am using the values like $p = 0.35$, $c = 0.1$, $q = 0.0205$, $r = 21.16679874$ and $b = 0.017781473$ (This is the case when (4.17) has a periodic solution. For the detail, see Section 4.7.4), (4.60) converges to the periodic solution as t gets larger if $f_4(t) \to 0$ as $t \to \infty$. The periodic solution is the same solution as the one (4.17) converges to as t gets larger. Linear equations like (2.6) has only one steady state which is zero. This is unique, so if this steady state is asymptotically stable (for instance, (2.11) when $b = 13/9$ and $w = 9/4$), as far as $f_2(t) \to 0$, they converge to zero by using Procedure 4.62. However, (2.6) can also have the periodic solutions but they are not unique (for instance, (2.11) when $b = 5/3$, $w = 3$). It is impossible for (2.6) to behave like (4.17).

Note : By taking Laplace transforms to

$$Y(t) \quad = \quad f_{10}(t) + \int_0^t k(t-\tau)Y(\tau)\,d\tau$$

it is possible to obtain $\bar{Y} = \bar{f}_{10}/(1-\bar{k})$ where $\bar{Y}$, $\bar{f}_{10}$ and $\bar{k}$ are the Laplace transforms of Y, f_{10} and $\bar{k}(\lambda)$ respectively. Then, $Y = \sum a_n e^{\lambda_n t}$ where

$$a_n = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\bar{y}(\lambda) = -\bar{f}_{10}(\lambda_n)/\bar{k}'(\lambda_n) \text{ and } \bar{k}(\lambda_n) = 1$$

So, f_{10} affects the numerical solutions of (2.6) obtained by using Procedure 4.62.

The nonlinear equations which have more than 2 positive steady states

The equation like

$$X(t) \quad = \quad \int_{-\infty}^t (w - b\sin a_1(t-\tau))e^{-a_2(t-\tau)}\sin(X(\tau))X(\tau)\,d\tau \tag{4.64}$$

(where a_2, w and b are all positive constants and $a_1 \in \Re$) has more than 1 positive steady state and they can be stable at the same values of a_1, a_2, w and b. Again, (4.64) is convertible to

$$X(t) \quad = \quad \int_0^t (w - b\sin a_1(t-\tau))e^{-a_2(t-\tau)}\sin(X(\tau))X(\tau)\,d\tau + f_7(t) \tag{4.65}$$

$$\text{where } f_7(t) \quad = \quad \int_{-\infty}^0 (w - b\sin a_1(t-\tau))e^{-a_2(t-\tau)}\sin(X(\tau))X(\tau)\,d\tau$$

$$\text{and } X(0) \quad = \quad \int_{-\infty}^0 (w - b\sin a_1(-\tau))e^{a_2(\tau)}\sin(X(\tau))X(\tau)\,d\tau$$

So, it seems to be possible to solve (4.64) by using Procedure 4.62 as I have done to (4.17). However, if you try to solve (4.64) by using Procedure 4.62, you will find some difficulties. Since the positive steady states can be locally stable at the same values of a_1, a_2, w and b, by choosing different $f_7(t)$, the solution can converge to different steady states. Moreover, (4.64) can have more than 1 limit cycles, so, it is possible that by choosing different $f_7(t)$, the numerical solutions can converge to different periodic solutions. Hence, you must be careful to choose f_7, when you want to see the solutions which converge to a particular steady state or a periodic solution. However, in general, I cannot say anything about how to choose. (4.64) is not a equation which applies population dynamics but in the real situations, nobody can tell how many steady states the equations must have.

Recall : In general, it is possible to obtain all the steady states of (1.3) when $F = F(\tau, N(t-\tau))$, namely,

$$N(t) \quad = \quad \int_0^\infty F(\tau, N(t-\tau))B(N(t-\tau))N(t-\tau)\, d\tau \tag{4.66}$$

by solving (4.1) for $N^{(s)}$ in the Section 4.2, it is possible to obtain the values of every steady state and (4.11) in the Section 4.3 is the linearized equation of (4.66). The local stability of the steady states will be obtained by the following 2 ways.

1. If at least one of the conditions of Conclusion 4.3.1.1 in the Section 4.3.1 is satisfied, $N^{(s)}$ of (4.66) is locally stable. (See Conclusion 4.3.1.3 as well).
2. As I obtained the local stability of $N^{(s)}$ of (4.17) in Section 4.4.2, by substituting $Y(t) = \epsilon e^{\lambda t}$ in to (4.11), I can have an algebraic equation of λ. Then, if the real values of λ are all negative, $N^{(s)}$ of (4.66) is locally stable.

When the algebraic equation of λ has purely imaginary roots, (4.66) may have a periodic solution. However, you need to use a method like Poincaré-Lindsted method which is discussed in Section 4.5 (Section 4.5.1, 4.5.3, 4.5.4, 4.5.5) to show that this is sub-critical or supercritical, since generally speaking, (4.66) is not an ordinary differential equation but a functional equation even if some of the special forms of (4.66) are reducible to systems of ordinary differential equations (Section 2.4.2).

Final comments

Application of the integral equations (1.3) and (1.4) In this thesis, I have done some mathematical works of (1.3) and (1.4). However, I have not done anything about the applications to the real situation. There is only one purpose of the mathematical model to the number of the population, which is to predict the number of the future population. As to the future work, the applications of (1.3) and (1.4) are the most important thing.

Formulation As I discussed in Comment 1.2.2.1 of Section 1.2.2 and Comment 1.3.3.1 of Section 1.3.3, it is possible to formulate the functions of birth rate and survival rate which could meet the real situation better than the F and B which I have formulated in Chapter 1 (Section 1.2 and Section 1.3). However, as I insisted in Comment 1.2.2.1 and Comment 1.3.3.1, it is not always true that the more concepts of the real situation included, the more accurate birth rates or survival rates are obtained.

Lyapunov functionals I introduced some Lyapunov functionals to obtain the asymptotic stability of the zero solutions of linear integral equations of (2.1) and (2.2) as well as to obtain the conditions of the local stability of the steady states of nonlinear integral equations (1.3) and (1.4) in Section 2.5.1, 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.6.1, 2.6.2, 2.7.1 and 2.7.2. I also introduced 2 Lyapunov functionals to obtain conditions of the global stability of the zero solution of nonlinear integral equations (3.5) and (3.2) which are the special forms of (1.3) and (1.4) when $F = F(\tau, N(t-\tau))$ respectively in Chapter 3. I think that it is possible to construct other types of Lyapunov functionals to obtain different conditions of the stability of zero solutions of (2.1) and (2.2), different conditions of the local stability of the steady states of (1.3) and (1.4), and moreover, different conditions of the global stability of the the steady states of (1.3) and (1.4).

Numerical methods In Section 4.7, I discussed about numerical solutions of (4.17) which is solved by using Procedure 4.62. However, I also mentioned about the some difficulties of applying Procedure 4.62 to the equations (1.3) in Section 4.7.5. Since (1.3) and (1.4) are not generally solvable analytically, it is important to obtain the numerical solutions to apply (1.3) and (1.4) to the real situations. So, the improvement of the numerical procedures is also important as to the future work.

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