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A Note on the Probability of Winning a Lottery when the Number of Competitors is a Binomial Random Variable

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Abstract:

Consider a model in which a consumer faces a lottery with j other people for a prize, so that the probability of winning the prize is $1/(j+1)$. Now let j be a random variable, determined by the binomial distribution. Specifically, let there be n potential competitors for the consumer in the lottery, each with an independent probability of π of being a competitor. In this note, we show how the resulting expression for the expected value of $1/(j+1)$ using binomial probabilities can be simplified by means of the binomial theorem.

Keywords: Binomial Distribution, Binomial Theorem, Lottery
JEL Classifications: C10, C16

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1. Introduction.

Consider a model in which a consumer faces a lottery with j other people for a prize, so that the probability of winning the prize is $1/(j+1)$. Now let j be a random variable, determined by the binomial distribution. Specifically, let there be n potential competitors for the consumer in the lottery, each with an independent probability of π of being a competitor. In this environment, the probability of the consumer winning the lottery is the expected value of $1/(j+1)$ which is given by

$$E\left[\frac{1}{j+1}\right] = \sum_{j=0}^{n} \frac{\binom{n}{j}\pi^{j}(1-\pi)^{n-j}}{j+1}. \tag{1}$$

The purpose of this note is to derive a simpler expression for Equation (1). We do this by proving a general theorem, for which this probabilistic example is a special case.

2. The Main Result.

Theorem 2:

For any positive variables, x and y and any positive integer, n, we have the following identity:

$$\sum_{j=0}^{n} \frac{\binom{n}{j} x^{j} y^{n-j}}{j+1} \equiv \frac{(x+y)^{n+1} - y^{n+1}}{(n+1)x}.$$

Proof:

Multiplying both sides by x we can restate the identity as

$$\sum_{j=0}^{n} \frac{\binom{n}{j} x^{j+1} y^{n-j}}{j+1} \equiv \frac{(x+y)^{n+1} - y^{n+1}}{(n+1)}. \tag{2}$$

This expression clearly holds as an equality in the limit as $x \to 0$. To show that it is an identity, it is sufficient to show that the derivative of both sides w.r.t. x are identically equal. Note that

$$\frac{\partial}{\partial x}\sum_{j=0}^{n}\frac{\binom{n}{j}x^{j+1}y^{n-j}}{j+1} = \sum_{j=0}^{n}\binom{n}{j}x^{j}y^{n-j}, \text{ and}$$

$$\frac{\partial}{\partial x}\left(\frac{(x+y)^{n+1}-y^{n+1}}{(n+1)}\right) = (x+y)^{n}.$$

Finally, from the binomial theorem, we have

$$\sum_{j=0}^{n}\binom{n}{j}x^{j}y^{n-j} = (x+y)^{n},$$

which establishes the result.

□

3. Application to the Binomial Distribution.

Applying Theorem 1 to Equation (1), we have

$$E\left[\frac{1}{j+1}\right] = \frac{1-(1-\pi)^{n+1}}{(n+1)\pi}. \tag{3}$$

Consider now a model in which the n potential competitors for the consumer in the lottery have a type described by some value x independently drawn from a subset of the real line according to the common distribution function, F. Further, let the model be such that one of the n other consumers will be a competitor in the lottery if and only if $x > \bar{x}$, for some critical value $\bar{x}$, so that the binomial probability is $(1-F(\bar{x}))$. In this case, we can write Equation (3) as

$$E\left[\frac{1}{j+1}\right] = \frac{1-(F(\bar{x}))^{n+1}}{(n+1)(1-F(\bar{x}))}. \tag{4}$$