DEPARTMENT OF ECONOMICS AND FINANCE

COLLEGE OF BUSINESS AND ECONOMICS

UNIVERSITY OF CANTERBURY

CHRISTCHURCH, NEW ZEALAND

The Fixed Price Offer Mechanism in *Trade Me* Online Auctions

Seamus Hogan, Hamish Kidd, Laura Meriluoto, and Andrew Smith

WORKING PAPER

No. 51/2010

Department of Economics and Finance
College of Business and Economics
University of Canterbury
Private Bag 4800, Christchurch
New Zealand

The Fixed Price Offer Mechanism in *Trade Me* Online Auctions

Seamus Hogan*, Hamish Kidd†, Laura Meriluoto‡, Andrew Smith§

August 13, 2010

Abstract:

The Fixed Price Offer (FPO) mechanism in *Trade Me* auctions allows sellers to make a take-it-or-leave-it offer at the conclusion of an unsuccessful auction. We investigate the effects of the FPO option on strategies and outcomes in independent-value auctions. The FPO option induces some bidders with a value above the seller's reserve to wait for an FPO instead of bidding. Overall, the FPO option increases the probability of sale but reduces expected seller revenue compared to a standard auction. The impact of the FPO option is reduced when the number of bidders increases.

Keywords: fixed price offer, private value auction, on-line auction, optimal reserve price, second chance offer

JEL classification number: D44

* University of Canterbury; seamus.hogan@canterbury.ac.nz
† Macquarie Capital Advisers; Hamish.Kidd@macquarie.com
‡ Corresponding author. University of Canterbury; laura.meriluoto@canterbury.ac.nz
§ andrewersmith@gmail.com

We would like to warmly thank Phil Gunby, Simon Loertscher, Ian MacDonald, Andrew McLennan, Phil Meguire, Ilke Onur, Matthew Ryan, Alan Woodfield, the seminar participants at Victoria University of Wellington, the participants of the 2009 Canadian Economic Association Conference in Toronto and the participants of the 2010 Australasian Economic Theory Workshop in Melbourne for their helpful comments.

Laura Meriluoto gratefully acknowledges funding from University of Canterbury, College of Business and Economics Research Committee.

This paper builds on the 2008 Honours project of Hamish Kidd and the 2009 Honours project of Andrew Smith.

The Fixed Price Offer Mechanism in *Trade Me* Online Auctions

1. Introduction.

Online auction sites are used as platforms of exchange by millions of individuals, households and firms worldwide. From humble beginnings in 1995 when the first major online auction sites began to emerge, the popularity of buying and selling goods online has expanded rapidly to the point where the world's largest and most well known auction site, *eBay*, now has over 86.3 million active users and facilitates nearly $60 billion of merchandise trade (*eBay*, 2008).

Trade Me is New Zealand's primary online auction site, having fended off non-trivial efforts by *eBay*, *Amazon* and *Yahoo!* to penetrate the local market. *Trade Me* is also available for Australian residents although *eBay* is Australia's primary online auction site. The auction structure of *Trade Me* is in many respects similar to those used by *eBay*, *Yahoo!* and *Amazon*. It is an ascending bid auction where, in theory, the winner pays a price that is marginally higher than the willingness to pay of the bidder with the second highest valuation. There are, however, some important differences between the online auction designs. For example, *eBay* auctions have a "hard close" end rule whereas *Amazon* and *Trade Me* auctions extend the end time if bids are made within the last few minutes of the auction to allow other bidders the opportunity to respond.

A substantial number of the goods listed on *Trade Me* do not sell successfully, either because there are no bids on the item or because bidding does not meet the reserve price set by the seller. Sell-through rates for a category of goods vary from as low as five percent (photographs and drawings) to as high as 70 per cent (iPods) (*Trade Me*, 2009b). As with other on-line auction sites, potential bidders can choose to become a "watcher" by bookmarking an item being sold. The number of watchers is known by the seller. In the event that an item is not sold during the auction the seller can abandon his attempt to sell the good, relist the good on *Trade Me* or another auction site, or make a fixed-price offer (hereafter FPO). An FPO is a take-it-or-leave-it offer at a dollar amount chosen by the seller that can be offered to any subset of the bidders and watchers of an auction. The first person to accept the FPO wins the auction. The FPO option changes the auction in two important ways. First, it removes the commitment value of the reserve price because the seller of a failed auction has an incentive to offer an FPO at a price that could be lower than the reserve price. Rational bidders are aware of this incentive and can adjust their bidding strategies accordingly.[1]

[1] See, for example, Menezes and Ryan (2005) for a discussion on commitment value of reserve prices.

Second, the FPO mechanism makes the auction nonstandard in the sense that the winner of the FPO part of the auction is not necessarily the bidder with the highest valuation.[2] This implies that the auction mechanism is inefficient and that the seller is unable to extract the maximum possible revenue from the auction.

eBay auctions have a feature called a "second-chance offer" that can be offered to the highest bidder at a price equal to her last bid, which is in contrast to the FPO that can be offered to all bidders and watchers at any price chosen by the seller. To win the second-chance offer, the bidder must not only bid but also be the highest bidder (and the reserve price must not be met) whereas winning the FPO requires no bidding. The two mechanisms are likely to yield quite different outcomes due to these differences.

Carney (2007) reports that 17% of all successful *Trade Me* auctions sell through the FPO mechanism. Despite the popularity of the FPO mechanism, no theoretical or empirical work has been done to establish the effects of the FPO option on auction outcomes. The aim of this paper is to investigate how the FPO option alters bidding behaviour, the probability of an auction selling, the level of the seller's optimal reserve price and ultimately the seller's expected revenue compared to a static second-price ascending auction with full commitment value of the reserve price. We also investigate how the number of bidders in the market affects the FPO outcomes.

While there is no previous literature on fixed-price offers, a number of papers have addressed how other forms of post-auction seller behaviour can affect auction outcomes.

Salmon and Wilson (2008) investigate the second-chance offer feature in multiple unit auctions. They find that with two bidders and two units being auctioned off only mixed strategy equilibria exist, because both bidders want to hide their true willingness to pay in hopes of being the lower bidder who receives the SCO. As far as we know, there is no literature on the use of SCO in single-good auctions, but it appears that the main results of this mechanism are easy to derive from the Revenue Equivalence Theorem formalised by Myerson (1981) and Riley and Samuelson (1981), and summarised by Klemperer (2004). As long as the bidders can rationally predict the secret reserve price, the SCO effectively makes single good *eBay* auctions first-price auctions until the reserve is met. The reason is that a proxy bid that specifies a maximum willingness to pay that is below the reserve price will always take the bid to that maximum bid, and thus if this is the highest bid at the end of the auction, the bidder would receive an SCO at her own maximum bid. If the reserve is met during the auction, the second-price feature is restored. Therefore, despite the SCO feature, the *eBay* auction can be classified as a standard auction where the winner is the bidder with the highest willingness to pay. If the bidders are risk neutral with values that are drawn from

[2] See, for example, Krishna (2002) for a definition of a standard vs. non-standard auction.

an atomless strictly increasing distribution where a bidder with the lowest value expects zero surplus, the revenue-equivalence theorem applies and the seller's expected revenue is the same as with any optimal standard static auction.

McAfee and Vincent (1997) and Grant et. al. (2006) investigate the impact on the auction of the potential for relisting. Grant et. al. investigate the impact of relisting on reserve prices in a setting where sellers can hold a potentially infinite sequence of auctions in which the draw of bidders is independent from the previous draws. In their model, the reserve price is set higher than it would be in absence of the possibility of relisting, since the opportunity cost of having the auction passed in is less.

McAfee and Vincent also consider a dynamic auction where the seller can hold an infinite series of second-price (and first-price) auctions, but with the same set of bidders participating in each run of the auction. As with the Grant et al paper, the opportunity cost of an auction being passed in is less than would be the case without the possibility of relisting. The equilibrium is also affected, however, because the opportunity cost to bidders from not initiating bidding is also less when passed-in goods can be relisted. In a broad class of distributions including the standard linear model, this second effect dominates, leading to the reservation price being lower than would be the case in a single-shot auction,

The model in this paper is similar to McAfee and Vincent's in the sense that the optimal response of the bidders to the possibility of post-auction activity turns out to be a crucial determinant of the model's equilibrium.

We use an independent-values model where the value of each bidder is drawn from a common distribution. To keep the model tractable and facilitate comparisons with other models, we assume a linear model. That is, we assume that the seller and all bidders are risk neutral and that the distribution of bidders valuations is uniform. We further impose the normalising assumption that the seller's value of the good is zero, and that the support of the bidders' distribution of valuations is the unit interval, [0,1].

For the most part, our model follows the institutional set up of actual *Trade Me* auctions, but with two minor exceptions.

First, we impose the rule that the seller sets the start price equal to the reserve. In real *Trade Me* auctions many goods are listed this way but others are listed with a secret reserve that is higher than the start price.[3] Even when the reserve is secret, bidders can inquire about its level and sellers are often willing to reveal it. Furthermore, the *Trade Me* auction site informs bidders when the highest bid is within 15% of the reserve. Last, the FPO mechanism

[3] In the former case on *Trade Me* auctions, a yellow flag appears by the start price indicating that the reserve will be met after the first bid.

is not tied to bids or even bidders (as it can be offered to watchers), and thus this assumption is not too restrictive.

Second, we assume that the FPO is the only post-auction activity available to the seller outside of consuming the good at utility zero. This assumption implies that sellers cannot relist the good and that bidders do not expect the good to be relisted. In actual *Trade Me* auctions sellers have an option to relist their product as an alternative to offering the FPO, but we wish to focus our analysis solely on the impact of the FPO in this paper to isolate its effects from the potentially similar effects of relisting.

Initially, we consider the model with only two bidders, and then extend it to the general case of *n* bidders. The two-bidder model gives, we hope, an easy introduction to the FPO auction that should be accessible to non-auction theorists, while the *n*-bidder game allows us to see how the thickness of the market affects the outcomes of the FPO auction.

The paper is organized as follows. We present the main assumptions of the model for *n* bidders in Section 2. We solve for the two-bidder model in Section 3 and compare the results with those from the two-bidder model without the FPO option. We present the *n*-bidder model in Section 4, where we also discuss how increasing *n* influences the outcomes of the FPO auction. Section 5 concludes.

2. The Model.

There is one risk-neutral seller with a value of zero for the good he is selling.

There are *n* bidders, indexed by $i = 1, 2, ., n$. The bidders have private values drawn from a common uniform distribution $\upsilon_i \sim U[0,1]$ with density equal to one. The distribution of private values is common knowledge to all bidders and the seller.

The game has three stages. In Stage 1, the seller chooses the level of his reserve price and sets the starting bid equal to this reserve price. In Stage 2, bidders choose bidding strategies, and in Stage 3, conditional on there being no bids, the seller chooses whether to make a fixed-price offer and, if so, the FPO price. The seller chooses the level of his reserve price, *R*, at Stage 1 to maximize his total expected revenue of the auction, which includes the expected revenue from the regular auction and FPO revenue if the regular auction fails to meet reserve.

Because bids below the reserve price are not possible, the auction is successful whenever there is at least one bid made during the Stage-2 auction. The good is then sold to the highest bidder at a price that equals either the second highest bid (plus a nominal margin which we approximate to zero) if the second highest bid is greater than or equal to *R*, or at a price equal to *R* if there were no other bids made.

If no bids were made in the auction, the auction is unsuccessful or passed in and the seller offers an FPO at Stage 3. We assume that the FPO is the only post-auction activity available to the seller, so we do not allow the seller to relist the item, for example. A bidder accepts the FPO if $\upsilon_i \geq p_f$. If j bidders have $\upsilon_i \geq p_f$ they each have a probability, 1/*j*, of winning the FPO, so there is a lottery component in the FPO auction.

A complete statement of the strategy space for bidders in Stage 2 is quite complicated since each bidder has to choose whether and how much to bid at any point in continuous time as a function of the history of bids to that point. The analysis, however, can be simplified by eliminating dominated strategies from consideration at the outset: First, given the assumption that the good cannot be relisted and has a value of zero to the seller, it is clearly a dominated strategy for him to not make a fixed-price offer. Second, because the auction rules permit fixed-price offers only if the Stage-2 auction fails to generate a single bid, it is a dominated strategy for bidders to watch rather than bid once the first bid is made. Finally, once the first-bid is made in Stage 2, the fact that there cannot be a fixed-price offer means that this is a standard second-price auction, for which it is well-known that it is a dominant strategy for each bidder to bid her true valuation. [4]

Restricting the game by eliminating these dominated strategies, implies that in Stage 2, each bidder *i* is choosing between two strategies, which we refer to as “bid” or “watch”, indexed by *B* and *W*:

Strategy *B*: place an auto bid, $b_i = \upsilon_i$, whether or not any other bidder bids;

Strategy *W*: add the auction to the watch list and place an auto bid, $b_i = \upsilon_i$, only if the reserve is met during the auction.

With these simplifications, the game is as follows: A strategy for the seller is a reservation price, *R*, and a fixed-price offer, p_f, to be revealed and offered in the event that no bids are made during the Stage-2 auction. A strategy for each bidder *i* is a choice between Strategy *B* and Strategy *W*, as a function of *R* and her own valuation, $\hat{\upsilon}_i$.

At the start of Stage 2, bidders do not know what will be the fixed-price offer p_f, although they may be able to infer it from equilibrium reasoning. At the start of Stage 3, the seller only knows if any bids were placed in Stage 2, and not the exact strategies chosen by the bidders. Accordingly, the game has exactly the same structure as if the seller were required to choose the value of the fixed-price offer should the Stage-2 auction attract no bids at the same time as bidders choose their strategies. Stages 2 and 3 therefore constitute a simultaneous subgame. An equilibrium for this subgame has bidders choosing the optimal strategy as a function of the other bidders’ strategies, the fixed-price offer and the reservation

[4] This result as it is well-known from Vickrey (1961) and the subsequent auction theory literature.

price, and the seller choosing the optimal fixed-price offer as a function of the bidders' strategies and the reservation price. A subgame-perfect equilibrium for the full game is one in which the seller and the bidders play the equilibrium strategy for this subgame as a function of the reservation price, and the reservation price is set to maximise the seller's expected payoff given equilibrium response to that price.

Our analysis of this game is simplified by the following result:

Theorem 1:

Any subgame perfect equilibrium in pure strategies has $p_f \leq R,$ and has bidder strategies described by a critical valuation, $\hat{\upsilon} \geq R,$ common to all bidders, such that bidders will bid in the Stage-2 auction, if and only $\upsilon \geq \hat{\upsilon}.$

Proof:

The proof of this result is somewhat technical and so is given in the Appendix.

With this result, we use backward induction to solve for the game, finding the optimal p_f as a function of *R* and $\hat{\upsilon},$ and the optimal $\hat{\upsilon}$ as a function of *R* and $p_f.$ From this, we can find the equilibrium values of p_f and $\hat{\upsilon}$ as a function of *R*, and hence solve for the optimal *R*. We first present the model with just two bidders in Section 3 and subsequently extend the model to *n* bidders in Section 4.

3. The Two-bidder Model.

3.1 Stage 3: The Optimal Fixed-Price Offer.

In this section we find the optimal FPO price conditional on the regular auction being unsuccessful. The fact of the auction not attracting any bids in the Stage 2 of the game implies, from Theorem 1, that none of the bidders has a valuation in excess of the cut-off valuation, $\hat{\upsilon}.$ The resulting truncated distribution is uniform, with $\upsilon / \hat{\upsilon}$ being the probability that any bidder has a valuation no greater than $\upsilon.$

If a seller makes an FPO offer at price, $p_f,$ the probability that any one bidder has a valuation less than $p_f,$ and hence will not accept the offer, is $p_f / \hat{\upsilon}.$ The probability that neither bidder will accept the offer is then $(p_f / \hat{\upsilon})^2,$ and with the remaining probability, $1-(p_f / \hat{\upsilon})^2,$ at least one of the two bidders will accept the FPO. Thus, the expected payoff of the seller in Stage 3 conditional on there being no bids in Stage 2 is

$$\pi_3 = \left(1 - \frac{p_f^2}{\hat{\upsilon}^2}\right) p_f .$$

Optimizing π_3 w.r.t. p_f gives the optimal price of the fixed price offer conditional on an unsuccessful auction:

$$p_f^* = \frac{\hat{\upsilon}}{\sqrt{3}}, \tag{1}$$

and the maximized Stage 3 profit is

$$\pi_3^* = \frac{2\hat{\upsilon}}{3\sqrt{3}}. \tag{2}$$

3.2 Stage 2: Optimal Bidder Strategies

We start by considering the optimal strategy choice by bidder *i* when the fixed-price offer is p_f^* and bidder *j* $(i \neq j)$ has a cutoff valuation, $\hat{\upsilon}$, such that she will play Strategy *B* if and only if $\upsilon_j > \hat{\upsilon}$.[5]

Note that if $\upsilon_j > \hat{\upsilon}$, bidder *j* will choose to initiate bidding if necessary, and there is no difference between Strategy *B* and Strategy *W* for bidder *i*. To find the optimal strategy for bidder *i*, therefore, we need only consider the case where $\upsilon_j \leq \hat{\upsilon}$.

Let $CS_i^W(\upsilon_i)$ be the surplus to bidder *i* from following Strategy *W*. Conditional on the information that $\upsilon_j \leq \hat{\upsilon}$, bidder *j* will have a valuation less than p_f with probability, $p_f / \hat{\upsilon}$, in which case she would not accept the FPO, and a valuation in excess of p_f with a probability of $(\hat{\upsilon} - p_f)/\hat{\upsilon}$. In the first case, bidder *i* would be guaranteed to be the first bidder to accept the FPO, if she wanted to; in the second, she would have a 0.5 probability of being the first to accept. If $\upsilon_i > p_f$, then, we have

$$CS_i^W(\upsilon_i) = \left(\frac{p_f^*}{\hat{\upsilon}} + \frac{(\hat{\upsilon} - p_f^*)}{2\hat{\upsilon}} \right)(\upsilon_i - p_f^*). \tag{3}$$

Now let $CS_i^B(\upsilon_i)$ be the surplus to bidder *i* from following Strategy *B*. If $\upsilon_i \geq R$, bidder *i* will win the auction and pay the reserve price, *R*, if $\upsilon_j \leq R$; she will win the auction and pay υ_j if $R < \upsilon_j \leq \upsilon_i$; and she will lose the auction if $\upsilon_i < \upsilon_j$. Conditional on the information that $\upsilon_j \leq \hat{\upsilon}$, the respective probabilities of these three outcomes are $R/\hat{\upsilon}$, $(\upsilon_i - R)/\hat{\upsilon}$, and $(\hat{\upsilon} - \upsilon_i)/\hat{\upsilon}$. We then have

$$CS_i^B(\upsilon_i) = \frac{R}{\hat{\upsilon}}(\upsilon_i - R) + \frac{(\upsilon_i - R)}{\hat{\upsilon}}\left(\upsilon_i - \frac{R + \upsilon_i}{2} \right) = \frac{(\upsilon_i^2 - R^2)}{2\hat{\upsilon}}. \tag{4}$$

The term, $(R + \upsilon_i)/2$ in Equation (4) is the expected value of υ_j, conditional on the information that $R < \upsilon_j \leq \upsilon_i$.

[5] As a convention, we assume that bidders indifferent between the two strategies will play strategy *W*. This is just for expositional convenience. With a continuous distribution of bidders' valuations, the choice of strategy when indifferent has no impact on any of the equilibrium variables derived in the paper.

Note that Equation (3) is linear and increasing in υ_i whereas Equation (4) is convex. The optimum cutoff valuation for bidder *i* is therefore given where $CS_i^W(\upsilon_i) = CS_i^B(\upsilon_i)$. Theorem 1 implies we only need search for symmetric equilibria, in which case $CS_i^W(\hat{\upsilon}) = CS_i^B(\hat{\upsilon})$ for interior solutions where $\hat{\upsilon} < 1$. Substituting in $\upsilon_i = \hat{\upsilon}$ to Equations (3) and (4) and equating gives

$$\hat{\upsilon}^2 - (p_f^*)^2 = \hat{\upsilon}^2 - R^2,$$

and hence that

$$p_f^* = R, \quad \text{if } \hat{\upsilon} < 1.$$

From Equation (1), we then have

$$\hat{\upsilon} = \sqrt{3}R, \quad \text{if } \hat{\upsilon} < 1. \tag{5}$$

When $R > 1/\sqrt{3}$, we have a corner solution, $\hat{\upsilon} = 1$, in which neither bidder will choose to initiate bidding in the Stage 2 auction, no matter what their valuation. The equilibrium value of $\hat{\upsilon}$ can therefore be given as,

$$\hat{\upsilon} = \min\{\sqrt{3}R, 1\}. \tag{6}$$

It is easy to see from Equation (6) that $\hat{\upsilon} > R$ when $R<1$, and thus that bidders with values above the seller's reserve may not bid during the regular auction.

3.3 Stage 1: The Optimal Reserve Price.

The expected payoff to the seller depends on the combination of valuations of the two bidders. If $\upsilon_i \le \hat{\upsilon}$ for both bidders, then neither will bid and the seller will receive the Stage-3 profit, π_3. If at least one of the two bidders, *i*, has $\upsilon_i > \hat{\upsilon}$, then the auction will complete in Stage 2, and the seller will receive either the reserve price or the lower of the two bidders' valuations, whichever is greater. Conditional on the lower valuation being in the range, $[R, \hat{\upsilon}]$, the expected value of this valuation is $(R + \hat{\upsilon})/2$. Conditional on both valuations exceeding $\hat{\upsilon}$, the expected value of the lower of the two valuations is $(2\hat{\upsilon} + 1)/3$. The four possibilities and their associated expected payoffs, π, and probabilities, *Prob*, are then:

- $\upsilon_i \le \hat{\upsilon}$, for both $i \in \{1,2\}$; $\pi = 2\hat{\upsilon}/(3\sqrt{3})$; $Prob = \hat{\upsilon}^2$.
- $\upsilon_i > \hat{\upsilon}$, for some $i \in \{1,2\}$, and $\upsilon_j \le R$, $j \ne i$; $\pi = R$; $Prob = 2R(1-\hat{\upsilon})$.
- $\upsilon_i > \hat{\upsilon}$, for some $i \in \{1,2\}$, and $R < \upsilon_j \le \hat{\upsilon}$, $j \ne i$; $\pi = (R + \hat{\upsilon})/2$; $Prob = 2(\hat{\upsilon} - R)(1-\hat{\upsilon})$.
- $\upsilon_i > \hat{\upsilon}$, for both $i \in \{1,2\}$; $\pi = (2\hat{\upsilon}+1)/3$; $Prob = (1-\hat{\upsilon})^2$.

Incorporating the outcomes and their probabilities gives the seller's expected payoff:

$$\pi_1 = \frac{1}{3} + R^2(1-\hat{\upsilon}) - \frac{\hat{\upsilon}^3}{3} + \frac{2\hat{\upsilon}^3}{3\sqrt{3}}. \tag{7}$$

Substituting in the equilibrium value of $\hat{\upsilon}$ from Equation (6) gives

$$\pi_1 = \begin{cases} \frac{1}{3} + R^2 - 2R^3(\sqrt{3}-1) & \text{if } R \le \frac{1}{\sqrt{3}} \\ \frac{2}{3\sqrt{3}} & \text{if } R > \frac{1}{\sqrt{3}} \end{cases}. \tag{8}$$

Optimising (8) w.r.t. R gives the seller's optimal reserve price:

$$R^* = \frac{1}{3(\sqrt{3}-1)} \approx .455,$$

so the interior solution for $\hat{\upsilon}$ is valid when the reserve price is chosen optimally, and there is a positive probability of the good selling during the regular auction. The seller's payoff at the optimum is

$$\pi_1^* = \frac{55\sqrt{3}-91}{3^3(\sqrt{3}-1)^3} \approx .402.$$

It is interesting to compare the optimal reserve price and seller's payoff to the model with no fixed-price offer. The no-FPO payoff as a function of R can be found by setting $\hat{\upsilon} = R$ and removing the last term (which is the Stage-3 payoff) in Equation (7). Optimising this payoff w.r.t. *R*, gives the familiar result that $R^* = 0.5$ and $\pi_1^* = 5/12 \approx .417$. The existence of the possibility of an FPO, then, lowers both the reserve price and the seller's payoff.

The effect on the reserve price is the result of two offsetting pressures. The ability to make an FPO lowers the opportunity cost to the seller of having an auction fail to attract any bids in Stage 2, and so creates an incentive to increase the reserve price. At the same time, however, the possibility of an FPO lowers the opportunity cost to bidders of not initiating bidding, creating an incentive to lower the reserve price in order to increase the probability of a Stage-2 sale where bidders compete up the sale price.

The first effect can be isolated by considering a hypothetical model in which the seller knows that he will make an FPO if necessary, but buyers do not adjust their bidding behaviour in response. The seller's payoff in this model can be found by setting $\hat{\upsilon} = R$ in Equation (7) but retaining the last term. Optimising this payoff w.r.t. *R* produces an optimal *R* in excess of the 0.5 found in the no-FPO model.

Overall, the second effect dominates, leading to a reduced reserve price as a result of the option for a FPO. In this respect results are analogous to those found by Grant et al (2006), and McAfee and Vincent (1997), referred to in the introduction, which considered a different form of post-auction behaviour—relisting. In the former paper, in a model constructed to have no endogenous response from bidder behaviour, the possibility of post-auction activity led to an increase in the reserve price; in the latter, the allowance for bidder response led to a reduction in the reserve price.

4. The *n*-Bidder Model.

In this section we will extend the results of the 2-bidder FPO auction to *n* bidders. The main goal of this section is to see how the size of the market affects the FPO auction outcomes.

4.1 Stage 3: The Optimal Fixed Price Offer.

If the auction fails to meet the reserve in Stage 2, the seller knows that all *n* bidders must have $\upsilon_i \le \hat{\upsilon}$. If a seller makes an FPO offer at price, p_f, it will not be accepted by any bidder at probability $p_f^n / \hat{\upsilon}^n$. With the remaining probability $1 - p_f^n / \hat{\upsilon}^n$. at least one of the bidders will accept the FPO. Thus, the expected payoff of the seller in Stage 3 is

$$\pi_3 = \left(1 - \frac{p_f^n}{\hat{\upsilon}^n}\right) p_f . \tag{9}$$

Optimizing (9) w.r.t. p_f gives the optimal fixed price offer conditional on an unsuccessful auction:

$$p_f^* = \frac{\hat{\upsilon}}{(n+1)^{\frac{1}{n}}} \tag{10}$$

and the maximized Stage 3 profit is

$$\pi_3^* = \frac{n\hat{\upsilon}}{(n+1)^{\frac{n+1}{n}}} . \tag{11}$$

4.2 Stage 2: Optimal Bidder Strategies.

As in the 2-bidder model, we are only interested in comparing the payoffs between Strategy *B* and Strategy *W* in the event that no other bidder initiates bidding, since that is the context in which the two strategies differ. We therefore consider the surplus to bidder *i* adopting each strategy, conditional on the information that the remaining *n*-1 bidders are playing Strategy *W* and hence have a valuation no greater than $\hat{\upsilon}$.

The surplus to bidder *i* playing Strategy *W* conditional on all other bidders having valuations no greater than $\hat{\upsilon}$ is

$$CS_i^W(\upsilon_i) = \left(\sum_{j=0}^{n-1} \frac{1}{j+1} \left(\binom{n-1}{j} \left(\frac{p_f^*}{\hat{\upsilon}} \right)^{n-j-1} \left(\frac{\hat{\upsilon} - p_f^*}{\hat{\upsilon}} \right)^j \right) \right) (\upsilon_i - p_f^*). \tag{12}$$

In Equation (12), $(\upsilon_i - p_f^*)$ is the surplus that would result from winning the FPO, 1/(*j*+1) is the probability of winning the FPO when there are exactly *j* other bidders willing to accept it, and the remaining term in parentheses is the binomial probability of there being exactly *j* other bidders with a valuation $\upsilon_j \in [p_f, \hat{\upsilon}]$ and thus be willing to accept.

Equation (12) can be rewritten as[6]

$$CS_i^W(\upsilon_i)=\left(\frac{\hat{\upsilon}^n-(p_f^*)^n}{n\hat{\upsilon}^{n-1}}\right)\frac{(\upsilon_i-p_f^*)}{(\hat{\upsilon}-p_f^*)}. \tag{13}$$

Let $\breve{\upsilon}_{-i}$ be the maximum valuation of the n-1 bidders other than i. If $\upsilon_i \geq R$, bidder i's surplus from playing Strategy B is $(\upsilon_i - R)$ if $\breve{\upsilon}_{-i} \leq R$, is $(\upsilon_i - \breve{\upsilon}_{-i})$ if $\breve{\upsilon}_{-i} \in (R, \upsilon_i)$, and is 0 if $\breve{\upsilon}_{-i} \geq \upsilon_i$. Let $\hat{F}^{n-1}(\breve{\upsilon})$ be the distribution of the maximum of n-1 valuations, conditional on none having a valuation greater than $\hat{\upsilon}$. For $\breve{\upsilon} \leq \hat{\upsilon}$, we have

$$\hat{F}^{n-1}(\breve{\upsilon})=\left(\frac{\breve{\upsilon}}{\hat{\upsilon}}\right)^{n-1}.$$

We can then write the surplus to bidder i from playing Strategy B as

$$CS_i^B(\upsilon_i)=\int_0^R(\upsilon_i-R)d\hat{F}^{n-1}(\breve{\upsilon})+\int_R^{\upsilon_i}(\upsilon_i-\breve{\upsilon})d\hat{F}^{n-1}(\breve{\upsilon}). \tag{14}$$

Integrating (14) by parts gives:

$$CS_i^B(\upsilon_i)=\int_R^{\upsilon_i}\hat{F}^{n-1}(\breve{\upsilon})d\breve{\upsilon} \tag{15}$$

If $\upsilon_i \leq \hat{\upsilon}$, Equation (15) can be written as

$$\begin{aligned}CS_i^B(\upsilon_i)&=\int_R^{\upsilon_i}\left(\frac{\breve{\upsilon}}{\hat{\upsilon}}\right)^{n-1}d\breve{\upsilon}\\ &=\frac{\upsilon_i^{\,n}-R^n}{n\hat{\upsilon}^{n-1}}.\end{aligned} \tag{16}$$

As with the analogous equations in the two-bidder case, Equation (14) is convex in υ_i whereas Equation (13) is linear, so again an interior solution for $\hat{\upsilon}$ is found where $CS_i^W(\hat{\upsilon})=CS_i^B(\hat{\upsilon})$. Substituting $\upsilon_i=\hat{\upsilon}$ into Equations (13) and (16) and equating gives

$$\hat{\upsilon}^n-(p_f^*)^n=\hat{\upsilon}^n-R^n, \tag{17}$$

and so, again, $p_f^*=R$ if $\hat{\upsilon}<1$.

Using Equations (10) and (17) we then have

$$\hat{\upsilon}=\min\{(n+1)^{1/n}R,1\}. \tag{18}$$

Again, so long as R<1, we have $\hat{\upsilon}>R$, implying that bidders with values above R may not bid during the regular auction, and thus that the auction may close without reserve having been met even when there are bidders on the market with values above R.

---

[6] See Hogan and Meriluoto (2010) for a proof of the equivalence between Equations (12) and (13).

We can also see from (18) that $\partial\hat{\upsilon}/\partial n < 0$ and that $\lim_{n\to\infty}\hat{\upsilon} = R$. That is, as the number of bidders in the auction increases, each bidder is less likely to choose to watch rather than initiate bidding, and in the limit as *n* goes to infinity, no bidder whose valuation meets the reserve will choose to simply watch.

4.3 Stage 1: The Optimal Reserve Price.

Let $\overline{\upsilon}$ be the valuation of the winning bidder, if there is one, in the Stage-2 auction. Let $\overline{F}^{n-1}(\breve{\upsilon}|\overline{\upsilon})$ be the distribution function of the maximum valuation, $\breve{\upsilon}$, of the remaining *n*-1 bidders. Since, by definition, $\overline{\upsilon}$ must be the highest of the *n* valuations, we have $\overline{F}^{n-1}(\breve{\upsilon}|\overline{\upsilon}) = (\breve{\upsilon}/\overline{\upsilon})^{n-1}$. Following the same approach as used to derive Equation (16), the expected price paid by the winning bidder, if there is one, must therefore be

$$
\begin{aligned}
E[p|\overline{\upsilon}] &= R\int_0^R dF^{n-1}(\breve{\upsilon}|\overline{\upsilon}) + \int_R^{\overline{\upsilon}} \breve{\upsilon}\, dF^{n-1}(\breve{\upsilon}|\overline{\upsilon}) \\
&= \overline{\upsilon} - \int_R^{\overline{\upsilon}} F^{n-1}(\breve{\upsilon}|\overline{\upsilon})\, d\breve{\upsilon} \\
&= \overline{\upsilon} - \frac{1}{n\overline{\upsilon}^{n-1}}\left(\overline{\upsilon}^n - R^n\right). \qquad (19)
\end{aligned}
$$

The seller's expected revenue from the Stage-2 auction, π_2, is this expected price, averaged over the possible values of $\overline{\upsilon}$ in the relevant range:

$$\pi_2 = \int_{\hat{\upsilon}}^1 E[p|\overline{\upsilon}]\, f^n(\overline{\upsilon})\, d\overline{\upsilon}, \qquad (20)$$

where $f^n(\overline{\upsilon})$ is the density function of the maximum valuation, $\overline{\upsilon}$. Since the distribution function of $\overline{\upsilon}$ is $\overline{\upsilon}^n$, we have

$$f^n(\overline{\upsilon}) = n\overline{\upsilon}^{n-1}. \qquad (21)$$

Putting Equations (19)-(21) together gives

$$
\begin{aligned}
\pi_2 &= \int_{\hat{\upsilon}}^1 \left((n-1)\overline{\upsilon}^n + R^n\right) d\overline{\upsilon} \\
&= \frac{n-1}{n+1}(1-\hat{\upsilon}^{n+1}) + (1-\hat{\upsilon})R^n. \qquad (22)
\end{aligned}
$$

The Stage-1 profit of the seller is then the sum of this Stage-2 profit and the Stage-3 profit in Equation (11) multiplied by $\hat{\upsilon}^n$, the probability of going to Stage 3:

$$\pi_1 = \frac{n-1}{n+1}(1-\hat{\upsilon}^{n+1}) + (1-\hat{\upsilon})R^n + \frac{n\hat{\upsilon}^{n+1}}{(n+1)^{\frac{n+1}{n}}}. \qquad (23)$$

Substituting for $\hat{\upsilon}$ from (18) and optimising with respect to R gives the optimal reserve price:

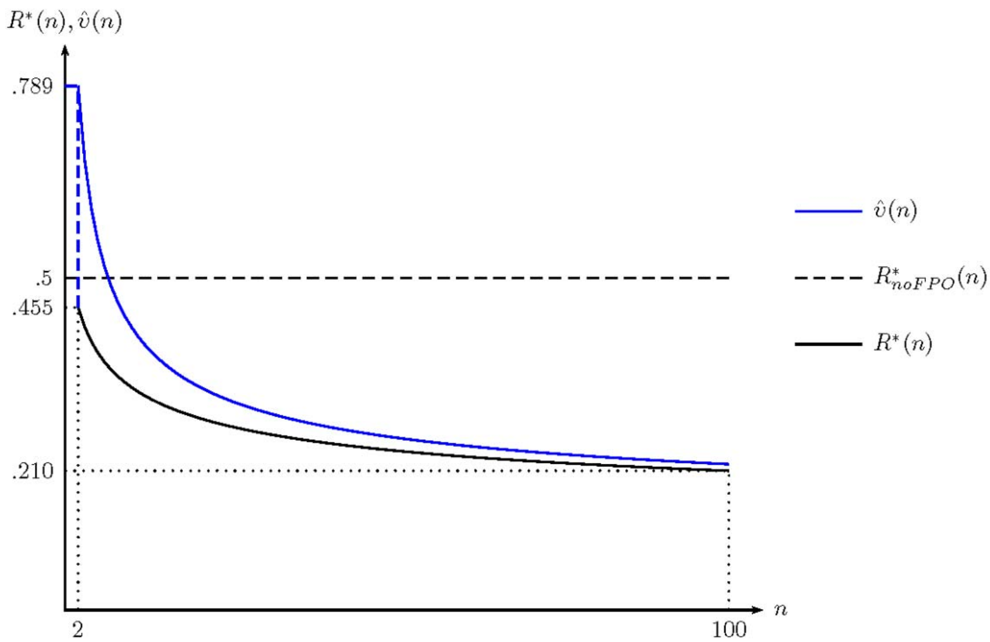
$$R^* = \frac{1}{(n+1)\left((n+1)^{\frac{1}{n}} - 1\right)}, \tag{24}$$

which, from (18), is always an interior solution. The value of marginal consumer, $\hat{\upsilon}$, is given by

$$\hat{\upsilon} = \frac{1}{(n+1)^{\frac{(n-1)}{n}}\left((n+1)^{\frac{1}{n}} - 1\right)}.$$

The optimal reserve and marginal-consumer valuation for different values of *n* are illustrated in Figure 1. Figure 1 also shows the optimal reserve price for a standard static auction with no FPO. This is easily derived as a special case of our model in which the seller maximises π_2 rather than π_1 and we set $\hat{\upsilon} = R$ in Equation (22). This produces the familiar result that $R^* = 0.5$, independent of *n*. In contrast, in the FPO model, Equation (24) shows that the optimal reserve has $R^* < 0.5 \ \forall n,$ with $\partial R^* / \partial n < 0,$ $\partial^2 R^* / \partial n^2 > 0,$ and $\lim_{n\to\infty} R^* = 0.$

Figure 1: Optimal Reserve and Marginal-Consumer Valuation.



We can again decompose the effect of the FPO on the reserve price into the effects due to seller response and bidder response. The effect of the FPO on seller behaviour can be isolated by setting $\hat{\upsilon} = R$ in Equation (23) (in effect, making the model one in which the seller knows that he will make an FPO if the auction fails to attract any bids in Stage 2, but bidders do not know this). In the this case, the optimal reserve price would be above the 0.5 obtained in the no-FPO model, and would be increasing in the number of bidders. This result is also found by Grant et al (2006) in their model of relisting in which there bidders' behaviour is not affected by the possibility of relisting.[7]

The effect on bidder behaviour can be isolated by assuming the seller maximises π_2 rather than π_1 (a model which assumes that the seller will not make an FPO but bidders believe he will). It is easy to show that this produces an optimal reserve price that is lower than the 0.5, and sharply decreasing in *n*. Overall, this effect dominates. Our finding that R^* decreases in *n* is in contrast to the results of relisting model of McAfee and Vincent (1997), where the optimal reserve price, while below that of the static one-shot model, increases in *n*. There seems to be no obvious intuition for this difference based on the distinction between an FPO and relisting as the form of post-auction activity. Rather the difference is just the result of the offsetting seller and bidder effects interacting in slightly different ways.

We can compare how often auctions will sell through the FPO process vs. during the regular auction. The probability that the auction sells during Stage 2 (regular auction) is

$$1-\hat{\upsilon}^{*n} = 1-\frac{1}{(n+1)^{(n-1)}\left((n+1)^{\frac{1}{n}}-1\right)^{n}}, \tag{25}$$

and the overall probability that the good is sold after Stage 3 is

$$1-\left(p_f^*\right)^n = 1-\frac{1}{(n+1)^{n}\left((n+1)^{\frac{1}{n}}-1\right)^{n}}. \tag{26}$$

Thus, the probability of an FPO sale is the difference between (26) and (25) or

$$\hat{\upsilon}^{*n}-\left(p_f^*\right)^n = \frac{n}{n+1}\hat{\upsilon}^n = \frac{n}{(n+1)^{n}\left((n+1)^{\frac{1}{n}}-1\right)^{n}}.$$

Figure 2 illustrates the three probabilities of sale as functions of *n*. It also shows, for comparison, the probability of a sale in the standard non-FPO auction of $(1-0.5^n)$. We can see that FPO sales are frequent in auctions with small number of bidders, but that the likelihood of an FPO sale diminishes fast as the number of bidders increases. The possibility of an FPO, however, does increase the overall probability of a sale for all *n*.

[7] The number of bidders in Grant et al is stochastic. By an increase in *n* in their model, therefore, we mean a shift in the distribution of possible *n* to a stochastically dominating distribution.

Finally, the seller's maximum expected revenue both with and without the FPO option can be obtained by substituting the optimal reserve price from (24) into (23) and $\hat{\upsilon} = R = 0.5$ into Equation (22).[8] In Figure 3 we show the ratio of the expected revenue from the FPO auction to expected revenue from an auction without the FPO option (or, equivalently, from an SCO auction). The FPO profit is smaller than the profit without the FPO option when *n* is small. For example, when *n*=2, the FPO auction generates 96.6% of the expected revenue that could be achieved with full reserve price commitment. This ratio improves fast, and becomes very close to one with 8 or more bidders. Thus, profit loss resulting from the FPO auction is relatively minor (at most 3.4% with our uniform distribution of bidder preferences) and this loss quickly disappears as the market becomes thicker.

Figure 2: Probabilities of Sale.

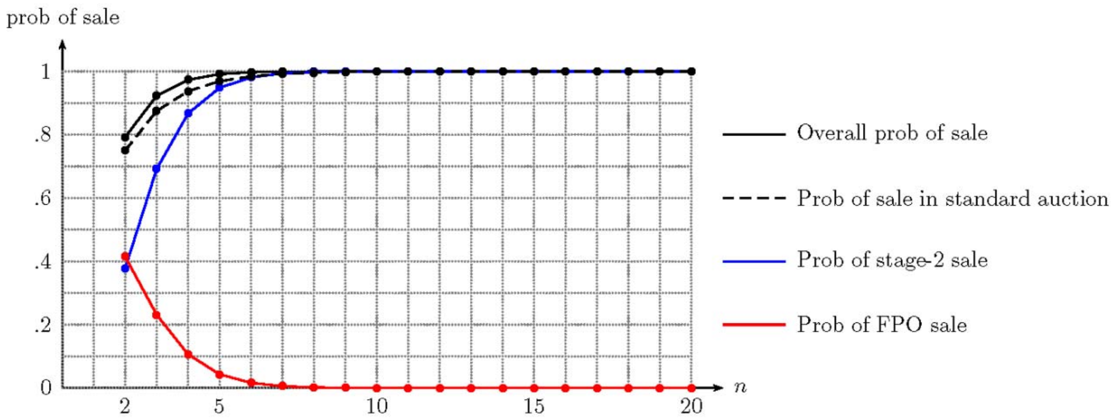
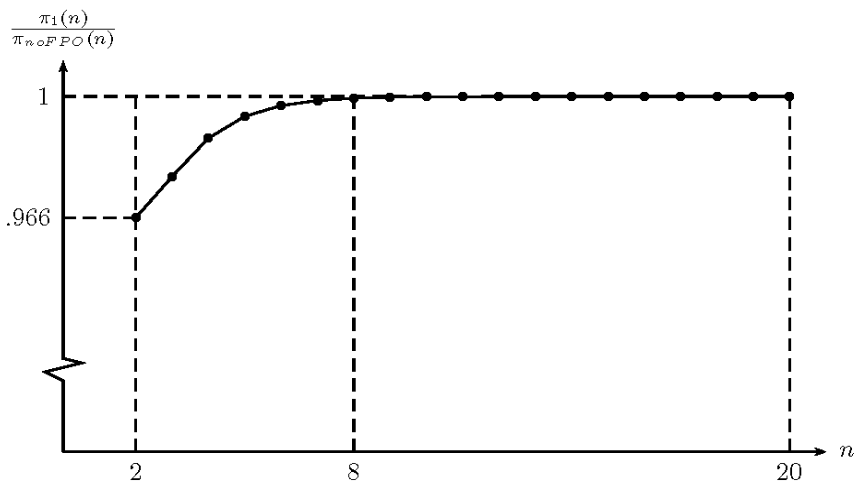


Figure 3: Expected Revenues.



[8] The formula for the maximized expected revenue is not shown here but is trivially derived.

The result that the seller's expected profit is lower with the FPO option than without it when bidders are fully rational is consistent with the literature on optimal auctions and optimal selling mechanism (see for example Riley and Samuelson (1981), Milgrom (1987) and Bulow and Klemperer (1996)). The main reason for this is that the FPO auction does not always allocate the good to the bidder with the highest willingness to pay (or highest marginal valuation) because of the lottery component that is present in the FPO stage of the game. Thus, by definition, this mechanism must be suboptimal to a static second-price ascending auction where the reserve price is set optimally.

5. Conclusion.

This paper investigates the effects on the bidder and seller strategies as well as the auction outcomes of a Fixed Price Offer (FPO) option at the end of a standard ascending bid auction that did not meet the reserve. The seller chooses the price of the FPO and can simultaneously offer it to all bidders and watchers of the auction.

One of the main results of the paper is that the presence of the FPO option induces some bidders with maximum willingness to pay above the seller's reserve to choose to follow a "wait-and-see" strategy of not bidding until the reserve is met. With this feature it is possible that the auction does not sell during the regular auction even if some bidders have values above the reserve price. This reduces the probability of the auction selling during the regular auction and, at a margin, reduces the optimal reserve price of the seller. However, the option to offer an FPO provides a form of insurance against the good not selling in the regular auction and thus, at a margin, provides the seller with an incentive to increase his reserve. The first effect dominates and the optimal reserve is set lower than in the absence of the FPO option. The optimal reserve is .455 with two bidders and falls (slowly) to zero as n goes to infinity.

The probability of the FPO sale diminishes quickly with the number of bidders. It is clear from these results that FPO option substantially affects the auction outcomes only when there are a relatively small number of bidders in the auction.

We show that the FPO option always reduces the expected seller profit compared to a static auction where the reserve price has full commitment value. This result is in line with the Revenue Equivalence Theorem because an auction where the winner is not always the bidder with the highest valuation must reduce expected profit compared to an optimal standard auction. The negative impact of the FPO option on expected seller revenue declines with the number of bidders in the auction. The FPO reduces the expected profit by 3.4% when there are two bidders but with 8 or more bidders this reduction is almost completely eliminated.

The *eBay* Second Chance Offer (SCO) differs from the *Trade Me* FPO offer because the former allows the seller to offer the SCO to the highest bidder at her highest bid. We claim that the SCO auction is effectively a first-price auction until the reserve is met after which it becomes a second-price auction. Thus, the Revenue Equivalence Theorem should still apply and thus the SCO auction should yield the same expected revenue as an optimal static auction. Therefore, the FPO mechanism is suboptimal to the SCO mechanism.

Our results have not clearly explained the existence of the FPO option in auctions. We have shown that the FPO option can at best be harmless and at worst reduces seller profit. Why, then, would an online auction site such as *Trade Me* have this design? One of our areas of future research is to examine if FPO could increase seller's expected revenue when bidders' entry into an auction is endogenous or when sellers are able to influence the bidders' perception of the likelihood of the FPO.

References

[1] Amir R, 1996, "Cournot oligopoly and the theory of supermodular games." *Games and Economic Behavior* 15, pp 132–148.

[2] Bulow, J. I. and P. D. Klemperer, 1996, "Auctions vs. negotiations." *American Economic Review* 86, pp. 180-194.

[3] Carney, M., 2007, *Trade Me Success Secrets.* Activity Press Limited.

[4] Coase, R., 1972, "Durability and Monopoly", *Journal of Law and Economics* 15, pp. 143-149.

[5] *eBay*, 2009, "2008 Annual Report." Retrieved September 2009 from http://investor.*eBay*.com/annuals.cfm.

[6] Grant, S., Kaiji, A., Menezes, F. and M. Ryan, 2006, "Auctions with options to re-auction." *International Journal of Economic Theory* 2, pp. 17-39.

[7] Hogan, S. and Meriluoto, L., 2010, "A Note on the Probability of Winning a Lottery when the Number of Competitors is a Binomial Random Variable", University of Canterbury Working Paper, 48/2010, available at http://www.econ.canterbury.ac.nz/RePEc/cbt/econwp/1048.pdf.

[8] Klemperer, P., 2004, *Auctions - Theory and Practice*, Princeton University Press.

[9] Krishna, V., 2002, *Auction Theory*, Academic Press.

[10] McAfee, R. P., and D. Vincent, 1997, "Sequentially optimal auctions." *Games and Economic Behavior* 18, pp. 24676.

[11] Menezes, F. and M. J. Ryan, 2005, "Reserve price commitments in auctions". *Economics Letters* 87, pp. 35-39.

[12] Milgrom, P., 1987, "Auction Theory." in T. Bewley (ed.), *Advances in Economic Theory: Fifth World Congress*, Cambridge University Press, pp. 1-32.

[13] Myerson, R. B., 1981, "Optimal auction design." *Mathematics of Operations Research* 6, pp. 58-73.

[14] Riley, J. G. and W. F. Samuelson, 1981, "Optimal Auctions." *American Economic Review* 71 (3), pp. 381-392.

[15] Salmon, T. C. and B. J. Wilson, 2008, "Second chance offers versus sequential auctions: theory and behavior." *Economic Theory* 34, pp. 47-67.

[16] *Trade Me*, 2009a, "Site stats." retrieved October 2009 from www.trademe.co.nz/Community/SiteStats.aspx/.

[17] *Trade Me*, 2009b, "Sell through rates by category." retrieved October 2009 from www.trademe.co.nz/Community/SellThroughRates.aspx/.

[18] Vickrey, W., 1961, "Counterspeculation, Auctions, and Competitive Sealed Tenders." *Journal of Finance* 16(1), pp. 8-37.

Technical Appendix:

This appendix contains the proof of Theorem 1, which is presented but not proved in the body of the paper.

Theorem 1:

Any subgame perfect equilibrium in pure strategies has $p_f \leq R,$ and has bidder strategies described by a critical valuation, $\hat{\upsilon} \geq R,$ common to all bidders, such that bidders will bid in the Stage-2 auction, if and only $\upsilon \geq \hat{\upsilon}.$

Proof:

The results in this paper are derived for the special case where the distribution of bidders' valuations is uniform defined over the interval, [0,1]. In this case, an implication of Theorem 1 is that the conditional distribution of bidders' valuations, given the information that none chooses to initiate bidding is also uniform in equilibrium. To prove this, however, requires us to consider non-equilibrium strategies, in which case the conditional valuation distribution need not be uniform. Accordingly, it is notationally more convenient in this appendix to prove the result for the more general case in which bidders valuations are independent draws from some general distribution, *F*, defined over an interval [*a*,*b*]. In this proof we simply assume that bidders who are indifferent between initiating bidding and watching will watch. This enables us to write the best-response correspondences as functions. The proof is easily extended to the case were the bidders can make either choice when indifferent, but at the expense of a lot more notation and no additional insight.

Let ϕ_i be the no-bid set of bidder *i*—that is, the set of valuations such that she will choose Strategy W. Let $\boldsymbol{\phi}_{-i}$ denote the set of other bidders' no-bid sets, and let the function $\phi_i(R, p_f, \boldsymbol{\phi}_{-i})$ denote the best-response function of bidder *i*. Similarly, let the function, $p_f(\boldsymbol{\phi}),$ denote the best response of the seller as a function of the set of all bidders' no-bid sets, $\boldsymbol{\phi},$ in choosing the fixed-price offer in Stage 3.

Note that Stages 2 and 3 of the complete game together define a simultaneous subgame. This is because, although the fixed-price offer is made after the Stage 2 auction, and then only in the event that no bidder bids at Stage 2, the seller cannot observe the strategies, $\boldsymbol{\phi},$ at the time of making the offer, just the fact that no-one has bid. Accordingly, the game has exactly the same structure as if the seller were required to choose the value of the fixed-price offer should the Stage-2 auction attract no bids at the same time as bidders choose their strategies.

A pure-strategy equilibrium for the Stage-2-and-3 subgame is a set of no-bid sets, $\boldsymbol{\phi}^*,$ and a fixed-price offer, $p_f^*,$ such that $\phi_i^* = \phi_i(R, p_f^*, \boldsymbol{\phi}_{-i}) \; \forall i$ and $p_f^* = p_f(\boldsymbol{\phi}^*).$ We wish to show that at any such equilibrium for this sub-game (and hence for any pure-strategy equilibrium of the full game) there exists $\hat{\upsilon}$ such that $\phi_i^* = [a, \hat{\upsilon}]\ \forall i$.

Let $\hat{F}_j(x|\phi_j)$ be the distribution of bidder *j*'s valuation, conditional on the information that she will not choose to initiate bidding, so that

$$\hat{F}_j(\upsilon|\phi_j) = \frac{\int_{x\in\phi_j\cap[0,\upsilon]} dF(x)}{\int_{x\in\phi_j} dF(x)}.$$

Let $\hat{F}_i^{n-1}(\upsilon|\boldsymbol{\phi}_{-i})$ be the distribution of the maximum valuation of the *n*-1 bidders who are not bidder *i*, conditional on none of those *n*-1 bidders initiating bidding, so that

$$\hat{F}^{n-1}(\upsilon|\boldsymbol{\phi}_{-i}) = \prod_{j\neq i} F_j(x|\phi_j). \tag{A1}$$

Let, $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$ be the expected surplus to bidder *i* if she chooses to initiate bidding, conditional on the information that no other bidder will initiate. If $\upsilon < R,$ then $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i}) = 0.$ If $\upsilon \geq R,$ then

$$CS_i^B(\upsilon|\boldsymbol{\phi}_{-i}) = \int_a^R (\upsilon - R)d\hat{F}_i^{n-1}(x|\boldsymbol{\phi}_{-i}) + \int_R^\upsilon (\upsilon - x)d\hat{F}_i^{n-1}(x|\boldsymbol{\phi}_{-i}) + \int_\upsilon^b 0 d\hat{F}_i^{n-1}(x|\boldsymbol{\phi}_{-i}). \tag{A2}$$

The first term in Equation (A2) reflects the fact that if no other bidder has a valuation greater than the reserve price, bidder *i* will pay the reserve. The second term reflects the nature of a 2nd-price auction, and the third term reflects the fact that if another bidder has a higher valuation, bidder *i* will not win the auction. Integrating Equation (A2) by parts, the full definition of $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$ is then

$$CS_i^B(\upsilon|\boldsymbol{\phi}_{-i}) = \begin{cases} 0 & \text{if } \upsilon < R \\ \int_R^\upsilon \hat{F}_i^{n-1}(x|\boldsymbol{\phi}_{-i})dx & \text{if } \upsilon \geq R \end{cases} \tag{A3}$$

Now let $\pi_i(k|p_f, \boldsymbol{\phi}_{-i})$ be the probability that exactly *k* of the bidders who are not bidder *i*, will have a valuation greater than p_f conditional on their all choosing not to initiate bidding. Let $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})$ be the expected surplus to bidder *i* if she chooses to watch rather than bid, in the hope of winning a fixed-price offer, again conditional on the information that no other bidder will initiate bidding. If $\upsilon < p_f$, then $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) = 0.$ If $\upsilon \geq p_f,$ then we have

$$CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) = (\upsilon - p_f)\left(\sum_{k=0}^{n-1}\left(\pi_i(k|p_f, \boldsymbol{\phi}_{-i})\frac{1}{k+1}\right)\right). \tag{A4}$$

Note that $\pi_i(0|p_f, \boldsymbol{\phi}_{-i}) = \hat{F}_i^{n-1}(p_f|\boldsymbol{\phi}_{-i})$ so we can restate $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})$ fully as

$$CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) = \begin{cases} 0 & \text{if } \upsilon < p_f \\ (\upsilon - p_f)\left(\hat{F}_i^{n-1}(p_f|\boldsymbol{\phi}_{-i}) + \sum_{k=1}^{n-1}\left(\pi_i(k|p_f,\boldsymbol{\phi}_{-i})\frac{1}{k+1}\right)\right) & \text{if } \upsilon \geq p_f \end{cases} \quad \text{(A5)}$$

The no-bid set, ϕ_i, is the set of valuations, υ, such that $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) \geq CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$.

Finally, let $\hat{F}^n(\upsilon|\boldsymbol{\phi})$ be the distribution of the maximum valuation of all *n* bidders conditional on none of those bidders initiating bidding. With this notation, p_f is the solution to

$$p_f(\boldsymbol{\phi}) = \arg\max_{p_f}\left\{p_f(1-\hat{F}^n(p_f|\boldsymbol{\phi})\right\}. \quad \text{(A6)}$$

Lemma 1:

a) If $p_f \leq R,$ then for each bidder, *i*, there exists $\hat{\upsilon}_i$ such that $\phi_i(R,\boldsymbol{\phi}_{-i},p_f) = [a,\hat{\upsilon}_i]\ \forall i.$
b) If $p_f > R,$ then for each bidder, *i*, either $\phi_i(R,\boldsymbol{\phi}_{-i},p_f) = [a,R]\bigcup[\overline{\upsilon}_i,\hat{\upsilon}_i],$ or there exists $\overline{\upsilon}_i > p_f$ such that $\phi_i(R,\boldsymbol{\phi}_{-i},p_f) = [a,R]\bigcup[\overline{\upsilon}_i,\hat{\upsilon}_i].$

Proof:

By definition, the no-bid set, $\phi_i(R,\boldsymbol{\phi}_{-i},p_f)$ includes the interval, [*a*,*R*] (since bidders cannot bid below the reservation price) so we are interested in the no-bid set for valuations greater than *R*.

From Equations (A3) and (A4), it is clear that $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$ is weakly convex and monotone increasing for $\upsilon \geq R,$ while $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})$ is linear and monotone increasing for $\upsilon \geq p_f$.

When $p_f < R,$ because we have

$$CS_i^W(R|\boldsymbol{\phi}_{-i}) > CS_i^B(R|\boldsymbol{\phi}_{-i}) = 0,$$

the convexity of $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$ implies that it can intersect with $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})$ at at most one value of $\upsilon,$ crossing from below. This is illustrated in Figure A1 below, showing a convex $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$ and a linear, $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})$.

When $p_f = R,$ we have

$$CS_i^W(R|\boldsymbol{\phi}_{-i}) = CS_i^B(R|\boldsymbol{\phi}_{-i}) = 0.$$

To show the existence of a cutoff valuation, $\hat{\upsilon},$ we need to show that $CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) > CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$ in a neighbourhood of υ above *R*. The result will then again follow from the convexity of $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})$. Note from Equation (A3) that

$$\left.\frac{\partial CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})}{\partial \upsilon}\right|_{\upsilon = R} = \hat{F}_i^{n-1}(R|\boldsymbol{\phi}_{-i}),$$

and from Equation (A5),

$$\left.\frac{\partial CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})}{\partial \upsilon}\right|_{\upsilon = p_f} = \hat{F}_i^{n-1}(p_f|\boldsymbol{\phi}_{-i}) + \sum_{k=1}^{n-1}\left(\pi_i(k|p_f,\boldsymbol{\phi}_{-i})\frac{1}{k+1}\right),$$

so that, when $\upsilon = p_f = R$,

$$\frac{\partial CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})}{\partial \upsilon} > \frac{\partial CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})}{\partial \upsilon}.$$

This is illustrated in Figure A2.

Finally, Part b of the Lemma follows automatically from the fact that, when $p_f > R$, $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i}) > CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) = 0$ in the interval, $(R, p_f]$. The two possible outcomes are shown in Figure A3. In one, $CS_i^B(\upsilon|\boldsymbol{\phi}_{-i}) > CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})$ for the entire interval (R,b], and so the bidder would always choose to initiate bidding as long as her valuation exceeded the reservation price. In the other, there are two disjoint sets of valuation under which the bidder would choose to not initiate bidding.

Figure A1: $p_f < R$.

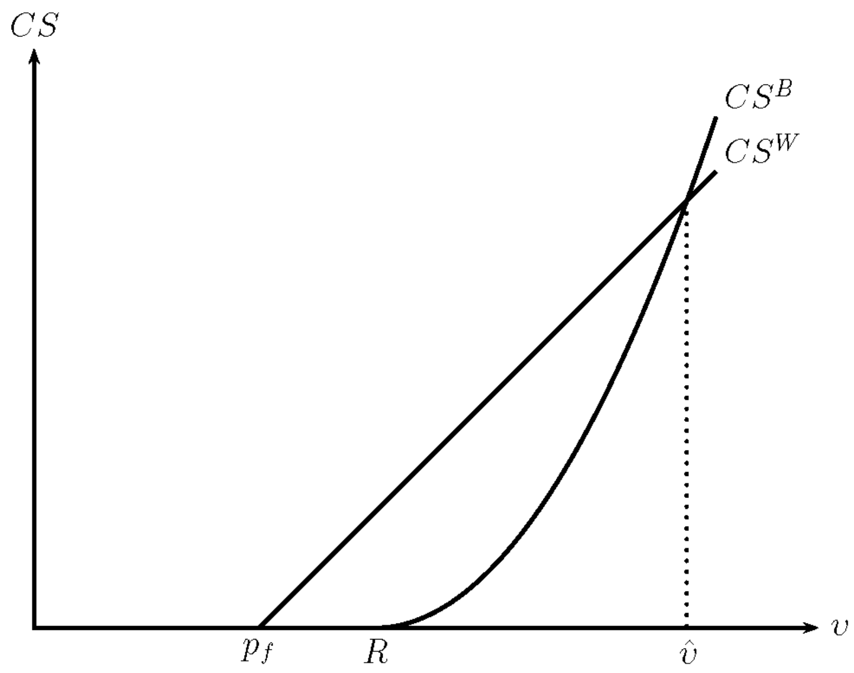


Figure A2: $p_f = R$.

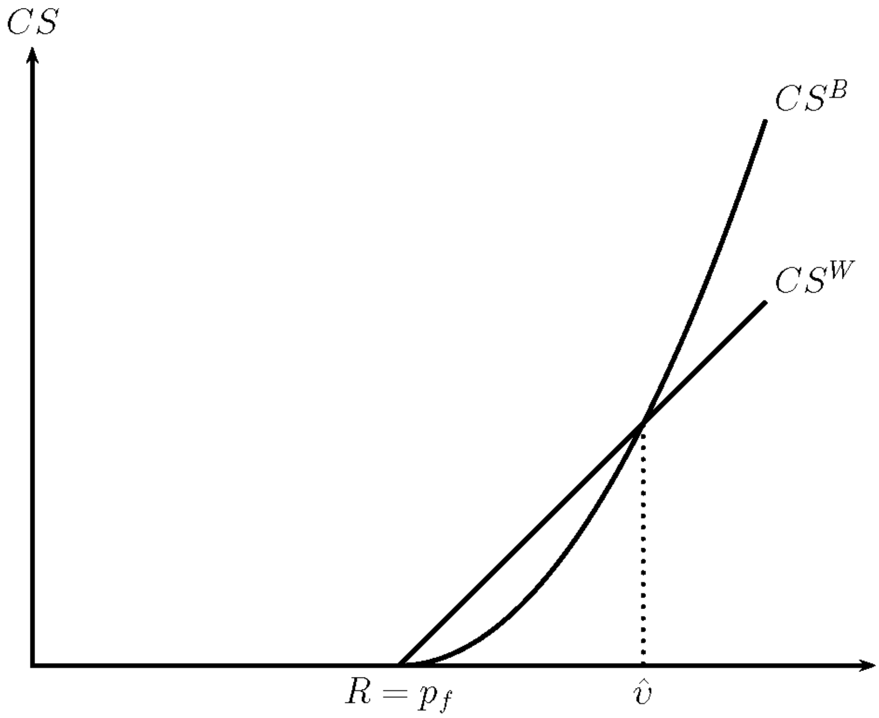
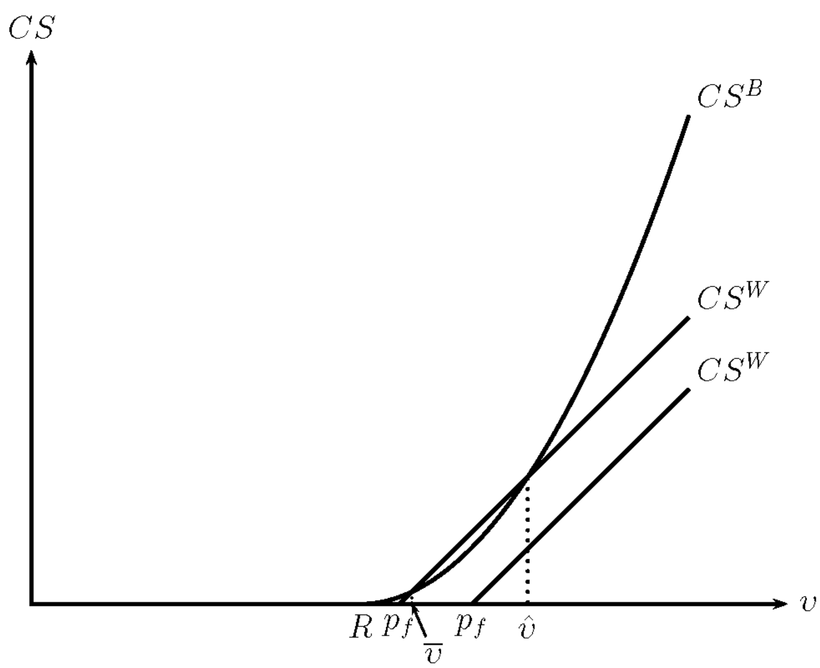


Figure A3: $p_f > R$.



Lemma 2:

There are no pure-strategy equilibria with $p_f > R$.

Proof:

First, consider the properties of $\hat{F}^n(\upsilon|\boldsymbol{\phi})$, the distribution function for the maximum of the n bidders' valuations, conditional on the information that none have chosen to initiate bidding. In the case where $\phi_i(R,\boldsymbol{\phi}_{-i},p_f)=[a,R]\ \forall i$—that is, if no bidder would choose to watch if her valuation exceeded the reservation price, then we would have $\hat{F}^n(\upsilon|\boldsymbol{\phi})=1\ \forall\,\upsilon\geq R.$ In this case, from Equation (A6), setting $p_f>R$ would not be a best response.

Now consider the other possibility in which there is at least one bidder whose no-bid set is $\phi_i(R,\boldsymbol{\phi}_{-i},p_f)=[a,R]\bigcup[\bar{\upsilon}_i,\hat{\upsilon}_i].$ In this case, let $\bar{\upsilon}^*$ be the minimum value of $\bar{\upsilon}_i$ amongst those bidders. This implies that any bidder with a valuation $\upsilon_i\in(R,\bar{\upsilon}^*),$ will choose to initiate bidding and so we will have

$$\frac{\partial\hat{F}^n(\upsilon|\boldsymbol{\phi})}{\partial\upsilon}=0\ \ \forall\,\upsilon\in(R,\bar{\upsilon}^*),$$

and so, from Equation (A6) we must have either $p_f\leq R$ or $p_f\geq\bar{\upsilon}^*$.

From Lemma 1, however, if $p_f>R,$ then $\bar{\upsilon}_i>p_f\ \ \forall i,$ and hence $\bar{\upsilon}^*>p_f.$ This contradiction establishes the proposition.

□

Lemma 1 and 2 together imply that any pure-strategy equilibrium must have $p_f\leq R,$ and hence that the strategies of each bidder i can be summarised by a cut-off valuation, $\hat{\upsilon}_i.$ We can now describe the game amongst bidders in terms of each bidder i choosing her cut-off valuation, $\hat{\upsilon}_i,$ as a function of the cut-off valuations of the other players.

To complete the proof of the Theorem 1, we just need to show that any pure-strategy equilibrium must be symmetric with all bidders having the same cut-off valuation in equilibrium.

Lemma 3:

The optimal cut-off valuation for any bidder i, is increasing in the cut-off valuation of any other bidder. That is,

$$\frac{\partial\hat{\upsilon}_i}{\partial\hat{\upsilon}_j}>0,\ \ \forall j\neq i.$$

Proof:

Define

$$H_i(\upsilon_i) = CS_i^B(\upsilon|\boldsymbol{\phi}_{-i}) - CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}), \tag{A7}$$

so that the optimal value of $\hat{\upsilon}_i$ is the solution to

$$H_i(\hat{\upsilon}_i) = 0. \tag{A8}$$

From the implicit function theorem, we wish to show that

$$-\frac{\partial H_i / \partial \hat{\upsilon}_j}{\partial H_i / \partial \hat{\upsilon}_i} > 0. \tag{A9}$$

As shown in the proof to Lemma 2, and as illustrated in Figures A1 and A2, we have

$$\partial H_i / \partial \hat{\upsilon}_i > 0, \tag{A10}$$

and so it remains to show that $\partial H_i / \partial \hat{\upsilon}_j < 0$ whenever $H_i(\hat{\upsilon}_i) = 0$.

Using Lemmas 1 and 2, we can now reexpress $\hat{F}_j(\upsilon|\phi_j)$ as

$$\hat{F}_j(\upsilon|\phi_j) = \begin{cases} \dfrac{F(\upsilon)}{F(\hat{\upsilon}_j)} & \text{if } \upsilon \le \hat{\upsilon}_j \\ 1 & \text{if } \upsilon > \hat{\upsilon}_j \end{cases},$$

so using Equation (A1) we have,

$$\frac{\partial \hat{F}_i^{n-1}(\upsilon|\boldsymbol{\phi})}{\partial \hat{\upsilon}_j} = -\frac{f_j(\hat{\upsilon}_j)}{F_j(\hat{\upsilon}_j)} \hat{F}_i^{n-1}(\upsilon|\boldsymbol{\phi}),$$

and hence,

$$\frac{\partial CS_i^B(\upsilon|\boldsymbol{\phi}_{-i})}{\partial \hat{\upsilon}_j} = -\frac{f(\hat{\upsilon}_j)}{F(\hat{\upsilon}_j)} CS_i^B(\upsilon|\boldsymbol{\phi}_{-i}). \tag{A11}$$

Now define $\pi_{ij}(k|p_f, \boldsymbol{\phi}_{-ij})$ as the probability that exactly *k* of the bidders who are not bidder *i* or *j*, will have a valuation greater than p_f conditional on their all choosing not to initiate bidding. Since the conditional probability that bidder *j* will have a valuation greater than p_f is $1-\left(F(p_f)/F(\hat{\upsilon}_j)\right)$, we can write

$$\pi_i(k|p_f, \boldsymbol{\phi}_{-i}) = \pi_{ij}(k|p_f, \boldsymbol{\phi}_{-ij}) \frac{F(p_j)}{F(\hat{\upsilon}_j)} + \pi_{ij}(k-1|p_f, \boldsymbol{\phi}_{-ij}) \left(1 - \frac{F(p_j)}{F(\hat{\upsilon}_j)}\right),$$

where we define $\pi_{ij}(k|p_f, \boldsymbol{\phi}_{-ij}) = 0$ for $k \notin \{0 \ldots n-2\}$.

We can then rewrite Equation (A4) as

$$CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) = (\upsilon - p_f)\left(\sum_{k=0}^{n-1}\left(\frac{\pi_{ij}(k|\cdot)\dfrac{F(p_j)}{F(\hat{\upsilon}_j)} + \pi_{ij}(k-1|\cdot)\left(1-\dfrac{F(p_j)}{F(\hat{\upsilon}_j)}\right)}{k+1}\right)\right).$$

We now have

$$\frac{\partial CS_i^W(\upsilon|\boldsymbol{\phi}_{-i})}{\partial\hat{\upsilon}_j} = -\frac{f(\hat{\upsilon}_j)}{F(\hat{\upsilon}_j)}(\upsilon - p_f)\left(\sum_{k=0}^{n-1}\left(\frac{\pi_{ij}(k|\cdot)\dfrac{F(p_j)}{F(\hat{\upsilon}_j)} - \pi_{ij}(k-1|\cdot)\left(\dfrac{F(p_j)}{F(\hat{\upsilon}_j)}\right)}{k+1}\right)\right)$$

$$= -\frac{f(\hat{\upsilon}_j)}{F(\hat{\upsilon}_j)}CS_i^W(\upsilon|\boldsymbol{\phi}_{-i}) + \frac{f(\hat{\upsilon}_j)}{F(\hat{\upsilon}_j)}\sum_{k=0}^{n-1}\left(\left(\pi_{ij}(k-1|\cdot)\right)\frac{1}{k+1}\right) \qquad \text{(A12)}$$

Putting Equations (A11) and (A12) into Equations (A7) and (A8) gives us

$$\partial H_i / \partial\hat{\upsilon}_i = -\frac{f(\hat{\upsilon}_j)}{F(\hat{\upsilon}_j)}\sum_{k=0}^{n-1}\left(\left(\pi_{ij}(k-1|\cdot)\right)\frac{1}{k+1}\right) < 0,$$

which is what we were seeking to show.

□

To complete the proof of Theorem 1 we need to show that all bidders will have the same cut-off valuation, $\hat{\upsilon}$. This follows automatically from Lemma 3 and the fact that all bidders are identical (see Amir, 1996).

□