Discrete Orthogonal Moment Features Using Chebyshev Polynomials

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Abstract

This paper introduces a new set of moment functions based on Chebyshev polynomials which are orthogonal in the discrete domain of the image coordinate space. Chebyshev moments eliminate the problems associated with conventional orthogonal image moments such as the Legendre moments and the Zernike moments. The theoretical framework of discrete orthogonal moments is given, and their superior feature representation capability is demonstrated.

Keywords: *Image Moment Functions, Orthogonal Moments, Chebyshev Polynomials*

1 Introduction

Moment functions are used in image analysis as feature descriptors, in a wide range of applications like object classification, invariant pattern recognition, object identification, robot vision, pose estimation and stereopsis. A general definition of moment functions Φ_{pq} of order $(p+q)$, of an image intensity function $f(x, y)$ can be given as follows:

$$\Phi_{pq} = \int_x \int_y \Psi_{pq}(x,y)\, f(x, y)\, \mathrm{d}x\, \mathrm{d}y, \qquad p, q = 0,1,2,3.... \tag{1}$$

where $\Psi_{pq}(x,y)$ is a continuous function of (x, y) known as the *moment weighting kernel* or the *basis set*. The simplest of the moment functions, with

$$\Psi_{pq}(x,y) = x^p y^q \tag{2}$$

were introduced by Hu [1] to derive shape descriptors that are invariant with respect to image plane transformations. Legendre and Zernike moments were later introduced by Teague [2] with the corresponding orthogonal functions as kernels. These orthogonal moments have been proved to be less sensitive to image noise as compared to geometric moments, and possess far better feature representation capabilities. The information redundancy measure is minimum in an orthogonal moment set. The computation of orthogonal moments of images pose two major problems [3, 4, 6]: (*i*) The image coordinate space must be normalized to the range (typically, −1 to +1) where the orthogonal polynomial definitions are valid. (*ii*) The continuous integrals in (1) must be approximated by discrete summations without loosing the essential properties associated with orthogonality.

This paper introduces a new set of moment functions based on Chebyshev (some times also written as "Tchebichef" [5]) polynomials that are orthogonal in the discrete domain of the image coordinate space. Chebyshev moments completely eliminate the two problems referred above, and preserve all the theoretical properties, since their implementation does not involve any kind of approximation. The superiority of Chebyshev moments over conventional orthogonal moments in terms of their feature representation capability can be conclusively established by using the inverse moment transform. Images reconstructed using the inverse transform of Chebyshev moments provide lower reconstruction errors compared to those obtained using Legendre and Zernike moments.

2 Chebyshev Moments

Given an NxN image, we first seek discrete orthogonal polynomials $\{t_n(x)\}$ that satisfy the condition

$$\sum_{x=0}^{N-1} t_m(x) t_n(x) = \rho(n,N)\delta_{mn}\,, \qquad m, n = 0,1,2, \ldots N-1. \tag{3}$$

where $\rho(n,N)$ is the squared norm of the polynomial set t_n. The classical discrete Chebyshev polynomials[5] satisfy the property of orthogonality (3), with

$$\rho(n,N) = \frac{N(N^2-1)(N^2-2^2)\ldots(N^2-n^2)}{2n+1}, \quad n = 0, 1, \ldots, N-1 \tag{4}$$

and have the following recurrence relation:

$$(n+1)t_{n+1}(x) - (2n+1)(2x-N+1)t_n(x) + n(N^2-n^2)t_{n-1}(x) = 0, \quad n = 1, \ldots, N-1. \tag{5}$$

However, the Chebyshev polynomials as defined above together with their norms, become numerically unstable for large values of N. It can be easily verified that the magnitudes of t_n grow at the rate of N^n. We therefore further scale the Chebyshev polynomials $t_n(x)$ by a factor N^{-n} to make them suitable for image analysis, and define the Chebyshev moments as follows (henceforth $t_n(x)$ denotes the scaled polynomials):

$$T_{pq} = \frac{1}{\rho(p,N)\rho(q,N)} \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} t_p(x) t_q(y) f(x,y), \quad p, q = 0, 1, \ldots N-1. \tag{6}$$

where

$$\rho(p,N) = \frac{N\left(1-\frac{1^2}{N^2}\right)\left(1-\frac{2^2}{N^2}\right)\ldots\left(1-\frac{p^2}{N^2}\right)}{2p+1} \tag{7}$$

and the scaled Chebyshev polynomials $t_p(x)$ are computed using the following recurrence relation:

$$t_0(x) = 1. \tag{8}$$

$$t_1(x) = \frac{2x+1-N}{N} \tag{9}$$

$$t_p(x) = \frac{(2p-1)t_1(x)t_{p-1}(x) - (p-1)\left\{1-\frac{(p-1)^2}{N^2}\right\}t_{p-2}(x)}{p}, \qquad p>1 \tag{10}$$

The classical Chebyshev polynomials modified as above do not lead to numerical overflows for large images. Both the polynomials and the associated moments do not show large variation in the dynamic range of values, as in the case of geometric moments. Since the Chebyshev polynomials are exactly orthogonal in the discrete coordinate space of the image, we further have the following theorem:

<u>*Theorem*:</u> The image intensity function $f(x, y)$ has a polynomial representation given by

$$f(x, y) = \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} T_{pq} t_p(x) t_q(y) \qquad (11)$$

where the coefficients T_{pq} are the Chebyshev moments defined in (6). The above result follows when the left-hand side of (3) is applied as an operator to both sides of equation (6).

3 Image Feature Representation

Equation (11) is the inverse Chebyshev moment transform, and provides an image reconstruction from a finite set of its moments. The reconstructed image is a measure of image features that are captured by the moment terms. The following figure shows the original image of the letter ‘E’ on a 20x20 pixel grid (N=20), and the reconstructed images using equation (11), with the maximum order of moments varied from 5 through 11.

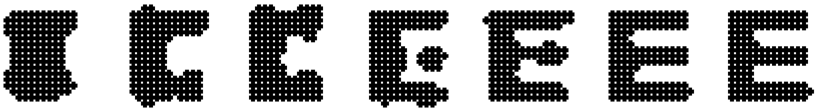

Figure 1: Image reconstruction using Chebyshev moments

There is also a close relationship between Chebyshev and Legendre moments arising from the fact that the Chebyshev polynomial values tend to the values of the Legendre polynomials evaluated at the corresponding points in the normalized coordinate space [-1,1], as the image size N tends to infinity. Indeed, the discrete approximation of Legendre moments[4] is very similar to the expression for Chebyshev moments. Legendre moments λ_{pq} of order (p+q) are defined as

$$\lambda_{pq} = \frac{(2p+1)(2q+1)}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} P_p\left(\frac{2i-N+1}{N-1}\right) P_q\left(\frac{2j-N+1}{N-1}\right) f(i, j) . \qquad (12)$$

where $P_n()$ denotes the Legendre polynomial of order n. The binary image reconstruction using the above moments is given in Fig. 2.

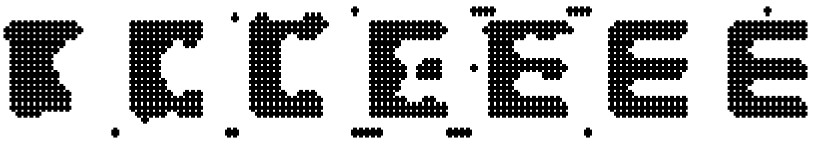

Figure 2: Image reconstruction using Legendre moments

A comparison plot of root-mean-square reconstruction error obtained from the above results is in Fig. 3. The superior feature representation capability of Chebyshev moments over Legendre moments is evident from this figure. The higher reconstruction error in Legendre moments is also partly due to the approximation of continuous moment integrals, which result in significant discretization errors when the image size is small.

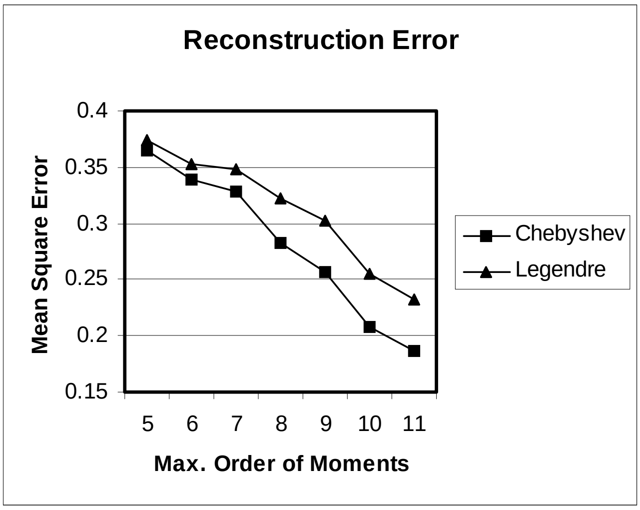

Figure 3: Comparison of reconstruction errors using Chebyshev and Legendre moments.

4 Computational Aspects

The symmetry property of Chebyshev polynomials can be made use of, to considerably reduce the time required for computing the associated moments. The scaled Chebyshev polynomials have the same symmetry property which the classical Chebyshev polynomials satisfy:

$$t_n(N-1-x) = (-1)^{-n} t_n(x) \tag{13}$$

The above relation suggests the subdivision the domain of an NxN image (where N is even) into four equal parts, and performing the computation of the polynomials only in the first quadrant where $0 \le x, y \le (N/2-1)$. The expression for Chebyshev moments in (6) can be modified with the help of (13), as follows:

$$T_{pq} = \frac{1}{\rho(p,N)\rho(q,N)} \sum_{x=0}^{(N/2)-1} \sum_{y=0}^{(N/2)-1} t_p(x) t_q(y) \left\{ \begin{array}{c} f(x,y) + (-1)^p f(N-1-x,y) \\ +(-1)^q f(x,N-1-y) \\ +(-1)^{p+q} f(N-1-x,N-1-y) \end{array} \right\} \tag{14}$$

In addition to reducing the computation time by a factor of 4, the symmetry property is also useful in minimizing the storage required for the scaled Chebyshev polynomials. The scaled Chebyshev polynomial $t_n(x)$ can be expressed as a polynomial of x, as given below. The polynomial expansion is useful in relating the Chebyshev moments to the discrete approximation of geometric moments.

$$t_n(x) = \frac{1}{\beta_n} \sum_{k=0}^{n} C_k(n,N) \sum_{i=0}^{k} s_k^{(i)} x^i , \tag{15}$$

where

$$C_k(n, N) = (-1)^{n-k} \frac{n!}{k!} \binom{N-1-k}{n-k} \binom{n+k}{n} \tag{16}$$

and $s_k^{(i)}$ are the Stirling numbers of the first kind [7], which satisfies the equation

$$\frac{x!}{(x-k)!} = \sum_{i=0}^{k} s_k^{(i)} x^i \tag{17}$$

5 Noise Effects

It is well known that image noise significantly affects the reconstruction error. Moments of higher orders can become more sensitive to image noise. The effect of noise was analyzed using a 60x60 binary image of a Chinese character shown in Fig. 4. The reconstructed images with the maximum order of moments varied from 18 to 30 in steps of 2 are also given in Fig. 4.

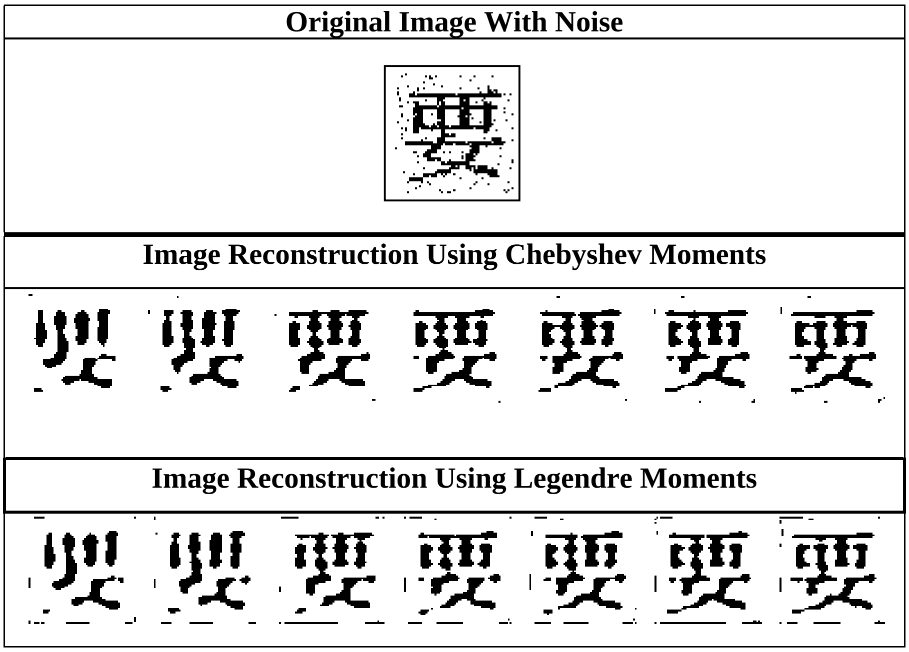

Figure 4: Reconstruction of an image after adding noise.

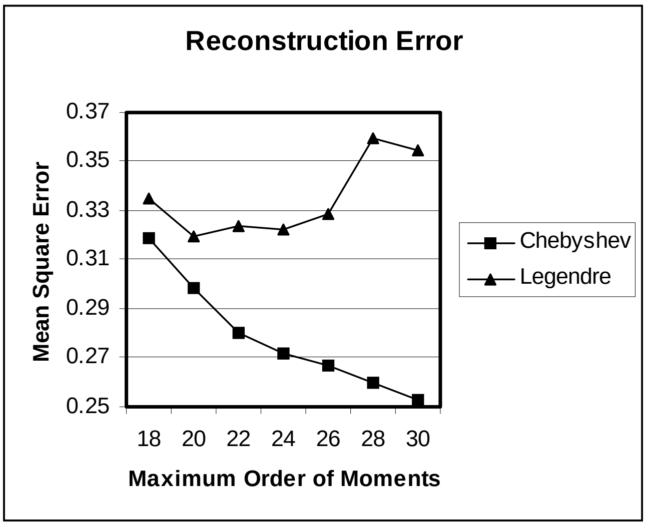

Figure 5: Effect of image noise on reconstruction error.

The root-mean-square error of reconstruction for both Chebyshev and Legendre moments are plotted in Fig. 5. The performance of Chebyshev moments is still better compared to that of Legendre moments. Chebyshev moments of higher orders are also less sensitive to image noise as can be seen from Fig. 5. As the order of Legendre moments are increased, the noise factor starts dominating, and causes an increase in the reconstruction error.

6 Conclusions

The paper has presented the theoretical framework for discrete orthogonal moments based on Chebyshev polynomials for a more accurate representation of image features than those obtained using continuous moment functions. The motivation for developing discrete orthogonal moments arises from the need for circumventing the commonly encountered problems of large discrete approximation errors and coordinate transforms associated with Legendre and Zernike moments. Images can be accurately reconstructed using the inverse Chebyshev moment transform. Image reconstruction from moments also demonstrate the superiority of the feature representation capability of Chebyshev moments over Legendre moments. Certain computational aspects of Chebyshev polynomials are also discussed.

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