A Comparison Between The Circuit Theory Model and Finite Element Model Reactive Components

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***Abstract*—This paper presents two different modelling techniques for the reactive components of a partial-core transformer; a circuit theory model and a finite element model. Each model was used to simulate a hypothetical partial-core transformer with a varying winding aspect ratio and the results compared. Analysis of the simulation results suggested modification of two parameters of the circuit theory model, the leakage function and the magnetising function. The models were then compared to test results from three different partial-core transformers that were built.**

I. INTRODUCTION

Partial-core transformers (PCT) have been designed as an alternative to full core transformers [1] [2], the difference being that the outer limbs and connecting yokes are absent from the PCT, (Figure 1). This means that the magnetic circuit for a PCT consists of the core and surrounding air which results in high reluctance and therefore low magnetising reactance when compared with similar full core transformers. Despite this, it is possible to design a PCT that performs comparably to a full core transformer under full load conditions while making significant savings on core material and weight.

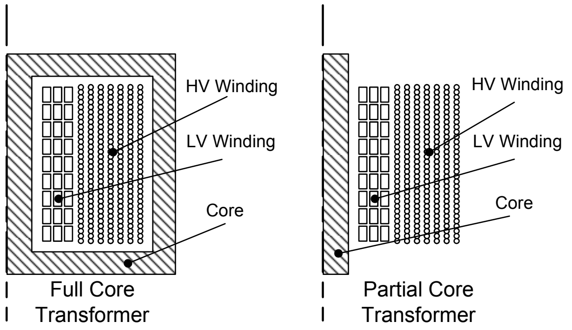


Fig. 1. A cross sectional view of the differences between full core and partial-core transformers

PCT have been built and used to test hydro generator stators in New Zealand [3] and Australia to test cable insulation at power frequencies. The inductance of the PCT is tuned to the capacitance of the insulation under test by moving the core. When used for this purpose they are referred to as partial-core resonant transformers. A superconducting PCT has also been built and tested for use as a power transformer [4].

A reverse design method [5] [6] is used to design a PCT where the physical dimensions and properties of the materials determine the performance of the transformer. A circuit theory model has been developed over some years to determine the components of the Steinmetz 'exact' transformer equivalent circuit. An updated model has since been developed using finite element analysis to model the reactive components [7].

In this paper a study has been undertaken to compare the two modelling techniques with respect to the winding aspect ratio of a partial core transformer. In the study, a hypothetical PCT is designed. It is a two winding transformer with 2000 turns per winding and a constant core weight.

II. REACTIVE COMPONENT MODELLING

The PCT modelling is based upon the Steinmetz 'exact' transformer equivalent circuit [8] depicted in Figure 2.

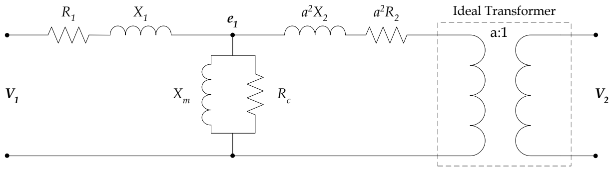


Fig. 2. Steinmetz 'exact' transformer equivalent circuit, referred to primary winding.

The individual components for the model are derived from the physical properties and dimensions of the materials used to build the transformer [9] [5]. The core loss and winding loss components of the model are those used in the reverse design method and are the same for the circuit theory model (CTM) and the finite element model (FEM).

A. Circuit Theory Model

1) Magnetising reactance: The reluctance of the magnetic circuit for a PCT is comprised of the reluctance of the core,

$\Re_c$, and the reluctance of the air path, $\Re_{air}$. The reluctance of the core is given by,

$$\Re_c = \frac{l_c}{\mu_0 \mu_r A_c} \tag{1}$$

where, l_c is the length of the core, A_c is the cross sectional area of the core, μ_0 is the permeability of free space ($4\pi \times 10^{-7}$) and μ_r is the relative permeability of the core material.

The reluctance of the air path is given by [10],

$$\Re_{\text{air}} = 338712 \left(\left(\frac{1}{A_c} \right)^{0.345} \left(\frac{1}{l_c} \right)^{0.31} \right) \tag{2}$$

So that the total reluctance is,

$$\Re_T = \Re_c + \Re_{air} \tag{3}$$

The total relative permeability of the magnetic circuit is derived from the total reluctance and a magnetising function. This magnetising function takes into account the winding aspect ratio of the PCT and scales the reactance accordingly [11]. The magnetising function is,

$$\gamma\left(\beta_a\right) = 1 - e^{\left(\frac{-\beta_a}{0.32\beta_a + 0.8}\right)} \tag{4}$$

where, β_α is the transformer winding aspect ratio (winding height over winding width).

The total relative permeability then becomes,

$$\mu_{rT} = \frac{l_c}{\mu_0 \Re_T A_c} \gamma\left(\beta_a\right) \tag{5}$$

and the magnetising reactance of the PCT is,

$$X_m = \frac{2\pi f N_1^2 \mu_0 \mu_{rT} A_c}{l_c} \tag{6}$$

2) Leakage Reactance: In the circuit theory model the leakage reactance for each winding is assumed equal.The total leakage reactance for a PCT is derived from,

$$\begin{aligned} X_1 &= a^2 X_2 \\ &= \frac{1}{2} \left(\frac{2\pi f \mu_0 N_1^2}{WH \times \Gamma\left(\beta_a\right)} \right) \left(\frac{l_1 d_1 + l_2 d_2}{3} + l_{12} \Delta d \right) \end{aligned} \tag{7}$$

where, WH is the winding height, $\Gamma(\beta_\alpha)$ is the leakage function, l_1 is the mean circumferential length of primary winding, d_1 is the primary winding thickness, l_2 is the mean circumferential length of secondary winding, d_2 is the secondary winding thickness, l_{12} is the mean circumferential length of inter-winding space and Δd is the inter-winding thickness.

The leakage function for a PCT is derived from the transformer aspect ratio and is given as [6],

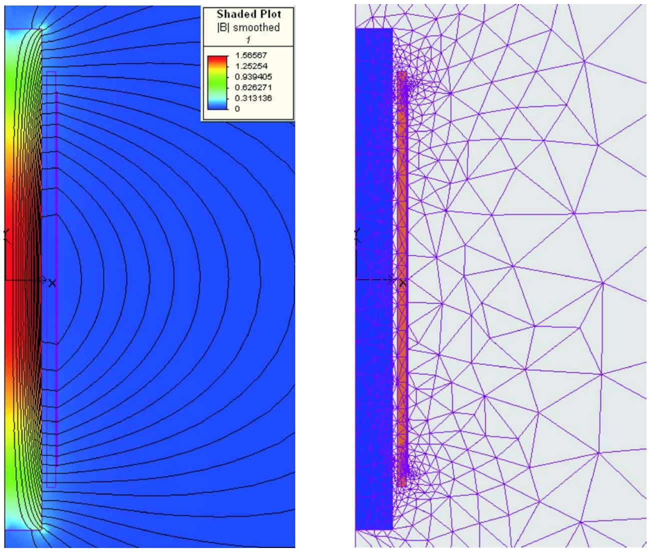
$$\Gamma\left(\beta_a\right) = 1 - e^{\left(\frac{-\beta_a}{0.4\beta_a + 1.59}\right)} \tag{8}$$

B. Finite Element Model

The finite element model (FEM) is based on 2D magnetostatic finite element analysis from a commercial computer simulation to model the reactive components of the Steinmetz 'exact' transformer equivalent circuit. This package removes the assumption of a uniform flux density and the empirical constants of the circuit theory model [7].

Each winding of the PCT was modelled as a block of solid copper that covers all turns of that winding. The insulation between windings was modelled as air. This model was considered sufficient as the windings were packed close together. Inter-winding eddy currents were assumed to have negligible effect on the global field distribution. Figure 3 shows an example of the FEM for a sample PCT.

The core was modelled as a single block of isotropic linear material with a relative permeability of 3000. This model does not take into account the non-linear effects such as core saturation and radial flux not being restricted to the lamination plane. These non-linearities were assumed to have minimal effect on the global field distribution.



(a) (b)

Fig. 3. Finite Element Model of a partial-core transformer: (a) Magnetic field plot for an open circuit test; (b) Final solution mesh for static 2D simulation;

The modelling of the PCT is considered an open-bounded problem because the flux return path is air. For this reason a simple truncation method was used where the outer air boundaries were located far from the transformer and a Dirichlet (flux tangential) constraint was applied.

1) Reactance parameters: The permeance matrix **P** for a PCT is defined as

$$\mathbf{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \tag{9}$$

where $P_{12} = P_{21}$

P is found using the finite element software by performing two simulations. In each simulation a unit turn winding was excited with unit current and the flux-linkage of both windings was found and used to calculate the magnetic permeance by,

$$P_{ij} = \frac{\lambda_i}{i_j}, \tag{10}$$

where λ_i is the flux linkage of winding i due to an excitation current i_j in winding j.

The inductance matrix $\mathbf{L}$ can then be calculated. Inductance is defined as

$$L_{ij} = N_i N_j P_{ij} \tag{11}$$

where N_i and N_j are the number of turns on windings i and j. The inductance matrix contains the self (L_1 L_2) and mutual (M_{12}) inductances for the transformer [12].

The inductances are transformed into the reactance components of the Steinmetz 'exact' transformer equivalent circuit using [7],

$$X_m = j\omega\left(aM_{12}\right) \tag{12}$$

$$X_1 = j\omega\left(L_1 - aM_{12}\right) \tag{13}$$

$$a^2 X_2 = j\omega a^2 \left(L_2 - \frac{1}{a} M_{12}\right) \tag{14}$$

where $a = N_1/N_2$ is the transformer turns ratio.

III. SIMULATION

A. Simulation Setup

A program has been written in Visual Basic which reads in data from an Excel spreadsheet on the dimensions and material properties of the transformer being designed. From this data the equivalent circuit elements are calculated for both the CTM model, and the FEM model via data calculated by MagNet including the reactive components discussed in Sections II-A1, II-A2 and II-B1.

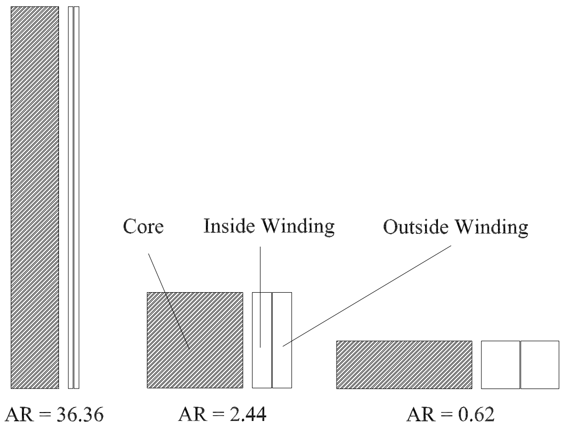


Fig. 4. cross sectional diagram of simulated PCTs for different winding aspect ratios. Note: only half of each transformer is shown for clarity

Seven different winding aspect ratios were investigated ranging from 36.36 to 0.62. Figure 4 shows an example of three of the simulated transformers. These were determined by increasing the number of layers on the simulated PCT while decreasing the winding height so that the total number of turns was constant. Also, to ensure that the different aspect ratio transformers were as similar as possible the core weight was kept constant. The simulations used a former thickness of 10 mm. Information on the material properties of the core is presented in Table I and the dimensions used for each of the winding aspect ratios are presented in Table II.

TABLE I
MATERIAL PROPERTIES OF THE CORE

Core Type	Partial-Core	
Core Shape	Circular	
Lamination Thickness	0.23	mm
Stacking Factor	0.95	
Relative permeability	3000	
Resistivity at 20 °C	160.0E-9	Ω−m
Thermal resistivity coefficient	6.0E-3	Ω−m/ °C
Operating Temperature	20	°C
Material Density	7833	kg/m^3

TABLE II
CORE DIMENSIONS FOR THE DIFFERENT ASPECT RATIO SIMULATIONS

Radius (mm)	**Length (mm)**	**Weight (kg)**
50.00	400	24.03
63.25	250	24.03
70.71	200	24.03
89.44	125	24.03
100.00	100	24.03
111.80	80	24.03
141.42	50	24.03

The simulations modelled a 1 to 1 PCT with 2000 turns on both the primary (inside) and secondary (outside) windings. To change the winding aspect ratio whilst keeping the number of turns constant, the height of the winding window was decreased while the number of layers was increased, (Table III). The wire size was chosen to be 1 mm in diameter with no inter-layer insulation, and a stacking factor of 1 for simplicity in calculating the number of turns per winding. There was a 1 mm gap between the two layers.

TABLE III
WINDING DATA

Length (mm)	**Primary Layers**	**Secondary Layers**	**Aspect Ratio**
400	5	5	36.36
250	8	8	14.71
200	10	10	9.52
125	16	16	3.79
100	20	20	2.44
80	25	25	1.57
50	40	40	0.62

B. Simulation Results

The results of the simulation (Table IV) are plotted in Figures 5, 6 and 7. Figure 5 shows how the magnetising reactance changes with respect to the winding aspect ratio. It can be noted that there is a significant difference between the CTM and the FEM. The CTM suggests the magnetising reactance will tend towards zero as the winding aspect ratio approaches zero. This is because of the magnetising function Equation 4 which is an exponential function forcing the reactance to zero for small aspect ratios. The FEM in contrast is tending towards infinity for small aspect ratios. For aspect ratios above 15 both models have a reasonably constant magnetising reactance although there is a 13% difference.

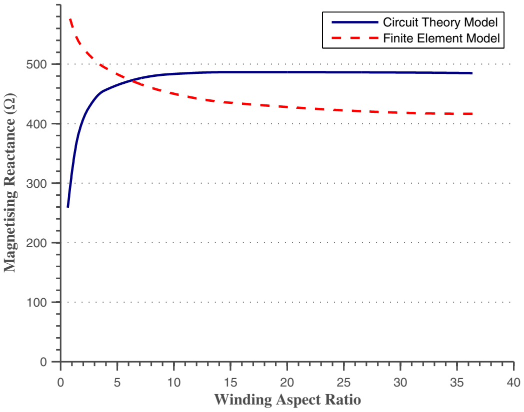


Fig. 5. Plot of simulation results for magnetising reactance against winding aspect ratio.

The primary side leakage reactance of Figure 6 shows good correlation between both models for aspect ratios above 10. For lower aspect ratios the CTM calculation increases rapidly as governed by the leakage function of Equation 8, whereas the FEM value increases much slower.

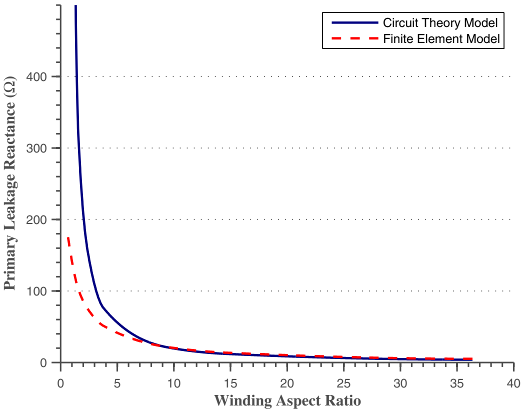


Fig. 6. Plot of simulation results for primary leakage reactance against winding aspect ratio.

For the secondary winding leakage reactance (Figure 7), the CTM calculation is exactly the same as that of the primary winding as defined by Equation 7. The FEM calculation however, is about 50% less at high aspect ratios and starts to increase from an aspect ratio below 5.

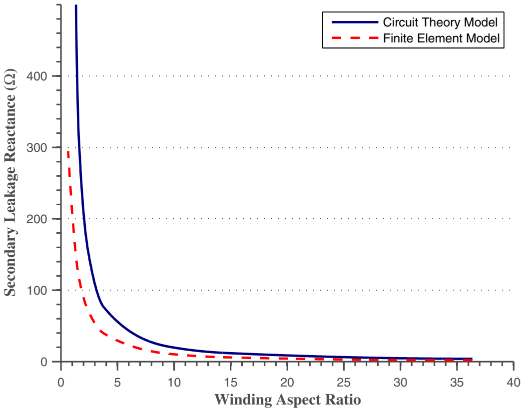


Fig. 7. Plot of simulation results for secondary leakage reactance against winding aspect ratio.

Also from Table IV, it is apparent that for large winding aspect ratios, the primary leakage reactance is larger than the secondary, but for small winding aspect ratios the opposite is true. The cross over point where X_1= X_2 occurs between aspect ratios of 2.44 and 1.57, which is similar to traditional full core transformer aspect ratios.

TABLE IV
RESULTS FROM SIMULATION

		CTM		
Aspect Ratio	**XM**	**X1**	**X2**	**X1+X2**
36.36	484.88	3.93	3.93	7.87
14.71	486.56	11.94	11.94	23.88
9.52	482.90	20.93	20.93	41.87
3.79	454.83	76.18	76.18	152.35
2.44	425.24	151.25	151.25	302.50
1.57	381.73	318.25	318.25	636.51
0.62	255.74	1845.92	1845.92	3691.84
		FEM		
Aspect Ratio	**XM**	**X1**	**X2**	**X1+X2**
36.36	416.53	5.16	1.73	6.89
14.71	435.38	13.64	5.93	19.57
9.52	452.40	21.36	10.92	32.28
3.79	495.54	51.20	39.47	90.67
2.44	517.20	73.86	70.22	144.08
1.57	538.91	102.40	119.45	221.85
0.62	591.11	179.35	299.20	478.55

The simulations were then repeated with the magnetising function (Equation 4) and the leakage function (Equation 8) set to 1. The results of those simulations are given in Figures 8, 9 and 10 and Table V.

With the magnetising function set to 1 the CTM magnetising reactance in Figure 8 is no longer forced to zero and increases with small aspect ratios. There is still a difference of about 18% between the two models at high aspect ratios.

Correlation between the two models has been improved with the leakage function set to 1 (Figures 9 and 10).

IV. TEST RESULTS

The results from the second simulation show improvement in the correlation between the CTM and FEM techniques. A

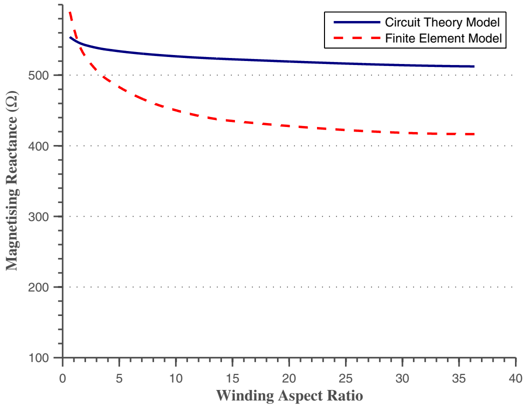


Fig. 8. Plot of simulation results for magnetising reactance against winding aspect ratio with magnetising function set to 1.

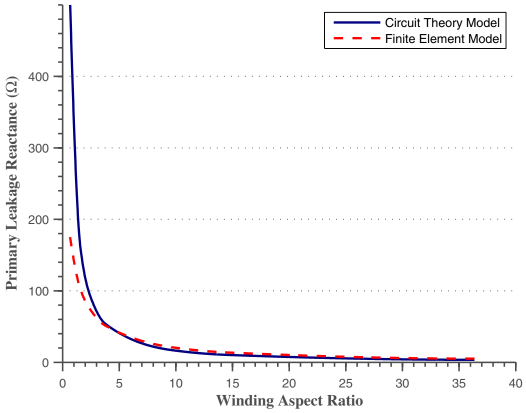


Fig. 9. Plot of simulation results for primary leakage reactance against winding aspect ratio with leakage function set to 1.

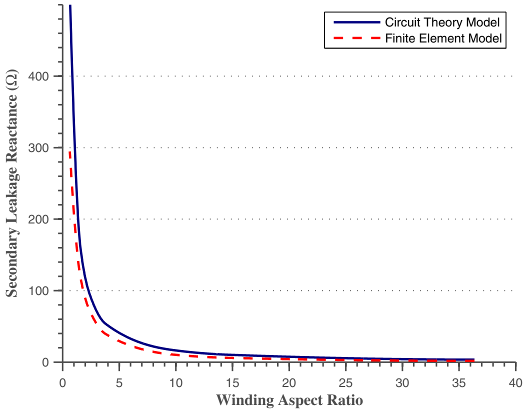


Fig. 10. Plot of simulation results for secondary leakage reactance against winding aspect ratio with leakage function set to 1.

series of tests were conducted on three PCT transformers to ascertain which modelling method is more accurate, Figure 11 shows testing of one of the PCT.

TABLE V
RESULTS FROM SIMULATION WITH MAGNETISING FUNCTION AND LEAKAGE FUNCTION SET TO 1

Aspect Ratio	**XM**	**CTM X1**	**X2**	**X1+X2**
36.36	512.40	3.52	3.52	7.04
14.71	522.73	10.27	10.27	20.55
9.52	527.26	17.35	17.35	34.69
3.79	536.49	53.68	53.68	107.37
2.44	540.81	92.80	92.80	185.59
1.57	545.12	161.38	161.38	322.76
0.62	554.25	526.84	526.84	1053.68
Aspect Ratio	**XM**	**FEM X1**	**X2**	**X1+X2**
36.36	416.53	5.16	1.73	6.89
14.71	435.58	13.64	5.93	19.57
9.52	452.40	21.36	10.92	32.29
3.79	495.54	51.20	39.47	90.67
2.44	517.20	73.86	70.22	144.09
1.57	538.91	102.40	119.45	221.85
0.62	591.11	179.35	299.20	478.55

Three partial core transformers have been built [13] and were used to test the accuracy of the modelling. The specifications of the PCT are outlined in Table VI.

Measurement of the applied voltage and current were taken on each of the PCT in both open circuit and short circuit configurations with a calibrated Fluke 41 meter. The measurements were used to determine the open circuit and short circuit impedance of the transformers and this was compared to simulation results from both the finite element model and the circuit theory models. These results are presented in Table VII where CTMm is the modified circuit theory model with the leakage and magnetising factors set to 1.

Fig. 11. Testing the Manapouri Partial-Core Transformer

The open circuit test gives an indication of the magnetising current and the core losses, as the leakage reactance and winding resistance of the excited winding tend to be small by comparison. From Table VII, it can be seen that the partial-core transformer has quite a low magnetising reactance when compared to the core loss component. The calculated impedance has generally underestimated the real component when compared to the measured impedance, by as much as

TABLE VI

PARTIAL CORE TRANSFORMER SPECIFICATIONS

		Lynch	**Matahina**	**Manapouri**
Core	Length	700	710	1200
	Radius	37.5	125	175
	Lamination Thickness	0.5	0.5	0.5
	Stacking Factor	0.96	0.95	0.95
Primary Winding (outside)	Length	600	680	900
	Layers	2	1	1
	Total Turns	160	65	62
	Voltage (V)	230	230	443
Secondary Winding (inside)	Length	700	735	995
	Layers	37	13	9
	Total Turns	1600	8840	502
	Voltage (V)	80000	30800	32000
Aspect Ratio		12.69	22.87	35.35

TABLE VII

COMPARISON BETWEEN TEST RESULTS AND SIMULATION RESULTS

		Lynch		**Matahina**		**Manapouri**	
		$\Re$	$\Im$	$\Re$	$\Im$	$\Re$	$\Im$
Parallel Impedance OCT(Ω)	Measured	62.05	3.25	9.37	0.65	31.40	1.10
	FEM	64.12	3.21	8.44	0.65	27.03	1.12
	CTM	54.34	3.02	9.82	0.72	21.79	0.97
	CTMm	58.82	3.24	10.82	0.76	23.32	1.02
% difference from measured	FEM	1.7	-1.2	-9.9	0	-13.9	1.8
	CTM	-12.4	-7.1	4.8	10.8	-30.6	-11.8
	CTMm	-5.2	-0.3	15.5	16.9	-25.7	-7.3
Series Impedance SCT(Ω)	Measured	0.164	0.159	0.045	0.014	0.041	0.015
	FEM	0.129	0.155	0.049	0.017	0.032	0.019
	CTM	0.124	0.140	0.049	0.017	0.031	0.013
	CTMm	0.124	0.120	0.049	0.015	0.031	0.012
% difference from measured	FEM	-21.3	-2.5	8.9	21.4	-22.0	26.7
	CTM	-24.4	-11.9	8.9	21.4	-24.4	-13.3
	CTMm	-24.4	-24.5	8.9	7.1	-24.4	-20.0

30%. The reactive component of the modelling was much closer to the measured value with the FEM model being the most accurate. The CTM models showed unusual results for the Matahina transformer. The real part of the impedance was overestimated for this transformer and underestimated for the other two transformers. This could be due to the relatively low impedance of the transformer compared with the other two.

The short circuit test results show the winding loss and leakage reactance. The results show a large percentage difference between measured and calculated values. This indicates that the models for the leakage reactance components are not as accurate as the models for the magnetising reactance. In practical terms, the model will give incorrect results for a short circuit test, but will be more accurate for a load test because the load impedance is much larger compared to the series impedance of the transformer.

V. CONCLUSIONS

This paper presents two different methods to model the reactive components for a partial core transformer, a circuit theory model and a finite element model. Both methods were used to model a hypothetical partial core transformer and compared to each other for different winding aspect ratios. The circuit theory model was then modified so that the leakage function and the magnetising functions were equal to 1 and the simulations were run again.

From the results of the simulations there is reason to suggest removal or modification of the leakage function and the magnetising function from the circuit theory model. These two functions are artificially forcing the magnetising reactance to zero and the leakage reactance to infinity for very low aspect ratios.

Open circuit and short circuit tests were performed on three partial core transformers that had been made for testing generator stator insulation. The measured impedance from each transformer was compared to the impedance from each modelling technique. The results from the comparison suggested that the finite element model was the most accurate for the open circuit test and that all three modelling techniques were fair for the short circuit impedance.

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