Normalization of a Fading Channel

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***Abstract*—In this paper we discuss the proper normalization of a fading channel model. Physically, radio channels correspond to passive circuits and follow the energy conservation law. The ratio of the received energy to the transmitted energy is the energy gain of the channel. The representative energy gain is defined as the average energy gain for a signal that is uniformly distributed in time, frequency and space. The major approaches for normalization include setting of either the average representative energy gain or the peak energy gain to unity. The peak energy gain of many fading models including Rayleigh fading is infinite, which is obviously impossible in a passive system where the peak energy gain should be less than or equal to unity. Our aim is to show that it is due to the normalization that in some cases the performance in a fading channel is better than in a nonfading channel.**

Keywords-energy conservation law; energy gain; passive systems; multipath fading; transmitter power control

I. INTRODUCTION

Rayleigh fading channel models have been used in wireless communications at least since the work by Price [1] and Turin [2]. The model is based on the central limit theorem. We assume that the channel is non-frequency-selective and its fading gain is $H(t)$. A countermeasure against fading is power control, which can be based either on water filling or channel inversion [5] (Fig. 1).

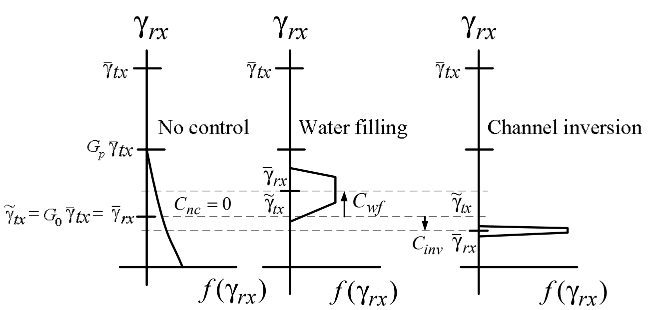

Figure 1. Comparison on water filling (wf) and channel inversion (inv) with no control (nc). The function $f(\cdot)$ refers to the probability density function.

In fading channels two alternative approaches are commonly used for performance measurements in terms of energy [14], [11]. Either the average transmitted energy per symbol $\bar{E}_{tx}$ or the average received energy per symbol $\bar{E}_{rx}$ is used, both usually normalized by the receiver noise spectral density N_0. This leads to the average transmitted signal-to-noise ratio (SNR) per symbol denoted by $\bar{\gamma}_{tx}$ [14], or the average received SNR per symbol $\bar{\gamma}_{rx}$ [11], respectively (Fig. 1) [3].

The ratio of the two SNR's is the average energy gain $G = \bar{\gamma}_{rx}/\bar{\gamma}_{tx} = \bar{E}_{rx}/\bar{E}_{tx}$ of the channel. In general G depends on the transmitted signal [3]. This is due to the strong correlation between the transmitted energy and the instantaneous energy gain $|H|^2$ of the channel in cases where transmitter power control is used. We have emphasized in [3] that it is the transmitted energy rather than the received energy that is the basic resource of an energy-limited transmission link.

The *representative energy gain* G_0 is defined as the average energy gain of a signal that is uniformly distributed in time, frequency and space [3]. In a non-frequency-selective channel the representative energy gain is $G_0 = E\{|H|^2\}$ which includes the path loss of the channel (Fig. 1). The *peak energy gain* is denoted by $G_p = \max(\gamma_{rx}/\gamma_{tx})$. In a non-frequency-selective channel the peak energy gain is $G_p = \max(|H|^2)$.

The transmitted SNR per symbol referred to the receiver is defined as $\tilde{\gamma}_{tx} = G_0\bar{\gamma}_{tx}$ (Fig. 1). In [3] we have shown that in a non-frequency-selective channel $\bar{\gamma}_{rx} = \tilde{\gamma}_{tx} + C$ where $C = \mathrm{Cov}(\gamma_{tx}, |H|^2)$ is a covariance that shows how well the transmitted energy is matched to the channel. Note that $\bar{\gamma}_{rx} = \tilde{\gamma}_{tx}$ only if $C = 0$. Since G is a function of the transmitted signal, we concluded in [3] that it is crucial to use the transmitted SNR rather than the received SNR for performance measurements. Due to power control the channel does not behave only as a scaling factor for the transmitted energy (Fig. 1).

The major approaches for normalization of the channel include normalization of G_0 or G_p to unity [4]. For brevity, we call them average $(G_0 = 1)$ and peak $(G_p = 1)$ normalization, respectively. Energy conservation holds for all physical systems. Therefore the output energy of a passive system cannot be larger than the input energy, and $G_p \le 1$ since usually a major part of the energy is lost in the channel.

In most of the literature on fading channels average normalization is used. This is usually a proper method since in link budgets we use the average energy [11]. The path loss is usually large. From Fig. 1 and from the results of [5], [6] we

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE GLOBECOM 2006 proceedings.

conclude that if average normalization is used, the system sometimes works better in a Rayleigh fading channel than in an AWGN channel. The reason is obviously the fact that the AWGN channel also includes path loss. If we were to use the transmitted SNR directly in comparisons and included the path loss in our model, there would not in general be such a problem. The probability that the peak energy gain would be larger than unity can in general be neglected when the path loss is large.

There is another motivation to use the peak normalization. Usually for convenience we neglect path loss in the channel model in link simulations and therefore performance in the fading channel can be better than in an AWGN channel. With average normalization some systems work apparently below the Shannon limit for the average transmitted SNR per bit, which has created some confusion [4].

Imagine that the energy gain of an additive white Gaussian noise (AWGN) channel is G_0. The Shannon limit of -1.6 dB is really valid for the received SNR per bit, but not for the transmitted SNR per bit. The *minimum possible transmitted SNR per bit* is -1.6 dB - $10\log_{10} G_0$ dB. In that case, due to the gain G_0, the received SNR per bit is -1.6 dB, which is just at the Shannon limit. If G_0 is increased, the minimum possible transmitted SNR is decreased. We do not call the limit for the transmitted SNR per bit the Shannon limit to avoid confusion.

In a similar way, the capacity of a Rayleigh fading channel can be larger than that of an AWGN channel at low transmitted SNR's if an adaptive transmitter knows the channel [5], [6]. If the received SNR is used, the fading effect will always degrade the capacity, but this does not really solve the problem since we have shown in [3] that in adaptive transmission the average transmitted SNR should be used in fair comparisons. We will show that in fact the confusion is due to the normalization process, and not to the use of the transmitted SNR.

Xiang and Pietropon [4] noticed that the peak energy gain of a linear time-invariant frequency-selective filter can be larger than unity if the average normalization is used. They proposed that the filter should be normalized by the peak energy gain, as otherwise the model does not represent a passive system. With the information in this paper we can generalize this idea to all linear systems. We note that the peak energy gain of the Rayleigh fading channel model is infinite. Therefore this model is not always a good model for radio channels, which are always passive systems and some care should be taken.

We approach the problem by using a model based on a finite noncoherent sum of equal-amplitude complex sinusoids, which has been considered in [7], [8]. For an extensive bibliography, see [9]. These authors did not use the sum explicitly for channel modelling, and the sum was not properly normalized for our purposes. Jakes [10] proposed a model for simulating fading channels with a sum of complex sinusoids. He also did not consider the normalization problem.

The rest of the paper is organized as follows. The system model is presented in Section II. The details of the channel model are covered in Section III. Our semianalytical method is presented in Section IV. The results are shown in Section V and there are some conclusions in Section VI.

II. SYSTEM MODEL

Usually, bits are transmitted by using a carrier that is modulated so that k bit blocks are mapped onto M-ary symbols a_n where $M = 2^k$ [5], [11]. We assume here quadrature amplitude modulation (QAM). We will transmit a block of $N_s = T_B / T$ symbols in an interval T_B where T is the symbol interval and N_s is assumed to be an integer. The interval T_B is the data block length and characterizes the delay from the transmitter to the receiver.

The channel is assumed to be randomly time-variant. We use adaptive power control where the energy of each symbol is selected according to the state of the channel (details below). As a special case, a system with no power control is used. All the random processes are assumed to be ergodic so that time averages are equal to statistical averages. A total of $N = kN_s$ bits are transmitted with total average transmitted energy $\overline{E}_B = E\{E_B\}$ within an interval T_B, in a bandwidth W, and having a bit error rate P_e.

We consider a single-input single-output (SISO) system with a slowly fading non-frequency-selective channel. The received complex baseband signal has the form

$$r(t) = \sum_{n=0}^{N_s-1} \sqrt{E_{n,tx}}\, a_n g(t - nT) H(nT) + w(t) \qquad (1)$$

where $E_{n,tx}$ is the energy of the n-th transmitted symbol given by the power control (averaged over a_n), a_n is the QAM symbol ($E\{|a_n|^2\} = 1$), $g(t)$ is the symbol waveform assumed to have unit energy ($\int_{-\infty}^{\infty} |g(t)|^2 dt = 1$), $H(t) = \nu(t)\exp[j\theta(t)]$ is the fading gain representing the channel response, $\nu(t) = |H(t)|$ and $\theta(t) = \arg[H(t)]$ are the magnitude and phase of the fading gain, respectively, and $w(t)$ is additive white Gaussian noise (AWGN) with two-sided power spectral density, N_0 and autocorrelation function $E\{w(t+\tau)w^*(t)\} = N_0\delta(\tau)$ where $\delta(\tau)$ is the unit impulse function.

By slow fading we mean that $H(t)$ does not significantly change during the transmission of a symbol waveform $g(t)$ so that over the n-th symbol interval we can use the approximation $H(t) \approx H(nT)$. Thus the symbol waveform is not distorted in the channel.

We drop the explicit time dependence (the index n), so the transmitted energy per symbol is denoted by E_{tx} and the received energy per symbol is denoted by E_{rx}. With adaptive power control the energy E_{tx} is changed according to the quality of the channel as determined by the ratio $\gamma_H = \overline{E}_{tx}|H|^2 / N_0$ [5] where $\overline{E}_{tx} = \overline{E}_B / N_s$, and we have dropped the argument t in $H(t)$. Power control algorithms can, in general, be divided into water filling and truncated channel inversion. If water filling is used the transmitted energy is $E_{tx} = \overline{E}_{tx}(1/\gamma_0 - 1/\gamma_H)$ for $\gamma_H \geq \gamma_0$ and zero otherwise where γ_0 is a cut-off value, which is found by numerically solving (4.15) in [5]. If truncated channel inversion is used, the transmitted energy is $E_{tx} = \overline{E}_{tx}(\sigma_0 / \gamma_H)$ for $\gamma_H \geq \gamma_0$ and zero otherwise where σ_0 is a constant selected so that the average transmitted energy is $\overline{E}_{tx}$. The cut-off value is found by numerically maximizing (4.22) in [5]. The cut-off value is $\gamma_0 = 0$ for full channel inversion.

Systems can be roughly divided into power limited and energy limited. In power limited systems such as in a base station connected to the electrical network the available power is limited but energy is infinite. In energy limited systems such as in a mobile station using a battery the available energy is limited. If the average transmitted power in a power limited system is $\overline{P}_{tx}$ and the outage probability is $\rho \equiv p(\gamma_H < \gamma_0)$, the average transmitted power above the cut-off value is $\overline{P}_{tx} /(1-\rho)$. On the other hand, if the average transmitted energy per symbol in an energy limited system is $\overline{E}_{tx}$, the same average is also obtained above the cut-off value since no energy and no symbols are transmitted during the outage. Thus optimization of power and energy limited systems differ.

III. NONCOHERENT SUM OF *N* COMPLEX EXPONENTIALS

The fading gain of the channel is represented by the sum

$$H = \sum_{n=0}^{N-1} \frac{1}{N} e^{-j\varphi_n} \tag{2}$$

where N is the number of complex equal-amplitude subpaths and φ_n is the phase of the n-th subpath [2]. If the phases are all equal, the sum (2) is a coherent sum. If $\varphi_n = 0$ for all n, we have $H = 1$. Thus peak normalization is used in (2). The amplitudes of the subpaths in (2) are identical, which is just a convenient selection for our numerical results. A similar selection was made in [7].

If the phases are random, independent and uniformly distributed, (2) is a noncoherent sum, which corresponds to a fading carrier. The peak amplitude of the noncoherent sum is unity with our normalization. We have therefore used peak normalization. The probability density function of the amplitude $|H|$ can be derived from the results presented in [8] for the values N = 2 and 3. It can be approximated with a truncated Rayleigh distribution. The probability density function of the energy gain can be derived from [7]. It can be approximated with a truncated exponential distribution. Its peak value is unity $(G_p = 1)$ and average value is $G_0 = 1/N$. The peak-to-average energy ratio is N.

We can alternatively use the average normalization. In that case we replace H in (2) by $\sqrt{N}H$, and therefore $G_p = N$ and $G_0 = 1$.

IV. SEMIANALYTICAL METHOD

We assume that both the receiver and the transmitter know the channel. The modulation method in the examples is antipodal (M = 2) and the coherent receiver is based on a matched filter. We first normalize $\overline{E}_{tx} = 1$. For a given $\overline{\gamma}_{tx}$ we compute $N_0 = 1/\overline{\gamma}_{tx}$. The N phases φ_n in (2) are selected randomly and independently from a uniform distribution in the interval $[0, 2\pi[$. If $\gamma_H \geq \gamma_0$ for the selected set of phases, the received SNR γ_{rx} is obtained by multiplying γ_H by the transmitted energy E_{tx} given by the power control law. Without power control $E_{tx} = 1$. The conditional bit error probability is computed from the equation $P_b = Q(\sqrt{2\gamma_{rx}})$. Below the cut-off value no bits and no energy are transmitted. This procedure is repeated for 100000 times for each $\overline{\gamma}_{tx}$ and then the bit error probabilities are averaged. The result is the average bit error probability for that $\overline{\gamma}_{tx}$.

V. RESULTS

A. Bit error rate without power control

We present error probability results for our system model with peak normalization in Figs. 2 and 3 for different values of N. For average normalization the results approach the Rayleigh fading curve when N is large. We note from Fig. 3 that for N = 2 the performance at large transmitted SNR is worse than for N = 3.

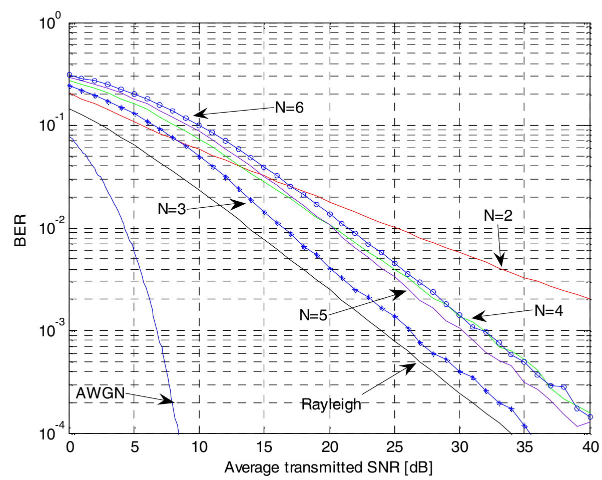

Figure 2. Bit error probability without transmitter power control as a function of average transmitted SNR for small N with peak normalization (average normalization for the Rayleigh fading channel).

The oscillation is continued so that better results are always obtained with an odd N rather than for the previous even N. When N is odd, it is improbable that the vector sum in (2) is close to zero and therefore [12] the bit error probability is not so bad as for the next N. This can be also seen from the energy distributions presented in [7]. When N is large, the loss for large transmitted SNR's is $10\log_{10}(N)$ dB compared to the Rayleigh fading channel with average normalization. This also implies that for average normalization the performance converges towards that of the Rayleigh fading channel.

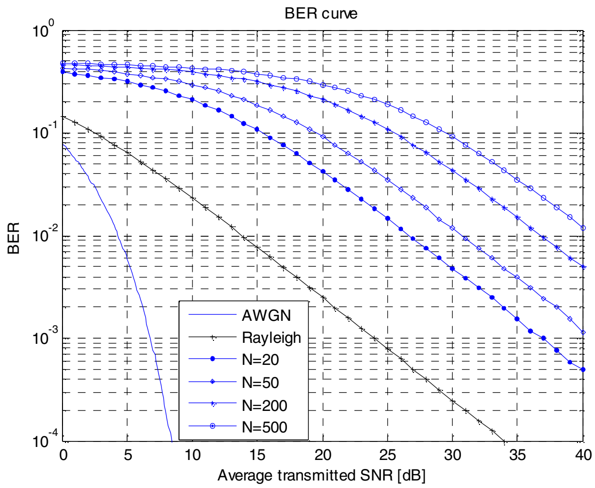

Figure 3. Bit error probability without transmitter power control as a function of average transmitted SNR for large N with peak normalization (average normalization for the Rayleigh fading channel).

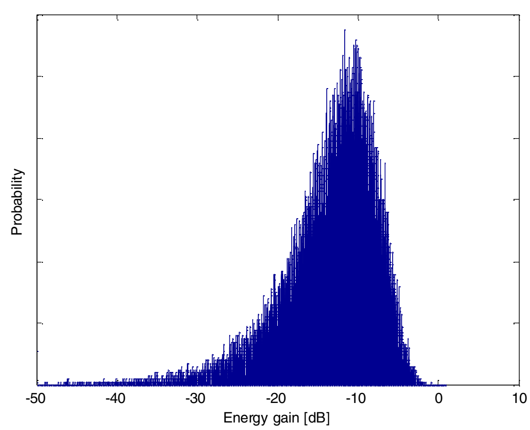

Figure 4. Distribution of the energy gain of the channel when N = 12 with peak normalization

In Fig. 4 we show the distribution of the energy gain of the channel model when peak normalization is used with N = 12. The distribution is approximately exponential, but the energy gain is shown in decibels, which explains the form of the distribution. The probability of the peak value is quite small. Therefore the maximum is -0.9 dB during 100000 samples. If the number of samples is increased, the maximum would approach 0 dB. The difference is negligible in the bit error rate results since the low instantaneous SNR's mainly determine the bit error rate [12].

B. Cut-off value, probability of outage, and covariance

The cut-off value γ_0 is presented in Fig. 5 and the probability of outage ρ is shown in Fig. 6. For water filling the cut-off value is always below 0 dB as shown in [5]. It is usually below that of truncated channel inversion. For the transmitted SNR of 20 dB the cut-off value is -1.3 dB for water filling and 4.3 dB for truncated channel inversion. The probability of outage is 0.08 dB for water filling and 0.27 for truncated channel inversion.

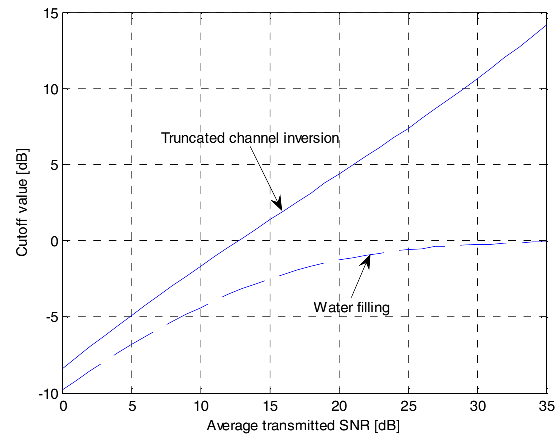

Figure 5. Cut-off value as a function of average transmitted SNR for N = 12 (peak normalization).

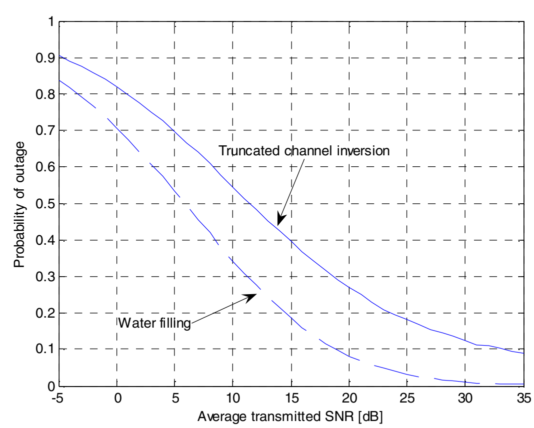

Figure 6. Probability of outage as a function of average transmitted SNR for N = 12 (peak normalization).

C. SNR distributions

In Figs. 7 through 10 we present the distributions for the transmitted SNR and received SNR for water filling and truncated channel inversion when the average transmitted SNR is 20 dB, averaged in the linear scale. Corresponding analytical results can be obtained from the results presented in [13, p. 97].

The distribution of the transmitted SNR is shown in Figs. 7 and 8. In water filling the maximum measured transmitted SNR is 21.3 dB, which is very close to the theoretical maximum obtained from knowledge of the cut-off value. In truncated channel inversion the maximum measured transmitted SNR is 25.7 dB, which is very close to the value obtained from the knowledge of $\sigma_0 = 9.9$ dB and $\gamma_0 = 4.3$ dB. The minimum measured transmitted SNR is close to the value $\sigma_0 = 9.9$ dB as it should be.

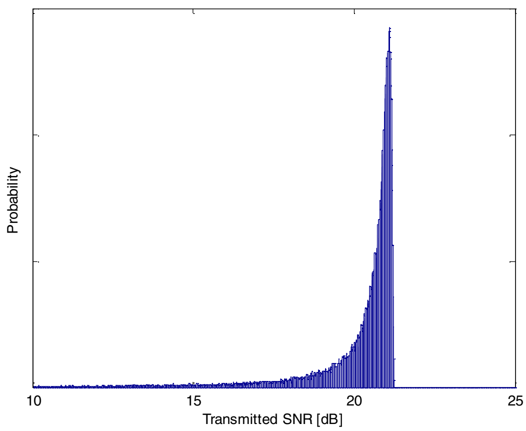

Figure 7. Distribution of the transmitted SNR in water filling for $N = 12$.

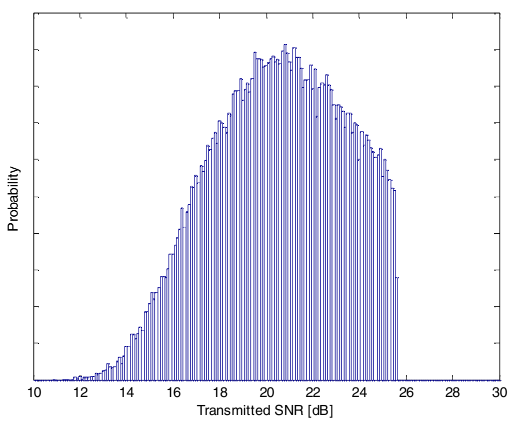

Figure 8. Distribution of the transmitted SNR in truncated channel inversion for $N = 12$ (peak normalization).

In Figs. 9 and 10 we show the distribution of the received SNR for the two power control rules when the average transmitted SNR is 20 dB. In water filling the distribution is almost exponential in the linear scale (see also Fig. 4), but for truncated channel inversion the distribution is an impulse at the value of $\sigma_0 = 9.9$ dB. Since the channel is peak normalized, the received SNR is always smaller than or equal to the maximum transmitted SNR. In water filling the maximum measured received SNR is 20.3 dB, which is slightly below the value 21.3 dB in the transmitter, but this is due to the same fact as in Fig. 4.

D. *Bit error rate with power control*

In our system the bit rate is constant above the cut-off value. Therefore the performance of water filling is quite poor and we do not present it due to space limitations. The performance of water filling can be significantly improved by using bit rate control in addition to power control. With truncated channel inversion the performance is good. The bit error rate performance for truncated channel inversion is presented in Fig. 11 with both average and peak normalization.

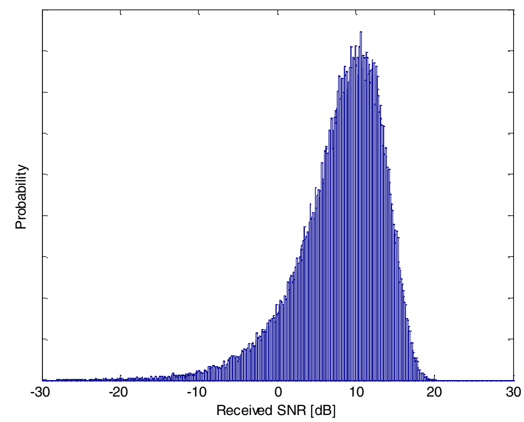

Figure 9. Distribution of the received SNR in water filling for $N = 12$ (peak normalization).

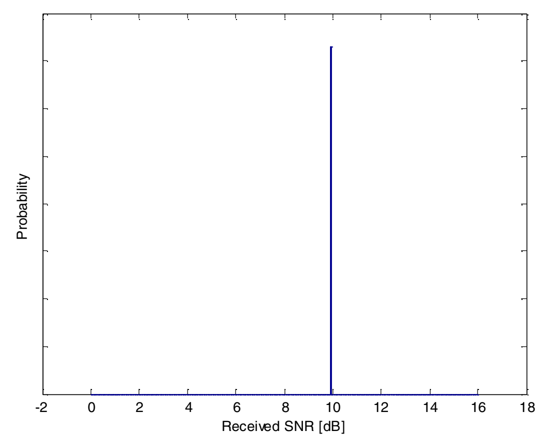

Figure 10. Distribution of the received SNR in truncated channel inversion for $N = 12$ (peak normalization).

The system was optimized as a power limited system. In an energy limited system we are also interested in the performance as a function of transmitted energy per bit. Due to the cut-off value, there are outages during which no energy and no bits are transmitted. If we replace the average transmitted SNR $\bar{\gamma}_{tx}$ by $\bar{\gamma}_{tx}/(1-\rho)$, we obtain the average transmitted SNR per actually transmitted bit. Fig. 11 is now replaced by Fig. 12. We see that now the BER curve with peak normalization no longer goes below the AWGN curve any more. In Fig. 13 we also show the covariance C as a function of transmitted SNR per bit $\bar{\gamma}_{tx}/(1-\rho)$. For water filling $C \geq 0$

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and for truncated channel inversion $C \leq 0$. This property was proved analytically in [3].

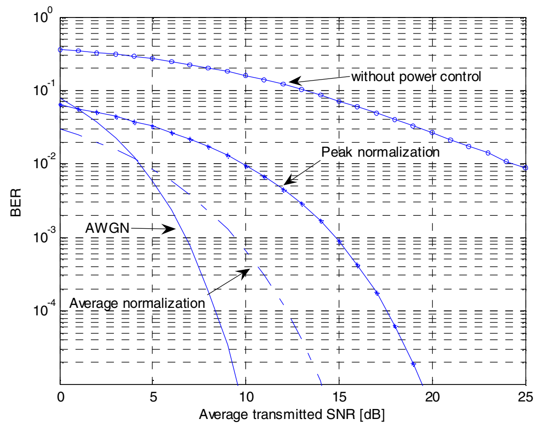

Figure 11. Bit error probability with truncated channel inversion as a function of average transmitted SNR for $N = 12$ with average and peak normalization.

VI. CONCLUSIONS

The Shannon limit was originally derived for the received SNR per bit. We conclude that by using the transmitted SNR and peak normalization, we can avoid confusing results and we know what the minimum transmitted SNR is. An alternative approach is to include the path loss in the channel model and to use average normalization. This is usually not convenient since the results would depend on the path loss. Our results can be easily generalized to time-variant frequency selective and multiple-input multiple-output (MIMO) channels. Details are omitted due to space limitations.

ACKNOWLEDGMENT

This work has been performed in the framework of the IST project IST-2003-507581 WINNER I, which was partly funded by the European Union, and the AWICS project, which is partly funded by the Academy of Finland.

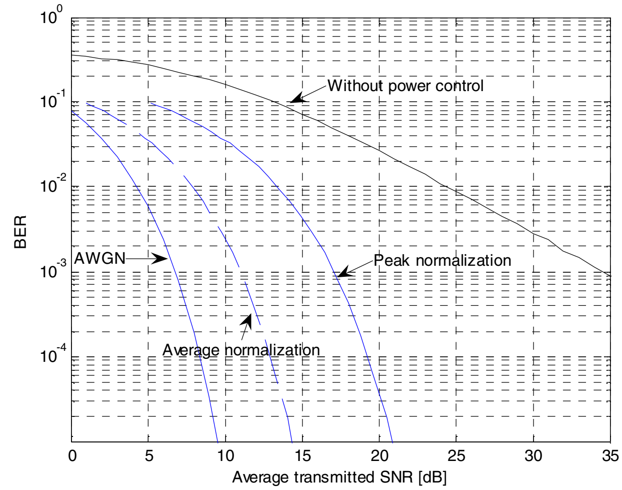

Figure 12. Bit error probability with truncated channel inversion as a function of average transmitted SNR per bit for $N = 12$ with average and peak normalization.

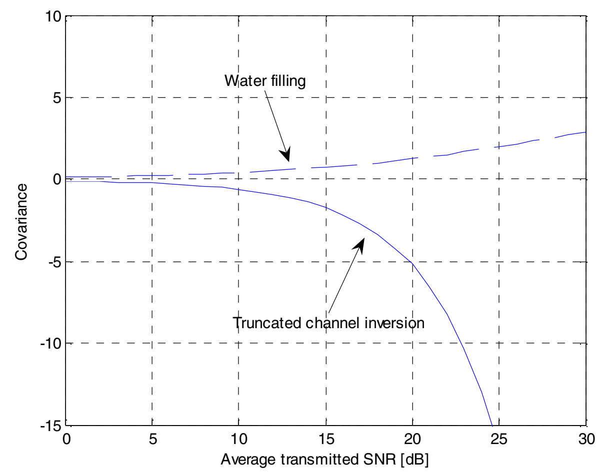

Figure 13. Covariance C as a function of average transmitted SNR per bit for $N = 12$.

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