

**A COMPARISON OF SKILLS
IN THE TRANSITION
FROM SECONDARY TO UNIVERSITY
MATHEMATICS**

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CONTENTS

CHAPTER	PAGE
ABSTRACT.....	1
I. INTRODUCTION.....	2
1. Transition from secondary to university study.....	2
2. Review of literature and theoretical framework.....	5
3. Student understanding	12
4. Mathematical understanding and critical thinking skills.....	14
5. Intuition	19
6. The research question.....	20
II. METHODOLOGICAL PERSPECTIVE AND CHOICE OF METHOD	21
1. Theoretical perspective.....	21
2. Choice of method	23
3. Validity and reliability	25
4. The institution and syllabus	26
III. SELECTION OF PARTICIPANTS AND TASKS.....	28
1. Selection of participants.....	28
2. Conduct of interviews	31
3. Questionnaire	31
4. The tasks	32
(1) Task 1: Calculus.....	34
(2) Task 2: Linear algebra.....	35

IV. THE STUDENTS	38
1. Low-entry students	39
2. High-entry students.....	42
V. ANALYSIS	46
1. Strategy	46
(1) Task 1(a) - Calculus	47
(2) Task 1(b) - Calculus	51
(3) Task 1(c) - Calculus.....	53
(4) Task 2(a) Matrix methods.....	55
(5) Task 2(b) Matrix methods.....	58
(6) Task 2(c) Matrix methods.....	61
(7) Task 2(d) Matrix methods.....	66
2. Structure	68
3. Progress monitoring.....	74
4. Content.....	78
(1) Calculus	78
(2) Matrix methods	82
5. Advance planning	88
VI. SUMMARY AND DISCUSSION	92
1. Summary	92
(1) Interviews.....	92
(2) Task and interviews combined.....	94
2. Discussion	98
REFERENCES	104
APPENDIX I.....	111

LIST OF FIGURES

FIGURE	PAGE
1. Elaborated model of student learning.(Biggs 1987a, p. 96).	10
2. Frequency of self-assessment of mathematical ability by high-entry and low-entry students.....	38
3. Frequency of self-assessed motivation for low-entry students who enrol in university mathematics.....	41
4. Alternative routes used by students to solve Task 1(a).....	48
5. Alternative routes used by students to solve Task 1(b).....	51
6. Alternative routes used by students to solve Task 1(c)	54
7. Alternative routes used by students to solve Task 2(a).....	56
8. Alternative routes used by students to solve Task 2(b).....	59
9. Alternative routes used by students to solve Task 2(c)(i).....	62
10. Alternative routes used by students to solve Task 2(c)(ii).	64
11. Alternative routes used by students to solve Task 2(d).....	66
12. Adjustment to Bigg's model (1987) depicted in Figure 1.....	100

LIST OF TABLES

TABLE	PAGE
1. Frequency of students using the various routes in Figure 4 to solve Task 1(a).....	48
2. Frequency of students using the various routes in Figure 5 to solve Task 1(b).....	52
3. Frequency of students using the various routes in Figure 6 to solve Task 1(c)	54
4. Frequency of students using the various routes in Figure 7 to solve Task 2(a).....	56
5. Frequency of students using the various routes in Figure 8 to solve Task 2(b).....	59
6. Frequency of students using the various routes in Figure 9 to solve Task 2(c)(i).	62
7. Frequency of students using the various routes in Figure 10 to solve Task 2(c)(ii).	64
8. Frequency of students using the various routes in Figure 11 to solve Task 2(d).	67
9. A comparison of characteristics between low-entry and high-entry students.....	94

ABSTRACT

In this study two groups of first year university mathematics students were qualitatively assessed for differences in task specific knowledge, understanding and problem solving skills. Individual interviews were conducted with 12 first year university students who obtained between 40% and 49% in a secondary school mathematics examination and nine first year university students who obtained between 85% and 89% in the same examination. These interviews took place after completion of two separate written tasks, one in calculus and the other in matrix methods. Results indicated that students in the group with the lower marks were more sensitive to institutional transition factors and exhibited isolated knowledge reinforced by a surface (or reproductive) approach to learning. The students with the higher marks used intuitive knowledge in problem solving and possessed a highly integrated knowledge base reinforced by a deep-achieving approach to learning. Differences in characteristics were consistent between the two groups for both tasks.

CHAPTER I

I. INTRODUCTION

1. TRANSITION FROM SECONDARY TO UNIVERSITY STUDY

Tertiary institutions give students more 'freedom to fail' than do secondary schools... It is well to remind ourselves that entering students are usually only a few months older than school-age students, and have not suddenly and miraculously acquired adult expertise in coping with the unexpected. (Bradley and Kemp 1993)

The transition from secondary to university study requires students to make social, cognitive, developmental and educational adjustments. This transition phase can be a larger hurdle in student intellectual development than the transition from primary to secondary school.

In moving from the secondary school to university mathematics, students face significant changes in the institutional environment and style of presentation. From the personalised interactive environment in secondary classes of up to 30 students, the students move to impersonal non-interactive large first year lectures with 250 or more other students. While struggling to cope with changes students can feel frustrated and overwhelmed with the rapid speed of presentation, the abundance of new complicated abstract concepts, and the reality of significant gaps in prior mathematical knowledge.

At university, students must find the self motivation for long hours of private study. Contact hours are fewer than at school and it is up to students to capitalise on the increased 'free' time available to them. The university expects

students to use private study to deepen their understanding and fill in any knowledge gaps. The student is assumed to be an active, self-motivated learner.

Additional difficulties facing some students in the transition to university mathematics are the possible rigidity of thought and rote learning procedures brought from the secondary schools. The author observed that many of her first year tertiary mathematics students believed that problem solving involved remembering a particular method and repeating that method in examinations. The suggestion that there were alternative approaches to solving mathematical problems, each with its own advantages and disadvantages, was often met with disbelief and resistance at the first year university level.

Work by William Perry (1970, 1984) can help explain some of the difficulty faced at university with student rigidity. Perry worked with the Harvard Bureau of Study Counsel, an undergraduate academic counselling unit designed to aid students with academic adjustment and study problems. From his work with university undergraduates, between 1953 and 1974, Perry identified nine developmental 'positions' beginning with dualism and ending with commitment. In summary:

- **Position 1:** The student sees the world in polar terms of we-right-good vs. other-wrong-bad. Right answers for everything exist in the absolute, known to authority whose role is to mediate (teach) them...
- **Position 2:** The student perceives diversity of opinion, and uncertainty, and accounts for them as unwarranted confusion in poorly qualified authorities or as mere exercises set by authority...

- **Position 3:** The student accepts diversity and uncertainty as legitimate but still temporary in areas where authority ‘hasn’t found the answer yet’.
- **Position 4:** The student perceives legitimate uncertainty (and therefore diversity of opinion) to be extensive and raises it to the status of ... ‘anyone has a right to his own opinion’ ... or ... the student discovers qualitative contextual relativistic reasoning as a special case of ‘what they want’ within authority’s realm.
- **Position 5:** The student perceives all knowledge and values (including authority’s) as contextual and relativistic.
- **Position 6:** The student apprehends the necessity of orientating himself in a relativistic world through some form of personal commitment.
- **Position 7:** The student makes an initial commitment in some area.
- **Position 8:** The student experiences the implications of commitment...
- **Position 9:** The student experiences the affirmation of identity among multiple responsibilities and realises commitment as an ongoing, unfolding, activity through which he expresses his life style.” (Perry 1970, p. 9 -10)

Perry’s stages have implications for students who come from the secondary system where authority is rarely questioned. These students enter a tertiary system that encourages the dissolution of dualism. This dissolution can be especially challenging in mathematics as first year students learn the limitations of the nicely polished algorithms presented at school, the uncertainty in problem solving and the need to use an intuitive plus reasoning approach to mathematics.

The most significant challenge to first year university mathematics students however, is the emphasis on the use of advanced thinking and reasoning skills required to deal with more complex and abstract mathematical concepts. Students with significant gaps in background knowledge and poor study techniques can find it difficult to adapt to abstract concepts and advanced thinking skills that require an intricate linking of new and prior knowledge.

2. REVIEW OF LITERATURE AND THEORETICAL FRAMEWORK

Many of the well-known cognitive theories that explain how individuals think and reason may be too simplistic and linear to account for the complicated adult thinking and reasoning processes displayed by university students. For this study an initial framework based on metacognitive research, especially one focussed on the style of approach rather than cognitive processes, could prove more successful in explaining why students experience difficulty with first year university mathematics.

In the field of cognitive science, Jean Piaget was the classic cognitive theorist whose work incorporated mathematical thought processes (Inhelder and Piaget 1958). Piaget outlined a developmental approach culminating with formal thinking. His stages of development began with a young child's concrete thought processes and progressed in a stepwise fashion to adolescent formal thinking. Piaget assumed that an individual's deductive reasoning and behaviour showed distinctive and consistent patterns and these ideas have since been challenged. For example, Karplus (1981) and his colleagues, categorised five different types of reasoning patterns. In contrast to Piaget's theory, the researchers found that

the same individual did not necessarily use consistent reasoning performance when confronted with different tasks.

Piaget's final stage of development was termed 'formal' thinking. However, there is some support in literature that *higher-order, post-formal* or *abstract* thinking skills may be more than an extension of his 'formal' stage. A major difficulty is that post-formal thinking is difficult to define (Ennis 1976, Quellmalz 1985, Sternberg and Baron 1985) and researchers cannot be certain they are measuring what they think they are measuring. This is even more confusing since Piaget's theory dealt with structures that appeared to be too simple and not clearly seen in the learning behaviour of university students or adults. In the field of cognitive theories some researchers turned to the study of adult thought processes rather than pre-adolescent thought to explain post adolescent thinking and reasoning (e.g. Rybush et al. 1986a, 1986b).

Although many aspects of adult cognition may be relevant to this study the author needed to begin with data extracted from the students themselves rather than fit data to established theories. Despite many criticisms of Piaget's theory, three aspects are particularly relevant to this study. Piaget assumed that student reasoning could be studied through student explanation and perception, he assumed that the learner actively constructed knowledge and he supported the idea that mechanical processing of concepts needed to be mastered before abstract ideas were understood.

In the area of metacognitive research where the focus was on university students' self-perceptions, the original investigator on the study of student approaches to learning was Marton (1981, 1989), especially in his earlier work (Marton and Saljo 1976a, 1976b). Other major research included Bain (1994), Biggs (1987a, 1987b, 1993, 1994), Biggs and Collis (1982), Biggs and Telfer (1987), Entwistle (1981, 1987), Entwistle and Entwistle (1992), Entwistle and Marton (1994), Entwistle and Ramsden (1983), Entwistle and Waterston (1988), Fransson (1977), Laurillard (1979, 1993), Ramsden (1984, 1992), Ramsden and Entwistle (1981), Ramsden et al. (1987, 1989), Saljo (1981, 1987), and Svensson (1977).

These researchers maintained that the way the student solved and reasoned through a problem was directly linked to how the students intended to learn concepts. Marton and his Swedish colleagues employed qualitative research to explore the way students approached learning. They tape recorded and transcribed student responses to questions. Their study used academic articles especially selected for their tight logical argument and lack of technical knowledge. Marton called his methodology *phenomenography*, the study of phenomena through student perceptions (Marton 1981). He also identified four different types of responses ranging from a summary of the main argument supplemented with personal understanding, to a few isolated points with confusion or misunderstanding (Saljo 1981).

From his work, Marton identified two main independent approaches to learning which he labelled as *deep* and *surface*. In the *deep* approach the

student intended to understand the meaning of the passage while in the *surface* approach the student intended to rote memorise parts of the passage. Marton found that students who adopted the *deep* approach had better detail recall after five weeks. The implication here was that the *deep* approach lead to deeper understanding. However, work by Svensson (1977) found that a *deep* approach did not necessarily lead to a deep level of understanding if prior knowledge was inadequate. Svensson also found that it was not possible for the *surface* approach to lead to *deep* understanding. Students using a rote learning technique often found this process so tedious and unrewarding that they eventually did less and less work. He noted that those students adopting the *surface* approach often ended the year by failing the examinations.

In the United Kingdom, another group of researchers, Ramsden, Entwistle and colleagues (Ramsden 1984, Ramsden and Entwistle 1981, Entwistle and Ramsden 1983) used a quantitative methodological approach to study how students approached learning. Their conclusions were similar to those of the Swedish researchers. Later, these British researchers extended Marton's work by using a combination of qualitative interviews and quantitative research involving inventories and questionnaires. They took several years to develop inventories based on earlier work by Entwistle (Entwistle and Wilson, 1970). Initially the inventories were used to predict levels of academic performance, to explore motivation and to contrast ways students approached studying. Later these inventories were used in a wide variety of other empirical studies (Entwistle 1981, Entwistle and Ramsden 1983, Watkins 1988, Watkins & Hattie 1981). For example, by sampling 2208 students across 66 academic departments, Entwistle and Ramsden (1983) confirmed Marton's work on *deep* and *surface* approaches

to learning using inventories but added a third factor that they called *achieving*. This third factor described the learning approach aimed at achieving a primary goal, such as a university degree. The researchers recorded a clear link between approaches to study, the level of understanding and outcome, both in experimental and natural environments. They also found evidence that while students used the *deep* approach in both science and humanities, the emphasis on detail and procedures in science could successfully encourage the *surface* approach via rote learning, while the emphasis on personal interpretation in the humanities encouraged *deep* understanding. They concluded that the discipline and method of teaching could influence the ways students tackled a task. The researchers also showed that students were capable of using a variety of approaches to learning in higher education. Different combinations of strategies suited different students who could use them successfully in one faculty but not so successfully in another, depending on the learning context. Although there have been attempts by researchers to add to this three-factor list, such as *disorganised surface* and *organised surface* approaches (Watkins 1988), the predominant empirical and qualitative evidence has favoured the *deep*, *surface* and *achieving* categories.

There have also been studies on the factors influencing the approach to learning. For example, it was easier to induce a *surface* approach to learning than a *deep* approach by adjusting the type of question asked (Marton and Saljo 1976a, 1976b). Students who took a *deep* approach easily adapted to surface-like questions while those who took a *surface* approach had great difficulty with *deep*-orientated questions. Likewise, students with anxiety, lack of interest or a perceived irrelevance of the topic were less likely to adopt a *deep* approach.

(Fransson 1977). On the other hand, high motivation and topic interest as perceived by the student, helped make the *deep* approach more likely to occur (Svensson 1977, Entwistle and Ramsden 1983).

In more recent research, Biggs (1987a) extended the three-factor *surface*, *deep* and *achieving* learning approach to a continual scale and included motive-strategy linkages. His model is summarised in Figure 1:

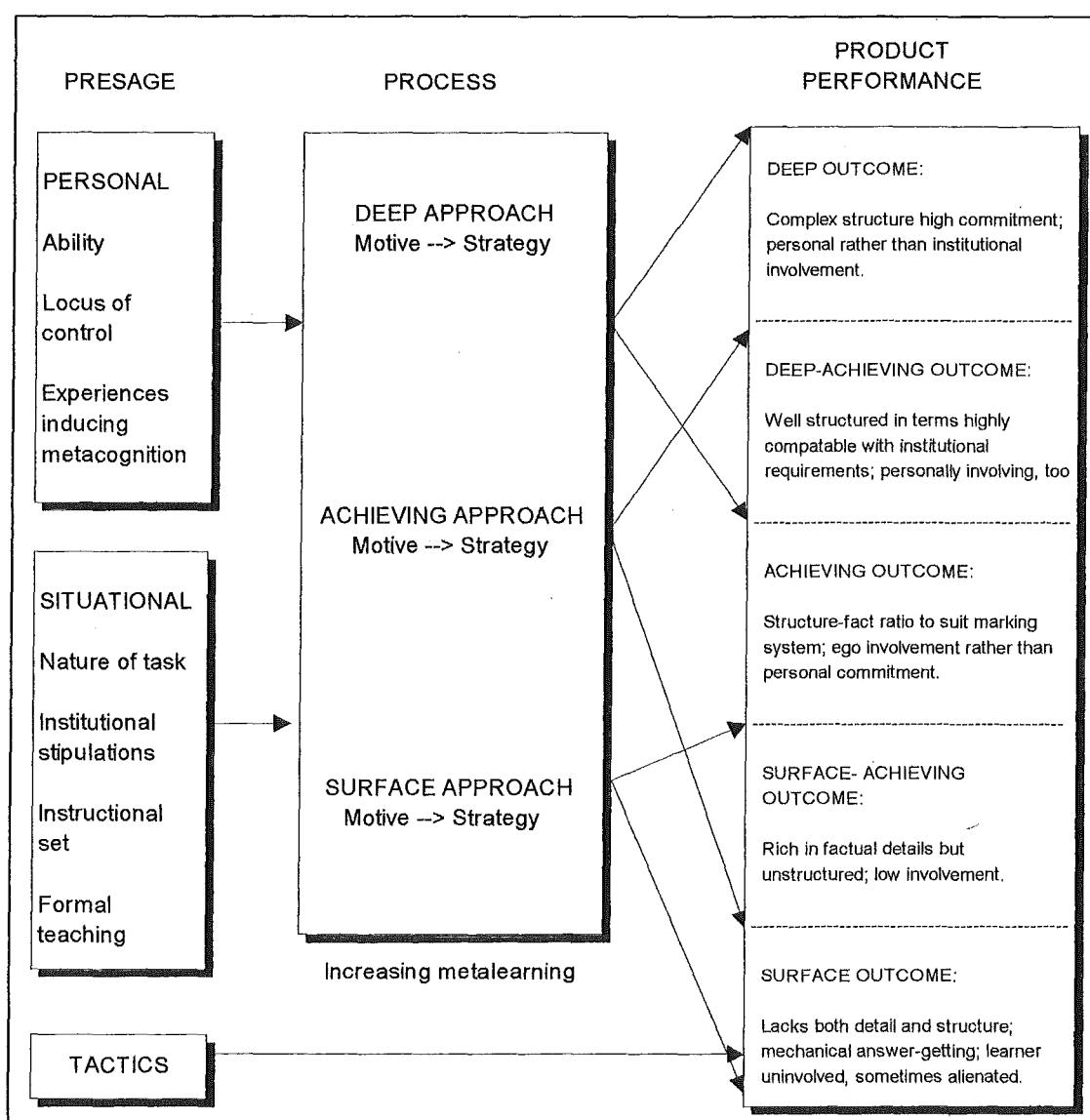


Figure 1: Elaborated Model of Student Learning. (Biggs 1987a, p. 96)

Biggs used factor analysis with questionnaire responses. In contrast to Marton's independent *surface* and *deep* approaches to learning, he placed *achieving*, *deep* and *surface* approaches on a continuous scale. The *deep* and *surface* approaches linked to how the student intended to deal with content, while the *achieving* approach referred to the way students organised the temporal and spatial contexts surrounding the task. That is, how the students went about structuring the *surface* or *deep* approach. It was therefore feasible to refer to learning approaches as *deep-achieving* or *surface-achieving*. In exploring the motivation to attend university, students could obtain qualifications with minimal understanding and effort (*surface*), they could actualise their interest (*deep*) or demonstrate their excellence publicly (*achieving*). Biggs concluded that students were capable of employing all of the motives to any extent. Students tended adopt the strategy most appropriate to their own motives and these strategies and motives could vary from subject to subject. Bigg's model has been confirmed by quantitative research (e.g. Andrews et al. 1994).

Biggs's model requires a little more explanation. In Figure 1, 'presage' variables were:

- 'personological' factors that predisposed individuals to select and to use effectively particular approaches to learning. (p. 94)

Biggs maintained that students favouring the *surface* approach were likely to be more influenced by factors such as style of lecturing, impersonal large classes, the type of task, assessment procedures and other situational factors. Those students favouring the *deep* approach were more likely to be influenced by personological factors such as information processing ability, a large pool of prior knowledge, maturity, high self-esteem and high motivation. Students favouring

the *achieving* approach were likely to be influenced more by personalogical than situational factors. Biggs acknowledged that his model was too simplistic compared to the complicated nature of student learning. For example, he noted that the relationship between *deep* and *achieving* approaches was stronger in preferred subjects. In addition, the *surface* approach could enhance performance while the *achieving* approach, especially *deep-achieving*, could lead to increasing complex performance and more positive student academic self-concepts.

More recent research on learning approaches in mathematics included work by Cox (1994). In the study, 17 to 27 first year mathematics students sat five multichoice one hour tests. Cox suggested that many students, possibly encouraged by content overload, adopted a strategic learning approach. That is, specific mechanical and routine topics were a priority when studying for examinations in preference to acquiring deeper knowledge. The students perceived these routine topics as a guarantee for the best examination results.

3. STUDENT UNDERSTANDING

Some recent research in Australia extended beyond the intended learning approach to the types of understanding the students actually attained. Bain's (1994) questionnaire study with third year psychology students concluded that there were three forms of understanding: *surface*, *reproductive* and *transformative*. *Surface* understanding referred to acquiring unstructured knowledge of basic terms, facts and ideas. This resulted from a *surface* approach to learning where students skimmed content and did not seek meaning. *Reproductive* understanding referred to knowledge structured according to

outside influences such as lectures and texts. It also referred to situations where students sought some meaning. Students with *transformative* understanding created their own structures that could reflect discipline understanding. The students that Marton and Biggs would describe as intending to use the *deep* or *deep-achieving* approaches to learning, Bain would describe as having acquired mainly *transformative* and *reproductive* understanding. The students may have used some form of repetitive learning or rote memorisation. Research evidence now points to a qualitative and quantitative difference in types of rote memorisation. In rote learning associated with *surface* learning the student has no intention to seek meaning. Repetitive learning, in association with *transformative* understanding could include retrieval, wise strategic choices for examinations and memorising to gain meaning (Biggs 1994). Therefore, in *reproductive* understanding a large amount of memorisation aided memory rather than understanding. In *transformative* understanding memorising was less frequent and provided support for concept learning (Bain 1994).

Why is mathematical problem solving often associated with memorisation? Recent research indicates that this may be the result of the way a problem is formed and presented (Bain 1994) or how the student perceived the task (Chi et al. 1989). A requirement for simple numerical answers would encourage *surface* and *reproductive* understanding. To encourage *transformative* understanding the student needs to reason qualitatively about what would happen under various conditions. Chi and her colleagues found that high achievers treated all tasks as problematic. These high achievers were aware of what they knew and did not know and could apply worked examples to other situations. Low achievers did

not perceive their task as a problem and were indiscriminate in applying worked examples to the problem task.

4. MATHEMATICAL UNDERSTANDING AND CRITICAL THINKING SKILLS

Can Biggs's *deep-surface-achieving* motivation-strategy model and Bain's three types of understanding be applied to mathematical thinking skills used in problem solving? Is it possible to distinguish successful mathematical thinkers from the not so successful in terms of level of concept understanding?

The main difficulty arises in defining what constitutes the thinking skills. In the general area of problem solving, especially in tertiary mathematics, the predominance of literature on *critical* thinking skills, *higher-order* reasoning or *abstract* thinking (e.g. Tall 1991, 1994, Sternberg and Baron 1985, Quellmalz 1985) does not seem to have a consensus on a definition. There is agreement, however, that such skills are used in problem solving. A successful problem solver often used skills of clarification, judgement and inference which researchers linked to cognitive strategies of analysis, comparison, inference and evaluation (Quellmalz 1985). For example, the strategies required to identify the central issues and appropriate knowledge in a problem involved skills of clarification and judgement using cognitive processes of analysis and comparison. The strategies of connecting and using the appropriate knowledge involved strategies of inference and skills of induction, deduction and judgement. Checking and judging the significance and soundness of the solution involved evaluation skills of deduction, judgement and prediction. Competence with these

skills could infer a *deep* approach to learning and an acquisition of *transformative* understanding.

Again, in the area of problem solving, Ennis (1976) outlined 13 dispositions of critical thinkers. These included open mindedness, and abilities to select suitable strategies or change strategies if appropriate, to take into account the whole situation, to seek to be well informed and to be flexible by considering alternative methods. Also included was the ability to seek and maintain a clear statement of the issue behind the problem. Therefore the able problem solver must have used a *deep* approach and attained *transformative* understanding about the concepts involved in the problem. Other dispositions such as orderly dealing with the problem and seeking as much precision as possible can reflect *reproductive* understanding rather than *transformative* understanding.

In the area of mathematical concept understanding, the main thrust of the literature has favoured duality or dichotomies of mathematical thinking (e.g. Skemp 1976, Heibert and Lefevre 1986, Kaplan 1987, Sfard 1991). For example, Kaplan (1987) concluded from his study on children aged between three and nine years that there were two types of mathematical thinkers that he called the *pro-mathematical* thinkers and the *anti-mathematical* thinkers. He found that *pro-math* thinkers persistently tackled a difficult challenge, took pleasure in describing solution strategies, automatically attempted to rework the problem if an error was spotted and exhibited focussed concentration. These students often did not need explicit instruction, they made connections between known concepts and new problems, and were capable of constructing new mathematical problems beyond

the task requirement. *Anti-math* thinkers gave up easily in the face of errors or challenges and used a *surface* approach to solving problems. That is, they came up with some perfunctory answer as a means of meeting some external obligation. The *anti-math* thinkers also spent a lot of energy avoiding tasks and waiting for instruction and judgement from others. Their mathematics remained at the counting level of thinking and the student often rigidly applied the same solution strategies that kept concepts as discrete and unrelated units. The *pro-math* thinkers could have acquired *transformative* understanding while the *anti-math* thinkers acquired *surface* understanding.

Other literature on mathematical thinking, knowledge and understanding refers to dichotomies. For example, Skemp (1976) distinguished relational understanding from instrumental understanding, and Hiebert and Lefevre (1986) distinguished conceptual from procedural knowledge. Sfard (1991) on the other hand, referred to a duality of operational and structural concept formation rather than a dichotomy and her proposals reflected some of Piaget's work. According to Sfard, mathematical concept learning progressed in a hierarchical spiral from computation to abstract understanding. Through a long and inherently difficult three step process the student first learned to become familiar with the processes that gave rise to a new concept (*interiorisation*). For example, a formula is applied to a problem situation. The next stage of concept development occurred when lengthy sequences of operations became condensed into more manageable units (*condensation*). That is, when selected operations are automatically connected to a particular problem situation. Sfard saw both these stages as being gradual and quantitative in nature. The *condensation* stage lasted as long as the new identity was applied to one particular process or

operation. In contrast, the third stage (*reification*) could occur instantly and was more qualitative. It was:

...a sudden ability to see something familiar in a totally new light.
(Sfard 1991, p. 19)

Abstract thinking occurred when the students saw concepts as a static structure or object, that is, they recognised the 'wholeness' of a concept. The third stage was the most difficult for students as they struggled with abstract ideas. The formation of a static cognitive *structure* meant the student seeing the concept in abstract form. Sfard maintained that high achievers possessed both operational processes and structural entities. In complex problem solving, these students would not only repeatedly switch between operational and structural approaches, but an optional skill they could also easily employ was visualisation.

There appear to be parallels between Sfard's three levels of concept development and Bain's three types of understanding. *Surface* understanding occurred when knowledge was superficially *internalised* as in the example of recalling information rote learned without understanding. Students' efforts to repeat operations (*reproductive* understanding) until they became familiar with the concept *condensed* knowledge so that it could be applied to particular situations. At the level of deep understanding knowledge was *transformed* into a complex, static *structure*.

This perspective on understanding explains the abundance of research on 'cognitive gaps', defined as the students' inability to operate spontaneously with or on the unknown (Herscovics and Linchevski 1994). For example, cognitive

gaps existed in the transition from arithmetic to algebra or in the translating of functions from tabular form into graphs and equations (Herscovics and Linchevski 1994). Cognitive gaps also existed in translating word problems into algebra (MacGregor and Stacy 1993).

At the university level, attempts to define the ideal characteristics and thinking skills of students who were mathematically well prepared for university work often highlight the institution's expectation that students already use a *deep* approach to learning and are capable of achieving *transformative* (or *structural*) understanding. For example, a preliminary statement released in 1989 (California Community Colleges 1989) outlined several achievements that the first year students should have gained from previous mathematics (secondary) courses.

- A sense of number and the ability to discern whether a proposed numerical answer to a problem is reasonable.
- The ability to use mathematical knowledge for unfamiliar problems (both in concrete and abstract situations) and an awareness of the analogy between mathematical structures and phenomena in the real world.
- The ability to discuss mathematical ideas in problems with other students and write coherently about mathematical topics and their interrelationships.
- Both informal and analytic reasoning abilities where mathematics is seen as an interplay between intuition and reason.
- General algebraic proficiency in manipulating algebraic expressions, the ability to check these manipulations and to have a feel for what manipulation is necessary to convert a complex algebraic expression to one that is manageable.

Before students could apply mathematical knowledge to unfamiliar abstract situations and before students had the ability to discuss mathematics ideas and write or discuss interrelationships within and between concepts, they must have already assimilated and linked both their prior and new knowledge to a depth beyond that achieved by a *surface* or *reproductive* understanding. At the secondary school level, it was possible for students to perceive success using *surface* strategies and *surface* or *reproductive* understanding. However studies have shown that at the tertiary level, *surface-achieving* and *deep-achieving* approaches were perceived by students as being successful (Cox 1994).

5. INTUITION

A sense of number, the ability to discern whether an answer is reasonable or realistic, and a 'feel for what manipulation is necessary' requires that the student achieve a knowledge-based *intuitive* feel for the problem. This *intuitive* knowledge must surely come from *transformative* understanding of many mathematical concepts assimilated and transformed as a structured object. In a cognitive theory on post-formal thought and adult cognition, Rybush et al. (1986a) mentioned this unconscious but knowledge-based *intuitive* approach:

..although expert problem solving seems firmly rooted in highly developed forms of declarative and procedural knowledge, these knowledge bases are not consciously brought to bear during the problem-solving process. Furthermore, experts may not necessarily be able to describe how they arrive at a decision... It seems that expertise within any domain gives rise to a set of intuitions that are derived from the automatic, unconscious processing of relevant information. (p. 148)

Again, a recent study by Entwistle and Marton (1994) using semi-structured questions to interview 11 students and explore their revision strategies for examination, found a recurring experience that they labelled *knowledge objects*.

A feeling that the material being revised had become so tightly integrated that it was experienced as an entity with form and structure. Only some aspects of these entities could be visualised but additional associated knowledge was readily available when needed. It was this recurring experience among the students which we came to describe as a 'knowledge object'. (p 166)

For these students there was a feeling of coherence and connection, a feeling of experienced wholeness, a quasi-nature perception said to be like *seeing* or knowing what to do without really thinking about it consciously. The triggers for this *knowledge object* could be anything from a visual image to related things not the focus of attention at the time. For this study the spontaneous *seeing* of what to do without conscious thought is termed *intuitive knowledge*. Such a quasi-perception would be dependent on tightly integrated and interconnected prior knowledge

6. THE RESEARCH QUESTION

In solving mathematical problems during the secondary to university transition phase, students bring prior knowledge and established learning approaches and strategies from the secondary to tertiary level. Are the lack of these skills and approaches at the root of mathematical difficulties in the first year at university? Are there any particular skills that we, as teachers, should encourage the students to develop so that they can better cope with the transition to tertiary mathematics?

CHAPTER II

II. METHODOLOGICAL PERSPECTIVE AND CHOICE OF METHOD

1. THEORETICAL PERSPECTIVE

This study seemed to be more suited to a qualitative than a quantitative approach since the research question itself dictated a qualitative answer. Two similar methodologies became the greatest influences in the investigation. These were *phenomenography* (Marton 1981) and one of the most rigorously structured qualitative methodologies proposed in the last few decades, *grounded theory* (Glaser and Strauss 1968, Chenitz and Swanson 1986, Strauss 1987, Strauss and Corbin 1990).

Phenomenography and *grounded theory* were similar in their approach to research. *Phenomenography* was a qualitative methodological approach used by Ference Marton in the 1970's, while *grounded theory* was an approach proposed in the mid-1960's by two sociologists, Barney Glaser and Anselm Strauss. While Marton's approach stressed the emergence of phenomena from a metacognitive perspective, Glaser and Strauss' approach seemed to come from a countermovement against the dominant empirical hypothesis-testing philosophy of the time. Their approach emphasised theory generation rather than the verification of existing 'great-man' theories.

Assumptions underlying both *grounded theory* and *phenomenography* included acknowledgment of a complex social world and the understanding that this world emerged from the unfolding data. Both approaches called for indepth description of categories (coding) with multi-linking between the categories. Comparisons of similarities and differences within and between categories aimed to create an ultimate 'core' category that gave an overview of the situations or phenomena under study.

The difference between the two methodological styles lay in the rigidity of data collection and analysis. *Phenomenography* involved straightforward collection and categorising of data from a metacognitive perspective while *grounded theory* used 'theoretical sampling' whereby the emerging data determined where the next data would be collected. While *grounded theory* involved a more exhaustive spiral process of data collection and analysis, its disadvantage over Marton's approach lay in the time and effort taken to achieve adequate saturation of the phenomena. Despite the danger of compromising reliability in the collection of subjective data, these methodologies appealed to the author. Both dealt with data qualitatively and both approaches had the potential to complement other forms of research.

Investigation into university student learning and reasoning in mathematics should have appropriate methodologies that delve more deeply into how students learn abstract thinking. Richardson (1987) pointed out that considerable danger could exist in using established theories based on low level thinking to explain higher level (critical and abstract) thinking. Some of these dangers could be in

the oversimplification of non-linear and complex situations. Therefore both methodological approaches were particularly suitable for situations where there was inadequate theory. Even when alternative theories existed both *phenomenography* and *grounded theory* emphasised avoidance of fitting data to established theories. In basing this study on *phenomenography* and *grounded theory* the researcher could concentrate on describing the perception of an individual's experience rather than describing the observed phenomena. This was especially pertinent for this study as university students not only express strong opinions but they also have the potential to be both vocal and astute in their perceptions.

Finally, a qualitative approach based on *phenomenography* or *grounded theory* can point to future research, both qualitative and quantitative. The piecing together of students' perceptions and experiences can give insight into difficulties students experienced in mathematics and can point to further investigations into how to alleviate these difficulties.

2. CHOICE OF METHOD

The author was not only interested in what strategies the students used to solve the two given tasks but also in the reasoning behind the approaches used. This type of qualitative data needed to come from the students' perspectives, especially through indepth interviews immediately after completion of each task. The selection of students and the number of case studies was restrained by the time available for the study.

Diagnostic Interviews

Written answers could highlight 'what' approaches students used. However, semi-structured diagnostic interviews were considered by the researcher to be the most appropriate medium for obtaining information on 'why' students used a particular approach to the problem tasks.

A diagnostic interview was considered by the researcher to be appropriate for this study for two reasons. First this type of interview was especially suited to the diagnosis of written tasks and second, the interviewer's influence was kept to a minimum. In a diagnostic interview the interviewer becomes an observer who guides the interviewee with general open-ended questions such as "how did you do the question?" or "why did you do the question this way?" Semi-structured interviews allow the interviewer to keep apart from the procedures and permit the students to explain their decisions and perceptions with minimal interviewer influence.

There were several other important advantages of diagnostic interviews relevant to this study. Not only can detailed explanations be obtained in these interviews but they have the advantage of allowing the researcher to assess students' feelings about topics through observing behaviour and facial expressions. Another advantage is that it is possible to alter interview questions to suit the specific needs of the student. That is, to probe deeper into areas that become worth pursuing. (Liedtke 1988). However, as Freud pointed out, an inherent danger in individuals making statements about themselves is the dominance of unconscious thoughts and feelings. What people say may be what

they perceive under the influence of factors that are not obvious to the observer.

For example, students selected for this study were likely to be close to Perry's stage of dualism and this could influence their perspective and decisions. The influence of this factor could be minimised by selecting students of similar age.

For this study, written solutions gave factual information about strategies. The interviews were used to check the reliability of student knowledge, to probe deeper into their skills in the selected topic and to gain information about unconscious or conscious feelings.

3. VALIDITY AND RELIABILITY

Internal validity was reinforced by devising two tasks that covered aspects of the secondary Seventh Form syllabus and first year university mathematics syllabus. Theoretically the tasks were familiar to participants. Students had sat the appropriate Bursary (a university matriculation and Seventh Form final examination) Mathematics with Calculus paper the previous year and were currently enrolled in first year mathematics courses, MATH 104 and MATH 106.

Triangulation reinforces both external validity and data reliability (Denzin 1978). With triangulation, the data are approached from many different sources. In this study, data were triangulated by using two independent groups of participants, by comparing new and familiar concepts and by using two different topics, one based on calculus and the other on linear algebra (in particular, matrix

methods). Data triangulation could be improved in further studies by selecting more groups and a wider variety of mathematical concepts.

Every method has its strengths and weaknesses. The aim of methodological triangulation was to harness the strengths and reduce the weaknesses. For this study the author used several methods to obtain the same data. Students gave written answers to tasks, they were interviewed and they completed a questionnaire. Student behaviour was observed and recorded at each session.

4. THE INSTITUTION AND SYLLABUS

Standard mathematics courses offered to first year students at the University of Canterbury covered calculus and linear algebra. Although other mathematics and statistics courses were offered at the first year level, enrolment in one of three basic mathematics courses was largely determined by a student's Seventh Form Mathematics with Calculus (Bursary) examination mark. Students who gained above 73% enrolled in MATH 104, those who gained between 50% and 72% enrolled in MATH 105, and the remaining students enrolled in MATH 106. MATH 104 and MATH 105 were 12 point courses while MATH 106 was a six point course. In 1994, approximately 130 students enrolled in MATH 104, 530 in MATH 105 and 330 in MATH 106.

The same core material existed for each of the three courses, and although there was some overlap with Seventh Form work the emphasis on content

differed. The focus in MATH 104 was towards more proof and abstract concepts while MATH 105 and MATH 106 emphasised practical applications. MATH 106 covered half the MATH 105 syllabus but at a slower pace. Students electing to take MATH 106 could reach the equivalent standard and content coverage of MATH 105 by enrolling in MATH 107 in their second year.

The calculus in each of the courses included differentiation, special functions, differential equations, functions of several variables, fundamental theorems of calculus and integration. The linear algebra syllabus covered linear equations and matrix algebra. In addition, MATH 104 and MATH 105 included vector geometry, vector spaces, determinants and complex polynomials. Requirements for MATH 104, MATH 105 and MATH 106 included formal lectures and informal tutorials on a weekly (or two-weekly for MATH 106) basis throughout the academic year. Lectures involved large class streams of up to 250 students and tutorials of ten to 15 students per tutor.

CHAPTER III

III. SELECTION OF PARTICIPANTS AND TASKS

1. SELECTION OF PARTICIPANTS

Students selected for this study met certain criteria. All potential participants completed at least four years secondary mathematics in a New Zealand school. At the time of the study a number of foreign students came to New Zealand for a Seventh Form year prior to entry into a New Zealand university. The aim of this criterion was to eliminate selection of these students to avoid dealing with extra factors such as potential language difficulties.

Another criterion for participation included the sitting of the Seventh Form Bursary *Mathematics with Calculus* examination paper in 1993. In setting this condition the author assumed that all students would be able to attempt tasks linked to the Seventh Form level. This also meant that students who had sat the Bursary examination prior to 1993 or had been absent from mathematics for a considerable number of years were automatically eliminated from participation in the study. This avoided dealing with extra factors such as 'rusty' knowledge and a possible mathematical cognitive gap caused by the delay between secondary school attendance and university enrolment. The author also wanted to avoid factors such as demographic, motivational and perceptual differences between mature and traditional students (e.g. Iovacchini et al. 1983). Students must also have enrolled in either MATH 104 or MATH 106 courses in 1994. The aim of this

criterion was to eliminate students not exposed to the level of mathematics required for the tasks.

Initially, the author contacted 27 students who obtained between 85% and 89% in the appropriate Bursary examination paper, and a further 32 students who obtained between 45% and 49% in the same examination. This selection formed the basis of two groups labelled *high-entry* and *low-entry*. This meant that students selected for the *high-entry* group had enrolled in MATH 104, while those selected for the *low-entry* group had enrolled in MATH 106.

For this study, initial contact to participants was by a letter handed out at an allocated tutorial time. In the letter, the author outlined the reason for the research and asked for volunteers to participate in the study. A further request for volunteers was made in lectures. Of the students in the *low-entry* group who obtained between 45% and 49% in the Bursary mathematics examination the previous year, over half could not be contacted since they did not attend tutorials. Seven of the students who did receive a letter eventually agreed to participate in the study. By extending one end of the range of the *low-entry* group to a 40% Bursary mark a further five students were added to the group. This resulted in a total of 12 students in the *low-entry* group.

Of the 27 original students who obtained marks within the 85% to 89% range, ten agreed to take part in the study. A further seven declined to participate and the remaining ten students had their letter withdrawn. The author discovered that these ten students were granted direct entry to second year

mathematics level in the Engineering School. They were not included in the study as the engineering mathematics syllabus differed from the topics selected for the set tasks. In addition, one student who initially agreed to participate in the study withdrew before the interviews commenced leaving a total of nine students in the *high-entry* group.

The emphasis in selecting students for this study was on voluntary participation rather than random selection. The author acknowledged the likelihood of bias. For example, many of the students within the 40% to 49% range declined to participate. Comments from some of these students indicated anxiety and insecurity with mathematics. The author further acknowledged that selection of students in the research could have gender or cultural bias. The *high-entry* group included four females and five males while the *low-entry* group contained four females and eight males. For the *low entry* group the gender ratio was close to the class ratio. Although all participants were New Zealand citizens, cultural background was not considered at any stage.

This study was subject to the University of Canterbury Human Ethics Committee approval and all participants gave their written consent before taking part in the research. Prior to obtaining written consents participants were informed of both the purpose of the study and confidentiality of data.

2. CONDUCT OF INTERVIEWS

Participants took part in two one-hour interviews two months after the university year began in March. All interviews were completed by mid-year break (June 1994).

Each session was conducted in a quiet room to avoid distraction or disturbance. Students understood that an explanation of their attempt was more important than a correct answer. Prior to the study the author estimated that each task should take from 20 to 40 minutes to complete for a hypothetical 'average' student. Time pressure was eliminated by limiting the size of the task and informing students that they had 'unlimited' time to perform each set task.

Taped interviews immediately followed the written tasks. In the interviews the students orally explained their solutions. The researcher used open ended questions to extract information on how the students reasoned through each task and why they used a particular style of approach. In addition, the author encouraged the students to talk about their mathematical background. The delay between first and second sessions ranged from two to four weeks and appointments were negotiated with individual students.

3. QUESTIONNAIRE

The first interview included questions such as "how would you describe your Seventh Form year in your school?". At the second session each participant

brought a completed questionnaire (Appendix I) which complemented demographic and motivational data obtained during the first interview. In the questionnaire students were asked to state the type of secondary school they attended, whether co-educational or single-gendered, private or public, large or small, city or rural. This question determined the background environment in which the student learned their secondary school mathematics.

The format of the interviews allowed for data to be gathered on extrinsic motivation. Students gave information about other courses taken concurrently with mathematics in 1994, the area in which the students intended to obtain their degrees and how much effort they put into mathematics relative to their other courses. The students also described their perception of their own ability in mathematics, the aspects of mathematics they found difficult, and what they felt were their strongest areas in mathematics. In terms of motivation it was also important for the author to determine whether students felt they had been encouraged or discouraged at any time in their study of mathematics.

4. THE TASKS

The curriculum base for the research involved two tasks. The first task was based on precalculus concepts and the second on basic linear algebra, in particular, matrix methods.

The students taking part in the study sat the same Bursary mathematics examination the previous year and it was assumed that all students covered a

similar syllabus during their secondary school years. The secondary syllabus weighted heavily in favour of precalculus concepts and some of the more advanced concepts from the secondary school syllabus formed the basis of the first task. The aim of the first task was to explore the knowledge and mathematical skills students brought with them to university.

The second task was centred on basic matrix methods covered in the university first term syllabus. Although solving a system of simultaneous equations was in the secondary syllabus, the approach of using matrix methods was new for most of the students. This topic provided a good opportunity to study how students coped with learning and organising new knowledge as they adapted to the university environment. Questions that highlighted reasoning skills proved difficult to formulate in the second task since matrix methods in the first weeks of university were more 'recipe' orientated and determinants were not included in the MATH 106 syllabus

Overall, elements in both the calculus and linear algebra tasks reflected either a recipe recall, interpretation of answers, or a situation where there could be several alternative ways to obtain solutions to questions that required the use of problem solving skills.

(1) Task 1: Calculus

The first question came from a 1993 MATH 106 test and was devised by the course lecturer, Dr David Robinson. This task was unfamiliar to the students prior to the study.

TASK 1

Let $f(x) = x^4 + 4x^3 - 44x^2$

- (a) Find the values of x for which $f'(x) = 0$.
(Leave as exact expressions.)
- (b) Find the tangent to the curve $y = f(x)$ at $x = 0$.
- (c) Describe how you would prove that the line $y = 96x - 576$ is tangent to the curve $y = f(x)$ at both $x = -6$ and $x = 4$.
Prove that the line is tangent at both values of x .

Part (a) was more than a recall of differentiation, that is,

$$f'(x) = nx^{n-1}$$

Students needed to find the value(s) of x for which this first derivative had a value of zero. Calculation involved removing a common factor and calculating the solution of a quadratic equation. To answer part (b) students needed to understand the meaning of a 'tangent' to a curve. Not only were students required to differentiate the function $f(x)$ and find the slope at $x = 0$, but they also needed to recall how to find y given x , and the equation of a line given a

point and slope. Even if students did not experience difficulty retrieving and organising known concepts, the solution of a horizontal slope at the origin was expected to cause hesitation and an opportunity for students to check their answers using alternative techniques. Part (c) contained the main thrust of the task and concentrated on reasoning skills. Students needed to show how they would reason through a question when the solution was given. Polya (1957) suggested that part of successful problem solving was the inclusion of a 'planning' stage. Therefore students were asked to outline how they would approach the task before beginning the calculation. Any difficulty in advance planning was expected to become clear in the interviews.

(2) Task 2: Linear Algebra

Task 2 was designed for the students by the author and included basic concepts from the MATH 106 syllabus. By the time the students attempted this second task, the author expected them to be familiar with matrix methods. One difficulty in devising this task was that the matrix methods did not include the determinant in the MATH 106 syllabus and eventually the focus of this task became a study of knowledge extension and how students coped with generalising to unfamiliar situations. Most of the concepts in Task 2 were basic enough for MATH 106 students and were similar to the work covered in MATH 104.

TASK 2

- (a) Solve the linear system:

$$u + v + w = 0$$

$$u + 2v + 3w = 0$$

$$3u + 5v + 7w = 1$$

What was the question asking you to do? Interpret your answer by describing the type of solution you found.

- (b) For which values of k do the two lines:

$$kx + y = 1$$

$$x + y = 1$$

have no solution, one solution, or infinitely many solutions?

- (c) Matrix A is invertible if there is an inverse (A^{-1}) such that

$$A A^{-1} = A^{-1} A = I.$$

If $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

- (i) Find A^{-1} and show that $A A^{-1} = A^{-1} A = I$.

- (ii) A^2 is $A A$ (or A times A). Find A^{-3} .

- (d) Given that $(A B)^{-1} = B^{-1} A^{-1}$ where A and B are any two $n \times n$ matrices, show that the inverse of

$$A B^{-1} C$$
 is $C^{-1} B A^{-1}$.

Part (a) could be solved by a recall of ‘recipe’ knowledge with row reduction of coefficients for a system of linear operations. Alternatively, students could recall techniques from their secondary school to solve the system using simultaneous equations by elimination or substitution. The answer to this section of the task was a “no solution” situation and it was how the students interpreted their solution and dealt with the situation that was important.

In part (b) the secondary school syllabus had already equipped students with the various tools needed to tackle this part of the task. In the Fourth Form secondary school students encountered the formula for an equation of a line in the form $y = mx + c$ where m was the slope of the line and c the point of the y intercept. In this task, the variable $(-)k$ represented the slope of a line. Since the y intercept was the same for both lines then the lines always intersected. A "no solution" scenario was not possible, and when $k = 1$ an infinite solution occurred as the two lines became equivalent. The variety of tools available to solve this task included a geometric approach (as just outlined), row reduction, substitution, or guesswork. To be able to describe the solutions fully, students needed to have some understanding of the types of possible solutions.

Part (c) (i) was a straightforward calculation of a two by two inverse matrix. However, since the matrix had a determinant value of one, students could easily have recalled a formula incorrectly from their secondary school education and yet achieved a correct result. Part (c) (ii) was an extension of a simple matrix multiplication concept. The author did not expect participants to have seen this question prior to the study and wanted to determine how students justified their answer.

In part (d) students were asked to generalise from a more abstract situation where both the question and answer were given. Although the researcher expected a 'substitution by example' approach from the *low-entry* group and a 'manipulation of variables' approach from the *high-entry* group, it was important to establish how students interpreted the question and justified their answer.

CHAPTER IV

IV. THE STUDENTS

Students taking part in the study were similar in age. The 12 *low-entry* and nine *high-entry* students were 18 years of age except for three *low-entry* students (two males and one female) aged 19 years, one *low-entry* female student aged 17 years, and one *high-entry* female student also aged 17 years. In 1993, five *high-entry* students enrolled in coeducational secondary schools and four *high-entry* students attended single-sexed schools. Eight *low-entry* students attended coeducational secondary schools and four *low-entry* students enrolled in private schools.

A component of metacognitive knowledge is *self-knowledge*, that is, a realistic assessment of one's own knowledge and capabilities (Biggs 1987b). The interviews and questionnaires formed the basis of data on student self-assessment of mathematical ability.

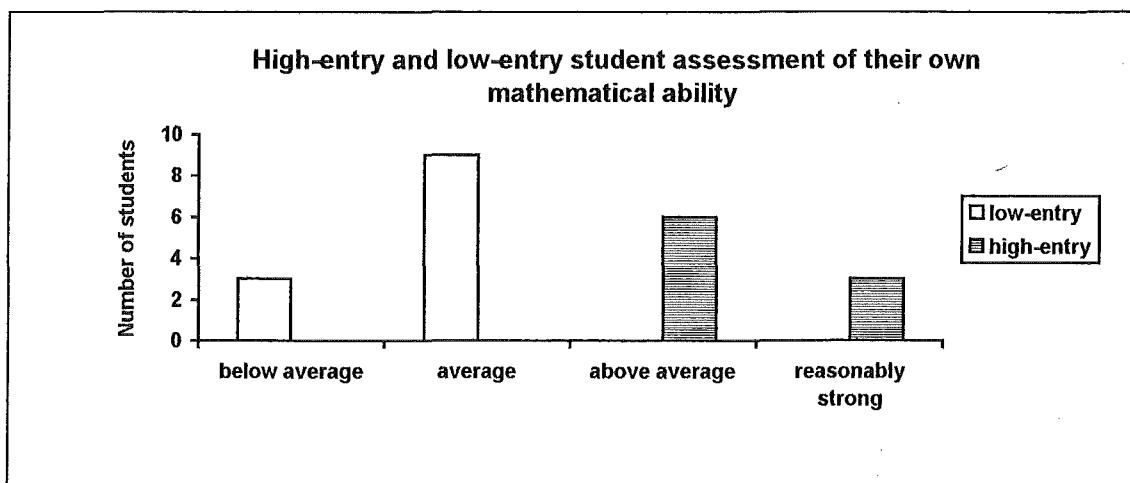


Figure 2. Frequency of self-assessment of mathematical ability by high-entry and low-entry students.

Figure 2 depicts student self-assessment. In the *high-entry* group six students assessed their mathematical ability as above average, two commented that they were reasonably strong in mathematics, and one stated that she had always been very good in secondary school mathematics and couldn't understand why others had difficulty. Nine of the *low-entry* students assessed their ability as average and these students added comments pointing to major difficulties in some areas, especially with calculus and word problems. Three students wrote that they used to be above average up to the Sixth and Seventh Form but now assessed their mathematics ability as below average.

1. LOW-ENTRY STUDENTS

Finding difficulty with Seventh (and Sixth) Form mathematics was mentioned by every *low-entry* student at some stage during the interview. Some comments indicated a predominance of more *surface-orientated* learning approaches based on reproductive understanding being successfully used in the first three to four years of secondary school. These same approaches then appeared to be inadequate at the Sixth and Seventh Form level. Typical comments included:

- The Third and Fourth Form was quite simple stuff as you didn't have to work. Because I didn't work, I missed out on the basic stuff. In the Fifth form it was all quite simple and it became learning equations and doing it because it was there and you didn't learn why you were doing it. I just couldn't do it in the Seventh Form.
- (In the Seventh Form) there is too much stuff to know. There are too many rules.

Other comments reflected a general dissatisfaction and negative attitude towards mathematics associated with the Seventh Form year at secondary school:

- Calculus used to be my best subject until the Seventh Form.
- The Seventh Form year felt like ten years.

Most *low-entry* students commented that they would have been better off without that Seventh Form year. Three low-entry students blamed themselves for being lazy and not working adequately in secondary mathematics, three blamed their teachers and four mentioned the high level of difficulty in the Seventh Form Bursary mathematics examination paper.

In the questionnaire students were asked for a self assessment on the past encouragement or discouragement experienced with mathematics. Nine students did not reply and three wrote that they were discouraged by the secondary teaching environment.

- We had three different teachers in mathematics last year.
- The teacher was retiring and had lost all interest. He expected everyone to know things already.

Although the *low-entry* students had not passed the Seventh Form Bursary mathematics examination their motives to enrol in first year mathematics courses fell into three categories (depicted in Figure 3). Enrolment in university mathematics courses was either as a prerequisite for engineering or economics, or to complement an intended major field, or because the students felt they could do better at university than they did in the Seven Form. This third group of five students were encouraged by peers to enrol in first year university mathematics as it was considered easier than the Seventh Form mathematics. An added incentive for enrolment was the availability of the MATH 106 paper which catered for students who did not pass the Seventh Form Bursary mathematics examination.

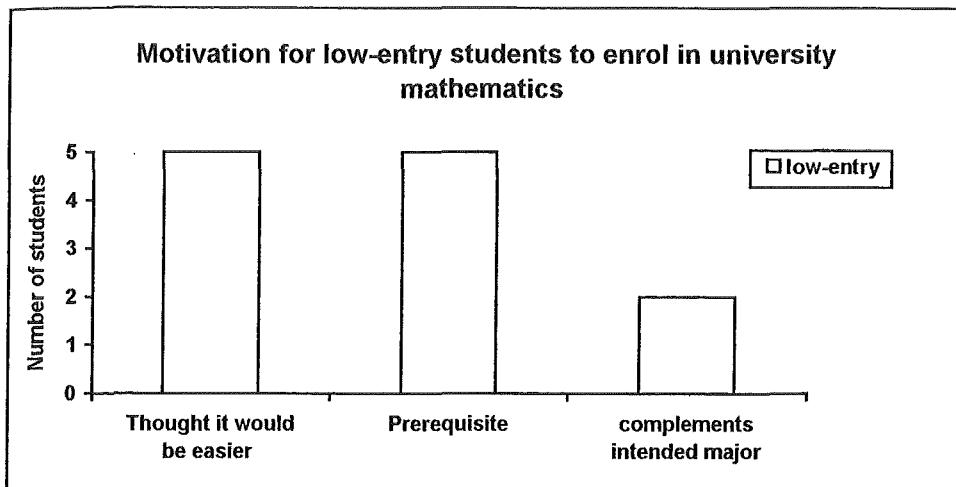


Figure 3: Frequency of self-assessed motivation for low-entry students who enrol in university mathematics.

By the end of the first term at university all *low-entry* students felt they were doing better with university mathematics than they did at school. However, their comments also pointed to a variety of difficulties centred on adjustment to university lecturing rather than mathematical content. In particular, they commented on the high speed of lecture presentation, lack of repetition in lectures and insecurity with asking for help in large classes.

- Everything seems a bit rushed at times and we can't ask questions.
- At school there were smaller classes. You put your hand up and ask for help. We can't do this in front of a lot of people (at university). Everyone would think we are stupid if we don't understand something. ... There is a lack of repetition of work. We have difficulty with what the lecturer says... he just lost us.
- The lecturers keep saying the work is easy when it is difficult.

All *low-entry* students stated that they often became lost in lectures but felt more comfortable with topics that were more 'recipe' orientated. Several cited row reduction of matrices as easier to 'do'.

Effort And Motivation

The effort students put into study at university was analysed from the interviews. Two students said they had tried to understand the topics this year and had invested a considerable amount of extra effort to do so. This effort involved revising lecture notes and using the textbook to do plenty of examples. Another student admitted he was not putting in any effort as he found the work

- ..boring as it is the same ideas as last year but different methods...It is mainly a revision of school mathematics.

The remaining nine students in the *low-entry* group found university mathematics very different from school. Mathematics appeared to be a low priority for them. It was noted that these nine students admitted to using minimal effort to understand topics they could not follow in lectures. All nine stated that any effort they put into mathematics was only for tutorial preparation, assignments or tests. The most common work technique was to rely on repetition of similar examples selected from the textbook and to ignore lecture notes. However, this system also seemed to be time-consuming.

- I do lots of examples. It takes a long time.
- I just look for something similar in the text book and do that. I usually adjust it to fit the assignment.
- I struggle to keep up with the assignments. I just need to find the time.
- I can't do much work in maths because of other tests and assignments.

2. HIGH-ENTRY STUDENTS

There was a distinct contrast in motivation and effort between the *low-entry* and *high-entry* students. The nine *high-entry* students were either the highest

scoring students in their secondary mathematics classes or near the top of an accelerated class. The students found secondary mathematics easy but repetitious.

- There wasn't much work done in school as I could understand it all.
- I thought it was boring. Forms Three, Four and Five repeated the same things.
- I always seemed to 'click' with maths at school.

Usually these *high-entry* students remained in the highest extended class with their peers throughout their secondary school. However, only one *high-entry* male student in the top class at his secondary school skipped from the Fourth to Sixth Form and later completed two years in the Seventh Form. He commented that his first attempt at Seventh Form work was very difficult for him, but with the second attempt he found the work easier.

The *high-entry* students generally had a positive attitude to mathematics. Seven of these students said they enjoyed mathematics. However, two female students commented:

- I don't know if I enjoy it. I didn't think I was any good. I kept doing real well at maths. It's just following instructions.
- I didn't like it (mathematics), but I just kept getting good marks.

Seven *high-entry* students cited strong encouragement by either parents, teachers or older students. Three of these students were also encouraged by their teachers to enter mathematics competitions and olympiads. The remaining two male *high-entry* students said they did not get any encouragement as they:

- .. didn't need much encouragement.

All nine students in the *high-entry* group intended to enrol for a double major at university. Eight of them wanted one major to be in mathematics and the other in engineering, economics or English. The student who chose not to major in mathematics still proposed to sit further mathematics courses even though her intended majors were in Chinese and law. Her incentive to continue in mathematics was based on advice from older students that employers were impressed by mathematics graduates.

It was previously noted that all students in the *high-entry* group were enrolled in the top first level mathematics paper at university, MATH 104. In this course, the emphasis was on abstract concepts and proofs. The transition difficulties experienced by these students reflected their concern with these proofs and abstract concepts. Transition factors of concern to students in the *low-entry* group, such as speed of lectures and insecurity in asking questions in large classes, were not considered as important by the students in the *high-entry* group.

- The topics are harder. There is more thinking needed.
- The lecturer uses a lot of long words and assumes you know what they mean.
- The concepts are more difficult... Although there is no chance to understand all the work, you can grasp bits.

Effort And Motivation

The *high-entry* students were asked what they would do if they did not understand topics covered in lectures. All nine students replied that they would spend a lot of time using the textbooks and reading through lecture notes.

- I can usually work most things out. If not, I use the library, notes and textbook. I usually work back from the answers.
- I would try to figure it out myself first. If not, I ask someone doing the course, and lastly the tutor.
- I prefer to work it out myself almost immediately. If not, you've forgotten what you didn't understand by the time you have the tutorial.

Therefore the *high-entry* students not only intended to actively understand concepts by transforming their knowledge, but they had more interest and motivation to sort out any difficulties with the mathematics content. It must be noted though that the volume of knowledge that could cause difficulty in understanding appeared to be considerably less for *high-entry* students than for *low-entry* students.

CHAPTER V

V. ANALYSIS

Five categories emerged from analysis of tasks and interviews. These were the techniques students used (*strategies*), the composition of the student solutions (*structure*), whether the students monitored their progress (*progress monitoring*), the amount of knowledge the students possessed on each topic (*content*), and whether the students planned before writing their solution (*advance planning*).

1. STRATEGY

Biggs and Telfer (1987) defined the term 'strategy' as long-term planning prior to problem solving. For this study 'strategy' was redefined as a mathematical approach to a task that resulted in a sequence of actions.

Most *low-entry* students consistently experienced more difficulty with all parts of the tasks than *high-entry* students. Although individual *low-entry* students had fewer choices of strategies, as a group these students confirmed that there were more ways to be incorrect than to be correct. At the time of the study the *low-entry* students were not consciously aware of alternative ways to approach the tasks and relied heavily on recall of 'recipes' or formulas. Some students occasionally attempted to fit the question to the recalled formula, even if that formula was inappropriate. This was in contrast to *high-entry* students who often selected their preferred strategy from a variety of alternative approaches.

Figures 4 to 11 depict a summary of the different routes taken by students in solving Tasks 1(a) to 2(d). Each route resulted in either a correct, incorrect or incomplete solution. Included in the figures are the number of *low-entry* and *high-entry* students taking each part of the route. For example, (2, 1) indicates two *low-entry* students and one *high-entry* student took a particular route. A non-attempt was classified as incomplete. Tables 1 to 8 compare the number of *low-entry* and *high-entry* students whose written work displayed the various routes taken.

(1) Task 1(a) - Calculus

Let $f(x) = x^4 + 4x^3 - 44x^2$

Find the values of x for which $f'(x) = 0$.

(Leave as exact expressions.)

Task 1(a) was based on basic secondary school calculus concepts. The student needed to calculate the derivative of a function and solve the resulting quadratic equation after taking out a common factor.

From Figure 4 and Table 1 it can be seen that 11 of the *low-entry* students either did not complete the task or gave incorrect answers. Seven of the *high-entry* students gave a correct solution, one gave an incorrect solution and one *high-entry* student solved most of the task before abandoning it. All *high-entry* students and four *low-entry* students used the predicted strategy of removing the common factor and solving a quadratic equation.

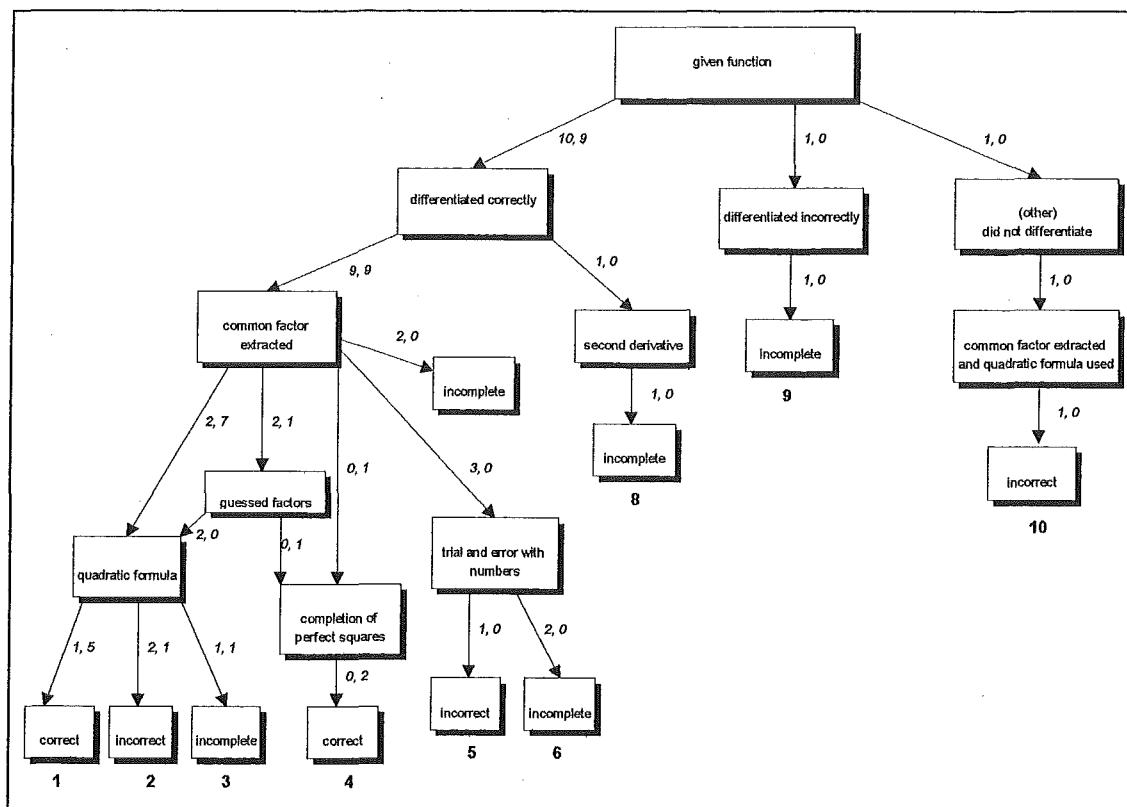


Figure 4: Alternative routes used by students to solve Task 1(a).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS	KEY
1	1	5	*** correct solution
4	0	2	***
2	2	1	** incorrect solution
5	1	0	**
10	1	0	*
3	1	1	* incomplete solution
6	2	0	*
7	2	0	*
8	1	0	*
9	1	0	*

Table 1: Frequency of students using the various routes In Figure 4 to solve Task 1(a).

Two *high-entry* students used the technique of completing the perfect square, and seven applied the quadratic formula. The four students with the incorrect solution failed to accurately recall the formula, and the *high-entry* student who eventually abandoned the task recognised early that his solution was not quite correct. He then tried unsuccessfully to complete the perfect square.

Many of the *high-entry* students were aware of, or briefly considered, other strategies but expressed preference for the strategy they eventually used.

- I prefer to use the formula as I used it at school. It is easier for me.
- I would just go straight to the quadratic formula here because I knew it.
- Yea, I considered using the formula but I have always found using the perfect square easier.

Eight of the *high-entry* students were confident in their strategy for solving the quadratic equation. The one student who was not sure of the quadratic formula went ahead with the working as if it was correct, but was aware the entire time that the formula 'felt' incorrect.

In contrast, the *low-entry* group gave the greatest variety of possible strategies. Eight students did not consider using the quadratic formula until prompted during the interview.

- I didn't think of using the formula. I suppose I could have.
- Yea, I had used it (the formula) before, but never thought of using it here. I don't know whether it is because I haven't done it in class recently or because maths was never my strong point. At school most things factorise nicely and it was assumed you knew how to do it. It created problems for those that couldn't remember.

The comments showed that these students associated formulas or operations with certain situations. Three *low-entry* students did not remove the common factor and tried unsuccessfully to solve a cubic equation. In each case, this approach resulted in a variety of trial and error attempts at solving a cubic equation. One *low-entry* student calculated the second derivative, another used the quadratic formula on a cubic equation, and the third searched for something familiar but confused the function with area under a graph. He wrote:

$$f(x) = x^4 + 4x^3 - 44x^2 \quad f'(x) = 4x^3 + 12x^2 - 88x = 0$$

Area under a graph . Rate of change
 () () ()

Four *low-entry* students gave incorrect solutions and a further seven students abandoned their solution before completion. The one student with the correct answer used exactly the same technique as five of the *high-entry* students. That is, after removing a common factor ($4x$) the resulting quadratic equation was solved using the standard quadratic formula. Of the four incorrect answers, two *low-entry* students experienced difficulty recalling the quadratic formula and one student used the quadratic formula without initially factorising. The remaining student used a trial and error method with various values of x being substituted into the cubic equation. Of the seven *low-entry* students who abandoned the question, one unsuccessfully attempted the first steps of the quadratic formula and two students used trial and error with numerical values. Two abandoned their solution after extracting the common factor and differentiating incorrectly, one calculated the second derivative and one student differentiated incorrectly.

(2) Task 1(b) - Calculus

Let $f(x) = x^4 + 4x^3 - 44x^2$

Find the tangent to the curve $y = f(x)$ at $x = 0$.

Task 1(b) was suitable for exploring student content-knowledge, especially linking knowledge, cue sensitivity, possible visualisation and the level of understanding in a specified topic, namely tangents to curves. Traditionally, students were expected to follow a set procedure of substituting the point (0,0) and the gradient (zero) into an equation of a line, to get $y=0$. All students attempted the task.

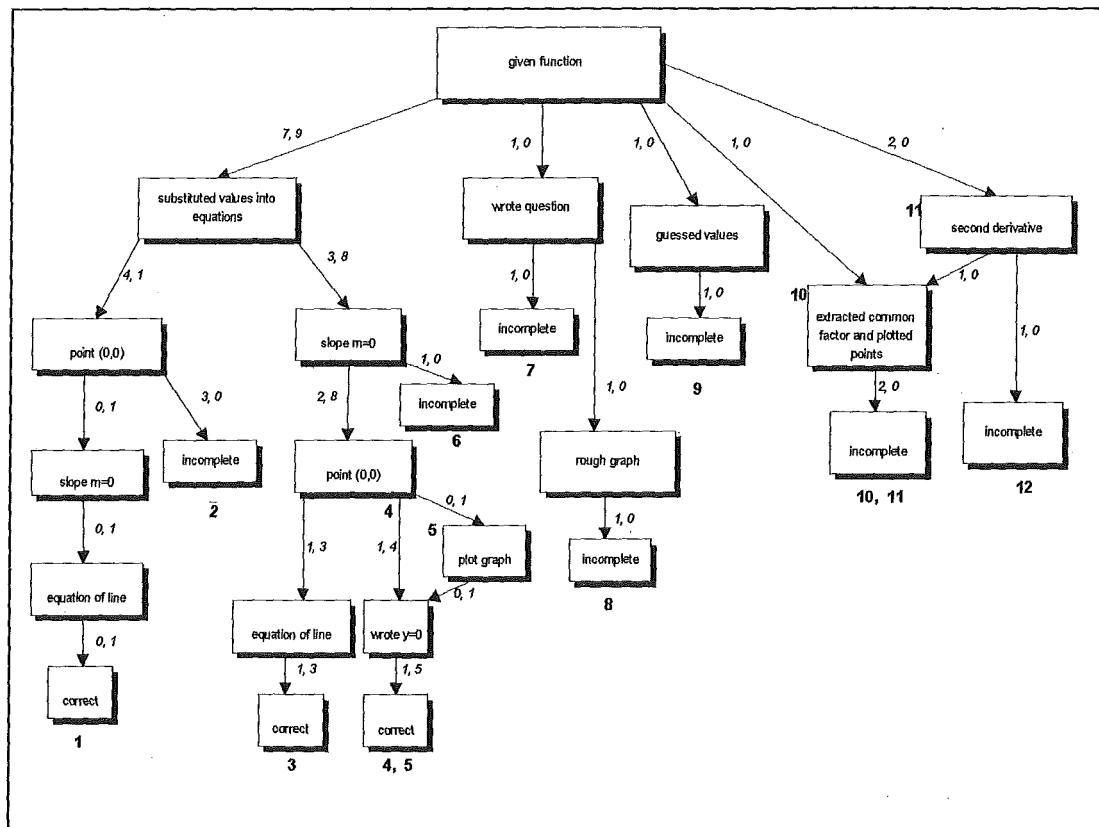


Figure 5: Alternative routes used by students to solve Task 1(b).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS	KEY
1	0	1	*** correct solution
3	1	3	***
4	1	4	***
5	0	1	***
2	3	0	* incomplete solution
6	1	0	*
7	1	0	*
8	1	0	*
9	1	0	*
10	1	0	*
11	1	0	*
12	1	0	*

Table 2: Frequency of students using the various routes in Figure 5 to solve Task 1(b).

As can be seen in Figure 5 and Table 2, the *low-entry* group displayed more ways to approach the task than the *high-entry* group. All *high-entry* students and two *low-entry* students obtained the correct answer using similar strategies. The remaining ten *low-entry* students abandoned their solutions at various stages and left gaps in their working. Despite their stated intention to do so, none of these ten *low-entry* students returned to complete or repeat the task. From Figure 5, it can be seen that two *low-entry* students calculated the second derivative but did not complete the task. One student abandoned the task after writing the given equation, one attempted to plot the graph by finding individual points and one guessed values. From their earlier comments many of the *low-entry* students appeared to search for a recall of 'recipes' that indicated reproductive understanding of this topic. Some of these students appeared to use isolated pieces of information that were remotely associated with Task 1(b).

- I thought of differentiating again but I needed a step in between that... I couldn't remember... I think I might have been thinking of something different other than finding the tangent... I'm not really sure.

(3) Task 1(c) - Calculus

Describe how you would prove that the line $y = 96x - 576$

is tangent to the curve $y = f(x)$ at both

$x = -6$ and $x = 4$.

Prove that the line is tangent at both values of x .

In Task 1(c) the opportunity was available to study how students reasoned through a question where the answer was presented within the task. By asking the student to 'describe how..' the students were expected to outline a plan before they began to write their solution. Any difficulties students had in advance planning could be highlighted. Results are shown in Figure 6 and Table 3.

All *high-entry* students and two *low-entry* students successfully solved this task. One of the *low-entry* students used a similar strategy to that employed by the *high-entry* students, that is, an algebraic approach, while the other *low-entry* student accurately plotted graphs. The remaining ten *low-entry* students abandoned the task at various stages. Of these ten students, three did not attempt the question while two tried unsuccessfully to draw the graphs by plotting points. They abandoned their attempt after ten minutes of writing. Another four students began with an algebraic approach but then changed tactics and found the relevant points on the line and curve. However they also abandoned the work before completion. Their comments indicated that they were unsure of what to do

next. The tenth *low-entry* student found the slopes but stated he did not know how to continue.

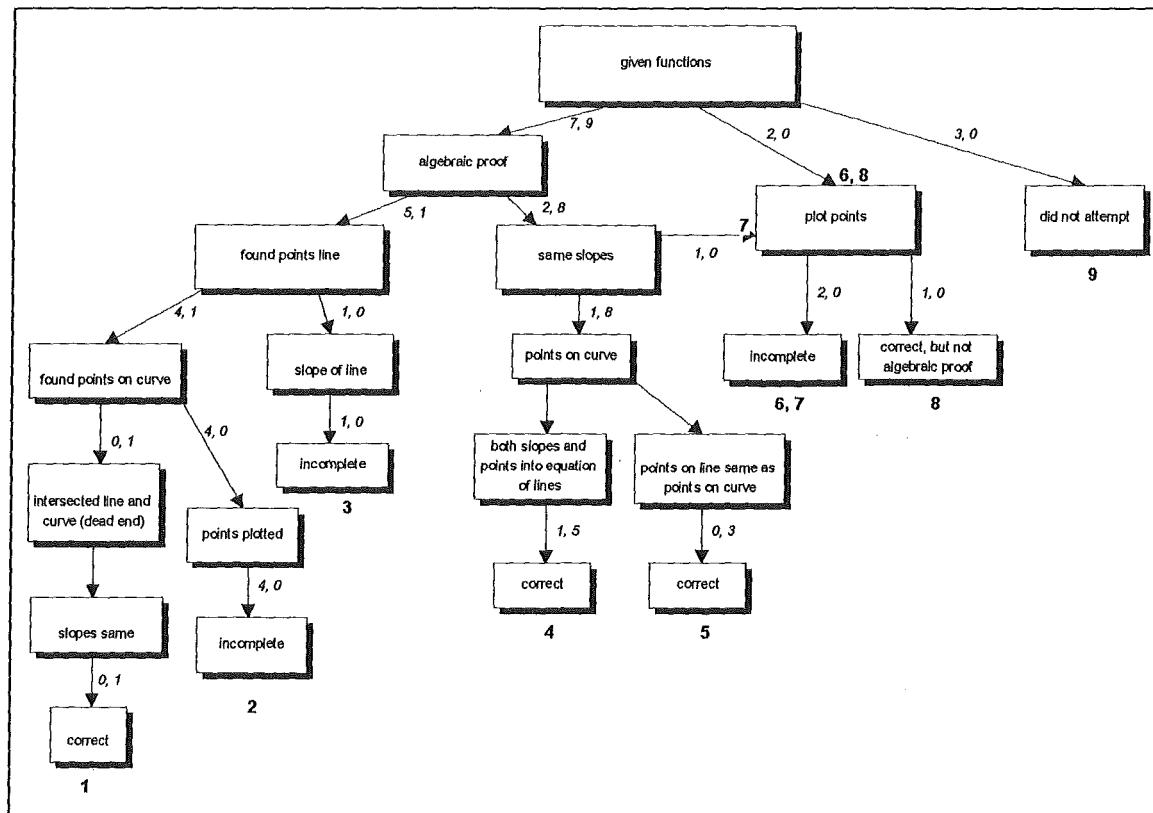


Figure 6: Alternative routes used by students to solve Task 1(c).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS	KEY
1	0	1	*** correct solution
4	1	5	***
5	0	3	***
8	1	0	***
2	4	0	* incomplete solution
3	1	0	*
6	1	0	*
7	1	0	*
9	3	0	*

Table 3: Frequency of students using the various routes in Figure 6 to solve Task 1(c).

In contrast, *high-entry* students performed well on this task both in their approach and calculation. The students used an algebraic approach and did not consider plotting points as an alternative strategy. Overall, the *high-entry* student solutions were logical and precise with minor variations within the group in the order in which the calculations were performed.

(4) Task 2(a) - Matrix Methods

Solve the linear system:

$$u + v + w = 0$$

$$u + 2v + 3w = 0$$

$$3u + 5v + 7w = 1$$

What was the question asking you to do? Interpret your answer by describing the type of solution you found.

Task 2(a) highlighted how students dealt with a 'recipe' matrix methods question and how they interpreted their result. The solution was deliberately designed to be 'inconsistent' to test students' ability with this type of solution. Students either used the matrix method techniques encountered in lectures or relied on the simultaneous equation techniques emphasised at school. Results are outlined in Figure 7 and Table 4. All *high-entry* students and three *low-entry* students performed well on this task. However, while these *high-entry* students could interpret their solutions correctly with verbal explanations the *low-entry* students who obtained correct solutions could not do so.

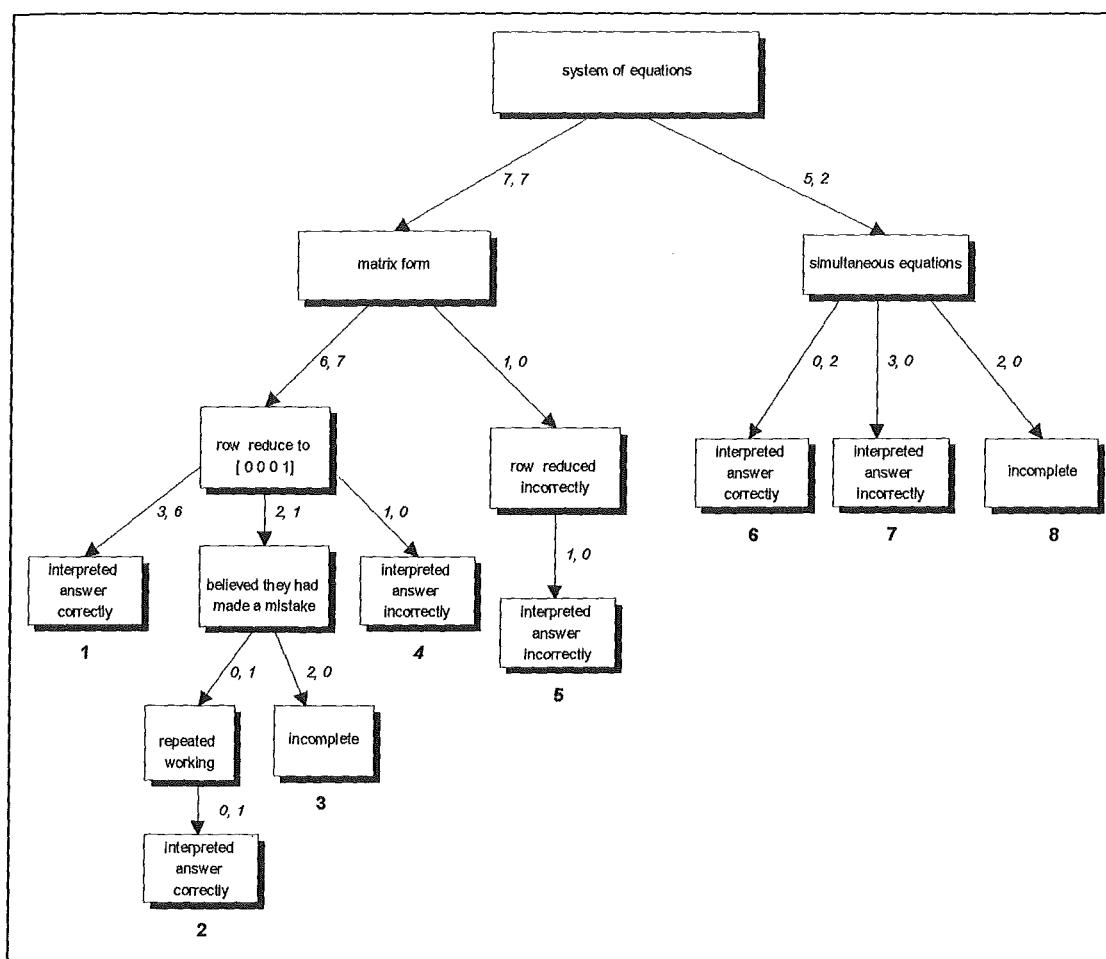


Figure 7: Alternative routes used by students to solve Task 2(a).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS	KEY
1	3	6	***correct solution
2	0	1	***
6	0	2	***
4	1	0	**..incorrect solution
5	1	0	**
7	3	0	**
3	2	0	* incomplete solution
8	2	0	*

Table 4: Frequency of students using the various routes in Figure 7 to solve Task 2(a).

Seven of the *high-entry* students used matrix methods and recognised an 'inconsistent' solution. One of these students, however, was initially unsure of his 'answer and repeated his row reduction. He accepted his 'inconsistent' solution when he recognised it early in his second attempt. He admitted that:

- I had to really think about the solution.

Another two *low-entry* students also had difficulty with accepting an 'inconsistent' solution. However, unlike the *high-entry* student, these students abandoned their work after completing most of the task. Of the seven *high-entry* students who used matrix methods, three made computational errors that they located and corrected. The students then repeated the task rather than follow the correction through the working. Only one *low-entry* student did not recognise that a computational error had occurred and two other *low-entry* students deliberately altered their calculations so that they obtained a unique solution. Two *high-entry* and five *low-entry* students used the techniques emphasised at school. That is, the elimination of variables using simultaneous equations. The reasons the two *high-entry* students gave for preferring this method rather than the matrix method was based on personal preference:

- I don't like matrices. I think we did them in the Fifth Form.
- I jumped in and saw that this equalled that. But it didn't work. So I went back and tried the normal technique of singling out one of the variables. I find matrices slower and I find I make more mistakes with them.

Of the five *low-entry* students who used simultaneous equations, two interpreted their answers as unique solutions and one finally abandoned his solution early.

(5) Task 2(b) - Matrix Methods

For which values of k do the two lines:

$$kx + y = 1$$

$$x + y = 1$$

have no solution, one solution, or infinitely many solutions?

The aim of this task was to study how students coped with variables. Geometric visualisation of the two lines was an efficient way to determine the values of k that gave the three types of solutions. The two given lines always intersected at the point $(0,1)$ and the solution became an infinite set of points if $k = 1$. The 'no solution' situation was not possible in this task. The results are outlined in Figure 8 and Table 5.

Although both groups of students employed a wide variety of strategies, only one student (*high-entry*) successfully visualised the question. Seven of the nine *high-entry* students stated that they found visualisation difficult and commented that it was easier to use numbers or to look at the algebra. However, the researcher noted that although the *high-entry* students commented that they could 'see' what to do in their heads, when asked to explain their answers several comments highlighted a geometrical interpretation.

- I rearranged into $y=mx+c$ form. When $k=0$, x didn't matter. Therefore $y=1$ and there is one solution. The lines had to cross, so k could have any value, and when $k=1$ then the lines lie on top of each other and the solution is infinite.

Two *low-entry* students tried to visualise the lines but were not so successful.

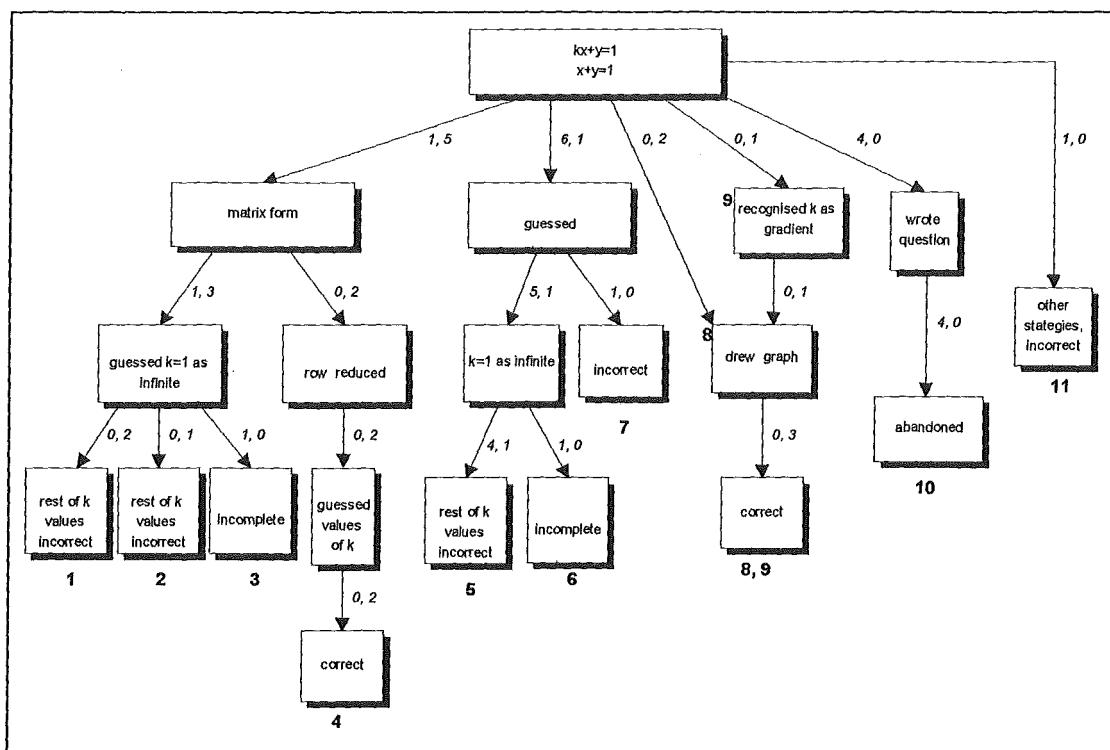


Figure 8: Alternative routes used by students to solve Task 2(b).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS	KEY
1	0	2	*** correct solution
4	0	2	***
8	0	2	***
9	0	1	***
2	0	1	** incorrect solution
5	4	1	**
7	1	0	**
11	1	0	**
3	1	0	* incomplete solution
6	1	0	*
10	4	0	*

Table 5: Frequency of students using the various routes in Figure 8 to solve Task 2(b).

The most common method used by the other students was to recognise that $k = 1$ gave an infinite solution, and to guess the values of k which gave the other solution types. Of the nine *high-entry* students, seven gave a correct solution. The two students who gave an incorrect solution found that $k = 1$ resulted in an infinite solution, but their values of k for a unique solution and 'no solution' scenarios were incorrect. However, in obtaining the other values of k , the *high-entry* students demonstrated structured guesswork. That is, even if *high-entry* students wrote an incorrect solution, often their guesswork was logically structured.

- It will have infinitely many solutions when $k=1$, because the supporting value for x would be the corresponding value for y . One solution if $k=0$, y has to equal 1 which makes $x=0$ in the second equation. Therefore, by trying to put in other values of k , then there is no solution if $k \neq 0$.

Of the twelve *low-entry* students, six gave incorrect answers and six abandoned their solutions before completion. Six *low-entry* students correctly wrote that $k = 1$ gave an 'infinite solution', four students incorrectly guessed the other values of k and two abandoned their solution after writing $k = 1$. Another four *low-entry* students wrote down the question and then left large gaps in their answer sheet.

One *low-entry* student attempted to use an inappropriate formula, namely $b^2 - 4ac$, when the equations were linear (see page 61). During the interview it became apparent that this student altered the equations to fit the formula she had chosen. She maintained that a 'solution' occurred where the graph intercepted the X -axis. She used two pages of working beginning with $kx + y = 1$ and

$b^2 - 4ac < 0$. Other students also mentioned this same definition of a solution but did not follow through with the associated calculation. A sample is inserted here:

$$\begin{aligned} kx + y &= 1 \\ x + y &= 1 \end{aligned}$$

$$\begin{aligned} b^2 - 4ac &< 0 \\ 1^2 - 4 \times k \times 1 &< 0 \\ 1 + 4k &< 0 \\ +4k &< 1 \\ \cancel{+4k} &\cancel{< 1} \\ k &< \frac{1}{4} \end{aligned}$$

$$b^2 - 4ac > 0$$

$b^2 - 4ac < 0$ have no solution

$$b^2 - 4ac = 0$$

$$\begin{aligned} b^2 - 4ac &< 0 \\ 1^2 - 4 \times 1 \times 1 &< 0 \\ 1 + 4 &< 0 \\ 5 &< 0 \end{aligned}$$

$$\begin{aligned} kx + y - 1 &= x + y - 1 \\ kx - x + y - y &= 0 \\ kx - x &= 0 \\ kx &= x \end{aligned}$$

(6) Task 2(c)- Matrix Methods

Matrix A is invertible if there is an inverse (A^{-1})

such that

$$A A^{-1} = A^{-1} A = I.$$

$$\text{If } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

(i) Find A^{-1} and show that $A A^{-1} = A^{-1} A = I$.

(ii) A^2 is $A A$ (or A times A). Find A^{-3} .

(a) Task 2(c)(i)

Task 2(c)(i) gave the opportunity to study how students coped with a routine calculation involving the inverse of a 2×2 matrix. Students were expected to use a technique covered in lectures, namely row reduction of $[A : I]$, where A was the given matrix and I was the equivalent $n \times n$ identity matrix. Results are outlined in Figure 9 and Table 6.

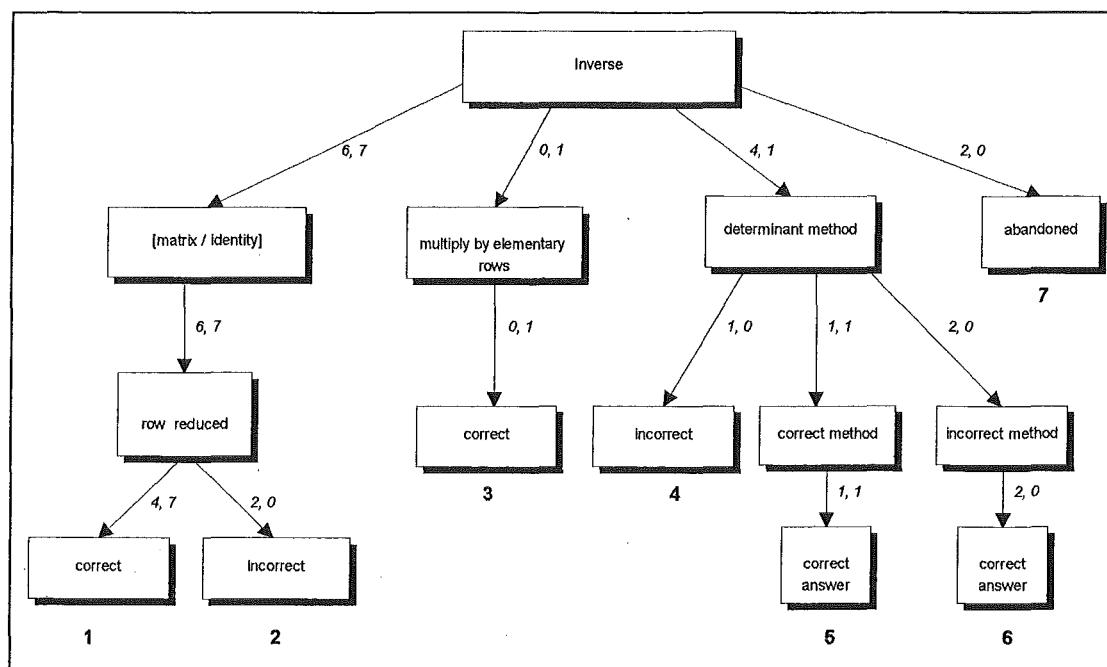


Figure 9: Alternative routes used by students to solve Task 2(c)(i).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS	KEY
1	4	7	*** correct solution
3	0	1	***
5	1	1	***
6	2	0	**
2	2	0	** incorrect solution
4	1	0	*
7	2	0	* incomplete solution

Table 6: Frequency of students using the various routes in Figure 9 to solve Task 2(c)(i).

One *high-entry* student found his solution by successfully multiplying the given matrix by elementary matrices. Another *high-entry* student and four *low-entry* students used a method from their Fifth Form at secondary school, namely:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

High-entry students who did not use this formula stated in the interview that they were aware of its existence as an alternative method but they did not choose to use it because:

- Maybe it wouldn't have been as accurate because you needed to remember it.

All nine *high-entry* students and seven of the *low-entry* students obtained the correct solution. Seven of these *high-entry* and six of the *low-entry* students used the expected strategy of row reducing a combination of the given matrix with the identity matrix. Only four *low-entry* students obtained the correct answer. The remaining *low-entry* students exhibited computational errors.

Two *low-entry* students with the correct answer exhibited an error by forgetting to include the scalar factor $\frac{1}{(ad - bc)}$. This scalar factor did not affect their final answer as it had a value of unity. These two *low-entry* students did not recognise or remember this part of the formula even when prompted. Another three *low-entry* students exhibited computational errors in either row reduction or calculation of the determinant. A further two *low-entry* students did not attempt the question as they had missed the relevant lectures and did not know how to begin.

(b) Task 2(c)(ii)

Task 2(c)(ii) was designed to highlight how students extended their knowledge to a possible ‘unfamiliar’ situation. Results are outlined in Figure 10 and Table 7.

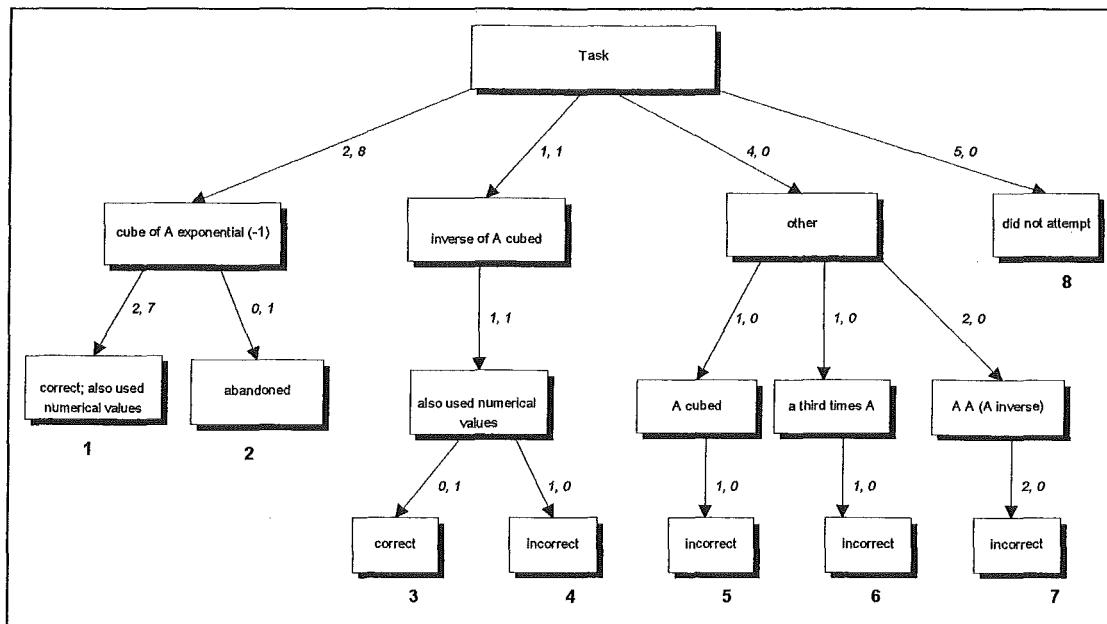


Figure 10: Alternative routes used by students to solve Task 2(c)(ii).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS
1	2	7
3	0	1
4	1	0
5	1	0
6	1	0
7	2	0
2	0	1
8	5	0

KEY

*** correct solution

** incorrect solution

**

**

* incomplete solution

*

Table 7: Frequency of students using the various routes in Figure 10 to solve Task 2(c)(ii).

Few students were expected to have encountered A^{-3} up to the time of the study. The *high-entry* students seemed to feel this task was straightforward. Seven of the nine students wrote

$$A^{-3} = (A^3)^{-1}$$

while one student wrote

$$A^{-3} = (A^{-1})^3$$

All checked with numerical values at the end of their solution and obtained the matrix for A^{-3} using the given matrix. Only one *high-entry* student abandoned the work because:

- I thought it would have a different answer if I multiplied on each side.

This student recognised that multiplication of two $n \times n$ matrices was not necessarily commutative and this knowledge caused enough confusion for her to abandon this part of the task.

Two *low-entry* students attempted part 2c(ii) and used the most common method employed by the *high-entry* students. Five *low-entry* students did not attempt the task:

- I just say, oh, that looks difficult, so I don't bother. I don't know how to do it.

Another four students used erroneous miscellaneous selection of other strategies such as, $A \times A \times A^{-1}$ or $A^{-3} = \frac{1}{3}A$. The author noted that they continued their calculation as if the errors did not exist.

(7) Task 2(d) - Matrix Methods

Given that $(A \ B)^{-1} = B^{-1} \ A^{-1}$ where A and B are any two $n \times n$ matrices,
show that the inverse of

$$A \ B^{-1} \ C \text{ is } C^{-1} \ B \ A^{-1}.$$

Two of the *high-entry* students mentioned that this type of question was similar to work they had done a long time ago. They commented that it had not taken them long to work out these sorts of questions at the time. This question was new territory for the *low entry* students. Results are outlined in Figure 11 and Table 8.

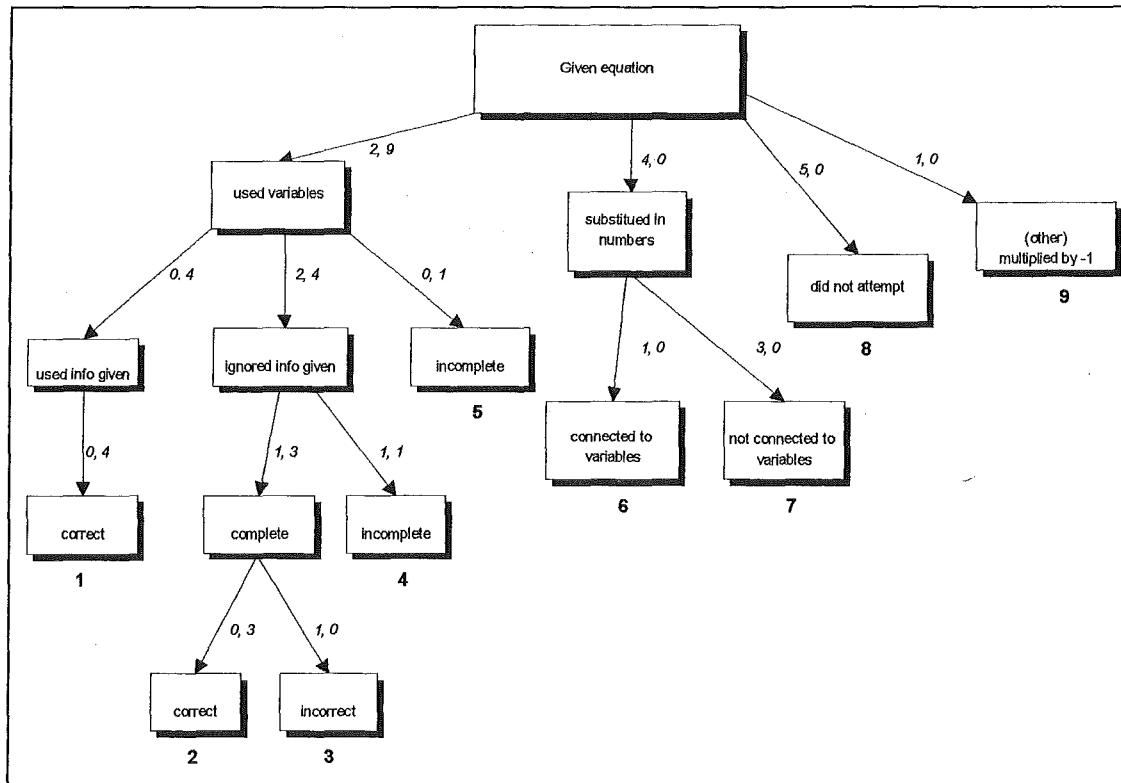


Figure 11: Alternative routes used by students to solve Task 2(d).

ROUTE	NUMBER OF LOW-ENTRY STUDENTS	NUMBER OF HIGH-ENTRY STUDENTS	KEY
1	0	4	*** correct solution
2	0	3	***
3	1	0	** incorrect solution
6	1	0	**
7	3	0	**
9	1	0	**
4	1	1	* incomplete solution
5	0	1	*
8	5	0	*

Table 8: Frequency of students using the various routes in Figure 11 to solve Task 2(d).

Seven of the *high-entry* students successfully completed the task. The two remaining students in the *high-entry* group who did not complete the task made an attempt before abandoning their work. All *high-entry* students manipulated variables although only four of these students successfully used the information that was given. Other *high-entry* students commented that they initially tried to use the information ‘in their heads’, but did not write it down. Of the five *high-entry* students who did not use the given information, the common method for showing that the inverse of $A \ B^{-1} \ C$ is $C^{-1} \ B \ A^{-1}$ was to assume that one was the inverse of the other and therefore that

$$(A \ B^{-1} \ C)(C^{-1} \ B \ A^{-1}) = I \quad \text{and} \quad (C^{-1} \ B \ A^{-1})(A \ B^{-1} \ C) = I.$$

All the *low-entry* students ignored the given information. Five did not attempt the task. One *low-entry* student demonstrated confusion with understanding exponential concepts when he consistently wrote: $(-1)A = A^{-1}$. This overall difficulty experienced by *low-entry* students was not surprising as they had not acquired the same exposure to this type of task as the *high-entry* students.

2. STRUCTURE

For this thesis, 'structure' was defined as the composition of a worked solution. Overall, *high-entry* students demonstrated a highly detailed and efficiently structured solutions. In contrast, *low-entry* students exhibited unstructured, inconsistent working that was often scattered and incomplete. Their solution structure improved considerably for 'recipe' type questions.

An obvious feature of *low-entry* student working was mathematical statements that were often left unfinished or incorrect. For example in Task 1(a) one *low-entry* student wrote:

$$f(x) = x^4 + 4x^3 - 44x^2$$

$$4x^3 + 12x^2 - 88x$$

$$12x^2 + 24x - 88^{\textcircled{1}}$$

Not only was "=" absent but the student did not indicate that the second line represented $f'(x)$ and the third line represented $f''(x)$. Again, in Task 1(a) another student wrote:

$$f(x) = x^4 + 4x^3 - 44x^2 = 4x + 12x^2 - 88$$

$$0 = x^4 + 4x^3 - 44x^2 \quad f' x$$

This second student also had difficulty with differentiation. *Low-entry* student confusion in the logic and coherence of written statements indicated a reliance on

^① The author felt that pertinent information would not be lost by typing the information from the student's script.

recall and reproduction of 'recipes'. There was very little evidence of transformative understanding. As two students stated

- I differentiated and put $f'(x)=0$ and then I tried to factorise it. It wouldn't factorise so... I didn't know what to do after that. I remembered $b^2 - 4ac$, but I wasn't sure where to put that in relation to the gradients.
- I was just thinking, like last year, when I get a formula, I just work through it. I just plug that into the formula...if it's not straightforward, that's where I get hung up... like for word problems.

Another obvious feature of *low-entry* student working was the isolation of partial solutions. For example in Task 1(b) one student wrote:

$$\begin{aligned}
 & x=0 \\
 & y = x^4 + 4x^3 - 44x^2 \\
 & \cancel{y} = 0^4 + 4\cancel{0}^3 - 440^2 \\
 & y=0
 \end{aligned}$$

tangent will be $(0,0)$

$$\begin{aligned}
 & y=mx \\
 & y = mx \\
 & m =
 \end{aligned}$$

\checkmark

$$\begin{aligned}
 & x=0 \\
 & \cancel{y} = 4x^3 + 12x^2 - 88x \\
 & y = 4(0)^3 + 12\cancel{0}^2 - 88\cancel{0} \\
 & y=c
 \end{aligned}$$

The student who wrote the above solution initially calculated the point $(0,0)$. He then tried to put this point into the equation of a line, but discovered he had too many variables. He finally differentiated the initial function, forgot to change y to y' and ended with $y=0$. This only added to his confusion. During the interview it

became apparent that the student did not know what he had found and resolved this dilemma by ignoring the initial point he calculated.

- I knew that to find the tangent to the curve you put it ($x=0$) into the differential equation, or either one, and it is zero. Once I got the answer $y=0$, I didn't know what to do.

In contrast, *high-entry* students consistently used systematic, efficient clear working interspersed with phrases that clarified their calculations. These students first considered all the information presented and then attempted to connect partial solutions into a conclusion statement. For example, for the same Task 1(b) given earlier a *high-entry* student wrote:

tangent $y = f(x)$ at $x = 0$.

gradient tangent at $x=0 = f'(0) = 0$.

when $x = 0 \quad y = 0$

\therefore equation tangent $\Rightarrow y = 0$.

This student wrote his version of the question, found the slope and the point, then concluded that the equation tangent was $y=0$. His verbal explanation, rather than his written work, showed a depth of understanding not found in the *low-entry* group. Likewise for Task 2(a):

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 3 & 5 & 7 & 1 \end{array} \right] \begin{array}{l} \text{Replace row 2 by row 2 - row 1} \\ \text{Replace row 3 by row 3 - 3row 1} \end{array}$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 1 \end{array} \right] \begin{array}{l} \text{Replace row 3 by row 3 - 2row 2} \\ \\ \end{array}$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

\leftarrow This line shows that the system of eqns is inconsistent. There is no solution to the set of eqns.

This intermingling of phrases and numerals was common for *high-entry* students while *low-entry* students commented that they rarely used phrases as part of their working. For this study, many of the *low-entry* students did write phrases but they stated that this was because of the task requirement to explain the solution. The only instance of clear, structured, detailed working by *low-entry* students was in 'recipe' type questions that the student had spent time memorising. For example, one student wrote for Task 2(c)(i):

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array} \right] \quad R_1 \leftarrow R_1 - R_2$$

$$R_2 \leftarrow R_2 - R_1$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right] \quad \text{The inverse matrix is}$$

$$R_2 \leftarrow R_2 - R_1$$

$$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

Although the working showed structure and detail, this same student demonstrated inconsistent, unstructured working in another task that required more understanding and employed more advanced reasoning skills.

Low-entry students consistently ignored information such as conditions or hints given in a task. They felt this type of information only confused the task as it was not relevant. For example, in Task 2(d) the given information was:

$$(A B)^{-1} = B^{-1} A^{-1}$$

Every *low-entry* student who attempted Task 2(d) ignored this information. All *high-entry* students stated that they tried to incorporate this information into their working even if they were not totally successful. Comments from the *high-entry* students who incorrectly used the information in their written work or abandoned the work were:

- I tried to find a pattern or something to do with the expressions I was shown. I tried different techniques in my head before I gave up.

The sole *high-entry* student who did not attempt to use the given information indicated an achievement-orientated approach to learning:

- This was beyond me. Although we did this type of question before, I didn't think it was important. This type of question didn't seem like things we had to know.

The way students persevered or abandoned tasks varied between the two groups. *High-entry* students abandoned a task only after several different strategies had been tried either mentally or in writing. On the other hand, a common strategy for *low-entry* students was to write the question and leave plenty of space for further working. Although they intended to return later to complete the work, they never did so. At times copying down the question was the only written work done. For other *low-entry* students this was supplemented with smaller writing as an aside. For example, in Task 1(b), one student wrote:

$$\begin{aligned}
 & y = 1 - kx \\
 & x + (1 - kx) = 0 \\
 & x + 1 - kx = 0 \\
 & x(1 - k) = -1 \\
 & x = \frac{-1}{1-k} \\
 & k = \frac{x}{x-1} \\
 \begin{matrix} kx + y = 1 \\ x + y = 1 \end{matrix} & \quad y = 1 - kx \\
 & x + (1 - kx) = 1 \\
 & x - kx = 0 \\
 & k = \frac{x}{x}
 \end{aligned}$$

From the interviews, it became clear that these students spent time searching for familiar associated knowledge. Again, for Task 2(a) one student stated:

- I don't really know what I was trying to do. I remember doing them at school. I think we always took row 1 away from row 2 and you did it like that. It was sort of like row reduction. I never even thought of doing it by row reduction.

This student abandoned the task after subtracting two rows. Her statement that she could not remember what to do next once again pointed to a surface-orientated learning approach and a reliance on recall.

Another student doing Task 2(a) was confused with the term 'solve'. To him, this meant that the three equations had to intersect. He made a computational error in his elimination of variables and found a unique solution that did not check out with all three equations. He then abandoned the task at this stage.

3. PROGRESS MONITORING

The three areas in which self checking could occur was in the way the student approached the task, continual progress self-monitoring and a check on the final answer. Self-monitoring could be either conscious or subconscious in that students could possess a subconscious inbuilt checking system without being aware of actively monitoring their work.

Both groups of students employed both conscious and subconscious self-monitoring in their approach to the tasks. Many of the *high-entry* students were confident that they had approached the tasks in a way that would result in the correct answer. This feeling appeared to be subconscious in that students could not explain why they knew it was correct and many could cite other equally acceptable ways. The *high-entry* students therefore selected from a range of relevant approaches.

- It felt right. I could have done it another way if I wanted.

In contrast, the *low-entry* students used a conscious trial and error approach.

- I did a long process. I tried to squash the values down. I thought it was going to work out to a nice round number, but it didn't.
- I tried three different ways, but I still didn't manage to get it to work. I didn't think I was right, so I would go on. I thought there was something that you did with the second derivative... and that didn't work out... I don't know what I was trying to do there. I felt what I was doing was totally irrelevant, so I just scrapped that.
- I tried to make sense of the question. First of all I thought it was area under the graph, then decided that was wrong.

When asked why they felt their approaches were incorrect, many of the *low-entry* students stated that they expected to get the wrong answer. This feeling appeared to be based on past experience.

- I usually get it wrong.

Continual self-monitoring of work in progress was often subconscious for *high-entry* students and non-existent for *low-entry* students. For example a *high-entry* student stopped halfway through her third matrix in Task 2(a):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 3 & 5 & 7 & 1 \end{array} \right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 1 \end{array} \right)$$

~~2nd row - 1st row~~

$$\left(\begin{array}{ccc|c} 3 & 5 & 7 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right) \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

~~3rd row - 2nd row~~

$$\left(\begin{array}{ccc|c} 3 & 5 & 7 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 1 \end{array} \right)$$

- Well...we did matrices a while ago and I forgot that when you replaced rows you replaced it by the row itself minus the other row instead of... because I started doing it the other row minus the row that I was replacing. I soon realised that that wasn't right so I did it the other way.

This student had recognised her error and promptly corrected it before she went any further. Occasionally some the *low-entry* students became conscious of errors soon after they occurred or if the calculation became too messy. However,

many *low-entry* students were oblivious to both computational and procedural errors. For example, a *low-entry* student doing Task 2(a) located a computational error when he reached the end of his working. He backtracked and incorrectly adjusted part of his working. He then redid the calculation. Another student (see below) did not realise he had made a computational error in Task 2(c) and continued his calculation even though the required checking indicated a computational error. When faced with evidence of an obvious mistake he abandoned the task rather than rechecking his calculations.

$$\begin{array}{r}
 \begin{array}{c}
 \left[\begin{array}{rr|rr} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array} \right] \\
 R_2 \rightarrow R_2 - 2R_1 \\
 \left[\begin{array}{rr|rr} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array} \right] \\
 R_1 \rightarrow R_1 - R_2 \\
 \left[\begin{array}{rr|rr} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{array} \right]
 \end{array}
 \end{array}$$

$$\begin{array}{l}
 \left[\begin{array}{rr|rr} 1 & 1 & 3 & 0 \\ 2 & 3 & -2 & 1 \end{array} \right] = \left[\begin{array}{rr} (3+1)(3+0) \\ (6+1)(0+3) \end{array} \right] \\
 I = \left[\begin{array}{rr} 1 & 3 \\ 0 & 3 \end{array} \right]
 \end{array}$$

$$\begin{array}{l}
 \left[\begin{array}{rr|rr} 3 & 0 & 1 & 1 \\ -2 & 1 & 2 & 3 \end{array} \right] = \left[\begin{array}{rr} (3+0)(3+0) \\ (-2+1)(-2+3) \end{array} \right] \\
 = \left[\begin{array}{rr} 3 & 3 \\ 0 & 1 \end{array} \right]
 \end{array}$$

The third area of progress monitoring was in the checking of final answers. Students in both groups did not check their final answers except where the task required them to do so. *High-entry* students appeared to have a subconscious estimation of the answer. They felt there was no need to check by substituting the answer back into the question because:

- It looked right.

Only one *high-entry* student commented that she would actively check the solution in an examination which pointed to problem solving strategy aimed at attaining better grades. In Task 2(b) all *high-entry* students checked their answers by multiplying the inverse matrix with the original matrix. This was a requirement of the task. Five *high-entry* students commented that:

- I would have checked anyway.
- If it hadn't have work out, I would have given it another shot.

In contrast, the *low-entry* students were content with whatever answer they found. While some commented that they did not know how to check their answer, others said that they rarely expected to get correct answers. The *low-entry* students did not show any evidence of consciously estimating answers, and most admitted they would not normally have checked their inverse matrix in Task 2(b). Two *low-entry* students who did check their inverse matrix found that the multiplication of the matrices did not equate to the identity, so they abandoned the task rather than find the error. Likewise, a number *low-entry* students who obtained solutions left their answers without any active checking. Again, some of these students stated that they did not know how to check their answers.

4. CONTENT

Content knowledge differed between the two groups but was similar within each group in both calculus and matrix methods topics. *Low-entry* students showed they possessed isolated pockets of knowledge that could only be accessed with the correct cues. This meant that students experienced considerable difficulty in tasks that asked for extensions of knowledge or interpretation of solutions. Isolated knowledge pockets may also explain why students did not have immediate access to a variety of strategies, did not readily recognise errors when working through a solution and left solutions scattered in incomplete pieces. In contrast, *high-entry* students appeared to have tightly interlinked and broader knowledge and skills that were not isolated.

(1) Calculus

The researcher discovered that by asking a series of appropriate questions during the interviews, all but one of the *low-entry* students could work through the tasks they abandoned. For example after a student abandoned most of Task 1(b) the following conversation took place:

What does $f'(x)$ represent?

- Oh, that is the slope.

How would you find the slope at a point on the curve?

- Oh, differentiate it and put in values of x . I feel so stupid. It would come to zero.

So, what about the point where the slope is zero?

- That is the origin. Oh,.. the answer is $y=0$. I thought the answer must have been harder than that.

Content-knowledge existed but seemed weakly linked or not linked at all. Only one *low-entry* student did not appear to have the knowledge necessary to work through a task with the author. It was possible that the appropriate cues were not accessed or else the difficulty lay in knowledge not being assimilated at all into the mind. By asking the appropriate questions, enough knowledge could be accessed to solve the tasks eventually for 11 of the 12 *low-entry* students. This forcing of the linkages did not necessarily mean that the students could then connect the knowledge themselves. When asked to repeat an identical task only six of the *low-entry* students could do so. But when given a task that was slightly different, those same six *low-entry* students once more experienced difficulty.

Content knowledge was not isolated for *high-entry* students. A typical response was to elaborate on the written solution without prompting. For example, in Task 1(a):

- $f'(x)$ is giving the gradient function. And making that equal to zero will give you the turning points... The gradient function goes from positive to negative.
- This (student points to working) found the tangent at $x=0$. This is the point where the gradient of the curve is equal to zero.

In contrast, the usual response from the *low-entry* student was to read the question aloud without any elaboration or personal interpretation. It took a lot of prompting from the author before the *low-entry* students continued to explain their written solution. A usual comment was:

- I wasn't sure why I did that.

Meanings of words, especially in calculus, could be a barrier for access to pockets of knowledge and subsequently problem solving. For one *low-entry* student the word 'tangent' was a barrier to his knowledge.

- As soon as I saw the word 'tangent' (Task 1(b)). I just went straight to 1(c). I would normally go back to this (1(b))...That was probably my downfall in calculus, the word 'tangent'.

When the researcher reworded the question by replacing 'tangent' with 'slope' the student stated that he now knew what to do. He would find the answer by plotting a curve. This would involve locating the maximum and minimum, substituting the points back into the original equation, then drawing the graph to see what happened at $x = 0$. This technique seemed to reflect a 'recipe' taught at school and a lack of understanding and flexibility with gradients. Even with prompting the student could not connect $f'(x) = 0$ with a horizontal gradient.

Two other *low-entry* students confused 'tangent' with 'normal'. When they were asked to draw the tangent line to a given curve, each drew a line orthogonal to the tangent line. One commented:

- Is the tangent at right angles?. I can never remember the more basic things.

Another two *low-entry* students could draw the tangent to a curve but stated:

- I'm not sure what a tangent is. It has something to do with rise over run.

Both these students selected two points on the curve and found the equation of the line between these two points. They had not distinguished 'secant' from 'tangent'.

- Well, it was the only formula I could remember actually, when finding the equation of something.

This type of barrier was not confined to *low-entry* students. One *high-entry* student said that a 'function' had to be:

- ..Curving around in a circle or something like that.

This student thought that the equation $y = 0$ represented the *Y*-axis and that a tangent had to be perpendicular to the curve. Despite these difficulties this *high-entry* student found the point, the gradient and the correct answer. However, to confirm her answer she drew a graph. This checking still caused a conflict because of her confusion with the equations of the *X* and *Y*-axes. She stated:

- I don't know. That's the only answer I could find.

After the researcher asked questions that guided this student to find the correct graph for $y = 0$, the student appeared to resolve the conflict quickly.

- Yea, I can see it now. It makes sense. The curve must now go like this (she draws a diagram) and the gradient is here. I was just a bit confused because of where I thought $y=0$ was.

For the other eight *high-entry* students, a solution was so obvious from the beginning that they felt they only needed to write the answer.

- I realised that in the equation $y=mx+c$, m is already zero. If that is zero and c has to be zero then the equation must also be zero. It is obvious.
- If the gradient is zero and the point is the origin, then obviously the line is $y=0$.

Although eight of the *high-entry* students stated that they did not visualise the tasks geometrically, it was noted that all *high-entry* students either roughly

sketched graphs or referred to some form of geometric interpretation during the interviews.

- I automatically did it. Because if you look at the graph and think about the graph, the tangent is there.

Overall, *high-entry* students as a group seemed to have their *content*-knowledge well connected. The students' own access to that knowledge was relatively quick and efficient during the interview. Many stated that they could 'see' what to do without writing anything.

(2) Matrix Methods

Similar evidence existed for differences between *low-entry* and *high-entry* students in matrix methods tasks.

Low-entry students had difficulty with solution interpretation, especially for 'recipe' tasks that resulted in a correct answer. For example in Task 2(a), one *low-entry* student who obtained the correct matrix, interpreted her solution as being infinite instead of inconsistent:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The question was asking you to find the values of u , v and w for which the given equations would be valid.

However, the last line of the matrix shows that there are infinitely many solutions.

Other students who used the matrix method for Task 2(a) also had difficulty, especially with the interpretation of the last line in the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

One *low-entry* student went back and changed the last line to

$$\begin{bmatrix} 0 & 0 & 2 & 1 \end{bmatrix}$$

so that the answer became unique. When questioned about his 'unique' answer, his interpretation was incorrect.

- All the lines would be the same.

Another *low-entry* student repeated Task 2(a) four times and obtained the same answer. He abandoned the task after three pages of calculation and stated that he kept getting the wrong answer. Once again, this student was certain the solution had to be unique.

Likewise, three of the *low-entry* students written the correct interpretation of their answer to Task 2(a). However the interviews highlighted one student's confusion in distinguishing types of solutions.

- They could be all parallel or not intersect at all, and that would be 'no solution'. Or they could intersect at different points and form a triangle or something, that would be a unique solution.

Another student with a correct solution stated:

- I'm not very good at this. Not the row reduction, but finding what it (the solution) meant.

Seven *low-entry* students did Task 2(c)(i) as requested, two ignored this piece of the task and two *low-entry* students did not understand what was meant by

$$A \ A^{-1} = A^{-1} A = I .$$

One student confused transpose and inverse:

- I had heard of the identity matrix but I didn't know what it was. I knew that you take rows and the rows become the columns and the columns become the rows.

Another student recognised a task similar to work covered in lectures but:

- I didn't know what he (the lecturer) was doing in lectures.

Unlike the calculus task, three quarters of *low-entry* students did not use a geometric interpretation in any part of the second task. For example, a *low-entry* student, who obtained a correct answer to Task 2(a), could not initially connect types of solutions to a geometric interpretation. However, when the researcher asked questions beginning with the unique case, the student could then interpret the 'infinitely many' solution and 'no solution' scenarios both geometrically and from the row reduced matrix. Most *low-entry* students recognised the result and connected it with words such as 'inconsistent', but from the interviews it became clear that this word had little meaning. A quarter of the students did take a geometric interpretation. In Task 2(b) a *low-entry* student looked at the two lines visually. However, he became confused in his interpretation:

- Obviously when $k=1$ it would end up with the same lines. There would be infinitely many solutions. When $k \neq 0$ there are two different lines... I wasn't sure here.

This student, like most of the other *low-entry* students, did not connect the variable $-k$ to the slope of the line. When prompted during the interview, these students did not have any problem explaining the 'infinite solution' case, but three students still experienced difficulty visualising the intersection of two lines. For example, with prompting, two students could comprehend that k altered the steepness of the line but the following comments indicate confusion with functions intercepting the X -axis:

- There was no solution as it (the line) was parallel to the X -axis, but it doesn't cut the X -axis, therefore there is no solution.

Another *low-entry* student stated that she never thought of looking at the tasks in terms of graphs, yet she answered most of the first task by plotting the graphs. This graphical interpretation was a result of the way the topics were learned rather than the use of graphs as an appropriate tool. Another *low-entry* student doing Task 2(b) could explain his answers but his working contradicted his explanation. He stated that:

- I looked for the easiest one, and the easiest one was infinitely many, because those two lines, to be the same k , have to be one. You have to have the y intercept different to have no solution. So it was not possible to have no solution. All the other values of k must give one solution.

However, he wrote:

for infinitely many solution

$k=1$ makes the equations

$$x+y=1$$

is the same line

this set of equation can not work for no solution

because no value of k can make the lines paraell.

For infinitely many solution

$k \neq 1$ because you can only have infinitely many solutions when you don't have no solutions or one solution. There is only 3 possible solutions.

High-entry students only used a geometrical interpretation when explaining their strategy in the interviews and did not display any evidence of this type of interpretation in their working. All *high-entry* students stated they were more comfortable with an algebraic strategy.

For many of the *low-entry* students it was the variable that created the initial difficulties in the task on matrix methods. For four students who just wrote down the task and little else, the presence of the variable k meant that

- I had done this before but didn't know what to do then or now.
- If I was sitting an exam and you know there is a way to do it, and you think, help I can't do that. I wasn't quite sure how I would keep k .

These students could not interpret what needed to be found let alone how to approach the question. The same students did not have any difficulty describing the types of solution when k was allocated numerical values.

It was not uncommon for *low-entry* students to substitute numerical values into variables that required a proof. For example, in Task 2(d):

- I wasn't too sure what to do, so I just made up matrices. I found the inverse of each and just multiplied them.

The student who substituted numerical values into the matrices did so because he thought it would be safer. Another student who used numerical values submitted two pages of working. In his work, this student committed several computational errors in his row-reduction. His justification for the working was achievement-orientated.

- I am not surprised I made a mistake in there somewhere. I would do this (working) in an exam as I would probably get marks for my working.

During the interviews, the preference for using numbers was also stated as a priority for four of the five *low-entry* students who did not attempt the task.

- I didn't understand what it was asking at all. I didn't get why it was like that. I would now probably make up matrices.

Another student used (1×2) matrices and ignored the part of the task that specified $n \times n$ matrices. Not only was his multiplication of matrices incorrect, but he also made two multiplication errors.

Like the *high-entry* students, two *low-entry* students were comfortable with variables. However, many of the *low-entry* students experienced conceptual problems with abstract variables due to their reliance on reproductive understanding. *Low-entry* students stated in their interviews that although their solution was probably incorrect they blindly attempted the more abstract task as this often gave them more marks in tests. The students remembered seeing 'this sort of thing' in textbooks and lectures and tried to recall parts of it. One student wrote three lines:

$$D D^{-1} = I$$

$$D = C B^{-1} A$$

$$D^{-1} = A^{-1} B C^{-1}$$

- I remembered I tried to learn off by heart this type of question. When I got right into it it just seemed to sort itself out. I didn't use this (information presented) as I couldn't see how they were related. It (the working) might not be strictly right, but it seemed easy.

A second student demonstrated difficulty with understanding exponentials.

He believed that A^{-1} was the result of multiplying matrix A by (-1). The following statement tends to confirm that the *low-entry* student was at Perry's (1970) stage of *dualism*.

- The lecturer referred to A inverse as being associated with negative one, and it simply stuck in my mind. I wouldn't have a clue on how to do anything else.

The *high-entry* students used variables for Task 2(d) and did not consider substituting examples into the general matrices. They could distinguish proofs from examples and their exposure to proofs in lectures gave them the advantage in this task over *low-entry* students.

- I didn't think of using examples. Although it can be helpful to use examples, it wasn't proving it.

5 ADVANCE PLANNING

Neither *low-entry* nor *high-entry* students actively planned a strategy to solve a task.

An interesting aspect highlighted in the analysis of Task 1(c) was that only one *low-entry* student and four *high-entry* students actively wrote out a plan on how they would solve the task. All the students in the research found this task extremely difficult. If a plan was written by the students, it was done because of the task requirement. The reasons for not planning in advance differed within the *low-entry* group and between the two groups of students. For example, one *low-entry* student had a different interpretation of the question:

- Because usually our maths teacher, when she said 'describe' she meant write it down in an equation form rather than write it verbally.

Other *low-entry* students did not write a plan because much of their strategy relied on random guess work.

- I fiddled around a bit and just substituted x into the line.
- I wasn't really sure, I just tried everything.
- I thought I would graph that to see what it would look like, so I just took a few point, then did the other graph as well.

One student said she planned this task mentally. However her comments pointed to a default strategy used by a number of the other *low-entry* students. That is, to initially substitute any given values of x into any given equations without understanding what is being calculated.

- Well, I sort of plan it out first, but I don't write it down. I just put $x=-6$ into there and $x=4$ into there. You just find the answer to y . But I started to differentiate this and got down to this... then I thought I was wrong, because that's differentiating that one (curve) and not that one (tangent line). I know I've got to differentiate but I don't know why.

The only *low-entry* student who did write a plan later made a computational error and abandoned the calculation stating that his plan did not work.

- I tried to write out a plan, but then I found it didn't work out that way. If it didn't work for -6 then it probably wouldn't work for 4.

Five *high-entry* students wrote a plan first but felt it was not natural for them.

- I did a mental think and wrote down all the steps as if I was being marked on it. It was quite hard. It was just the writing what you are doing instead of thinking as you are doing it. I would have done the mathematics instead of thinking about what to do. I knew where I was going and I don't actually plan it step by step, but I work it out a bit at a time.

This aversion to planning in advance was reinforced by other *high-entry* students who ignored the request for a written statement of intent.

- It said to describe how you would do it. I'd be more likely to just go and do it. In really thinking about what I was doing I became confused.

These students who ignored the request for a plan found it easier to go straight into the question.

- I read the whole question. I thought it would be easier to do it first than to figure out how I was going to do it. I get things worked out but with numbers before I could work out exactly what it meant.
- I try it first and then I write it down. I really don't know what works until I try it.

Each of these *high-entry* students mentioned a vague indescribable intuitive feeling of knowing what to do but they could not describe how they knew. The students also found that details did not become clear until after they commenced the task, or in their words they:

- ..did the mathematics.

Both groups of students used guesswork in selecting strategies to solve the tasks. For the *high-entry* students this guesswork was so accurate it appeared to

be directed by this structured intuitive knowledge. *Low-entry* students often used guesswork that at times had no connection to the question, or was based on vague recall of information indicative of reproductive understanding. For example, comments included:

- I knew what the question was sort of getting at, but I couldn't remember what the actual things were, so I just put... I knew it had something to do with zero being either negative or non-negative, so I just put an infinite number of solutions for any natural number of k . For one solution, I put $k=0$. I'm not sure why. I just put that anyway.

The inaccurate random guesswork and reliance on recall of isolated knowledge by *low-entry* students points to an absence of this intuitive knowledge in tasks that require an extension of knowledge or application to a new situation.

Most of the *high-entry* students mentioned having to be taught by their teachers on how to write and organise their problem solving solutions. The *low-entry* students stated they had some recollection of being taught these skills but felt it did not mean very much. Although most of *low-entry* students did not show any sign of intuitive knowledge defined as stemming from a highly integrated structured knowledge, one *low-entry* student displayed signs of partially developed intuitive knowledge by the way he reasoned through Task 1(b). However, further investigation of this apparent atypical *low-entry* student showed that he was not confident of his answers and would tend to abandon a task if it did not work the first time. Unlike the *high-entry* students, this student did not consider alternative methods. In addition, although the student could reason just as well as the *high-entry* students he often left partial solutions in isolation.

CHAPTER VI

VI. SUMMARY AND DISCUSSION

1. SUMMARY

The qualitative approach used to compare the problem solving skills of *high entry* and *low-entry* first year university students highlighted major differences between the two groups. Although this case study approach involved a small number of participants (nine and 12 students respectively), the analysis directed the researcher to areas of further investigation.

(1) Interviews

Initially major categories that emerged from analysis of interviews were metacognitive where students gave self-assessments on their mathematical ability, attitudes to mathematics, motivation to learn, sensitivity to transition factors and approach to learning. Students based their assessment of their own mathematical abilities on support and school experiences. In particular they commented on the influence of teacher assessment and success or failure in examinations.

High-entry students spent most of their secondary mathematics education as one of the best students in a top stream and modestly assessed their abilities as above average to reasonably strong. They possessed positive attitudes to mathematics reinforced by teacher encouragement or attainment of high examination marks. In the transition to university study, these students were not highly sensitive to institutional factors but displayed initial concern with

understanding mathematical abstract concepts and proofs. In the approaches to learning mathematics, the *high-entry* students aimed at achievement-orientated transformative understanding rather than *deep* understanding of fundamental topics as defined by Marton. This would equate with Biggs' (1987) 'deep-achieving' approach. Cox's (1994) analysis of first year university tests on retention of core topics found that

- ..even the best students appear to optimize their performance by strategic learning. (p. 11)

The *high-entry* students in this study used textbooks, peers and tutors to assist with their understanding of topics.

In contrast, *low-entry* students gauged their ability as either average or below average. These students felt their Seventh Form secondary year did more damage to their self-esteem and self-assessment of ability than anything. With an examination failure in mathematics the previous year, these *low-entry* students exhibited a negative attitude to mathematics on reaching university. They enrolled in university mathematics either to fulfil a compulsory prerequisite for another discipline or because they were influenced by older students stating that university mathematics was easier than secondary Seventh Form mathematics. The *low-entry* students were also more sensitive to institutional transition factors, especially speed of lectures and intimidation in large impersonal classes. In their approach to learning they demonstrated a strategic approach (Cox 1994) that was more *surface*-orientated (Biggs's model Figure 1). This often led to both rote and reproductive understanding (Bain 1994). Evidence for this learning approach lay in the student comments that many intended to learn basic topics by repeatedly performing plenty of similar examples.

(2) Task and interviews combined

The five categories that emerged from analysis based on a combination of tasks and interviews were *strategy*, *structure*, *monitoring progress*, *content* and *advance planning* (Table 9). Although strategy, progress and content relate to metacognitive knowledge (Biggs 1987b) the emphasis in this study was broader than metacognitive knowledge as it also extracted data from worked solutions.

	LOW-ENTRY STUDENTS	HIGH-ENTRY STUDENTS
STRATEGY	Relied on recall of recipes and formulas.	Recall of recipes and formulas only where appropriate.
	Did not consider alternative strategies.	Considered alternative strategies but own preference selected.
STRUCTURE	Unstructured, inconsistent working.	Systematic, efficient, clear working.
	Ignored information given.	Rarely ignored information given.
	Partial solutions not connected.	Partial solutions connected into whole.
	Easily abandoned work; intention to return later .	Persevered; only abandoned work after trying alternatives.
PROGRESS MONITORING	Knew approach was incorrect. Rarely checked work in progress. Accepted any answer.	Knew approach was correct. Prompt corrections. Felt answers were correct without actively checking.
CONTENT	Not well developed.	Well connected and interlinked.
	Variables and abstract concepts ignored.	Comfortable with variables and abstract concepts.
	Interpretation difficulty.	Little difficulty with interpretation.
ADVANCE PLANNING	Did not plan, rarely knew where to begin.	Did not consciously plan; intuitive feel for the problem.
	Guesswork random.	Guesswork directed.

Table 9: A comparison of characteristics between low-entry and high-entry students.

As a group, *low-entry* students exhibited a greater variety of strategies in each part of the tasks than the *high-entry* students. Individually, the choice of strategies from which the *low-entry* students could select was limited. They proved that there were more ways to be incorrect than to be correct. Many of the *low-entry* strategies were random and reliant on recall of recipes and formulas. Occasionally students even attempted to fit the task to the formula rather than the reverse and this invariably resulted in the use of inappropriate formulas. In addition, *low-entry* students consistently employed inefficient strategies that often resulted in incomplete or incorrect solutions. In contrast, *high-entry* students could choose from a variety of strategies and their eventual selection was based on personal preference. The *high-entry* students also dealt equally well with all types of tasks whether they were recipe-orientated, extensions of knowledge or manipulation of variables.

Analysis of the structure of student work pointed to major differences between the two groups of students. All *low-entry* students displayed unstructured, inconsistent working that was often scattered and incomplete. Mathematical statements were too often left unfinished or incorrect and students rarely used phrases as part of the working. In contrast, *high-entry* students consistently used systematic, efficient, clear working interspersed with phrases that gave further explanation. However, the *high-entry* students did not give many details and often their written solution was sketchy since they felt the work was obvious and did not require too much detailed written explanation. Several *high-entry* students admitted that they had considerable guidance from secondary school teachers in learning to express their knowledge in a coherent form.

Low-entry students often ignored relevant information such as conditions or hints given in a task. They felt that this type of information only confused the task. In contrast, the *high-entry* students were aware of all conditions which they gave as top priority. A major characteristic of *low-entry* students was the unfinished isolated partial solutions scattered throughout the work. The students did not collect these solutions into a whole solution and many did not understand what they had calculated. *High-entry* students, on the other hand, usually connected partial solutions together with a conclusion at the end of each task.

The *low-entry* group experienced difficulty in interpreting solutions and often ignored abstract concepts and variables. They also persevered with details in 'recipe-type' questions but often abandoned other types of tasks. For tasks considered particularly difficult, the *low-entry* students would write an abridged form of the question before abandoning the work. For most abandoned tasks, *low-entry* students left more than adequate space with the intention of returning later, but never did so. In contrast, the *high-entry* students usually persevered using alternative ways to approach the task. They did not abandon the task easily and experienced little difficulty interpreting solutions. They were equally comfortable using abstract concepts and variables and knew which was appropriate for the tasks.

There were three areas in which self-checking could occur. These were the approach to the problem, progress through the solution and the final answer. Most of the *low-entry* students stated they were probably doing the task the wrong way but could not think of any other way to approach the problem. Their

comments confirmed a lack of confidence in their mathematical ability based on past experience. These students rarely checked their work. Computational errors, if recognised by these students, were methodically traced back and overwritten on the existing calculations. Any answers that emerged from a calculation were accepted as the final answer. For the *high-entry* students the choice of strategy was obvious and any errors were corrected soon after they were made. With computational errors, *high-entry* students consistently chose to repeat the calculation rather than correct the existing solution.

Initially it seemed that a characteristic in common with *high-entry* and *low-entry* students was that neither group tended to check their answers to the tasks. However, it became apparent from the interviews that there was a difference. While the *high-entry* students felt they did not need to check answers because the answer usually 'felt right', the *low-entry* students often did not know how to check their final answers.

Neither of the two groups actively planned a strategy before they solved a problem. For the *low-entry* students this was because they rarely knew where to begin and resorted to random guesswork to see if they could hit the 'correct' strategy. For the *high-entry* students, advance planning was not considered natural. Many reported that they knew what to do but could not describe how they knew. There was evidence of intuitive knowledge that gave a vague overview without details. These details came naturally as the *high-entry* students worked on the task. Any guesswork was directed by this intuition.

2. DISCUSSION

Many of the *low-entry* students left large gaps in a number of the questions. These gaps did not necessarily indicate lack of knowledge. Although incorrect learning was evident at times, pockets of knowledge often existed correctly but in isolation. By asking a series of appropriate questions the researcher could often help the student access these knowledge pockets. However, this access did not necessarily mean that the student then understood the work. At the interviews, the *low-entry* students could not redo the task using a slightly different but similar problem.

This study confirmed that for *low-entry* students many of the mathematical concepts learned at secondary school and in the first term at university were still at early levels of understanding, namely reproductive (Bain 1994) or operational stages (Sfard 1991). Although there was some evidence of erroneous procedural knowledge and algebraic manipulation problems (Orton 1983a, 1983b) the basic difficulty displayed by the *low-entry* students was a lack of conceptual understanding and intuitive knowledge. The students had not actively transformed or assimilated most of the concepts into unique structures where they could 'see' ideas as a whole. The investigation also confirms that the *low-entry* students consistently used a *surface* approach to learning by relying on repetition of similar questions accompanied by little understanding of the concepts. This approach subsequently lead to the acquisition of rote or reproductive understanding, reinforced the isolation of knowledge pockets and lead to the inability of students to deal with variables (generalisation) or extension of knowledge. This isolation of knowledge and reliance on operational rather than structural conceptual learning may account for the random guesswork

students used in most parts of the tasks. It could also explain the non-systematic way students answered questions or ignored relevant information. The isolated knowledge pockets could also explain why the students selected inappropriate techniques such as calculating the second derivative when the task required the student to find a tangent.

In contrast, the *high-entry* students often elaborated on written solutions by giving detailed explanations of both relevant and appropriate related knowledge. This related information indicated extended knowledge about the topic often found in verbal explanations rather than in written solutions. Visualisation was used as an aid rather than a strategy (Moses, 1980). The students appeared to have developed the ability to transform information into a tightly interconnected mesh of knowledge. Retrieval of the knowledge for each task was not only relevant, but accurate in detail. New concepts were assimilated and transformed into existing knowledge as these students coped equally well with the new work. This strongly interconnected knowledge meant that students could easily recall alternative methods, many of their decisions and methods were relevant to the task and they could extend their knowledge to unfamiliar situations. This interconnection of knowledge and formation of mathematical concepts as structural entities also meant the *high-entry* students were equally comfortable switching from numbers to variables and vice versa. In addition, these *high-entry* students developed much of the work in their heads rather than in writing. They could 'see' what to do but could not explain how or why they knew what to do. They had developed an intuitive knowledge for mathematical concepts.

With some adjustment the findings in this study could be incorporated into Biggs's elaborated model of learning (Figure 1).

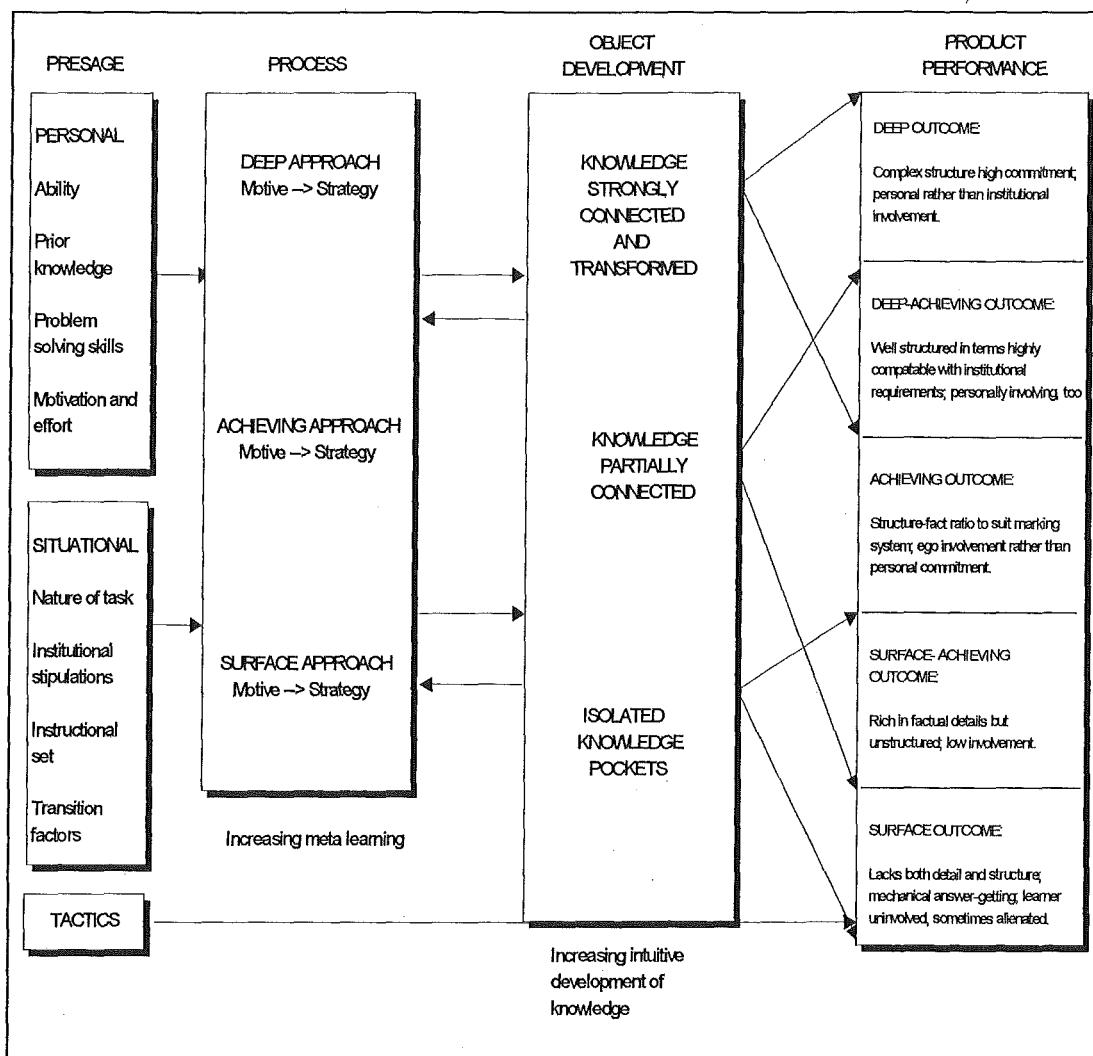


Figure 12: Adjustment to Bigg's model (1987a) depicted in Figure 1.

An additional feature, labelled 'object development' (Entwistle and Marton 1994), was added to the model to include the isolated knowledge pockets reinforced by a surface-orientated approach and reproductive understanding displayed by *low-entry* students. Also included is the highly interconnected intuitive knowledge reinforced by the *deep-achieving* approach and transformative understanding displayed by *high-entry* students. Since all students within each group varied

slightly in their degree of interconnected or isolated knowledge and this appeared to be task specific, a continuous scale of knowledge object development seemed suitable. In this context, intuitive knowledge could exist at a deeper subconscious level without the student being conscious of the details necessary to solve the problem. Once this knowledge object or structural entity was brought forward to the conscious, the details could be filled in by highly interlinked transformative understanding. Absence or inadequate development of these knowledge objects could account for the random guesswork displayed by *low-entry* students.

The link between process and object development should be reversible. *Low-entry* students at university with already established isolated knowledge are likely to find themselves committed to a *surface* learning and problem solving approach leading to rote or reproductive understanding of topics. *High-entry* students at university, who have already acquired a broad highly interconnected range of knowledge and skills, are likely to continue learning new concepts by transforming their knowledge and seeing concepts as abstract entities. However, for these students reproductive understanding could be also an option if deemed appropriate for gaining higher marks.

The analysis in this thesis did not conflict with the research cited earlier. Any conflict would lie in the assumption that all students can adopt all of the approaches to learning and understanding in mathematics to any extent. At an earlier stage in mathematical concept understanding, students can usually solve problems that are a repeat of the work they have just been taught. However, to acquire skills in solving problems for unfamiliar situations or extensions of

knowledge, an intuitive approach may be needed. Considering the strong connection found between approaches to learning and the level of isolated/connected knowledge, it seemed that for *low-entry* first year mathematics students the knowledge mesh and structures needed to solve 'non-recipe' problems intuitively would be a long difficult route requiring a lot of motivation. In this study the *low-entry* students did not exhibit the interest or motivation to achieve this goal. Many researchers have proposed ideas to motivate students to think more abstractly in problem solving situations (e.g. Sweller et al. 1983, Sewller 1989, Ward and Sweller 1990, Schoenfeld, 1992) but their emphasis has not been on the development of intuitive knowledge.

There is a major assumption here that students with a lower level of understanding, who employed repetitive learning or surface approaches to learning, can be helped to develop intuitive knowledge. Contrasting examples such as those outlined in this study, point to distinct differences between two extreme groups of students. The dissimilarities bear a striking resemblance to the literature on the differences between novices and experts. The question could now be whether these differences would always exist even if the two groups theoretically exhibited similar motivation, effort and prior knowledge. That is, do the more able students possess something more than just highly developed skills? This study showed that the able students possessed intuitive knowledge, but why did they, and not the other students, have this knowledge? Were the more able students faster, more efficient and more highly motivated learners or is there some fundamental aptitude that they possess which is absent in the other students? Should we, as teachers motivate, encourage and develop this intuitive knowledge when the students would never be able to achieve it anyway?

If it was a clear case of the more able mathematical students being good all-rounders and the contrasting students being below average at everything, then perhaps a general aptitude could be considered a major factor. Even though this aspect was not pursued in this study, the more able students did appear to pick up ideas faster and more efficiently. However, it needs to be noted that the study techniques of the other students did not foster development of these skills.

The author has recorded the occasional examples of students who failed with their first attempt in a mathematics course, but who returned three years later to top the class in the same course. Is maturity a factor for some students or had these students suddenly acquired this intuitive ability, helped by the prior knowledge and motivation that they now possessed?

Further investigation is needed to determine how this intuitive knowledge is developed and what factors influence its development. Is the acquisition of intuitive knowledge a gradual process or does it involve a sudden 'insight' as proposed by Sfard (1991)? Just how much does this intuitive knowledge contribute to the 'general intelligence' of an individual? Is it possible to develop intuitive knowledge in the classroom and how do we, as teachers go about achieving this?

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APPENDIX I

Questionnaire: Please complete this questionnaire and bring to your interview. You may use the back of this sheet if necessary.

1. Name.....
2. Gender..... 3. Age.....
4. Type of secondary school attended (co-ed, private, etc).....
5. Part-time or full-time university student?.....
6. What other study (i.e. courses) are you doing this year?.....
.....
7. What area(s) do you intend to major in at university?.....
8. How many hours per week do you spend on each of you mathematics paper(s)? (Please elaborate on how this time is allotted.)
.....
.....
9. How would you describe your ability in mathematics?
.....
.....
10. Are there any aspect of mathematics (topics, exams, study, etc) that you find particularly difficult? Please elaborate.
.....
.....
11. What are your strongest areas in mathematics? Give reasons.
.....
.....
12. Have you received any particular encouragement or discouragement in your study of mathematic either at school or university? Briefly outline.
.....
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