where $\mathbf{R}_S = \mathbf{r}_S - \mathbf{r}$ and the integrals I_1, I_2, and I_9 are given by

$$I_1 = \int_0^1 h\left(\frac{k^2}{f^3} - \frac{3jk}{f^4} - \frac{3}{f^5}\right)\tau\, d\tau \tag{27}$$

$$I_2 = \int_0^1 h\left(\frac{k^2}{f^3} - \frac{3jk}{f^4} - \frac{3}{f^5}\right) d\tau \tag{28}$$

$$I_9 = \int_0^1 h\left(\frac{-jk}{f^2} - \frac{1}{f^3}\right) d\tau \tag{29}$$

with $f = f(\tau) = |\tau\mathbf{R} - \mathbf{R}_S|$ and $h = h(\tau) = \exp(-jk(\tau R + f))$. The results of the analytical evaluations of these integrals are given in [7, (51), (56), (61)]. Inserting these expressions for the integrals into (26), the final expression in (14) is obtained.

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Polynomial Approximations to Bessel Functions

R. P. Millane and J. L. Eads

***Abstract*—A polynomial approximation to Bessel functions that arises from an electromagnetic scattering problem is examined. The approximation is extended to Bessel functions of any integer order, and the relationship to the Taylor series is derived. Numerical calculations show that the polynomial approximation and the Taylor series truncated to the same order have similar accuracies.**

***Index Terms*—Approximation, Bessel functions, polynomial.**

I. Introduction

Bessel functions appear in numerous physical problems, and play an important role in many electromagnetic scattering problems. There is no closed form expression for Bessel functions so that approximations suitable for numerical evaluation are necessary in applications. Gross [1] has derived interesting polynomial approximations to the zeroth- and first-order Bessel functions of the first kind for small arguments, that arise from an integral that occurs in an electromagnetic scattering problem. We study here in detail properties of these approximations. First we extend the analysis in [1] to derive corresponding polynomial approximations for Bessel functions of any integer order. Second we show that as the degree of the polynomial approximation increases, it converges to the Taylor series expansion. Third we compare the accuracy of the polynomial approximations to that of the truncated Taylor series of the same order.

II. Background

Gross [1] begins by considering the integral

$$f_{2n}(k) = \frac{2}{\pi}\int_0^\delta \frac{\cos x \cos(2nx)}{\sqrt{k^2 - \sin^2 x}} dx, \qquad 0 \le k \le 1 \tag{1}$$

where

$$k = \sin\delta, \qquad 0 \le \delta \le \frac{\pi}{2} \tag{2}$$

that occurs in the expression for the current density on a conducting strip grating illuminated by a plane electromagnetic wave [2]. The integral is evaluated as [1]

$$f_{2n}(k) = \sum_{m=0}^{n} b_m k^{2n-2m} \tag{3}$$

where

$$b_m = \frac{n(-1)^{m+n}(2n-m-1)!2^{2n-2m}}{m!\,((2n-2m)!!)^2} \tag{4}$$

and $(2n)!! = 2 \cdot 4 \cdot 6 \cdots 2n$. Making the substitution $x = w\delta$ in (1) gives

$$f_{2n}(\sin\delta) = \frac{2}{\pi}\int_0^1 \frac{\cos(\delta w)\cos(2n\delta w)}{\sqrt{\sin^2\delta - \sin^2(\delta w)}} \delta dw \tag{5}$$

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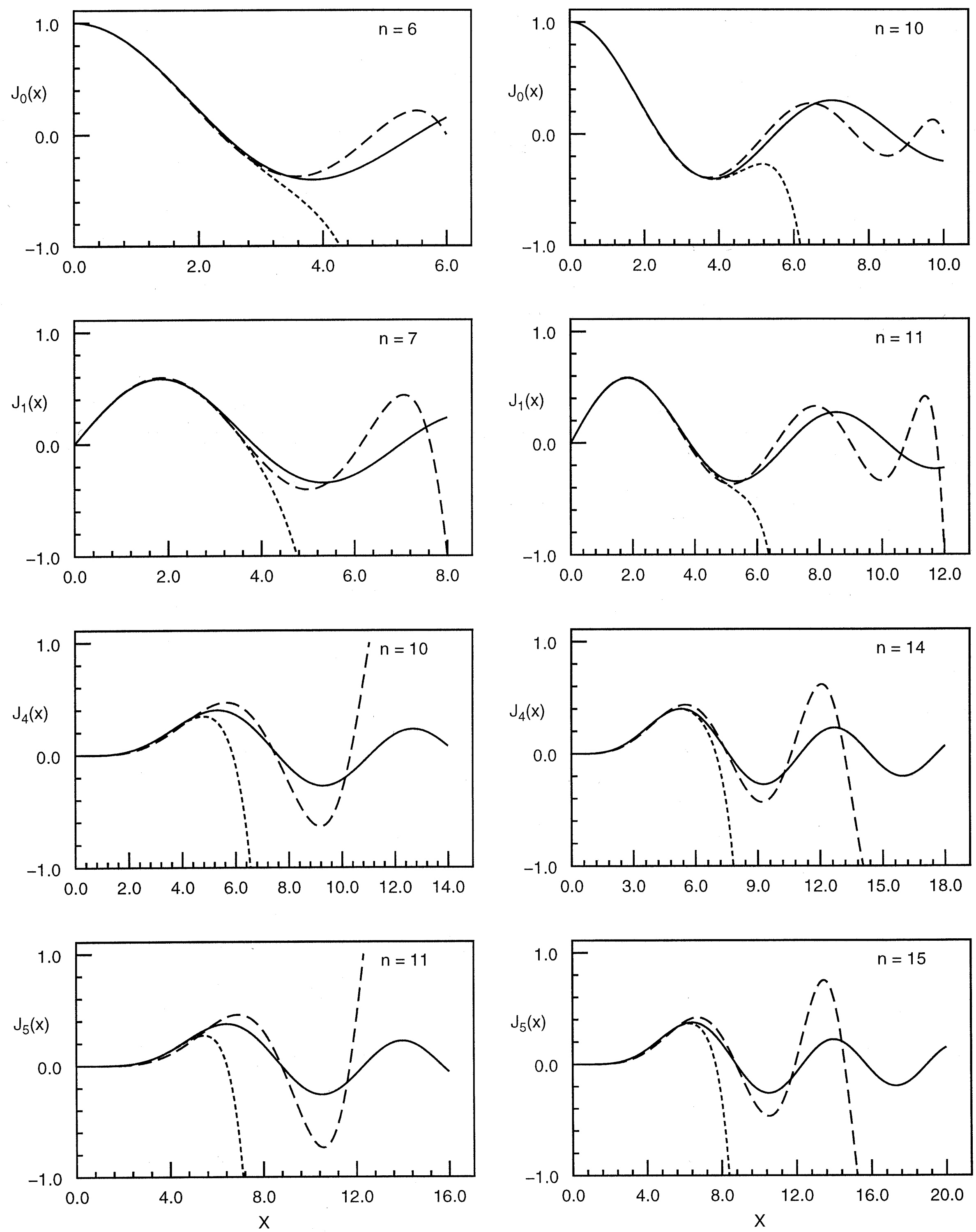

Fig. 1. Bessel functions $J_p(x)$(———), the polynomial approximations $J_p^{G,n}(x)$(- - -), and the truncated Taylor series $J_p^{T,n}(x)(\cdots)$, on the interval $0 < x < n + p$ and for orders x^n as shown.

and taking the limit $\delta \to 0$ on the left- and right-hand sides of (5) and using (2) shows that for large n

$$f_{2n}(\delta) \simeq \frac{2}{\pi} \int_0^1 \frac{\cos(2n\delta w)}{\sqrt{1-w^2}} dw. \tag{6}$$

Using the integral form of the Bessel functions [3] then shows that

$$J_0(x) \simeq f_{2n}\left(\frac{x}{2n}\right) \qquad 0 \le x \le 2n \tag{7}$$

the approximation improving with increasing n. Using (3) and (4), (7) provides a polynomial approximation to $J_0(x)$. Note that the domain on which the approximation (7) is defined increases with increasing n. Using (7) and the properties of Bessel functions also gives the approximation

$$J_1(x) \simeq -\frac{d}{dx} f_{2n}\left(\frac{x}{2n}\right) \tag{8}$$

on the same domain. Numerical evaluation of (7) and (8) for $2n = 10$ and 20, and 0< x <5, shows that they are substantially more accurate than the first term of the Taylor series for the Bessel functions [1].

III. Approximations for Any Bessel Order

The approach described above can be extended to any Bessel order as follows. Substituting (3) into (7), rearranging the summation, and using (4) allows the approximation (7) it to be written as

$$J_0(x) \simeq \sum_{m=0}^{n} c_{nm} x^{2m}, \qquad 0 \le x \le 2n \tag{9}$$

where

$$c_{nm} = \frac{(-1)^m n^{1-2m}(n+m-1)!}{2^{2m}(n-m)!(m!)^2}. \tag{10}$$

Applying the relationship [3]

$$J_{p+1}(x) = -x^p \frac{d}{dx}\left(x^{-p} J_p(x)\right) \tag{11}$$

recursively to (9) gives

$$J_p(x) \simeq \sum_{m=p}^{n} d_{pnm} x^{2m-p}, \qquad 0 \le x \le 2n \tag{12}$$

where

$$d_{pnm} = \frac{(-1)^{p+m} 2^{p-2m} n^{1-2m}(n+m-1)!}{(m-p)!(n-m)!m!}. \tag{13}$$

We denote by $J_p^{G,n}(x)$ the approximation (12) for which the highest order is x^n, i.e.

$$J_p^{G,n}(x) = \sum_{m=p}^{\frac{n+p}{2}} d_{p,\frac{n+p}{2},m} x^{2m-p} \qquad 0 \le x \le n+p. \tag{14}$$

IV. Relationship to the Taylor Series

The Taylor series for $J_p(x)$ is [3]

$$J_p(x) = \sum_{m=0}^{\infty} t_{pm} x^{2m+p} \tag{15}$$

where

$$t_{pm} = \frac{(-1)^m}{2^{2m+p} m!(m+p)!}. \tag{16}$$

We denote the Taylor series truncated to order x^n by $J_p^{T,n}(x)$ and, to ease comparison with (14), write it as

$$J_p^{T,n}(x) = \sum_{m=p}^{\frac{n+p}{2}} t_{p,m-p} x^{2m-p}. \tag{17}$$

We show that as the order of the polynomial approximation (14) increases, it approaches the Taylor series on a term-by-term basis, i.e.

$$\lim_{n\to\infty} J_p^{G,n}(x) = J_p^{T,n}(x) \qquad 0 \le x < \infty. \tag{18}$$

Referring to (14) and (17) shows that (18) is equivalent to

$$\lim_{n\to\infty} e_{pnm} = 1 \tag{19}$$

where

$$e_{pnm} = \frac{d_{p,\frac{n+p}{2},m}}{t_{p,m-p}}. \tag{20}$$

Using, (13) and (16), changing the factorials to gamma functions and simplifying, shows that

$$e_{pnm} = \frac{2^{2m-1}(n+p)^{1-2m}\Gamma\left(\frac{n+p}{2}+m\right)}{\Gamma\left(\frac{n+p}{2}-m+1\right)}. \tag{21}$$

Using (21) and the relationship [3] $\lim_{x\to\infty}\Gamma(x+a) = \Gamma(x)x^a$ for $x \in \mathbb{R}^+$ shows that e_{pnm} satisfies (19), confirming (18). The polynomial approximation therefore approaches the Taylor series, on a term-by-term basis.

V. Numerical Comparison With the Taylor Series

The polynomial and truncated Taylor series approximations of the same order were calculated, on the domain for which the former is defined, using (14) and (17), respectively, and compared with the actual values of the Bessel functions. All of the approximations were evaluated numerically using Horner's method [4]. The results of these calculations are presented in Fig. 1 for Bessel functions of order $p =$ 0, 1, 4, and 5, and for two different orders n of the approximations that correspond to 4 and 6 terms in the series. Inspection of the figure shows consistent behavior in the relative accuracies of the two approximations. The truncated Taylor series approximation $J_p^{T,n}(x)$ is slightly more accurate than the polynomial approximation $J_p^{G,n}(x)$ in the region $0 \le x \lesssim n/2$. The Taylor series diverges rapidly from the true value of $J_p(x)$ for $x \gtrsim n/2$. The polynomial approximation is more accurate than the Taylor series for $n/2 \lesssim x \lesssim n$, although it is probably not usefully accurate in this region. For $x \gtrsim n$ the polynomial approximation also diverges rapidly from the Bessel function.

VI. Conclusion

The polynomial approximations [1] have been extended to Bessel functions of any integer order, and they approach the Taylor series expansion as the order of the polynomial increases. Although the polynomial approximation [1] has integer coefficients, both the polynomial approximation and the Taylor series have good numerical stability if they are evaluated using standard numerical methods. Comparison of the accuracies of the two approximations (of identical orders) shows that the polynomial approximation has no practical advantage over the Taylor series, and that such approximations must be used with caution.

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