

Expounding the Mathematical Seed: A translation of Bhāskara I on the Mathematical Chapter of the Āryabhaṭīya

By Agathe Keller. Basel, Boston, Berlin (Birkhäuser). 2006. ISBN: 978-3-7643-7299-6. 2 vols. 412 pp. \$169.00

Agathe Keller's latest publication, *Expounding the Mathematical Seed*, is an ambitious two-volume work which provides a translation and detailed commentary on a mathematical text composed in the 7th century by the Indian mathematician Bhāskara I, the content of which is itself a commentary that elucidates and clarifies the mathematical section of a 5th-century work by Indian mathematician Āryabhaṭa. Cleverly and perhaps cryptically selected, Keller's title immediately evokes Bhāskara's own sentiment (vol. I, p. xli) that the rules given in a mathematical work are “seeds” (*bija*) that a commentator must bring to light. Perhaps the title is also intended to play somewhat humorously upon the fact that the word *bija* in some mathematical contexts can also refer to computations with “unknowns” (*bijaganita*). Keller is then insinuating, even at the very outset, at the sort of hermeneutic nightmare for the commentator, who attempts to describe on the one hand ‘a rule’, but on the other hand what is indeed ‘unknown’!

Play upon words, extended metaphors, and double-entendre are no strangers even to mathematical compositions in the Indian tradition and these are but a handful of the challenges faced by Keller in her task of translating, explaining, and contextualizing the mathematical content and circumstances of this work. It may seem surprising to the reader unfamiliar with the Sanskrit mathematical tradition that a mere thirty-three mathematical ‘verses’ could generate so much attention, not only from ancient scholars, but modern ones too. Indeed, in addition to the translation and clarification of the mathematical content, perhaps the most difficult tasks for the historian are the investigation of the mathematical results themselves and the examination of the possible ways in which the mathematician arrived at the results that he has presented. For Indian mathematicians were seldom inclined to reveal how they had arrived at their rules, nor did they offer any justifications for why they believed the result in question to be valid.

To this end, to assist in the comprehension of the text, numerous commentaries were composed. These, in contrast to the base text, were almost always in prose. The commentary was intended to enlighten the reader as to the contents of the base text and would attempt to do so by any of the following techniques: restating the rule in prose, unravelling the grammatical complexities, offering synonyms for obscure or ambiguous words, deriving the parameters, giving a worked example, or quoting other texts with relevant material, to name a few. However, in practice, the existence of a commentary is not always helpful in clearing up the difficulties presented in the verses. Many a commentator had little interest in explaining the theoretical bases, the rules, or deriving the formulae. In fact, most commonly, they would simply provide an obvious restatement of the rules and perhaps in addition give a trivial example to show that the rule worked. Many failed in even this task, perhaps a testament to the obscurity of the text, or else an insight into the somewhat questionable abilities of the commentator himself. However, some commentaries are of an exceptionally high quality, and greatly clarify the text; without these, modern historians would have been at a loss to situate and explain the meanings of the verses.

Bhāskara is one of the latter commentators, who comments thoroughly and expertly on Āryabhaṭa’s original text. Keller’s work is extremely important because it makes this commentary available to a wider modern audience. As her detailed analysis shows, in addition to the sound mathematical content, there are also ambiguities and inconsistencies in Āryabhaṭa’s original text as well as explanations, worked examples, references, claims, counter-claims, corrections, and more in Bhāskara’s commentary. Indeed, Keller’s commentary could be viewed as a ‘hyper-commentary’ on Bhāskara’s commentary, very much in the spirit of continuing the ancient tradition. The task of translating and commenting is thus an enormous undertaking. The result is commendable; neither novice nor experienced readers of Sanskrit mathematical texts will come away from her book disappointed.

Keller has sensibly split her material into two separate volumes. The first contains a detailed introduction and translation of Āryabhaṭa’s original text accompanied by Bhāskara’s commentary. Visually, the layout is very well-defined; Āryabhaṭa’s original text is translated in boldface with Bhāskara’s commentary interwoven in usual regular type, and Keller has adopted various other editorial devices to demarcate the material where appropriate. Readers will find the introduction most informative and useful. The second volume contains the mathematical supplements for the translation presented in the first volume with helpful diagrams, formulae, useful glossaries containing Sanskrit words and their translations, and other vital information. One only wishes that such a conscientious individual had put her hand to an additional third volume that contained a critical edition of the text to increase the appeal of this book to indologists, philologists, and other relevant specialists. To this end though, she has added the Sanskrit transliteration in brackets for technical and significant terms, so that the reader can be informed as to which Sanskrit word is being referred to. In general, it is a beautifully formatted book with just a few distracting proof-reading issues.¹

Appropriately, Keller carefully situates the commentary in its historical and mathematical context and describes the intellectual climate, heavily emphasizing the oral tradition in which such a text was composed and revealing some of the reasons why the mathematics contained within the commentary and the original text has the particular features it does. She highlights significant arithmetical and geometrical features of Bhāskara’s work, and in particular, her expertise in the visual diagrams included as part of the treatise gives the reader a glimpse of the way in which particular mathematical concepts may have been originally conceived and visualized. She makes links with mathematics and astronomy, as well as detailing the ‘tools’ of the trade with her fascinating look at the various descriptions of a mathematical compass (vol. II, pp. 75–78). Mathematicians will be particularly interested by the oldest testimony of the sine derivation in India (vol. I, p. xxx), the accurate approximation to π , the algorithms for the computation of squares, cubes and their roots, and the indeterminate analysis, to name but a few.

What is especially well done is Keller’s analysis of the role of commentaries themselves, which is contained within the introduction. She gives a summary of the techniques and tools of a

¹Some of these include: the retroflex ś has not appeared properly on many occasions (for example, vol. I, p. xxxiv, line 7, should read *rāśi*; vol. II, p. 92, should read *Sampātaśarau*, and elsewhere); vol. II, p. 48, “textbf” appears in translation and as a result the translation is not in bold; p. 36, line 5, “whith” should presumably be “with”; vol. II, p. 37, the final ‘a’ in Āryabhaṭa should be in boldface; vol. I, p. 3, line 2, the Sanskrit should read *astadhā*.

commentator and at pertinent points pauses to reflect on relevant questions raised. Her questions all point to the dearth of scholarship on the history of the tradition of the commentary and she notes the lack of any systematic study of commentaries themselves both within mathematical tradition and also the broader Indian intellectual traditions. Beyond their attempts to explain the texts, commentaries, both good and bad, are valuable for a number of reasons. They reveal to the historian some insight into the ways in which mathematicians responded to mathematical problems and sometimes uncover the methods by which the material was presented and explained to students. Importantly, they also indicate the status of a text in the commentator's generation and give a glimpse into how the practitioners thought about mathematics—perhaps the closest thing to a consideration of the more philosophical aspects of mathematics. In this respect, Keller notes that Bhāskara as a commentator is significant because, in addition to the usual features of a commentary, he attempts to set out what a good commentary should contain (vol. I, p. xlvi) and gives an evaluation of what a good rule is (vol. I, p. xli).

There are many points of mathematical interest in this text—too many to highlight here, but a few will be mentioned. Keller has done a remarkable job of unravelling rules and procedures which initially seem quite impenetrable, but indeed some of the text still refuses to relinquish its mysterious contents! A particularly noticeable inconsistency in Āryabhaṭa's text is his rule for determining the volume of a “pyramid”—or at least the volume of a solid which has “six edges” (*śadasṛī* – a bahuvrīhi compound) (vol. I, pp. 30–31, vol. II, pp. 27–28), given as:

$$V = \frac{1}{2}A \times ah$$

where A is the area of the base and ah is the perpendicular ‘height’ of the six-edged solid. Keller notes that this is incorrectly formulated, commenting that it may be a mistaken assumed continuity between the two dimensional case of a triangle and the three dimensional field that is responsible for Āryabhaṭa's error. Bhāskara seems unconcerned that anything is wrong here, and carefully glosses the words.

The rule for the volume of the sphere (also notably ‘incorrect’) and Bhāskara's subsequent discussion is interesting as it reveals something of the nature of the criteria by which Indian mathematicians judged their results (vol. I, pp. 35–37, vol. II, pp. 32–34). Āryabhaṭa gives the formula for the volume of a sphere as follows:

$$V = A \times \sqrt{A}$$

where A is the area of the circle produced from bisecting the sphere, and Bhāskara follows this up by introducing another relation for the volume of a sphere:

$$V = \frac{9 \times (\frac{D}{2})^3}{2}$$

where D is the diameter. Bhāskara comments that this rule is ‘practical’ (*vyāvahārika*), highlighting that Indian mathematicians did discriminate between the idea of a formula being ‘practical’, i.e. approximate, and one that was ‘exact’ or ‘accurate’. In this case, it seems from Bhāskara's own critique (vol. I, p. 35) that the criteria for an ‘accurate’ result as opposed to a ‘practical’ one is due to the fact that the square root of a number is not always easily expressible. Bhāskara does not mention that in Āryabhaṭa's rule the volume is computed by

means of the area, which is only obtained approximately itself.

Keller includes a table in the introduction, which lists in order the mathematical content of Āryabhaṭa’s rules. It may come as some surprise to many that the order does not necessarily reflect increasing complexity. That is, some material which is seemingly fundamental to other areas comes after them. For example, the value for π is given in verse ten, after several verses which give rules for areas and volumes of regular solids, many of which rely upon the relationship of the diameter to its circumference. Similarly, an expression for the so-called Pythagorean relationship and computations for the inner segments in a circle come five verses after the derivation of sines and sine differences.

One final notable feature is the relation of Āryabhaṭa’s mathematical rules to astronomy. A particular rule Āryabhaṭa gives is to compute the “arrows penetrating the circle” (vol. I, pp. 92–92; vol. II, pp. 105–106), that is, a pair of formulae that give the two ‘amounts’ of the intersection of two circles along their common diameter. Keller notes that Bhāskara interprets this rule as relating to eclipses, where the two circles can be viewed as the ‘object being eclipsed’ and the ‘eclipser’. Indeed, the language used by Āryabhaṭa is reminiscent of the terms used to describe eclipses which commonly reflect various inflections of the root *grah*, (literally, to seize); here he has employed the term *grāsa* to indicate one of the given components in the rule, a term commonly used to represent the magnitude of the eclipse. Keller mentions that the result of this rule may be used to deduce the “the extent of the eclipse” (vol. II, p. 106), where the ‘extent’ is the line segment perpendicular to the *grāsa* connecting the points of intersection of the two circumferences. This is a little unusual as the so-called ‘extent’ bears little relevance to the actual way in which Indian astronomers traditionally measured the magnitude of an eclipse.

Furthermore, the two portions of the *grāsa*, that is, the resulting amounts of Āryabhaṭa’s rules, do not have any apparent astronomical (or astrological) significance or utility either, so it is unclear how or why astronomers would have been interested in this rule when calculating the various parameters associated with eclipses. The magnitude of the eclipse, the *grāsa*, which is the given element in Āryabhaṭa’s rule, is precisely one of the parameters which they must compute. In keeping with the Greek tradition, Indian astronomers would measure the magnitude of the eclipse along the diameter of the eclipsed body, and not perpendicular to this as Keller notes. In fact, the perpendicular cannot be a measure of the magnitude or ‘extent’ of the eclipse as this line segment reaches a maximum at just over half of the disk being obscured and then decreases again, whereas the magnitude of an eclipse increases until the entire disk is obscured.

In *Expounding the Mathematical Seed*, Agathe Keller has produced an exemplary book which epitomizes the level of detail and analysis required to convey the complexity, nuances, and technical aspects of these ancient texts. It will no doubt be frequently consulted and referenced by scholars who prepare translations and commentaries on related texts within the Indian tradition. We look forward to Keller’s future publications with anticipation.

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