Hedging the Value of Waiting*

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Abstract

We analyze the optimal hedging policy of a firm that has flexibility in the timing of investment. Conventional wisdom suggests that hedging adds value by alleviating the underinvestment problem associated with capital market frictions. However, our model shows that hedging also adds value by allowing investment to be delayed in circumstances where the same frictions would cause it to commence prematurely. Thus, hedging can have the paradoxical effect of reducing investment. We also show that greater timing flexibility increases the optimal quantity of hedging, but has a non-monotonic effect on the additional value created by hedging. These results may help explain the empirical findings that investment rates do not differ between hedgers and non-hedgers, and that hedging propensities do not depend on standard measures of growth opportunities.

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Hedging the Value of Waiting

1 Introduction

Firms enhance shareholder wealth by making productive investments. Building on this simple observation, Bessembinder (1991) and Froot, Scharfstein and Stein (1993) show that hedging allows firms to alleviate the under-investment problem associated with imperfect capital markets. When external financing is costly, a shortfall in internal funds can lead to a sub-optimal reduction in investment. By ensuring that internal funds are sufficient to allow profitable investment projects to proceed, hedging adds to firm value.

The models of Bessembinder (1991) and Froot et al. (1993) assume that projects are available at a single future date and thus do not allow for managerial flexibility in the timing of investment. As Dixit and Pindyck (1994) point out, most investment decisions have some degree of timing flexibility, although this may be constrained by factors such as competition and financing uncertainty.[1] In this paper, we examine the role of hedging when the firm's investment opportunities are available for some finite period of time, a simple extension to a dynamic world that has significant implications for the optimal hedging policy. When the timing of investment is flexible, a firm subject to external financing restrictions faces the risk that its ability to finance investment may disappear in the future, perhaps exactly at the time it would wish to invest. Consequently, the firm may decide to invest in states where it has sufficient funds to do so, even when the optimal unconstrained policy would entail delay of investment. By reducing the risk of future funding difficulties, hedging allows the firm to improve the timing of investment; without hedging, the firm might have to rush into investment and sacrifice some of the project's value. Or, to put it another way, hedging adds value not only because it allows investment to occur, as in the "now-or-never" environment of Froot et al., but also because it allows investment to be delayed. Thus, hedging not only permits more investment, but also paradoxically encourages less investment.

The need for hedging in a dynamic world with investment timing flexibility differs from its static counterpart in two contrasting ways. First, by extending the period of time in which investment can occur, timing flexibility increases the probability of the firm accumulating sufficient funds for financing that investment and thus lowers the need for hedging. Second, by creating valuable options that also require protection, timing flexibility increases the need for hedging. These two effects change the sensitivity of hedging demand to the firm's stochastic environment. For example, in the absence of any flexibility, a 1% increase in firm cash flow volatility has approximately the same effect on the optimal hedge as does a 1% fall in the correlation between firm cash flow and project value. With flexibility, however, hedging demand becomes significantly more sensitive to the former.

As well as Bessembinder (1991) and Froot et al. (1993), our work is also related to several

[1] On these issues, see, for example, Pindyck (1993) and Boyle and Guthrie (2003) respectively.

other studies. Grossman and Vila (1992) examine the optimal trading strategy of an investor subject to a financing constraint, but do not explicitly consider the roles of hedging and investment timing flexibility. Mello et al. (1995) consider the situation of a multinational firm that can exit production in one or more countries and, like Froot et al., conclude that hedging allows continued investment in circumstances where exit would otherwise have occurred. Mello and Parsons (2000) analyze the role of hedging in protecting the value of a project's abandonment option for a firm subject to capital market frictions, and compare the performance of alternative hedging rules, but do not address the initial decision to invest. McDonald and Siegel (1986) provide the seminal treatment of investment timing flexibility, but assume that markets are frictionless and thus allow no role for hedging. Boyle and Guthrie (2003) extend the McDonald and Siegel model by introducing external financing restrictions, but do not allow the firm to hedge its cash flow risk and thus do not consider the optimal hedging policy or its interaction with investment.

In the next section, we outline a simple example that intuitively captures the role of investment timing flexibility in determining the optimal hedge position. We then develop a more general model of investment and hedging in Section 3. In Sections 4 to 6, we show that the optimal hedging policy protects not only the firm's ability to undertake investment, but also its ability to delay investment, and that this additional role may have implications for our understanding of empirical work on hedging. Finally, we summarize our findings and offer some concluding remarks in Section 7.

2 A simple example

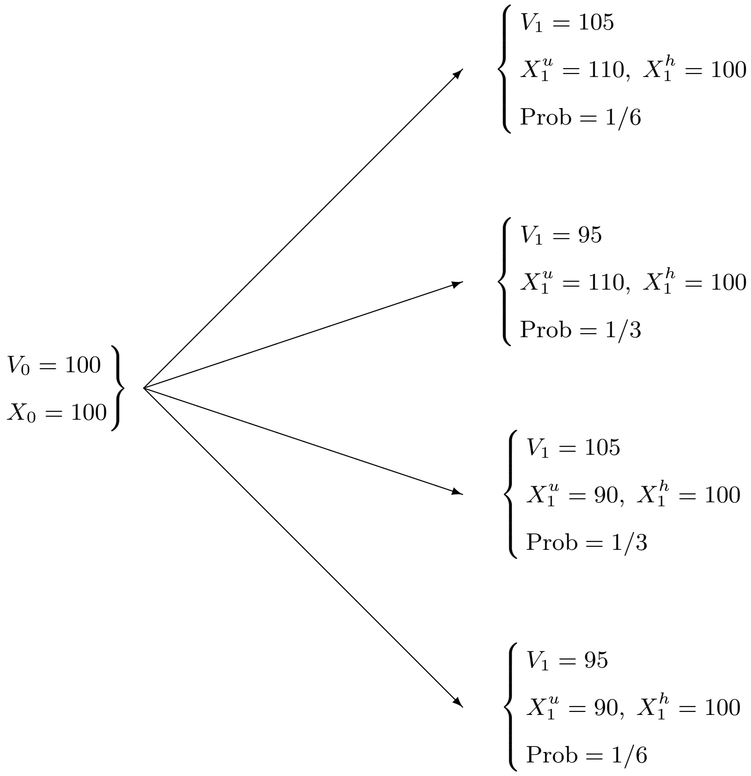
We begin with a simple numerical example of the firm's hedging decision when the timing of investment is flexible. Although this example is highly stylized, it illustrates the basic insight of our story in a transparent and easily-understood manner.

A firm has the rights to a \$96 project and it must finance this from internal funds. There are two dates; we ignore discounting. Currently (at date 0), the project value V is \$100 and the firm's cash stock X is \$100. Subsequently (at date 1), V rises or falls by 5% and X rises or falls by \$10.[2] Both X and V are as likely to rise as fall, and they are twice as likely to move in opposite directions as they are to move in the same direction. The possible outcomes for these variables are shown in Figure 1 as V_1 and X_1^u.

The firm can either invest immediately (and receive a payoff equal to $100 - 96 = 4$) or it can delay this decision until date 1. By waiting, the firm potentially obtains a higher payoff (equal to $105 - 96 = 9$), but runs the risk that its internal funds will be insufficient to pay for the project (when $X_1^u = 90$).

[2]The distinction between multiplicative shocks to V and additive shocks to X anticipates our more general model of Section 3. Unlike project market value, the absolute magnitude of shocks to the firm's cash stock should not depend on its initial value.

Figure 1: Evolution of project value V and cash stock X with and without hedging



Notes. At date 0, the project is worth \$100 and is 5% higher or lower at date 1. The firm's cash stock X is \$100 at date 0 and, if left unhedged (X_1^u), is \$10 higher or lower at subsequent dates. In this case, the expected payoff from date 1 investment is \$1.5, less than the \$4 payoff available from immediate investment, so the firm invests at date 0. The firm can hedge date 1 cash flow by holding h units of an instrument that offers payoffs that are perfectly negatively correlated with cash stock innovations. In this diagram, the firm sets $h = 1$ and thus guarantees sufficient funds for investment at date 1. Now the payoff from date 1 investment is \$4.5, so the firm delays investment at date 0 and firm value is \$0.5 higher than in the unhedged case. Hedging has added value by allowing investment to be delayed.

This financing risk can be mitigated by hedging. At date 0, the firm can hedge date 1 cash flow by holding h units of an instrument that offers payoffs that are perfectly negatively correlated with cash stock innovations. As a result, the date 1 cash stocks are either

$$X_1 = 100 + 10(1 - h) \quad \text{or} \quad X_1 = 100 - 10(1 - h) \tag{1}$$

depending on whether the unhedged cash stock rises or falls respectively. For $h = 1$, the state-contingent values of hedged cash flow appear in Figure 1 as X_1^h.

Given these state-contingent payoff and cash flow realizations, we determine the optimal investment policy in various scenarios and show that hedging not only eliminates under-investment, but also, in so doing, eliminates inefficient early investment at date 0. That is, hedging permits investment to be delayed in circumstances where it would otherwise occur prematurely. As a result, hedging can have the effect of *reducing* investment at date 0.

As a benchmark case, suppose that the financing constraint can be ignored. At date 1, the firm has a 50% probability of obtaining a payoff equal to 9 and a 50% probability of obtaining a zero payoff. From the perspective of date 0, this has an expected present value of $0.5(9) = 4.5$. Since $4.5 > 4$, the value of waiting and retaining the investment option exceeds the date 0 payoff from investment, so the optimal investment policy is to wait until date 1.

However, incorporating the financing constraint yields a different outcome. From Figure 1, we see that the date 1 payoff of 9 is achievable in only one of the two states that it occurs, since the firm has insufficient funds to pay the investment cost of 100 in the other state. Moreover, the latter state is twice as likely as the viable state. As a result, date 1 investment now has a date 0 expected present value of only $(1/6)9 = 1.5$. Since $1.5 < 4$, the firm chooses to invest at date 0.

Although this decision is optimal given the financing constraint, firm value is lower than in the unconstrained case. From the perspective of date 0, the value of the project rights is 4.5 if investment is unconstrained, but is equal to 4 in the presence of the constraint. Thus, the financing constraint reduces firm value by 0.5. That is, relative to the first-best unconstrained outcome, the firm adopts a sub-optimal early investment policy.

In this situation, hedging at date 0 is valuable because it saves the firm from having to invest prematurely. With $h = 1$, the firm has sufficient cash to invest in all date 1 states; in particular, it has sufficient cash in both states where $V_1 = 105$. Now date 1 investment once again has a present value of $(1/2)9 = 4.5$, just as in the unconstrained case, and the optimal policy is to delay investment at date 0. With hedging, the risk of losing the ability to finance the project at date 1 disappears, so the firm can delay investment at date 0 confident in the knowledge that sufficient funding will be available at date 1 should it wish to invest at that date. Hedging thus restores the value of the investment option by eliminating the need for premature investment.

The principal lesson of this example is straightforward. When the firm has flexibility in investment timing, hedging allows it, should it so choose, to retain its investment option. Without hedging, the firm may have to accelerate investment in a sub-optimal fashion and thus give up its option; hedging allows it to retain flexibility.

Although this example provides helpful intuition, and demonstrates the fundamental point that hedging helps avoid both inefficient under-investment *and* inefficient early investment, it has obvious limitations. Not only is the firm able to perfectly hedge cash flow risk, but also investment can occur at only two dates. With greater intertemporal opportunities and imperfect hedging capacity, the tradeoff between under-investment and inefficient early investment, and the relationship of this tradeoff to optimal hedging, is potentially much more complex than captured above. In particular, the dynamic nature of a multi-date investment timing problem introduces an additional element into the hedging decision. At each date where it does not invest, the firm with timing flexibility knows that the optimal decision at the next date may be to invest, or it may be to delay further. Optimal hedging therefore requires that both alternatives be protected, i.e., hedging not only protects a firm's investment projects, but also the options on those projects.

The example also leaves a number of other questions unanswered. For example, what determines the optimal quantity of hedging in the presence of timing flexibility, and how does this differ from its static counterpart? Is hedging more or less valuable in a dynamic world, and how does this depend on the degree of timing flexibility? And does cash flow risk (including the correlation of cash flow with project value) have the same implications for optimal hedging as in a static framework? In the next section, we develop a more general model of the joint determination of investment and hedging that allows us to address these and other issues.

3 The model

The firm owns the rights to a project that first becomes available at date T_1. These rights give it an option to invest in the project at any date t in the interval $[T_1, T_2]$. During this time, the project cannot be undertaken by any other firm, so at each date in $[T_1, T_2)$ the firm can either exercise the option and invest, or delay investment and retain the option. If the firm invests, it pays a fixed amount I and receives a project worth V. The risk-neutral process for V is

$$dV = (\mu - \kappa)V\,dt + \sigma V\,d\eta, \tag{2}$$

where μ, σ, and κ are constants, and η is a Wiener process. The parameter μ is the expected rate of growth in V, σ is the standard deviation, and κ is the market price of risk associated with V, i.e., $\mu - \kappa$ is the "certainty-equivalent" expected growth in V.[3]

Our timing setup contains other common specifications as special cases. For example, the case where $T_1 = T_2$ corresponds to that analysed by Froot et al. (1993). That is, the project becomes available at some future date and no delay is possible. On the other hand, the case where $T_1 = 0$ and $T_2 = \infty$ corresponds to the perpetual option analysed by McDonald and Siegel (1986) and Boyle and Guthrie (2003) in the context of investment timing without hedging. For some fixed T_1, varying T_2 allows for differing degrees of timing flexibility; higher values of T_2 correspond to greater flexibility, reflecting, for example, the firm's competitive position. For ease of comparison with the static model of Froot et al., we focus on the case $1 = T_1 \leq T_2$ in subsequent sections, but for the moment allow these dates to remain arbitrary.

In contrast to standard models of the investment timing decision (see, for example, McDonald and Siegel (1986) and Dixit and Pindyck (1994)), where the firm has unlimited costless access to capital markets, we assume that the firm is restricted to financing the project with internal funds.[4] Specifically, at the date the firm wishes to exercise its investment option, it can do so if

[3] If V is a traded asset, then κ is its market-determined risk premium. See, for example, Hull (2003, ch. 21).

[4] It is straightforward to allow the firm access to external funding (see Boyle and Guthrie, 2003), but as this has no qualitative effect on our results, we maintain the simpler structure here. Numerous articles in the financial press refer to the external funding difficulties faced by many firms in recent years. See, for example, Alsop (2001), Anonymous (2002), Chung (2002), and Zellner et al. (2003).

and only if its cash balance is greater than or equal to I.[5] Prior to that date, however, it can enter into hedging contracts that alter the future distribution of its internal funds.

To model this constraint, we assume that the firm begins with an initial cash stock X which, over time, is augmented in three ways, which we denote dX_1, dX_2, and dX_3 respectively. First, if the firm does not launch the project, X is invested in riskless securities which earn interest at the rate r. That is

$$dX_1 = rXdt.$$

Second, the firm has existing activities that generate a perpetual cash flow stream which can be used for investment in the project. Each dollar invested in these existing assets generates operating cash flow with risk-neutral dynamics $rdt + \phi d\zeta$ where $\phi > 0$ is a constant and ζ is a Wiener process with $d\eta\, d\zeta = \rho dt$. Letting G denote the total value of the firm's existing activities (i.e., the present value of a claim to the perpetual cash flow), the second innovation to the firm's cash stock is then described by[6]

$$dX_2 = G(rdt + \phi d\zeta).$$

Third, the firm hedges its operating cash flow. For this purpose, we suppose there is a futures contract for which the futures price P follows the process

$$dP = \psi P d\epsilon,$$

where $\psi > 0$ is a constant and ϵ is a Wiener process satisfying $d\epsilon\, d\zeta = \lambda dt$ and $d\epsilon\, d\eta = \lambda\rho dt$ for some constant $\lambda \in (-1, 1)$.[7] Thus, the change in the futures price is imperfectly correlated with shocks to the firm's cash flow, so the firm cannot eliminate all cash flow volatility. A hedging policy consists of a choice of h futures positions; if $h > 0$, h is the number of long positions; if $h < 0$, $-h$ is the number of short positions. Consequently, the firm's third source of cash is given by

$$dX_3 = hdP = \psi hPd\epsilon.$$

To summarize, over the time interval dt, the firm's beginning cash stock yields interest dX_1, the firm's existing assets generate cash flow dX_2, while marking-to-market the futures contracts earns dX_3. Combining these sources of cash, the dynamics of the change in the firm's total cash stock are given by

$$dX = r(X + G)dt + \phi G d\zeta + \psi hPd\epsilon.$$

[5]The source of this constraint is immaterial for our purposes, but it could be due to severe information or agency problems, or because the firm does not wish to reveal information to competitors about the project at the investment stage.

[6]Note that the perpetual nature of operating cash flow means that G is a constant (see Boyle and Guthrie, 2003), the value of which reflects the size, nature and productivity of the firm's existing activities, as well as the financial commitments created by its capital structure policy. The only role of these activities in our model is to serve as a source of cash.

[7]This can be motivated by assuming that $d\epsilon = \lambda d\zeta + \sqrt{1-\lambda^2}d\theta$ for some Wiener process θ satisfying $d\theta\, d\zeta = d\theta\, d\eta = 0$.

Note that hedging affects only future values of the cash stock via the marking-to-market process. Thus, the firm cannot bypass the financing constraint by using the futures contract to raise funds directly.

It is convenient to rewrite the process for X in an equivalent form which involves only one Wiener process; that is,

$$dX = r(X+G)dt + (\phi^2 G^2 + 2\lambda\phi\psi G H + \psi^2 H^2)^{1/2} d\xi, \tag{3}$$

where $H = hP$ and ξ is a Wiener process satisfying

$$d\xi\, d\eta = \frac{\rho(\phi G + \lambda\psi H)}{(\phi^2 G^2 + 2\lambda\phi\psi G H + \psi^2 H^2)^{1/2}}\, dt.$$

Given the firm's choice of hedging policy h, its cash stock evolves according to (3). Together, equations (2) and (3) represent a system with state variables (X, V) and the control H. Consequently, the firm's hedging policy can be described by the function $H(X, V, t)$. Given this function, the corresponding hedge position comprises $h = H(X, V, t)/P$ long positions.

The firm can invest at date t if and only if $T_1 \le t \le T_2$ and $X \ge I$; if $t < T_1$ or $X < I$, then the firm cannot invest. Hence, the value $F(X, V, t)$ of the investment option depends on X and t as well as V. If the firm invests at date t, $F(X, V, t) = V - I$. Otherwise, it satisfies the partial differential equation (see the appendix for details; subscripts on F denote partial derivatives)

$$\begin{aligned} rF = \sup_H \Big\{ & F_t + \frac{1}{2}\left(\phi^2 G^2 + 2\lambda\phi\psi G H + \psi^2 H^2\right) F_{XX} \\ & + \rho\sigma V(\phi G + \lambda\psi H) F_{XV} + \frac{1}{2}\sigma^2 V^2 F_{VV} + r(X+G)F_X + (\mu - \kappa) V F_V \Big\}. \end{aligned} \tag{4}$$

Intuitively, the left-side of this equation is the expected return required by the firm in order to hold the project rights. The right-side is the expected return obtainable from holding the rights, given the optimal hedging strategy.

The value function $F(X, V, t)$ and the optimal hedge function $H^*(X, V, t)$ are determined simultaneously by solving the nonlinear system comprising, from (4),

$$H^*(X, V, t) = \frac{-\lambda\phi G}{\psi} - \frac{\rho\lambda\sigma V F_{XV}}{\psi F_{XX}} \tag{5}$$

and

$$\begin{aligned} 0 = & F_t + \frac{1}{2}\left(\phi^2 G^2 + 2\lambda\phi\psi G H^* + \psi^2 H^{*2}\right) F_{XX} + \rho\sigma V(\phi G + \lambda\psi H^*) F_{XV} + \frac{1}{2}\sigma^2 V^2 F_{VV} \\ & + r(X+G)F_X + (\mu-\kappa) V F_V - rF. \end{aligned} \tag{6}$$

The functions H^* and F are thus inextricably linked. We focus on this link between optimal investment and hedging policies in the next section and then turn to a more detailed examination of the determinants of hedging in Section 5.

4 Hedging and the optimal investment policy

For projects with timing flexibility, the optimal rule is to invest if project value V exceeds some minimum threshold, but otherwise wait. Because investment is allowed if and only if $X \geq I$, the value of X places restrictions on the future states in which the investment option can be exercised, so the threshold is a function of X. Moreover, because stochastic fluctuations in future X can be altered by hedging, the threshold also depends on H. Thus, investment is justified if and only if

$$V \geq \hat{V}(X, t; H),$$

where $\hat{V}$ is the investment threshold function.

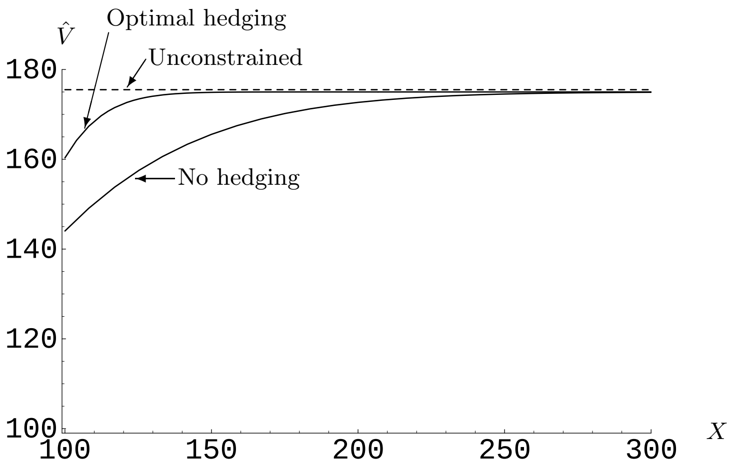
Unfortunately, the complexity of the system (5) and (6) prevents us finding an analytical solution for $\hat{V}$. In order to shed some light on the relationship between hedging and the optimal investment policy, we obtain a numerical solution for $\hat{V}(X, t; H)$ and calculate this for different hedging policies. For the purposes of this exercise, we assume $I = 100$, $\sigma = 0.2$, $\mu - \kappa = 0$, $r = 0.03$, $\rho = 0.5$, $\phi = 0.6$, $G = 100$, $\psi = 0.2$, $\lambda = 0.9$, $T_1 = 1$, and $T_2 = 5$, and evaluate the threshold when $t = 1$.[8] Using these benchmark parameter values, we write the partial differential equation describing $F(X, V, t)$ as a partial difference equation and solve this using the explicit finite difference method. $\hat{V}(X, 1; H)$ is set equal to the smallest value of V that satisfies $F(X, V, 1) = V - I$. Further details of this procedure appear in the appendix.

Figure 2 illustrates the effect of hedging on the optimal investment policy. The bottom curve depicts the situation where the firm is unable to hedge. At low levels of X, the firm adopts a low threshold, reflecting the risk of future funding shortfalls. As this risk recedes with higher levels of X, the threshold increases, eventually reaching that of the corresponding unconstrained firm (depicted by the dashed line). When the firm follows the optimal hedging policy (top curve), the relationship between $\hat{V}$ and X is similar, but the threshold now lies above its unhedged counterpart for low and medium values of X; only when X is approximately 2.5 times the investment cost I do the thresholds converge.[9] Thus, hedging leads to a higher investment threshold, reflecting its role in reducing the risk of future funding shortfalls. By hedging, the firm can confidently delay investment in circumstances where it would otherwise have had to invest prematurely. In other words, hedging allows the firm to improve the timing of its investment, as well as the quantity.

[8] Although greater timing flexibility further enhances the value of hedging, increasing T_2 beyond 5 has little effect on the optimal investment and hedging policies described below. The remaining parameter values are based on other authors who obtain numerical solutions to similar models, e.g., Mauer and Triantis (1994), Milne and Whalley (2000), and Boyle and Guthrie (2003). Different values have little qualitative effect on our results, a point discussed further below.

[9] At that point, the low risk of future cash shortfalls means that the investment threshold is the same as that for a financially-unconstrained firm, regardless of hedging. At lower levels of X, the threshold for the optimally-hedging firm is less than the unconstrained threshold because the imperfect nature of the hedging instrument leaves the firm with some residual funding risk.

Figure 2: The investment threshold function: hedged and unhedged.



Notes. The top curve depicts the value of the firm's investment threshold function when the optimal hedging policy is followed; the bottom curve corresponds to the case where there is no hedging; the dashed line shows the unconstrained threshold. Hedging decreases the risk that the firm will have insufficient cash to finance the project in the future, thereby increasing the value of waiting and raising the investment threshold. We use the parameter values $I = 100$, $\sigma = 0.2$, $\mu - \kappa = 0$, $r = 0.03$, $\rho = 0.5$, $\phi = 0.6$, $G = 100$, $\psi = 0.2$, $\lambda = 0.9$, $T_1 = 1$, and $T_2 = 5$, and evaluate the threshold when $t = 1$.

In confirming the findings of our Section 2 examples, this outcome highlights a new motivation for hedging. When projects are of the "now-or-never" variety, as in Froot et al. (1993), cash has value because it allows the firm to invest, so insufficient hedging may cause the firm to forgo investment. In terms of Figure 2, the investment threshold is the horizontal axis ($\hat{V} = 100$) whenever $X \geq 100$, but is plus infinity if $X < 100$. Hedging reduces the probability of the latter occurring and thus causing the firm to miss the investment opportunity altogether. More succinctly, hedging adds value because it allows investment to proceed. By contrast, when there is flexibility in the timing of projects, cash also has value because it allows the firm to retain the option to invest, so insufficient hedging may cause the firm to invest prematurely. Thus, hedging adds value because it allows investment to be delayed.

5 Determining the optimal hedge

5.1 The effect of investment timing flexibility on hedging

Having seen that investment timing flexibility creates an additional role for hedging, we now turn our attention to the implications of this for the optimal hedging policy. Although not a closed-form solution, equation (5) is helpful for addressing this issue, insofar as it identifies the fundamental components of dynamic hedging. To explore the implications of (5), we first provide a general intuitive discussion of its properties, then illustrate these with a numerical solution of the type described in Section 4.

In general, hedging reduces cash flow volatility by shifting cash from high-cash states to low-cash states. In equation (5), the first term represents the 'volatility-minimizing' (henceforth VM) hedge position, i.e., the hedge that minimizes the variance of fluctuations in the firm's cash flow. However, shifting cash from a high-cash state to a low-cash state is counter-productive if the marginal value of cash is high in the former state, but low in the latter state. Consequently, the optimal hedging policy does not shift cash from all high-cash states to all low-cash states, but only from high-cash states in which the marginal value of cash is low to low-cash states in which the marginal value of cash is high. The second term in (5) reflects these considerations. When the firm has investment timing flexibility, the marginal value of cash is given by F_X and the optimal hedging policy moves cash from states where F_X is low to states where F_X is high.[10]

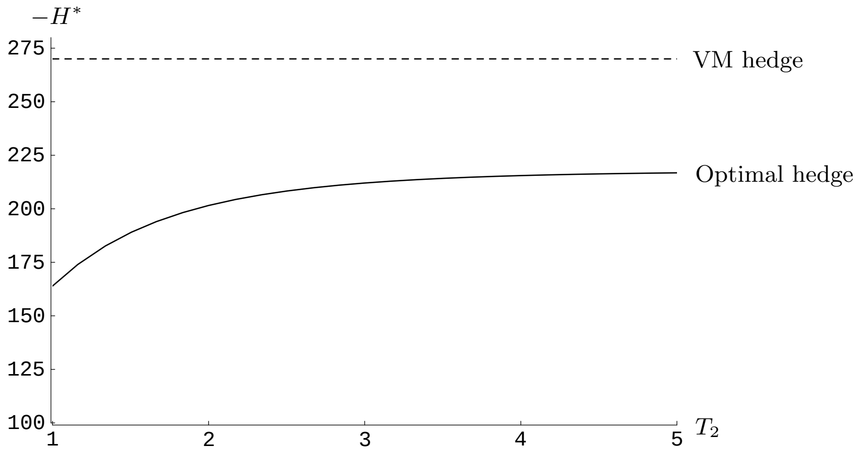
The crucial aspect of equation (5) is the dependence of H^* on the value function F. Since F reflects in part the value of delaying investment, this implies that the optimal hedging policy of a firm with investment timing flexibility will generally differ from that of a firm without this flexibility. When a project disappears if not taken at a particular date, the firm hedges to protect its ability to realize the project's value at the specified date. By contrast, if the timing of the project is flexible, then the firm wishes to protect not only the immediate payoff from investment, but also the value of retaining the option to invest at a later date. The optimal hedge therefore has to consider both sources of value, not just the former.

Investment timing flexibility thus has two effects on the optimal quantity of hedging. On the one hand, allowing the firm to choose the timing of its investment reduces its need for hedging since it does not lose the project if funding is not available on a given date. On the other hand, the need to protect the value of the investment option increases the quantity of required hedging. For example, suppose X and V are positively correlated such that, at the next date, X exceeds I whenever V exceeds I. Then the optimal hedge is zero if the firm can invest only at that date. But for firms with an ongoing option to invest, the value of this option is positive even when $V < I$ and, moreover, this value is enhanced by additional X, since this increases the probability of eventual exercise. Consequently, firm value would be increased by a hedge that moved cash from states where X is more than sufficient to finance investment to states where X is low. This way, the firm not only has sufficient funds to invest if it wishes to do so, but it also maximizes the value of its investment option should it choose to retain it.

We provide a concrete illustration of these points by explicitly calculating the optimal hedge position when the firm has no discretion in the timing of investment, and then determining how this choice is affected by allowing for varying degrees of flexibility. For the purposes of this exercise, we set $t = 0$ and $T_1 = 1$ so as to best facilitate comparison with a static analysis of hedging where the project becomes available at a single future date. Otherwise, we use the same parameter values as in Section 4, but allow T_2 to vary. For the case with no timing flexibility, we

[10]It is straightforward to show that the hedging policy given by (5) minimizes the variance of F_X. Mello and Parsons (2000) also emphasize this characteristic of optimal hedging policies.

Figure 3: The optimal hedging policy.



Notes. The solid curve depicts $-H^*$ (the optimal short position) as a function of the firm's investment timing flexibility (T_2). Greater flexibility increases the need for hedging in order to protect the valuable timing options created by this flexibility, but this effect levels off because greater flexibility also increases the probability that internal funds will be sufficient to finance investment. The dashed line indicates the volatility-minimizing hedge, given by the first term on the right-side of equation (5). We use the parameter values $I = 100$, $\sigma = 0.2$, $\mu - \kappa = 0$, $r = 0.03$, $\rho = 0.5$, $\phi = 0.6$, $G = 100$, $\psi = 0.2$, $\lambda = 0.90$, and $T_1 = 1$, and evaluate H^* at $X = V = I$ and $t = 0$.

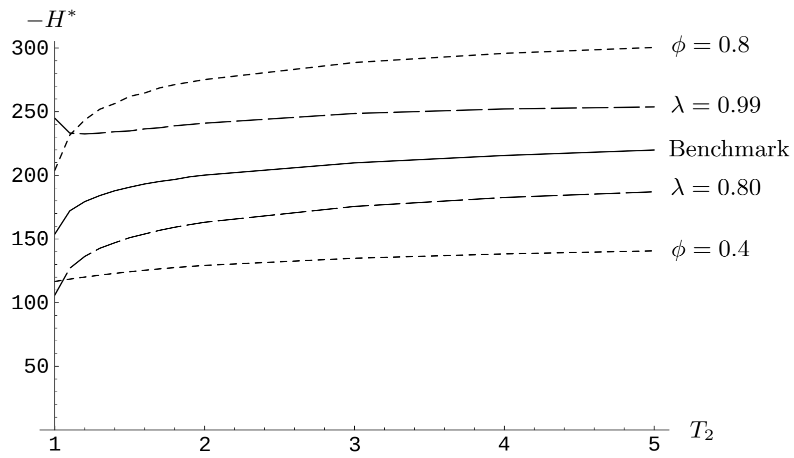
set $T_2 = T_1$, solve the partial differential equation describing $F(X, V, t)$ using the explicit finite difference method, and then use (5) to obtain the optimal hedge position when $X = V = I$. We then repeat this procedure for successively higher values of T_2, each corresponding to a greater degree of flexibility.

The results from this exercise are depicted in Figure 3.[11] The VM hedge position is a constant, but the optimal hedge monotonically increases in the degree of timing flexibility, reflecting the need to protect the valuable options created by this flexibility. However, this effect becomes smaller as T_2 increases, reflecting the lower funding risk afforded by a longer potential investment period. For example, in the case depicted in Figure 3, increasing T_2 from 1 to 2 raises optimal hedging by 30%, but further increasing T_2 to 3 raises $-H^*$ by only another 5%.[12]

[11] Setting $\lambda = 0.9$ means that hedging instrument returns are positively correlated with operating cash flow. As a result, the optimal hedge position is a short one, so we report its value as $-H^*$.

[12] By contrast, Mello et al. (1995) show that, for a given financial structure (given by X in our model), the size of the optimal hedge is a *decreasing* function of the multinational firm's flexibility in switching production between countries. Similarly, MacKay (2003) finds that greater investment flexibility increases debt capacity, thus implicitly weakening financing constraints and reducing the need for hedging. These conclusions contrast with our results because of differing forms of flexibility. In Mello et al., for example, greater flexibility reduces the probability that the firm will need to exit the industry, so hedging against this possibility is less necessary. In our model, greater flexibility increases the probability that the firm will wish to delay investment, so more hedging is needed to ensure that this is possible.

Figure 4: The optimal hedging policy under different parameter values.



Notes. The solid curve depicts $-H^*$ (the optimal short position) as a function of the firm's investment timing flexibility (T_2). The benchmark case is the same as in Figure 3. The remaining curves show the effects of changing one parameter at a time.

To check that these results are not simply an artifact of a particular example, we repeat our calculations for different parameter values and provide an overview of the results in Figure 4.[13] Overall, the hedging-flexibility relationship described above is very robust. For example, higher values of ϕ increase the optimal hedge ratio for all T_2, but the relationship between this ratio and the degree of timing flexibility is unaffected. That is, although different values of ϕ shift the curve up or down it maintains the same concave shape. In general, the same is true of changes to the other parameters. The one partial exception concerns the hedging effectiveness parameter λ. If this is particularly high, then $-H^*$ initially *falls* with greater timing flexibility before resuming the shape portrayed in Figure 3. Intuitively, this is unsurprising. Greater timing flexibility takes the pressure off a firm to invest at a particular date, so it can afford less hedging if the hedge instrument is particularly effective. By contrast, a less effective hedge (e.g., the benchmark case) offers no such luxury, so the need to protect the greater options provided by flexibility dominates the hedging calculation. Of course, as timing flexibility increases further, even an effective hedge is used primarily to protect these flexibility options, so its use rises just as in the benchmark case.

We also consider the sensitivity of hedging to the firm's underlying stochastic cash flow structure, and the dependence of this sensitivity on the degree of timing flexibility. Specifically, for various values of T_2 we calculate the percentage change in optimal hedging $-H^*$ in response to a 1% increase in each of firm cash flow volatility ϕ, the correlation between firm cash flow and project value ρ, and the correlation between firm cash flow and the hedging instrument price λ. Intuitively, these parameters represent, respectively, the risk of being unable to fund the project,

[13] We also experiment with different combinations of high, medium and low values of V, X, ρ and σ, but these too affect only the position, and not the general appearance, of the curves in Figure 3.

Table 1: The optimal hedging policy.

T_2	1	2	3	4	5
$-H^*$	153.89	200.18	209.80	215.60	219.88
Elasticities					
λ	4.088	2.054	1.795	1.619	1.520
ϕ	0.664	1.085	1.098	1.104	1.104
ρ	−0.593	−0.405	−0.322	−0.271	−0.233

Notes. This table illustrates the sensitivity of the optimal hedge to changes in parameters describing the stochastic structure of firm cash flow. When $T_2 = 1$, the firm can invest only at date 1; otherwise investment can occur at any time during the interval $[1, T_2]$. The first row thus gives the optimal hedge position for varying degrees of timing flexibility. Each cell in the following three rows shows the percentage change in the optimal hedge resulting from a 1% change in the corresponding parameter. The elasticity calculations use the same parameter values as in Figure 3.

the extent to which the project offers natural protection against this risk, and the effectiveness of the hedging instrument in protecting against this risk. As Table 1 shows, hedging demand is most sensitive to the effectiveness of the hedging instrument; without timing flexibility (i.e., $T_1 = 1$, as in Froot et al., 1993), a 1% increase in λ raises optimal hedging by 4.1%. However, this elasticity declines with T_2, reflecting the fact that even an imperfect hedging instrument is able to ensure sufficient funds are available to allow investment at least once during a long window. Unsurprisingly, higher cash flow volatility also raises the demand for hedging, and this effect increases with the degree of timing flexibility, reflecting the greater potential for adverse cash flow shocks over a long period. Finally, greater correlation between V and X reduces the need for hedging (as suggested by Froot et al.), but the absolute value of this effect declines with timing flexibility. Overall, in comparison to the static environment of Froot et al. (where $T_1 = T_2 = 1$), it seems that timing flexibility can have a significant effect on both the optimal quantity of hedging and on the relative importance of the determinants of this quantity.

The role of the correlation ρ in determining the optimal quantity of hedging merits further attention. For a firm without timing flexibility, Froot et al. (1993) emphasize the importance of this parameter for the natural hedge properties of the project. If ρ is positive, then cash tends to be high precisely when it is most needed (i.e., when V is high). This pattern of volatility is more desirable than one in which cash is independent of its potential uses, so the optimal hedge is less than the VM hedge. On the other hand, if ρ is negative then cash tends to be low in states where it is most valuable, so the optimal quantity of hedging is greater than that needed for the VM hedge. This is precisely the pattern observed in Table 1: higher ρ leads to a lower $-H^*$.

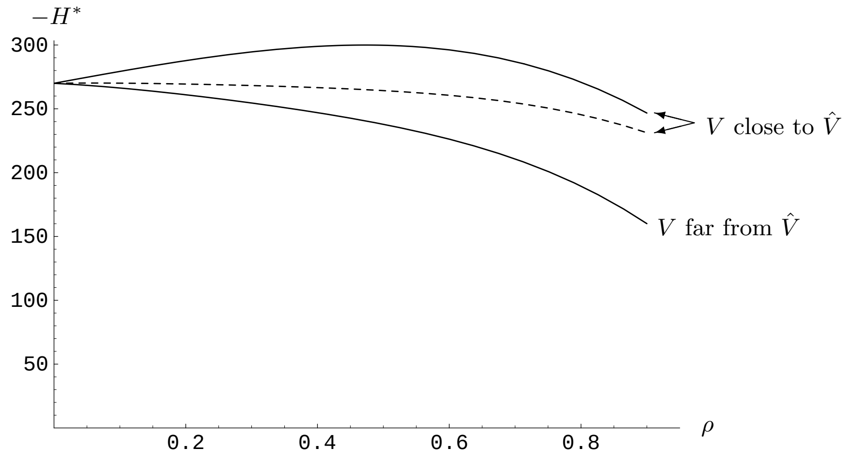
However, this outcome is not entirely general. Equation (5) shows that in a dynamic world, the direction in which the optimal hedge deviates from the VM hedge actually depends not on the sign of ρ, but rather on the sign of ρF_{XV}.[14] That is, positive correlation between X and V reduces the need for hedging if and only if $F_{XV} > 0$. This latter condition is automatically satisfied when investment cannot be delayed, since additional cash allows a financially-constrained firm to move from a payoff of zero to a payoff of $V - I$, the value of which is clearly increasing in V. Consequently, the sign of ρ is sufficient to determine the direction of the optimal deviation from the VM hedge. However, this need not be the case when investment can be delayed. Suppose, for example, that the firm would prefer not to invest immediately (even though it has sufficient funds to do so), but faces a high risk of funding shortfalls in the future. Faced with this risk, the firm may choose to invest now. Extra cash in these circumstances reduces the risk of future funding difficulties, thereby allowing the firm to delay investment. Without such a cash injection, the firm invests immediately, so $F = V - I$ and a \$1 increase in project value V increases F by an amount equal to the change in project value, i.e., by \$1. With a cash injection, the firm can afford to delay investment and retain the option, so F satisfies (4) and a \$1 increase in project value V increases F by an amount less than the change in project value, i.e., by less than \$1. As a result, F_V is smaller with the cash injection than without it, i.e., $F_{XV} < 0$.

If $F_{XV} < 0$, then the marginal value of cash F_X is high when project value V is low, and vice versa. In that case, a positive correlation between X and V tends to deliver high (low) cash when the value of that cash is low (high), i.e., the correlation does a poor job of protecting the investment option F. Consequently, higher ρ can *increase* the need for hedging in a dynamic world, in contrast to its effect in a static world.

To illustrate the different ways in which the optimal hedge can depend on ρ, we apply our earlier benchmark example to three different cases. In the first two cases, we focus on the hedging decision immediately before investment becomes possible ($t = 1$). In one case, the project is in a breakeven situation ($V = 100$); in the other case the project payoff is significantly positive ($V = 150$). As the solid curves in Figure 5 show, the optimal quantity of hedging is decreasing in ρ in the first case, but non-monotonic in the second, a difference that reflects the considerations discussed above. In the first case, the project value is a long way from the investment threshold, so the firm chooses to delay investment and both X and V contribute positively to the value of the investment option that the firm continues to hold. Consequently, a positive correlation between X and V increases the value of this option and reduces the need for hedging. In the second case, however, the project value is near the investment threshold and the firm's calculation is different. Near-term investment is now a possibility, the likelihood of which depends on the ultimate realisation of X and V. If X turns out to be low (but still greater than the investment cost of 100), then further waiting is risky and the firm has a strong incentive to invest. That being the case, the best outcome is a high value of V. On the other hand, a high realisation of X means the firm can afford to delay, so a high value of V is less important. In these circumstances, optimal hedging entails reversing any positive correlation

[14] The second order condition for the maximization problem in (4) ensures that $F_{XX} < 0$.

Figure 5: The optimal hedging policy and correlation ρ.



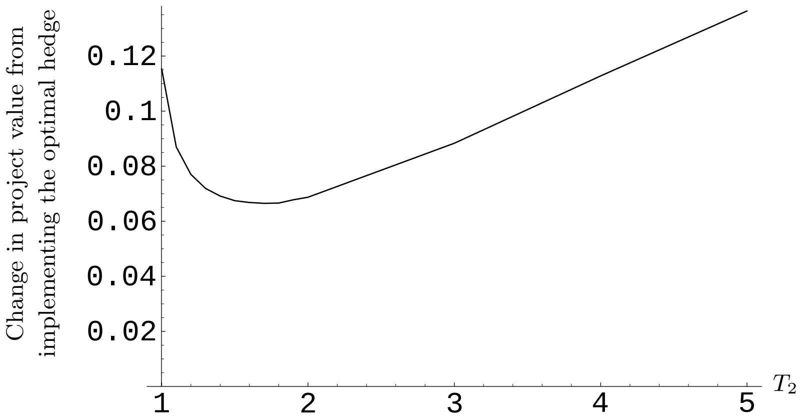
Notes. Each curve depicts $-H^*$ (the optimal short position) as a function of the correlation (ρ) between project value (V) and firm cash (X). The dashed curve plots $-H^*$ at $t = 0$, and the solid curves plot $-H^*$ evaluated at $t = 1$. 'V close to $\hat{V}$' corresponds to $X = 150$ and $V = 150$. 'V far from $\hat{V}$' corresponds to $X = 150$ and $V = 100$. In all cases the project is available between dates $T_1 = 1$ and $T_2 = 5$; other parameters are the same as in Figure 2. Although the relationship between $-H^*$ and ρ is usually negative, it can be positive when project value approaches the investment threshold (V close to $\hat{V}$) and investment is immediately available ($t = 1$).

between X and V, so the optimal quantity of hedging initially increases with the size of this correlation. However, if ρ becomes high enough, then the risk of being unable to fund the project from internal sources diminishes, so the investment threshold rises and the incentive for immediate investment recedes. Consequently, the desire for negative correlation between X and V disappears and the usual negative relationship between the quantity of hedging and ρ re-emerges.

Our third case focuses on the hedging decision when investment is currently impossible ($t = 0$), depicted by the dashed curve in Figure 5. Here, the project value is near the investment threshold, but the optimal quantity of hedging falls monotonically with ρ, in contrast to the case when $t = 1$. Intuitively, the greater time between today ($t = 0$) and when the first investment decision can be made ($T_1 = 1$) allows the firm to take advantage of any natural hedge properties of the project in order to minimize the risk of being unable to fund the project when it becomes available or of having to invest prematurely. Thus, the considerations that arise when the firm is confronted with either of these possibilities (as is the case when $t = 1$) do not apply, so greater ρ permits less hedging.

Together, these three cases emphasize that the dynamic relationship between hedging and ρ is more complex than envisaged by the standard static framework. In Froot et al. (1993), greater ρ lowers the marginal value of cash in all states and hence reduces the need for hedging. By contrast, the dynamic relationship can be strongly negative, weakly negative, or even positive, depending on the immediacy of the investment decision, the distance between V and $\hat{V}$, and the

Figure 6: The value of hedging.



Notes. The curves depict the proportional increase in $F(X, V, t)$ when the firm hedges optimally over its value when the firm does not hedge, as a function of the firm's investment timing flexibility (T_2). Greater flexibility initially reduces the value of hedging, but as flexibility increases, more and more of the project's value originates from timing options and hedging's ability to protect these options increases its value to the firm. We use the parameter values $I = 100$, $\sigma = 0.2$, $\mu - \kappa = 0$, $r = 0.03$, $\rho = 0.5$, $G = 100$, $\psi = 0.2$, $\lambda = 0.90$, and $T_1 = 1$, and evaluate $F(X, V, t)$ at $X = V = I$ and $t = 0$.

value of ρ itself.

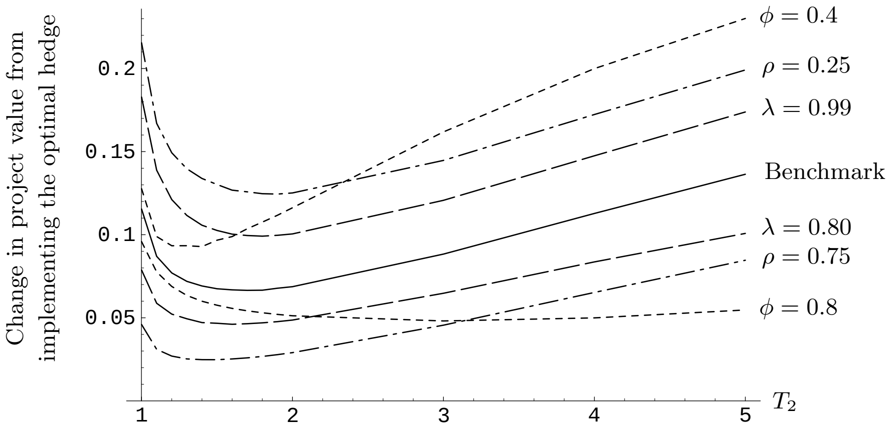
5.2 The value of hedging

Having described the interaction between the optimal investment and hedging policies, the remaining issue of interest is the implications of this interaction for project value. More prosaically, what effect does hedging have on the value of a firm with investment timing flexibility, and how does this differ from a firm without such flexibility?

To address this issue, we continue with the benchmark parameter values used in previous sections and numerically calculate the value added by hedging. That is, for varying degrees of flexibility (different values of T_2), we calculate the project value $F(X, V, t)$ when the firm follows the optimal investment and hedging policies, and compare this with the corresponding value under a no-hedging policy. As Figure 6 indicates, the value of hedging is related to timing flexibility in a non-monotonic fashion. For $T_2 = T_1$ (i.e., no timing flexibility), the optimal hedge increases project value by 12%, but higher values of T_2 are initially associated with smaller hedging effects. As T_2 continues to increase, however, the optimal hedge has a proportionately bigger effect on project value. Thus, small amounts of timing flexibility reduce the value of hedging, but significant flexibility has the opposite effect.

This non-monotonicity is due to the competing effects discussed in Section 5.1. When the project is "now-or-never", insufficient funds for investment mean that it is lost forever. In these

Figure 7: The value of hedging under different parameter values.



Notes. The curves depict the proportional increase in $F(X, V, t)$ when the firm hedges optimally over its value when the firm does not hedge, as a function of the firm's investment timing flexibility (T_2). The benchmark case is the same as in Figure 6. The remaining curves show the effects of changing one parameter at a time.

circumstances, hedging is relatively valuable because it reduces the probability of permanent abandonment. Allowing for timing flexibility alters this relationship in two ways. First, by eliminating the need to invest at just one date, the risk of abandonment falls and so hedging is less valuable. In effect, flexibility can be substituted for hedging. Second, flexibility in the choice of investment date increases the options available to the firm, thereby making hedging more valuable. That is, flexibility and hedging are complementary. When timing flexibility is low, the proportion of project value that is due to timing options is relatively small, so the first effect dominates and flexibility reduces the value of hedging. However, as flexibility increases, more and more of the project's value emanates from timing options, so the second effect dominates and flexibility increases the value of hedging. Consequently, the relationship between timing flexibility and the value of hedging has the U-shape depicted in Figure 6.

Finally, we examine the sensitivity of this conclusion to different parameter values and show the results in Figure 7. Although the general U-shape is unaffected, higher ϕ lowers the value of hedging for all T_2 and raises the critical value of T_2 at which the two factors affecting hedging value exactly offset each other. Intuitively, a high ϕ means that the risk of being unable to invest remains high even with the optimal hedge in place, so hedging is less effective and thus less valuable. Other parameters have similar effects. For example, higher ρ and lower λ both reduce the value of hedging for all T_2 and lower the critical value of T_2, but leave the basic U-shape unchanged.

6 Empirical Evidence

Our analysis identifies two fundamental principles of dynamic hedging and investment. First, when the timing of investment is flexible, optimal hedging requires that this flexibility be protected (Sections 5.1 and 5.2). Second, in facilitating the efficient timing of investment, hedging can lead to less investment by eliminating the need to invest prematurely (Section 4). Both of these potentially assist our interpretation of existing empirical work on hedging.

On the first principle, empirical studies of the determinants of corporate hedging generally proceed by regressing some measure of hedging activity on a range of variables designed to capture factors that theoretical research has shown to be important. However, several studies report little support for the hypothesis that more valuable investment opportunities enhance the propensity to hedge; see for example, Nance, Smith and Smithson (1993), Berkman and Bradbury (1996), Mian (1996), Howton and Perfect (1998), and Graham and Rogers (1999). Although Géczy et al. (1997) and Gay and Nam (1998) obtain different results, it is clear that the empirical evidence for this hypothesis is, at best, mixed.

A possible explanation for this finding is simply that many firms are not hedging optimally. However, our model raises the possibility that hedging rates can be unrelated to standard measures of investment opportunities even if firms *are* optimally hedging these opportunities. As we have emphasized, the value generated by hedging a given set of investment projects depends in part on the timing flexibility offered by those projects. When timing is flexible, the optimal hedging strategy protects not only the projects themselves, but also the option to delay those projects until conditions are more favorable. With "now-or-never" projects, only the former consideration applies. Thus, firms with similar investment opportunities may adopt very different hedging policies, reflecting their different timing flexibilities. Consequently, in the absence of reliable controls for differences in timing flexibility, there is no reason to expect a robust relationship between hedging propensity and standard measures of investment opportunities.

Rectifying this 'omitted-variable' problem represents a considerable challenge for empirical research as there is no obvious way of distinguishing firms that have significant timing flexibility from those that do not. One possibility is to identify firms with considerable market power, as these face little competitive pressure to rush investment and thus have greater scope for delay. Another possibility is to identify firms that tend to make large and infrequent investments, as these have a strong incentive to time such investments in an optimal fashion. Differentiating these two types of firms from others may help to improve the explanatory power of empirical models of hedging and investment.

Turning to the second principle, the notion that hedging can result in less investment may seem a somewhat counter-intuitive and unlikely proposition, but it is consistent with a puzzling feature of data on the investment rates of hedging and non-hedging firms. To the extent that hedging mitigates the under-investment problem, one might expect hedging firms to invest more than non-hedging firms, all else being equal. However, evidence obtained by empirical researchers provides little support for this view. For example, after adjusting for size differences, Graham and Rogers (1999, Table 4) and Allayannis and Mozumdar (2000, Table 1) report little difference in the level of investment between hedgers and non-hedgers, while Géczy, Minton and Schrand (1997, Table III) find that non-hedgers invest more than hedgers.

Of course, there are some obvious explanations for these findings. Non-hedgers in the above researchers' samples may (i) have better investment opportunities than hedgers, or (ii) have more internal funds than hedgers, or (iii) face lower costs of external funding than hedgers. However, none of these is an unambiguous feature of the data: in the samples of Géczy et al. (1997), Graham and Rogers (1999), and Allayannis and Mozumdar (2000), hedgers (i) have similar (or in some cases higher) research and development expenditure, Tobin's Q, and/or market-to-book ratios, than non-hedgers, and (ii) are larger, have less or similar leverage, higher dividend yields, and higher operating cash flow. Thus, hedgers in these samples appear to have similar investment opportunities and face weaker financial constraints than non-hedgers, so neither of these factors can be unequivocal explanations for the independence of investment and hedging.

Our model suggests an alternative explanation: that hedging has an ambiguous effect on the incentive to invest. On the one hand, hedging allows firms to undertake more investment by reducing the number of states in which there is a funding shortfall. On the other hand, it also reduces the risk of future funding shortfalls and thus makes waiting more attractive. Since these two effects work in opposite directions, investment can be unrelated to hedging over any finite period of time; although hedging allows the firm to undertake more investment, it also allows it to delay more investment.

7 Concluding remarks

The fundamental goal of hedging for a firm with valuable investment opportunities that are available only at single future dates is to protect the ability to fund these projects as they become available. For projects that offer a choice of investment dates, however, hedging has an additional goal: to protect the ability of the firm to utilize this timing flexibility. Without hedging, the firm may have to rush into investment because of the risk that waiting exposes it to the risk of future funding shortfalls. By reducing this risk, hedging is valuable. Not only does it allow investment to occur, but it also allows investment to be delayed.

This simple observation implies that the value of hedging depends not only on a firm's investment opportunities and financing constraints, but also on its investment timing flexibility, a conclusion that has two corollaries. First, it reinforces the view that incorporating dynamic features in capital investment models can yield predictions significantly different to those generated by the static, one-period, framework. Second, it has implications for our interpretation of empirical work on hedging. In particular, it suggests that the failure of empirical work to find significant relationships between hedging rates, investment rates and investment opportunities is not, after all, particularly surprising.

Although our framework captures the interaction between hedging and investment timing in a fairly simple and appealing manner, some important issues remain unresolved. First, we consider the hedging-investment decision with respect to a single project in isolation. In reality, most firms juggle several such decisions simultaneously. The correlations between these projects are likely to introduce an additional layer of complexity into the optimal hedging and investment decisions, particularly when the feedback effect of each project's introduction on the firm's cash position is considered (a phenomenon ignored by our model). Second, we examine the hedging-investment decision of a single firm and thus ignore dynamic equilibrium complications in a world where many firms are simultaneously facing the same problem. Third, the source of the financial constraint is unspecified. This could be important if, for example, moral hazard conditions were present, in which case the firm could optimally choose to reverse-hedge when cash is particularly low.

Appendix

Derivation of the partial differential equation for $F(X, V, t)$

In the region where the firm delays investment, it holds a portfolio consisting of physical assets worth G, the project rights worth $F(X, V, t)$, a cash stock of X, and h futures positions. Since G is a constant and the instantaneous value of the futures position is zero due to marking-to-market, Itô's Lemma implies that the change in the value of this portfolio over the time interval dt (with respect to the risk-neutral probability measure) is

$$\begin{aligned} dR &= dF + dX \\ &= \Big(F_t + \frac{1}{2}\left(\phi^2 G^2 + 2\lambda\phi\psi G H + \psi^2 H^2\right) F_{XX} + \rho\sigma V(\phi G + \lambda\psi H) F_{XV} \\ &\qquad + \frac{1}{2}\sigma^2 V^2 F_{VV} + r(X+G)F_X + (\mu - \kappa) V F_V + r(X+G) \Big)\, dt \\ &\quad + (\phi^2 G^2 + 2\lambda\phi\psi G H + \psi^2 H^2)^{1/2}(1 + F_X)\, d\xi + \sigma V F_V\, d\eta, \end{aligned}$$

where $H = hP$. The firm's expected rate of return with respect to the risk-neutral probability measure equals the risk free interest rate, so

$$E[dR] = r(F + G + X)dt.$$

Since H is a control variable, this implies the following partial differential equation for F:

$$\begin{aligned} rF = \sup_H \Big\{ & F_t + \frac{1}{2}\left(\phi^2 G^2 + 2\lambda\phi\psi G H + \psi^2 H^2\right) F_{XX} \\ & + \rho\sigma V(\phi G + \lambda\psi H) F_{XV} + \frac{1}{2}\sigma^2 V^2 F_{VV} + r(X+G)F_X + (\mu-\kappa) V F_V \Big\}. \end{aligned}$$

Numerical solution of our model

At any date t, the value of the investment option depends on whether or not the firm decides to invest. Option pricing problems of this kind can be formulated as systems of variational inequalities (see Duffie, 1996, pp. 175–178). For our problem, F must satisfy

$$F \geq g, \quad \mathcal{D}(F) \leq 0, \quad (F - g)\mathcal{D}(F) = 0,$$

when $t \in [T_1, T_2]$ and $X \geq I$, and $\mathcal{D}(F) = 0$ otherwise, for $g(V) = V - I$ and

$$\begin{aligned} \mathcal{D}(F) &= \frac{\partial F}{\partial t} + \frac{1}{2}\left(\phi^2 G^2 + 2\lambda\phi\psi G H^* + \psi^2 (H^*)^2\right)\frac{\partial^2 F}{\partial X^2} + \rho\sigma V(\phi G + \lambda\psi H^*)\frac{\partial^2 F}{\partial X \partial V} \\ &\quad + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} + r(X+G)\frac{\partial F}{\partial X} + (\mu-\kappa) V \frac{\partial F}{\partial V} - rF. \end{aligned}$$

The system of variational inequalities is solved on a grid with nodes $\{(X_k, V_j, t_n) : j = 1, \ldots, J,\ k = 1, \ldots, K,\ n = 0, \ldots, N\}$, where $X_k = k\,\Delta X$, $V_j = j\,\Delta V$, $t_n = n\,\Delta t$, and N is chosen so that $t_N = T_2$. We use $F^n_{j,k}$ to denote $F(X_k, V_j, t_n)$. Using the backward difference to approximate $\partial F/\partial t$, the finite difference approximation of $\mathcal{D}(F)$ at (X_j, V_k, t_n) is

$$\begin{aligned}\mathcal{D}(F)\big|_{(X_j,V_k,t_n)} &= \frac{1}{\Delta t}\Big(-F^{n-1}_{j,k} + \alpha^n_{-1,j,k}\,F^n_{j-1,k-1} + \alpha^n_{0,j,k}\,F^n_{j,k-1} + \alpha^n_{1,j,k}\,F^n_{j+1,k-1} \\ &\qquad +\beta^n_{-1,j,k}\,F^n_{j-1,k} + \beta^n_{0,j,k}\,F^n_{j,k} + \beta^n_{1,j,k}\,F^n_{j+1,k} \\ &\qquad + \gamma^n_{-1,j,k}\,F^n_{j-1,k+1} + \gamma^n_{0,j,k}\,F^n_{j,k+1} + \gamma^n_{1,j,k}\,F^n_{j+1,k+1}\Big),\end{aligned}$$

where

$$\begin{aligned}\alpha^n_{-1,j,k} &= \frac{\rho\sigma V_j(\phi G + \lambda\psi H^n_{j,k})\Delta t}{4(\Delta X)(\Delta V)}, \\ \alpha^n_{0,j,k} &= \frac{(\phi^2 G^2 + 2\lambda\phi\psi G H^n_{j,k} + \psi^2 (H^n_{j,k})^2)\Delta t}{2(\Delta X^2)} - \frac{r(X_k+G)\Delta t}{2\Delta X}, \\ \alpha^n_{1,j,k} &= -\frac{\rho\sigma V_j(\phi G + \lambda\psi H^n_{j,k})\Delta t}{4(\Delta X)(\Delta V)}, \\ \beta^n_{-1,j,k} &= \frac{\sigma^2 V_j^2 \Delta t}{2(\Delta V)^2} - \frac{(\mu-\kappa)V_j\Delta t}{2\Delta V}, \\ \beta^n_{0,j,k} &= 1 - r\Delta t - \frac{(\phi^2 G^2 + 2\lambda\phi\psi G H^n_{j,k} + \psi^2 (H^n_{j,k})^2)\Delta t}{(\Delta X)^2} - \frac{\sigma^2 V_j^2 \Delta t}{(\Delta V)^2}, \\ \beta^n_{1,j,k} &= \frac{\sigma^2 V_j^2 \Delta t}{2(\Delta V)^2} + \frac{(\mu-\kappa)V_j\Delta t}{2\Delta V}, \\ \gamma^n_{-1,j,k} &= -\frac{\rho\sigma V_j(\phi G + \lambda\psi H^n_{j,k})\Delta t}{4(\Delta X)(\Delta V)}, \\ \gamma^n_{0,j,k} &= \frac{(\phi^2 G^2 + 2\lambda\phi\psi G H^n_{j,k} + \psi^2 (H^n_{j,k})^2)\Delta t}{2(\Delta X^2)} + \frac{r(X_k+G)\Delta t}{2\Delta X}, \\ \gamma^n_{1,j,k} &= \frac{\rho\sigma V_j(\phi G + \lambda\psi H^n_{j,k})\Delta t}{4(\Delta X)(\Delta V)}.\end{aligned}$$

Therefore

$$\begin{aligned}F^{n-1}_{j,k} &= \max\{V_j - I, \alpha^n_{-1,j,k}\,F^n_{j-1,k-1} + \alpha^n_{0,j,k}\,F^n_{j,k-1} + \alpha^n_{1,j,k}\,F^n_{j+1,k-1} \\ &\qquad + \beta^n_{-1,j,k}\,F^n_{j-1,k} + \beta^n_{0,j,k}\,F^n_{j,k} + \beta^n_{1,j,k}\,F^n_{j+1,k} \\ &\qquad + \gamma^n_{-1,j,k}\,F^n_{j-1,k+1} + \gamma^n_{0,j,k}\,F^n_{j,k+1} + \gamma^n_{1,j,k}\,F^n_{j+1,k+1}\}\end{aligned} \tag{A-1}$$

when $T_1 \le t_{n-1} < T_2$ and $X_k \ge I$, and

$$\begin{aligned}F^{n-1}_{j,k} &= \alpha^n_{-1,j,k}\,F^n_{j-1,k-1} + \alpha^n_{0,j,k}\,F^n_{j,k-1} + \alpha^n_{1,j,k}\,F^n_{j+1,k-1} \\ &\qquad + \beta^n_{-1,j,k}\,F^n_{j-1,k} + \beta^n_{0,j,k}\,F^n_{j,k} + \beta^n_{1,j,k}\,F^n_{j+1,k} \\ &\qquad + \gamma^n_{-1,j,k}\,F^n_{j-1,k+1} + \gamma^n_{0,j,k}\,F^n_{j,k+1} + \gamma^n_{1,j,k}\,F^n_{j+1,k+1}\end{aligned} \tag{A-2}$$

otherwise.

We start by using the terminal condition to solve for $F^N_{j,k}$ for all j and k. In particular, $F^N_{j,k} = 0$ if $X_k < I$ or $V_j < I$, and $F^N_{j,k} = V_j - I$ at all other nodes. We then work backwards through time in a series of steps, solving for $\{F^{n-1}_{j,k} : j = 1, \ldots, J,\ k = 1, \ldots, K\}$ for $n = N$, $n = N-1$, and so on. Each time step is broken into two stages.

1. In the first stage, we calculate $H_{j,k}^n \equiv H(X_k, V_j, t_n)$ for all j and k. For interior points, if the second order condition $\partial^2 F/\partial X^2 < 0$ holds at (X_k, V_j, t_n), we set $H_{j,k}^n$ equal to the finite difference approximation of equation (5), otherwise we set $H_{j,k}^n = 0$. For all other points we set $H_{j,k}^n = 0$.

2. In the second stage, we use the (explicit) finite difference approximation of the variational equalities, together with the solution $\{(F_{j,k}^n, H_{j,k}^n) : j = 1, \ldots, J,\ k = 1, \ldots, K\}$ at the previous step. If $T_1 \leq t_{n-1} < T_2$ and $X_k \geq I$, so that investment is feasible, then $F_{j,k}^{n-1}$ is given by (A-1); otherwise $F_{j,k}^{n-1}$ is given by (A-2).

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