Capacity and Fairness of MIMO Broadcast Algorithms in Shadow Fading Environments

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***Abstract*— For MIMO broadcast systems the effects of shadowing on the channel capacity and the fairness of the system in sharing the resources amongst multiple users are important issues that need to be addressed. In this paper we consider a variety of capacity-approaching algorithms for MIMO broadcast channels. We compare the performance of these algorithms in terms of their ability to approach the sum-capacity and their fairness in sharing the channel resources amongst the multiple users. We also model distance-based attenuation effects and shadowing, yielding valuable insights into the relative performances of the algorithms under varying SNR conditions. Using a novel approach of mapping the broadcast channel to an "equivalent' single-user channel followed by power scaling, we derive closed-form analytical approximations for the capacity of MIMO broadcast channels. For the more evenly distributed SNR case our approximations are quite accurate, suggesting an analytical lower bound for the well known iterative waterfilling solution. Finally, our Monte Carlo simulations comparing the algorithms point to an inherent tradeoff between sum-capacity and fairness.**

I. Introduction

There is increasing interest in understanding the capacity of multi-user MIMO systems. A communication system where multiple uncoordinated transmitters send their information to a common receiver (e.g., the uplink of a cellular network) is referred to as a multiple access channel (MAC). The dual of this uplink channel is the broadcast channel (BC) [1].

Computing the capacity regions of the MIMO-MAC and the MIMO-BC has attracted much interest. An achievable region for the capacity of a MIMO-BC channel was found in [2], and this region was shown to attain the sum-rate capacity in [1]. The capacity of the uplink MAC is easier to find than the downlink BC, which is a non-convex optimization problem. A duality technique in [1] transforms the non-convex downlink problem into a convex sum-power uplink problem. The duality establishes that the dirty paper [3] rate region for the MIMO-BC is equal to the capacity region of the MIMO-MAC. This in turn implies that the sum-capacities of the MIMO-BC and MIMO-MAC are equal to each other. This duality also provides a method to convert the uplink covariance matrices to equivalent downlink covariance matrices.

The great majority of the work on the MIMO-BC has assumed that the users have equal SNR and has concentrated on the sum-rate capacity. In this paper, we have a different focus. In the MIMO-BC, the multiple users are usually located separately. Hence, they experience different SNRs due to distance and shadowing effects. This has a considerable impact on the capacity and on the relative merits of broadcast algorithms. Furthermore, with the variation in SNR comes an increased likelihood that the capacity allocations are unequal. Fairness, therefore, becomes a more important issue. Hence, we focus on the effects of shadowing and on fairness issues.

In [4], an iterative waterfilling (ITWF) technique is used to find the sum-rate capacity of the MIMO-MAC. In [5] the iterative algorithms are utilized to find the maximum sum-rate of a MIMO-BC channel. These ITWF methods are complex and do not result in a fair sharing of the resources at lower SNRs. Recently researchers have derived the capacity for the MIMO-BC where each user achieves the same capacity [6]. This is the perfectly fair approach. Again, the approach is complex and in scenarios with variable SNR, it might offer considerably less capacity than the higher SNR users would expect. Alternatively, ITWF can be modified using a weighted sum-rate maximization to increase fairness as done in [7]. However, such an approach may not be suitable in the highly shadowed environments that we study. Hence, we are motivated to look at simpler techniques that could approach ITWF capacity but with relatively low complexity and reasonable fairness to all users.

Beamforming (BF) is one alternative in which multiple antennas at the transmitter and receiver can be used to provide array and diversity gains in place of capacity gains [8]. In BF, transmission takes place over one or more eigenchannels. As another alternative, antenna subset selection is a cost effective solution to the increased hardware needs of MIMO systems [9]. Antenna selection can be applied to either or both ends of the MIMO link. In [10] we have considered transmit and receive antenna selection methods for a single user system and have provided a method for analyzing the capacity of the selection schemes via a simple power scaling factor. The transmit/receive antenna selection schemes in [10] can be applied to multiple users. Finally, we consider the baseline case where the transmitter sends independent signals to each user with equal power. The four methods provide a useful hierarchy of complexity with ITWF being the most complex, requiring full channel feedback and extensive processing at the transmitter.

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Beamforming has lighter demands for feedback and processing and the selection methods are simple in comparison. The baseline case requires no feedback and no extra processing.

To compare these methods, we adopt the analysis philosophy of [10]. Here, the idea is to derive approximately "equivalent" single-user MIMO capacities so that we can compare the schemes using simple power scaling and system size parameters. Such an approximate analysis is particularly useful since performance results on the MIMO-BC are rare.

The key results and contributions of this paper are:

- We provide a simple way to compare techniques through simple power scaling and system size parameters.
- We show the extent to which ITWF is "unfair" in variable SNR environments.
- We demonstrate that suboptimal approaches can be close to ITWF, yet remain fairer and simpler.
- We find that suboptimal approaches can be even closer to ITWF in the presence of shadowing.

II. System Model for MIMO-BC Channel

Consider a MIMO-BC system with a single transmitter (TX) and K receivers (RX). The TX has M antennas and receiver i has r_i antennas. The link equation for user i is given by [1]

$$\boldsymbol{y}_i = \boldsymbol{H}_i \boldsymbol{x} + \boldsymbol{n}_i, \quad \text{for } i = 1, \ldots, K \tag{1}$$

where $\boldsymbol{y}_i$ is the $r_i \times 1$ received signal vector, $\boldsymbol{x}$ is the complex $M \times 1$ transmitted signal vector, $\boldsymbol{n}_i$ is an $r_i \times 1$ additive white complex Gaussian noise vector with unit modulus variance, and $\boldsymbol{H}_i$ is an $r_i \times M$ complex channel matrix. Throughout, we assume that the entries of $\boldsymbol{H}_i$ are i.i.d. zero-mean complex Gaussian variables with modulus variance given by the shadow fading variable, denoted Γ_i. Standard lognormal shadowing is assumed so that $\Gamma_i = A L_i d_i^{-\gamma}$, where A is a constant, L_i is lognormal with a standard deviation of σdB, d_i is the distance of user i from the TX and γ is the path-loss exponent. It is often convenient to write $\boldsymbol{H}_i$ in its standardized form, $\boldsymbol{H}_i = \sqrt{\Gamma_i}\boldsymbol{U}_i$, where the entries of $\boldsymbol{U}_i$ are i.i.d. unit variance complex Gaussians and Γ_i is the SNR for user i. Matrices or vectors containing such elements are denoted i.i.d. $\mathcal{CN}(0,1)$. Note that the analysis which follows can be extended to Ricean, Nakagami and correlated Rayleigh channels.

The transmitted signal is the superposition of the signals intended for the K users. Hence, $\boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_K$, where $\boldsymbol{x}_i$ is intended for user i and has the transmit covariance matrix Σ_i. We assume a sum-power constraint at the TX, so that $\sum_{i=1}^{K} \text{trace}(\Sigma_i) \leq P$.

Analysis of the MIMO-BC is facilitated by considering the dual MIMO-MAC system, where the K users transmit to the M-element array and the system equation is [1]

$$\boldsymbol{y} = \sum_{i=1}^{K} \boldsymbol{H}_i^{\dagger} \boldsymbol{x}_i + \boldsymbol{n} \tag{2}$$

where $\boldsymbol{y}$ is the $M \times 1$ received signal vector, $\boldsymbol{x}_i$ is the complex $r_i \times 1$ transmitted signal vector from user i and $\boldsymbol{n}$ is an $M \times 1$ i.i.d. $\mathcal{CN}(0,1)$ additive noise vector. The channel matrix $\boldsymbol{H}_i$ is as defined for the BC channel and $\dagger$ denotes conjugate transposition. In the MIMO-MAC system, the i-th user employs the covariance matrix $\boldsymbol{Q}_i$ for signal $\boldsymbol{x}_i$.

III. Performance Metrics and Algorithms

A. Performance Metrics

In this paper we consider performance metrics for both throughput and fairness. For throughput, we consider the sum-capacity of ITWF and also the achievable sum-rates offered by the sub-optimal approaches. From the viewpoint of the dual MIMO-MAC, the sum-capacity can be written as [4],[5]

$$C = \max_{\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_K} \log_2 \left[\det \left(\boldsymbol{I}_M + \sum_{i=1}^{K} \boldsymbol{H}_i^{\dagger} \boldsymbol{Q}_i \boldsymbol{H}_i \right) \right] \quad \text{b/s/Hz} \tag{3}$$

where the maximization in (3) is performed over all positive, semi-definite matrices, $\boldsymbol{Q}_i$, that satisfy $\sum_{i=1}^{K} \text{trace}(\boldsymbol{Q}_i) \leq P$. When suboptimal algorithms are used to select the $\boldsymbol{Q}_i$ matrices in (3), then an achievable rate is obtained.

The second family of metrics gauges the fairness of the strategy proposed. We consider the proportion of users in active communication, the proportion of the total number of spatial channels used, the proportion of the transmit power allocated to the dominant user and the minimum rates achieved.

B. Iterative Waterfilling

Iterative waterfilling has been shown in [5] to achieve the sum-capacity of the MIMO-BC channel. Note that the duality of the MIMO-MAC and MIMO-BC channels discussed in [1] allows us to perform iterative waterfilling in the MAC domain and to convert the resulting optimal $\boldsymbol{Q}_i$ matrices to optimal $\boldsymbol{\Sigma}_i$ matrices for use at the BC transmitter.

C. Equal Power Independent Uncorrelated Transmission

As a simple suboptimal power allocation method we consider equal power independent uncorrelated transmission (EPIUT) in the MIMO-MAC, where $\boldsymbol{Q}_i = \frac{P}{Kr_i}\boldsymbol{I}_{r_i}$, or in the MIMO-BC, where $\boldsymbol{\Sigma}_i = \frac{P}{KM}\boldsymbol{I}_M$. These are the baseline (do nothing) approaches and act as convenient benchmarks for the other approaches. Note that we have found EPIUT in the MIMO-MAC to give better rate values for all systems considered, and hence we use this technique in this paper.

D. Beamforming Techniques

As in the previous sub-section, beamforming can be implemented in either the MIMO-BC or MIMO-MAC. Again, we have found that the implementation in the MIMO-MAC achieves higher rates, and so we only consider this version. The advantage of employing EPIUT and BF in the MAC makes intuitive sense. In the BC, communicating with multiple users suffers from inherent problems of interference. At the MAC end, the approaches are automatically decoupled and the duality then provides a method for implementation at the BC end. Hence, throughout this paper, we develop methods for the MIMO-MAC, and the MIMO-BC uses the duals of these methods. In the MIMO-MAC, beamforming consists of user

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i transmitting ℓ_i symbols along the ℓ_i principal eigenvectors. Specifically, user i selects the top $\ell_i \leq \min(r_i, M)$ eigenchannels, which have eigenvalues $\lambda_1^{(i)} > \lambda_2^{(i)} > \ldots > \lambda_{\ell_i}^{(i)}$. These eigenvalues are the ordered eigenvalues of the $r_i \times r_i$ matrix $\boldsymbol{H}_i\boldsymbol{H}_i^\dagger$. We denote the eigenvector of $\boldsymbol{H}_i\boldsymbol{H}_i^\dagger$ corresponding to $\lambda_j^{(i)}$ by $\boldsymbol{v}_j^{(i)}$. Then, beamforming results if we use a $\boldsymbol{Q}_i$ matrix defined by

$$\boldsymbol{Q}_i = \left[\boldsymbol{v}_1^{(i)}\boldsymbol{v}_2^{(i)}\ldots\boldsymbol{v}_{\ell_i}^{(i)}\right]\boldsymbol{P}_i\left[\boldsymbol{v}_1^{(i)}\boldsymbol{v}_2^{(i)}\ldots\boldsymbol{v}_{\ell_i}^{(i)}\right]^\dagger \quad (4)$$

where $\boldsymbol{P}_i = \text{diag}(P_{i1}, P_{i2}, \ldots, P_{i\ell_i})$ is a diagonal matrix which allocates powers to the eigenchannels.

1) Power Allocation: We consider both P_{ij} constant and $P_{ij} \propto \lambda_j^{(i)}$ in this paper.

2) Selecting Eigenchannels: A key decision in the beamforming approach is the number of eigenchannels to be employed by each user. Hence, we consider the family of "fair" approaches, denoted BF2, where each user selects their largest L eigenchannels and $L \leq \min(r_1, r_2, \ldots, r_K, M)$. An alternative approach, denoted BF1, is performance based and selects the best L' eigenchannels irrespective of the user. As an example combination of *1)* and *2)*, for BF2 and proportional power, we have $\boldsymbol{P}_i = P(K\sum_{j=1}^{L}\lambda_j^{(i)})^{-1}\text{diag}(\lambda_1^{(i)}, \ldots, \lambda_L^{(i)})$, and for constant power allocation, we have $\boldsymbol{P}_i = P(KL)^{-1}\boldsymbol{I}_L$.

E. Selection Techniques

With selection methods we are again faced with the trade-off between fairness and performance. We consider the following approaches:

1) Selecting Users: The optimum way to select an individual user is to select the link with the largest single-user waterfilling capacity. Although optimal, this approach requires substantial feedback and is complex. A simpler approach is to select the user with the greatest link gain, i.e., $\max_{i=1,\ldots,K}||\boldsymbol{H}_i||^2$. We denote this approach, SLG. This is simpler computationally and requires much less feedback. Once the user is selected, we assume they employ EPIUT. This aproach can be amended to cater for different link dimensions.

2) Selecting Antennas: We consider the family of "fair" approaches, denoted S2, where each user selects their best L antennas and $L \leq \min(r_1, r_2, \ldots, r_K)$. The alternative approach, denoted S1, is performance based and selects the best L' antennas irrespective of the user. For simplicity, antennas are ranked on the basis of the link gain rather than capacity calculations [10]. Hence, user i ranks antenna j first if row j of $\boldsymbol{H}_i$ has the largest row norm, denoted $\Gamma_i\alpha_{ij} = ||(\boldsymbol{H}_i)_{j.}||^2$, where $(\boldsymbol{H}_i)_{j.}$ is row j of $\boldsymbol{H}_i$. The ordered row norms are denoted by $\Gamma_i\alpha_{i(1)} > \cdots > \Gamma_i\alpha_{i(r_i)}$. As in the beamforming approaches, there remains the issue of power allocation, and we employ the same two basic approaches here. A selected antenna for user i, say antenna j, can be allocated power P_{ij}, where P_{ij} is either constant or $P_{ij} \propto \alpha_{ij}$.

IV. Equivalent Single-User System Analysis

The optimal iterative waterfilling approach can be simulated, but little analytical progress appears possible. For most of the other approaches, considerable insight can be achieved by converting the systems to the "equivalent" single-user systems where capacity results are well known and established results can be employed [11]. We denote the dimension of a single-user MIMO system with n_T antennas at the transmitter and n_R antennas at the receiver by (n_T, n_R).

A. Equal Power Independent Uncorrelated Transmission

Using the EPIUT approach in the MIMO-MAC gives $\boldsymbol{Q}_i = \frac{P}{Kr_i}\boldsymbol{I}_{r_i}$. In the case where $r_i = r$ and $\Gamma_i = \Gamma$ for all users, then (3) gives the rate R, where

$$R = \log_2\left[\det\left(\boldsymbol{I}_M + \frac{P\Gamma}{Kr}\boldsymbol{U}\boldsymbol{U}^\dagger\right)\right] \quad \text{b/s/Hz} \quad (5)$$

with $\boldsymbol{U} = \left[\boldsymbol{U}_1^\dagger \ldots \boldsymbol{U}_K^\dagger\right]$. Hence, the resulting rate is that of a single-user MIMO system with M RX antennas, Kr TX antennas (i.e., a (Kr, M) system) and equivalent SNR $= P\Gamma$ for an i.i.d. Rayleigh channel. In more general cases with differing antenna numbers or link gains, Γ_i, we can also express the rate in terms of a single-user system, but the channel is no longer i.i.d. Instead, we have

$$R = \log_2\left[\det\left(\boldsymbol{I}_M + \boldsymbol{U}\boldsymbol{D}_1\boldsymbol{U}^\dagger\right)\right] \quad \text{b/s/Hz} \quad (6)$$

with $\boldsymbol{U}$ defined as before and diagonal matrix $\boldsymbol{D}_1$ made up of K diagonal matrices of the form $\frac{P\Gamma_i}{Kr_i}\boldsymbol{I}_{r_i}$. The mean rate for such a system can be computed as discussed in [12].

B. Beamforming Techniques

A coarse approximation to beamforming can be constructed in the following way. For simplicity, we only show the approach for the BF1 case with $L = 1$, where each user communicates over their maximal eigenchannel. For this case, (4) collapses to $\boldsymbol{Q}_i = v_1^{(i)}P_{i1}v_1^{(i)\dagger}$. We require the singular value decomposition (SVD) $\boldsymbol{H}_i = \boldsymbol{V}_i\boldsymbol{\Lambda}_i\boldsymbol{S}_i$, where $\boldsymbol{V}_i$, $\boldsymbol{S}_i$ are unitary matrices, $\boldsymbol{\Lambda}_i$ is diagonal with principal $r_i \times r_i$ submatrix given by $\text{diag}\left(\sqrt{\lambda_1^{(i)}}, \ldots, \sqrt{\lambda_{r_i}^{(i)}}\right)$ and $v_1^{(i)}$ is the first column of $\boldsymbol{V}_i$. Substituting $\boldsymbol{Q}_i$ and the SVD for $\boldsymbol{H}_i$ into (3) gives

$$\begin{aligned} R &= \log_2\left[\det\left(\boldsymbol{I}_M + \sum_{i=1}^{K}\boldsymbol{S}_i^\dagger\,\text{diag}(P_{i1}\lambda_1^{(i)}, 0, \ldots, 0)\,\boldsymbol{S}_i\right)\right] \\ &= \log_2\left[\det\left(\boldsymbol{I}_M + \boldsymbol{S}^\dagger\boldsymbol{D}_2\,\boldsymbol{S}\right)\right] \quad \text{b/s/Hz} \end{aligned} \quad (7)$$

where $\boldsymbol{S}^\dagger = \left[(\boldsymbol{S}_1^\dagger)_{.1}\ (\boldsymbol{S}_2^\dagger)_{.1}\ \cdots\ (\boldsymbol{S}_K^\dagger)_{.1}\right]$ is an $M \times K$ matrix containing the first columns of the $\boldsymbol{S}_i^\dagger$ matrices and $\boldsymbol{D}_2 = \text{diag}(P_{11}\lambda_1^{(1)}, \ldots, P_{K1}\lambda_1^{(K)})$. Now $\boldsymbol{S}^\dagger$ contains independent columns with column norm equal to 1. Hence, as a very coarse approximation, we might replace (7) by

$$R = \log_2\left[\det\left(\boldsymbol{I}_M + \boldsymbol{U}^\dagger\boldsymbol{D}_3\,\boldsymbol{U}\right)\right] \quad \text{b/s/Hz} \quad (8)$$

where $\boldsymbol{U} : K \times M$ is i.i.d. $\mathcal{CN}(0,1)$ and $\boldsymbol{D}_3 = \boldsymbol{D}_2/M$. Note that in (8) we replace the matrix $\boldsymbol{S}$, containing singular vectors, by an i.i.d. Gaussian matrix with the same mean column power.

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C. Selection Techniques

In principle, the user selection method based on waterfilling can be analyzed approximately by using results on Gaussian order statistics. However, this approach depends on the mean and variance of the capacity, and these are not known for waterfilling. Hence, we do not provide any analysis here.

User selection based on link gain (SLG) can be handled using the approach developed in [10]. For simplicity, we consider the case where $r_i = r$ for all users. The link gain for user i is then $\Gamma_i \alpha_i = \Gamma_i \sum_{j=1}^{r} \alpha_{ij} = \Gamma_i Y_i$, where Y_i has a complex χ^2 distribution with rM degrees of freedom. Hence, the link gain of the selected user is $g_{\max} = \max(\Gamma_1 Y_1, \ldots, \Gamma_K Y_K)$. The approach in [10] is to create an equivalent $r \times M$ channel matrix for the chosen user, $\boldsymbol{H}_{\text{equiv}} = \sqrt{\mathrm{E}(g_{\max})/(rM)}\,\boldsymbol{U}$, which has the same mean link gain and for which $\boldsymbol{U}$ is i.i.d. $\mathcal{CN}(0,1)$. This gives the rate

$$R = \log_2 \left[\det \left(\boldsymbol{I}_M + \frac{\mathrm{E}(g_{\max})}{rM} \boldsymbol{U}^\dagger \boldsymbol{U} \right) \right] \quad \text{b/s/Hz}. \tag{9}$$

Note that $\mathrm{E}(g_{\max})$ can be obtained from standard order statistic results for independent, non-identical random variables [13].

A similar approach holds for the antenna selection methods. We describe the approach for the S2 method with $L = 1$, with a similar methodology possible for the other techniques. In S2 with $L = 1$, user i selects antenna j if $\alpha_{ij} = \max(\alpha_{i1}, \ldots, \alpha_{ir_i})$ and allocates power P_{ij}. The corresponding $\boldsymbol{Q}_i$ matrix is $\boldsymbol{Q}_i = \mathrm{diag}(0, \ldots, P_{ij}, \ldots, 0)$ with the non-zero entry in position j. For the $L = 1$ case, we can simplify the notation, dropping the j subscript in P_{ij}, denoting the j-th row of $\boldsymbol{H}_i$ by $\boldsymbol{h}_i$ and using the order statistic notation $\alpha_{i(1)}$ for the maximum row norm. Substituting for $\boldsymbol{Q}_i$ in (3) gives

$$R = \log_2 \left[\det \left(\boldsymbol{I}_M + \sum_{i=1}^{K} P_i \boldsymbol{h}_i^\dagger \boldsymbol{h}_i \right) \right]. \tag{10}$$

Following [10] again, we replace the rows $\boldsymbol{h}_i$ by $\sqrt{\Gamma_i E(\alpha_{i(1)})}\,\boldsymbol{u}_i$, where $\boldsymbol{u}_i$ is an i.i.d. $\mathcal{CN}(0,1)$ vector and (10) becomes

$$R = \log_2 \left[\det \left(\boldsymbol{I}_M + \boldsymbol{U}^\dagger \boldsymbol{D}_4 \boldsymbol{U} \right) \right] \quad \text{b/s/Hz}. \tag{11}$$

In equation (11), $\boldsymbol{U}$ is a $K \times M$ i.i.d. $\mathcal{CN}(0,1)$ matrix and $\boldsymbol{D}_4 = \mathrm{diag}[P_1 \Gamma_1 \mathrm{E}(\alpha_{1(1)}), \ldots, P_K \Gamma_K \mathrm{E}(\alpha_{K(1)})]$. Note that the values of $\mathrm{E}(\alpha_{i(1)})$ can be obtained from standard order statistic results for i.i.d. random variables [13].

V. Results and Monte Carlo Simulations

A. System Comparison

Of the equivalent systems constructed, only (5) and (9) are in the form of rates for i.i.d. MIMO channels. The remaining equivalent systems, e.g., (6), (7), (8) and (11), have diagonal $\boldsymbol{D}_i$ matrices in the quadratic forms. This form of log determinant can be analyzed [12], but greater insight is achieved by converting all cases to approximate single-user MIMO systems in i.i.d. channels. Going further, we can also approximate these systems by replacing $\boldsymbol{D}_i$ by $\mathrm{E}\{\mathrm{trace}(\boldsymbol{D}_i)/\nu_i\}\,\boldsymbol{I}_{\nu_i}$, where ν_i is the dimension of $\boldsymbol{D}_i$. With this approach, we can compare the different algorithms considered in terms of equivalent SNR and dimension, as shown in Table I. Note that the terms, *equal* and *proportional* used in Table I refer to the power allocation method used, which can be equal or proportional to the eigenvalues (in BF1 and BF2) or the antenna link gains (in S1 and S2). Using Table I, we have the approximations

$$R \approx \log_2 \left[\det \left(\boldsymbol{I}_M + \frac{\mathrm{SNR}}{n_T} \boldsymbol{U}^\dagger \boldsymbol{U} \right) \right] \quad \text{b/s/Hz} \tag{12}$$

where SNR and n_T are from Table I and $\boldsymbol{U}$ is $n_T \times M$. Note that the derivations and resulting equivalent MIMO systems for S1 and BF1 have been omitted for reasons of space.

TABLE I

Equivalent MIMO Systems

Method	Dimension (n_T, n_R)	Equivalent SNR
EPIUT ($r_i = r, \Gamma_i = \Gamma$)	(Kr, M)	$P\Gamma$
EPIUT (general case)	$\left(\sum_{i=1}^{K} r_i, M\right)$	$\frac{P}{K}\sum_{i=1}^{K}\Gamma_i$
BF2 (equal)	(KL, M)	$\frac{P}{K}\sum_{i=1}^{K}\Gamma_i \mathrm{E}\left[\frac{\sum_{j=1}^{L}\lambda_j^{(i)}}{ML}\right]$
BF2 (proportional)	(KL, M)	$\frac{P}{K}\sum_{i=1}^{K}\Gamma_i \mathrm{E}\left[\frac{\sum_{j=1}^{L}\lambda_j^{(i)2}}{M\sum_{j=1}^{L}\lambda_j^{(i)}}\right]$
SLG	(r_i, M)	$\mathrm{E}(g_{\max})/M$
S2 (equal)	(KL, M)	$\frac{P}{K}\sum_{i=1}^{K}\Gamma_i \mathrm{E}\left[\frac{\sum_{j=1}^{L}\alpha_{i(j)}}{ML}\right]$
S2 (proportional)	(KL, M)	$\frac{P}{K}\sum_{i=1}^{K}\Gamma_i \mathrm{E}\left[\frac{\sum_{j=1}^{L}\alpha_{i(j)}^2}{M\sum_{j=1}^{L}\alpha_{i(j)}}\right]$

B. Performance Results

We have simulated three users randomly located in a single hexagonal cell of radius 500m, where $M = 4$ and $r_i = 2$ for all users. No specific cellular structure or inter-cellular interference is considered. The shadow fading is parameterized by a path-loss exponent of 3, and the standard deviation of the lognormal variable is 8dB. The constant, A in the definition of Γ_i is chosen so that the mean SNR received by an arbitrary user is $\mathrm{SNR}_{\text{av}} \in \{0\text{dB}, 10\text{dB}\}$. The methods simulated include ITWF, EPIUT, S1 (best 3 antennas), S2 (best antenna per user), BF1 (best 3 eigenchannels) and BF2 (best eigenchannel per user). It is important to keep in mind the types of SNR used: SNR_{av} is defined above, the equivalent SNR is given in Table I and the SNR of user i is Γ_i.

Figures 1 and 2 show a comparison of the simulated sum-rate CDFs with those obtained from the equivalent systems in Table I. Considering the simplicity of the equivalent systems, BF2 (Fig. 1) and S2 (Fig. 2) show a surprisingly good agreement over the whole CDF at both SNR levels for the non-shadowing case. Similar accuracy is found for the EPIUT, BF1 and S1 approaches. With shadowing, the equivalent SNR averages over the unequal links, and this is beneficial to the sum-rate. Hence, we observe optimistic performance from the equivalent system, although the mean values are quite similar. Increased accuracy is expected through the use of the non-i.i.d equivalent systems, e.g., (6), (7), (8) and (11). It is clear from the SNR column in Table I that large discrepancies in the Γ_i

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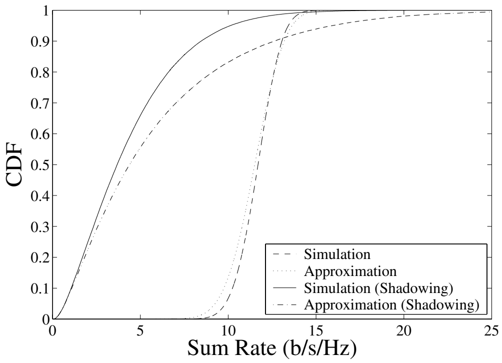

Fig. 1. Sum-rate distributions for BF2 ($\mathrm{SNR}_{\mathrm{av}}$ = 10dB).

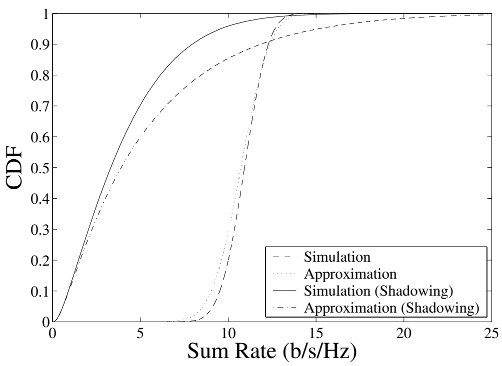

Fig. 2. Sum-rate distributions for S2 ($\mathrm{SNR}_{\mathrm{av}}$ = 0dB).

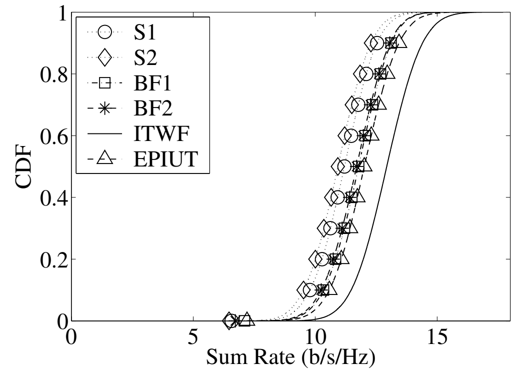

Fig. 3. Sum-rate distributions with no shadowing (SNR = 10dB).

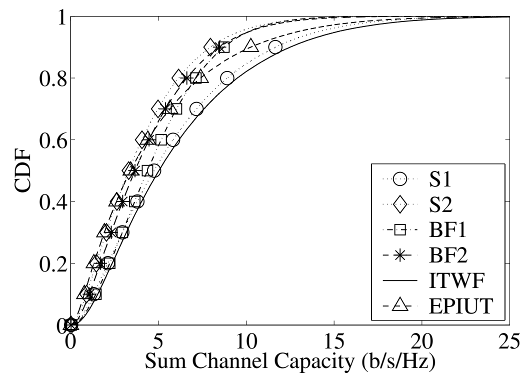

Fig. 4. Sum-rate distributions with shadowing ($\mathrm{SNR}_{\mathrm{av}}$ = 10dB).

values will lead to some users dominating the performance, and this relationship to the shadowing takes the same form for EPIUT, BF2 and S2. Also, the eigenvalues in BF2 and the row norms in S2 contribute in the same way to the equivalent SNR. Hence, BF2 (or S2) becomes efficient when the leading eigenchannels (or row norms) are dominant, and this property is accentuated by proportional power allocation. Furthermore, since the leading eigenvalues tend to dominate more than the leading row norms, we see that BF2 will tend to give a higher equivalent SNR than S2 and improved sum-rate performance.

Figures 3 and 4 compare the simulated sum-rate CDFs for the case with no shadow fading (SNR = 10dB in Fig. 3) and shadowing ($\mathrm{SNR}_{\mathrm{av}}$ = 10dB in Fig. 4). In the absence of shadowing we observe the well-known approximately Gaussian CDF shapes. With shadowing, the averaging over the Γ_i variables results in a very different shape (see Fig. 4) similar to those found in [14]. Figures 3, 4 and 5 also allow us to compare the different algorithms at different SNR levels and with/without shadowing. Figure 3 shows the CDF of the sum-rate when SNR=10dB for each user. ITWF offers the largest capacity, but the simpler EPIUT and the two BF approaches are not far behind. This is to be expected because waterfilling advantages over equal power are more prominent under low SNR conditions. EPIUT has a higher dimensionality than the BF approaches (sending 6 rather than 3 symbols), yielding a better performance. The selection antennas curves have the same dimensionality as the BF curves but are weaker due to their lower equivalent SNRs. Note that S1 and BF1 are superior to S2 and BF2 as expected, but the difference is slight.

Figures 4 and 5 show the sum-rate CDFs with $\mathrm{SNR}_{\mathrm{av}} \in \{0\mathrm{dB}, 10\mathrm{dB}\}$. The relative performances in this case have changed with S1 and BF1 performing almost as well as ITWF. With the shadowing effects, skewing the channel in favor of particular users, concentrating the array gain in certain directions or using a subset of antennas yields excellent sum-rate gains. Note that S1 may provide a convenient lower bound on ITWF, since an equivalent MIMO system can be used to generate analytical approximations to the mean and variance. Hence, in the presence of shadowing we have the possibility of developing a tight analytical lower bound to the mean sum-rate and potentially to the CDF. This depends on an accurate equivalent system and is the subject of further research.

C. Fairness Results

In Sec. III-A we defined four metrics for investigating the (un)fairness of various broadcast algorithms. These are now compared in different environments over a range of SNR values for the ITWF approach.

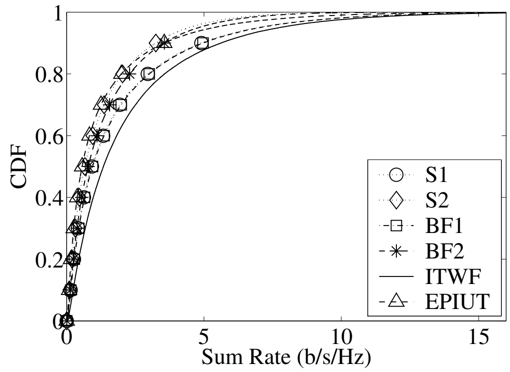

Fig. 5. Sum-rate distributions with shadowing ($\mathrm{SNR}_{\mathrm{av}}$ = 0dB).

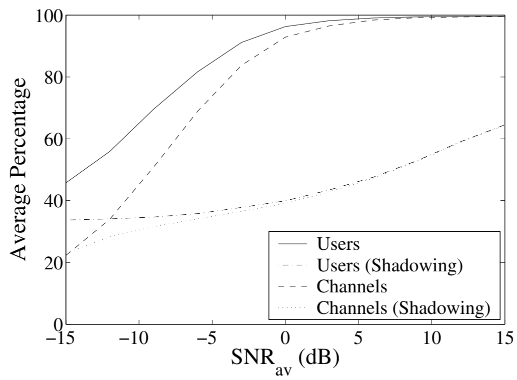

Fig. 6. Fairness of ITWF: percentage of active spatial channels and users, with and without shadowing.

1) Percentage of Active Users and Channels: Figure 6 shows the increasing fairness of ITWF as $\mathrm{SNR}_{\mathrm{av}}$ increases. At high $\mathrm{SNR}_{\mathrm{av}}$, ITWF is essentially fair with each user getting a portion of the available power. However, for lower $\mathrm{SNR}_{\mathrm{av}}$ values and in shadowing, ITWF is inherently unfair. This can cause one or more users to be "shutout" and not receive any signal for the time slot. In a fast-fading environment the lengths of these "shutouts" are minimal due to rapid changes in the channel parameters. In a slow-fading environment, these lengths of time can be significant. Note that in this particular simulation there were three users, and thus the percentage is lower bounded by 33%. Hence, the shadowed case is experiencing communication with only 1 user extremely frequently. Figure 6 also indicates that the proportion of open spatial channels is very similar to that of active users. Note that S2 and BF2 by definition communicate with all 3 users, and simulations show that S1 and BF1 communicate with around 70% of users at $\mathrm{SNR}_{\mathrm{av}}$ = 0dB and 10dB.

2) The Dominant User and Minimum Rates: Simulations at $\mathrm{SNR}_{\mathrm{av}}$ = 0dB and 10dB show that the dominant user receives approximately 94% and 83% of the transmit power in ITWF in shadowing and 47% without shadowing. This power imbalance and the tendency of ITWF to select only 1 or 2 users means that, although the the minimum rate is usualy zero, the minimum rate of a user in active communication ranges from 53% at 10dB to 85% at 0dB. The other methods are less sensitive to $\mathrm{SNR}_{\mathrm{av}}$. For both S1 and BF1, the minimum rate of a user in active communication is approximately 15% and this drops to 6% for S2 and BF2.

VI. Conclusion

We have studied the effects of shadowing and the fairness properties of a variety of capacity-approaching algorithms for MIMO broadcast channels. We have found that shadowing can have a large impact on the relative performances of the algorithms as well as their fairness in distributing the channel resources. ITWF always achieves the best capacity, but at the cost of fairness, as it tends to allocate all of the resources to the best user, particularly in shadowing conditions. In contrast, we have found that relatively simple selection algorithms can approach the capacity while still maintaining a better level of fairness among the users than ITWF for varying SNR conditions. Finally, we have developed a new analytical approximation method for the MIMO-BC capacity using power-scaled equivalent single-user channels which should prove to be a useful tool in better understanding these channels.

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