Compactness Under Constructive Scrutiny

Hannes Diener

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Abstract

The aim of this thesis is to understand the constructive scope of compactness.

We show that it is possible to define, constructively, a meaningful notion of compactness in a more general setting than the uniform/metric space one. Furthermore, we show that it is not possible to define compactness constructively in a topological space.

We investigate exactly what principles are necessary and sufficient to prove classically true theorems about compactness, as well as their antitheses.

We develop beginnings of a constructive theory of differentiable manifolds.

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Contents

Chapter 1

Introduction

> Ah, compactness! What a wonderful property ... everything works smoother, easier, more complete as long as we have a compact space, manifold etc. [31]

It was in his PhD thesis of 1906 that Fréchet gave a name to a concept that today is at the very heart of topology. He called a space **compact** if any infinite bounded subset contains a point of accumulation [24]—something known today as sequential compactness. As so often, the basic idea had been lurking around the mathematical community for a while and it is not possible to attribute its discovery to one single person as easily as we can its naming [22]. Fréchet's definition generalised the idea of what is known today as the **Bolzano-Weierstrass** theorem.

> Every bounded sequence has a convergent subsequence.

A proof for this theorem was first given by Bolzano, lost and forgotten, and it was later re-proved by Weierstrass; hence resulting in its hyphenated name. It is no surprise that Fréchet's definition of compactness appeared in the same thesis that introduced the notion of metric spaces;[1] for it was this new, more general

[1] The name 'metric space', though, is due to Hausdorff.

type of space that provided the language to look at the Bolzano-Weierstrass result from the other side, and use it not as a theorem but as a definition.

A similar development led to the notion of open-cover compactness. In 1914, eight years after Fréchet's seminal thesis, Hausdorff gave the definition of what he called a topological space, but now is known as a Hausdorff space. And another eight years later Kuratowski generalised this definition and introduced topological spaces as mathematicians know and love them today. By then it was long known that a continuous function on a closed interval is also uniformly continuous. Around 1850, Dirichlet gave a proof of this theorem implicitly using an open cover that was refined to a finite one. Later Borel and Lebesgue recognised the more general content of this and other proofs, and showed that a space is complete and totally bounded if and only if

(1.1) every open cover admits a finite subcover.

In the same way that Fréchet's introduction of metric spaces widened the applicability of Bolzano's and Weierstrass's ideas, the introduction of topological spaces made it possible to use property (1.1) as a definition for compactness.

The story of compactness is a story of success; such a broad success that it is almost tempting to finish this short historical overview with the words: "and compactness and the mathematician lived happily ever after." But every child knows that life is more complicated than a fairy tale. And so we cannot stop here. Unfortunately, the Bolzano-Weierstrass definition of compactness does not work from a constructivist's point of view; not because it is inherently contradictory, but because we cannot hope to prove that any nontrivial space is sequentially compact. Its constructive dubiousness was established by Mandelker, who showed in [35] that the Bolzano-Weierstrass principle is equivalent to the non-constructive **limited principle of omniscience**

> **LPO:** For any binary sequence $(\alpha_n)_{n\geqslant 1}$, either $\alpha_n = 0$ for all $n \in \mathbb{N}$, or there exists $n \in \mathbb{N}$ such that $\alpha_n = 1$.

The open-cover definition of compactness is not quite as problematic as the Bolzano-Weierstrass one. In fact, if we accept additional principles such as Brouwer's fan theorem, then we can show that any totally bounded and complete metric space satisfies (1.1). However, other (semi-)constructive principles such as the Church-Markov-Turing thesis imply that we can explicitly cover the unit interval with countably many open intervals $(I_n)_{n \geqslant 1}$ such that for any natural number N, we can construct a real number $x \in [0,1]$ such that $x \notin \bigcup_{n=1}^{N} I_n$ [2, p. 68]. That means that under such assumptions it is not only not possible to show that the unit interval satisfies (1.1), it is provably false. Thus, unless we accept additional principles, constructively the only useful definition for compactness is totally boundedness together with completeness.

The aim of this thesis is to understand the constructive meaning of compactness. Some constructive research has been done on compactness. This includes the paper by Mandelker [35], which we already mentioned. Others, such as [29, 30], will be cited later, once they become relevant to our investigations. It is almost unnecessary to stress that it is impossible to cover all aspects of compactness within the limited scope of a PhD thesis. One such missing aspect, for example, is a discussion of compactness in the theory of formal (or point-free) topology. Some details on this approach can be found in [37, 38].

Many of the themes, ideas, and issues that were mentioned in this introduction will re-appear throughout this thesis, some more than once. Before we can scrutinise compactness though, we need to introduce several notions and make necessary definitions in the following chapter. We then start by asking and answering, in Chapter 3, the (seemingly naive) question, whether there might be another constructive approach to compactness; one that is not tied to metric/uniform spaces. Chapter 4 rigorously investigates principles and theorems linked with compactness. At first glance, the last chapter might seem disconnected from the rest of this thesis; it contains beginnings of a constructive theory of differentiable manifolds. However, there are two connections to our main theme—compactness. The first connection is that the development of a constructive theory of differentiable manifolds is where this thesis started initially. It was the work on differentiable manifolds that motivated a closer, constructive look at compactness. The second connection is a negative one. Chapter 5 shows how, when working constructively, not having access to compactness arguments can complicate constructive proofs severely. Nevertheless, we would like to add that we believe that this complication is worth the effort, for it clarifies the constructive content of results and, in Bishop's words, maintains meaningful distinctions.

1.1 Notable results

Although we hope that the reader will find a well rounded, accessible and interesting thesis throughout, we would like to draw his attention to the following notable selection of results.

- *Chapter 3.* The notion of 'neatly located' sets in an apartness space is neat. Although classically all sets are neatly located, if LPO is provably false then the class of neatly located sets coincides with the class of totally bounded ones, as seen is in Corollary 3.1.7. The notion of 'neatly compact' is less elegant. Nevertheless it generalises compactness to a large class of spaces (Corollary 3.1.14). Furthermore, by means of a counterexample given in Proposition 3.2.2, it is shown that we cannot hope to generalise compactness any further.

- *Chapter 4.* It is in this chapter that for the (to our knowledge) first time the various versions of Brouwer's fan theorem have been systematised. It is shown that $\mathbf{FT}_{\Pi^0_1}$ implies the uniform continuity theorem (Theorem 4.1.6). Also very interesting is the proof for Proposition 4.1.5, showing that $\mathbf{FT}_c$ implies $\mathbf{FT}_{\Pi^0_1}$ under the assumption of $\mathbf{BD}$-$\mathbb{N}$. The chapter discusses many equivalents of the various versions of the fan theorem. An overview can be found in Figure 4.6.

- *Chapter 5.* The section on connectedness in this chapter contains the technically complicated Lemma 5.1.15, which states that path-wise connectedness is an equivalence relation on a differentiable manifold. It also contains an intuitionistic counterexample illustrating the difficulties of connectedness constructively (Proposition 5.1.14).

Chapter 2

Preliminaries

2.1 Bishop-style constructive mathematics

Many introductions to the philosophical motivations for constructive mathematics can be found in the literature. There is no reason to add another one to the list. We assume that the reader is familiar with the basic aspects of constructive mathematics. Nevertheless a couple of words need to be said.

Brouwer is commonly considered to be the founder of constructive mathematics[1]. Leaving philosophical issues aside, there are two main features to the way Brouwer did mathematics. The first feature is showing that many results in classical mathematics imply nonconstructive principles and are thus unacceptable to a constructivist. The second feature is recreating mathematics without the use of these constructively dubious principles, but with the use of substitute principles such as bar induction and continuous choice. Almost ironically it is the first, *destructive* rather than constructive, feature of intuitionism nonconstructivists are most aware of. This misconception extends much further to the common impression that Brouwerian counterexamples is the only thing constructivists of any school produce. One might only suspect that it is the use of unfamiliar concepts, which are often inconsistent with classical mathe-

[1] Nowadays Kronecker gets his fair share of attention.

matics, that deters non-constructivists from the positive, constructive side of intuitionism.

The most popular school of constructivism, next to Brouwer's intuitionism, is the one of Markov, which developed in the Soviet Union after the second world war. Markov favoured a recursive approach to mathematics, based on the acceptance of the Church-Markov-Turing thesis. Both Brouwer's intuitionism and Markov's Russian recursive mathematics use intuitionistic logic, which is easiest described as classical logic minus the law of excluded middle (**LEM**)—the principle that asserts that for any syntactically correct statement P we can decide whether P or its truth-functional negation $\neg P$ holds.

Since its beginnings constructivism was stuck with the dilemma of either having to accept additional principles or apparently not being able to prove anything deep at all. It was Bishop who managed to achieve the unexpected: recreating entire branches of mathematics within a minimal system; that is, without the aid of omniscience principles such as the law of excluded middle, but also without constructive substitutes for them. If Hilbert said that taking away the law of excluded middle from a mathematician is comparable to taking away the telescope from an astronomer, then Bishop discovered new galaxies without *any* astronomical equipment.

Bishop-style constructive mathematics is an informal system. Working informally has advantages as well as disadvantages. The advantages, which include clearness, readability and generality, outweigh—in our opinion—any disadvantages. It is our belief that proofs written in Bishop's style can easily be adapted into algorithms for many formal systems [1].

With complete disregard for philosophical aspects, we can very conveniently view classical mathematics (**CLASS**), Brouwer's intuitionistic mathematics (**INT**) and recursive constructive mathematics of Markov et al. (**RUSS**) as models of Bishop-style constructive mathematics (**BISH**).

2.2 Notation and definitions

> Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.[2] —L. Kronecker

We adapted the same logical and set theoretic notations as any textbook on analysis or topology such as [39]. Beyond notation we followed the notions and basic constructions of [13,19]. However, some definitions vary slightly across the constructive community, and it is these notions that we will explicitly define in the following.

A set $D \subset X$ is **decidable** if for every $x \in X$ we can decide whether $x \in D$ or $x \notin D$. A set X is called **inhabited** if there exists $x \in X$. If X is inhabited, we write $X \neq \emptyset$. In the same spirit, for two points x, y of a metric space (X, d), $x \neq y$ will mean

$$\exists \varepsilon > 0 \ (d(x, y) > \varepsilon) \, ,$$

rather than $\neg(x = y)$.

The reader should notice that our notions often diverge from Bishop's definitions, who for example calls 'continuous' what we would refer to as 'uniformly continuous on compact sets'. There are numerous definitions of continuity commonly in use, that, unlike in classical mathematics, do not turn out to be equivalent. We call a function $f : X \to Y$ between two metric spaces (X, d) and (Y, d') **continuous at** $x \in X$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x' \in X$

$$d(x, x') < \delta \Rightarrow d'(f(x), f(x')) < \varepsilon.$$

A function $f : X \to Y$ is **continuous** if it is continuous at every point $x \in X$. Finally a function is **uniformly continuous** if the choice of δ is independent of x.

Moving on to constructive notions, we call a subset $A \subset X$ of a metric space

[2] God made the integers, all else is the work of man.

(X, d) **located** if for any $x \in X$,

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

exists. If a set $A \subset X$ is located we can form its **metric complement**

$$-A = \{x \in X : d(x, A) > 0\},$$

which is not necessarily located [19, Exercise 2.11]; if, in addition, $-A$ is located, we call A **colocated.**

Slightly weaker, but constructively equally dubious as **LPO** is the **lesser limited principle of omniscience,**

> **LLPO:** For any binary sequence $(\alpha_n)_{n\geqslant 1}$, with at most one term nonzero, either $\alpha_{2n} = 0$ for all $n \in \mathbb{N}$ or $\alpha_{2n+1} = 0$ for all $n \in \mathbb{N}$.

A—from a constructive point of view—rather innocently looking principle is H. Ishihara's principle **BD**-$\mathbb{N}$. A subset S of the natural numbers is called **pseudobounded** if for any sequence $(a_n)_{n\geqslant 1}$ in S there exists $N \in \mathbb{N}$ such that

$$\forall i \geqslant N \left(\frac{a_i}{i} < 1\right).$$

> **BD-$\mathbb{N}$:** Any countable, pseudobounded subset of the natural numbers is bounded.

BD-$\mathbb{N}$ holds in any of the three big varieties: CLASS, INT and RUSS. It requires a fair amount of effort to produce a model of BISH in which it fails. To this day there are only two models of BISH in which **BD**-$\mathbb{N}$ fails [32, p. 67].

2.3 Our choice of choice

The axiom of choice (**AC**) comes in many equivalent formulations. For example in this rather innocent looking one:

> If A and B are two non-empty sets and R is a subset of the Cartesian product $A \times B$ and is such that for all $a \in A$ there exists $b \in B$ such that $(a, b) \in R$, then there exists a (extensional) *function* $f : A \to B$ such that $(a, f(a)) \in R$ for all $a \in A$.

It is a well known result, proved by Cohen and Gödel, that the axiom of choice is independent of the axioms of ZF set theory [26]. It is therefore a matter of taste for a classical working mathematician whether to use it or not. However, even those classical mathematicians who accept it seem to have some doubts:

> "Since, in our opinions, it is more desirable to have a constructive proof rather than a non-constructive proof, the axiom of choice and other non-constructive principles should be avoided whenever possible. At times, of course, non-constructive principles are unavoidable and the least the mathematician can do at these times is to declare their use in his/her proof." [26, p. xi]

A constructivist does not have a choice. In [25] Goodman and Myhill showed that the axiom of choice implies the law of excluded middle; it is therefore unacceptable to a constructive mathematician—but even if it did not imply the law of excluded middle, its constructive content would be doubtful.

Generally accepted by constructivists, however, are weaker versions, such as the axiom of *countable* choice, which restricts the set A in the formulation above to the set of natural numbers or the axiom of *unique* choice, which restricts the relation R in the formulation above to functional relations. Stronger than these two versions, yet still accepted by constructivists in general, is the axiom of dependent choice, which states:

> Let X be a set, $a \in X$ and R a subset of $X \times X$ such that for each $x \in X$ there exists $y \in X$ with $(x, y) \in R$; then there exists a function $f : \mathbb{N} \to X$ such that $f(0) = a$ and $(f(n), f(n+1)) \in R$ for each $n \in \mathbb{N}$.

We will make use of these principles, but make it a good habit to mention explicitly when doing so.

2.4 Cantor space

An important space in constructive mathematics is Cantor space $2^{\mathbb{N}}$ the space of all binary sequences. Note that Cantor space is a metric space, with metric d defined by

$$d(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha_0 \neq \beta_0 \\ \inf\left\{2^{-n-1} : \forall k \leqslant n\,(\alpha_k = \beta_k)\right\} & \text{if } \alpha_0 = \beta_0 \end{cases}$$

Cantor space's little brother is 2^* the set of all binary, finite sequences. For two elements $u, v \in 2^*$ we define the **concatenation** $u * v$ to be

$$(u_1, \ldots, u_k, v_1, \ldots, v_m)\,.$$

In the same way, for $u \in 2^*$ and $\alpha \in 2^{\mathbb{N}}$ we define the **concatenation** $u * \alpha$ to be the infinite binary sequence

$$(u_1, \ldots, u_k, \alpha_1, \alpha_2, \ldots).$$

For any element $\alpha \in 2^{\mathbb{N}}$ and any natural number $n \in \mathbb{N}$ we write $\overline{\alpha}(n)$ to be the finite sequence consisting of the first n elements of α. Furthermore, we will often identify 0 and 1 with the one-element sequences (0) and (1).

A subset $B \subset 2^*$ is called a **bar** if for each $\alpha \in 2^{\mathbb{N}}$ there exists a natural number $n \in \mathbb{N}$ such that $\overline{\alpha}(n) \in B$. A bar is called **uniform** if there exists a natural number M such that for each $\alpha \in 2^{\mathbb{N}}$ there is a natural number $n \leqslant M$ such that $\overline{\alpha}(n) \in B$.

2.5 Uniform spaces

Uniform spaces are a natural generalisation of metric spaces. For a classical introduction see [39].

Let X be a inhabited set. We define certain associated subsets of $X \times X$ as follows:

$$U \circ V = \{(x, y) : \exists z \in X \ ((x, z) \in U \wedge (z, y) \in V)\},$$

$$U^1 = U, U^{n+1} = U \circ U^n,$$

$$U^{-1} = \{(x, y) : (y, x) \in U\}.$$

We say that U is **symmetric** if $U = U^{-1}$. The **diagonal** of $X \times X$ is the set

$$\Delta = \{(x, x) : x \in X\}.$$

If $U \subset X \times X$ and $x \in X$, we define

$$U[x] = \{y \in X : (x, y) \in U\}.$$

A family $\mathcal{U}$ of subsets of $X \times X$ is called a **uniform structure** on X if the following conditions hold.

U1 **(i)** Every finite intersection of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.

(ii) Every subset of $X \times X$ that contains a member of $\mathcal{U}$ is in $\mathcal{U}$.

U2 Every member of $\mathcal{U}$ contains both the diagonal Δ and a symmetric member of $\mathcal{U}$.

U3 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$.

U4 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$\forall x \in X \times X \ (x \in U \vee x \notin V).$$

An element $U \in \mathcal{U}$ is called an **entourage**. If a set $\mathcal{U}' \subset X \times X$ is such that

$$\mathcal{U} = \{U \subset X \times X : \exists U' \in \mathcal{U}\ (U' \subset U)\}$$

is a uniform structure, then we call the set $\mathcal{U}'$ a **base of entourages** for $\mathcal{U}$, and we say that the uniform structure $\mathcal{U}$ is **generated** by the entourages $\mathcal{U}'$. A n**-chain of entourages** is a family of n entourages $U_1, \ldots, U_n$ such that for all $1 \leqslant i < n$ we have $U_i^2 \subset U_{i+1}$ and

$$X \times X = U_i \cup \neg U_{i+1}.$$

It is elementary to prove that for any entourage $U \in \mathcal{U}$ and any natural number n there always exists a n-chain of entourages with $U = U_1$.

The canonical example of a uniform space is a metric space (X, d) equipped with the uniform structure generated by the sets $(V_{2^{-n}})_{n \geqslant 1}$, where V_ε denotes the set

$$V_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

If not explicitly mentioned otherwise, this is the uniform structure we have in mind when talking about metric spaces.

We call an uniform space $(X,\mathcal{U})$ **totally bounded** if for each $U \in \mathcal{U}$ there exists a **finite** U**-approximation** to X, that is if there exist finitely many points $x_1, \ldots, x_n$ such that

$$X = \bigcup_{i=1}^{n} U[x_i].$$

Equivalently if for every $x \in X$ there exists $1 \leqslant i \leqslant n$ such that $(x, x_i) \in U$. A sequence $(x_n)_{n \geqslant 1}$ in a uniform space $(X,\mathcal{U})$ **converges** to a point $x \in X$ if for $U \in \mathcal{U}$ there exists N such that $(x, x_n) \in U$ for all $n \geqslant N$; it is a **Cauchy sequence** if for $U \in \mathcal{U}$ there exists N such that $(x_n, x_m) \in U$ for all $n, m \geqslant N$. Finally a uniform space $(X,\mathcal{U})$ is called **complete** if every Cauchy sequence converges, and **compact** if it is both totally bounded and complete. All of these notions coincide with their respective counterpart for metric spaces if the standard uniform structure defined above is used.

Chapter 3

Compactness and apartness spaces

3.1 Neat compactness

This section contains results published in [20].

With open-cover compactness failing from a constructive point of view, and totally bounded + completeness being limited to metric/uniform spaces, we investigate whether it is possible to define a meaningful notion of compactness in a general setting.

What properties would an ideal notion of compactness exhibit? The holy grail of all constructive compactness notions—let us call it **HGC**—would have the following properties.

(i) **HGC** would be defined for, and in the language of, topological spaces.

(ii) **HGC** would be classically equivalent to the notion of open-cover compactness.

(iii) Within BISH a complete and totally bounded metric/uniform space would satisfy **HGC**.

(iv) We would be able to prove deep and meaningful theorems assuming that the underlying space satisfies **HGC**. (For example, we would be able to prove the existence of the supremum of the image under some sort of continuous function into the reals.)

One central idea is to use the last of these requirements on **HGC** as a starting point. We begin by noting the following proposition from Bishop's book [6]:

Proposition 3.1.1. *Let A be an inhabited set of real numbers that is bounded above. Then* $\sup A$ *exists if and only if for all real a, b with $a < b$, either b is an upper bound of A or there exists $x \in A$ such that $a < x$.*

We could change the notation of the last disjunction to "either $A \subset (-\infty, b)$ or $A \cap (a, \infty) \neq \emptyset$".

We would like to generalise the idea of actually having the computational information provided by the statement "$a < b$". There seems to be no way to encode this information in the language of topological spaces. However, we can very conveniently do so in an apartness space. The theory of pre-apartness spaces has been developed over the last seven years in a series of papers [8, 10, 14, 18],[1] and offers a very promising constructive approach to various topological concepts. In that theory we have an inhabited set X with an inequality $\neq$, and a symmetric[2] binary relation $\bowtie$ of **pre-apartness**. These relations give rise to two of the following three complements of a subset S of X:

- the **logical complement** $\neg S = \{x \in X : x \notin S\}$,
- the **complement** $\sim S = \{x \in X : \forall y \in S\ (x \neq y)\}$,
- the **apartness complement** $-S = \{x \in X : \{x\} \bowtie S\}$.

We require the pre-apartness to satisfy these four axioms:

B1 $X \bowtie \emptyset$.

[1]A systematic treatment of apartness spaces is in preparation. [15]

[2]The symmetry of $\bowtie$ can be dropped at the cost of amplifying the axioms; see [15].

B2 $A \bowtie B \Rightarrow A \subset\sim B$

B3 $(A \bowtie (B \cup C) \Leftrightarrow (A \bowtie B \wedge A \bowtie C))$

B4 $-A \subset\sim B \Rightarrow -A \subset -B$.

The standard example for a pre-apartness spaces is a uniform space $(X,\mathcal{U})$ with the apartness relation $\bowtie$ defined by

$$S \bowtie T \Leftrightarrow \exists U \in \mathcal{U}\ (S \times T \subset \neg U)$$

and the inequality defined by

$$x \neq y \Leftrightarrow \{x\} \bowtie \{y\}\,.$$

Remember that when dealing with a metric space (X, d), we consider the natural uniform structure defined in Section 2.5. So in a metric space two sets A, B are apart if and only if

$$\exists \varepsilon > 0\, \forall a \in A\, \forall b \in B\ (d(a, b) > \varepsilon)\,.$$

The natural topology on a uniform space $(X,\mathcal{U})$ is given by the base of neighbourhoods

$$\{U[x] : U \in \mathcal{U}, x \in X\}\,.$$

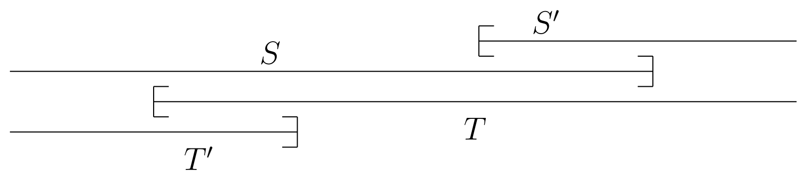
This is the topology we have in mind when talking about topological concepts like “open” and “topological continuity”. (In the case of a metric space it is the usual topology.) The natural topology on a pre-apartness space is given by the base of neighbourhoods

$$\{-S : S \subset X\}\,.$$

In the case of uniform space the topology induced by the uniform structure coincides with the topology induced by the apartness relation [15].

Bearing in mind the image of pairs of intervals whose complements cover the whole real line, we say that two subsets S, T of a pre-apartness space X are, or form, a **neat covering** if there exist S', T' such that $S' \bowtie T'$, $X = T \cup T'$ and $X = S \cup S'$.

It may be time for a picture.



For example if (X, d) is a metric space and $\xi \in X$ is a fixed point, then for $a < b < c < d$ the sets

$$S = \{x \in X : d(x, \xi) > a\} \text{ and } T = \{x \in X : d(x, \xi) < d\}$$

form a neat cover, because the sets

$$S' = \{x \in X : d(x, \xi) < b\} \text{ and } T' = \{x \in X : d(x, \xi) > c\}$$

are such that

$$X = S \cup S', \ X = T \cup T' \text{ and } S' \bowtie T'.$$

We call an inhabited subset A of a pre-apartness space X **neatly located** if for all neat coverings S, T of X, either $A \subset S$ or $A \cap T \neq \emptyset$.

Proposition 3.1.2. *In a metric space (X, d), an inhabited, neatly located subset A is located.*

Proof. Consider an arbitrary $\xi \in X$ and fix $x_0 \in A$. The set

$$D = \{d(\xi, x) : x \in A \wedge d(\xi, x) \leqslant d(\xi, x_0)\}$$

is inhabited and bounded. Furthermore, let $a < d$ be arbitrary and choose $b, c \in \mathbb{R}$ such that $a < b < c < d$. Let S, S', T and T' be as in the example above. Then either $A \subset S$ or there exists $y \in A \cap T$. In the case that $A \subset S$, $d(\xi, x) > a$ for all $x \in A$ and therefore $z > a$ for all $z \in D$. In the case that $y \in A \cap T$, $d(\xi, y) < d$, that is, there exists $z \in D$ with $z < d$. Hence we can use Proposition 3.1.1 to conclude that $\inf D$ exists which means that $d(\xi, A)$ exists. □

Every totally bounded subset of a metric space is located [6, p. 95]. Similarly we can prove:

Proposition 3.1.3. *A totally bounded set in a uniform space is neatly located.*

Proof. Let A be a totally bounded subset of a uniform space $(X, \mathcal{U})$, and consider a neat cover S, T of X. Let S', T' be such that $S' \bowtie T'$, $X = T \cup T'$, and $X = S \cup S'$. Choose $U \in \mathcal{U}$ such that

$$\forall s \in S' \, \forall t \in T' \, \left((s,t) \notin U\right).$$

Construct a finite U-approximation $\{x_1, \ldots, x_n\}$ to A. Either there exists $1 \leqslant j \leqslant n$ such that $x_j \in T$ and we are done, or for all $1 \leqslant i \leqslant n$ we have $x_i \in T'$. For an arbitrary x in A, let j be such that $(x_j, x) \in U$. If $x \in S'$, then $(x, x_j) \notin U$, a contradiction. Hence $A \subset \neg S' \subset S$. □

Evidence that locatedness is a helpful notion is the fact that it is preserved under the natural morphisms between pre-apartness spaces. We call a mapping $f : X \to Y$ between pre-apartness spaces **strongly continuous** if for all subsets A, B of X,

$$f(A) \bowtie f(B) \Rightarrow A \bowtie B.$$

Proposition 3.1.4. *If $f : X \to Y$ is a strongly continuous mapping between two pre-apartness spaces, and A is a neatly located subset of X, then $f(A)$ is neatly located.*

Proof. Let S, T form a neat cover of $f(A)$. There exist S' and T' such that $S' \bowtie T'$, $Y = T \cup T'$, and $Y = S \cup S'$. The sets $f^{-1}(S)$ and $f^{-1}(T)$ form a neat cover of X. For $S' \bowtie T'$ and f is strong continuous so $f^{-1}(S') \bowtie f^{-1}(T')$; also

$$X = f^{-1}(S) \cup f^{-1}(S'), \text{ and } X = f^{-1}(T) \cup f^{-1}(T').$$

Hence either $A \subset f^{-1}(S)$ or $A \cap f^{-1}(T) \neq \emptyset$, and so either $f(A) \subset S$ or $f(A) \cap T \neq \emptyset$. □

Proposition 3.1.4 shows a definite advantage of the notion of neat locatedness, since without it the only known way to ensure that the image of a space under a continuous function is located is to assume that the space is totally bounded and the function is uniformly continuous. As uniformly continuity implies strong continuity [15], and totally boundedness implies neat locatedness, we have weakened *both* the previous requirements. The question remains whether this generalised notion of locatedness helps us with our quest for HGC. The surprising answer is "yes". More surprisingly, the underlying reason for this is a result that looks *prima facie* rather disappointing to a constructive mathematician.

Proposition 3.1.5. *The interval* $(0, \infty)$ *is neatly located if and only if* **LPO** *holds.*

Proof. Let $(a_n)_{n \geqslant 1}$ be an increasing binary sequence. Let

$$\begin{aligned} S &= \bigcup \{(-\infty, n+3) : a_n = 0\}, \\ S' &= \bigcap \{(n+2, \infty) : a_n = 0\}, \\ T &= \bigcap \{(n, \infty) : a_n = 0\}, \\ T' &= \bigcup \{(-\infty, n+1) : a_n = 0\}. \end{aligned}$$

Then $S' \bowtie T'$, $\mathbb{R} = T \cup T'$, and $\mathbb{R} = S \cup S'$. So S and T form a neat covering of $\mathbb{R}$. If $(0, \infty) \subset S$, then $a_n = 0$ for all n; if $(0, \infty) \cap T \neq \emptyset$, then we can find n

such that $a_n = 1$. We conclude that **LPO** holds.

Conversely, assume **LPO** and let S and T form a neat cover of X. Since every totally bounded set is neatly located, as shown in Proposition 3.1.3, we can construct a binary sequence $(a_n)_{n\geqslant 1}$ such that

$$a_n = 0 \Rightarrow (0, n) \subset S,$$
$$a_n = 1 \Rightarrow (0, n) \cap T \neq \emptyset.$$

Since we are assuming **LPO**, either $a_n = 0$ for all n and therefore

$$(0, \infty) \subset \bigcup_{i\in\mathbb{N}} (0, i) \subset S,$$

or else there exists $n \in \mathbb{N}$ such that $(0, n) \cap T \neq \emptyset$ and therefore $(0, \infty) \cap T \neq \emptyset$. □

We now state our central result, which will ultimately lead us to a new notion of compactness.

Proposition 3.1.6. *Let $(X, \mathcal{U})$ be a neatly located, separable uniform space. For each $U \in \mathcal{U}$, either there exist finitely many points $x_1, \ldots, x_n$ of X such that $X = \bigcup_{k=1}^{n} U[x_k]$, or else* **LPO** *holds. In the latter case, there exists a countable open cover of X that cannot be refined to a finite one.*

Proof. Let $(x_n)_{n\geqslant 1}$ be a countable dense subset of X, and let $U \in \mathcal{U}$ be arbitrary. Fix an 8-chain of entourages $(U, U_1, \ldots, U_7)$. We show that for each n, the sets

$$A_n = \bigcup_{i=1}^{n} U[x_i] \text{ and } B_n = \bigcap_{i=1}^{n} \neg U_3[x_i]$$

form a neat cover. Define

$$A'_n = \bigcap_{i=1}^{n} \neg U_1[x_i] \text{ and } B'_n = \bigcup_{i=1}^{n} U_2[x_i].$$

To see that $X = A_n \cup A'_n$, let $x \in X$ be arbitrary. Either $x \notin U_1[x_i]$ for all $i \leqslant n$, in which case $x \in A'_n$; or else there exists $k \leqslant n$ such that $x \in U[x_k]$, in which case $x \in A_n$. A similar argument shows that $X = B_n \cup B'_n$. To show that $A'_n \bowtie B'_n$, we observe that this is a simple consequence of the fact that for any $x \in X$ the sets $\neg U_1[x]$ and $U_2[x]$ are apart.

We now define an increasing sequence $(k_n)_{n\geqslant 1}$ of natural numbers, and an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$, such that

$$\lambda_n = 1 \Rightarrow \forall i \geqslant n \left(x_{k_i} = x_{k_{n-1}}\right) \wedge X \subset A_{k_{n-1}},$$
$$\lambda_n = 0 \Rightarrow x_{k_n} \in \bigcap_{i=1}^{k_{n-1}} \neg U_4[x_i].$$

Setting $k_0 = 1$ and $\lambda_0 = 0$, assume that we have constructed both k_n and λ_n. If $\lambda_n = 1$, set $\lambda_{n+1} = 1$ and $k_{n+1} = k_n$. If $\lambda_n = 0$, then either $X \subset A_{k_n}$ or else there exists $y \in X \cap B_{k_n}$. In the first case set $\lambda_{n+1} = 1$ and $k_{n+1} = k_n$. In the second case set $\lambda_{n+1} = 0$ and choose N such that $y \in U_4[x_N]$. If $x_N \in \bigcup_{i=1}^{k_n} U_4[x_i]$, then $y \in \bigcup_{i=1}^{k_n} U_3[x_i]$, which is a contradiction. Hence $x_N \in \bigcap_{i=1}^{k_n} \neg U_4[x_i]$, which also shows that $N > k_n$. We now set $k_{n+1} = N$.

Next, let

$$S = \bigcup_{i:\lambda_{i+1}=0} \bigcup_{j=1}^{k_i} U_4[x_j] \quad \text{and} \quad T = \bigcap_{i:\lambda_{i+1}=0} \neg \bigcup_{j=1}^{k_i} U_7[x_j].$$

It is easy to see that S and T form a neat cover. So either $X \subset S$ or else there exists $y \in X \cap T$. In the first case we must have $\lambda_n = 0$ for all n: for if we assume that there exists N such that $\lambda_{N+1} = 1$, losing no generality by taking $\lambda_N = 0$, then

$$x_{k_N} \in X \subset S = \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{k_i} U_4[x_j] = \bigcup_{i=1}^{k_{N-1}} U_4[x_i],$$

which contradicts the fact that

$$x_{k_N} \in \bigcap_{i=1}^{k_{N-1}} \neg U_4[x_i].$$

In the case where there exists $y \in X \cap T$, choose M such that $y \in U_7[x_M]$. Assuming that $\lambda_{M+2} = 0$, we easily see that $k_{M+1} > M$; whence

$$y \in T \subset \neg \bigcup_{j=1}^{k_{M+1}} U_7[x_j] \subset \neg \bigcup_{j=1}^{M} U_7[x_j],$$

which is a contradiction. Thus $\lambda_{M+2} = 1$. We conclude that either there exists a finite U-approximation to X or else the sequence $(x_{k_n})_{n\geqslant 1}$ is such that

$$x_{k_n} \in \bigcap_{i=1}^{k_{n-1}} \neg U_4[x_i]$$

for all n. We show that in the latter case, **LPO** holds. To do so, let $(a_n)_{n\geqslant 1}$ be an increasing binary sequence. Define

$$E = \bigcup_{i:a_i=0} \bigcup_{j=1}^{k_i} U_4[x_j] \text{ and } F = \bigcap_{i:a_i=0} \neg \bigcup_{j=1}^{k_i} U_7[x_j].$$

Then (E, F) is a neat cover of X. Either $X \subset E$, in which case we must have $a_n = 0$ for all n; or else there exists $z \in X \cap F$ which enables us to find n such that $z \in U_7[x_n]$. For this n we must have $a_n = 1$. Thus **LPO** holds, and the interiors of the sets

$$\bigcup_{i=1}^{k_n} (U_4[x_i])^\circ \quad (n \geqslant 1)$$

form a countable open cover of X that cannot be refined to a finite one. □

Corollary 3.1.7. *In* INT *and* RUSS*, a separable uniform space is totally bounded if and only if it is neatly located.*

Proof. Using our main result Proposition 3.1.6, we see that for any entourage U

either there exists a U-approximation or else **LPO** holds. The latter alternative is ruled out, since **LPO** is provably false in both INT and RUSS. We conclude that the set is totally bounded. Conversely, we proved in 3.1.3 that a totally bounded set is neatly located. □

Corollary 3.1.8. *In* INT *and* RUSS*, the supremum of the image of an inhabited, neatly located set under a real-valued strongly continuous mapping exists.*

Proof. As we saw in Proposition 3.1.4, the image of a neatly located inhabited set is neatly located and inhabited. Using the previous corollary, we see that the image is totally bounded and therefore that its supremum exists. □

In general, we cannot rule out the second possibility of Proposition 3.1.6 as easily as we can do it in INT and RUSS. Thus one (admittedly not very elegant) way of defining a more general notion of compactness would be as follows. Call a subset A of a pre-apartness space X **neatly compact** if and only if

- A is inhabited,
- A is neatly located, and
- **LPO** implies that there does not exist a sequence of open sets $(U_n)_{n\geqslant 1}$ such that $A \subset \bigcup_{n\geqslant 1} U_n$ and $A \cap \neg U_n \neq \emptyset$ for each n.

The following holds.

Corollary 3.1.9. *A separable, neatly compact uniform space* $(X,\mathcal{U})$ *is totally bounded.*

Proof. This follows from our central theorem (3.1.6). □

Lemma 3.1.10. *If* $f : X \to Y$ *is a strongly continuous and topologically continuous*[3] *mapping between two pre-apartness spaces, and* A *is a neatly compact subset of* X*, then* $f(A)$ *is neatly compact.*

[3]Notice that in the case of a uniform space strong continuity implies topological continuity.

Proof. As we saw earlier, the image of a neatly located set under a strongly continuous map is neatly located. Assume that **LPO** holds. Assume also that there exist $U_1 \subset U_2 \subset U_3 \cdots$ such that

$$f(A) \subset \bigcup_{n \geqslant 1} U_n$$

and for all $n \in \mathbb{N}$

$$f(A) \cap \neg U_n \neq \emptyset. \tag{3.1}$$

Then

$$f^{-1}(U_1) \subset f^{-1}(U_2) \subset \cdots,$$

$$A \subset \bigcup_{n \geqslant 1} f^{-1}(U_n),$$

and for all $n \in \mathbb{N}$,

$$A \cap \neg f^{-1}(U_n) \neq \emptyset.$$

Since A is neatly compact, we get the desired contradiction. □

Corollary 3.1.11. *Let A be a neatly compact subset of a pre-apartness space X, and $f : X \to \mathbb{R}$ a strongly continuous and topologically continuous map. Then* $\sup f(A)$ *exists.*

Proof. The argument is like that in the proof of Corollary 3.1.8, with the difference that the second possibility is ruled out by our third assumption on a neatly compact set. □

Note that we cannot hope to weaken the condition "strongly continuous" in this corollary: even if we consider only the compact space $[0,1]$ and weaken "strongly continuous" to "(pointwise) continuous", it is easy to get a recursive counterexample like the one constructed in Section 4.5.

Proposition 3.1.12. *A compact subset of a uniform space with a countable base of entourages is neatly compact.*

Proof. Take a compact subset A of a uniform space $(X, \mathcal{U})$ with a countable base of entourages $(V_n)_{n \geqslant 1}$. By Proposition 3.1.3, A is neatly located. Next, assume that **LPO** holds. Assume also that there exist open sets $U_1 \subset U_2 \cdots$ such that $A \subset \bigcup_{n \geqslant 1} U_n$, and, without loss of generality, (3.1) holds. Using countable choice, we can define a sequence $(x_n)_{n \geqslant 1}$ such that

$$x_n \in A \cap U_{n+1} \cap \neg U_n$$

for each n. We show that there is a Cauchy subsequence of $(x_n)_{n \geqslant 1}$. To apply diagonalisation, construct for each $k \geqslant 0$ a subsequence $(x_{\sigma^k(n)})_{n \geqslant 1}$ such that

- for natural numbers $p \leqslant q$, $(x_{\sigma^q(n)})_{n \geqslant 1}$ is a subsequence of $(x_{\sigma^p(n)})_{n \geqslant 1}$ and
- for all $i, j \geqslant n$, $(x_{\sigma^n(i)}, x_{\sigma^n(j)}) \in V_n$.

First set $x_{\sigma 0(n)} = x_n$ for each n. Now assume that we have constructed $(x_{\sigma^k(n)})_{n \geqslant 1}$. Since A is totally bounded, there exist $y_1, \ldots, y_m$ such that the sets

$$V_{k+2}[y_1], \ldots, V_{k+2}[y_m]$$

cover A. **LPO** now implies that one of these sets contains infinitely many elements of $(x_{\sigma^k(n)})_{n \geqslant 1}$. (A proof of this not-totally-obvious fact can be found in [12]). So there is a subsequence $(x_{\sigma^{k+1}(n)})_{n \geqslant 1}$ such that for all $i, j > k$,

$$(x_{\sigma^{k+1}(i)}, x_{\sigma^{k+1}(j)}) \in V_{k+2}^2 \subset V_{k+1}.$$

We now get our final subsequence by setting $x_i' = x_{\sigma^i}(i)$ for each i. This gives a Cauchy sequence, since for $n \geqslant m \geqslant k$ there exists $n' \geqslant n$ such that

$$(x_n', x_m') = (x_{\sigma^m}(n'), x_{\sigma^m}(m)) \in V_m \subset V_k.$$

As A is complete, this sequence converges to a limit $x \in A$. We now compute N, $M > N$, and $k > N$ such that $x_k \in V_M[x] \subset U_N$, a contradiction to the fact that $x_k \in \neg U_{k-1} \subset \neg U_N$. □

To complete the circle of ideas and get a notion fully equivalent to "totally bounded and complete", we introduce the following notions.

A sequence $(x_n)_{n\geqslant 1}$ in a pre-apartness space X is a **neat Cauchy sequence** if and only if for a finite number of neat covers $(S_i,T_i)_{1\leqslant i\leqslant m}$, there exists N such that either $x_N \in T_i$, for every $1 \leqslant i \leqslant m$, or else $x_n \in S_j$ for some $1 \leqslant j \leqslant m$ and for all $n \geqslant N$.

A pre-apartness space is **neatly complete** if every neat Cauchy sequence $(x_n)_{n\geqslant 1}$ in X **converges neatly** to a limit $x \in X$, in the sense that for a finite number of neat covers $(S_i,T_i)_{1\leqslant i\leqslant m}$, either $x \in T_i$, for every $1 \leqslant i \leqslant m$ or else $x \in S_j$ for some $1 \leqslant j \leqslant m$ and there exists N such that $x_n \in S_j$ for all $n \geqslant N$.

Having defined this variant of completeness, we can easily proof the following proposition.

Proposition 3.1.13. *A totally bounded uniform space* $(X,\mathcal{U})$ *is neatly complete if and only if it is complete.*

Proof. Suppose that X is neatly complete, and consider an arbitrary Cauchy sequence $(x_n)_{n\geqslant 1}$ in X. Let $(S_i,T_i)_{1\leqslant i\leqslant m}$, be a finite number of neat covers. By definition, for every $1 \leqslant i \leqslant m$ there exists S_i' and T_i' such that $S_i' \bowtie T_i'$, $X = S_i \cup Si'$, and $X = T_i \cup T_i'$. So there exists $U_i \in \mathcal{U}$ such that $S_i' \times T_i' \subset \neg U_i$. Find $U \in \mathcal{U}$ such that $U \subset U_i$ for $1 \leqslant i \leqslant m$. Choose a positive integer N such that $(x_n, x_m) \in U$ for all $n, m \geqslant N$. Either $x_N \in T_i$ for all $1 \leqslant i \leqslant m$ and we are done, or else $x_N \in T_j'$ for some $1 \leqslant j \leqslant m$. Then, because $x_n \in U[x_N]$ for all $n \geqslant N$, the assumption that $x_n \in S_j'$ for some $n \geqslant N$ leads to a contradiction; so $x_n \in S_j$ for all $n \geqslant N$. Hence $(x_n)_{n\geqslant 1}$ is also a neat Cauchy sequence, and therefore there exists x such that $(x_n)_{n\geqslant 1}$ converges neatly to x. We show that $(x_n)_{n\geqslant 1}$ also converges to x in the usual sense. Accordingly, let $U \in \mathcal{U}$ be arbitrary, and construct a 4-chain (U, U_1, U_2, U_3) of entourages. As so often before, we use the fact that the sets $U[x]$ and $\neg U_3[x]$ form a neat cover. Then either $x \in \neg U_3[x]$, which leads to a contradiction. or else there exists N such that $x_n \in U[x]$ for all $n \geqslant N$. This completes the proof that neat completeness implies completeness.

To prove the converse, assume that X is complete, and consider an arbitrary neat Cauchy sequence $(x_n)_{n\geqslant 1}$ in X. We show that it also a Cauchy sequence in the usual (uniform space) sense. Let $U \in \mathcal{U}$ be arbitrary, and construct a 5-chain (U, U_1, U_2, U_3, U_4) of entourages of X. Construct also points $y_1, \ldots, y_m$ such that $X = \bigcup_{i=1}^{m} U_4[y_i]$. There exists a natural number N such that either $x_N \in \neg U_4[y_i]$ for each $1 \leqslant i \leqslant m$, or else there exist $1 \leqslant j \leqslant m$ such that that $x_n \in U_1[y_j]$ for all $n \geqslant N$. The first case is ruled out by the fact that $X = \bigcup_{i=1}^{m} U_4[y_i]$. Hence the second case holds. Thus for all $n, m \geqslant N$ we have $(x_n, x_m) \in U_1^2 \subset U$. Since we assumed that X is complete, there exists $x \in X$ that is the limit of $(x_n)_{n\geqslant 1}$ in the normal sense. To see that the sequence also converges neatly to x, let $(S_i, T_i)_{1\leqslant i\leqslant m}$, a finite number of neat covers. By definition for every $1 \leqslant i \leqslant m$ there exists S_i' and T_i' such that $S_i' \bowtie T_i'$, $X = S_i \cup Si'$, and $X = T_i \cup T_i'$. So there exists $U_i \in \mathcal{U}$ such that $S_i' \times T_i' \subset \neg U_i$. Find $U \in \mathcal{U}$ such that $U \subset U_i$ for $1 \leqslant i \leqslant m$. Either $x \in T_i$ for every $1 \leqslant i \leqslant m$ and we are done, or else $x \in T_j'$ for some $1 \leqslant j \leqslant m$. In the second case, choose N such that $(x, x_n) \in U$ for all $n \geqslant N$. Then for such n, the assumption $x_n \in S_j'$ leads to a contradiction; whence $x_n \in S_j$. □

Corollary 3.1.14. *In a uniform space* $(X,\mathcal{U})$ *with a countable base of entourages the following are equivalent:*

- *X is separable, neatly compact and neatly complete.*
- *X is totally bounded and complete.*

Proof. Use Proposition 3.1.13 together with Corollary 3.1.9 and Proposition 3.1.12. □

3.2 Counterexample topology

We have shown that it *is* possible to define a sensible notion of compactness in a pre-apartness space. Unfortunately, this is as far we can go, as we will see in the following example.

Proposition 3.2.1. *In* RUSS *there exists a uniform structure on Cantor space* $2^{\mathbb{N}}$ *that induces the usual topology, but which is not totally bounded.*

Proof. Let B be a decidable bar such that for each $n \in \mathbb{N}$ there exists $u \in 2^*$ with $|u| \geqslant n$ but $u \notin B$. A suitable construction for such a bar is described in [13, ch. 3]. Let

$$U_n = \bigcup_{\substack{u \in B \\ |u| \geqslant n}} I_u \times I_u \subset 2^{\mathbb{N}} \times 2^{\mathbb{N}},$$

where for each $u \in 2^*$

$$I_u = \left\{ \alpha \in 2^{\mathbb{N}} : \overline{\alpha}(|u|) = u \right\}.$$

We show that $(U_n)_{n \geqslant 1}$ generates a uniform structure. We omit the straightforward verification of axioms **U1** and **U2**. The interesting axioms to prove are **U3** and **U4**. We do this by showing that for each $n \in \mathbb{N}$,

- $U_n^2 \subset U_n$ and
- For each $x \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$

$$x \in U_n \vee x \notin U_n.$$

These properties are actually stronger than required. Let $\alpha, \beta, \gamma \in 2^{\mathbb{N}}$ be such that $(\alpha, \beta) \in U_n$ and $(\beta, \gamma) \in U_n$. That means that there are natural numbers n_0 and n_1, as well as $u_0, u_1 \in B$, such that $n_0, n_1 \geqslant n$, $\overline{\alpha}(n_0) = \overline{\beta}(n_0) = u_0$ and $\overline{\beta}(n_1) = \overline{\gamma}(n_1) = u_1$. For $n = \min\{n_0, n_1\}$ we get $\overline{\alpha}(n) = \overline{\gamma}(n)$ and $\overline{\alpha}(n) \in B$. Hence $(\alpha, \gamma) \in U_n$.

To prove the second property, consider arbitrary $\alpha, \beta \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$. Since

B is is a decidable bar, we can find

$$N = \min\{i \in \mathbb{N} : \overline{\alpha}(i) \in B\}.$$

Now either $\overline{\alpha}(N) \neq \overline{\beta}(N)$, in which case $(\alpha, \beta) \notin U_n$, or else $\overline{\alpha}(N) = \overline{\beta}(N)$. In the second case, either $n \leqslant N$, which means that $(\alpha, \beta) \in U_n$, or else $n > N$. If $n > N$, then either $\overline{\alpha}(n) \neq \overline{\beta}(n)$ and therefore $(\alpha, \beta) \notin U_n$ or $\overline{\alpha}(n) = \overline{\beta}(n)$ and therefore $(\alpha, \beta) \in U_n$. In any case we can decide that

$$(\alpha, \beta) \in U_n \vee (\alpha, \beta) \notin U_n.$$

As B is a bar, given any $\alpha \in 2^{\mathbb{N}}$, we can find $n \in \mathbb{N}$ such that $\overline{\alpha}(n) \in B$. By definition of the topology induced by a uniform structure, a base of neighbourhoods for α is given by $(U_i[\alpha])_{i \geqslant n}$, which is the same base of neighbourhoods as $(I_{\overline{\alpha}(i)})_{i \geqslant n}$. We see that this base is equivalent to a base of neighbourhoods given by the usual topology $(I_{\overline{\alpha}(i)})_{i \geqslant 1}$.

Now assume that $(2^{\mathbb{N}}, \mathcal{U})$ is totally bounded. Hence there exists $\alpha_1, \dots, \alpha_n$ such that

$$2^{\mathbb{N}} = \bigcup_{i=1}^{n} U_1[\alpha_i].$$

By the definition of U_1, this means that there exist natural numbers $m_1, \dots, m_n$ such that $\overline{\alpha}(m_i) \in B$ for any $1 \leqslant i \leqslant n$; moreover

$$2^{\mathbb{N}} = \bigcup_{i=1}^{n} I_{\overline{\alpha}(m_i)}.$$

Let

$$N = \max\{m_1, \dots, m_n\}.$$

We can find $u \in 2^*$ such that $|u| > N$ and $u \notin B$. Find $1 \leqslant j \leqslant n$ such that

$$u * 0 * \cdots \in I_{\overline{\alpha}(m_j)}.$$

Then $\overline{u}(m_j) \in B$, and since B is closed under extensions, $u \in B$—a contradiction. We conclude that B is not totally bounded. $\square$

We can extend this example to metric spaces.

Proposition 3.2.2. *In* RUSS *there exists a metric on Cantor space* $2^{\mathbb{N}}$ *that induces the usual topology but is not totally bounded.*

Proof. Let $\mathcal{U}$ be the uniform structure constructed in the previous proof. For convenience, set $U_0 = 2^{\mathbb{N}} \times 2^{\mathbb{N}}$. For $\alpha, \beta \in 2^{\mathbb{N}}$ let

$$d(\alpha, \beta) = \inf \left\{ 2^{-n} : n \geqslant 0, (\alpha, \beta) \in U_n \right\}.$$

We claim that d is a metric on $2^{\mathbb{N}}$. As so often, the only interesting axiom to verify is the triangle inequality. To this end let $\alpha, \beta, \gamma \in 2^{\mathbb{N}}$ be arbitrary. Assume that there is $N \in \mathbb{N}$ such that

$$d(\alpha, \gamma) > d(\alpha, \beta) + d(\beta, \gamma) + 2^{-N}. \tag{3.2}$$

This means, in particular, that $(\alpha, \gamma) \notin U_N$. Hence we can find $M < N$ such that $d(\alpha, \gamma) = 2^{-M}$. Now either $(\alpha, \beta) \notin U_{M+1}$ or $(\beta, \gamma) \notin U_{M+1}$, since the assumption that both $(\alpha, \beta) \in U_{M+1}$ and $(\beta, \gamma) \in U_{M+1}$ implies that $(\alpha, \gamma) \in U_{M+1}$ and therefore $d(\alpha, \gamma) \leqslant 2^{-M-1}$, a contradiction. If either $(\alpha, \beta) \notin U_{M+1}$ or $(\beta, \gamma) \notin U_{M+1}$, it follows that either $d(\alpha, \beta) \geqslant 2^{-M}$ or $d(\beta, \gamma) \geqslant 2^{-M}$, which in any case contradicts (3.2). Hence we conclude that

$$d(\alpha, \gamma) \leqslant d(\alpha, \beta) + d(\beta, \gamma).$$

$\square$

The spaces we have constructed show that it is impossible to define a notion of compactness solely in terms of a topology.

Theorem 3.2.3. *There is no notion that satisfies all the condition for* **HGC**.

Proof. Such a notion of compactness would imply (constructively) that any uniform structure or metric structure on such a topological space would be totally bounded. □

Neat compactness taken together with neat completeness satisfies almost all of the requirements that we set for **HGC**. Nevertheless we have also seen that the holy grail does not exist—but who believed it would anyway?

Chapter 4

Compactness and constructive reverse mathematics

As we have seen in the introduction, the classical open-cover definition of compactness is of no use in a constructive setting. In this chapter we investigate exactly what principles are necessary and sufficient to prove classically true theorems about compactness, as well as their antitheses.

4.1 A hierarchy of fan theorems

This section contains a number of results and definitions which will be of great importance for the rest of the chapter.

There are currently four versions of Brouwer's fan theorem in common use. All of them enable one to conclude that a given bar is uniform. The difference between them lies in the required complexity of the bar. This ranges from the very strongest requirement—decidable—to no restriction on the bar at all.

A set $C \subset 2^*$ is called a ***c*-set** if there exists a decidable set $C' \subset 2^*$ such that

$$u \in C \Leftrightarrow \forall w \in 2^* \ (u * w \in C')\,.$$

A set $P \subset 2^*$ is called $\mathbf{\Pi}^1_0$ if there exists a set $S \subset 2^* \times \mathbb{N}$ with the following properties,

- if $(u, n) \in S$ for some $u \in 2^*$ and $n \in \mathbb{N}$, then $(u * 0, n) \in S$ and $(u * 1, n) \in S$;
- $u \in P$ if and only if $(u, n) \in S$ for all $n \in \mathbb{N}$.

Naturally a bar $B \subset 2^*$ is called a ***c*-bar** if it is a c-set, and a **Π^0_1-bar** if it is a Π^0_1-set. We can now state four interesting versions of the fan theorem.

$\mathbf{FT}_\Delta$: Every decidable bar is uniform.

$\mathbf{FT}_c$: Every c-bar is uniform.

$\mathbf{FT}_{\Pi^0_1}$: Every Π^0_1-bar is uniform.

$\mathbf{FT}_{\mathrm{Full}}$: Every bar is uniform.

The following implications hold:

$$\boxed{\mathbf{FT}_\Delta} \Longleftarrow \boxed{\mathbf{FT}_c} \Longleftarrow \boxed{\mathbf{FT}_{\Pi^0_1}} \Longleftarrow \boxed{\mathbf{FT}_{\mathrm{Full}}}$$

It is one of the big open question in constructive reverse mathematics whether for any of these implications either the converse holds or one can prove they are strict. We can, however, give additional conditions that guarantee that *some* converses hold.

First we prove a nice little lemma about decidable bars which has two corollaries.

Lemma 4.1.1. *If*

$$B = \{u : P(u)\}$$

is a bar such that for each u *there exists a binary sequence* $(a(u)_n)_{n \geqslant 1}$ *with the property*

$$\exists n \in \mathbb{N}\ ((a(u)_n = 1) \Leftrightarrow P(u))\,,$$

then there exists a decidable bar B' *which is uniform only if* B *is.*

Proof. Let $B = \{u : P(u)\}$ an arbitrary bar and $(a(u)_n)_{n \geqslant 1}$ as above. Let

$$B' = \{u \in 2^* : \exists i \in \mathbb{N} \, \exists v \in 2^* \ (i \leqslant |u| \wedge v = \overline{u}(|v|) \wedge a(v)_i = 1)\} .$$

To see that B' is a bar, let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Since B is a bar, there exists n such that $\overline{\alpha}(n) \in B$, which means that $P(\overline{\alpha}(n))$. Hence there exists j such that $a(\overline{\alpha}(n))_j = 1$. Let $M = \max\{j, n\}$; then $\overline{\alpha}(M) \in B'$.
Now assume that B' is uniform. Since B' is closed under extensions, there exists N such that $\overline{\alpha}(N) \in B_0$ for all $\alpha \in 2^{\mathbb{N}}$. Hence there exist $k, l \leqslant N$ with $a(\overline{\alpha}(l))_k = 1$, which means that $\overline{\alpha}(l) \in B$. □

Corollary 4.1.2. *Every decidable bar is uniform if and only if every countable bar is.*

This result is not new and was proved in [13, Lemma 6.2.4], but we believe that the proof above is more elegant. Lemma 4.1.1 has another curious corollary, that depends on **Kripke's schema**.

> For each proposition P there exists an increasing binary sequence $(a_n)_{n \geqslant 1}$ such that P holds if and only if $a_n = 1$ for some n.

Kripke's schema is accepted by most intuitionists. It is most prominently invoked to refute Markov's principle in intuitionism [23, p. 350–352].

Corollary 4.1.3. *Under the assumption that Kripke's schema holds,* $\mathbf{FT}_{\text{Full}}$ *is equivalent to* $\mathbf{FT}_{\Delta}$*.*

This corollary might not be a surprise, since Kripke's schema is a very powerful principle. Nevertheless we can see that if we want to provide a model that separates *any* of the fan theorems, it must be in a model in which Kripke's schema fails to hold.

A more interesting result assumes Ishihara's principle **BD**-$\mathbb{N}$. It uses an equivalence that links $\mathbf{FT}_c$ with sequences.

Let A be a subset of a metric space (X, d). We say that a sequence $(x_n)_{n \geqslant 1}$ in X is

- **eventually bounded away from each point of** A, if

$$\forall x \in A \, \exists \delta > 0 \, \exists N \in \mathbb{N} \, \forall n \geqslant N \; (d(x_n, x) > \delta) \, ,$$

and

- **eventually bounded away from** A if

$$\exists \delta > 0 \, \exists N \in \mathbb{N} \, \forall n \geqslant N \, \forall x \in A \; (d(x_n, x) > \delta) \, .$$

The existence of a **(strong) Specker sequence**, that is a sequence of real numbers in the unit interval that is bounded away from every point of the unit interval (but not the interval itself), is one of the fascinating facets of RUSS. The first description of such a sequence was given in [41]. We say that A has the **anti-Specker** property if any sequence $(x_n)_{n \geqslant 1}$ in X that is eventually bounded away from every point of A is eventually bounded away from A.

Berger and Bridges have shown that $\mathbf{FT}_c$ is equivalent to the unit interval satisfying the anti-Specker property [4]. Along the same lines, we prove that $\mathbf{FT}_c$ *implies* the anti-Specker property for Cantor space.

Proposition 4.1.4. $\mathbf{FT}_c$ *implies the anti-Specker property for* $2^{\mathbb{N}}$*, relative to its one-point compactification* $2^{\mathbb{N}} \cup \{\omega\}$*.*

Proof. Let $(\alpha_n)_{n \geqslant 1}$ in $2^{\mathbb{N}} \cup \{\omega\}$ be a sequence that is bounded away from every point in $2^{\mathbb{N}}$. Define a decidable set

$$C' = \left\{ u \in 2^* : u \neq \overline{\alpha_{|u|}}(|u|) \right\} ,$$

and a c-set

$$C = \{ u \in 2^* : \forall w \in 2^* \; (u * w \in C') \} \, .$$

To prove that C is a bar, let $\alpha \in 2^{\mathbb{N}}$ arbitrary. Since $(\alpha_n)_{n \geqslant 1}$ is bounded away from every point in $2^{\mathbb{N}}$, there exist $N, M \in \mathbb{N}$ such that $\overline{\alpha}(M) \neq \overline{\alpha_i}(M)$ for all

$i \geqslant N$. Set $N' = \max\{N, M\}$. Then for each $i \geqslant N'$ and for each $w \in 2^*$

$$\overline{\alpha}(N') * w \neq \overline{\alpha_i}(i),$$

which means that $\overline{\alpha}(N') \in C$. Hence C is a bar.

Since we assumed $\mathbf{FT}_c$, we can find a natural number K such that $\overline{\alpha}(K) \in C$ for any $\alpha \in 2^{\mathbb{N}}$. The assumption that there is $j \geqslant K$ such that $\alpha_j \in 2^{\mathbb{N}}$ implies that $\overline{\alpha_j}(j) \notin C'$. Hence $\overline{\alpha_j}(j) \notin C$; a contradiction; and therefore $\alpha_i = \omega$ for each $i \geqslant K$. □

Using Proposition 4.1.4 we can prove

Proposition 4.1.5. $\mathbf{BD}$-$\mathbb{N}$ + $\mathbf{FT}_c$ *implies* $\mathbf{FT}_{\Pi^0_1}$.

Proof. Given a Π^0_1-bar B, pick a decidable set $S \subset 2^* \times \mathbb{N}$ such that

- if $(u, n) \in S$, then $(u * 0, n) \in S$ and $(u * 1, n) \in S$, and
- $u \in P$ if and only if $(u, n) \in S$ for each $n \in \mathbb{N}$.

Let

$$K = \{n \in \mathbb{N} : \exists u \in 2^* \, \exists i \in \mathbb{N} \, (|u| = n \wedge (u, i) \notin S)\} .$$

Then K is countable. In order to apply $\mathbf{BD}$-$\mathbb{N}$, we show that K is also pseudobounded. To this end, let $(a_n)_{n \geqslant 1}$ be a sequence in K. By countable choice[1], there exists a sequence $(u_n)_{n \geqslant 1}$ in 2^* such that for each n, $|u_n| = a_n$ and $u_n \notin B$. Define a sequence $\alpha_n \in 2^{\mathbb{N}} \cup \{\omega\}$ by

$$\alpha_n = \begin{cases} (u_n * 0 * \cdots) & \text{if } |u_n| \geqslant n, \\ \omega & \text{if } |u_n| < n. \end{cases}$$

Then $(\alpha_n)_{n \geqslant 1}$ is bounded away from every point in $2^{\mathbb{N}}$. For if $\alpha \in 2^{\mathbb{N}}$, then, since B is a bar, there exists N such that $\overline{\alpha}(N) \in B$; also, since every Π^0_1 bar is closed under extensions,

$$\forall w \in 2^* \; (\overline{\alpha_x}(N) * w \in B) . \tag{4.1}$$

[1] This is avoidable if we take u_n to be minimal for some total ordering on 2^*.

Now assume that there exists $i \geqslant N$ such that $\alpha_i \in 2^{\mathbb{N}}$ and $\overline{\alpha_i}(N) = \overline{\alpha}(N)$. Since $\alpha_i \in 2^{\mathbb{N}}$, we have $|u_i| \geqslant i \geqslant N$. Hence $\overline{u_i}(N) = \overline{\alpha}(N)$ and therefore $u_i \in B$—a contradiction. We conclude that for each $i \geqslant N$ either $\alpha_i = \omega$ or $\overline{\alpha_i}(N) \neq \overline{\alpha}(N)$, which means that the sequence $(\alpha_n)_{n \geqslant 1}$ is eventually bounded away from α. Using Proposition 4.1.4, we conclude that there exists $M \in \mathbb{N}$ such that $\alpha_i = \omega$ for all $i \geqslant M$. Hence $a_i = |u_i| < i$ for all $i \geqslant M$. We have now shown that K is pseudobounded. Since we are assuming **BD**-$\mathbb{N}$, it follows that K is bounded, which immediately yields that B is a uniform bar. □

$\mathbf{FT}_c$ was introduced by Berger in [3], with the intention of pinning down the exact strength of the uniform continuity theorem. In that paper, he showed that $\mathbf{FT}_c$ is equivalent to the following statement:

> $\mathbf{UCT}_{\mathbb{N}}$ Every continuous map $f : 2^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.

It is trivial that $\mathbf{UCT}_{\mathbb{N}}$ is *implied* by the **uniform continuity theorem**:

> **UCT** Every continuous map $f : 2^{\mathbb{N}} \to \mathbb{R}$ is uniformly continuous.

The next section shows that this principle, surprisingly, is equivalent to more general formulations of itself.

So **UCT** implies $\mathbf{FT}_c$. To relate **UCT** fully to the hierarchy of fan theorems we will also prove that it is implied by $\mathbf{FT}_{\Pi^0_1}$.

Theorem 4.1.6. $\mathbf{FT}_{\Pi^0_1}$ *implies* **UCT.**

Proof. Assuming $\mathbf{FT}_{\Pi^0_1}$, let $f : 2^{\mathbb{N}} \to \mathbb{R}$ be a pointwise continuous function. Given $\varepsilon > 0$ and using countable choice, construct a binary function λ on $2^* \times 2^*$ such that

$$\lambda(u,v) = 1 \Rightarrow |f(u) - f(v)| < \varepsilon,$$
$$\lambda(u,v) = 0 \Rightarrow |f(u) - f(v)| > \frac{\varepsilon}{2}.$$

Let S be the set of all ordered pairs $(u,n) \in 2^* \times \mathbb{N}$ with this property:

If v, w are elements of 2^* each with length at most $n - |u|$, then

$$\lambda(u * v, u * w) = 1.$$

It is clear that S is decidable. Let

$$B = \{u \in 2^* : \forall n \in \mathbb{N} \ ((u, n) \in S)\}.$$

Clearly, B satisfies the second of the two defining properties of a Π_1^0-set. To show that it satisfies the first, consider an arbitrary (u, n) in S. For all elements v, w of 2^* with lengths at most $n - |u| - 1$, we have

$$\lambda(u * 0 * v, u * 0 * w) = 1.$$

Hence $(u * 0, n) \in S$; likewise, $(u * 1, n) \in S$. This completes the proof that B is a Π_1^0-set. To prove that it is a bar, consider an arbitrary $\alpha \in 2^{\mathbb{N}}$. By the pointwise continuity of f, there exists N such that for all $v, w \in 2^*$,

$$|f(\overline{\alpha}(N) * v) - f(\overline{\alpha}(N) * w)| < \frac{\varepsilon}{2};$$

whence $\lambda(\overline{\alpha}(N) * v, \overline{\alpha}(N) * w) \neq 0$ and therefore

$$\lambda(\overline{\alpha}(N) * v, \overline{\alpha}(N) * w) = 1. \tag{4.2}$$

It follows that (4.2) holds in particular for each n and all elements v, w of 2^* with lengths at most $n - N$. Hence $\overline{\alpha}(N) \in B$, and B is a Π_1^0-bar.

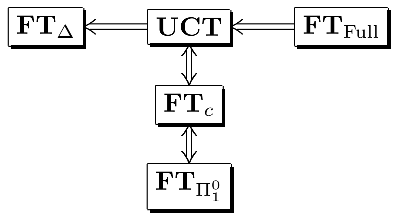
Applying $\mathbf{FT}_{\Pi_1^0}$, we compute N such that $\overline{\alpha}(N) \in B$ for each $\alpha \in 2^{\mathbb{N}}$. For such α and all $n \in \mathbb{N}$ we then have $(\overline{\alpha}(N), n) \in S$. It follows that for all $v, w \in 2^*$, condition (4.2) holds and therefore

$$|f(\overline{\alpha}(N) * v) - f(\overline{\alpha}(N) * w)| < \varepsilon.$$

Thus f is uniformly continuous. $\square$

It seems convenient to add **UCT** as a new principle to our hierarchy of fan theorems even though strictly speaking it is not concerned with bars. The full chain of implications now is:

Under the assumption of **BD**-$\mathbb{N}$ this collapses to:



If we assume Kripke's schema, then, as shown above, all of these principles collapse into one single principle.

4.2 The Heine-Borel theorem

It is time to return to the open-cover definition of compactness. The reader may note that similar work and results can be found in various places [13, 29, 30].

As we mentioned in the introduction, there is no hope of proving that, even for the unit interval, any open cover admits a finite subcover in BISH. This is because in RUSS we can explicitly define a countable cover of the unit interval with open intervals that does not admit a finite subcover, as we have seen in the introduction. However, the Heine-Borel theorem is provable in both INT and classical mathematics. Naturally this leads us once again to the various fan theorems. Our suspicion that the Heine-Borel theorem is equivalent to some fan theorem is confirmed by the fact that the following version of the Heine-Borel theorem is equivalent, over BISH, to the fan theorem $\mathbf{FT}_\Delta$.

HB$_0$: If $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are sequences of real numbers such that

$$[0,1] \subset \bigcup_{i=1}^{\infty} (a_i, b_i),$$

then there exists N such that

$$[0,1] \subset \bigcup_{i=1}^{N} (a_i, b_i).$$

An easy proof of this equivalence relies on the equivalence of $\mathbf{FT}_\Delta$ with the following principle.

> **POS:** If $f : [0,1] \to \mathbb{R}$ is a uniformly continuous and positive valued function, then there exists $c > 0$ such that $f(x) > c$ for all $x \in [0,1]$.

We can now prove:

Proposition 4.2.1. $\mathbf{HB}_0$ *is equivalent to* **POS**.

Proof. Assuming **POS**, let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ satisfy

$$[0,1] \subset \bigcup_{i=1}^{\infty} (a_i, b_i).$$

It is elementary to see that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^{\infty} 2^{-i} d(x, -(a_i, b_i))$$

is uniformly continuous. For an arbitrary $x \in [0,1]$, choose $i \in \mathbb{N}$ such that $x \in (a_i, b_i)$. Then $d(x, -(a_i, b_i)) > 0$ and therefore

$$f(x) \geqslant d(x, -(a_i, b_i)) > 0.$$

Applying **POS** yields $c > 0$ such that $f(x) > c$ for all $x \in [0,1]$. Choose $N \in \mathbb{N}$

such that $2^{-N} < c$. For $x \in [0,1]$, the assumption

$$\sum_{i=1}^{N} 2^{-i} d(x, -(a_i, b_i)) < 2^{-N-1}$$

implies that $f(x) < 2^{-N}$, a contradiction. Therefore

$$\sum_{i=1}^{N} 2^{-i} d(x, -(a_i, b_i)) > 2^{-N-2},$$

so there exists $j \in \mathbb{N}$ such that $1 \leqslant j \leqslant N$ and $d(x, -(a_j, b_j)) > 0$. Hence $x \in (a_j, b_j)$.

For the converse, $\mathbf{HB}_0 \Rightarrow \mathbf{POS}$, assume that $f : [0,1] \to \mathbb{R}$ is uniformly continuous and that $f(x) > 0$ for every $x \in [0,1]$. Since $[0,1]$ is totally bounded and f is uniformly continuous, for each $n \in \mathbb{N}$ there exist $\delta_n > 0$, finitely many $x_{n,1}, \ldots, x_{n,m_n} \in [0,1]$, and $\lambda_{n,1}, \ldots, \lambda_{n,m_n} \in \{0,1\}$ such that the following conditions hold for any $i \in \mathbb{N}$ with $1 \leqslant i \leqslant m_n$

$$[0,1] \subset \bigcup_{i=1}^{m_n} I_{n,i},$$

$$\lambda_{n,i} = 0 \Rightarrow f(I_{n,i}) < 2^{-n+1},$$

$$\lambda_{n,i} = 1 \Rightarrow f(I_{n,i}) > 2^{-n},$$

where $I_{n,i}$ denotes the open interval $(x_{n,i} - \delta_n, x_{n,i} + \delta_n)$. Since f is continuous and positive valued,

$$[0,1] \subset \bigcup_{\substack{n \geqslant 1, \\ i=1\ldots m_n, \\ \lambda_{n,i}=1}} I_{n,i}.$$

Hence we can apply $\mathbf{HB}_0$ to get $N \in \mathbb{N}$ such that

$$[0,1] \subset \bigcup_{\substack{n=1\ldots N, \\ i=1\ldots m_n, \\ \lambda_{n,i}=1}} I_{n,i}.$$

It is now easy to see that $f(x) > 2^N$ for all $x \in [0,1]$. □

Ishihara has produced more general results of a similar type by linking the complexity of bars with the complexity of the index set of the cover with open intervals [29]. We will take a different route, linking the complexity of the open sets to the complexity of bars.

We call a subset $A \subset X$ of a metric space (X,d) **(uniformly) cozero** if there exists a (uniformly) continuous function $f : X \to \mathbb{R}$ such that

$$A = \{x \in X : f(x) > 0\}.$$

By continuity, cozero sets are open. With these definitions we can, as follows, identify four more versions of the Heine-Borel theorem, in each of which we assume that X is an arbitrary compact metric space and $(U_n)_{n \geqslant 1}$ is a sequence of open sets such that $X = \bigcup_{i=1}^{\infty} U_i$.

HB$_1$: If U_n is uniformly cozero for each $n \in \mathbb{N}$, then there exists N such that $X = \bigcup_{i=1}^{N} U_i$.

HB$_2$: If U_n is colocated for each $n \in \mathbb{N}$, then there exists N such that $X = \bigcup_{i=1}^{N} U_i$.

HB$_3$: If U_n is cozero for each $n \in \mathbb{N}$, then there exists N such that $X = \bigcup_{i=1}^{N} U_i$.

HB$_4$: There exists N such that $X = \bigcup_{i=1}^{N} U_i$.

Proposition 4.2.2. *The following are equivalent:*

(i) $\mathbf{FT}_\Delta$

(ii) $\mathbf{HB}_i$*, where* $i = 0, 1, 2$

Proof. As we mentioned earlier, $\mathbf{FT}_\Delta$ is equivalent to **POS**. Proposition 4.2.1 now shows that $\mathbf{FT}_\Delta$ is equivalent to $\mathbf{HB}_0$. It is clear that $\mathbf{HB}_2$ implies $\mathbf{HB}_0$. Next, let $(U_n)_{n \geqslant 1}$ be a uniformly cozero cover of X, and let $(f_n)_{n \geqslant 1}$ be a sequence of uniformly continuous functions such that

$$U_i = \{x \in X : f_i(x) > 0\}$$

for each n; then we can virtually repeat the proof of Proposition 4.2.1, replacing $d(x, -(a_i, b_i))$ by $f_i(x)$, to prove that $\mathbf{FT}_\Delta$ implies $\mathbf{HB}_1$. Lastly, to see that $\mathbf{HB}_1$ implies $\mathbf{HB}_2$, simply note that if U is a colocated subset of a metric space, then

$$U = \{x \in X : f(x) > 0\},$$

where the uniformly continuous function f is defined by $f(x) = d(x, -U)$. $\square$

The following equivalence shows that not all versions of the Heine-Borel theorem are equivalent to $\mathbf{FT}_\Delta$.

Corollary 4.2.3. $\mathbf{HB}_3$ *is equivalent to* **UCT**.

Proof. First note that **UCT** implies that a cozero subset of a compact metric space is uniformly cozero; whence $\mathbf{UCT} \Rightarrow \mathbf{HB}_3$. For the reverse implication, we refer to the next section[2] and the fact that for a continuous function $f : X \to \mathbb{R}$,

$$X = \bigcup_{i=1}^{\infty} \{x \in X : f(x) < i\},$$

where each of the sets in the union on the right-hand side is cozero. $\square$

In a well known paper [36] Moerdijk showed that, if we restricted $\mathbf{HB}_4$ to the unit interval, we could not hope to prove the reverse implication of the following theorem.

Proposition 4.2.4. $\mathbf{HB}_4$ *is equivalent to* $\mathbf{FT}_{\mathrm{Full}}$.

Proof. A proof that $\mathbf{FT}_{\mathrm{Full}}$ implies $\mathbf{HB}_4$ can be found in [13]. To show that the converse holds let $B \subset 2^*$ be a bar. Given $u \in 2^*$, define an open set

$$U_u = \left\{\alpha \in 2^{\mathbb{N}} : u \in B \wedge \overline{\alpha}(|u|) = u\right\}.$$

Then

$$2^{\mathbb{N}} = \bigcup_{u \in 2^*} U_u.$$

[2]The reader is asked to excuse this un-mathematical forward reference, and may be assured that there are no circularities stemming from it.

Applying $\mathbf{HB}_4$ yields N such that

$$2^{\mathbb{N}} = \bigcup_{\substack{u \in 2^*, \\ |u| \leqslant N}} U_u.$$

Thus for each $\alpha \in 2^{\mathbb{N}}$, there exists $u \in 2^*$ such that $|u| \leqslant N$ and $\alpha \in U_u$. Furthermore, $u \in B$ and $\overline{\alpha}(|u|) = u$; whence $\overline{\alpha}(|u|) \in B$. $\square$

4.3 Pseudocompactness

This section contains results produced together with D.S. Bridges and which are published in [9].

In classical topology, the image of a compact space under a continuous map is compact. A real-valued continuous function with a compact (topological) space as a domain is therefore bounded. The interesting question is whether the converse also holds: if every real-valued continuous map on a metric space X is bounded, is X compact?

A metric space X is called **pseudocompact** if every pointwise continuous mapping of X into $\mathbb{R}$ is bounded.

Theorem 4.3.1. *In classical mathematics, for metric spaces, open-cover compactness is equivalent to pseudocompactness.*

Proofs can be found in many textbooks on topology, such as [39]. In this section we investigate the pseudocompactness of [0,1] within Bishop-style constructive mathematics.

First we notice that in the recursive model of BISH there exists a function $f : [0, 1] \to \mathbb{R}$ such that f is unbounded. (A simple construction for such a function can be found in Section 4.5.) In INT, every continuous function defined on a compact metric space is uniformly continuous, and hence bounded. This combination of results suggests, yet again, fan-theoretic principles. The main result of this section shows that the pseudocompactness of the interval $[0, 1]$, or Cantor space $2^{\mathbb{N}}$, is equivalent to the uniform continuity theorem.

We begin with some technical material that will enable us to connect pointwise continuous, real-valued functions on $[0,1]$ with such functions on Cantor space $2^{\mathbb{N}}$.

Recall that the **Cantor set** C consists of all real numbers of the form $\sum_{n=1}^{\infty} a_n 3^{-n}$ with each $a_n \in \{0,2\}$. For each $\alpha \in 2^{\mathbb{N}}$ we define

$$F(\alpha) = \sum_{n=0}^{\infty} 2\alpha_n 3^{-n-1},$$

to produce a mapping F of $2^{\mathbb{N}}$ onto the Cantor set. The following elementary lemma, whose proof we omit, shows that F is one-one.

Lemma 4.3.2. *Let* $x = \sum_{n=0}^{\infty} a_n 3^{-n-1}$ *and* $y = \sum_{n=0}^{\infty} b_n 3^{-n-1}$, *where* $a_n, b_n \in \{0,2\}$ *for each* n, *and let* N *be a positive integer. If* $|x-y| < 3^{-N-1}$, *then* $a_n = b_n$ *for* $0 \leqslant n \leqslant N$.

Thus, given a pointwise continuous mapping $f : 2^{\mathbb{N}} \to \mathbb{R}$, we can define a mapping $\widetilde{f} : C \to \mathbb{R}$ by

$$\widetilde{f}(F(\alpha)) = f(\alpha) \quad \left(\alpha \in 2^{\mathbb{N}}\right).$$

In the construction of the Cantor set, at the N^{th} iteration of the process of removing "middle thirds" of intervals, we obtain $2^{N+1}-1$ subintervals

$$I_{N,0}, I_{N,1}, \ldots, I_{N,2^{N+1}-2}$$

of $[0,1]$ such that the following properties hold:

- the left endpoint of $I_{N,0}$ is 0;
- the right endpoint of $I_{N,2^{N+1}-2}$ is 1;
- for $k = 0, \cdots, 2^{N+1}-3$, the right endpoint of $I_{N,k}$ is the left endpoint of $I_{N,k+1}$;
- for each even k, $I_{N,k}$ is closed, has endpoints in C, and has length 3^{-N};

- for each odd k, $I_{N,k}$ is open and lies in $[0,1] - C$.

For given N and k there exist unique $\alpha_{N,k}, \beta_{N,k} \in 2^{\mathbb{N}}$ such that the left and right endpoints of $I_{N,k}$ are $F(\alpha_{N,k})$ and $F(\beta_{N,k})$ respectively.

In order to extend $\widetilde{f}$ to a mapping on $C \cup (I - C))$, consider an arbitrary natural number N and an arbitrary odd natural number k. Each point x of $I_{N,k}$ can be written uniquely in the form

$$tF(\alpha_{N,k}) + (1-t)\,F(\beta_{N,k})$$

with $t \in (0,1)$; we define

$$\widetilde{f}(x) = tf(\alpha_{N,k}) + (1-t)\,f(\beta_{N,k}).$$

Note that for each $\varepsilon > 0$, if $x \in I_{N,k}$ is sufficiently close to an endpoint of $I_{N,k}$, then $\widetilde{f}(x)$ is within ε of the value of f at that endpoint.

Lemma 4.3.3. *Let $f : 2^{\mathbb{N}} \to \mathbb{R}$ be pointwise continuous. Then $\widetilde{f}$ is continuous at each point of C.*

Proof. Fixing $x = F(\alpha) \in C$ and $\varepsilon > 0$, choose a positive integer N such that

$$|f(\alpha) - f(\overline{\alpha}(N) * \gamma)| < \varepsilon/2$$

for all $\gamma \in 2^{\mathbb{N}}$. Then find the even k with $x \in I_{N+2,k}$. Since

$$|F(\alpha_{N+2,k}) - F(\beta_{N+2,k})| = 3^{-N-2} < 3^{-N-1},$$

Lemma 4.3.2 shows that $\overline{\alpha_{N+2,k}}(N) = \overline{\alpha}(N) = \overline{\beta_{N+2,k}}(N)$. So

$$|f(\alpha) - f(\alpha_{N+2,k})| < \varepsilon/2 \text{ and } |f(\alpha) - f(\beta_{N+2,k})| < \varepsilon/2.$$

Choose $\delta > 0$ such that if $y \in I_{N+2,k-1}$ and $|y - F(\alpha_{N+2,k})| < \delta$, then

$$\left|\widetilde{f}(y) - f(\alpha_{N+2,k})\right| < \varepsilon/2,$$

and such that if $y \in I_{N+2,k+1}$ and $|y - F(\beta_{N+2,k})| < \delta$, then

$$\left|\widetilde{f}(y) - f(\beta_{N+2,k})\right| < \varepsilon/2.$$

Consider $y \in C \cup ([0,1] - C)$ such that $|x - y| < \min\left\{\delta, 3^{-N-2}\right\}$. If $y = F(\beta) \in C$, then by Lemma 4.3.2, $\overline{\alpha}(N) = \overline{\beta}(N)$; so, by our choice of N, $\left|\widetilde{f}(x) - \widetilde{f}(y)\right| < \varepsilon/2$. If $y \in [0,1] - C$, then reference to the bullet points above shows that $y \in I_{N+2,k-1} \subset [0,1] - C$ or $y \in I_{N+2,k}$ or $y \in I_{N+2,k+1} \subset [0,1] - C$. In the first case,

$$\left|\widetilde{f}(x) - \widetilde{f}(y)\right| \leqslant |f(\alpha) - f(\alpha_{N+2,k})| + \left|f(\alpha_{N+2,k}) - \widetilde{f}(y)\right| < \varepsilon.$$

A similar argument gives $\left|\widetilde{f}(x) - \widetilde{f}(y)\right| < \varepsilon$ in the third case. In the second case, again with reference to the bullet points above, we see that there exist $m > N + 2$ and an odd j with $1 \leqslant j \leqslant 2^{m+1} - 3$ such that $y \in I_{m,j} \subset I^{\circ}_{N+2,k}$. Then

$$y = tF(\alpha_{m,j}) + (1-t)\,F(\beta_{m,j})$$

for a unique $t \in (0,1)$, and

$$\begin{aligned}\left|\widetilde{f}(y) - f(\alpha)\right| &\leqslant t\,|f(\alpha_{m,j}) - f(\alpha)| + (1-t)\,|f(\beta_{m,j}) - f(\alpha)| \\ &< t\frac{\varepsilon}{2} + (1-t)\,\frac{\varepsilon}{2} = \frac{\varepsilon}{2}.\end{aligned}$$

It follows that $\left|\widetilde{f}(x) - \widetilde{f}(y)\right| < \varepsilon$ holds in all three cases for $y \in [0,1] - C$. □

Lemma 4.3.4. *Under the hypotheses of* Lemma *4.3.3, for each* $x \in [0,1]$ *and each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $y, z \in [0,1] - C$ *and*

$$\max\{|x-y|, |x-z|\} < \delta,$$

then $\left|\widetilde{f}(y) - \widetilde{f}(z)\right| < \varepsilon$.

Proof. If $x \in C$, the desired conclusion follows almost immediately from Lemma

4.3.3. For a general $x \in [0,1]$, since C is compact, we can apply Bishop's Lemma [19, Proposition 3.1.1], to construct $\alpha \in 2^{\mathbb{N}}$ such that if $x \neq F(\alpha)$, then $d(x,C) > 0$. There exists $t > 0$ such that if $y, z \in [0,1] - C$ and

$$\max\{|F(\alpha) - y|, |F(\alpha) - z|\} < t, \tag{4.3}$$

then $\left|\widetilde{f}(y) - \widetilde{f}(z)\right| < \varepsilon$. Either $|x - F(\alpha)| < t/2$ or $x \neq F(\alpha)$. In the first case, if $y, z \in [0,1] - C$ and $\max\{|x-y|, |x-z|\} < t/2$, then (4.3) holds, so $\left|\widetilde{f}(y) - \widetilde{f}(z)\right| < \varepsilon$ and we can take $\delta = t/2$. In the second case, $d(x,C) > 0$, so there exist N and an odd k such that $x \in I_{N,k}$; since the function $\widetilde{f}$ is linear on $I_{N,k}$, it is clear that there exists $\delta > 0$ with the required property. □

Proposition 4.3.5. *For each pointwise continuous $f : 2^{\mathbb{N}} \to \mathbb{R}$, the function $\widetilde{f}$ extends to a pointwise continuous mapping of $[0,1]$ into $\mathbb{R}$ such that f is bounded on $2^{\mathbb{N}}$ if and only if $\widetilde{f}$ is bounded on $[0,1]$.*

Proof. Consider $x \in [0,1]$. Since $[0,1] - C$ is dense in $[0,1]$, there exists a sequence $(x_n)_{n\geqslant 1}$ of points of $[0,1] - C$ converging to x. It readily follows from Lemma 4.3.4 that

$$\widetilde{f}(x) = \lim_{n\to\infty} \widetilde{f}(x_n)$$

exists, is independent of the choice of sequence $(x_n)_{n\geqslant 1}$ converging to x, is pointwise continuous at x, and coincides with our original value of $\widetilde{f}(x)$ when $x \in C \cup ([0,1] - C)$. □

Classically, a metric space is pseudocompact if and only if it is compact 4.3.1; however, in the recursive model of BISH, the interval $[0,1]$ is compact but not pseudocompact.

Lemma 4.3.6. *If $[0,1]$ is pseudocompact, then so is every compact metric space.*

Proof. Let X be a compact metric space. The work on pages 103–106 of [13] shows that there exists a uniformly continuous map h of $2^{\mathbb{N}}$ onto X. If $g : X \to \mathbb{R}$ is pointwise continuous, then so is $g \circ h$. By Proposition 4.3.5, $g \circ h$ extends to a pointwise continuous map $\widetilde{g \circ h}$ on $[0,1]$ which is bounded if and only if $g \circ h$ is bounded. So if $[0,1]$ is pseudocompact, then $g \circ h$, and therefore g, is bounded. □

The proof of the next proposition is found in [16].

Proposition 4.3.7. *If X is a separable pseudocompact metric space, then every pointwise continuous map of X into $\mathbb{R}$ has totally bounded range.*

Our next Lemma is a variant of Theorem 2.2.13 of [19]. It requires the following corollary of that theorem [19, Corollary 2.2.12].

Proposition 4.3.8. *If X is a totally bounded metric space, then for each $\varepsilon > 0$ there exist totally bounded sets $K_1, \ldots, K_n$, each of diameter less than or equal to ε, such that $X = \bigcup_{i=1}^{n} K_i$.*

Lemma 4.3.9. *Suppose that $[0,1]$ is pseudocompact. Let X be a compact metric space, and F a pointwise continuous mapping of $X \times X$ into $\mathbb{R}$. Then for all but countably many $r \in \mathbb{R}$, the set*

$$\{x \in X \times X : F(x) \leqslant r\} \tag{4.4}$$

is either compact or empty.

Proof. Using Proposition 4.3.8, for each positive integer k, cover $X \times X$ by finitely many compact subsets $X_{k,j}$, where $1 \leqslant j \leqslant n_k$, each with diameter less than $1/k$. In view of Lemma 4.3.6, each $X_{k,j}$ is pseudocompact; whence, by Proposition 4.3.7, $F(X_{k,j})$ is totally bounded. It follows from Proposition 2.2.5 of [19] that

$$c_{k,j} = \inf \{F(x) : x \in X_{k,j}\}$$

exists. To complete the proof that the set (4.4) is totally bounded, we now use an argument virtually identical to that in the proof of Theorem 2.2.13 of [19]; we omit the details. Finally, since F is pointwise continuous, the set (4.4) is closed in X and hence complete. □

Lemma 4.3.10. *Suppose that* $[0,1]$ *is pseudocompact. Let* X *be a compact metric space, and* f *a pointwise continuous, positive-valued mapping of* X *into* $\mathbb{R}$*. Then the mapping* $1/f$ *is bounded.*

Proof. By Lemma 4.3.6, X is pseudocompact. The result now follows since $1/f$ is pointwise continuous on X. □

This brings us to the main result of this section.

Theorem 4.3.11. *The following conditions are equivalent.*

(i) $[0,1]$ *is pseudocompact.*

(ii) Every pointwise continuous mapping of $[0,1]$ *into* $\mathbb{R}$ *is uniformly continuous.*

(iii) $2^{\mathbb{N}}$ *is pseudocompact.*

(iv) Every pointwise continuous mapping of $2^{\mathbb{N}}$ *into* $\mathbb{R}$ *is uniformly continuous.*

(v) **UCT.**

Proof. We prove that

$$\text{(v)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)} \Rightarrow \text{(v)}.$$

It is clear that (v) $\Rightarrow$ (ii) and that (iv) $\Rightarrow$ (iii). The implication from (iii) to (i) is a simple consequence of the existence of a uniformly continuous mapping from $2^{\mathbb{N}}$ onto $[0,1]$ [13, Theorem 1.4]. Suppose that (ii) holds. If $f : 2^{\mathbb{N}} \to \mathbb{R}$ is pointwise continuous, then by Lemma 4.3.3 and (ii), $\widetilde{f}$ is uniformly continuous; since $f = \widetilde{f} \circ F$ and F is uniformly continuous, we conclude that f is uniformly continuous. Thus (ii) implies (iv).

Supposing that $[0,1]$ is pseudocompact, let X be a compact metric space, and f a pointwise continuous map of X into a metric space Y. Given $\varepsilon > 0$, and referring to Lemma 4.3.9, we may assume that

$$S = \{(x,x') \in X \times X : d\left(f(x), f(x')\right) \geqslant \varepsilon\}$$

is totally bounded. Then the mapping $h : X \to \mathbb{R}$ defined by

$$h(x) = d\left((x, x), S\right) = \inf\left\{d\left((x, x), (s, t)\right) : (s, t) \in S\right\}$$

is well defined on X. Since the function that maps (x, x') to $d\left(f(x), f(x')\right)$ is pointwise continuous on $X \times X$, for each $x \in X$ we have $h(x) > 0$. By Lemma 4.3.10, there exists $\delta > 0$ such that $h(x)^{-1} < 1/\delta$, and therefore $h(x) > \delta$, for all $x \in X$. Now consider a point (x, x') in the product space $X \times X$ such that $d(x, x') < \delta$. If $(x, x') \in S$, then

$$d\left((x, x), S\right) \leqslant d\left((x, x), (x, x')\right) = d(x, x') < \delta,$$

a contradiction. Hence $(x, x') \notin S$ and therefore $d(f(x), f(x')) \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have proved that (i) $\Rightarrow$ (v).

□

4.4 Equicontinuity and Ascoli's theorem

This section contains results produced in cooperation with I. Loeb and which are currently awaiting publication [21].

Ascoli's theorem shows how strong a concept compactness is, as the compactness of a space is inherited by a sequence of functions defined on this space. The theorem is deeply related to the notions of equicontinuity and uniform equicontinuity, which generalise their counterparts for continuity. We say that a sequence $(f_n)_{n \geqslant 0}$ of real-valued functions on $[0, 1]$ is

- (pointwise) **equicontinuous** if for each x in $[0, 1]$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that if y in $[0, 1]$ and $|x - y| < \delta$, then

$$|f_n(x) - f_n(y)| < \varepsilon \text{ for all } n; \qquad (4.5)$$

- **uniformly equicontinuous** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in [0,1]$ and $|x-y| < \delta$, then (4.5) holds.

We can also generalise notions of convergence to sequences of functions. We say that a sequence $(f_n)_{n \geqslant 0}$ of real-valued functions on $[0,1]$ is

- (pointwise) **convergent** if there exists a real-valued function f from $[0,1]$ such that for each x in $[0,1]$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geqslant m; \tag{4.6}$$

- **uniformly convergent** if there exists a real-valued function f from $[0,1]$ such that for each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that (4.6) holds for all $x \in [0,1]$.

We then call f the **limit** of the sequence.

The metric we use for function spaces is the metric associated with the supremum norm:

$$\|f\| = \sup \{|f(x)| : x \in X\}$$

for all uniformly continuous functions f on a totally bounded space X into $\mathbb{R}$. This is the metric we have in mind when we talk about a sequence of functions being totally bounded.

Before we can prove two equivalences involving **UCT** and $\mathbf{FT}_{\Pi_1^0}$, we need three technical lemmas and the following construction. Given a real-valued continuous function f on $[0,1]$, we construct an equicontinuous sequence $(f_n)_{n \geqslant 0}$ of real-valued uniformly continuous functions on $[0,1]$ that converges to f. To construct the n^{th} function of this sequence, the key idea is to define

$$f_n(i2^{-n}) = f(i2^{-n})$$

for each $i \leqslant 2^n$, and to interpolate linearly between these points.

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For each $n \in \mathbb{N}$ and each $i < 2^n$

define a real number $m_{n,i}$ by:

$$m_{n,i} = f((i+1)2^{-n}) - f(i2^{-n}).$$

For each n define a function

$$g_n : \bigcup_{0 \leqslant i \leqslant 2^n - 1} [i2^{-n}, (i+1)2^{-n}] \to \mathbb{R}$$

by setting

$$g_n(x) = 2^n m_{n,i} x + f(i2^{-n}) - i m_{n,i}$$

for $x \in [i2^{-n}, (i+1)2^{-n}]$. The n^{th} **linear approximation to** f, is the uniformly continuous mapping $f_n : [0,1] \to \mathbb{R}$ that extends g_n to the domain $[0,1]$ [6, Lemma 3.7]. We call $(f_n)_{n \geqslant 0}$ the **sequence of linear approximations to** f**.**

Lemma 4.4.1. *Let f be a real-valued continuous function on $[0,1]$ and let $(f_n)_{n \geqslant 0}$ be its sequence of linear approximations. Then*

(i) the sequence $(f_n)_{n \geqslant 0}$ is convergent;

(ii) the sequence $(f_n)_{n \geqslant 0}$ is equicontinuous;

(iii) the function f_n is uniformly continuous for each n;

(iv) for each $\varepsilon > 0$ there exists an ε-approximation $\{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ such that the set

$$\{(f_i(x_0), f_i(x_1), \ldots, f_i(x_n)) : i \in \mathbb{N}\}$$

is totally bounded.

Proof. Given $x \in [0,1]$ and $\varepsilon > 0$, compute a positive integer m such that if $y \in [0,1]$ and $|x - y| \leqslant 2^{-m}$, then $|f(x) - f(y)| < \frac{\varepsilon}{3}$. Taking $p > m$, compute $k \in \mathbb{N}$ such that

$$\frac{k}{2^p} \leqslant x \leqslant \frac{k+2}{2^p}.$$

Then for $l \in \{k+1, k+2\}$,

$$\begin{aligned}\left|f_p\left(\frac{k}{2^p}\right) - f_p\left(\frac{l}{2^p}\right)\right| &\leqslant \left|f_p\left(\frac{k}{2^p}\right) - f(x)\right| + \left|f(x) - f_p\left(\frac{l}{2^{-p}}\right)\right| \\ &= \left|f\left(\frac{k}{2^p}\right) - f(x)\right| + \left|f(x) - f\left(\frac{l}{2^p}\right)\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.\end{aligned}$$

Thus (by the definition of f_p)

$$\left|f_p\left(x\right) - f_p\left(\frac{k}{2^p}\right)\right| < \frac{2\varepsilon}{3}.$$

It follows that

$$\begin{aligned}|f_p(x) - f(x)| \leqslant &\left|f(x) - f\left(\frac{k}{2^p}\right)\right| + \left|f_p(x) - f_p\left(\frac{k}{2^p}\right)\right| + \\ &\left|f\left(\frac{k}{2^p}\right) - f_p\left(\frac{k}{2^p}\right)\right| \\ &< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} + 0 = \varepsilon.\end{aligned}$$

Hence the sequence $(f_n)_{n\geqslant 1}$ converges to f.

To prove (ii), with x and ε as before and using the continuity of f at x, compute a positive integer m such that if $y \in [0,1]$ and $|x-y| \leqslant 2^{-m}$, then $|f(x) - f(y)| < \frac{1}{4}\varepsilon$. For each $i \leqslant m$, there exists δ_i such that if $y \in [0,1]$ and $|x-y| \leqslant \delta_i$, then $|f_i(x) - f_i(y)| < \varepsilon$. Let $p = m+1$, and pick $k \in \mathbb{N}$ such that

$$\frac{k}{2^p} < x < \frac{k+2}{2^p}.$$

Define

$$\delta_\infty = \min\left\{\left|x - \frac{k}{2^p}\right|, \left|x - \frac{k+2}{2^p}\right|\right\},$$

and

$$\delta = \min\left\{\delta_0, \delta_1, \ldots, \delta_m, \delta_\infty\right\}.$$

Consider any $y \in [0,1]$ with $|x-y| < \delta$, and any $n \in \mathbb{N}$. We claim that

$|f_n(x) - f_n(y)| < \varepsilon$. In the case $n \leqslant m$, the claim follows immediately from the definition of δ_n and δ. In the case $n > m$, noting that

$$\frac{k}{2^p} < x, y < \frac{(k+2)}{2^p}$$

and using an argument similar to the one in the first paragraph of this proof, we see that $|f_n(x) - f_n(\frac{k}{2^p})| < \frac{1}{2}\varepsilon$, $|f_n(y) - f_n(\frac{k}{2^p})| < \frac{1}{2}\varepsilon$ and therefore $|f_n(x) - f_n(y)| < \varepsilon$. This completes the proof of (iii).

Conclusion (iii) follows from the definition of "n$^{\text{th}}$ linear approximation". To prove (iv) compute a positive integer m such that $2^m < \varepsilon$. Then

$$S = \left\{ \frac{k}{2^{m+1}} : 0 \leqslant k \leqslant 2^{m+1} \right\}$$

is an ε-approximation to $[0, 1]$. Moreover, $f_n(S)$ is totally bounded, since $f_j(x) = f_k(x)$ for all $j, k \geqslant 2^{m+1}$ and all $x \in S$. □

Lemma 4.4.2. *Every uniformly equicontinuous convergent sequence of real-valued functions on* $[0, 1]$ *has a uniformly continuous limit.*

Proof. Let $(f_n)_{n \geqslant 0}$ be a uniformly equicontinuous sequence of real-valued functions on $[0, 1]$ with limit f, and let $\varepsilon > 0$. There exists $\delta > 0$ such that for each $n \in \mathbb{N}$, if $x, y \in [0, 1]$ and $|x - y| < \delta$ then $|f_i(x) - f_i(y)| < \frac{1}{3}\varepsilon$. Given such x and y, compute positive integers m_0, m_1 such that $|f_n(x) - f(x)| < \frac{1}{3}\varepsilon$ for all $n \geqslant m_0$, and $|f_n(y) - f(y)| < \frac{\varepsilon}{3}$ for all $n \geqslant m_1$. Setting

$$p = \max\{m_0, m_1\},$$

we see that

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - f_p(x)| + |f_p(x) - f_p(y)| + |f_p(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus f is uniformly continuous on $[0, 1]$. □

Lemma 4.4.3. *Every totally bounded sequence of uniformly continuous real-valued functions on* $[0, 1]$ *is uniformly equicontinuous.*

Proof. Let $(f_n)_{n\geqslant 0}$ be a totally bounded sequence of uniformly continuous real-valued functions on $[0, 1]$, and let $\varepsilon > 0$. Compute m such that for each $k \in \mathbb{N}$, there exists $j \leqslant m$ such that $\|f_k - f_j\| < \frac{1}{3}\varepsilon$. For each $i \leqslant m$, there exists δ_i such that if $x, y \in [0, 1]$ and $|x - y| < \delta_i$ then $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$. Define

$$\delta = \min\{\delta_1, \ldots, \delta_m\}.$$

Consider any $n \in \mathbb{N}$ and any $x, y \in [0, 1]$ with $|x - y| < \delta$. Pick $j \leqslant m$ such that $\|f_n - f_j\| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned} |f_n(x) - f_n(y)| &\leqslant |f_n(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f_n(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence the sequence $(f_n)_{n\geqslant 0}$ is uniformly equicontinuous. □

We now introduce a variant of **UCT** for sequences of functions.

> **UECT**$_0$: For each equicontinuous sequence $(f_n)_{n\geqslant 0}$ of real-valued continuous functions on $[0, 1]$, if $\{f_i(x) : i \in \mathbb{N}\}$ is totally bounded for each $x \in [0, 1]$, then $(f_n)_{n\geqslant 0}$ is uniformly equicontinuous.

Theorem 4.4.4. *The following statements are equivalent:*

(i) **UCT**.

(ii) *Every totally bounded sequence of real-valued continuous functions on* $[0, 1]$ *is uniformly equicontinuous.*

(iii) **UECT**$_0$

Proof. Lemma 4.4.3 shows that (i) implies (ii). For the reverse implication, given a continuous mapping $f : [0, 1] \to \mathbb{R}$, set $f_n = f$ for all $n \in \mathbb{N}$. The sequence $(f_n)_{n\geqslant 0}$ is totally bounded, so if (ii) holds, it is uniformly equicontinuous; whence f is uniformly continuous.

To prove that (i) implies (iii), let $(f_n)_{n\geqslant 0}$ be an equicontinuous sequence of real-valued functions on $[0,1]$ such that $\{f_n(x) : n \in \mathbb{N}\}$ is totally bounded. Then for all $x, y \in [0,1]$,

$$\{|f_n(x) - f_n(y)| : n \in \mathbb{N}\}$$

is totally bounded, so

$$f(x,y) = \sup\{|f_n(x) - f_n(y)| : n \in \mathbb{N}\}$$

exists. We show that the function $f : [0,1]\times[0,1] \to \mathbb{R}$ so defined is continuous. Given $\varepsilon > 0$ and points $x_1, x_2 \in [0,1]$, for $k = 1, 2$ pick δ_k such that for each n, if $x \in [0,1]$ and $|x - x_k| < \delta_k$, then $|f_n(x) - f_n(x_k)| < \varepsilon/2$. Setting

$$\delta = \min\{\delta_1, \delta_2\},$$

consider $(x_1', x_2') \in [0,1]\times[0,1]$ such that

$$|x_1' - x_1| + |x_2' - x_2| < \delta. \tag{4.7}$$

For each n we have

$$\begin{aligned} |f_n(x_1) - f_n(x_2)| &\leqslant |f_n(x_1) - f_n(x_1')| + |f_n(x_2) - f_n(x_2')| + |f_n(x_1') - f_n(x_2')| \\ &\leqslant |f_n(x_1) - f_n(x_1')| + |f_n(x_2) - f_n(x_2')| + f(x_1', x_2') \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + f(x_1', x_2') = \varepsilon + f(x_1', x_2'). \end{aligned}$$

It follows that

$$f(x_1, x_2) \leqslant \varepsilon + f(x_1', x_2'),$$

and therefore

$$f(x_1, x_2) - f(x_1', x_2') \leqslant \varepsilon.$$

Similarly

$$f(x_1', x_2') - f(x_1, x_2) \leqslant \varepsilon.$$

Hence

$$|f(x_1, x_2) - f(x_1, x_2)| \leqslant \varepsilon. \tag{4.8}$$

Thus f is continuous. Assuming (i), and recalling that **UCT** implies that every continuous function from a compact metric space into a metric space is uniformly continuous, we now see that f is uniformly continuous on $[0,1] \times [0,1]$; so we may assume that (4.8) holds for all x_1, x_1', x_2, x_2' in $[0,1]$ to which (4.7) applies. It follows that if $x, y \in [0,1]$ and $|x - y| < \delta$, then for each n,

$$|f_n(x) - f_n(y)| \leqslant f(x, y) = |f(x, y) - f(x, x)| < \varepsilon.$$

Thus (iii) holds.

Finally to prove that (iii) implies (i), we argue as we did when proving that (ii) implies (i). □

Having proved the equivalence of **UCT** and $\mathbf{UECT}_0$, we introduce a variant of **UCT** that turns out to be equivalent to $\mathbf{FT}_{\Pi_1^0}$.

> $\mathbf{UECT}_1$**:** Every equicontinuous sequence $(f_n)_{n \geqslant 0}$ of real-valued functions on $2^{\mathbb{N}}$ is uniformly equicontinuous.

Theorem 4.4.5. *The following statements are equivalent:*

(i) $\mathbf{FT}_{\Pi_1^0}$

(ii) $\mathbf{UECT}_1$

Proof. Let $(f_n)_{n \geqslant 0}$ be an equicontinuous sequence of real-valued maps on $2^{\mathbb{N}}$. Assuming $\mathbf{FT}_{\Pi_1^0}$, we see from Theorem 4.1.6 that f_n is uniformly continuous for each $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary. Construct a mapping

$$\lambda : 2^* \times \mathbb{N} \to \{0, 1\}$$

such that

$$\lambda(u,n) = 0 \Rightarrow \sup\{|f_n(u*v) - f_n(u*w)| : v,w \in 2^*\} > \frac{\varepsilon}{2},$$
$$\lambda(u,n) = 1 \Rightarrow \sup\{|f_n(u*v) - f_n(u*w)| : v,w \in 2^*\} < \varepsilon.$$

and

$$\lambda(u,n) = 1 \Rightarrow \lambda(u*0,n) = \lambda(u*1,n) = 1.$$

Let

$$S = \{(u,i) \in 2^* \times \mathbb{N} : \lambda(u,i) = 1\},$$

and

$$B = \{u \in 2^* : \forall n \in \mathbb{N}\, (u,n) \in S\}.$$

To show that B is a Π^0_1-bar, consider any $\alpha \in 2^{\mathbb{N}}$. Since the sequence $(f_n)_{n\geqslant 0}$ is equicontinuous, there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$

$$\sup\{|f_n(\overline{\alpha}(N)*v) - f_n(\overline{\alpha}(N)*w)| : v,w \in 2^*\} < \frac{\varepsilon}{2},$$

and therefore $\lambda(\overline{\alpha}(N), n) = 1$. Hence $\overline{\alpha}(N) \in B$, and therefore B is a bar. The other requirements that make B into a Π^0_1-bar are satisfied by the definition of B and our choice of λ.

As we are assuming $\mathbf{FT}_{\Pi^0_1}$, B is a uniform bar. Thus there exists M such that for any $\alpha \in 2^{\mathbb{N}}$ and any $n \in \mathbb{N}$,

$$\lambda(\overline{\alpha}(M), n) = 1.$$

It follows that

$$\sup\{|f_n(\overline{\alpha}(M)*v) - f_n(\overline{\alpha}(M)*w)| : v,w \in 2^*\} < \varepsilon$$

and hence that $(f_n)_{n\geqslant 0}$ is uniformly equicontinuous. Thus (i) implies (ii).

For the converse implication, suppose that (ii) holds, and let B be a Π^0_1-bar.

Then there exists a decidable subset S of 2^* such that

$$u \in B \Leftrightarrow \forall n \in \mathbb{N}\ ((u, n) \in S).$$

For each n let

$$f_n(\alpha) = \min\{i \in \mathbb{N} : (\overline{\alpha}(i), n) \in S\}.$$

Property (ii) implies that the sequence $(f_n)_{n \geqslant 0}$ is uniformly equicontinuous, so we can find $N \in \mathbb{N}$ such that for all $\alpha, \beta \in 2^{\mathbb{N}}$ and all $n \in \mathbb{N}$,

$$\overline{\alpha}(N) = \overline{\beta}(N) \implies |f_n(\alpha) - f_n(\beta)| < 1.$$

Furthermore, the set $\{f_n(\alpha) : n \in \mathbb{N}\}$ is bounded, so for any $u \in 2^*$ with $|u| = N$ there exists $M_u \in \mathbb{N}$ such that

$$|f_i(u)| + 1 < M_u$$

for all $n \in \mathbb{N}$. Set

$$M = \max\left\{M_u : u \in 2^N\right\}.$$

Then for all $n \in \mathbb{N}$ and all $\alpha \in 2^{\mathbb{N}}$ we have $|f_n(\alpha)| < M$. If $(\overline{\alpha}(M), i) \notin S$, then $|f_i(\alpha)| > M$, a contradiction; since S is decidable it follows that $(\overline{\alpha}(M), n) \in S$. Hence B is a uniform bar. $\square$

It seems reasonable to hope that, using the same embedding techniques as in Section 4.3, we could replace $2^{\mathbb{N}}$ by $[0, 1]$ in the previous theorem. That would be a desired result, as (by part (iii) of Theorem 4.4.4) it would show more clearly the contrast in strength between **UCT** and $\mathbf{FT}_{\Pi^0_1}$. The construction in Section 4.3 is intuitive, but technically involved. We will not prove an altered version of the previous theorem for the interval $[0, 1]$ here.

The most prominent place that equicontinuity takes in classical mathematics, is in the classical Ascoli's theorem:

> Let $(f_n)_{n\geqslant 0}$ be a sequence of real-valued continuous functions on $[0,1]$. If $(f_n)_{n\geqslant 0}$ is totally bounded and equicontinuous, then it has a convergent subsequence.

If we take constant functions, we see that this theorem implies the Bolzano-Weierstrass theorem. As already mentioned in the introduction, the Bolzano-Weierstrass theorem cannot be proved within Bishop-style constructive mathematics [27, 35]; and hence, neither can Ascoli's theorem.

Consider the following two constructive variants of Ascoli's theorem. The first is the Bishop-Ascoli theorem [6]; the second is a constructive version of the de Swart-Ascoli theorem [42].

> Let $(f_n)_{n\geqslant 0}$ be an uniformly equicontinuous sequence of real-valued functions on $[0,1]$. If for each $\varepsilon > 0$ there exists an ε-approximation $\{x_0, x_1, \ldots, x_n\}$ to $[0,1]$ such that
>
> $$\{(f_i(x_0), f_i(x_1), \ldots, f_i(x_n)) : i \in \mathbb{N}\}$$
>
> is totally bounded, then $\{f_i : i \in \mathbb{N}\}$ is totally bounded.

> Let $(f_n)_{n\geqslant 0}$ be a uniformly equicontinuous sequence of real-valued functions on $[0,1]$. If $\{f_i(x) : i \in \mathbb{N}\}$ is totally bounded for each $x \in [0,1]$, then for each $\varepsilon > 0$ there is a finite covering of $(f_n)_{n\geqslant 0}$ by sets of diameter less than or equal to ε.

We will consider two strengthenings of these statements: a version of the Bishop-Ascoli theorem in which the hypothesis is weakened and a version of the de Swart-Ascoli theorem with a stronger conclusion. We will see that neither of these is provable in Bishop-style constructive mathematics, since the former is equivalent to **UCT** and the latter implies **LPO**.

We start with a strengthening of the Bishop-Ascoli theorem, which can also be found in [42]. If we replace "uniformly equicontinuous" by "equicontinuous", our theorem is equivalent to **UCT** over BISH:

Theorem 4.4.6. *The following two statements are equivalent:*

(i) **UCT**.

(ii) Let $(f_n)_{n\geqslant 0}$ *be an equicontinuous sequence of real-valued functions on* $[0,1]$ *such that for each* $\varepsilon > 0$ *there exists an* ε*-approximation* $\{x_0, x_1, \ldots, x_n\}$ *to* $[0,1]$ *such that*

$$\{(f_i(x_0), f_i(x_1), \ldots, f_i(x_n)) : i \in \mathbb{N}\}$$

is totally bounded. Then $\{f_i : i \in \mathbb{N}\}$ *is totally bounded.*

Proof. Assume (i). Let $(f_n)_{n\geqslant 0}$ be an equicontinuous sequence of real-valued functions on $[0,1]$ such that for each $\varepsilon > 0$ there exists an ε-approximation $\{x_0, x_1, \ldots, x_n\}$ to $[0,1]$ for which the set

$$\{(f_i(x_0), f_i(x_1), \ldots, f_i(x_n)) : i \in \mathbb{N}\}$$

is totally bounded. Then $\{f_n(x) : n \in \mathbb{N}\}$ is totally bounded for each $x \in [0,1]$. Hence, by (iii) of Theorem 4.4.4, the sequence $(f_n)_{n\geqslant 0}$ is uniformly equicontinuous. It follows from the Bishop-Ascoli theorem, that $\{f_n : n \in \mathbb{N}\}$ is totally bounded. Thus (i) implies (ii).

For the reverse implication, given a real-valued continuous function f on $[0,1]$, construct its sequence of linear approximations $(f_n)_{n\geqslant 0}$. By Lemma 4.4.1, this sequence is equicontinuous, and for each $\varepsilon > 0$ there exists an ε-approximation $\{x_0, x_1, \ldots, x_n\}$ to $[0,1]$ such that

$$\{(f_i(x_0), f_i(x_1), \ldots, f_i(x_n)) : i \in \mathbb{N}\}$$

is totally bounded. It follows from Lemmas 4.4.3 and 4.4.2, that f is uniformly continuous. Thus (ii) implies (i). □

We now introduce a strong form of the de Swart-Ascoli theorem:

For each uniformly equicontinuous sequence $(f_n)_{n\geqslant 0}$ of real-valued functions on $[0,1]$, if $\{f_n(x) : n \in \mathbb{N}\}$ is totally bounded for each $x \in [0,1]$, then the set $\{f_n : n \in \mathbb{N}\}$ is totally bounded as well.

This appears to be a dramatic change, as the following theorem shows:

Theorem 4.4.7. *The strong de Swart-Ascoli theorem implies* **LPO**.

Proof. Let $\alpha \in 2^{\mathbb{N}}$. Define a sequence $(f_n)_{n\geqslant 0}$ of real-valued functions on $[0,1]$ as follows:

- $f_0(x) = x$ and $f_1(x) = 1 - x$;
- if $\alpha_{n-2} = 0$, then: $f_n(x) = x$; and
- if $\alpha_{n-2} = 1$, then: $f_n(x) = \max\{x, x-1\}$.

It is easy to check that the sequence $(f_n)_{n\geqslant 0}$ is (uniformly) equicontinuous, and that the set $\{f_n(x) : n \in \mathbb{N}\}$ is totally bounded for every $x \in \mathbb{N}$. Now assume that the sequence $(f_n)_{n\geqslant 0}$ is totally bounded (equivalently, that for each $\varepsilon > 0$, there exists a finite ε-approximation $\{x_0, x_1, \ldots, x_n\}$ to $[0,1]$ such that the set

$$\{(f_i(x_0), f_i(x_1), \ldots, f_i(x_n)) : i \in \mathbb{N}\}$$

is totally bounded in $\mathbb{R}^n$). Taking $\varepsilon = 1$, pick $m \in \mathbb{N}$ such that for all $j \in \mathbb{N}$ there exists $n \leqslant m$ such that $\|f_j - f_n\| < 1$. If $\alpha_{n-2} = 0$ for all $n \leqslant m$, then $\alpha_{n-2} = 0$ for all $n \in \mathbb{N}$. It follows that either $\alpha_n = 0$ for all n or there exists n such that $\alpha_n = 1$. ☐

Theorem 4.4.7 shows that we have no hope of proving the strong de Swart-Ascoli theorem from any variant of the fan theorem, or from any other intuitionistic principle.

4.5 Antitheses of compactness

Russian Recursive Mathematics not only constitutes an informal, computational model of BISH, it also enables one to construct some of the stranger objects in constructive mathematics. In this section we identify and investigate some principles that are equivalent to strong noncompactness principles. This will be done in the very same way as in constructive reverse mathematics, only with a different focus from usual.

Recall that a bar $B \subset 2^*$ is said to **admit arbitrarily long paths** if for each $n \in \mathbb{N}$, there exists $u \in B$ with $|u| \geqslant n$ but $u \notin B$.

Motivated purely by symmetry reasons, we will define the following principles that are analogues of the fan theorems described in Section 4.1:

Anti-FT$_\Delta$: There exists a decidable bar that admits arbitrarily long paths.

Anti-FT$_c$: There exists a c-bar that admits arbitrarily long paths.

Anti-FT$_{\Pi^0_1}$: There exists a Π^0_1-bar that admits arbitrarily long paths.

Anti-FT$_{\mathrm{Full}}$: There exists a bar that admits arbitrarily long paths.

The following implications hold:

$$\boxed{\textbf{Anti-FT}_\Delta} \Longrightarrow \boxed{\textbf{Anti-FT}_c} \Longrightarrow \boxed{\textbf{Anti-FT}_{\Pi^0_1}} \Longrightarrow \boxed{\textbf{Anti-FT}_{\mathrm{Full}}}$$

One might expect a mirrored picture of equivalences to the one painted in the previous three sections. This expectation is met by the fact that we can show that the antitheses of **FT**$_\Delta$ is equivalent to antitheses of equivalences of **FT**$_\Delta$.

Proposition 4.5.1. *The following are equivalent within BISH.*

(i) **Anti-FT**$_\Delta$*;*

(ii) There exist two compact[3] *subsets A, B of a metric space such that*

$$\forall a \in A \, \forall b \in B \ (d(a, b) > 0)$$

[3]We remind the reader that we defined 'compact' as 'totally bounded and complete'.

but

$$\inf \{d(x,y) : x \in A, y \in B\} = 0.$$

(iii) There is a uniformly continuous map $f : X \to \mathbb{R}^+$ *with a compact metric space* X *as a domain and with* $\inf f(X) = 0$.

Proof. A proof that (i) implies (ii) can be found in [13]. Assume there exist two compact subsets $A, B \subset X$ of a metric space (X, d) as in (ii). Corollary 4.4 of [43] shows that there exist uniformly continuous, surjective mappings $f : 2^{\mathbb{N}} \to A$ and $g : 2^{\mathbb{N}} \to B$. For a sequence $\alpha = (\alpha_0, \alpha_1, \ldots) \in 2^{\mathbb{N}}$ let α^e denote the sequence of all even terms; that is the sequence $(\alpha_0, \alpha_2, \ldots)$. Similarly define α^o to be the sequence of all odd terms of α. We claim that the map G that maps $\alpha \in 2^{\mathbb{N}}$ to $d(f(\alpha^e), g(\alpha^o))$ fulfills the requirement of (iii). By definition $G(2^{\mathbb{N}}) \subset \mathbb{R}^+$. Furthermore, G is uniformly continuous, since it is the composition of uniformly continuous mappings. Let $\varepsilon > 0$ be arbitrary. By our assumptions on A and B, there exist $a \in A$ and $b \in B$ such that $d(a, b) < \varepsilon$. Since the mappings f, g are both surjective, there exist $\alpha = (\alpha_0, \alpha_1, \ldots) \in 2^{\mathbb{N}}$ and $\beta = (\beta_0, \beta_1, \ldots) \in 2^{\mathbb{N}}$ such that $f(\alpha) = a$ and $g(\beta) = b$. Let $\gamma \in 2^{\mathbb{N}}$ be the sequence

$$\gamma = (\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots).$$

Then

$$G(\gamma) = d(f(\gamma^e), g(\gamma^o)) = d(f(\alpha), g(\beta)) = d(a, b) < \varepsilon.$$

Since ε is arbitrary we conclude that $\inf G(2^{\mathbb{N}}) = 0$. Thus we have shown that (ii) implies (iii).

To prove that (iii) implies (i), assume there exists a uniformly continuous map $f : X \to \mathbb{R}^+$ with a compact metric space (X, d) as a domain, such that $\inf f(X) = 0$. Again using Corollary 4.4 of [43], we see that there exists a uniformly continuous surjective map $F : 2^{\mathbb{N}} \to X$. Construct a decidable set $B \subset 2^*$ such that

$$u \in B \Rightarrow \sup \left\{ f(F(u * \beta)) : \beta \in 2^{\mathbb{N}} \right\} > 2^{-(|u|+1)},$$

$$u \notin B \Rightarrow \sup\left\{f(F(u * \beta)) : \beta \in 2^{\mathbb{N}}\right\} < 2^{-|u|}.$$

Without loss of generality we may assume that B is closed under extensions. Let $\alpha \in 2^{\mathbb{N}}$. Since $f \circ F$ is continuous and $f(F(\alpha)) > 0$, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\forall \beta \in 2^{\mathbb{N}} \ (f(F(\overline{\alpha}(n) * \beta)) > \varepsilon).$$

Choose $M \in \mathbb{N}$ such that $2^{-M} < \varepsilon$ and $M > n$. It is impossible that $\overline{\alpha}(M) \notin B$, since $f(F(\overline{\alpha}(M) * \beta)) > \varepsilon > 2^{-M}$ for all $\beta \in 2^{\mathbb{N}}$. Therefore, since B is decidable, $\overline{\alpha}(M) \in B$. To see that B admits arbitrary long paths, let $n \in \mathbb{N}$. Since $\inf f(X) = 0$, we can find $x \in X$ such that $f(x) < 2^{-(n+1)}$. Furthermore, since F is surjective, there is $\alpha \in 2^{\mathbb{N}}$ such that $f(F(\alpha)) < 2^{-(n+1)}$. The assumption $\overline{\alpha}(n) \in B$ implies that

$$\sup\left\{f(F(\overline{\alpha}(n) * \beta)) : \beta \in 2^{\mathbb{N}}\right\} > 2^{-(|u|+1)}$$

and therefore, in particular, $f(F(\alpha)) > 2^{-(n+1)}$. This is a contradiction. Hence $\overline{\alpha}(n) \notin B$. □

We call a function $f : X \to \mathbb{R}$ **unbounded** if for each $n \in \mathbb{N}$ there exists $x \in X$ with $f(x) > n$. Furthermore, we call a function $f : X \to Y$ between two metric spaces (X, d) and (Y, d') **uniformly discontinuous** if there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exist $x, x' \in X$ such that $d(x, x') < \delta$ and $d'(f(x), f(x')) > \varepsilon$. In the presence of countable choice, the latter property coincides with the antithesis of uniform sequential continuity [11]. This coincidence explains—at least partially—the following surprising result.

Proposition 4.5.2. *The following are equivalent within BISH.*

(i) $\mathbf{Anti\text{-}FT}_c$.

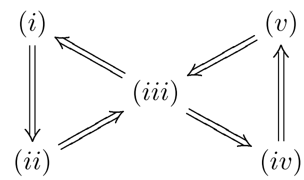
(ii) $\mathbf{Anti\text{-}FT}_{\text{Full}}$

(iii) There exists a strong Specker sequence in $[0, 1]$.

(iv) *There exists a continuous function* $f : [0,1] \to \mathbb{N}$ *that is unbounded.*

(v) *There exists a continuous function* $f : [0,1] \to [0,1]$ *that is uniformly discontinuous.*

Proof. We prove the following implications



By definition, (i) implies (ii). To see that (ii) implies (iii), let $B \subset 2^*$ be a bar that admits arbitrarily long paths. Using countable choice, we can find a sequence $(u_n)_{n \geqslant 1}$ in 2^* with $|u_n| > n$ and $u_n \notin B$. Let $F : 2^{\mathbb{N}} \to [0,1]$ be the natural embedding of Cantor space into the Cantor set, as in Section 4.3. We prove that the sequence $(F(u_n))$ is bounded away from every point in $[0,1]$. Let $x \in [0,1]$ be arbitrary. By Bishop's lemma [19, Proposition 3.1.1], we can construct a binary sequence α_x such that

$$F(\alpha_x) \neq x \Rightarrow d(x, F(2^{\mathbb{N}})) > 0.$$

Since B is a bar, there exists $N \in \mathbb{N}$ such that $\overline{\alpha_x}(N) * w \in B$ for all $w \in 2^*$. Now either $d(F(\alpha_x), x) > 3^{-N-3}$, which means that x is bounded away from the Cantor set and therefore the entire sequence $(F(u_n))$, or else $d(F(\alpha_x), x) < 3^{-N-2}$. In the latter case, for each $n \geqslant N$ we have $d(F(u_n), x) > 3^{-N-3}$, since the assumption $d(F(u_n), x) < 3^{-N-2}$ implies that $d(F(\alpha_x), F(u_n)) < 3^{-N-1}$. Using Lemma 4.3.2, we see that implies that $\overline{u_n}(N) = \overline{\alpha_x}(N)$ and therefore $u_n \in B$, a contradiction.

To see that (iii) implies (i), we simply refer to Theorem 6.a of [4] to get a construction of a c-bar from a Specker sequence. (Note that [4] shows that the antithesis of (iii) is equivalent to $\mathbf{FT}_c$.)

To prove that (iii) implies (iv), assume that $(x_n)_{n \geqslant 1}$ is a sequence in $[0,1]$ that is bounded away from every point in $[0,1]$. For $x \in [0,1]$ and $\varepsilon > 0$ let

$t_{x,\varepsilon} : [0,1] \to \mathbb{R}$ be the spike function centered around x and with width ε, defined by

$$t_{x,\varepsilon}(z) = \max\left\{0, 1 - \varepsilon^{-1}|x - z|\right\}.$$

We claim that the function

$$f = \sum_{n\in\mathbb{N}} n t_{x_n, 2^{-n}}$$

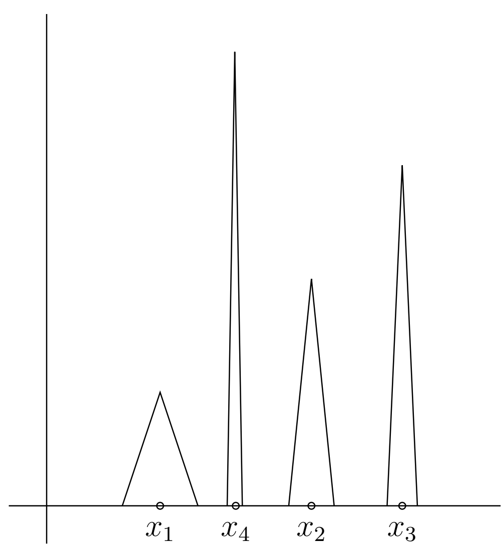
is well defined, continuous, and unbounded. To see that it is well defined and continuous, note that for each $z \in [0,1]$ there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$f(x) = \sum_{n=0}^{N} n t_{x_n, 2^{-n}}(x)$$

for all $x \in B_z(\varepsilon)$. Furthermore, for each $n \in \mathbb{N}$,

$$f(x_n) \geqslant n t_{x_{n,2^{-n}}}(x_n) = n,$$

which means that f is unbounded. We have this picture:



For the implication (iv) ⇒ (v) assume that $f : [0,1] \to \mathbb{R}$ is a continuous but unbounded function. Then $\sin \circ f$ is a continuous function that is uniformly discontinuous.

Finally, to get the last implication, (v) $\Rightarrow$ (iii), let $f : [0,1] \to [0,1]$ be a continuous but uniformly discontinuous function. Then there exists $\varepsilon > 0$ such that for each $\delta > 0$, there exist $x, x' \in [0,1]$ such that $|x - x'| < \delta$ and $|f(x) - f(x')| > \varepsilon$. Using countable choice, fix two sequences of real numbers $(x_n)_{n\geqslant 1}, (x'_n)_{n\geqslant 1}$ such that $|x_n - x'_n| < 2^{-n}$, but $|f(x_n) - f(x'_n)| > \varepsilon$ for each $n \in \mathbb{N}$. We claim that $(x_n)_{n\geqslant 1}$ is bounded away from every point in $[0,1]$. Let $x \in [0,1]$ be arbitrary. Since f is continuous, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/2$. Compute N such that $2^{-N} < \delta$. For each $n > N$, the assumption $|x - x_n| < 2^{-N-1}$ implies that $|x - x'_n| < 2^{-N}$ and therefore that

$$|f(x_n) - f(x'_n)| \leqslant |f(x_n) - f(x)| + |f(x'_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is a contradiction. Hence $|x - x_n| > 2^{-N-2}$ for all $n > N$, and we are done. □

This means that the “Anti”-side of constructive reverse mathematics only seems to have *two* different-strength interesting principles.

4.6 Epilogue: the full picture

We can now put all results of this chapter together.

Figure 4.6 shows that there are seven categories of equivalent principles (**FT**$_\Delta$, **FT**$_\mathbf{c}$,**UCT**,**FT**$_{\Pi^0_1}$,**FT**$_\mathbf{Full}$, **Anti-FT**$_\Delta$,**Anti-FT**$_\mathrm{Full}$). These seven categories of equivalences are neither fully nonconstructive, like the law of excluded middle, nor fully accepted by all constructive varieties. This makes it interesting for the program of constructive reverse mathematics to associate theorems with them. We can say slightly more about what kind of theorem is likely to fall into which category. The five fan theorems are concerned with principles that allow one to pass from a pointwise property to a global/uniform property;

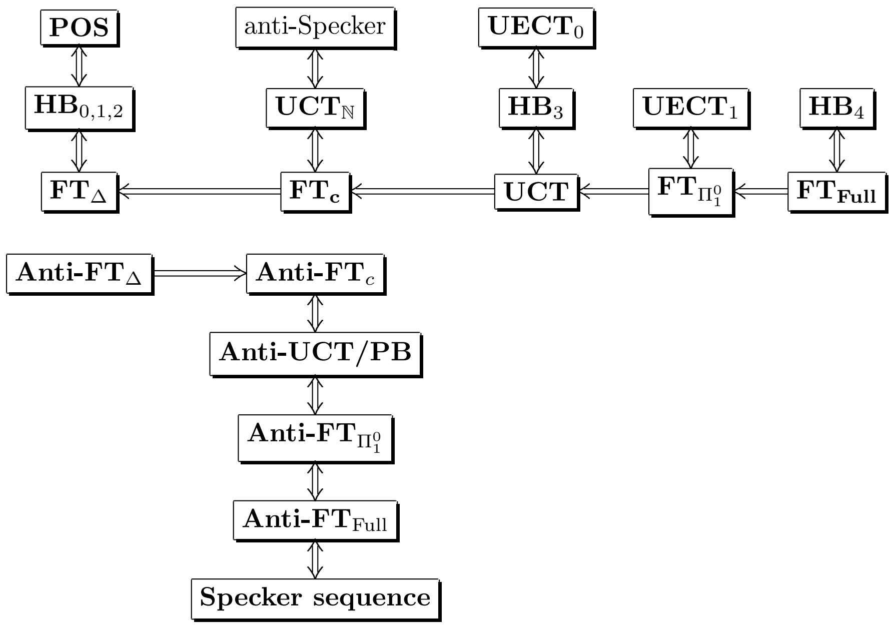


Figure 4.1: An illustration of the relationship between most of the principles in this section.

contrary to the two anti-principles, which are concerned with the existence of a structure having certain pointwise properties but failing to have respective global/uniform properties. Schuster has shown in [40] that many *unique* existence theorems are equivalent to $\mathbf{FT_\Delta}$. The $\mathbf{FT_c}$ category seems to attract mainly theorems about sequences and sequential continuity. This is no surprise, given that $\mathbf{FT_c}$ is equivalent to the antithesis of Specker's Theorem [4]. Even less surprising is that theorems about uniform continuity fall into the **UCT** category [9, 21]. The principle $\mathbf{FT_{\Pi^0_1}}$ looks like a natural one from a logician's point of view, but fails to have as many known interesting equivalents as the other three.

We emphasise that it is unknown whether *any* of the implications between forms of the fan theorem are strict. None of the "big three" varieties of Bishop-style constructive mathematics—namely classical mathematics, intuitionistic mathematics and Russian recursive mathematics—separate between them. Furthermore there is no model to our knowledge that does. However the results of this chapter affect the search for such a model.

Remark 4.6.1. *If a model of* BISH *separates any of* $\mathbf{FT}_{\cdot}$*, Kripke's schema must fail to hold in that model. If a model of* BISH *separates between any of* $\mathbf{FT}_c$*,* **UCT** *and* $\mathbf{FT}_{\Pi^0_1}$*, then* **BD-**$\mathbb{N}$ *must fail to hold in that model.*

The collapse of the hierarchy on the "anti"-side has exciting implications. We say that a model of BISH **separates strongly** between two fan principles $\mathbf{FT}_A$ and $\mathbf{FT}_B$ if it satisfies $\mathbf{FT}_A$ and $\textbf{Anti-FT}_B$.

Proposition 4.6.2. *Assuming countable choice, there cannot be a model of BISH that strongly separates* $\mathbf{FT}_c$*,* **UCT***,* $\mathbf{FT}_{\Pi^0_1}$*, and* $\mathbf{FT}_{\mathrm{Full}}$*.*

Proof. Any such model would satisfy $\mathbf{FT}_c$ and $\textbf{Anti-FT}_{\mathrm{Full}}$. Proposition 4.5.2 shows that $\textbf{Anti-FT}_{\mathrm{Full}}$ implies $\textbf{Anti-FT}_c$; a contradiction. □

Notice that countable choice does not play an essential part in proving Proposition 4.5.2. It seems likely that if one constructed a model in which $\textbf{Anti-FT}_{\mathrm{Full}}$ held, one would be able to specify a sequence such that $\textbf{Anti-FT}_{\mathrm{Full}}$ is satisfied.

Chapter 5

Differentiable manifolds

In this last chapter we develop a constructive theory of differentiable manifolds—or at least beginnings of such a theory. A classical introduction to differentiable manifolds can be found in [7]. In the spirit of the rest of the thesis, we will try to work with as few assumptions as possible, thus being able to clearly identify important, necessary principles on the way.

Throughout the entire chapter let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denote the projection(s) onto the i^{th} coordinate and $(e_i)_{i=1}^n$ the canonical base vectors in $\mathbb{R}^n$. A function $f : D \to \mathbb{R}$ defined on some set $D \subset \mathbb{R}$ is said to be **differentiable at a point** $x \in D$ if there exists a real number ξ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $y \in D$

$$|x - y| < \delta \Rightarrow |f(x) - f(y) - \xi(x - y)| < \varepsilon |x - y| . \tag{5.1}$$

The function f is said to be **differentiable on** D if there exists a function $f' : D \to \mathbb{R}$, called the **derivative of** f, such that for each $x \in D$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $y \in D$

$$|x - y| < \delta \Rightarrow |f(x) - f(y) - f'(x)(x - y)| < \varepsilon |x - y| . \tag{5.2}$$

The real number ξ in (5.1) is unique. Hence, if we assume unique choice, then f is differentiable on D iff it is differentiable at each point $x \in D$. Now consider a function $f : D \to \mathbb{R}$ defined on some subset $D \subset \mathbb{R}^n$ and $1 \leqslant i \leqslant n$. If the function $D_i f = f_i : D \to \mathbb{R}$ is the derivative of the function defined by

$$t \mapsto f(x + te_i)$$

then we call it a **partial derivative of** f. Let

$$\{1, \ldots, n\}^*$$

denote the set of all finite sequences in $\{1, \ldots, n\}$. We denote higher order derivatives, if they exist, as usual, by f_α or $D_\alpha f$. The function f is said to be $\mathcal{C}^\infty$ or a $\mathcal{C}^\infty$**-function**, if f_α exists for any $\alpha \in \{1, \ldots, n\}^*$. A function $f : D \to \mathbb{R}^m$ defined on some subset $D \subset \mathbb{R}^n$ is called differentiable ($\mathcal{C}^\infty$), if $\pi_i \circ f$ is differentiable ($\mathcal{C}^\infty$) for each $1 \leqslant i \leqslant m$. Finally a homeomorphism $\varphi : X \to X'$ is called a **diffeomorphism**, if both φ and φ^{-1} are $\mathcal{C}^\infty$.

Consider a topological space X. A n**-dimensional atlas** for X is a family of **charts**, that is pairs of open sets and injective mappings $(U_i, \varphi_i)_{i \in I}$, such that

(i) $\varphi : U_i \to \mathbb{R}^n$ for each $i \in I$,

(ii) $X = \bigcup_{i \in I} U_i$ and

(iii) if $i, j \in I$ are such that $U_i \cap U_j \neq \emptyset$ then $\varphi_j \circ \varphi_i^{-1}$ is a diffeomorphism on $\varphi_i(U_i \cap U_j)$.

The pair $(X, (U_i, \varphi_i)_{i \in I})$ (or just X if no confusion is likely to arise) is called a **(**n**-dimensional) differentiable manifold**, or simply **manifold**.

If we do not state otherwise, we will assume that X is a n-dimensional differentiable manifold with atlas $(U_i, \varphi_i)_{i \in I}$.

5.1 Topology

For convenience, and without loss of generality, we assume from now on that any neighbourhood is open.

5.1.1 The induced topology

There is a natural topology arising on a manifold X. If

$$\tau_i = \left\{ U \subset X : \exists V \subset \mathbb{R}^n \left(V \text{ is open in } \mathbb{R}^n \wedge U = \varphi_i^{-1}(V) \right) \right\},$$

then τ_i is the coarsest topology such that φ_i is continuous. Let now $\mathcal{B} = \bigcup_{i \in I} \tau_i$. To prove that this set is closed under intersection, we need the following Lemma.

Lemma 5.1.1. *If $i, j \in I$ and $V \subset \mathbb{R}^n$ is open, then the set*

$$\varphi_i(U_i \cap \varphi_j^{-1}(V))$$

is open.

Proof. Let $i, j \in I$ be arbitrary, $V \subset \mathbb{R}^n$ be open and $p \in \varphi_i(U_i \cap \varphi_j^{-1}(V))$. Then $\varphi_i^{-1}(p) \in U_i \cap U_j$ and hence $\varphi_i \circ \varphi_j^{-1}$ is a homeomorphism. Therefore

$$\varphi_i \circ \varphi_j^{-1}(V \cap \varphi_j(U_i \cap U_j))$$

is open. We conclude that

$$\varphi_i \circ \varphi_j^{-1}(V \cap \varphi_j(U_i \cap U_j)) = \varphi_i(U_i \cap U_j \cap \varphi_j^{-1}(V))$$

is open. It is also a neighbourhood of p, and

$$\varphi_i(U_i \cap U_j \cap \varphi_j^{-1}(V)) \subset \varphi_i(U_i \cap \varphi_j^{-1}(V)).$$

Hence $\varphi_i(U_i \cap \varphi_j^{-1}(V))$ is open. □

Lemma 5.1.2. *$\mathcal{B}$ is closed under intersection.*

Proof. Let now $U, V \subset \mathbb{R}^n$ be open sets, and $i, j \in I$ be arbitrary. It suffices to prove that $\varphi_i(\varphi_i^{-1}(U) \cap \varphi_j^{-1}(V))$ is open. We get

$$\begin{aligned}\varphi_i(\varphi_i^{-1}(U) \cap \varphi_j^{-1}(V)) &= \varphi_i(U_i \cap \varphi_i^{-1}(U) \cap \varphi_j^{-1}(V)) \\ &= U \cap \varphi_i(U_i \cap \varphi_j^{-1}(V)),\end{aligned}$$

which is open, as $\varphi_i(U_i \cap \varphi_j^{-1}(V))$ is open by Lemma 5.1.1. □

So $\mathcal{B}$ is a base for a topology τ, which is easily seen to be the coarsest topology such that all φ_i are continuous at the same time. We will call this topology the **induced topology on** X.

Proposition 5.1.3. *Let X be a manifold equipped with a topological structure τ'. Then the induced topology described above coincides with the original topology if and only if the charts φ_i are homeomorphism with respect to the topology τ'.*

Proof. Assume that $\tau = \tau'$. Then, as above, the mappings φ_i are continuous. For let $U \in \mathcal{B}$ be arbitrary; it is enough to show that $\varphi_i(U \cap U_i)$ is open in $\mathbb{R}^n$. There exists j such that $U \in \tau_j$, which means by definition that there is an open subset V of $\mathbb{R}^n$ such that $U = \varphi_j^{-1}(V)$. By Lemma 5.1.1

$$\varphi_i(U_i \cap \varphi_j^{-1}(V))$$

is open in $\tau = \tau'$.

Conversely, assume that all the φ_i are homeomorphisms. Obviously $\tau \subset \tau'$. So let $U \in \tau'$. Since

$$U = \bigcup_{i \in I} \varphi_i^{-1}(\varphi_i(U \cap U_i)),$$

to prove that $U \in \tau$ it is enough to show that $\varphi_i(U \cap U_i)$ is open in $\mathbb{R}^n$, which is the case as all the φ_i are τ'-homeomorphisms and therefore τ'-open mappings. □

In view of this let us—as most authors do anyway—assume that the topological structure on X is the induced topology.

Proposition 5.1.4. *The induced topology on a manifold satisfies the first axiom of countability—that is every point has a countable base of neighbourhoods.*

Proof. Let $x \in X$. Choose $i \in I$ such that $x \in U_i$. Then

$$\mathcal{N}_x = \left\{ \left(\varphi_i^{-1} \left(\varphi_i(U_i) \cap B_{\varphi_i(x)} \left(\frac{1}{n} \right) \right) \right) : n \in \mathbb{N} \right\}$$

is a countable base of neighbourhoods for x. □

Proposition 5.1.5. *If a manifold has a countable atlas, then the induced topology satisfies the second axiom of countability—that is there exists a countable basis.*

Proof. Let $\mathcal{Q}$ be a countable basis for the topology on $\mathbb{R}^n$. Then

$$\mathcal{B} = \bigcup_{i \in I} \left\{ \varphi_i^{-1}(A \cap \varphi_i(U_i)) : A \in \mathcal{Q} \right\}$$

has the desired property. □

The converse would be a consequence of **Lindelöf's theorem** [7, p. 10]:

> If a space satisfies the second axiom of countability, then every open covering contains a countable subcover.

Ishihara has shown in [28] that Lindelöf's theorem can be proved using a version of the Church-Markov-Turing thesis; it seems unlikely to hold in BISH, though.

5.1.2 Local decomposability and T1

A topological property that is very useful, when working constructively, is **local decomposability**.

LD: $\forall x \in X \, \forall U \in \tau \; (x \in U \Rightarrow \exists V \in \tau \, \forall y \in X \, (y \in U \vee y \notin V))$

Not every manifold constructively has this property, as the following example shows:

Proposition 5.1.6. *If every manifold satisfies* **LD** *then the law of excluded middle holds.*

Proof. Consider any syntactically correct statement P, and let a, b be two points such that $(a = b) \Leftrightarrow P$. Let $U_1 = \mathbb{R} \times \{a\}$ and $U_2 = \mathbb{R} \times \{b\}$. Let $\varphi_i : U_i \to \mathbb{R}$ be defined by $\varphi_i((x_1, \alpha)) = x_1$. Then it is easy to see that $X = U_1 \cup U_2$ is a manifold.

Now assume that X satisfies **LD** Consider the neighbourhood

$$U = (-1, 1) \times \{a\}$$

of $(0, a)$. Assume there exists a neighbourhood V as in **LD**. If $(0, b) \notin V$ then $\neg P$ holds, since if P holds then $a = b$ and therefore $(0, b) = (0, a) \in V$, a contradiction. If $(0, b) \in U \subset U_1$ then $a = b$ and therefore P holds. □

Proposition 5.1.7. *If* $(X, (U_i, \varphi_i)_{i \in I})$ *is a manifold then for each* $i \in I$ *the set* U_i *satisfies* **LD**.

Proof. Let $x \in U_i$ and U be any neighbourhood of x. Then $\varphi_i(U_i \cap U)$ is a neighbourhood of $x' = \varphi_i(x)$. It is easy to see that $\mathbb{R}^n$ satisfies **LD**. Therefore there exists a neighbourhood $V' \subset \mathbb{R}^n$ of x' such that

$$\mathbb{R}^n = \varphi_i(U_i \cap U) \cup \neg V'.$$

Let $V = \varphi_i^{-1}(V')$ be another neighbourhood of x, and consider an arbitrary $y \in U_i$. Either $\varphi_i(y) \in \varphi_i(U_i \cap U)$ and hence $y \in U$ or else $\varphi_i(y) \notin V'$ and hence $y \notin V$. □

It often is enough to assume the following property, which is a simple consequence of **LD**:

LD′: $\forall x \in U_i \, \exists x \in V \in \tau \, \forall y \in X \; (y \in U_i \vee y \notin V).$

However, we have

Proposition 5.1.8. **LD** *is equivalent to* **LD′**.

Proof. As already mentioned it is almost trivial that **LD** implies **LD′**, since U_i is a neighbourhood of any $x \in U_i$. For the converse, let $x \in U_i$ and let U be any neighbourhood of x. By **LD′**, there is a neighbourhood W such that

$$\forall y \in X (y \in U_i \vee y \notin W).$$

By Proposition 5.1.7 there is also a neighbourhood W' such that

$$\forall y \in U_i \; (y \in U \vee y \notin W') \, .$$

Set $V = W \cap W'$, which is a neighbourhood of x. Consider an arbitrary $y \in X$. Either $y \notin W$ and therefore $y \notin V$ or $y \in U_i$. In this second case, again either $y \notin W'$ and therefore $y \notin V$, or else $y \in U$. □

If we assume **LD′** we can prove that our manifold satisfies a principle we will call **T1**. We have not said anything about the inequality on X yet. In view of the following we should at least assume that

$$(w \neq z \wedge (\exists i \in I \, (w \in U_i \wedge z \in U_i))) \Rightarrow \varphi_i(w) \neq \varphi_i(z). \tag{5.3}$$

This means that φ_i is strongly extensional. We will not make any more assumptions on the inequality. We say that a manifold satisfies **T1**, if for each pair of points $x, x' \in X$ such that $x \neq x'$ there exists a neighbourhood $x \in U$ such that $x' \notin U$.

Proposition 5.1.9. *If* **LD′** *holds, then the induced topology on a manifold satisfies* **T1**.

Proof. Consider points $x_1, x_2 \in X$ with $x_1 \neq x_2$. Choose $i \in I$ such that $x_1 \in U_i$. Using **LD′**, we see that either $x_2 \in U_i$ as well or else there exists a V such that **T1** is satisfied. In the first case we can use the property (5.3). □

5.1.3 Partitions of unity and T2

A family of nonnegative continuous maps $(f_p)_{p\in P}$ defined on a manifold X is called a **partition of unity** if the following properties hold:

PU1 The **support**

$$C_p = \overline{\{x \in X : f(x) \neq 0\}}$$

of f_p is contained in some U_i, and $\varphi_i(C_p)$ is compact.

PU2 The family $(C_p)_{p\in P}$ is locally finite: that is, for each $x \in X$ the set $\{p \in P : x \in C_p\}$ is finite.

PU3 For each $x \in X$, $\sum_{p\in P} f_p(x) = 1$.

If a partition of unity exists for a manifold X, we say that X **admits** a partition of unity.

Lemma 5.1.10. *If $(f_p)_{p\in P}$ is a partition of unity for the manifold X, then*

$$\forall x \in X\, \exists p \in P\ (f_p(x) > 0)\,.$$

Proof. This is a simple consequence of **PU2** and **PU3**. □

We will say that a manifold is **Hausdorff**, or that it satisfies **T2**, if for each pair of points $x, x' \in X$ such that $x \neq x'$ there exists neighbourhoods $x \in U$ and $x' \in U'$ for both, such that $U \cap U' = \emptyset$. Just as in the classical theory we get:

Proposition 5.1.11. *If a manifold admits a partition of unity, then it is Hausdorff.*

Proof. Assume $x_1 \neq x_2$. Using Lemma 5.1.10, we can find $p \in P$ such that $f_p(x_1) = 5\varepsilon > 0$. Now either $f_p(x_2) > \varepsilon$ or $f_p(x_2) < 2\varepsilon$. In the first case,

$$x_1, x_2 \in C_p \subset U_i$$

for some $i \in I$ and we are done, since U_i is homeomorphic to some open subset of $\mathbb{R}^n$. In the second case, $f_p^{-1}(B_{f_p(x_2)}(\varepsilon))$ and $f_p^{-1}(B_{f_p(x_1)}(\varepsilon))$ are disjoint neighbourhoods of x_1 and x_2 respectively. □

Proposition 5.1.12. *If a manifolds admits a partition of unity, then it satisfies* **LD**.

Proof. Let $x \in U_i$. Using Lemma 5.1.10, choose $p \in P$, such that $f_p(x) = 3\varepsilon > 0$. Find $j \in I$ such that $C_p \subset U_j$. Let $U = U_i \cap U_j$ and $W = f_p^{-1}((2\varepsilon, \infty))$. For each $y \in X$ either $f_p(y) > \varepsilon$ and therefore $y \in C_p \subset U_j$, or else $f_p(y) < 2\varepsilon$ and therefore $y \notin W$. Since, by Proposition 5.1.7, U_j satisfies **LD**, there exists W' such that

$$\forall y \in U_j (y \in U \vee y \notin W').$$

Let now $V = W \cap W'$. Then V is a neighbourhood of x, and as in the proof above,

$$\forall y \in X (y \in U \vee y \notin V).$$

Finally, since $U \subset U_i$

$$\forall y \in X (y \in U_i \vee y \notin V).$$

□

5.1.4 Connectedness

As in the classical theory of differentiable manifolds, things get interesting once we include connectedness in our considerations. It is trivial that if $\varphi_i(U_i)$ is pathwise-connected then so is U_i. It is almost as trivial that every manifold is locally pathwise connected. But things get more complicated from here on. The classical definition of a connected topological space is obviously of no constructive use whatsoever, as it implies the law of excluded middle whenever the set is inhabited [17]. The type of connectedness that we are going to consider is a wonderful notion introduced by Mandelker [34]: a topological space X is said to be **M-connected** if whenever U, V are inhabited, open sets such that

$X = U \cup V$, $U \cap V$ is inhabited.

Lemma 5.1.13. *Let U_i, U_j be two open, M-connected sets such that $U_i \cap U_j$ is inhabited. Then $U_i \cup U_j$ is M-connected.*

Proof. Let U, V be inhabited, open sets such that $U_i \cup U_j \subset U \cup V$. Pick $x \in U$, $y \in V$ and $z \in U_i \cap U_j$. If $x \in U_i$ and $y \in U_i$, then by the M-connectedness of U_i,

$$(U_i \cap U) \cap (U_i \cap V) \neq \emptyset$$

We can deal similarly with the case $x \in U_j$ and $y \in U_j$. So we may assume that $x \in U_i$ and $y \in U_j$. Either $z \in U$ or $z \in V$. In the first case, the M-connectedness of U_j implies that

$$(U_j \cap U) \cap (U_j \cap V) \neq \emptyset;$$

whereas in the second case, the M-connectedness of U_i implies that

$$(U_i \cap U) \cap (U_i \cap V) \neq \emptyset.$$

The case $x \in U_j$ and $y \in U_i$ is dealt with analogously. In every case we have $U \cap V \neq \emptyset$. □

Classically, every connected manifold is also pathwise-connected. The admittedly very neat proof is unfortunately of no constructive use at all, since it uses the classical notion of connectedness. We might hope that we could at least get some weak counterpart using M-connectedness and strong conditions; but as the following counterexample shows even assuming compactness, we have no hope of doing this.

For the following counterexample we work in INT. In particular, we use the fact that $\mathbf{FT}_{\mathrm{Full}}$ holds in INT, and therefore $\mathbf{HB}_4$ is provable (as shown in Section 4.1). Furthermore, **UCT** holds in INT.

Proposition 5.1.14. *The statement "every M-connected, locally pathwise-connected, compact space is pathwise-connected" is provably false in* INT.

Proof. Let $(a_n)_{n \geqslant 1}$ a binary sequence with at most one term equal 1. Let

$$a = \sum_{n=0}^{\infty} \frac{a_{2n}}{2^n} \text{ and } a' = \sum_{n=0}^{\infty} \frac{a_{2n+1}}{2^n}.$$

For these two real numbers

$$\neg \left(a \neq 0 \wedge a' \neq 0\right).$$

Define

$$U_1 = [-1, 1] \times \{1\},$$
$$U_2 = [-1, 1] \times \{-1\},$$
$$U_3 = \{-1\} \times [-1, -a]$$
$$U_4 = \{-1\} \times [a, 1]$$
$$U_5 = \{1\} \times [-1, -a]$$
$$U_6 = \{1\} \times [a, 1]$$

Let

$$X = \bigcup_{i=1}^{6} U_i.$$

Figure 5.1.4 is a sketch of the set X. Clearly X is locally pathwise-connected. It is also easily seen to be totally bounded. Hence the completion $\overline{X}$ is compact and locally pathwise-connected. To show that $\overline{X}$ is M-connected, let U, V be two inhabited, open sets such that $X = U \cup V$. Let

$$\mathcal{U} = \left\{(p, q) : p, q \in \mathbb{Q}^2 \wedge \left((p, q) \cap \overline{X} \subset U \vee (p, q) \cap \overline{X} \subset V\right)\right\}.$$

Then $\mathcal{U}$ is easily seen to be an open covering of $\overline{X}$. By $\mathbf{HB}_4$, there are intervals $(p_1, q_1), \ldots, (p_N, q_N)$ all contained in U, and intervals $(p'_1, q'_1), \ldots, (p'_M, q'_M)$ all

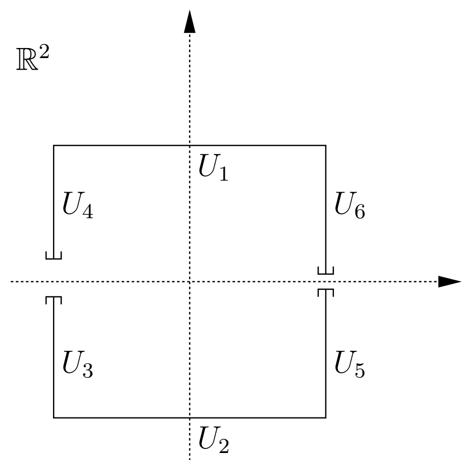


Figure 5.1: X is the unit square centered around the origin. If there is a n such that $a_n = 1$, then a small piece is missing around the x-axis.

contained in V, such that

$$\overline{X} = \bigcup_{i=1}^{N} (p_i, q_i) \cup \bigcup_{i=1}^{M} (p'_i, q'_i).$$

Since we are dealing with intervals with rational endpoints, a contradiction argument shows that there exists a point $r \in X$ and indices k, l such that $r \in (p_k, q_k) \cap (p'_l, q'_l)$; hence $r \in U \cap V$. Assume that here exists a continuous map $h : [0,1] \to X$ such that $h(0) = (0, -1)$ and $h(1) = (0, 1)$. Since we are working within INT, h is uniformly continuous. Thus we can find $N \in \mathbb{N}$ such that for $x, y \in [0,1]$

$$|x - y| < 2^{-N} \Rightarrow \|h(x) - h(y)\| < \frac{1}{8}.$$

Now there exists $0 \leqslant i \leqslant 2^{-N} - 3$, such that

$$\pi_2(h(i2^{-N})) > 0 \text{ and } \pi_2(h((i+3)2^{-N})) < 0$$

Either $\pi_1(h(i2^{-N})) < 1$ or $\pi_1(h(i2^{-N})) > 1$. In the first case, the assumption $a_{2n} = 1$ leads to a contradiction, so $a_{2n} = 0$ for all $n \in \mathbb{N}$. Similarly, in the second case, $a_{2n+1} = 0$ for all $n \in \mathbb{N}$. As $(a_n)_{n \geqslant 1}$ was arbitrary, we conclude that if we can show if that every M-connected, locally pathwise-connected, compact space is pathwise-connected, then **LLPO** holds. □

So what assumptions are sufficient to assure that a manifold is pathwise-connected? To answer this question, we need to prove some technically involved lemmas.

Lemma 5.1.15. *The relation $m \sim m'$ between two points of X, if there exists a path from m to m' is an equivalence relation.*

Proof. The reflexiveness and symmetry of $\sim$ is clear; so the only thing left to prove is transitivity. Let $m_1, m_2, m_3 \in X$, such that there exist continuous functions $h, g : [0,1] \to X$ with $h(0) = m_1$, $h(1) = g(0) = m_2$ and $g(1) = m_3$. Choose $i \in I$ such that $m_2 \in U_i$, and then $\varepsilon > 0$ such that $B_{\varphi_i(m_2)}(\varepsilon) \subset \varphi_i(U_i)$. Since $\varphi_i \circ h$ is continuous at 1 and $\varphi_i \circ g$ is continuous at 0, we can choose $\delta > 0$

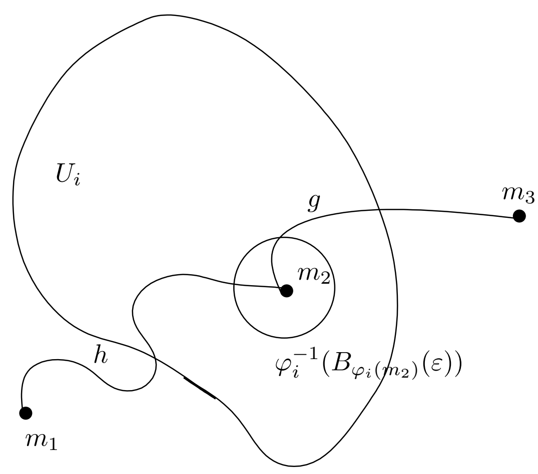


Figure 5.2:

such that for each $y \in [0,1]$

$$|1 - y| < \delta \Rightarrow h(y) \in U_i \wedge |\varphi_i(h(y)) - \varphi_i(m_2)| < \varepsilon$$

and

$$|y| < \delta \Rightarrow g(y) \in U_i \wedge |\varphi_i(g(y)) - \varphi_i(m_2)| < \varepsilon.$$

For any $z \in [0,1]$ such that $|z - \frac{1}{2}| < \frac{1}{2}\delta$, using countable choice, define a sequence $\lambda_n(z) \in \{-1, 0, 1\}$ such that if

$$\lambda_n(z) = \begin{cases} -1, & \text{then } |\frac{1}{2} - z| > \frac{\delta}{2^{n+2}} \\ 0, & \text{then } |z - \frac{1}{2}| < \frac{\delta}{2^{n+1}} \\ 1, & \text{then } |z - \frac{1}{2}| > \frac{\delta}{2^{n+2}}. \end{cases}$$

Now let $\alpha_n(z) \in U_i$ be defined by

$$\alpha_n(z) = \begin{cases} \varphi_i(h(2z)) & \text{if } \lambda(z)_n = -1 \\ \varphi_i(m_2) & \text{if } \lambda(z)_n = 0 \\ \varphi_i(g(2z-1)) & \text{if } \lambda(z)_n = 1. \end{cases}$$

The sequence $(\alpha_n(z))_{n\geqslant 1}$ is easily seen to be Cauchy. We denote its limit by $\alpha_\infty(z)$. This limit is independent of the choice of the sequence $\lambda_n(z)$, as two such sequences differ on at most one term. Now define a function $f : [0,1] \to X$ by

$$f(x) = \begin{cases} h(2x) & \text{if } |\frac{1}{2} - x| > \frac{\delta}{4} \\ \varphi_i^{-1}(\alpha_\infty(x)) & \text{if } |\frac{1}{2} - x| < \frac{\delta}{2} \\ g(2x-1) & \text{if } |x - \frac{1}{2}| > \frac{\delta}{4}. \end{cases}$$

We show that f is well-defined and continuous. Consider $x \in [0,1]$ such that $\frac{1}{4}\delta < |\frac{1}{2} - x| < \frac{1}{2}\delta$. Then there exists N such that for all $i \geqslant N$, $|\lambda_i(x)| = 1$ and therefore

$$\varphi_i^{-1}(\alpha(x)_\infty) = h(2x) \text{ or } \varphi_i^{-1}(\alpha(x)_\infty) = g(2x-1).$$

Thus f is well-defined. To see that f is continuous, let $x \in [0,1]$ and $V \subset X$ be an open set containing $f(x)$. Note that

$$f(x) = h(2x) \text{ if } x \in \left[0, \frac{1}{2}\right), \text{ and } f(x) = g(2x-1) \text{ if } x \in \left(\frac{1}{2}, 1\right]. \tag{5.4}$$

Hence we may assume that $|\frac{1}{2} - x| < \frac{1}{2}\delta$, which means that $f(x) \in U_i$. Let $\varepsilon' > 0$ be such that

$$B_{\varphi_i(f(x))}(\varepsilon') \subset \varphi_i(V \cap U_i). \tag{5.5}$$

Choose $\delta' > 0$ such that for $y \in [0,1]$,

$$|1 - y| < \delta' \Rightarrow h(y) \in U_i \wedge |\varphi_i(h(y)) - \varphi_i(m_2)| < \varepsilon'/2$$

and

$$|y| < \delta' \Rightarrow g(y) \in U_i \wedge |\varphi_i(g(y)) - \varphi_i(m_2)| < \varepsilon'/2.$$

As earlier we may assume that

$$\|\varphi_i(m_2) - \varphi_i(f(x))\| < \varepsilon'/2,$$

since the assumption that $\|\varphi_i(m_2) - \varphi_i(f(x))\| > 0$ implies that $|x - \frac{1}{2}| > 0$. Furthermore, we may assume that $|x - \frac{1}{2}| < \delta'/2$. Let now $y \in [0,1]$ be such that $|x - y| < \delta'/2$. Then

$$\begin{aligned}\|\varphi_i(f(x)) - \varphi_i(f(y))\| &\leqslant \|\varphi_i(f(x)) - \varphi_i(f(m_2))\| + \|\varphi_i(f(m_2)) - \varphi_i(f(y))\| \\ &\leqslant \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'.\end{aligned}$$

Together with (5.5) that means that if $|x - y| < \delta'/2$ then $f(y) \in V$. Finally also $f(0) = h(0) = m_1$ and $f(1) = g(1) = m_3$, whence $m_1 \sim m_3$. □

Although the proof of the previous lemma is technical and lengthy, there seems to be no easier way of glueing two paths together. The construction in its proof has corollaries which can be useful, not only for manifolds.

Corollary 5.1.16. *Pathwise-connectedness is an equivalence relation on* $\mathbb{R}^n$*.*

Proof. Simply note that $\mathbb{R}^n$ is a manifold. □

Corollary 5.1.17. *Let* $g : [a,b] \to \mathbb{R}^n$ *and* $h : [b,c] \to \mathbb{R}^n$ *be continuous functions such that* $g(b) = h(b)$*. Then there exists a continuous function* $f : [a,c] \to \mathbb{R}^n$ *such that* $f(x) = g(x)$ *for each* $x \in [a,b]$*, and* $f(x) = h(x)$ *for each* $x \in [b,c]$*.*

Proof. The construction in proof of theorem for $\mathbb{R}^n$ as a manifold produces a function f with the desired properties. □

Lemma 5.1.18. *Let* $(X, (\varphi_i, U_i)_{i\in I})$ *be a M-connected manifold, such that* I *is finite, and for each* $i \in I$*,* U_i *is pathwise connected. If* i *and* j *are such that both* U_i *and* U_j *are inhabited, then there exists* $k \leqslant n$ *points* $x_1, \ldots, x_k$ *of* X *and an injection*

$$\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$$

such that $\sigma(1) = i, \sigma(k) = j$*, and*

$$x_l \in U_{\sigma(l)} \cap U_{\sigma(l+1)} \quad (l = 1 \ldots k-1).$$

Proof. Choose m_1, m_2 such that $m_1 \in U_i$ and $m_2 \in U_j$. Without loss of generality $i = 1$ and $j = n$. Then the sets U_1 and $\bigcup_{r=2}^{n} U_r$ form an inhabited, open covering of the M-connected space X. Hence there exists $x_1 \in U_1 \cap \bigcup_{r=2}^{n} U_r$. Let $\sigma(1) = 1$, and choose $\sigma(2) \in \{2, \ldots, n\}$ such that $x_1 \in U_{\sigma(2)}$. Either $\sigma(2) = n$ and we are done, or else we continue this process iteratively as follows. Assume we have constructed $\sigma(1), \ldots, \sigma(l)$ and $x_1, \ldots, x_{l-1}$. Then the sets

$$\bigcup_{k=1}^{l-1} U_{\sigma(k)} \quad \text{and} \quad \bigcup_{\substack{k=1 \\ k\neq\sigma(1),\ldots,\sigma(l)}}^{n} U_k$$

form an open inhabited covering of M and hence there is a x_l which lies in both sets. So choose $\sigma(l+1) \in \{1, \ldots, n\} \setminus \{\sigma(1), \ldots, \sigma(l)\}$, such that $x_l \in$

$U_{\sigma(l)} \cap U_{\sigma(l+1)}$. Now either $\sigma(l+1) = n$ and we are done, or else we continue the iteration. Clearly this process ends after at most n steps. □

Corollary 5.1.19. *Let $(X, (\varphi_i, U_i)_{i \in I})$ be an M-connected manifold such that I is finite, and $\varphi_i(U_i)$ is pathwise connected for each $i \in I$. Then X is pathwise-connected.*

Proof. Let $m_1, m_2 \in X$ arbitrary. Choose $i, j \in I$ such that $m_1 \in U_i$ and $m_2 \in U_j$. By the previous lemma, there exist $k \leqslant n$ points $x_1, \ldots, x_k$ and an injection

$$\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$$

such that $\sigma(1) = i, \sigma(k) = j$ and for each $1 \leqslant l \leqslant k-1$

$$x_l \in U_{\sigma(l)} \cap U_{\sigma(l+1)}.$$

Since for each $i \in I$ the set $\varphi_i(U_i)$ is pathwise-connected, so is U_i. Finally we get the chain of equivalences

$$m_1 \sim x_1 \sim \cdots \sim x_k \sim m_2,$$

and hence by Lemma 5.1.15 the desired $m_1 \sim m_2$. □

The work in this section shows again, how almost trivial classical proofs can turn into constructive nightmares.

In view of the space constructed in Proposition 5.1.14, it is also clear that we cannot dispense with the finiteness of the atlas, or only M-connectedness of the sets $\varphi_i(U_i)$. This is one of many times, that assuming a complete atlas—as the classical theory often does—loses valuable information.

5.2 Some topics related to differentiation

5.2.1 Extension of differentiable functions

If $U \subset \mathbb{R}^n$ is an open set containing some a, then we define the set

$$U_{a,i} = \{x \in U : \pi_i(x) = \pi_i(a)\} \cup \{x \in U : \pi_i(x) \neq \pi_i(a)\}$$

for each $1 \leqslant i \leqslant n$.

The following lemma is reminiscent of the glueing results in the previous subsection. It is needed later, to prove that the dimension of the tangent space exists.

Lemma 5.2.1. *Let $a \in \mathbb{R}^n$, and let U be an open set containing a. If $f : U_{a,i} \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function, then there exists a $\mathcal{C}^\infty$-function $F : U \to \mathbb{R}$ which coincides with f on $U_{a,i}$.*

Proof. Let $x \in \mathbb{R}^n$ be arbitrary and

$$x' = (\pi_1(x), \ldots, \pi_{i-1}(x), \pi_i(a), \pi_{i+1}(x), \ldots, \pi_n(x)).$$

Define a binary sequence $(\lambda_n)_{n \geqslant 1}$ such that

$$\lambda_n = 0 \Rightarrow |\pi_i(x) - \pi_i(a)| < \frac{1}{n},$$
$$\lambda_n = 1 \Rightarrow |\pi_i(x) - \pi_i(a)| > \frac{1}{n+1}.$$

Next, define a sequence $(x_n)_{n \geqslant 1}$ by

$$x_n = \begin{cases} f(x') & \text{if } \lambda_n = 0, \\ f(x) & \text{if } \lambda_n = 1. \end{cases}$$

Let $\varepsilon > 0$ be arbitrary, and choose $\delta > 0$ such that $|f(x) - f(x')| < \frac{\varepsilon}{2}$ whenever $|\pi_i(x) - \pi_i(a)| < \delta$. Choose N such that $\frac{1}{N} < \delta$. Then either $|\pi_i(x) - \pi_i(a)| >$

$\frac{1}{N+1}$, in which case $x_n = f(x)$ for any $n > N$, or else

$$\|x - x'\| \leqslant |\pi_i(x) - \pi_i(a)| < \frac{1}{N} < \delta.$$

In the latter case, for all $n, m > N$

$$|x_n - x_m| \leqslant |x_n - f(x')| + |x_m - f(x)'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence the sequence $(x_n)_{n\geqslant 1}$ is Cauchy, and its limit exists in the complete space $\mathbb{R}$. Furthermore, the limit is independent of the choice of the sequence $(\lambda_n)_{n\geqslant 1}$, and therefore the function $F : U \to \mathbb{R}$ defined by

$$F(x) = \lim_{n\to\infty} x_n$$

is well defined.

By definition, F coincides with f on $U_{a,i}$. We can repeat the construction and extend f_α to F_α for each $\alpha \in \{1, \ldots, n\}^k$. Since f is $\mathcal{C}^\infty$, F is $\mathcal{C}^\infty$ and has derivatives F_α. □

Corollary 5.2.2. *There is a* $\mathcal{C}^\infty$*-function* $g : \mathbb{R} \to [0, 1]$*, such that*

$$x \leqslant 0 \Rightarrow g(x) = 0,$$
$$x > 0 \Rightarrow g(x) > 0,$$
$$x \geqslant 1 \Rightarrow g(x) = 1.$$

Proof. Apply Lemma 5.2.1 to the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ \exp(-\frac{1}{x^2}) & \text{if } x > 0. \end{cases}$$

The important thing to prove is that the limit

$$\lim_{h>0, h\to 0} \frac{\exp^{(k)}(-\frac{1}{(h)^2})}{h}$$

exists for each $k \geqslant 0$. Using a simple induction argument, we can show that

$$\exp^{(k)}\left(-\frac{1}{h^2}\right) = \exp\left(-\frac{1}{h^2}\right) P_k\left(\frac{1}{h}\right),$$

where P_k is some polynomial. Since $\lim_{x\to\infty}\exp(-x^2)P(x) = 0$ for any polynomial P, we are done. □

Lemma 5.2.3. *Consider a $\mathcal{C}^\infty$-function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ defined on some open set U. For any $a \in U$ and $1 \leqslant i \leqslant n$, there exists a $\mathcal{C}^\infty$-function $g_i : U \to \mathbb{R}$ such that*

$$f(x) = f(a) + g_i(x)(\pi_i(x) - \pi_i(a)).$$

Proof. Consider the function g_i defined by

$$g_i(x) = \begin{cases} f(x) & \text{if } \pi_i(x) = \pi_i(a), \\ \frac{f(x)-f(a)}{\pi_i(x)-\pi_i(a)} & \text{if } \pi_i(x) \neq \pi_i(a). \end{cases}$$

We can apply Lemma 5.2.1 to this function to obtain a $\mathcal{C}^\infty$ function which coincides with g_i on $U_{a,i}$. □

Corollary 5.2.4. *Consider a $\mathcal{C}^\infty$-function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ defined on some open set U. Then for any $a \in U$, there exist n $\mathcal{C}^\infty$-functions $g_i : \mathbb{R}^n \to \mathbb{R}$ such that*

$$f(x) = f(a) + \sum_{i=1}^{n} g_i(x)(\pi_i(x) - \pi_i(a))$$

Proof. Applying the preceding lemma n times gives us:

$$\begin{aligned} f(x) &= f(a) + (\pi_1(x) - \pi_1(a))g_1(x) \\ &= f(a) + (\pi_1(x) - \pi_1(a))g_1(a) + (\pi_1(x) - \pi_1(a))(\pi_2(x) - \pi_2(a))g_2(x) \\ &\ \ \vdots \\ &= f(a) + \sum_{i=1}^{n} \widetilde{g}_i(x)(\pi_i(x) - \pi_i(a)), \end{aligned}$$

where

$$\widetilde{g}_i(x) = \prod_{j=1}^{i-1} g_i(x)\,(\pi_j(x) - \pi_j(a))$$

for $1 \leqslant i \leqslant n$. □

We call a function $f : D \to \mathbb{R}$, defined on some subset D of a manifold $(X, (U_i, \varphi_i)_{i\in I})$ **differentiable** ($\mathcal{C}^\infty$), if for each $i \in I$

$$f \circ \varphi_i^{-1} : \varphi_i(D \cap U_i) \to \mathbb{R}$$

is differentiable ($\mathcal{C}^\infty$).

Corollary 5.2.5. *Consider a $\mathcal{C}^\infty$-function $f : V \to \mathbb{R}$ defined on some open subset V of a manifold $(X, (U_i, \varphi_i)_{i\in I})$. For any $m \in V$ and each $j \in I$ with $m \in U_j$, there exist a neighbourhood W of m, and $\mathcal{C}^\infty$-functions $g_i : W \to \mathbb{R}$, such that for $x \in W$,*

$$f(x) = f(m) + \sum_{i=1}^{n} g_i(x)\,(\pi_i(\varphi_j(x)) - \pi_i(\varphi_j(m)))$$

Proof. Apply Corollary 5.2.4 to the functions

$$f \circ \varphi_j^{-1} : \varphi_j(V \cap U_j) \to \mathbb{R}.$$

□

5.2.2 Special differentiable functions and T3

Classically, we can extend a function on a manifold that is defined locally to a global function. The proof depends on finding, for each point $x \in X$ and each neighbourhood U of x, another neighbourhood $V \subset U$ of x and a differentiable function h_x such that

$$y \notin U \Rightarrow h_x(y) = 0$$
$$y \in V \Rightarrow h_x(y) = 1.$$

We call such a function a **differentiable Urysohn function for x and U**. Classically, the existence of such a function is assured by the existence of a partition of unity, which itself exists if every component of the manifold is Hausdorff and satisfies the second axiom of countability. There seems to be no hope to ensure constructively that a partition of unity exists, unless very strong conditions hold. This means that we have to take a different approach.

We say that a topological space satisfies **T3** if the closed neighbourhoods form a neighbourhood basis, that is, if for each point $x \in X$ and each neighbourhood U of x, there exists a neighbourhood V of x such that

$$x \in V \subset \overline{V} \subset U.$$

Furthermore, we call a space **regular** if it is Hausdorff and satisfies **T3**.

Proposition 5.2.6. *If there exists a differentiable Urysohn function for every point $m \in X$ of a locally decomposable manifold X and every neighbourhood U of m, then the manifold is regular.*

Proof. Let m, m' be points in X such that $m \neq m'$. Choose $i \in I$ such that $m \in U_i$. By Lemma 5.1.8, there exists a neighbourhood $V \subset U_i$ of m such that

$$X = U_i \cup \neg V.$$

Now assume that there exist a neigbourhood $V' \subset V$ of m and a differentiable function h_m such that $h_m(y) = 1$ for each $y \in V'$ and $h_m(y) = 0$ for each $y \in V$. If $m' \in U_i$, then we are done, since U_i is homeomorphic to some open subset of $\mathbb{R}^n$. In the case that $m' \in \neg V$, we know that $h_m(m') = 0$ and therefore $h_m^{-1}((-\infty, \frac{1}{3}))$ and $h_m^{-1}((\frac{2}{3}, \infty))$ are two disjoint neighbourhoods. To see that the manifold is **T3**, consider an arbitrary point m and some neighbourhood U. By **LD** there exists another neighbourhood W of m such that

$$X = U \cup \neg W.$$

Assume there exists yet another neighbourhood V of m and a function h_m such that $h_m(y) = 1$ for $y \in V$ and $h_m(y) = 0$ for $y \in \neg W$. The set $F = h_m^{-1}(1)$ is closed, since h_m is continuous. Furthermore, if $y \in F$, then $y \in U$, since the possibility $y \in \neg W$ is ruled out. Hence

$$x \in W \subset \overline{U} \subset F = \overline{F} \subset U.$$

□

Not every manifold that satisfies the Hausdorff condition is regular, as the following example shows.

Example 5.2.7. *Let*

$$\begin{aligned} U_1 &= \mathbb{R} \times \{0\}, \\ U_2 &= \mathbb{R} \times \{1\}, \\ U_3 &= (\mathbb{R} \times \{0\} \setminus \{(0,0)\}) \cup \{(0,1)\} \end{aligned}$$

Let $X = \bigcup_{i=1}^{3} U_i$. *Then* $(X, (U_i, \pi_1)_{i=1,2,3})$ *is a manifold. A basis for the induced topology is given by*

$$\begin{aligned} \mathcal{B} = &\{(p,q) \times \{a\} : p < q, a \in \{0,1\}\} \\ &\cup \{((p,0) \cup (0,q)) \times \{0\} \cup (0,1) : p < 0 < q\}. \end{aligned}$$

This manifold is easily seen to be Hausdorff. But it is not regular, since the closure of every neighbourhood V *of* $(0,1)$ *such that*

$$V \subset ((-1,0) \cup (0,1)) \times \{0\} \cup \{(0,1)\}$$

also contains the point $(0,0)$.

We can prove that the converse holds as well.

Lemma 5.2.8. *Consider a locally decomposable manifold that satisfies* **T3**. *Then for every open neighbourhood $m \in U \subset U_i$, there are neighbourhoods V, V' of m such that*

$$\overline{V} \subset V' \subset \overline{V'} \subset U \subset U_i,$$

$X = U \cup \neg V'$, and $X = V' \cup \neg V$. Furthermore, there exists a differentiable function $h_m : M \to \mathbb{R}_0^+$ such that $h(y) = 1$ for $y \in \overline{V}$, $h(y) > 0$ for $y \in V'$, and $h(y) = 0$ for $y \notin V'$.

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be the function constructed in Lemma 5.2.2. Given a $m \in X$, we see from **LD** that there is a neighbourhood W such that $X = U \cup \neg W$. Choose $\rho > 0$ such that

$$B_{\varphi_i(m)}(3\rho) \subset \varphi_i(W).$$

Let $V = \varphi_i^{-1}(B_{\varphi_i(m)}(\rho))$. Then V is a neighbourhood of m, and $\overline{V} \subset U_i$. Similarly, $V' = \varphi_i^{-1}(B_{\varphi_i(m)}(2\rho))$ is a neighbourhood of m. The decompositions $X = U \cup \neg V'$, and $X = V' \cup \neg V$ hold by **LD** and trichotomy in $\mathbb{R}$. Now define $h : X \to \mathbb{R}_0^+$ by

$$h(x) = \begin{cases} g\left(\frac{2\rho^2 - \|\varphi_i(x) - \varphi_i(m)\|^2}{\rho^2}\right) & \text{if } x \in U_i, \\ 0 & \text{if } x \notin W. \end{cases}$$

The function h has the desired properties. To show that it is well-defined, consider $y \in U_i$ and $y \notin W$. Then $g(y) = 0$, since $\|\varphi_i(y) - \varphi_i(m)\| \geqslant 3\rho$ implies that

$$2\rho^2 - (\|\varphi_i(y) - \varphi_i(m)\|)^2 \leqslant 0.$$

□

If we assume **T3** we can prove the following extension result.

Proposition 5.2.9. *Assume that $g : D \to \mathbb{R}^n$ is a differentiable function defined on some open subset $D \subset X$, and let m be some point in this domain. Then there exists a differentiable global function $G : X \to \mathbb{R}^n$ which coincides with g*

on some neighbourhood of m.

Proof. Choose i such that $m \in U_i$. Using Lemma 5.2.8, choose neighbourhoods V and V' such that $V' \subset D \cap U_i$, and a differentiable function h_m with the properties described. Define G by

$$G(x) = \begin{cases} g(x)h_m(x) & \text{if } x \in U, \\ 0 & \text{if } x \notin V'. \end{cases}$$

This G satisfies our requirements. □

5.3 Differentiation on a manifold

This last section of this thesis contains plenty of definitions which are important for proving an imbedding result at the end.

Consider a function $f : V \subset X \to X'$ between two manifolds, defined on some open subset V of X. If (φ, U) and (ψ, U') are charts of X and X' respectively, then we call

$$F = \psi \circ f \circ \varphi^{-1} : \varphi(V \cap U) \to \mathbb{R}^n$$

a **coordinate representative** of f. We say that f is **differentiable at** $x \in V$ if F is differentiable at $\varphi(x)$. Note that this definition is independent of the choice of the charts. We call a function **differentiable (or $\mathcal{C}^\infty$-differentiable)** if it is differentiable (or $\mathcal{C}^\infty$-differentiable) at every point of its domain. These definitions coincide with the ones for $\mathbb{R}^n$, if we consider $\mathbb{R}^n$ as a manifold equipped with the atlas consisting of only one chart $(\mathbb{R}^n, \mathrm{id})$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

Assume φ is a chart with domain U on some manifold X and $f : V \to \mathbb{R}$ a differentiable function. We define **partial derivatives** by

$$\frac{\partial f}{\partial \varphi_k} = D_k\left(f \circ \varphi^{-1}\right) \circ \varphi : U \cap V \to \mathbb{R}.$$

Proposition 5.3.1. *If f, g are real-valued differentiable functions defined on X, and $\alpha, \beta \in \mathbb{R}$, then*

$$\frac{\partial}{\partial \varphi_k}(\alpha f + \beta g) = \alpha \frac{\partial f}{\partial \varphi_k} + \beta \frac{\partial g}{\partial \varphi_k},$$
$$\frac{\partial}{\partial \varphi_k}(fg) = \frac{\partial f}{\partial \varphi_k} g + f \frac{\partial g}{\partial \varphi_k}.$$

Proof. These results follow by straightforward calculations, which are therefore omitted. □

Let $\mathcal{F}(m)$ denote the set of all real-valued differentiable functions on some manifold X whose domains include a given point m. An $\mathbb{R}$-linear functional on $\mathcal{F}(m)$ is called a **linear operator** on $\mathcal{F}(m)$. A **derivation** is a linear operator $\Lambda : \mathcal{F}(m) \to \mathbb{R}$ such that

$$\Lambda(fg) = (f(m))(\Lambda g) + (\Lambda f)(g(m))$$

for all $f, g \in \mathcal{F}(m)$. Proposition 5.3.1 shows that for any chart (U, φ) and for any point $m \in U$, the function

$$\left(\frac{\partial}{\partial \varphi_k}\right)_m : \mathcal{F}(m) \to \mathbb{R}$$

that maps $f \in \mathcal{F}(m)$ to $(\partial f / \partial \varphi_k)$ is a derivation on $\mathcal{F}(m)$. The set of all derivations on $\mathcal{F}(m)$ has $\mathbb{R}$-linear structure. It becomes a real vector space, which is called the **tangent space** $T_m M$ **at** m. Any derivation is an element of the tangent space and is called a **tangent vector** at m.

Theorem 5.3.2. *The tangent space has dimension n. More precisely,*

$$\left(\frac{\partial}{\partial \varphi_k}\right)_{k=1\ldots n}$$

is a basis of $T_m M$.

Proof. We use the fact that

$$\frac{\partial}{\partial \varphi_k}(\pi_l(\varphi)) = \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}$$

Assume that

$$\sum_{k=1}^{n} \alpha_k (\partial/\partial \varphi_k) = 0$$

for some real numbers α_k. Then

$$\alpha_j = \sum_{k=1}^{n} \alpha_k \frac{\partial \varphi_j}{\partial \varphi_k} = \left(\sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial \varphi_k}\right)(\varphi_j) = 0$$

for $j = 1 \ldots n$. Hence the set

$$\left\{\frac{\partial}{\partial \varphi_k} : k = 1, \ldots, n\right\}$$

is linearly independent. To see that it is also a generating system of the tangent space, let $\Lambda \in T_m X$ and $f \in \mathcal{F}(m)$. By Corollary 5.2.5, there exist a neighbourhood V of m and $\mathcal{C}^\infty$ functions $g_1, \ldots, g_n$ such that for $x \in V$,

$$f(x) = f(m) + \sum_{i=1}^{n} (\pi_i(\varphi(x)) - \pi_i(\varphi(m))) g_i(x).$$

We conclude that

$$\begin{aligned} \Lambda f &= \Lambda(f(m) + \sum_{i=1}^{n} (\pi_i(\varphi(x)) - \pi_i(\varphi(m))) g_i(x)) \\ &= \sum_{i=1}^{n} (\Lambda \pi_i(\varphi(x))) g_i(m) + (\Lambda g_i(x))(\pi_i(\varphi(m)) - \pi_i(\varphi(m))) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} (\Lambda \pi_i(\varphi(x))) g_i(m) \\
&= \sum_{i=1}^{n} (\Lambda \pi_i(\varphi(x))) \left(\sum_{j=1}^{n} \left(\frac{\partial \pi_j(\varphi)}{\partial \varphi_i} \right)_m g_i(m) \right) \\
&= \sum_{i=1}^{n} (\Lambda \pi_i(\varphi(x))) \left(\frac{\partial f}{\partial \varphi_i} \right)_m .
\end{aligned}
$$

Since f was arbitrary,

$$
\Lambda = \sum_{k=1}^{n} (\Lambda \varphi_k) \left(\frac{\partial}{\partial \varphi_k} \right)_m .
$$

□

Suppose that m is a point in the domain of a differentiable function

$$
\Phi : X \to X',
$$

and set $m' = \Phi(m)$. If $f \in \mathcal{F}(m')$, then $f \circ \Phi \in \mathcal{F}(m)$. Hence a vector $v \in T_m X$ determines a function $f \mapsto v\,(f \circ \Phi)$. This function is easily seen to be a derivation on $\mathcal{F}(m')$. It is therefore a vector in $T_{m'} X'$, which we denote by $\Phi_{*m} v$. The function

$$
\Phi_{*m} : T_m X \to T_{m'} X'
$$

defined in this way is easily seen to be linear. It is called the **derived linear function** on $T_m X$.

Proposition 5.3.3. *Consider two differentiable functions $\Phi : X \to X'$ and $\Psi : X' \to X''$. If m lies in the domain of Φ, and $\Phi(m)$ in the domain of Ψ, then*

$$
(\Psi \circ \Phi)_{*m} = \Psi_{*(\Phi(m))} \circ \Phi_{*m}.
$$

Proof. Let $v \in T_m X$ be arbitrary, and $f \in \mathcal{F}(\Psi(\Phi(m)))$. Then

$$
\begin{aligned}
(\Psi \circ \Phi)_{*m}(v)(f) &= v(f \circ \Psi \circ \Phi) \\
&= \left[\Phi_{*m}(v)\right](f \circ \Psi)
\end{aligned}
$$

$$= \left[\Psi_{*(\Phi(m))}(\Phi_{*m}v)\right] f.$$

□

The union of all the tangent spaces T_mX for $m \in X$ is called the **tangent bundle** TX. It is the set of all tangent vectors. A differentiable function $\Phi : X \to X'$ determines a function $\Phi_* : TX \to TX'$ by $v \mapsto \Phi_{*m}v$ for $v \in T_mM$. We define the **rank** of a differentiable function $\Phi : X \to X'$ at a point m to be the rank of the derived linear function Φ_{*m}—that is the dimension of the range of Φ_{*m}. That is, of course, only if the dimension exists, which it does not always, as the following example shows.

Proposition 5.3.4. *If the rank of any differentiable function at any point exists, then* **LPO** *holds.*

Proof. Let $(a_n)_{n\geqslant 1}$ be a binary sequence, and define a real number

$$a = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

Consider the manifold $X = \mathbb{R}^2$, equipped with the identity chart. Define $\Phi : M \to M$ by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

If Φ has rank 1 (at any point), then $a = 0$. Similarly, if Φ has rank 2, then $a \neq 0$. In the first case $a_n = 0$ for all n, whereas in the second case $a_n = 1$ for some n. Hence **LPO** holds. □

We call a differentiable function $\Psi : X' \to X$ an **immersion** if its rank exists and is equal to the dimension of X' at each point of its domain. A manifold X' is called a **submanifold** of X if it is a subset of X and the natural injection is an immersion. An immersion of X_1 into X which is an injection is called an **imbedding** of X_1 into X. We can now prove an imbedding theorem.

Proposition 5.3.5. *Let $(X, (U_i, \varphi_i)_{i=1,\dots,n})$ be a differentiable manifold with a finite atlas. Assume that there are sets V_k such that*

$$V_k \subset U_k \text{ and } X = \bigcup_{i=1}^{m} V_i,$$

and differentiable functions f_k such that $y \in V_k$ implies that $f_k(y) = 1$, and $y \notin U_k$ implies that $f_k(y) = 0$. Then there exists a natural number N such that X can be imbedded into $\mathbb{R}^N$.

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be the function constructed in Lemma 5.2.2. For $1 \leqslant k \leqslant n$ let $\alpha_k : M \to \mathbb{R}^n$ be defined by

$$\alpha_k(x) = \begin{cases} g(2f_k(x) - 1)\varphi_k(x) & \text{if } f_k(x) > \frac{1}{3}, \\ 0 & \text{if } f_k(x) < \frac{1}{2}. \end{cases}$$

These functions are well-defined, since $g(2f_k(x) - 1) = 0$ for $x \in X$ such that $f_k(x) \leqslant \frac{1}{2}$. Also, for $x \in V_k$, $g(2f_k(x) - 1) = 1$ and therefore $\alpha_k(x) = \varphi_k(x)$. Now define a mapping $h : X \to \mathbb{R}^m \times \mathbb{R}^{n \times m}$ by

$$h : x \mapsto (f_1(x), \dots, f_m(x), \alpha_1(x), \dots, \alpha_m(x)).$$

The function h is $\mathcal{C}^\infty$, because g, f_k and φ_k are. Now assume that $h(x) \neq h(x')$, and choose k such that $x \in V_k$. Then $f_k(x') = 1$ and therefore $\varphi_k(x) = \varphi_k(x')$. Since φ_k is injective, also $x = x'$; whence h is injective. Let $x \in X$ an arbitrary point and choose k such that $x \in V_k$. Let p be the projection from $\mathbb{R}^N$ onto the $m(k+1)^{th}$ to the $(m(k+2)-1)^{th}$ coordinate. Then $p \circ h(x) = \varphi_k(x)$ for each $x \in V_k$. By Proposition 5.3.3,

$$p_{*h(x)} \circ h_{*x} = (\varphi_k)_{*x}.$$

It follows that the rank of h_{*x} is the same as the rank of $(\varphi_k)_{*x}$.

□

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