outer bound is tight for some new class of broadcast channels that may perhaps include the BSSC.

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Instantaneous Capacity of OFDM on Rayleigh-Fading Channels

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***Abstract*—For a power limited orthogonal frequency-division multiplexing (OFDM) system transmitting a large number N of subcarriers over a Rayleigh-fading channel, the distribution of the instantaneous capacity is shown to be approximately Gaussian. The mean and variance of the approximating distribution are derived. It is also shown that, in the limit as $N \to \infty$, the capacity approaches a constant value equal to the capacity of the infinite-bandwidth Gaussian channel.**

***Index Terms*—Channel capacity, orthogonal frequency-division multiplexing (OFDM), Rayleigh-fading channels.**

I. INTRODUCTION

Orthogonal frequency-division multiplexing (OFDM) systems employing a large number of subcarriers are being considered for ultra-wideband applications [1]. We consider the capacity of an OFDM system transmitting N subcarriers over a Rayleigh-fading frequency-selective channel. Furthermore, we presume that N is large, so that a large bandwidth is occupied with low spectral efficiency. Recent similar results concerning the capacity of such wideband systems include [2]–[5], while a comprehensive review of wideband capacity is provided by [6] and the references therein, and [7] gives an overview of recent results. However, we restrict ourselves to OFDM systems, and consider the distribution of the instantaneous capacity during transmission of each OFDM block, assuming a slow fading channel. We show that the use of OFDM with equal power transmitted on all subcarriers does not incur any capacity reduction.

The overall capacity is the sum of the individual subchannel capacities. Due to fading, this is a random variable for each OFDM block. Since the subchannel gains are correlated, we cannot employ the classical central limit theorem to estimate the distribution of capacity, as a function of the channel gains. However, for an OFDM system with large N and finite power, we show the distribution of the instantaneous capacity is approximately Gaussian, using a central limit theorem. This result may be used to construct confidence intervals on the capacity of an OFDM system employing a large number of subcarriers. In addition, for fixed power and unlimited bandwidth we show that the spectral efficiency, as $N \to \infty$, approaches a constant dependent only on the signal-to-noise ratio (SNR).

In the following section, we outline key assumptions and notation used in the description of OFDM systems and Rayleigh-fading channels. We show the approximating and asymptotic distributions of the capacity in Section III, and compare this to simulated results in Section IV. Section V provides conclusions.

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II. Preliminaries

A. OFDM System

We assume familiarity with OFDM systems [8], [9] and summarize notation only. An N subcarrier OFDM system transmits data symbol $S_{n,k}$ on the kth subcarrier, during the nth discrete time interval, for $n \in \mathbb{Z}, k \in \{0, 1, \ldots, N-1\}$, where $S_{n,k} \in \mathbb{R}^2$ is from some two-dimensional symbol constellation. We place no further restriction on the symbols $S_{n,k}$ although to approach or achieve the channel capacity the symbols would need be restricted by some strong coding scheme, with appropriate demodulation and decoding at the receiver. We refer to the superposition of all N modulated subcarriers during the nth time interval as the nth OFDM block. We assume each subcarrier occupies a subchannel of bandwidth Δf (hertz), such that the total bandwidth $B = N\Delta f$, with block duration $T = \frac{1}{B}$. We denote the center frequency of each subchannel as f_k, so that $f_{k+1} - f_k = \Delta f$. Furthermore, each subcarrier symbol is transmitted with equal energy E_0 such that the total average transmitted energy is $E_N = N\,E_0$.

The transmitted time-domain OFDM signal during the nth block is denoted $s_n(t)$, for $(n-1)T < t \leq nT$. We obtain samples of $s_n(t)$ at rate B using the inverse discrete Fourier transform of the subcarrier symbols. We may write the samples $s_{n,i}$ as

$$s_{n,i} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_{n,k} \exp\left(j2\pi \frac{ik}{N}\right) \tag{1}$$

for $i = 1, \ldots, N$. We assume a time-varying channel impulse response $h(t,\tau)$ [10]. However, the channel is sufficiently slowly fading such that over the period T of an OFDM block being transmitted, $h(t,\tau)$ is constant in t. We also assume a Gaussian noise process $n(t)$ with power spectral density $\frac{N_0}{2}$ (watts per hertz), so that the receiver obtains

$$r(t) = s(t) \otimes h(t,\tau) + n(t) \tag{2}$$

where $\otimes$ denotes convolution by τ. We assume perfect synchronization in time and frequency and sufficient guard interval for inter-block interference and inter-carrier interference to be negligible. Then, at the receiver we may sample $r(t)$ at rate B, and perform a discrete Fourier transform to obtain symbols [11]

$$R_{n,k} = H_{n,k}S_{n,k} + N_{n,k} \tag{3}$$

where $N_{n,k} \in \mathbb{R}^2$ is an independent and identically distributed (i.i.d.) complex Gaussian random variable with one-dimensional variance $\frac{N_0}{2}$, and $H_{n,k} \in \mathbb{R}^2$ is the kth complex subchannel gain. We then obtain the time-varying channel frequency response, denoted $H(t,f)$, via the Fourier transform of $h(t,\tau)$. In [11, Sec 2.6] it is shown that $H_{n,k} = H(nT, f_k)$, assuming a slow fading channel such that $H(t,f)$ changes negligibly with respect to time over any interval $(n-1)T < t \leq nT$. At the receiver, the SNR for the kth subcarrier, nth time interval is then $\gamma_{n,k} = |H_{n,k}|^2 \frac{E_0}{N_0} = \gamma_0 |H_{n,k}|^2$, where $\gamma_0 = \frac{E_0}{N_0}$. The total SNR is given by $\frac{E_N}{N_0}$.

B. Multipath Channel

We assume familiarity with frequency-selective Rayleigh-fading channels, and use the well-known Jakes' model [10]. We make the usual assumption that $h(t,\tau)$ describes a frequency-selective Rayleigh-fading channel, that is wide-sense stationary with uncorrelated, isotropic scattering. Furthermore, we presume the delay autocorrelation function may be described as

$$\frac{1}{2}\mathbb{E}[h(t,\tau_1)h^*(t,\tau_2)] \equiv \frac{1}{\tau_d} \exp\left(-\frac{\tau_1}{\tau_d}\right) \delta(\tau_1 - \tau_2) \tag{4}$$

where * denotes the complex conjugate, which defines an exponential delay power profile with root mean square (rms) delay τ_d. Then, from [10] we may write the k_1th subchannel gain during time $(n_1-1)T < t \leq n_1 T$, and the k_2th subchannel gain during time $(n_2-1)T < t \leq n_2 T$, as

$$\begin{aligned} H_{n_1,k_1} &= X_{n_1,k_1} + jY_{n_1,k_1} \quad \text{and} \\ H_{n_2,k_2} &= X_{n_2,k_2} + jY_{n_2,k_2} \end{aligned} \tag{5}$$

where $X_{n_1,k_1}, Y_{n_1,k_1}, X_{n_2,k_2}$, and Y_{n_2,k_2} are identically distributed zero-mean Gaussian random variables. Without loss of generality, we may set $\mathbb{E}[X_{n,k}^2] = \mathbb{E}[Y_{n,k}^2] = \frac{1}{2}$, for all n, k. Following [10] we may then write the cross-correlation properties

$$\begin{aligned} \mathbb{E}[X_{n_1,k_1}X_{n_2,k_2}] &= \mathbb{E}[Y_{n_1,k_1}Y_{n_2,k_2}] \\ &= \frac{1}{2}\frac{J_0(2\pi f_d |n_1 - n_2| T)}{1 + (2\pi\tau_d \Delta f \Delta k)^2} \\ \mathbb{E}[X_{n_1,k_1}Y_{n_1,k_1}] &= \mathbb{E}[X_{n_2,k_2}Y_{n_2,k_2}] = 0 \\ \mathbb{E}[X_{n_1,k_1}Y_{n_2,k_2}] &= -\mathbb{E}[X_{n_2,k_2}Y_{n_1,k_1}] \\ &= -(2\pi\Delta f \Delta k\, \tau_d)\mathbb{E}[X_{n_1,k_1}X_{n_2,k_2}] \end{aligned} \tag{6}$$

where $\Delta k = |k_1 - k_2|$, f_d is the maximum Doppler shift and $J_0(\cdot)$ is the zero-order Bessel function of the first kind. $\mathbb{E}[X_{n_1,k_1}Y_{n_2,k_2}]$ decreases only as $\frac{1}{\Delta k}$, and this *strong dependence* [12] prohibits the use of classical limit theorems for functions of independent or weakly dependent random variables [12].

Note, from (5), that the marginal distribution of each channel gain $|H_{n,k}|^2$ follows an exponential distribution with

$$\mathbb{E}[|H_{n,k}|^2] = 1, \mathrm{var}[|H_{n,k}|^2] = 1$$

and correlation coefficient [13]

$$\rho\left(|H_{n,k}|^2, |H_{n,k+\Delta k}|^2\right) = \frac{1}{1 + (2\pi\Delta f \Delta k\, \tau_d)^2}. \tag{7}$$

We may write the channel gain as a function of the underlying Gaussian random variables $X_{n,k}$ and $Y_{n,k}$, as

$$|H_{n,k}|^2 \triangleq \mathrm{h}\left(X_{n,k}^2, Y_{n,k}^2\right) = X_{n,k}^2 + Y_{n,k}^2. \tag{8}$$

The Hermite rank of a function is the index of the first nonzero coefficient in its Hermite polynomial expansion, defined in Appendix I and [14]. The Hermite rank of $\mathrm{h}(\cdot)$ is denoted $\varphi(\mathrm{h})$, and is readily shown to be at least two, using the method of Appendix I; a necessary result for a forthcoming lemma.

III. OFDM System Capacity Distribution

A. Subchannel Capacity Distribution

The capacity of each subchannel is a function of the subchannel spacing, SNR per subcarrier, and the channel gain. We ignore any reduction in capacity due to the OFDM guard interval, or cyclic prefix, and may then write the subchannel capacity as [15]

$$C_{n,k} = \frac{\Delta f}{\ln 2} \ln\left(1 + \gamma_0 |H_{n,k}|^2\right) \text{ bit/s.} \tag{9}$$

Note that we have restricted our system so that each subcarrier transmits the same amount of energy. We may equivalently write (9) as a function of the underlying Gaussian variables $X_{n,k}$ and $Y_{n,k}$, as

$$\begin{aligned} C_{n,k} &\triangleq \mathrm{c}(X_{n,k}, Y_{n,k}) \\ &= \frac{\Delta f}{\ln 2} \ln\left(1 + \gamma_0 \left[X_{n,k}^2 + Y_{n,k}^2\right]\right) \text{ bit/s.} \end{aligned} \tag{10}$$

The probability density function (pdf) of $C_{n,k}$, since $|H_{n,k}|^2$ follows an exponential distribution with unity mean, is then

$$f_{C_{n,k}}(x) = \frac{\ln 2}{\gamma_0 \Delta f} \exp\left(\frac{x \ln 2}{\Delta f}\right) \exp\left(\frac{1}{\gamma_0} - \frac{\exp\left(\frac{x\ln 2}{\Delta f}\right)}{\gamma_0}\right) \tag{11}$$

with mean capacity [16]

$$\mathbb{E}[C_{n,k}] = \int_0^\infty \frac{\Delta f}{\ln 2}\ln(1+\gamma_0 y)\exp(-y)dy = -\frac{\Delta f}{\ln 2}\exp\left(\frac{1}{\gamma_0}\right)\mathrm{Ei}\left(-\frac{1}{\gamma_0}\right) \quad (12)$$

where $\mathrm{Ei}(\cdot)$ is the exponential integral function [17]. This expression is also obtained in [18], [3]. Numerically calculable expressions for the mean-squared capacity $\mathbb{E}[C_{n,k}^2]$ and the correlation $\mathbb{E}[C_{n,k_1}C_{n,k_2}]$ between subchannel capacities at time n are found in Appendix II.

B. System Capacity Distribution

We define the capacity of the overall OFDM system during the nth time interval as $C_n = \sum_{k=1}^N C_{n,k}$. We then find an approximation to the distribution of C_n for large, finite N. Note that as N increases, the bandwidth $B = N\Delta f$ increases. We write the capacity during the nth time interval as

$$C_n = \sum_{k=1}^N \mathrm{c}\left(X_{n,k}, Y_{n,k}\right). \quad (13)$$

Consider the following limit theorem, proofs of which are found in [19], [20].

Theorem 1 (Arcones-de Naranjo): Let $\{\boldsymbol{X}_j\}_{j=1}^\infty$ be a stationary mean-zero sequence of Gaussian vectors in $\mathbb{R}^d$. Set $\boldsymbol{X}_j = (X_j^{(1)}, \ldots, X_j^{(d)})$. Let f be a function on $\mathbb{R}^d$ with Hermite rank (Appendix I) $\varphi(f)$ such that $1 \le \varphi(f) < \infty$. Define

$$r^{(p,q)}(k) = \mathbb{E}\left[X_m^{(p)}X_{m+k}^{(q)}\right] \quad (14)$$

for $k \in \mathbb{Z}$, where m is any number large enough that $m \ge 1$ and $m + k \ge 1$. Suppose that

$$\sum_{k=-\infty}^{\infty}\left|r^{(p,q)}(k)\right|^{\varphi(f)} < \infty \quad (15)$$

for all $1 \le p \le d$ and $1 \le q \le d$. Then, as $N \to \infty$

$$\frac{1}{\sqrt{N}}\sum_{j=1}^N (f(\boldsymbol{X}_j) - \mathbb{E}[f(\boldsymbol{X}_j)]) \xrightarrow{D} \mathcal{N}(0,\sigma^2) \quad (16)$$

where "$\xrightarrow{D}$" denotes "convergence in distribution," and

$$\sigma^2 = \mathbb{E}\left[(f(\boldsymbol{X}_1) - \mathbb{E}[f(\boldsymbol{X}_1))^2\right] + 2\sum_{k=1}^\infty \mathbb{E}[(f(\boldsymbol{X}_1) - \mathbb{E}[f(\boldsymbol{X}_1)])) \times (f(\boldsymbol{X}_{1+k}) - \mathbb{E}[f(\boldsymbol{X}_{1+k}))]. \quad (17)$$

We apply this theorem to the stationary mean-zero sequence of complex Gaussian subchannel gains at time n, that is, $\{(X_{n,1}, Y_{n,1}), (X_{n,2}, Y_{n,2}), \ldots, (X_{n,N}, Y_{n,N})\}$, with correlation properties described in (6). The function of interest is the subchannel capacity $\mathrm{c}(X_{n,k}, Y_{n,k})$. It is shown in Appendix I that $\mathrm{c}(\cdot)$ has Hermite rank $\varphi(\mathrm{c}) \ge 2$. Thus, using the correlation properties of (6) we may write

$$\sum_{\Delta k=-\infty}^{\infty}|\mathbb{E}[X_{n,k}X_{n,k+\Delta k}|^{\varphi(\mathrm{c})} = \sum_{\Delta k=-\infty}^{\infty}|\mathbb{E}[Y_{n,k}Y_{n,k+\Delta k}|^{\varphi(\mathrm{c})} = \sum_{\Delta k=-\infty}^{\infty}\left|\frac{1}{2}\frac{1}{1+(2\pi\Delta f\Delta k)^2}\right|^{\varphi(\mathrm{c})} < \infty$$

$$\sum_{\Delta k=-\infty}^{\infty}|\mathbb{E}[X_{n,k}Y_{n,k+\Delta k}]|^{\varphi(\mathrm{c})} = \sum_{\Delta k=-\infty}^{\infty}|\mathbb{E}[Y_{n,k}X_{n,k+\Delta k}]|^{\varphi(\mathrm{c})} = \sum_{\Delta k=-\infty}^{\infty}\left|\frac{1}{2}\frac{(2\pi\Delta f\Delta k\tau_D)}{1+(2\pi\Delta f\Delta k)^2}\right|^{\varphi(\mathrm{c})} < \infty \quad (18)$$

for $k, k+\Delta k \ge 1$, since $\varphi(\mathrm{c}) \ge 2$. Requirement (15) is then satisfied, and we apply the theorem to write

$$\frac{1}{\sqrt{N}}\sum_{k=1}^N \{\mathrm{c}(X_{n,k}, Y_{n,k}) - \mathbb{E}[\mathrm{c}(X_{n,k}, Y_{n,k})]\} \xrightarrow{D} \mathcal{N}\left(0, \Omega_c^2\right) \quad (19)$$

as $N \to \infty$, where

$$\begin{aligned}\Omega_c^2 &= \mathbb{E}\left[(\mathrm{c}(X_{n,1}, Y_{n,1}) - \mathbb{E}[\mathrm{c}(X_{n,1}, Y_{n,1})])^2\right] \\ &\quad + 2\sum_{k=1}^\infty \mathbb{E}\left[(\mathrm{c}(X_{n,1}, Y_{n,1}) - \mathbb{E}[\mathrm{c}(X_{n,1}, Y_{n,1})])\right. \\ &\quad \times (\mathrm{c}(X_{n,k+1}, Y_{n,k+1}) - \mathbb{E}[\mathrm{c}(X_{n,k+1}, Y_{n,k+1})])] \\ &= \mathrm{var}[\mathrm{c}(X_{n,1}, Y_{n,1})] \\ &\quad + 2\sum_{k=1}^\infty \mathrm{cov}[\mathrm{c}(X_{n,1}, Y_{n,1}), \mathrm{c}(X_{n,k+1}, Y_{n,k+1})] \end{aligned} \quad (20)$$

with the variance and covariance terms readily calculable using the expressions in Appendix II. The convergence in distribution described in (19) clearly motivates the following approximation. For large finite N, the distribution of the instantaneous capacity C_n may be approximated by a Gaussian random variable with mean $N\mathbb{E}[C_{n,k}]$ and variance $N\Omega_c^2(N)$, where

$$\Omega_c^2(N) = \mathrm{var}[\mathrm{c}(X_{n,1}, Y_{n,1})] + 2\sum_{k=1}^N \mathrm{cov}[\mathrm{c}(X_{n,1}, Y_{n,1}), \mathrm{c}(X_{n,k+1}, Y_{n,k+1})]. \quad (21)$$

Note that since capacity is nonnegative, the Gaussian approximation to the distribution of C_n is invalid at $C_n < 0$. However, for moderately large SNR and N, $\mathrm{P}(C_n < 0)$ becomes negligible, and the deviation from the Gaussian approximation is small, as demonstrated in Section IV.

We now apply this Gaussian approximation to specify the distribution of the instantaneous capacity for systems with very large bandwidth, such as ultra-wideband systems [1], and fixed total average transmitted energy E_N. We set the average transmitted energy per subcarrier to be $E_0 = \frac{E_N}{N}$, such that $\gamma_0 = \frac{E_N}{N N_0}$. We may substitute this expression for γ_0 into (12), and (41) and (42), to obtain $N\mathbb{E}[C_{n,k}]$ and $N\Omega_c^2(N)$, respectively. Simulation results in the following section show this to be a good approximation to the distribution of the instantaneous capacity for a very large bandwidth, power-limited OFDM system. We would expect this approximation to be tighter for larger N, and this is demonstrated by the simulations.

We now consider the case of a power-limited OFDM system with fixed Δf, and we let the number of subcarriers N approach infinity, so that the bandwidth also approaches infinity. For such power-limited systems E_N is fixed, so that $E_0 = \frac{E_N}{N} \to 0$, as $N \to \infty$. We show that the limiting capacity C_∞ of such a system converges in probability to a constant.

Lemma 1: Assuming $\mathbb{E}[|H_{n,k}|^2] = 1$, the distribution of the arithmetic average subchannel gain converges to a degenerate distribution, as $N \to \infty$, such that

$$\frac{1}{N}\sum_{k=1}^N |H_{n,k}|^2 \xrightarrow{P} 1 \quad (22)$$

where "$\xrightarrow{P}$" denotes "convergence in probability."

Proof: Let $\Omega_H^2(N) = \mathrm{var}[\frac{1}{N}\sum_{k=1}^{N}|H_{n,k}|^2]$. Then, $\Omega_H^2(N)$ can be expanded as

$$\begin{aligned}\Omega_H^2(N) &= \frac{1}{N^2}\left\{\sum_{k=1}^{N}\mathrm{var}[|H_{n,k}|^2] + \sum_{r=1}^{N-1}2(N-r)\mathrm{cov}[|H_{n,1}|^2, |H_{n,1+r}|^2]\right\}\\ &\le \frac{1}{N}\mathrm{var}[|H_{n,k}|^2] + \frac{2}{N}\sum_{r=1}^{N-1}\mathrm{cov}[|H_{n,1}|^2, |H_{n,1+r}|^2].\end{aligned} \tag{23}$$

From (7), the covariance between $|H_{n,1}|^2$ and $|H_{n,1+r}|^2$ vanishes as $r \to \infty$ with order r^{-2}. Hence, the right-hand side of (23) converges to zero as $N \to \infty$, and we have $\Omega_H^2(N) \to 0$ as $N \to \infty$. Since $\mathbb{E}[\frac{1}{N}\sum_{k=1}^{N}|H_{n,k}|^2] = 1$ for all N, and $\mathrm{var}[\frac{1}{N}\sum_{k=1}^{N}|H_{n,k}|^2] \to 0$, it follows that

$$\frac{1}{N}\sum_{k=1}^{N}|H_{n,k}|^2 \xrightarrow{ms} 1 \tag{24}$$

where "$\xrightarrow{ms}$" denotes mean-square convergence. This mean-square convergence then implies $\frac{1}{N}\sum_{k=1}^{N}|H_{n,k}|^2 \xrightarrow{P} 1$, as required.

We use this lemma to write the asymptotic capacity C_∞ of a power-limited OFDM system, with $E_0 = \frac{E_N}{N}$, as follows.

Lemma 2: As N approaches infinity, the capacity of an infinite-bandwidth, power-limited OFDM system converges, in probability, to the constant

$$C_\infty = \frac{\Delta f E_N}{N_0 \ln 2}\mathbb{E}[|H_{n,k}|^2]. \tag{25}$$

That is, the limiting capacity is dependent only on the SNR, subchannel separation and mean channel gain. Since we have set $\mathbb{E}[|H_{n,k}|^2] = 1$, we may then write

$$C_\infty = \frac{\Delta f E_N}{N_0 \ln 2}. \tag{26}$$

Proof: When $E_0 = \frac{E_N}{N}$, we may write the instantaneous capacity as

$$C_n = \sum_{k=1}^{N}\frac{\Delta f}{\ln 2}\ln\left(1 + \frac{E_N}{NN_0}|H_{n,k}|^2\right). \tag{27}$$

From [17] we may write

$$z - \frac{z^2}{1+z} \le \ln(1+z) \le z \tag{28}$$

for $z > -1$. Then, using (27) and (28) we may write

$$\frac{\Delta f}{\ln 2}\sum_{k=1}^{N}\left\{\frac{E_N}{NN_0}|H_{n,k}|^2 - \frac{\left(\frac{E_N}{NN_0}|H_{n,k}|^2\right)^2}{1+\frac{E_N}{NN_0}|H_{n,k}|^2}\right\} \le C_n \le \frac{\Delta f}{\ln 2}\sum_{k=1}^{N}\frac{E_N}{NN_0}|H_{n,k}|^2. \tag{29}$$

We now show that the above lower and upper bounds converge in probability to the same limit. Consider the lower bound, which we may write as

$$\frac{\Delta f}{\ln 2}\sum_{k=1}^{N}\frac{E_N}{NN_0}|H_{n,k}|^2 - \frac{\Delta f}{\ln 2}\sum_{k=1}^{N}\frac{\left(\frac{E_N}{NN_0}|H_{n,k}|^2\right)^2}{1+\frac{E_N}{NN_0}|H_{n,k}|^2}. \tag{30}$$

The second term in (30) satisfies

$$\frac{\Delta f}{\ln 2}\sum_{k=1}^{N}\frac{\left(\frac{E_N}{NN_0}|H_{n,k}|^2\right)^2}{1+\frac{E_N}{NN_0}|H_{n,k}|^2} \le \frac{\Delta f}{\ln 2}\frac{E_N^2}{N_0^2}\sum_{k=1}^{N}\frac{|H_{n,k}|^4}{N^2}. \tag{31}$$

The random variables $|H_{n,k}|^2, k = 1, \ldots, N$, are marginally exponentially distributed, and thus nonnegative with finite second moments. Hence, as $N \to \infty$, we may write

$$\mathbb{E}\left[\frac{1}{N^2}\sum_{k=1}^{N}|H_{n,k}|^4\right] \to 0 \tag{32}$$

and

$$\begin{aligned}\mathrm{var}\left[\frac{1}{N^2}\sum_{k=1}^{N}|H_{n,k}|^4\right] &= \frac{1}{N^4}\mathrm{var}\left[\sum_{k=1}^{N}|H_{n,k}|^4\right]\\ &\le \frac{1}{N^4}N^2\mathrm{var}|H_{n,k}|^4]\\ &\to 0.\end{aligned} \tag{33}$$

The properties in (32) and (33) imply that $\frac{1}{N^2}\sum_{k=1}^{N}|H_{n,k}|^4 \xrightarrow{ms} 0$, so that we may then write

$$\frac{1}{N^2}\sum_{k=1}^{N}|H_{n,k}|^4 \xrightarrow{P} 0. \tag{34}$$

Thus, the right-hand side of (31) converges in mean square to zero as $N \to \infty$, so that the expression in (30) converges in mean square to the first term only, as $N \to \infty$. Therefore, both the upper and lower bounds in (29) converge to the same limit. From Lemma 1, we also have $\frac{1}{N}\sum_{k=1}^{N}|H_{n,k}|^2 \xrightarrow{P} 1$, and we substitute this into (29) to write

$$C_n \xrightarrow{P} C_\infty = \frac{\Delta f}{\ln 2}\frac{E_N}{N_0}\mathbb{E}[|H_{n,k}|^2]. \tag{35}$$

We have thus verified that OFDM systems can achieve the fading wideband channel spectral efficiency $\frac{C_\infty}{\Delta f}$ derived by [2]. Moreover, $\frac{C_\infty}{\Delta f}$ is equal to the spectral efficiency of an unlimited bandwidth system transmitting over a flat Rayleigh-fading channel [18], or the infinite bandwidth additive white Gaussian noise (AWGN) system [21].

IV. Simulations

We simulate the normalized capacity $\frac{C_n}{\Delta f}$ of two example systems, and compare the observed instantaneous capacity distributions with approximating distributions calculated from (19) and (20). System A is a 1024 subcarrier system and system B is a 32768 subcarrier system. Both systems have subcarrier separation $\Delta f = 0.3125$ MHz, and SNR $\frac{E_N}{N_0} = 30$ dB. Systems A and B thus occupy bandwidths of 320 MHz and 10.24 GHz, respectively. We assume an exponential power delay profile with mean delay of 50 ns, and a receiver velocity of 100 km/h.

In Fig. 1 we plot the analytical approximating distributions and simulated instantaneous capacity distributions for the fading channel response during transmission of 500 000 blocks. Observe that we obtain

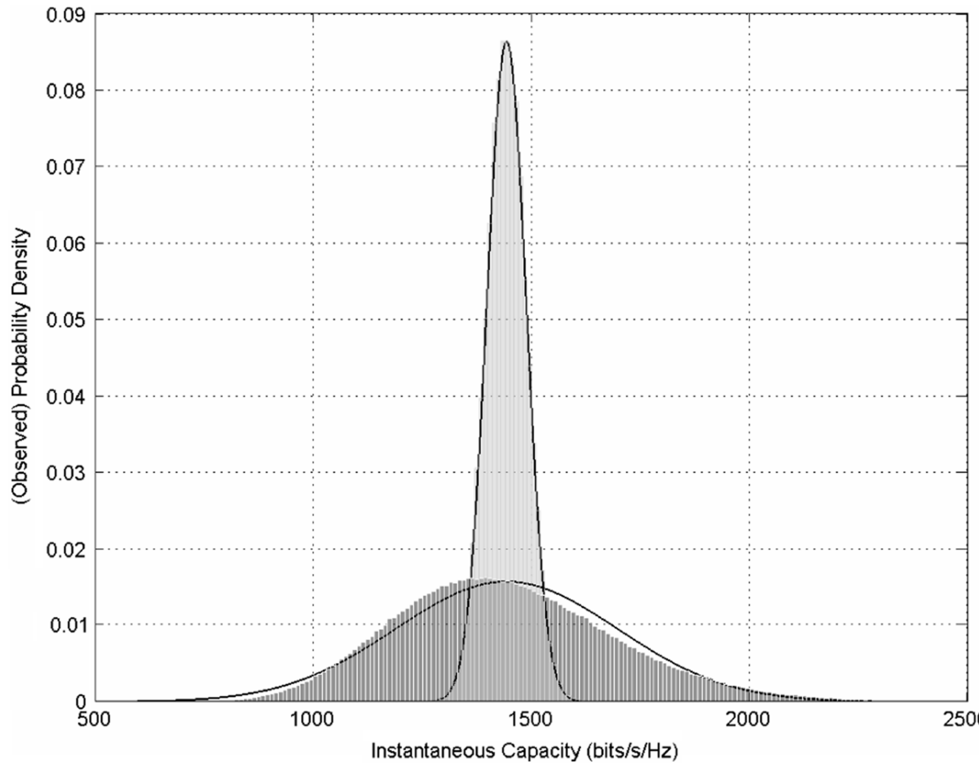

Fig. 1. Simulated (bars) and theoretical (solid line) distributions of instantaneous capacity, normalized by Δf, for 1024 subcarrier system (larger variance) and 32768 subcarrier system (smaller variance). Both systems have $\Delta f = 0.3125$ MHz and SNR $\frac{E_N}{N_0} = 30$ dB.

a reasonable analytical approximation for System A, and a closer approximation for the system with more subcarriers, as expected. Furthermore, the variance of the capacity of System B is much smaller than that of System A, consistent with Lemma 2.

V. Concluding Remarks

Using a central limit theorem, we have shown that the instantaneous capacity of an OFDM system transmitting a large, finite number of subcarriers over a Rayleigh-fading channel is well approximated by a Gaussian distribution. The mean and variance of the Gaussian distribution are readily calculable. The theoretical distributions closely approximate simulated results for 1024- and 32768-subcarrier systems. As the number of subcarriers approaches infinity, with no bandwidth constraint, the capacity approaches the well-known capacity of an infinite-bandwidth frequency-selective fading channel [4] or infinite-bandwidth flat-fading channel [18]. We then conclude that power-limited, infinite-bandwidth systems employing OFDM can achieve the capacity of Jakes' model frequency-selective Rayleigh-fading channels.

Appendix I
Hermite Rank of $\mathrm{c}(\cdot)$

The Hermite rank $\varphi(f)$ of a measurable function $f : \boldsymbol{X} \to \mathbb{R}$ for the zero mean Gaussian vector $\boldsymbol{X} = \{X_1, \ldots, X_d\} \in \mathbb{R}^d$, where f has finite second moment, is defined as

$$\varphi(f) = \inf \Bigg\{ \tau \colon \exists l_j \quad \text{with } \sum_{j=1}^{d} l_j = \tau \quad \text{and } \mathbb{E}\left[(f(\boldsymbol{X}) - \mathbb{E}[f(\boldsymbol{X})]) \prod_{j=1}^{d} H_{l_j}(X_1)) \right] \neq 0 \Bigg\} \tag{36}$$

where H_{l_j} is the (l_j)th-order Hermite polynomial [14]. Equivalently [19], we may define $\varphi(f)$ as

$$\inf \Bigg\{ \varphi(f) : \exists \text{ polynomial } P \text{ of degree } \varphi(f) \text{ with } \mathbb{E}\left[(f(\boldsymbol{X}) - \mathbb{E}[f(\boldsymbol{X})]) \cdot P(X_1, \ldots, X_d) \right] \neq 0 \Bigg\}. \tag{37}$$

We show that the Hermite rank $\varphi(\mathrm{c})$ of $\mathrm{c}(X_1, X_2) = B \ln(1 + A[X_1^2 + X_2^2])$, for constants $A, B \in \mathbb{R}$, is at least two by showing that it is neither zero nor unity. Consider first a zero-order polynomial in $P_1(X_1, X_2) = \alpha_0$, then

$$\begin{aligned} &\mathbb{E}[(\mathrm{c}(X_1, X_2) - \mathbb{E}[\mathrm{c}(X_1, X_2))P_1(X_1, X_2)] \\ &= \alpha_0 \mathbb{E}[\mathrm{c}(X_1, X_2)] - \alpha_0 \mathbb{E}[\mathrm{c}(X_1, X_2)] \\ &= 0, \text{for all } \alpha_0, \end{aligned} \tag{38}$$

thus, $\varphi(\mathrm{c}) \neq 0$. Now consider a first-order polynomial, $P_2(X_1, X_2) = \alpha_2 X_1 + \alpha_1 X_2 + \alpha_0$, then

$$\begin{aligned} &\mathbb{E}[(\mathrm{c}(X_1, X_2) - \mathbb{E}[\mathrm{c}(X_1, X_2)])P_2(X_1, X_2)] \\ &= \mathbb{E}[(\mathrm{c}(X_1, X_2) - \mathbb{E}[\mathrm{c}(X_1, X_2)])(\alpha_2 X_1 + \alpha_1 X_2 + \alpha_0)] \\ &= \alpha_2 B \, \mathbb{E}\left[X_1 \ln\left(1 + A\left[X_1^2 + X_2^2\right]\right)\right] \\ &\quad + \alpha_1 \, \mathbb{E}\left[X_2 \ln\left(1 + A\left[X_1^2 + X_2^2\right]\right)\right] \end{aligned} \tag{39}$$

since $\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$. Since X_1 and X_2 are i.i.d. Gaussian random variables, with mean zero and variance σ^2, we may write

$$\begin{aligned} &\mathbb{E}\left[X_1 \ln\left(1 + A\left[X_1^2 + X_2^2\right]\right)\right] \\ &= \mathbb{E}\left[X_2 \ln\left(1 + A\left[X_1^2 + X_2^2\right]\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_2 \ln\left(1 + A\left[X_1^2 + X_2^2\right]\right) \\
&\quad \times \exp\left(-\frac{X_1^2+X_2^2}{2\sigma^2}\right) dX_2\, dX_1 \\
&= 0
\end{aligned} \tag{40}
$$

since the integrand is the product of an odd function and two even functions in X_2. Thus, $\varphi(\mathrm{c}) \neq 1$, and it follows that $\varphi(\mathrm{c}) \geq 2$.

Appendix II
Capacity Correlation

The mean-squared capacity $\mathbb{E}[C_{n,k}^2]$ may be expressed as

$$
\mathbb{E}[C_{n,k}^2] = \left(\frac{\Delta f}{\ln 2}\right)^2 \int_0^{\infty} \left[\ln\left(1+\gamma_0 y\right)\right]^2 \exp(-y) dy \tag{41}
$$

which is readily numerically evaluated. We may write the correlation between the capacity of subchannels k_1 and k_2 in time interval n as

$$
\begin{aligned}
\mathbb{E}[C_{n,k_1}C_{n,k_2}] = \left(\frac{\Delta f}{\ln 2}\right)^2 \int_0^{\infty}\int_0^{\infty} \ln\left(1+\gamma_0 x\right) \\
\times \ln(1+\gamma_0 y) f_{H_1,H_2}(x,y)\, dx dy
\end{aligned} \tag{42}
$$

where $f_{H_1,H_2}(x,y)$ is the joint pdf of two correlated exponential random variables. Specifically [13]

$$
f_{H_1,H_2}(x,y) = \kappa \exp(-\alpha x - \alpha y)\, \mathrm{I}_0(\theta\sqrt{xy}) \tag{43}
$$

where

$$
\begin{aligned}
\alpha = \kappa &= \frac{1}{1-\left[\rho\left(|H_{n,k_1}|^2, |H_{n,k_2}|^2\right)\right]^2} \\
\theta &= \frac{\rho\left(|H_{n,k_1}|^2, |H_{n,k_2}|^2\right)}{1-\left[\rho\left(|H_{n,k_1}|^2, |H_{n,k_2}|^2\right)\right]^2}
\end{aligned} \tag{44}
$$

and $\mathrm{I}_0(\cdot)$ is the modified zero-order Bessel function of the first kind [17]. Substituting (43) into (42) we obtain

$$
\begin{aligned}
\mathbb{E}\left[C_{n,k_1}C_{n,k_2}\right] &= \kappa \int_0^{\infty}\int_0^{\infty} \left(\frac{\Delta f}{\ln 2}\right)^2 \ln(1+\gamma_0 x)\ln(1+\gamma_0 y) \\
&\quad \times \exp(-\alpha(x+y))\, \mathrm{I}_0(\theta\sqrt{xy})\, dx\, dy \\
&= \kappa \left(\frac{\Delta f}{\ln 2}\right)^2 \sum_{i=0}^{\infty} \left\{ \frac{\theta^{2i}}{4^i (i!)^2} \right. \\
&\quad \left. \times \left[\int_0^{\infty} x^i \ln(1+\gamma_0 x)\exp(-\alpha x)\, dx\right]^2 \right\}
\end{aligned} \tag{45}
$$

using the series expansion [17] for $\mathrm{I}_0(\cdot)$. Consider the integral in the above expression. After substituting $u = 1+\gamma_0 x$, using the binomial expansion of $(u-1)^i$ and after some manipulation we obtain

$$
\begin{aligned}
\int_0^{\infty} x^i \ln(1+\gamma_0 x) e^{-\alpha x}\, dx &= \frac{1}{\gamma_0^{i+1}} \exp\left(\frac{\alpha}{\gamma_0}\right) \sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} \\
&\quad \times \int_1^{\infty} \ln(u)\, u^r \exp\left(-\frac{\alpha\, u}{\gamma_0}\right) du.
\end{aligned} \tag{46}
$$

We then integrate by parts and use a result from [16, Sec 3.381] to write

$$
\begin{aligned}
&\int_1^{\infty} \ln(u) u^r \exp\left(-\frac{\alpha u}{\gamma_0}\right) du \\
&= \int_1^{\infty} \exp\left(-\frac{\alpha\, u}{\gamma_0}\right) \left[\frac{\gamma_0\, u^{r-1}}{\alpha} + \sum_{k=1}^{r} r(r-1) \right. \\
&\quad \left. \times\ (r-2)\ldots(r-k+1)\left(\frac{\gamma_0}{\alpha}\right)^{k+1} u^{r-k-1}\right] du \\
&= \left(\frac{\gamma_0}{\alpha}\right)^{r+1} \left[\Gamma\left(r, \alpha\gamma_0^{-1}\right) + \sum_{k=1}^{r} \frac{r!}{(r-k)!}\Gamma\left(r-k, \alpha\gamma_0^{-1}\right)\right]
\end{aligned} \tag{47}
$$

where $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function [16]. We may substitute (47) and (46) into (45) to then write

$$
\begin{aligned}
\mathbb{E}[C_{n,k_1}C_{n,k_2}] &= \kappa \left(\frac{\Delta f}{\ln 2}\right)^2 \exp\left(\frac{\alpha}{\gamma_0}\right) \sum_{i=0}^{\infty} \frac{\theta^{2i}}{4^i\, i!} \\
&\quad \times \left\{ \sum_{r=0}^{i} \frac{1}{(i-r)!} \frac{(-\gamma_0)^{r-i}}{\alpha^{r+1}} \right. \\
&\quad \left. \times \left[\frac{1}{r!}\Gamma\left(r, \alpha\gamma_0^{-1}\right) + \sum_{k=1}^{r} \frac{\Gamma\left(r-k, \alpha\gamma_0^{-1}\right)}{(r-k)!}\right] \right\}^2
\end{aligned} \tag{48}
$$

for $k_1 \neq k_2$. This series representation is rapidly convergent, and may be used to numerically calculate the variance of the instantaneous capacity distribution.

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Trellis Complexity of Short Linear Codes

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***Abstract*—An extended table of Shuurman's bounds on the state complexity of short binary linear codes is presented. Some new lower and upper bounds are obtained. Most of the newly found codes are based on the so-called double zero-tail termination (DZT) construction.**

***Index Terms*—Block codes, double zero-tail terminated convolutional codes (DZT-codes), minimal span form, minimal trellis, trellis complexity.**

I. Introduction

A trellis representation of linear block codes may be used in order to organize efficient trellis-based soft decoding. The Viterbi decoding algorithm is one example of such decoding procedures. Its computational complexity is upper-bounded by the state complexity of the code. BEAST [1], [2] is a soft-decoding algorithm which is more efficient than the Viterbi algorithm. BEAST exploits a tree structure of the code but as shown in [1] its decoding complexity also depends on the code trellis complexity.

The trellis structure of block codes has been intensively studied, see for example [3]–[9], etc. The best known asymptotic lower bounds on the state complexity are given in [8], the asymptotic upper bounds are presented in [10]. A detailed analysis of short codes (of length $n \le 24$) has been performed by Schuurman in [11].

Short block codes with soft-decision decoding can be efficiently used in concatenated constructions [12]. In this case, they provide coding gains close to those of turbo schemes but with reduced complexity and coding delay. Therefore, it is important to find low-complexity trellises for a wider range of code parameters than in [11].

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II. Definitions and Problem Statement

Denote by (n, k, d) a binary linear block code of length n with dimension k and minimum distance d. Let

$$G = \begin{pmatrix} \boldsymbol{g}_1 \\ \cdots \\ \boldsymbol{g}_k \end{pmatrix} = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{k1} & \cdots & g_{kn} \end{pmatrix} \tag{1}$$

be its generator matrix. Denote by

$$d_{\mathrm{H}}(\boldsymbol{x}, \mathcal{C}) = \min_{\boldsymbol{c} \in \mathcal{C}} \{d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{c})\}$$

the Hamming distance between any binary vector $\boldsymbol{x}$ and the code $\mathcal{C}$, then

$$d_{\mathrm{H}}(\mathbf{0}, \mathcal{C} \setminus \{\mathbf{0}\}) = d$$

and let ρ be the covering radius of code $\mathcal{C}$, that is,

$$\rho = \max_{\boldsymbol{x} \in \{0,1\}^n} \{d_{\mathrm{H}}(\boldsymbol{x}, \mathcal{C})\}.$$

The trellis diagram, state complexity profile, and state complexity of the corresponding code can be easily obtained from G.

For a binary n-tuple $\boldsymbol{x} = (x_1, \dots, x_n) \in \{0,1\}^n$, we denote by $\mathrm{start}(\boldsymbol{x})$ and $\mathrm{end}(\boldsymbol{x})$ the first and the last nonzero position, respectively.

We say that G is in its minimal span form [13] if

$$\mathrm{start}(\boldsymbol{g}_i) \ne \mathrm{start}(\boldsymbol{g}_j), \quad i \ne j$$

and

$$\mathrm{end}(\boldsymbol{g}_j) \ne \mathrm{end}(\boldsymbol{g}_j), \quad i \ne j.$$

When G is in minimal span form, at most two branches diverge and at most two branches remerge at each node in the trellis.

All positions in the codeword $\boldsymbol{x}$ which belong to $[\mathrm{start}(\boldsymbol{x}), \mathrm{end}(\boldsymbol{x})]$ are called *nontrivial*.

A row $\boldsymbol{g}$ is called *active* in the ith position if $i \in [\mathrm{start}(\boldsymbol{g}), \mathrm{end}(\boldsymbol{g}))$.

The *state complexity* at the ith position s_i coincides with the number of rows which are active at the ith level of trellis diagram [10], and

$$s = \max_{i=1,\dots,n} \{s_i\} \tag{2}$$

is the state complexity of the code. A generator matrix in minimal span form yields a trellis of minimal state complexity.

Example 1: Consider the $(6, 3, 3)$-code with generator matrix

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The start positions of the three rows are $1, 2,$ and 3 and the end positions are $4, 6,$ and 5. The matrix G is in minimal span form and it has the state complexity profile $(1, 2, 3, 2, 1, 0)$. The state complexity is equal to 3. The trellis diagram for the minimal span form is shown in Fig. 1.

Notice that the trellis diagram is a direct product of elementary trellises corresponding to the rows of the generator matrix [9].

Next we introduce the notion of the sectionalized complexity. Split the set of indices $\{1, \dots, n\}$ into L nonintersecting subsets of consecutive indices: $I_1 = \{1, \dots, c_1\}$, $I_2 = \{c_1 + 1, \dots, c_1 + c_2\}, \dots,$ $I_L = \{c_1 + \cdots + c_{L-1} + 1, \dots, c_1 + \cdots + c_L\}$, such that, $\cup_{i=1}^{L} I_i = \{1, \dots, n\}$, $\sum_{i=1}^{L} c_i = n$, and the rows can start only in the first position and end only in the last position of I_i. We refer to these subsets as *sections* and to c_i as *section lengths* or *branch lengths*.