The known finite Minkowski planes — a characterization in terms of Klein–Kroll types

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What is a Minkowski plane?

A \((B^*)\)-geometry or hyperbola structure \(\mathcal{M} = (P, C, \mathcal{G}_1 \cup \mathcal{G}_2)\) is an incidence structure consisting of a point set \(P\), a circle set \(C\), elements of which are subsets of \(P\) with at least three points, and two different partitions \(\mathcal{G}_1\) and \(\mathcal{G}_2\) of \(P\), whose members are called generators of \(\mathcal{M}\), such that the following three axioms are satisfied:

\((G)\) Each generator in \(\mathcal{G}_1\) intersects each generator in \(\mathcal{G}_2\) in a unique point.

\((C)\) Each circle intersects each generator in precisely one point.

\((J)\) Three points no two of which are on the same generator are joined by a unique circle.

A Minkowski plane is a \((B^*)\)-geometry that also satisfies the axiom

\((T)\) The circles which touch a fixed circle \(C\) at \(p \in C\) partition \(P \setminus \{p\}\).
Models of Minkowski planes

The *miquelian Minkowski plane* over a field $\mathbb{F}$ is obtained as the geometry of non-trivial plane sections of a ruled quadric in 3-dimensional projective space over $\mathbb{F}$.
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An equivalent description

A \((B^*)\)-geometry corresponds to a sharply 3-transitive set \(\Sigma\) of permutations on a generator \(G\); circles are the graphs of permutations in \(\Sigma\).

\[
\begin{array}{c}
C_0 \\
\downarrow \\
\bullet p \\
\downarrow \\
G \\
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The miquelian Minkowski plane over the field \(\mathbb{F}\) corresponds to the group \(\text{PGL}(2,\mathbb{F})\) of linear fractional maps acting on \(\mathbb{F} \cup \{\infty\}\).
The known finite Minkowski planes

All known finite Minkowski planes are of the form $\mathcal{M}(q, \alpha)$ obtained from sharply 3-transitive sets

$$G(q, \alpha) = \text{PSL}(2, q) \cup (\text{PGL}(2, q) \setminus \text{PSL}(2, q))\alpha$$

where $q$ is a prime power and $\alpha$ is an automorphism of GF$(q)$.

Circles are the graphs of permutations in $G(q, \alpha)$ on GF$(q) \cup \{\infty\}$.

- The miquelian Minkowski planes are obtained when $\alpha = \text{id}$.
- $G(q, \alpha)$ is a group if and only if $\alpha$ has order at most 2.
- A finite hyperlola structure is a Minkowski plane.

A finite Minkowski plane has order $n$ if each generator and circle has precisely $n + 1$ points.
Derived incidence structures and consequences

The derived incidence structure $\mathcal{M}_p$ at a point $p$ of a Minkowski plane $\mathcal{M}$ is an affine plane.
A circle $C$ not passing through $p$ induces an oval in the projective extension of $\mathcal{M}_p$ by removing the points $C \cap [p]$ and adding the points at infinity of lines that come from generators of $\mathcal{M}$.

**Theorem**

- A finite Minkowski plane of even order is miquelian. (Heise 1974)
- A finite Minkowski plane of odd order with a Desarguesian derivation is miquelian. (Chen, Kaerlein 1973, Payne, Thas 1976)
- A finite Minkowski plane of order at most 8 is miquelian.
- There are precisely two finite Minkowski planes of order 9, up to isomorphism. These planes correspond to the two sharply 3-transitive groups of degree 10. (S. 1992)
G-translations

- An automorphism of a Minkowski plane $\mathcal{M}$ is a permutation of the point set such that generators are mapped to generators and circles are mapped to circles.

- A $G$-translation of $\mathcal{M}$ is an automorphism of $\mathcal{M}$ that either fixes precisely the points of the generator $G$ or is the identity; it induces a translation in the derived affine plane of $\mathcal{M}$ at any point of $G$.

- A group $\Gamma$ of automorphisms of $\mathcal{M}$ is said to be $G$-transitive if $\Gamma$ contains a subgroup of $G$-translations that acts transitively on each circle minus its point of intersection with $G$. 
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M. Klein and H.-J. Kroll [1989] considered the set $\mathcal{E}(\Gamma)$ of all generators $G$ for which a group $\Gamma$ of automorphisms of $\mathcal{M}$ is G-transitive. They found six types for $\Gamma$, labelled A to F.
The six Klein–Kroll types w.r.t. $G$-translations

**Theorem (Klein, Kroll, 1989)**

If $Z = Z(\Gamma)$ denotes the set of all generators $G$ for which a group $\Gamma$ of automorphisms of a hyperbola structure is $G$-transitive, then exactly one of the following statements is valid for $Z$:

A. $Z = \emptyset$;
B. $|Z| = 1$;
C. $Z = \{[p]_1, [p]_2\}$ for some point $p$;
D. $Z = G_1$ or $Z = G_2$;
E. $Z = G_1 \cup \{G_2\}$ or $Z = G_2 \cup \{G_1\}$ where $G_i \in G_i$;
F. $Z = G_1 \cup G_2$.

There are examples of groups of automorphisms of miquelian Minkowski planes for each of the six types.
The Klein–Kroll type of a Minkowski plane

The type of a Minkowski plane $\mathcal{M}$ is the type of the (full) automorphism group of $\mathcal{M}$.

**Question:** Which types do occur as the type of a (finite) Minkowski plane?

The planes $\mathcal{M}(q, \alpha)$ are of type F. Each map

$$(x, y) \mapsto (\gamma_1(x), \gamma_2(y))$$

where $\gamma_1, \gamma_2 \in \text{PSL}(2, q)$ is an automorphism of $\mathcal{M}(q, \alpha)$. Those automorphisms with $\gamma_i = \text{id}$ and $\gamma_{3-i}$ fixing precisely one point are $\mathcal{G}$-translations of $\mathcal{M}(q, \alpha)$. 

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A characterization of the known finite Minkowski planes
Type at least D and 2-transitive groups

Lemma

Let $\mathcal{M}$ be a Minkowski plane whose automorphism group is $G$-transitive for each $G \in \mathcal{G}_1$. Then the group generated by all $G$-translations for $G \in \mathcal{G}_1$ acts 2-transitively on $\mathcal{G}_1$ and trivially on $\mathcal{G}_2$. Furthermore, the stabilizer of three points no two of which are on the same generator in $\mathcal{G}_1$ is trivial.

Theorem (Feit 1960, Ito, Suzuki 1962)

If $\Pi$ is a 2-transitive permutation group of even degree $n + 1$ such that only the identity fixes more than two points, then one of the following occurs:

1. $\Pi$ is sharply 2-transitive (and isomorphic to the group of all permutations $x \mapsto xa + b$, where $a \neq 0$, of a nearfield of order $n + 1$).
2. $\Pi \cong \Gamma L(1, n + 1)$ where $n = 2^q - 1$ and $q$ is a prime.
3. $\Pi$ contains $PSL(2, n)$ as a normal subgroup of index at most 2.
Type E

Theorem
Let $\mathcal{M}$ be a finite Minkowski plane whose automorphism group is $G$-transitive for each $G \in \mathcal{G}_i$. If the group $\Delta$ generated by all $G$-translations for $G \in \mathcal{G}_i$ is non-solvable, then the order of $\mathcal{M}$ is a prime power $q$ and $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(q, \alpha)$.

Theorem
Let $\mathcal{M}$ be a finite Minkowski plane whose automorphism group contains a group of type E. Then the order of $\mathcal{M}$ is a prime power $q$ and $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(q, \alpha)$.

Corollary
There is no finite Minkowski plane of type E.

There are infinite Minkowski planes of types A, B, C, D and F.
The characterization and a conjecture

**Theorem**

*The Minkowski planes \( M(q, \alpha) \) are precisely the finite Minkowski planes of Klein–Kroll type at least \( E \).*

**Conjecture**

*There is no finite Minkowski plane of type \( D \).*

The conjecture will follow if the Prime Power Conjecture for finite projective planes and the longstanding conjecture that a projective plane of prime order is desarguesian are both true.