Three-dimensional connected groups of automorphisms of toroidal circle planes

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Abstract

We contribute to the classification of toroidal circle planes and flat Minkowski planes possessing three-dimensional connected groups of automorphisms. When such a group is an almost simple Lie group, we show that it is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Using this result, we describe a framework for the full classification based on the action of the group on the point set.

1 Introduction

In incidence geometry, Minkowski planes are one of the three types of Benz planes, the other two are Möbius (inversive) and Laguerre planes. There are different ways to consider Minkowski planes. They appear in the context of finite geometries, with applications in coding theory, cf. [9]. In the work of Benz [2], Minkowski planes were studied from an algebraic point of view. Perhaps this and the influence of Salzmanns's work on $\mathbb{R}^2$-planes (cf. [14]) leads to the development of Minkowski planes in a topological setting.

Under suitable topological assumptions, a Minkowski plane defined over the reals is called a flat Minkowski plane (cf. Subsection 2.1). In contrast to the scarcity of known models defined over other fields, there are many examples of flat Minkowski planes, cf. [13] or [8] and references therein. The classical model of a flat Minkowski plane is the geometry of plane sections of the standard nondegenerate ruled quadric in real 3-dimensional projective space. An algebraic description of the classical flat Minkowski plane can be found in [13, Subsection 4.1.5].

Schenkel [16] showed that the automorphism group of a flat Minkowski plane is a Lie group with respect to the compact-open topology and has dimension at most 6. One way to investigate flat Minkowski planes is to describe all possible planes with automorphism groups of a given dimension n. For convenience, we call this number n the *group dimension* of the plane. It is known that planes of group dimension at least 5 are isomorphic to the classical real Minkowski plane, which has group dimension 6. Also in [16], Schenkel determined planes of group dimension 4 or with 3-dimensional kernels. In case of flat Minkowski planes of group dimension 3, despite many examples, a full classification for this dimension is not complete.

Toroidal circle planes are a generalisation of flat Minkowski planes in the sense that they are required to satisfy all but one geometric axiom for flat Minkowski planes. For precise definitions of these incidence structures and related terminology, we refer the reader to Subsection 2.1. So far, the only examples of toroidal circle planes that are not flat Minkowski planes are those constructed by Polster [12]. In [4], we showed that the automorphism group of a toroidal circle plane is also a Lie group with dimension at most 6. This allows us to extend the ongoing classification of flat Minkowski planes to the more general toroidal circle planes.

Also in the same paper above, we showed that toroidal circle planes of group dimension at least 4 or with 3-dimensional kernels are precisely the flat Minkowski planes that Schenkel described. As it can be applied for any group dimension, the machinery used by Schenkel plays an important role in our framework. The action of a group of automorphisms G is dictated by its action on each factor $\mathbb{S}^1$ of the torus. Groups acting on 1-manifolds are known by Brouwer's Theorem (cf. Theorem 2.1), and when the group G is large enough, its structure can be fully described. This is how Schenkel's result (for flat Minkowski planes) and [4, Theorem 1.2] (for toroidal circle planes) were obtained.

Concerning the next open case of toroidal circle planes with 3-dimensional groups of automorphisms, we face the following two objectives:

(A) Narrow down the possibilities for a 3-dimensional connected group of automorphisms of a toroidal circle plane.

(B) Given a group G that can possibly occur as in (A), determine the existence and characterise toroidal circle planes admitting G as its group of automorphisms.

In this paper, we present results in the direction of (A) thereby laying the groundwork to address (B) systematically in the future. The paper is organized as follows. From Brouwer's Theorem, we first obtain an initial list of possibilities for a 3-dimensional group. Next, in the special case when such a group is an almost simple Lie group, we show that it is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. We then determine how $\mathrm{PSL}(2,\mathbb{R})$ acts as a group of automorphisms. This result on almost simple Lie groups is independent from the group dimension, and we state it as our first main theorem.

Theorem 1.1. *Let S be an almost simple connected group of automorphisms of a toroidal circle plane $\mathbb{T}$. Then $S \cong \mathrm{PSL}(2,\mathbb{R})$. Furthermore, exactly one of the following occurs.*

1. *S fixes either every $(+)$-parallel class or every $(-)$-parallel class. In both cases, $\mathbb{T}$ is isomorphic to a half-classical Minkowski plane $\mathcal{M}(f, id)$ (described below), where f is an orientation-preserving homeomorphism of $\mathbb{S}^1$.*
2. *S acts diagonally on the point set. The diagonal D is a circle, and S fixes D.*

Half-classical Minkowski planes can be described briefly as follows. Let f and g be two orientation-preserving homeomorphisms of $\mathbb{S}^1$. The circle set $\mathcal{C}(f,g)$ of a *half-classical Minkowski plane* $\mathcal{M}(f,g)$ consists of sets of the form

$$\{(x, \gamma(x)) \mid x \in \mathbb{S}^1\},$$

where $\gamma \in \mathrm{PSL}(2,\mathbb{R}) \cup g^{-1}(\mathrm{PGL}(2,\mathbb{R})\backslash\mathrm{PSL}(2,\mathbb{R}))f$. For a reference, cf. [13, p. 239].

With the aid of Theorem 1.1 for almost simple groups and previous work in the literature for other cases, we determine all possible geometric configurations a 3-dimensional group can fix. We denote the connected component of the affine group $\mathrm{AGL}_1(\mathbb{R})$ by

$$\mathrm{L}_2 = \{x \mapsto ax + b \mid a, b \in \mathbb{R}, a > 0\}.$$

We also define the following subgroups of $\mathrm{AGL}_2(\mathbb{R})$:

$$\Phi_\infty := \{(x, y) \mapsto (x + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\},$$

and

$$\Phi_d := \{(x, y) \mapsto (ax + b, a^d y + c) \mid a, b, c \in \mathbb{R}, a > 0\},$$

for $d \in \mathbb{R}$. In the second half of the paper, we prove the following.

Theorem 1.2. *Let Σ be a 3-dimensional connected group of automorphisms of a toroidal circle plane $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$. Let $\Delta^\pm$ be the kernel of the induced action of Σ on $\mathcal{G}^\pm$. Then exactly one of the following occurs.*

1. *Σ fixes no parallel classes but fixes and acts transitively on exactly one circle. In this case $\Sigma \cong \mathrm{PSL}(2, \mathbb{R})$ and it acts diagonally on the torus, under suitable coordinates. The diagonal is the fixed circle under this action.*
2. *Σ fixes no points but fixes and acts transitively on either every $(+)$-parallel class or every $(-)$-parallel class. In this case $\Sigma \cong \mathrm{PSL}(2, \mathbb{R})$ and $\mathbb{T}$ is isomorphic to a half-classical Minkowski plane $\mathcal{M}(f, id)$, where f is an orientation-preserving homeomorphism of $\mathbb{S}^1$.*
3. *Σ fixes no points but fixes and acts transitively on exactly one parallel class π. In this case $\Sigma \cong \mathrm{L}_2 \times \mathrm{SO}(2, \mathbb{R})$. Assume π is a $(+)$-parallel class. Then the factor group Σ/Δ^- is isomorphic and acts equivalently to $\mathrm{SO}(2, \mathbb{R})$ on $\mathcal{G}^-$. Also, the factor group Σ/Δ^+ is isomorphic and acts equivalently to L_2 on $\mathcal{G}^+ \backslash \{\pi\}$. The case when π is a $(-)$-parallel class is analogous.*
4. *Σ fixes exactly two parallel points. The coordinates may be chosen such that the two fixed points are (∞, ∞) and $(0, \infty)$. In this case $\Sigma \cong \Phi_d$, for some $d \leq 0$, and the action of Σ is described by the maps*

   $$\{(x, y) \mapsto (ax, by + c) \mid a, b > 0, c \in \mathbb{R}\},$$

   when $\Sigma \cong \Phi_0$, and

   $$\{(x, y) \mapsto (a\ sgn(x) \cdot |x|^b, b^d y + c) \mid a, b > 0, c \in \mathbb{R}\},$$

   when $\Sigma \cong \Phi_d$, when $d < 0$.
5. *Σ fixes exactly one point. The coordinates may be chosen such that the fixed point is $p = (\infty, \infty)$. In this case the derived plane $\mathbb{T}_p$ is Desarguesian and $\Sigma \cong \Phi_d$, for some $d \in \mathbb{R} \cup \{\infty\}$. The action of Σ is described by the standard action of Φ_d on $\mathbb{R}^2$.*

Remark 1.3. Besides Case 2 which is fully determined, there are examples of flat Minkowski planes of group dimension 3 for Cases 1 and 5. Two families of flat Minkowski planes admitting 3-dimensional groups fixing no points but fixing and acting transitively on a circle were constructed by Steinke [17] and [18]. An Artzy-Groh plane $\mathcal{M}_{AG}(f, g)$ (cf. [1]) admits the group Φ_1 with its standard action. It is currently unknown if there are toroidal circle planes satisfying the conditions in Cases 3 and 4.

2 Preliminaries

2.1 Toroidal circle planes, flat Minkowski planes and derived planes

A *toroidal circle plane* is a geometry $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$, whose

point set $\mathcal{P}$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$, circles (elements of $\mathcal{C}$) are graphs of homeomorphisms of $\mathbb{S}^1$,

(+)-parallel classes (elements of $\mathcal{G}^+$) are the verticals $\{x_0\} \times \mathbb{S}^1$,

(−)-parallel classes (elements of $\mathcal{G}^-$) are the horizontals $\mathbb{S}^1 \times \{y_0\}$,

where $x_0, y_0 \in \mathbb{S}^1$.

We denote the (±)-parallel class containing a point p by $[p]_\pm$. When two points p, q are on the same (±)-parallel class, we say they are *(±)-parallel* and denote this by $p \parallel_\pm q$. Two points p, q are *parallel* if they are (+)-parallel or (−)-parallel, and we denote this by $p \parallel q$.

Furthermore, a toroidal circle plane satisfies the following

Axiom of Joining: three pairwise non-parallel points p, q, r can be joined by a unique circle $\alpha(p, q, r)$.

A toroidal circle plane is called a *flat Minkowski plane* if it also satisfies the following

Axiom of Touching: for each circle C and any two nonparallel points p, q with $p \in C$ and $q \notin C$, there is exactly one circle D that contains both points p, q and intersects C only at the point p.

The *derived plane* $\mathbb{T}_p$ *of* $\mathbb{T}$ *at the point* p is the incidence geometry whose point set is $\mathcal{P} \backslash ([p]_+ \cup [p]_-)$, whose lines are all parallel classes not going through p and all circles of $\mathbb{T}$ going through p. For every point $p \in \mathcal{P}$, the derived plane $\mathbb{T}_p$ is an $\mathbb{R}^2$-plane and even a flat affine plane when $\mathbb{T}$ is a flat Minkowski plane, cf. [13, Theorem 4.2.1].

2.2 The automorphism group

An *automorphism of a toroidal circle plane* $\mathbb{T}$ is a permutation of the point set $\mathcal{P}$ such that parallel classes are mapped to parallel classes and circles are mapped to circles. With respect to composition, the set of all automorphisms of a toroidal circle plane is an abstract group, which we call the *automorphism group of* $\mathbb{T}$, denoted by $\mathrm{Aut}(\mathbb{T})$. Every automorphism of $\mathbb{T}$ is continuous and thus a homeomorphism of the torus, cf. [13, Theorem 4.4.1]. With respect to the compact-open topology, the group $\mathrm{Aut}(\mathbb{T})$ is a Lie group of dimension at most 6, cf. [4, Theorem 1.1].

The automorphism group $\mathrm{Aut}(\mathbb{T})$ has two distinguished normal subgroups, the *kernels* $T^\pm$ consisting of all automorphisms of $\mathbb{T}$ that fix every (±)-parallel class. For convenience, we refer to these two subgroups as the *kernels* $T^\pm$ *of the plane* $\mathbb{T}$. The connected component of $T^\pm$ of a flat Minkowski plane is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$, L_2, $\mathrm{SO}(2, \mathbb{R})$, $\mathbb{R}$, or the trivial group $\{id\}$, cf. [13, Proposition 4.4.9]. The same result holds for toroidal circle planes, since their automorphism groups are Lie groups.

Another fact that we frequently use is that automorphisms fixing three pairwise non-parallel points have order at most 2. In particular, if such an automorphism takes (+)-parallel classes to (+)-parallel classes, then it is the identity map. A proof can be found in [13, Lemma 4.4.2].

2.3 Some results on transformation groups

The following theorem describes all possible transitive and effective actions of transformation groups on 1-manifolds. We refer to this result as Brouwer's Theorem throughout this paper.

In some sources, a weaker version (for Lie groups) is stated, without a name, as a consequence of a result by Sophus Lie on Lie algebras, cf. [10], [5, p.348] and [11, Theorem 2.1, p.218]. According to [15, 96.30], this theorem is proved by Brouwer [3], and a sketch of proof is provided there.

Theorem 2.1 (Brouwer's Theorem)**.** *Let G be a locally compact, connected, effective and transitive transformation group on a connected 1-dimensional manifold M. Then G has dimension at most 3.*

(a) If $M \cong \mathbb{S}^1$, then G is isomorphic and acts equivalently to the rotation group $\mathrm{SO}(2,\mathbb{R})$ or a finite covering group $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ of the projective group $\mathrm{PSL}(2,\mathbb{R})$.

(b) If $M \cong \mathbb{R}$, then G is isomorphic and acts equivalently to $\mathbb{R}$, the connected component L_2 of the affine group of $\mathbb{R}$, or the simply connected covering group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ of $\mathrm{PSL}(2,\mathbb{R})$.

Often, groups of automorphisms are assumed to be connected, so that they map $(\pm)$-parallel classes to $(\pm)$-parallel classes. Consequently, they have induced actions on $\mathcal{G}^{\pm}$, which are homeomorphic to $\mathbb{S}_1$. It is then important to know how these groups act on $\mathbb{S}_1$ and $\mathbb{R}$ (the only connected 1-manifolds). The following is helpful for this purpose.

Lemma 2.2 (cf. [15] 96.29)**.** *Consider a connected group Γ acting effectively on $\mathbb{R}$ or $\mathbb{S}^1$.*

(a) If Γ has no fixed point, then Γ is transitive.

(b) Any non-trivial compact subgroup of Γ acts freely on $\mathbb{S}^1$; it cannot act on $\mathbb{R}$.

(c) If Γ is compact, then $\Gamma = \{id\}$, or $\Gamma \cong \mathrm{SO}(2,\mathbb{R})$ and Γ is sharply transitive on $\mathbb{S}^1$.

Lemma 2.3 (cf. [15] 93.12)**.** *If Δ is a closed subgroup of the locally compact, connected group Γ, and if $\dim \Delta = \dim \Gamma < \infty$, then $\Delta = \Gamma$.*

Proofs of the main theorems rely on arguments with the dimension of orbits and stabilisers, which are based on the following.

Lemma 2.4 (cf. [15] 96.10, [13] Theorem A.2.3.6)**.** *If the Lie group G acts on a manifold M, then*

$$\dim G = \dim G_p + \dim G(p),$$

where G_p and $G(p)$ are the stabiliser and orbit, respectively, of the point $p \in M$.

We will usually refer to Lemma 2.4 as the dimension formula.

3 Proof of Theorem 1.1

Let $\mathbb{T}$ be a toroidal circle plane with automorphism group $\mathrm{Aut}(\mathbb{T})$. We denote the connected component of $\mathrm{Aut}(\mathbb{T})$ by Γ and let S be a non-trivial almost simple connected Lie subgroup of Γ. Let $K^{\pm}$ be the kernel of the action of S on $\mathcal{G}^{\pm}$.

The overall structure of the proof of Theorem 1.1 is as follows. We first show in Lemma 3.2 that S is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. This implies S is isomorphic to either the universal covering group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ or a finite covering group $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$. In particular, the centre $Z(S)$ is cyclic. Next, we prove that S is in fact isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ by showing that the centre $Z(S)$ is trivial in Lemma 3.4. Finally, in Lemma 3.5, we apply Brouwer's Theorem (Theorem 2.1) to determine possible actions of S on the torus.

We start with the following observation.

Lemma 3.1. *If S contains a subgroup H isomorphic to $\mathrm{SO}(2,\mathbb{R})$, then S acts transitively on at least one of $\mathcal{G}^{\pm}$.*

Proof. Since $T^+ \cap T^- = \{id\}$, the subgroup H cannot be contained in both $T^{\pm}$. Without loss of generality, we assume $H \not\subset T^+$. This means H cannot act trivially, and therefore, acts transitively on $\mathcal{G}^+ \cong \mathbb{S}^1$ (cf. Lemma 2.2). □

Lemma 3.2. *S is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.*

Proof. We first note that S cannot have dimension 6, because this implies $\mathbb{T}$ is the classical flat Minkowski plane (cf. [4, Theorem 2]) and $S \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, which is not almost simple. From the list of almost simple groups of low dimensions (cf. [13, Theorem A2.2.6] or [15, 94.33]), S is locally isomorphic to either $\mathrm{SO}(3,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{R})$.

Suppose for a contradiction that S is locally isomorphic to $\mathrm{SO}(3,\mathbb{R})$, that is, S is isomorphic to either $\mathrm{SO}(3,\mathbb{R})$ or $\mathrm{SU}(2,\mathbb{C})$. Then S is 3-dimensional, compact and contains a subgroup isomorphic to $\mathrm{SO}(2,\mathbb{R})$. By Lemma 3.1, we may assume S is transitive on $\mathcal{G}^+$. By Brouwer's Theorem and the compactness of S, the factor group S/K^+ is isomorphic to $\mathrm{SO}(2,\mathbb{R})$. But this implies $\dim K^+ = 2$, which contradicts the assumption S is almost simple. □

Let κ be a generator of $Z(S)$.

Lemma 3.3. *$Z(S)$ is contained in at least one of $K^{\pm}$.*

Proof. In parts 1) to 5) we show that $\kappa(p) \parallel p$ for every point $p \in \mathcal{P}$. The final step, part 6), yields the lemma. Suppose for a contradiction that there exists a point p such that $\kappa(p)$ is not parallel to p.

1) We show $\dim S_{[p]_\pm} = 2$. From Lemma 2.4,

$$3 = \dim S = \dim S_{[p]_+} + \dim S([p]_+).$$

Since $\dim S([p]_+)$ is either 0 or 1, $\dim S_{[p]_+}$ is either 3 or 2. If $\dim S_{[p]_+} = 3$, then by Lemma 2.3, $S = S_{[p]_+}$. This cannot be true however, since κ does not fix $[p]_+$. Hence $\dim S_{[p]_+} = 2$. Similarly, $\dim S_{[p]_-} = 2$.

2) We show that $S_{[p]_\pm}$ fixes at least one point on $[p]_\pm$. If $S_{[p]_+}$ is transitive on $[p]_+$, then by Brouwer's Theorem $S_{[p]_+}/K^- \cong \mathrm{SO}(2,\mathbb{R})$. Since $\dim S_{[p]_+} = 2$, $\dim K^- = 1$. But this contradicts the fact that S is almost simple. Hence $S_{[p]_+}$ is not transitive on $[p]_+$. By Lemma 2.2, $S_{[p]_+}$ fixes at least one point on $[p]_+$. The same argument holds for $S_{[p]_-}$ on $[p]_-$.

3) We show that either $S_p = S_{[p]_-}$ or $S_p = S_{[p]_+}$. Following part 2), let q be a point such that $[q]_\mp$ is fixed by $S_{[p]_\pm}$. Since $S_p \leq S_{[p]_+} \cap S_{[p]_-}$, $S_p = S_{p,q}$. Because $\kappa \in Z(S)$, we also have $S_p = S_{p,\kappa(p)}$, and so $S_p = S_{p,q,\kappa(p)}$. If q is nonparallel to p and $\kappa(p)$, then S_p is trivial and $\dim S_p = 0$. From the dimension formula, we get $\dim S(p) = \dim S - \dim S_p = 3$, which cannot be true. Hence either $q \parallel p$ or $q \parallel \kappa(p)$.

If $q \parallel_+ p$, then $[q]_+ = [p]_+$, so that $S_{[p]_-}$ fixes $[p]_+$. If $q \parallel_+ \kappa(p)$, then $S_{[p]_-}$ fixes $\kappa([p]_+)$, and since κ commutes with every element of S, we see that $S_{[p]_-}$ also fixes $[p]_+$. In both cases, $S_p = S_{[p]_-}$.

In the cases $q \parallel_- p$ and $q \parallel_- \kappa(p)$, we obtain $S_p = S_{[p]_+}$ in a similar manner.

4) We show that there exists $r \in S(p)$ such that $r \nparallel p$ and $r \nparallel \kappa(p)$. From part 1) and 3), $\dim S_p = 2$, so that $\dim S(p) = 1$. Also from part 3), we can assume $S_p = S_{[p]_-}$, and so $S_{\kappa(p)} = S_{\kappa([p]_-)}$.

Suppose for a contradiction that for every $\alpha \in S$, either $\alpha(p) \parallel p$ or $\alpha(p) \parallel \kappa(p)$. From the assumption $S_p = S_{[p]_-}$, we see that $\alpha(p) = p$ if $\alpha(p) \parallel_- p$. Similarly, if $\alpha(p) \parallel_- \kappa(p)$, then α fixes $\kappa(p)$. It follows that the orbit $S(p)$ consists of points in $[p]_+ \cup [\kappa(p)]_+$. But this is impossible, since $S(p)$ is connected.

5) Let r be as in part 4). From the dimension formula, we have

$$\dim S_p = \dim S_{p,r} + \dim S_p(r) \leq \dim S_{p,r,\kappa(p)} + 1 = 1,$$

a contradiction. This proves that $\kappa(p) \parallel p$ for every point $p \in \mathcal{P}$.

6) Suppose that there are two nonparallel points p, q such that $\kappa(p) \neq p, \kappa(q) \neq q$, $\kappa(p) \parallel_+ p$, $\kappa(q) \parallel_- q$. Let $r = [p]_- \cap [q]_+$. Then $\kappa(r) \parallel_- \kappa(p)$ and $\kappa(r) \parallel_+ \kappa(q)$, compare Figure 1. Since $\kappa(p) \neq p$ and $\kappa(q) \neq q$, we see that $\kappa(r)$ is nonparallel to r, a contradiction. This shows that $Z(S)$ is contained in at least one of $K^{\pm}$. $\square$

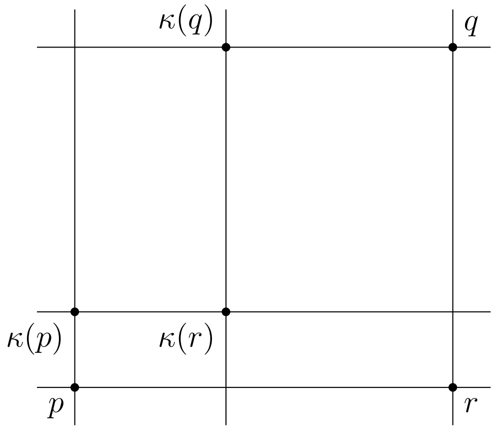

Figure 1

Lemma 3.4. *$Z(S)$ is trivial. In particular, $S \cong \mathrm{PSL}(2, \mathbb{R})$.*

Proof. Since $K^{\pm}$ is a normal subgroup of S, the dimension of $K^{\pm}$ is either 0 or 3. If either $\dim K^{\pm} = 3$, then by Lemma 2.3, the list of possible connected groups in a kernel (cf. [13, Proposition 4.4.9]), and the fact that S is almost simple, it follows that $S = K^{\pm} \cong \mathrm{PSL}(2, \mathbb{R})$ and thus $Z(S)$ is trivial.

In the remainder of the proof, we deal with the case that both $\dim K^{\pm} = 0$. Following Lemma 3.3, without loss of generality, we assume $Z(S) \leq K^+$. As a discrete normal subgroup of S, $K^+ \leq Z(S)$ and hence $Z(S) = K^+$. Also, $K^- \leq Z(S)$ and since $K^- \cap K^+ = \{id\}$, we see that K^- is trivial.

Suppose for a contradiction that $Z(S)$ is non-trivial.

1) We show that the action of S/K^+ on $\mathcal{G}^+$ is equivalent to the standard action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{S}^1$. We first note that $S/K^+ = S/Z(S) \cong \mathrm{PSL}(2,\mathbb{R})$ so that S/K^+ contains a subgroup H isomorphic to $\mathrm{SO}(2,\mathbb{R})$. By Lemma 2.2, H acts transitively on $\mathcal{G}^+$, as the preimage of H in S is not a subgroup of K^+. Hence S/K^+ is transitive on $\mathcal{G}^+$ and its action is derived from Brouwer's Theorem.

2) Since $S \neq K^-$, there exists a point x such that $\dim S([x]_-) = 1$. From the dimension formula, $\dim S_{[x]_-} = 2$. Similar to part 2) in the proof of Lemma 3.3, $S_{[x]_-}$ fixes a point p on $[x]_-$. This implies $S_{[x]_-} \leq S_p$, and in particular, $2 \leq \dim S_p \leq 3$. Furthermore, $\dim S_p \neq 3$, otherwise from Lemma 2.3, $S_p = S$, a contradiction to the transitivity of S/K^+ on $\mathcal{G}^+$. Hence $\dim S_p = 2$ and $\dim S(p) = 1$. For the rest of the proof, we will fix such a point p.

3) Denote the connected component of S_p by S_p^1.

If S_p^1 is not transitive on $\mathcal{G}^+\backslash\{[p]_+\}$, then it fixes a point $q \in [\kappa(p)]_-$ nonparallel to p. From the dimension formula, for a point $r \in S(p)$ nonparallel to p and q, we have

$$2 = \dim S_p^1 = \dim S_{p,q}^1 = \dim S_{p,q,r}^1 + \dim S_{p,q}^1(r),$$

which implies $\dim S_{p,q}^1(r) = 2$, contradicting the fact that $\dim S(p) = 1$. Hence S_p^1 is transitive on $\mathcal{G}^+\backslash\{[p]_+\}$. Since $S_p^1 \cap K^+$ is trivial, Brouwer's Theorem implies that S_p^1 is isomorphic and acts equivalently to L_2 on $\mathcal{G}^+\backslash\{[p]_+\}$.

Let $R \cong \mathbb{R}$ be a 1-dimensional orbit of S_p^1 on $[p]_+$. Let R^- be the kernel of the action of S_p^1 on R. By Brouwer's Theorem, either $S_p^1/R^- \cong \mathbb{R}$ or $S_p^1/R^- \cong \mathrm{L}_2$. If $S_p^1/R^- \cong \mathbb{R}$, then R^- is isomorphic to $\mathbb{R}$, and, as a 1-dimensional normal subgroup of $S_p^1 \cong \mathrm{L}_2$, acts transitively on $\mathcal{G}^+\backslash\{[p]_+\}$. However, this implies $[x]_-\backslash\{x\} \in S(p)$ for each $x \in R$, which cannot occur, since $\dim S(p) = 1$. Hence, $S_p^1/R^- \cong \mathrm{L}_2$. In particular, R^- is trivial and S_p^1 acts equivalently to L_2 on R.

It follows that S_p^1 has a 1-dimensional orbit $Q \cong \mathbb{R}$ on $R \times \mathcal{G}^+\backslash\{[p]_+\}$, which corresponds to the diagonal under the identification $R \times \mathcal{G}^+\backslash\{[p]_+\} \cong \mathbb{R}^2$, cf. Figure 2. Note that any two distinct points in Q are non-parallel so that S_p^1 acts equivalently to L_2 on Q.

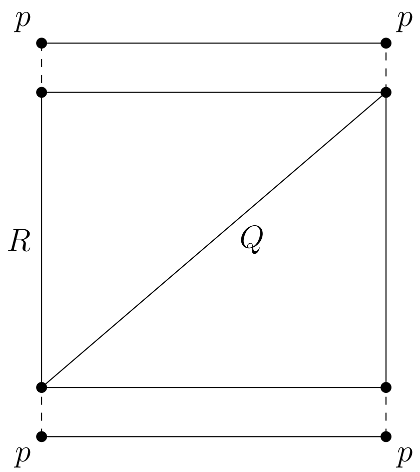

Figure 2

4) For a given point $x \in Q$, let $\mathcal{C}_{p,x}$ be the set of circles going through p and x. Let $\mathcal{C}_{p,x}^+$ and $\mathcal{C}_{p,x}^-$ be the subsets of $\mathcal{C}_{p,x}$ consisting of circles described by orientation-preserving and orientation-reversing homeomorphisms of $\mathbb{S}^1$ to itself. Define

$$\phi : [\kappa(p)]_-\backslash\{\kappa(p), [\kappa(p)]_- \cap [x]_+\} \to \mathcal{C}_{p,x} : y \mapsto \alpha(p, x, y).$$

The map ϕ is induced from the operation Joining, which is a homeomorphism, cf. [8]. In particular, ϕ is also a homeomorphism and maps each connected component of its domain onto one of $\mathcal{C}^{\pm}_{p,x}$.

5) We now make two observations on $S^1_{p,x}$, the connected component of the 2-point stabilizer $S_{p,x}$, for each $x \in Q$. Firstly, from part 3), $S^1_{p,x}$ is isomorphic to $\mathbb{R}$ and acts sharply transitively on each connected component of $Q \backslash \{x\}$.

Secondly, $S^1_{p,x}$ is also sharply transitive on each of $\mathcal{C}^{\pm}_{p,x}$. This comes from the action of S^1_p on the fixed parallel class $[\kappa(p)]_-$ and the identification of points on $[\kappa(p)]_-$ with circles in $\mathcal{C}^{\pm}_{p,x}$ described in part 4).

6) Let $q \in Q$ be a point and U, V be the two connected components of $Q \backslash \{q\}$. Let $u \in U, v \in V$ and let A, B be the circles $A := \alpha(p, q, u)$ and $B := \alpha(p, q, v)$, cf. Figure 3.

By part 3), S^1_p is 2-set transitive (2-homogeneous) on Q, and so there exists $\gamma \in S^1_p$ such that $\gamma(\{v, q\}) = \{q, u\}$. In particular, $\gamma(A) = B$. Since S^1_p is connected, both A, B belong to either $\mathcal{C}^+_{p,q}$ or $\mathcal{C}^-_{p,q}$. Assume $A, B \in \mathcal{C}^+_{p,q}$.

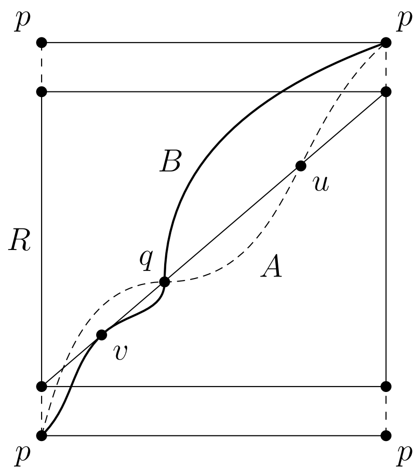

Figure 3

From part 5), there exists a unique $\sigma \in S^1_{p,q}$ such that $\sigma(B) = A$. This implies $\sigma(v) \in A$, and from the action of $S^1_{p,q}$ on Q in part 5), it follows that $\sigma(v) \in V$. Since $\sigma(v)$ and q are in the same connected component of $Q \backslash \{u\}$, there exists a unique $\tau \in S^1_{p,u}$ such that $\tau(q) = \sigma(v)$. In other words, τ fixes $p, u \in A$ and maps $q \in A$ to $\sigma(v) \in A$, so τ must fix A. This implies $\tau = id$. But this is impossible, because $\sigma(v) \neq q$. This completes the proof. ☐

We now determine the action of S on the torus.

Lemma 3.5. *If $S \cong \mathrm{PSL}(2, \mathbb{R})$, then exactly one of the following occurs.*

1. *Either S fixes every $(+)$-parallel class or S fixes every $(-)$-parallel class. In both cases, $\mathbb{T}$ is isomorphic to a half-classical Minkowski plane $\mathcal{M}(f, id)$, where f is an orientation-preserving homeomorphism of $\mathbb{S}^1$.*
2. *S acts diagonally on the point set. The diagonal D is a circle of $\mathbb{T}$, and S fixes D.*

Proof. Since $K^{\pm}$ are normal subgroups of the simple group S, either $K^{\pm} = S$ or $K^{\pm} = \{id\}$. If $K^+ = S$, then S fixes every $(+)$-parallel class, and likewise when $K^- = S$. In both cases, one of the kernels of $\mathbb{T}$ is 3-dimensional. From [4, Theorem 1.2] and [13, Theorem 4.4.10], $\mathbb{T}$ is determined.

We now consider the case that both $K^\pm = \{id\}$. Since S contains a subgroup $H \cong \mathrm{SO}(2,\mathbb{R})$ which is not contained in $K^\pm$, by Lemma 2.2, H and thus S acts transitively on $\mathcal{G}^\pm$. Hence S acts equivalently to $\mathrm{PSL}(2,\mathbb{R})$ on both $\mathcal{G}^\pm$. It follows that S has two orbits: the diagonal $D \cong \mathbb{S}^1$, in a suitable coordinate system, and its complement $\mathcal{P}\backslash D$.

It only remains to show that D is a circle of $\mathbb{T}$.

1) As seen in the proof of Lemma 3.4, for any pair of distinct points $p, q \in D$, the stabiliser $S_{p,q}$ is isomorphic to $\mathbb{R}$. The orbits of points in $\mathcal{P}\backslash\{[p],[q]\}$ under $S_{p,q}$ are 1-dimensional and partition $\mathcal{P}\backslash\{[p],[q]\}$. Under a suitable coordinate system with $p = (\infty,\infty)$ and $q = (0,0)$, these orbits can be represented as sets of the form $\{(x,ax)|x > 0\}$ or $\{(x,ax)|x < 0\}$, where each $a \neq 0$ determines such an orbit.

2) Let $p,q,r \in D$ and $C \coloneqq \alpha(p,q,r)$. We claim that there exists an orbit $O \coloneqq S_{x,y}(\xi)$, where $x, y \in \{p,q,r\}, x \neq y$, for some $\xi \in \mathcal{P}$, intersecting C at least two times. If C intersects D at an additional point v, then depending on the position of v, the orbit O can be chosen as $S_{p,q}(r)$, $S_{p,r}(q)$, or $S_{q,r}(p)$, cf. Figure 4a.

Otherwise, at least one intersection of C and D is not transversal. By changing the roles of p,q,r if necessary, we can assume this intersection is r, cf. Figure 4b. Coordinatise the plane as in part 1) so that the orbits of points under $S_{p,q}$ are rays emanating from q. Let C_0 be the connected component of $C\backslash\{p,q\}$ that contains r. Let C_1, C_2 be the two connected components of $C_0\backslash\{r\}$. Since the orbits $S_{p,q}(\xi)$ depend continuously on ξ and approach $S_{p,q}(r) \subseteq D$ as ξ tends to r, the intermediate value theorem implies that there exists an orbit that meets C in at least two points, one in C_1 and one in C_2.

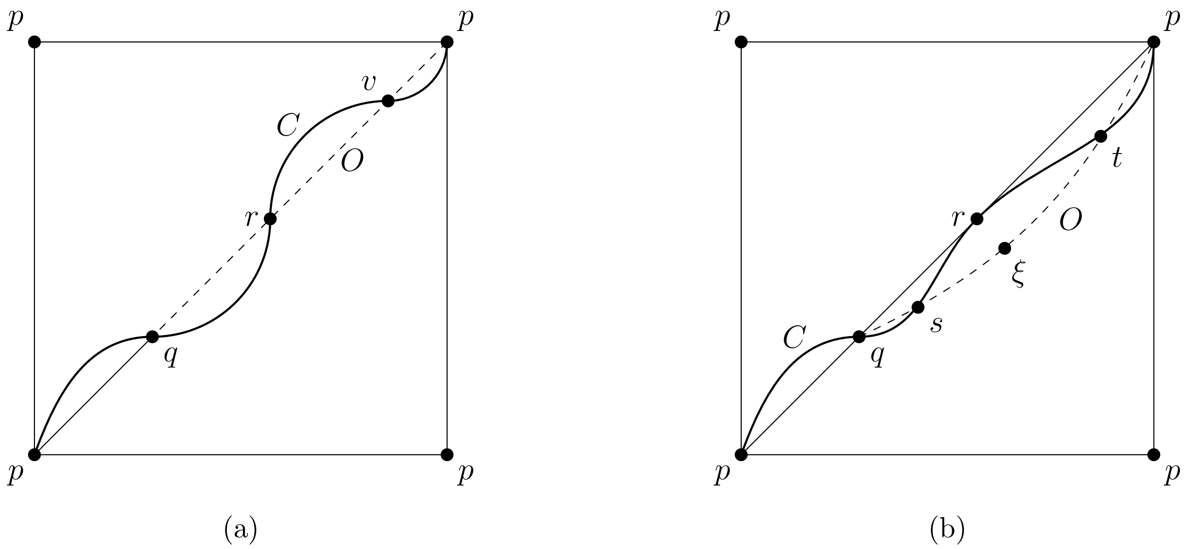

Figure 4

3) Following part 2), without loss of generality, assume $O \coloneqq S_{p,q}(\xi)$ for some $\xi \in \mathcal{P}$. Let R be the connected component of $D\backslash\{p,q\}$ that contains r. We show that R intersects C infinitely many times.

Let $s,t \in O \cap C$, $s \neq t$. Then there exists $\sigma \in S_{p,q}$ such that $\sigma(s) = t$. This implies $\sigma(C) = C$. But since $s \neq t$, we have $\sigma \neq id$, so that $\sigma(r) \neq r$ and $\sigma(r) \in R \cap C$. Since σ has infinite order, the points $\sigma^n(r)$ are distinct and belong to $R \cap C$.

4) By repeating the arguments in part 2) and 3) for different pairs of points in $D\backslash\{p\}$, it follows that the set $D \cap C$ is dense on D. From the compactness of D and C, we have $D \subseteq C$. As there is no proper subset of $\mathbb{S}^1$ homeomorphic to $\mathbb{S}^1$, it follows that $D = C$. This completes the proof. □

This proves Theorem 1.1. We also obtain the following results.

Corollary 3.6. *Let Σ be a 3-dimensional connected group of automorphisms of a toroidal circle plane. Then Σ is either solvable or isomorphic to* $\mathrm{PSL}(2, \mathbb{R})$.

Corollary 3.7. *Let Σ be a connected semi-simple group of automorphisms of a toroidal circle plane. Then Σ is isomorphic to either* $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ *or* $\mathrm{PSL}(2, \mathbb{R})$.

4 Proof of Theorem 1.2

In this section, let Σ be a 3-dimensional connected group of automorphisms of a toroidal circle plane $\mathbb{T}$ with kernels $\Delta^\pm$ on $\mathcal{G}^\pm$. We prove Theorem 1.2 in three parts via Lemmas 4.1, 4.2 and 4.3.

Lemma 4.1. *Exactly one of the following occurs.*

1. *Σ fixes at least one parallel class.*
2. *Σ fixes no parallel classes but fixes and acts transitively on exactly one circle. In this case $\Sigma \cong \mathrm{PSL}(2, \mathbb{R})$ and, under suitable coordinates, it acts diagonally on the torus.*

Proof. If at least one of $\Sigma/\Delta^\pm$ is not transitive on the corresponding set $\mathcal{G}^\pm$, then Σ fixes a parallel class.

We now assume both $\Sigma/\Delta^\pm$ are transitive on $\mathcal{G}^\pm$. By Brouwer's Theorem, $\Sigma/\Delta^\pm$ is isomorphic and acts equivalently to either $\mathrm{SO}(2, \mathbb{R})$ or a finite covering group of $\mathrm{PSL}(2, \mathbb{R})$. Furthermore, it cannot be the case that both $\Sigma/\Delta^\pm$ are isomorphic to $\mathrm{SO}(2, \mathbb{R})$, otherwise $\dim \Delta^\pm = 2$, which in turn implies $\dim \Delta^+\Delta^- = 4 > \dim \Sigma$, a contradiction. If Σ/Δ^+ is isomorphic to $\mathrm{PSL}^k(2, \mathbb{R})$, then $\dim \Delta^+ = 0$ and so Σ is almost simple. By Theorem 1.1, Σ is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ and acts diagonally on the torus under a suitable coordinate system. The diagonal is a circle, as proved in Lemma 3.5. Finally, Σ cannot fix another circle because it only has one 1-dimensional orbit on the torus. □

Lemma 4.2. *Assume Σ fixes at least one parallel class. Without loss of generality, let π be the fixed $(+)$-parallel class. Then exactly one of the following occurs.*

1. *Σ fixes at least one point.*
2. *Σ fixes no points but fixes and acts transitively on every $(+)$-parallel class. In this case $\Sigma \cong \mathrm{PSL}(2, \mathbb{R})$ and $\mathbb{T}$ is isomorphic to a half-classical Minkowski plane $\mathcal{M}(f, id)$, where f is an orientation-preserving homeomorphism of $\mathbb{S}^1$.*
3. *Σ fixes no points but fixes and acts transitively on exactly one parallel class, which is π. In this case $\Sigma \cong \mathrm{L}_2 \times \mathrm{SO}(2, \mathbb{R})$. The factor group Σ/Δ^- is isomorphic and acts equivalently to $\mathrm{SO}(2, \mathbb{R})$ on $\mathcal{G}^-$. Also, Σ/Δ^+ is isomorphic and acts equivalently to L_2 on $\mathcal{G}^+\backslash\{\pi\}$.*

Proof. Since Σ fixes the $(+)$-parallel class π, Σ/Δ^+ is not transitive on $\mathcal{G}^+$. If Σ/Δ^- is not transitive on $\mathcal{G}^-$, then Σ fixes at least one point.

We proceed by assuming that Σ/Δ^- is transitive on $\mathcal{G}^-$. By Brouwer's Theorem, Σ/Δ^- is isomorphic and acts equivalently to either $\mathrm{SO}(2,\mathbb{R})$ or $\mathrm{PSL}^k(2,\mathbb{R})$ for some k. We consider these 2 cases separately.

Case 1: $\Sigma/\Delta^- \cong \mathrm{PSL}^k(2,\mathbb{R})$. Then $\dim\Delta^- = 0$, and so Σ is almost simple. From the first case in Theorem 1.1, $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$ and the plane $\mathbb{T}$ is determined.

Case 2: $\Sigma/\Delta^- \cong \mathrm{SO}(2,\mathbb{R})$. If Σ/Δ^+ acts trivially on $\mathcal{G}^+$, then $\Sigma = \Delta^+$. But since $\Delta^+ \cap \Delta^- = \{id\}$, it follows that $\Delta^- = \{id\}$, a contradiction to the dimensions of Σ and Σ/Δ^-. Hence Σ/Δ^+ acts non-trivially on $\mathcal{G}^+$. By Lemma 2.2, there is an open subset $I \cong \mathbb{R}$ of $\mathcal{G}^+$ on which Σ/Δ^+ acts transitively. On the other hand, Σ is isomorphic to a subgroup of $\Sigma/\Delta^+ \times \Sigma/\Delta^-$, so that

$$2 \le \dim\Sigma/\Delta^+ \le 3.$$

From Brouwer's Theorem and Theorem 1.1, it follows that $\Sigma/\Delta^+ \cong \mathrm{L}_2$. By Lemma 2.3, $\Sigma \cong \Sigma/\Delta^+ \times \Sigma/\Delta^- \cong \mathrm{L}_2 \times \mathrm{SO}(2,\mathbb{R})$.

Since Δ^- fixes at most two $(+)$-parallel classes, so does Σ. Suppose that Σ fixes exactly two $(+)$-parallel classes π and π_2. Let π_3 be a $(+)$-parallel class different from π and π_2. From the dimension formula,

$$2 = \dim\Delta^- = \dim\Delta^-_{\pi,\pi_2} = \dim\Delta^-_{\pi,\pi_2,\pi_3} + \dim\Delta^-_{\pi,\pi_2}(\pi_3) \le 0 + 1 = 1,$$

a contradiction.

Hence Σ fixes exactly one $(+)$-parallel class π. In particular, $I = \mathcal{G}^+\backslash\{\pi\}$ and Σ/Δ^+ acts equivalently to L_2 on $\mathcal{G}^+\backslash\{\pi\}$, by Brouwer's Theorem. □

Lemma 4.3. *Assume Σ fixes at least one point. Under a suitable coordinate system, let $p = (\infty,\infty)$ be a fixed point. Then exactly one of the following occurs.*

1. *Σ fixes exactly two parallel points. In this case $\Sigma \cong \Phi_d$, for some $d \le 0$. The coordinates may be chosen such that the second fixed point is $(0,\infty)$ and the action of Σ is described by the maps*

   $$\{(x,y) \mapsto (ax, by + c) \mid a,b > 0, c \in \mathbb{R}\},$$

   when $\Sigma \cong \Phi_0$, and

   $$\{(x,y) \mapsto (a\ sgn(x)\cdot|x|^b, b^d y + c) \mid a,b > 0, c \in \mathbb{R}\},$$

   when $\Sigma \cong \Phi_d$, for some $d < 0$.

2. *Σ fixes exactly one point, which is p. In this case the derived plane $\mathbb{T}_p$ is Desarguesian and $\Sigma \cong \Phi_d$, for some $d \in \mathbb{R}\cup\{\infty\}$. The coordinates may be chosen such that the action of Σ is the standard action of Φ_d on $\mathbb{R}^2$.*

Proof. There are 3 cases depending on the transitivity of $\Sigma/\Delta^\pm$ on $\mathcal{G}_\pm\backslash\{[p]_\pm\}$.

Case 1: Neither of $\Sigma/\Delta^\pm$ is transitive on $\mathcal{G}_\pm\backslash\{[p]_\pm\}$. This implies there is an additional fixed point q nonparallel to p. From the dimension formula, for a point r nonparallel to p and q, we have

$$3 = \dim\Sigma_p = \dim\Sigma_{p,q} = \dim\Sigma_{p,q,r} + \dim\Sigma_{p,q}(r) \le 2,$$

a contradiction. Hence this case cannot occur.

Case 2: Σ/Δ^- is transitive on $\mathcal{G}^-\backslash\{[p]_-\}$, Σ/Δ^+ is not transitive on $\mathcal{G}^+\backslash\{[p]_+\}$, or vice versa. By Lemma 2.2, Σ fixes an additional point $q \in [p]_-$. We show Σ fixes at most two points. By changing the coordinates if necessary, we can assume $q = (0, \infty)$. Suppose to the contrary that Σ fixes three points on $[p]_-$. Then Δ^- fixes three $(+)$-parallel classes pointwise and so must be trivial. By Brouwer's Theorem, $\Sigma \cong \Sigma/\Delta^- \cong \mathrm{PSL}(2, \mathbb{R})$, which contradicts Theorem 1.1.

Hence Σ fixes precisely two parallel points p and q. On the derived plane $\mathbb{T}_p$, Σ induces a group of automorphisms that fixes precisely a line. By [7, Theorem 7.5B], Σ is isomorphic to either $\mathbb{R} \times \mathrm{L}_2 \cong \Phi_0$ or Φ_d, for some $d < 0$, and the action of the group is described as in the statement of the lemma.

Case 3: Both $\Sigma/\Delta^\pm$ are transitive on $\mathcal{G}^\pm\backslash\{[p]_\pm\}$. In this case, p is the only point fixed by Σ. Brouwer's Theorem implies that $\Sigma/\Delta^\pm$ is isomorphic to $\mathbb{R}$, L_2 or $\mathrm{PSL}(2, \mathbb{R})$. From Theorem 1.1, we can rule out the last case. Since Σ is isomorphic to a subgroup of $\Sigma/\Delta^+ \times \Sigma/\Delta^-$, it cannot be the case that both $\Sigma/\Delta^\pm \cong \mathbb{R}$. This leads to 2 subcases.

Subcase 3A: $\Sigma/\Delta^+ \cong \mathbb{R}$ and $\Sigma/\Delta^- \cong \mathrm{L}_2$, or vice versa. By Lemma 2.3, $\Sigma = \Sigma/\Delta^+ \times \Sigma/\Delta^- \cong \mathbb{R} \times \mathrm{L}_2$. By Brouwer's Theorem, the action of $\Sigma/\Delta^\pm$ is standard, and so the action of Σ on $\mathcal{P}$ is described by the maps

$$\{(x, y) \mapsto (x + b, ay + c) \mid a > 0, b, c \in \mathbb{R}\},$$

in suitable coordinates. In particular, $\Sigma \cong \Phi_\infty$.

When $\Sigma/\Delta^- \cong \mathbb{R}$ and $\Sigma/\Delta^+ \cong \mathrm{L}_2$, we obtain $\Sigma \cong \Phi_0$ in a similar fashion.

Subcase 3B: $\Sigma/\Delta^\pm \cong \mathrm{L}_2$. We show that $\Sigma \cong \Phi_d$ for some $d \in \mathbb{R} \cup \{\infty\}$.

Let $\overline{\mathcal{P}} := \mathcal{P}\backslash([p]_+ \cup [p]_-) \cong \mathbb{R}^2$. We consider the action of $\Delta^+\Delta^-$ on $\overline{\mathcal{P}}$. We have $\dim \Delta^+ = 1$. If Δ^+ fixes a parallel class $[q]_- \in \mathcal{G}^-\backslash\{[p]_-\}$, then, as a normal subgroup, it fixes the orbit of $[q]_-$ pointwise. But since Σ/Δ^- is transitive on $\mathcal{G}^-\backslash\{[p]_-\}$, Δ^+ is then trivial, which is impossible. Hence Δ^+ is transitive on $\mathcal{G}^-\backslash\{[p]_-\}$. By Brouwer's Theorem, Δ^+ is isomorphic and acts equivalently to $\mathbb{R}$. With the same reasoning for Δ^-, it follows that $\Delta^+\Delta^-$ is the translation group $\mathbb{R}^2$ (in suitable coordinates) and is sharply transitive on $\overline{\mathcal{P}}$.

Denote $o := (0, 0)$. By [15, Proposition 91.2], $\Sigma = \Delta^+\Delta^-\Sigma_o$. Since $\Delta^+\Delta^-$ is a normal subgroup of Σ and $\Delta^+\Delta^- \cap \Sigma_o$ is trivial, $\Sigma = \Sigma_o \ltimes \Delta^+\Delta^-$. From the dimensions of Σ and $\Delta^\pm$, $\dim \Sigma_o = 1$. By [15, Corollary 94.39], $\Sigma_o \cong \mathbb{R}$. The action of Σ_o is then described by the maps

$$\{(x, y) \mapsto (x, ay) \mid a > 0\},$$

or

$$\{(x, y) \mapsto (ax, a^d y) \mid a > 0\}.$$

This shows that $\Sigma \cong \Phi_d$ for some $d \in \mathbb{R} \cup \{\infty\}$.

In both subcases, $\Delta^+ \times \Delta^-$ contains a normal subgroup that is sharply transitive on $\overline{\mathcal{P}}$. Hence Σ induces a 3-dimensional group of automorphisms acting transitively on the point set of $\mathbb{T}_p$. The list of possibilities for $\mathbb{T}_p$ is given by [6, Main Theorem 2.6] and a case by case check shows that $\mathbb{T}_p$ is Desarguesian, cf. [8, Subsection 7.1.1]. □

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