ZERO-CYCLES OF DEGREE ONE ON SKOROBOGATOV'S BIELLIPTIC SURFACE

BRENDAN CREUTZ

Abstract. Skorobogatov constructed a bielliptic surface which is a counterexample to the Hasse principle not explained by the Brauer-Manin obstruction. We show that this surface has a 0-cycle of degree 1, as predicted by a conjecture of Colliot-Thélène.

Consider the smooth projective surface $\mathcal{S}/\mathbb{Q}$ given by the affine equations,

$$\mathcal{S} : (x^2+1)y^2 = (x^2+2)z^2 = 3(t^4 - 54t^2 - 117t - 243)\,.$$

Skorobogatov [Sko99] showed that the set $\mathcal{S}(\mathbb{Q})$ of rational points on $\mathcal{S}$ is empty, despite there being adelic points on $\mathcal{S}$ that are orthogonal to all elements in the Brauer group of $\mathcal{S}$. A fortiori the Brauer group does not obstruct the existence of $\mathbb{Q}$-rational 0-cycles of degree 1 on $\mathcal{S}$. In this short note we show that, as predicted by a conjecture of Colliot-Thélène [CT99], the surface $\mathcal{S}$ does in fact possess a $\mathbb{Q}$-rational 0-cycle of degree 1.

To wit,

$$\begin{aligned} x_0 &= \theta^2 + 1 \\ y_0 &= 3357\theta^2 - 2133\theta + 4851 \\ z_0 &= 2826\theta^2 - 2025\theta + 4158 \\ t_0 &= -42\theta^2 + 24\theta - 54\,, \end{aligned}$$

where θ satisfies $\theta^3 + \theta + 1 = 0$ are the coordinates of a closed point of $\mathcal{S}$ whose residue field is the cubic number field $L = \mathbb{Q}[\theta]$. As there are obviously 0-cycles of degree 4 on $\mathcal{S}$, this shows that there is a 0-cycle of degree 1 on $\mathcal{S}$.

We note, however, that the far more important question of whether the Brauer-Manin obstruction to the existence of 0-cycles of degree 1 is the only one for bielliptic surfaces remains open.

Let us briefly explain how this point was discovered. Consider the genus one curves

$$\mathcal{C} : U^2 = g(T) = 3(T^4 - 54T^2 - 117T - 243)\,,$$

and

$$\mathcal{D} : \begin{cases} Y^2 = p(X) = X^2 + 1 \\ Z^2 = q(X) = X^2 + 2\,. \end{cases}$$

There are actions of $\mu_2 = \{\pm 1\}$ on $\mathcal{C}$ and $\mathcal{D}$ given by

$$(T, U) \mapsto (T, -U) \quad \text{and} \quad (X, Y, Z) \mapsto (X, -Y, -Z)\,.$$

The quotient of $\mathcal{C} \times \mathcal{D}$ by the diagonal action of μ_2 is the μ_2-torsor $\rho : \mathcal{C} \times \mathcal{D} \to \mathcal{S}$ given by $s^2 = g(t)$ over $\mathcal{S}$. After a change of variables, $\mathcal{S}$ may be written

$$\mathcal{S} : \begin{cases} y^2 = g(t)p(x) \\ z^2 = g(t)q(x) \end{cases}$$

and ρ is defined by $x = X, t = T, y = UY, z = UZ$.

Lemma. There is a 0-cycle of degree 1 on $\mathcal{S}$ if and only if there is an odd degree number field K and $a \in K^\times$ such that genus one curves over K given by

$$\mathcal{C}^a : aU^2 = g(T) \quad \text{and} \qquad \mathcal{D}^a : \begin{cases} aY^2 = p(X) \\ aZ^2 = q(X) \end{cases}$$

both possess a K-rational point.

Proof. It is clear that there is a 0-cycle of degree 1 on $\mathcal{S}$ if and only if $\mathcal{S}(K) \neq \emptyset$ for some odd degree number field K. Let $\rho_K : (\mathcal{C} \times \mathcal{D})_K \to \mathcal{S}_K$ denote the base change to K. By descent theory every element of $\mathcal{S}(K) = \mathcal{S}_K(K)$ lifts to a K-rational point on some twist of the μ_2-torsor $\rho_K : (\mathcal{C} \times \mathcal{D})_K \to \mathcal{S}_K$ by a cocycle in $H^1(K, \mu_2) = K^\times/K^{\times 2}$. A straightforward computation shows that the twist of $(\mathcal{C} \times \mathcal{D})_K$ by $a \in K^\times/K^{\times 2}$ is the product of the curves in the statement of the proposition. $\square$

The quotient of $\mathcal{D}$ by μ_2 is the μ_2-torsor $\phi : \mathcal{D} \to \mathcal{D}'$ given by $U^2 = p(X)$ over the genus one curve $\mathcal{D}'$ given by $W^2 = p(X)q(X)$. This curve has two rational points at infinity. Fixing either as the identity endows $\mathcal{D}'$ with the structure of an elliptic curve such that the other point is 2-torsion. There are rational points on $\mathcal{D}$ above both of these, and so we may view $\phi : \mathcal{D} \to \mathcal{D}'$ as an isogeny of elliptic curves. The twists $\mathcal{D}^a$ which have a K-rational point correspond to the image of $\mathcal{D}'(K)$ under the connecting homomorphism δ_ϕ in the exact sequence

$$1 \to \mu_2 \to \mathcal{D}(K) \xrightarrow{\phi} \mathcal{D}'(K) \xrightarrow{\delta_\phi} H^1(K, \mu_2) \simeq K^\times/K^{\times 2}\,.$$

In particular, for a given K, one can compute by means of an explicit 2-isogeny descent a finite set $A \subset K^\times/K^{\times 2}$ such that every twist with a K-point is $\mathcal{D}^a$ for some $a \in A$. Thus the determination of $\mathcal{S}(K)$ is reduced to the determination of the set of K-rational points on an explicit finite set of genus 1 curves over K, namely the curves $\mathcal{C}^a$ and $\mathcal{D}^a$ with $a \in A$. Conjecturally, this is a finite computation. For K of small degree and discriminant it is often possible in practice.

We carried out these computations for various number fields using the Magma Computational Algebra System [BCP97]. There are no points on $\mathcal{S}$ defined over the cubic field L_1 of smallest absolute discriminant. The map $\phi : \mathcal{D}(L_1) \to \mathcal{D}'(L_1)$ is surjective and so the only twist with L_1-points is $\mathcal{D} = \mathcal{D}^1$. But as $\mathcal{C} = \mathcal{C}^1$ has no points over any odd degree number field we see that $\mathcal{S}(L_1) = \emptyset$. The point $(x_0, y_0, z_0, t_0) \in \mathcal{S}(L)$ is defined over the cubic field of second smallest absolute discriminant. The group $\mathcal{D}'(L)$ has rank 1 and is generated by the points with X-coordinate equal to $x_0 = \theta^2 + 1$. This corresponds to the twist by $a = 6\theta^2 - 4\theta + 9 \in L^\times$ and one finds that $\mathcal{C}^a$ has a point with T-coordinate equal to $t_0 = -42\theta^2 + 24\theta - 54$. In fact, $\mathcal{C}^a(L)$ and $\mathcal{D}^a(L)$ are both infinite, showing that $\mathcal{S}(L)$ is Zariski dense in $\mathcal{S}$.

Acknowledgements. The author would like to thank J.-L. Colliot-Thélène for helpful comments and suggestions.

References

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). ↑2

[CT99] Jean-Louis Colliot-Thélène, *Conjectures de type local-global sur l'image des groupes de Chow dans la cohomologie étale*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 1–12 (French). ↑1

[Sko99] Alexei N. Skorobogatov, *Beyond the Manin obstruction*, Invent. Math. **135** (1999), no. 2, 399–424. ↑1

School of Mathematics and Statistics, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand

E-mail address: `brendan.creutz@canterbury.ac.nz`

URL: `http://www.math.canterbury.ac.nz/~bcreutz`