The Drinker Paradox and its Dual

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Abstract

The Drinker Paradox is as follows.

> In every nonempty tavern, there is a person such that if that person is drinking, then everyone in the tavern is drinking.

Formally,

$$\exists x\big(\varphi \to \forall y \varphi[x/y]\big) \ .$$

Due to its counterintuitive nature it is called a paradox, even though it actually is a classical tautology. However, it is not minimally (or even intuitionistically) provable. The same can be said of its dual, which is (equivalent to) the well-known principle of *independence of premise*,

$$\varphi \to \exists x \psi \ \vdash \ \exists x(\varphi \to \psi)$$

where x is not free in φ.

In this paper we study the implications of adding these and other formula schemata to minimal logic. We show first that these principles are independent of the law of excluded middle and of each other, and second how these schemata relate to other well-known principles, such as Markov's Principle of unbounded search, providing proofs and semantic models where appropriate.

1 Introduction

Minimal logic [11] provides, as its name suggests, a minimal setting for logical investigations. Starting from minimal logic, we can get to intuitionistic logic, by adding *ex falso quodlibet* (EFQ), and to classical logic, by adding *double negation elimination* (DNE).[1] Therefore, every statement proven over minimal logic can also be proven in intuitionistic logic and classical logic. In addition, minimal logic has many structural advantages, and is easier to analyse. Of course, there is a price one has to pay for working within a weak framework. The price is that fewer well-known statements are provable outright, which leads to the question how they relate. This is a very similar question to the one considered in constructive reverse mathematics (CRM; [8, 5]), whose aim it is to find some ordering in a multitude of principles, over intuitionistic logic. CRM has been around for some decades now, and some even trace the origins back to Brouwerian counterexamples. Most results in CRM are focused on analysis, where most theorems can be classified into being equivalent to about ten major principles. It is a natural question to ask whether we can find similar results in the absence of EFQ. Previous work by a subset of the authors [6] has investigated the case of propositional schemata, but has left the predicate case

[1]Either as a rule, or an axiom scheme. See below for details.

untouched. Similar work can also be found in [7, 10]. A more detailed approach, but again focused on the propositional case can be found in [9], where it was studied exactly which instances of an axiom scheme are required to prove a given instance of another axiom scheme over minimal logic. In this paper we will make first steps in the predicate case. As is so often the case, the first-order analysis is subtler and technically more difficult to deal with.

For the sake of brevity and readability we have only included non-trivial proofs. The missing proofs, which are in natural deduction style, have been put into an appendix. A version of this paper including that appendix will be made available on `arxiv.org` under the same title.

2 Technical Preliminaries

We will generally follow the notation and definitions found in [11].

An n-ary *scheme* $\mathrm{SCH}(X_1, \dots, X_n)$ is a formula SCH containing n propositional variables $X_1, \dots, X_n$. An instance $\mathrm{SCH}(\Phi_1, \dots, \Phi_n)$ is obtained by replacing the variables with formulae $\Phi_1, \dots, \Phi_n$. A scheme is *derivable* in a logical system if every instance is derivable in that system. A scheme is *minimal* (*constructive*) (*classical*) if it is derivable in minimal (intuitionistic) (classical) logic.

Example 1. *The* law of excluded middle, $\mathrm{LEM}(\Phi) := \Phi \vee \neg\Phi$ *is a classical unary scheme.*

A logical system can be extended by adding that certain schemata are derivable in the system. In the case of natural deduction and minimal logic, an extension by LEM is an addition of a deduction rule

$$\frac{}{\alpha \vee \neg\alpha}\ \mathrm{LEM}(\alpha)$$

for every formula α. This produces a subsystem of classical logic.

More general, if a formula Φ is derivable over minimal (intuitionistic) logic extended by schemata $A_0, A_1, \dots, A_n$, then we write

$$\vdash_{A_0+A_1+\cdots+A_n} \Phi$$

($\vdash_{i+A_0+A_1+\cdots+A_n} \Phi$). Moreover, if B is an n-ary scheme, and for all formulae $\alpha_0, \dots, \alpha_n$,

$$\vdash_{A_0+A_1+\cdots+A_n} B(\alpha_0, \dots, \alpha_n)\ ,$$

then we write

$$\vdash_{A_0+A_1+\cdots+A_n} B\ .$$

Extending a logic by a scheme differs from allowing undischarged assumptions of instances of the scheme. For example, it should follow from LEM that every predicate is decidable. Consider the proof of $\vdash_{\mathrm{LEM}} \forall x(Px \vee \neg Px)$:

$$\dfrac{\dfrac{}{Px \vee \neg Px}\ \mathrm{LEM}}{\forall_x\,(Px \vee \neg Px)}\ \forall\mathrm{I}$$

The proof uses $\mathrm{LEM}(Px)$. However,

$$\mathrm{LEM}(Px) \nvdash \forall x(Px \vee \neg Px)\ ,$$

since the rule $\forall$I requires that x is not free in any open assumptions.[2]

It is trivial to check that the following holds.

[2]If we defined LEM as the axiom scheme $\forall\vec{x}P\vec{x} \vee \neg P\vec{x}$, there would be no difference between adding it as a rule or an assumption. This trick is the same as used in [11, page 14] for EFQ and stability.

Proposition 2. *Define* DNE$(\Phi) := \neg\neg\Phi \to \Phi$, *and EFQ*$(\Phi) := \bot \to \Phi$. *For all (finite) collections of schemata S and T,*

$$S \vdash_i T \iff S \vdash_{EFQ} T$$

and

$$S \vdash_c T \iff S \vdash_{DNE} T\ .$$

A preorder $\supset$ may be defined on finite collections of schemata by considering derivability over extensions of minimal logic.

Definition 3. For schemata A_0, A_1, ..., A_n and B_0, B_1, ..., B_m, we write

$$\{A_0, A_1, \ldots A_n\} \supset \{B_0, B_1, \ldots B_m\}$$

if

$$\vdash_{A_0+A_1+\cdots+A_n} B_i$$

for $0 \leqslant i \leqslant m$. For simplicity, when $m = 0$ we write $\{A_0, A_1, \ldots A_n\} \supset B$, and say that B is *reducible* to $A_0, \ldots A_n$. Where unambiguous, if $n = 0$ we write $A \supset B$.

To demonstrate that $\{A_0, A_1, \ldots A_n\} \not\supset B$, we exhibit a Kripke model (see Section 5.3 of [4] for more details on Kripke semantics[3]) in which an instance of B does not hold, but where $A_0, \ldots A_n$ hold for every formula. A full model, as described in [6], is sufficient. A full model is one where we can freely create predicates, as long as they satisfy the usual monotonicity requirements. So it is full in the sense that everything that potentially is the interpretation of a predicate actually is one. In Section 5 we will have to consider non-full models. An *intuitionistic Kripke model* is one where $\bot$ is never forced at any world. These are exactly the Kripke models that force EFQ.

Proposition 4. *If* $\vdash_{B_0+\cdots+B_m} \Phi$, *and* $\{A_0, A_1, \ldots A_n\} \supset \{B_0, B_1, \ldots B_n\}$, *then* $\vdash_{A_0+\cdots+A_n} \Phi$.

Proof. Consider a natural deduction proof of $\vdash_{B_0+\cdots+B_m} \Phi$. For each k, replace each instance of the rule B_k with a proof of $\vdash_{A_0+\cdots+A_n} B_k$. This produces the required derivation. $\square$

We examine relative strengths of a selection of schemata by considering their relations under '$\supset$'. The renaming of bound variables in a scheme should not affect its strength. For simplicity of notation, it is therefore assumed that when working a predicate Px, any variables other than x which appear in quantifiers are bound in Px. We write Py as shorthand for $Px[x/y]$.

3 Principles

In addition to DNE, LEM, and EFQ, which are included below for convenience, we examine the following principles as axiom schemata over minimal logic:

DNE$(A) := \neg\neg A \to A$ (Double Negation Elimination[4])

EFQ$(A) := \bot \to A$ (Ex Falso Quodlibet[5])

[3]While technically speaking the Kripke semantics described in [4] are for the intuitionistic case, we can use them in the minimal one, by not forcing and condition on $\bot$—that is treating it just like some fixed propositional symbol.

[4]Also known as "Stability".

[5]Also known as "explosion".

$\mathrm{LEM}(A) := A \vee \neg A$ (Law of Excluded Middle[6])

$\mathrm{WLEM}(A) := \neg A \vee \neg\neg A$ (Weak Law of Excluded Middle)

$\mathrm{DGP}(A, B) := (A \to B) \vee (B \to A)$ (Dirk Gently's Principle[7])

$\mathrm{DP}(Px) := \exists y(Py \to \forall x Px)$ (Drinker Paradox)

$\mathrm{H}\varepsilon(Px) := \exists y(\exists x Px \to Py)$ (Schematic Form of Hilbert's Epsilon)

$\mathrm{GMP}(Px) := \forall x \neg Px \to \exists y \neg Py$ (General Markov's Principle)

$\mathrm{GLPO}(Px) := \forall x \neg Px \vee \exists x Px$ (General Limited Principle of Omniscience)

$\mathrm{GLPO}'(Px) := \forall x Px \vee \exists x \neg Px$ (Alternate General Principle of Omniscience)

$\mathrm{DNS}_\exists(Px) := \forall x \neg\neg Px \to \neg\neg\forall x Px$ (Existential Double Negation Shift)

$\mathrm{DNS}_\forall(Px) := \neg\neg\exists x Px \to \exists x \neg\neg Px$ (Universal Double Negation Shift)

$\mathrm{UD}(Px, Q) := \forall x(Px \vee \exists x Q) \to \forall x Px \vee \exists x Q$ (Universal Distribution)

$\mathrm{IP}(Px, Q) := (\exists x Q \to \exists x Px) \to \exists x(\exists x Q \to Px)$ (Independence of Premise)

These principles are *all* classically derivable. That is, DNE implies all of these principles in the sense of $\supset$.

Principles UD and IP are also stated as

$$\mathrm{UD}(P, Q) \equiv \forall x(Px \vee Q) \to \forall x Px \vee Q$$
$$\mathrm{IP}(P, Q) \equiv (Q \to \exists x Px) \to \exists x(Q \to Px)$$

where x is not free in Q. These forms are syntactically equivalent to the definitions above for such Q, but the variable freedom condition is not convenient to work with when classifying schemata.

4 The Drinker Paradox and Hilbert's Epsilon

The *drinker paradox*, which was popularised by Smullyan in his book of puzzles [12], is the scheme

$$\mathrm{DP}(P) := \exists_y(Py \to \forall_x Px) \ .$$

[6] Also known as the "principle of excluded middle" and as "tertium non datur".

[7] The name DGP was introduced in [6], and is a literary reference to the novel [1], whose main character believes in "the fundamental interconnectedness of all things". DGP is otherwise also known as weak linearity.

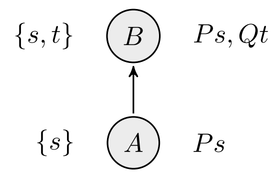


Figure 1: Kripke countermodel for DP(P) and Hε(Q)

Liberally interpreted, it states that (in every nonempty tavern) there exists a person such that if that person is drinking, then everyone (in the tavern) is drinking.

Classically this is true because there is always a *last* person to be drinking, and it is true for that person. Due to various non-classical interpretations of "there is", however, countermodels may be formed (see below). Notably, the constructivist may object that it is not always clear who is the last to drink—except in the case of a tavern in which the number of patrons is an enumerable positive integer amount.

The drinker paradox can alternatively be stated as

$$\exists_y \forall_x (Py \to Px)\ .$$ [8]

The dual of the drinker paradox is the scheme

$$\mathrm{H}\varepsilon(P) := \exists_y (\exists_x Px \to Py)\ ,$$

or alternatively,

$$\exists_y \forall_x (Px \to Py)\ .$$

Hε resembles an axiom scheme form of Hilbert's Epsilon operator [2]. In particular, within a natural deduction proof, from $\exists_x Px$ it allows a temporary name for a term satisfying P to be introduced. It is equivalent to *Independence of Premise*

$$\mathrm{IP}(P, Q) := (\exists x Q \to \exists x Px) \to \exists x (\exists x Q \to Px)\ .$$

This does not have the same power as Hilbert's Epsilon operator, however.[9]

We will now characterise (full Kripke) models in which DP and/or Hε hold, and use these to separate the two schemata. We will ignore models containing disconnected states (i.e. models where there are pairs of states such that every state related to one is unrelated to the other), as these can be examined by the characteristics of the individual components.

First consider a model with states $A \preceq B$ where there is a term $t \in T(B) \setminus T(A)$ (for example Figure 1). Create a predicate P with $A \Vdash Ps$ for all $s \in T(A)$ (and take the upwards closure). Now $B \nVdash Pt$, so $A \nVdash Ps \to \forall_x Px$, so DP fails. Furthermore, create a predicate Q with $B \Vdash Qt$ (and take the upwards closure). Then $B \Vdash \exists_x Qx$, but $B \nVdash Qs$ for any $s \in T(A)$. Thus Hε fails at A. Hence any model for either DP or Hε must have the same terms known at every related pair of states. We will from now on consider only these models. Moreover, note that a system with only one term at each state trivially models DP and Hε.

Now consider a model with a branch in it, i.e. there are states A, B, C such that $A \preceq B$, $A \preceq C$, and B is not related to C. Assume there are at least two distinct terms understood at

[8] For a proof that this is actually an equivalent formulation see the appendix.

[9] Milly Maietti has communicated to us the—currently unpublished—result that Hilbert's Epsilon operator implies the drinker paradox. Thus, together with our results in this paper this shows that the operator version of Hε is stronger than the scheme version.

A. Let t be one such term. Then create a predicate P with $B \Vdash Pt$, and $C \Vdash Ps$ for all terms $s \in T(A) : s \neq t$ (and any other states forcing these atomic formulae as required to maintain upwards closure). Certainly neither B nor C force $\forall_x Px$, but for every $u \in T(A)$ either B or C forces Pu, so DP fails at A. Furthermore if $u \in T(A)$ then either B or C will fail to force Pu, but both states force $\exists_x Px$, so Hε also fails at A. Hence any model for DP or Hε with more than two terms must have no branches, i.e. be totally ordered.

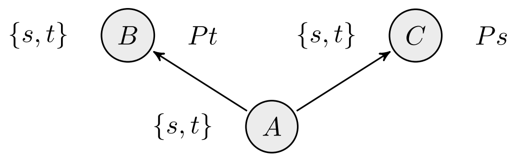


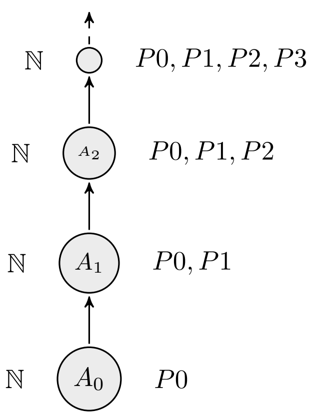
Figure 2: Kripke countermodel for both DP(P) and H$\varepsilon(P)$

Consider then a linear model with $n \in \mathbb{N}$ terms. Given a predicate Q if every state forces Qt for every term, or if every state does not force Qt for any term, then both DP and Hε trivially hold (by applying the classical reasoning), so we may suppose that this is not the case. For each term t, assign a set $U_t = \{A \in \Sigma | A \nVdash Qt\}$. By upwards closure (and the assumed linearity), if t and s are terms then either $U_t \subseteq U_s$ or $U_s \subseteq U_t$, meaning these sets are totally ordered with respect to the subset relation. There are finitely many of them, so there must be a maximal set $U_{t_{\max}}$ with associated term $t_{\max}$. Suppose a state A forces $Qt_{\max}$. Then $A \notin U_{t_{\max}}$, and so $A \notin U_s$ for every term s. Thus A forces Qs. Hence $Q_{t_{\max}} \to \forall_x Qx$ holds in the model, and so this is a model for DP. A similar argument shows Hε also holds, using sets $V_t = \{A \in \Sigma | A \Vdash Qt\}$, and in particular the maximal set V_{t_0}, to show that $\exists_x Qx \to Q_{t_0}$ is forced everywhere.

We now know that to separate DP and Hε we require linear models with infinitely many terms.

Proposition 5. *Hε does not imply DP in intuitionistic logic.*

Proof. Consider the (intuitionistic) Kripke model with infinitely many worlds below. In general, $A_n \preceq A_{n+1}$ and $A_n \Vdash P0 \dots Pn$.



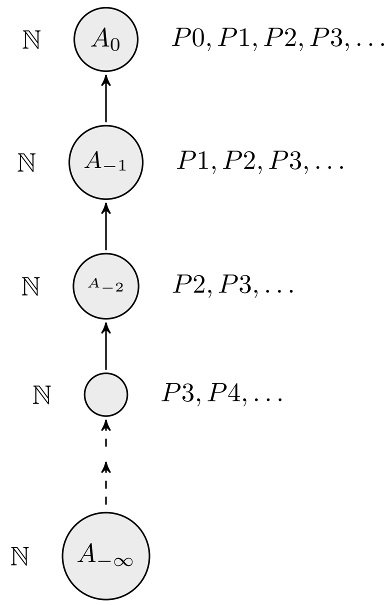
No state forces $\forall_x Px$, but for any term $t \in T(A_0)$ we have $A_0 \preceq A_t$ and $A_t \Vdash Pt$. Therefore $A_0 \nVdash \exists_y(Py \to \forall_x Px)$, i.e. DP does not hold in this model. (In fact, this argument works for any state.)

Now consider any predicate Qx in this model. If there is no state forcing Qt for some $t \in \mathbb{N}$, then trivially every state forces $\exists_x Qx \to Q0$, and it follows that Hε is forced. On the other hand, if there are $i, t \in \mathbb{N}$ such that $A_i \Vdash Qt$, then choose a pair i, t with minimal i. Then, by upwards closure, $\exists_x Qx \to Qt$ is forced by every state. Hence every state forces Hε. $\square$

The above model is also a countermodel for DNS$_\forall$. As $\bot Pt$ is not forced at any world for any t, $A_0 \Vdash \forall x \neg\neg Px$. However $A_0 \Vdash \neg \forall x Px$, so $A \nVdash \mathrm{DNS}_\forall(P)$.

Proposition 6. *DP does not imply Hε in intuitionistic logic.*

Proof. Consider the (intuitionistic) Kripke system with states $A_0 \succeq A_{-1} \succeq A_{-2} \succeq \ldots \succeq A_{-\infty}$. Let $T(B) = \mathbb{N}$ for every state B. Set $F(A_{-\infty}) = \emptyset$, and $F(A_{-n}) = \{Pn, P(n+1), P(n+2), \ldots\}$.



Let $t \in T(A_{-\infty})$. Then $A_{-(t+1)} \nVdash Pt$. However, $A_{-(t+1)} \Vdash P(t+1)$, so $A_{-(t+1)} \Vdash \exists_x Px$. Therefore $A_{-(t+1)} \nVdash \exists_x Px \to Pt$. Thus $A_{-\infty} \nVdash \exists_y(\exists_x Px \to Pt)$, so H$\varepsilon$ does not hold in this model.

Now consider any predicate Qx in this model. If every state forces $\forall_x Px$, then trivially they also force $\exists_y(Py \to \forall_x Qx)$. On the other hand, if there are $i, t \in \mathbb{N}$ such that $A_{-i} \nVdash Qt$ then choose a pair i, t with minimal i (i.e. maximal A_{-i}). Then by upwards closure, whenever Qt is forced, $\forall_x Qx$ is also forced. Hence every state forces $Qt \to \forall_x Px$, and so also forces DP. $\square$

In general, if a model contains an infinite sequence of states $A_0 \preceq A_1 \preceq \cdots$, then a predicate P can be constructed as in 5 in order to contradict DP. On the other hand if no such sequence exists then every sequence of related states has a maximal element. Following reasoning in 6 shows that DP will hold in such a model.

Conversely, if a model contains an infinite sequence of states $B_0 \succeq B_{-1} \succeq \cdots$, along with an element $B_{-\infty}$ which precedes every state in the sequence, then P may be constructed as in 6, contradicting Hε.

If, on the other hand, no such states exist, then every set of related states either contains a minimal element or has no lower bound, i.e. every set of states contains its infimum. Let A be a state in such a model. Now consider

$$S = \{B \in \Sigma \mid A \preceq B \wedge B \Vdash \exists_x Px\}.$$

If $S = \emptyset$, then vacuously $A \Vdash \exists_t Px \to Pt$ for every term t, so A forces Hε. Otherwise, note that A is certainly a lower bound for S. By the above assumption, S must have a minimum element B. Now $B \Vdash \exists_x Px$ so $B \Vdash Pt$ for some t. By upwards closure, $C \Vdash Pt$ for every $C \succeq B$, and so specifically for all $C \in S$. Thus whenever $A \preceq C$ and $C \Vdash \exists_x Px$, we have $C \in S$, so $C \Vdash Pt$. Then $A \Vdash \exists_x Px \to Pt$, and so A forces Hε. Hence Hε is forced by every state, and so holds in this model.

We now have a characterisation for models of DP and Hε. They are the models wherein every state has exactly one term, or otherwise,

- the model is linear, and
- all terms are known at all states, and
- (to model DP) every set of states has a maximal element, and/or
- (to model Hε) every set of states contains its infimum.

Where T is the set of terms (at every state):

	$\lvert T\rvert = 1$	$\lvert T\rvert \in \mathbb{N}$	$\lvert T\rvert \geq \lvert\mathbb{N}\rvert$
Branched	DP, Hε	Neither	Neither
Linear	DP, Hε	DP, Hε	Indeterminate
Linear, $\max\Pi$ exists for all $\Pi \subset \Sigma$	DP, Hε	DP, Hε	DP
Linear, $\inf\Pi \in \Pi$ for all $\Pi \subset \Sigma$	DP, Hε	DP, Hε	Hε
Both of the two above	DP, Hε	DP, Hε	DP, Hε

If a model has graph-like connectedness, where all related pairs of states have finitely many states between them (and so finite paths between them), then it cannot fall under the third or fourth rows, and so cannot separate DP and Hε.

The models are evocative of the intuitions. For, recall the "last drinker in the tavern" reason for accepting DP as true; similarly Hε can be justified by pointing to "the first person to drink".

Corollary 7. *DP and Hε are independent of each other in minimal logic with LEM (and so certainly over decidable predicates).*

Proof. Recall the Kripke systems in 6 and 5. Considering them now as minimal Kripke systems, and forcing $\perp$ at every state forces LEM everywhere, but their respective separations still hold. □

5 Separations without full models

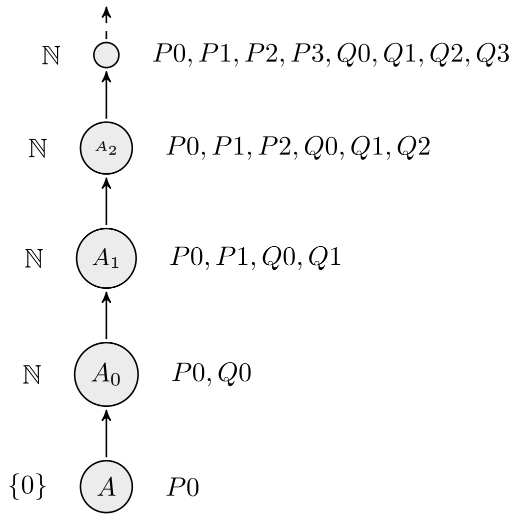
The principle of *Universal Distribution* is

$$\mathrm{UD}(P, Q) := \forall x(Px \vee \exists xQ) \to \forall xPx \vee \exists xQ\ .$$

Consider a full Kripke model in which all related worlds have the same domain. For a world A, if $A \Vdash \forall x(Px \vee \exists xQ)$ then $A \Vdash Pt \vee \exists xQ$ for all t in the domain. If $A \nVdash \exists xQ$, then $A \Vdash Pt$, and so $A \Vdash \forall xPx$. Therefore this is a model for UD. Hence any full Kripke countermodel for UD must have related worlds with different domains, and so must also be a countermodel to Hε (from the section above).

However, we cannot conclude H$\varepsilon \supset$ UD, as restriction to full Kripke models does not preserve completeness of Kripke semantics. To see that $\nvdash_{\mathrm{H}\varepsilon}$ UD(P, Q), we require a non-full countermodel to UD in which Hε holds. Therefore, a notion of an axiom scheme holding in a non-full model is needed. For every formula Φ in the model, H$\varepsilon(\Phi)$ should be forced. Formulae in the model should be at least closed with respect to the logical operations '$\to$', '$\wedge$', '$\vee$', '$\exists$', and '$\forall$', and '$\perp$' must also be a formula. The constants in the domain of the root world may also appear in formulae, but no others.

Consider the following infinite model:



We have $A \nVdash$ UD(P, Q).

Hε holds trivially for propositions. It remains to confirm that Hε holds for all predicates which exist in this model. Predicates are definable by combining 'Px' and 'Qx', with each other and with propositions, using the binary logical operations. Clearly, combining a predicate with itself in this manner is trivial. The propositions available are only $P0$, $Q0$, $\perp$, since

$$\begin{aligned} \forall xPx &\equiv \perp \\ \forall xQx &\equiv \perp \\ \exists xPx &\equiv P0 \\ \exists xQx &\equiv Q0 \end{aligned}$$

and $P0$, $Q0$, $\perp$ are closed under the binary logical operations (with respect to equivalence in this model). First,

$$\begin{aligned} Px \to Qx &\equiv Qx \\ Qx \to Px &\equiv P0 \\ Px \vee Qx &\equiv Px \\ Px \wedge Qx &\equiv Qx \end{aligned}$$

Now, with $P0$,

$$\begin{aligned}
Px \to P0 &\equiv P0 \\
P0 \to Px &\equiv Px \\
Px \vee P0 &\equiv P0 \\
Px \wedge P0 &\equiv Px \\
Qx \to Q0 &\equiv P0 \\
Q0 \to Qx &\equiv Qx \\
Qx \vee Q0 &\equiv Q0 \\
Qx \wedge Q0 &\equiv Qx.
\end{aligned}$$

With $Q0$,

$$\begin{aligned}
Px \to Q0 &\equiv Q0 \\
Q0 \to Px &\equiv Qx \\
Px \vee Q0 &\equiv P0 \\
Px \wedge Q0 &\equiv Qx \\
Qx \to Q0 &\equiv P0 \\
Q0 \to Qx &\equiv Qx \\
Qx \vee Q0 &\equiv Q0 \\
Qx \wedge Q0 &\equiv Qx
\end{aligned}$$

Finally, with $\perp$,

$$\begin{aligned}
Px \to \perp &\equiv \perp \\
\perp \to Px &\equiv P0 \\
Px \vee \perp &\equiv Px \\
Px \wedge \perp &\equiv \perp \\
Qx \to \perp &\equiv \perp \\
\perp \to Qx &\equiv P0 \\
Qx \vee \perp &\equiv Qx \\
Qx \wedge \perp &\equiv \perp.
\end{aligned}$$

Thus, Px and Qx really are the only predicates in this model. $A \Vdash \mathrm{H}\varepsilon(P), \mathrm{H}\varepsilon(Q)$, so we have a non-full model for $\mathrm{H}\varepsilon$ where UD fails.

6 From first-order to propositional schemata

Some first-order schemata are infinitary forms of propositional schemata. Viewing universal and existential generalisation as conjunction and disjunction on propositional symbols A and B, the drinker paradox becomes

$$(A \to (A \wedge B)) \vee (B \to (A \wedge B)),$$

and so DGP follows. A formal proof requires embedding A and B in a single predicate. For example, over the domain of natural numbers, a predicate P such that

$$P(0) \leftrightarrow A$$

$$P(Sn) \leftrightarrow B$$

gives $\mathrm{DP}(P) \vdash \mathrm{DGP}(A, B)$. However, such an embedding is not possible if the domain contains a single element. It was shown above that DP holds in models with branches if the domain contains only one term, while in [6] it is shown that DGP holds only in v-free models. Therefore there can be no way of deriving instances of DGP from DP without an embedding using two or more elements in the domain.

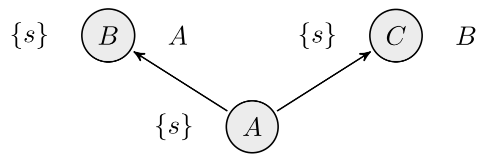


Figure 3: Kripke countermodel for $\mathrm{DGP}(A, B)$ where DP holds

Domain is a semantic concept. In order to derive an instance of DGP using DP, we require syntax corresponding to the existence of more than one (distinct) term.

Definition 8. Natural deduction can be extended by adding term names 0 and 1, a unary predicate D, and the rules

D0:

$$\frac{}{D0}\ \mathrm{D0}$$

D1:

$$\frac{}{\neg D1}\ \neg D1$$

Dx:

$$\frac{}{\forall_x\,(Dx \vee \neg Dx)}\ \mathrm{Dx}$$

D serves to make a weak distinction between the constants named by 0 and 1. [10]

Minimal (intuitionistic) logic extended by these rules is *two-termed* minimal (intuitionistic) logic, in which case we write '$\vdash_{TT}$' in place of '$\vdash$'.

Semantically, an intuitionistic Kripke model for TT is one in which there are two constants 0 and 1, $D0$ holds at every world, and Dn is not forced for $n \neq 0$. For minimal Kripke models, it is also possible instead that there is only one term, and $\perp$ holds everywhere.

In general, given propositional symbols A and B, we want to define a predicate P such that $\forall x Px \vdash A \wedge B$ and $\exists x Px \vdash A \vee B$.

We recover

$$\mathrm{DP}((Dx \to A) \wedge (\neg Dx \to B)) \vdash_{EFQ,TT} \mathrm{DGP}(A, B)$$

$$\mathrm{H}\varepsilon((Dx \to A) \wedge (\neg Dx \to B)) \vdash_{EFQ,TT} \mathrm{DGP}(A, B)$$

$$\mathrm{DP}((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)) \vdash_{TT} \mathrm{WLEM}(A)$$

$$\mathrm{H}\varepsilon((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)) \vdash_{TT} \mathrm{WLEM}(A)$$

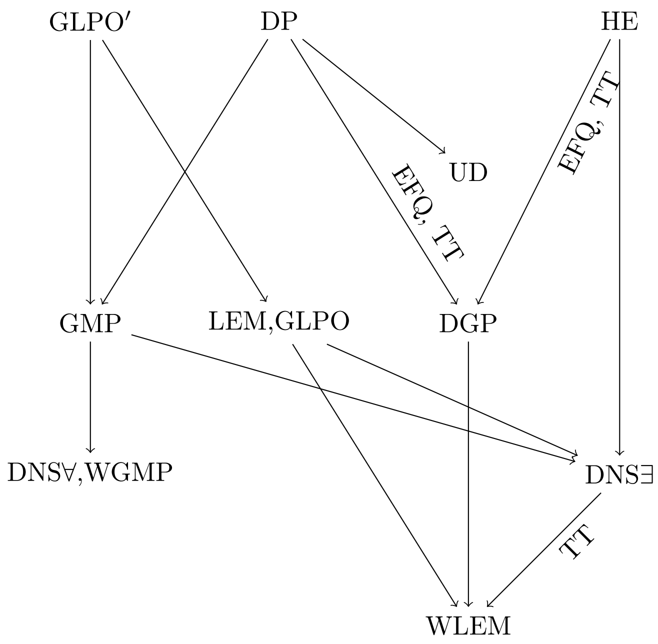
$$\mathrm{GMP}((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)) \vdash_{TT} \mathrm{WLEM}(A)$$

$$\mathrm{DNS}_{\exists}((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)) \vdash_{TT} \mathrm{WLEM}(A)\,.$$

[10] Bell in [3] suggests this "modest 'decidability' condition" in the form of a decidable equality for a single constant a, along with a constant $b \neq a$.

7 Hierarchy

The preorder from '$\supset$' produces a hierarchy. Arrows labelled with schemes indicate that those schemes must be taken together with the scheme at the tail to produce the scheme at the head.

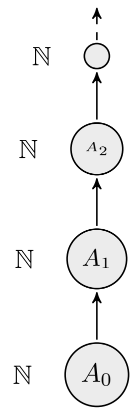


This hierarchy is complete in the sense the sense that no other unlabelled arrows may be added (see below). Moreover, for arrows labelled with at least one of EFQ, TT, the remaining open questions are if $\mathrm{GMP}, \mathrm{EFQ} \supset \mathrm{UD}$ and/or $\mathrm{GMP}, \mathrm{EFQ}, \mathrm{TT} \supset \mathrm{UD}$.

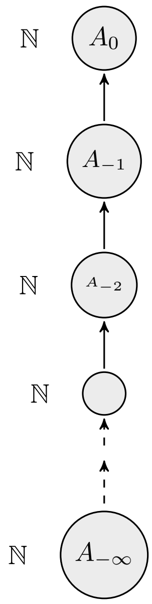
8 Semantics

In addition to the Kripke model analysis presented earlier, the following full models give all possible separations of the schemes under investigation. In cases where models should have TT, we omit labelling $D0$ on every world for the sake of brevity.

In [6], it is shown that DGP and WLEM hold in all v-free models, EFQ holds in a model if and only if $\perp$ is not forced anywhere, and LEM holds if only one world does not force $\perp$. Revisiting the countermodels (and previously given reasoning) for DP and Hε, we have

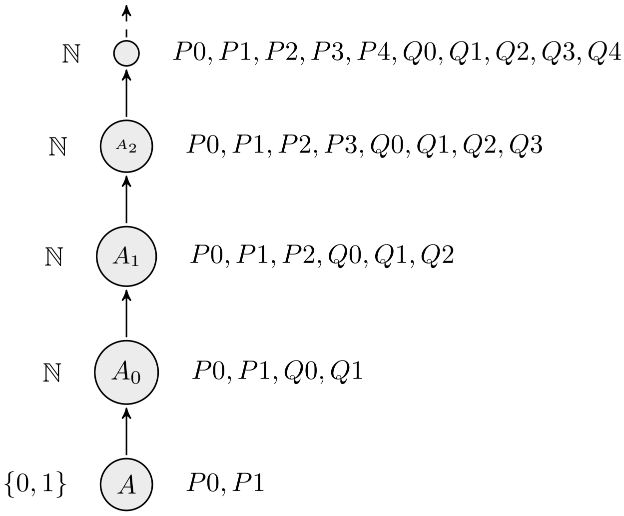


is a model for EFQ, TT, Hε, DGP, WLEM, UD, and a countermodel for DP, LEM, DNS$_\forall$, while

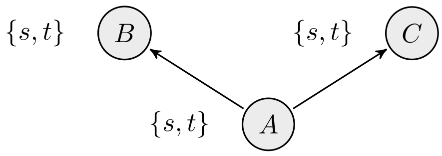


is a model for EFQ, TT, DP, DGP, WLEM and a countermodel for Hε, LEM.

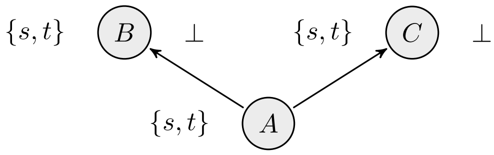
It is trivial to check that model presented in 5 can be modified as follows, to model both Hε and TT while still being a countermodel to UD.



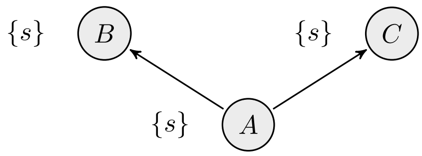
It is straightforward to check whether a scheme holds or fails in a given finite full model; as only (few and) finitely many upwards closed labellings of worlds are possible, and these may be checked exhaustively. We therefore present the remaining models without comment.



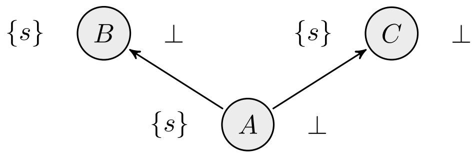
is a model for EFQ, TT, DNS$_\forall$, UD and a countermodel for DP, Hε, DGP, WLEM, DNS$_\exists$.



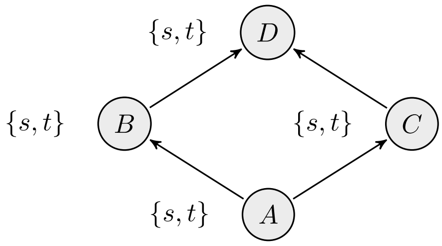
is a model for GLPO$'$, LEM and a countermodel for DP, Hε, DGP.



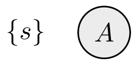
is a model for EFQ, DP, Hε and a countermodel for DGP, WLEM.



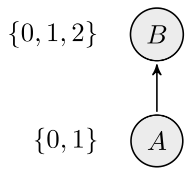
is a model for TT, DP, Hε, GLPO$'$ and a countermodel for DGP.



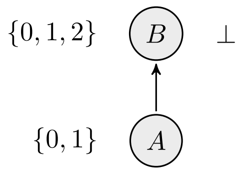
is a model for EFQ, TT, WLEM, GMP and a countermodel for DP, Hε, DGP.



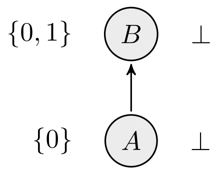
is a model for LEM, WLEM, DGP, GLPO$'$, GMP, DP, Hε, DNS$_\forall$, DNS$_\exists$, UD, EFQ and a countermodel for TT.



is a model for TT, EFQ, DGP, WLEM, DNS$_\forall$ and a countermodel for DNS$_\exists$, UD.



is a model for TT, LEM and a countermodel for EFQ, GMP, UD, DNS$_\forall$, DP, Hε.



is a model for TT, DGP, GMP, GLPO$'$ and a countermodel for EFQ, UD, Hε, DP.

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Appendix

Proposition 9. *DNE* $\supset$ *LEM*

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DNE}
\UnaryInfC{$\neg\neg(A \lor \neg A) \to (A \lor \neg A)$}
\AxiomC{}
\UnaryInfC{$\neg(A \lor \neg A)$}
\AxiomC{}
\UnaryInfC{$\neg(A \lor \neg A)$}
\AxiomC{}
\UnaryInfC{A}
\RightLabel{$\lor$I}
\UnaryInfC{$A \lor \neg A$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg A$}
\RightLabel{$\lor$I}
\UnaryInfC{$A \lor \neg A$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg(A \lor \neg A)$}
\RightLabel{$\to$E}
\BinaryInfC{$A \lor \neg A$}
\end{prooftree}
$$

$\square$

Proposition 10. *DNE* $\supset$ *EFQ*

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DNE}
\UnaryInfC{$\neg\neg A \to A$}
\AxiomC{}
\UnaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\to$E}
\BinaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$\bot \to A$}
\end{prooftree}
$$

$\square$

Proposition 11. *LEM*, *EFQ* $\supset$ *DNE*

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{LEM}
\UnaryInfC{$A \lor \neg A$}
\AxiomC{}
\UnaryInfC{A}
\AxiomC{}
\RightLabel{EFQ}
\UnaryInfC{$\bot \to A$}
\AxiomC{}
\UnaryInfC{$\neg\neg A$}
\AxiomC{}
\UnaryInfC{$\neg A$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$E}
\BinaryInfC{A}
\RightLabel{$\lor$E}
\TrinaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A \to A$}
\end{prooftree}
$$

$\square$

Proposition 12. *H*ε $\supset$ *IP*

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{HE}
\UnaryInfC{$\exists_y\,(\exists_x Px \to Py)$}
\AxiomC{}
\UnaryInfC{$\exists_x Px \to Py$}
\AxiomC{}
\UnaryInfC{$\exists_x A \to \exists_x Px$}
\AxiomC{}
\UnaryInfC{$\exists_x A$}
\RightLabel{$\to$E}
\BinaryInfC{$\exists_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{Py}
\RightLabel{$\to$I}
\UnaryInfC{$\exists_x A \to Py$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x\,(\exists_x A \to Px)$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists_x\,(\exists_x A \to Px)$}
\RightLabel{$\to$I}
\UnaryInfC{$(\exists_x A \to \exists_x Px) \to \exists_x\,(\exists_x A \to Px)$}
\end{prooftree}
$$

□

Proposition 13. *IP* $\supset$ *Hε*

Proof.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{IP}
\UnaryInfC{$(\exists_x Px \to \exists_x Px) \to \exists_x (\exists_x Px \to Px)$}
\AxiomC{}
\UnaryInfC{$\exists_x Px$}
\RightLabel{$\to$I}
\UnaryInfC{$\exists_x Px \to \exists_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\exists_x (\exists_x Px \to Px)$}
\AxiomC{}
\UnaryInfC{$\exists_x Px \to Px$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_y (\exists_x Px \to Py)$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists_y (\exists_x Px \to Py)$}
\end{prooftree}$$

□

Proposition 14. *LEM* $\supset$ *GLPO*

Proof.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{LEM}
\UnaryInfC{$\exists_x Px \lor \neg\exists_x Px$}
\AxiomC{}
\UnaryInfC{$\exists_x Px$}
\RightLabel{$\lor$I}
\UnaryInfC{$\forall_x \neg Px \lor \exists_x Px$}
\AxiomC{}
\UnaryInfC{$\neg\exists_x Px$}
\AxiomC{}
\UnaryInfC{Px}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg Px$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall_x \neg Px$}
\RightLabel{$\lor$I}
\UnaryInfC{$\forall_x \neg Px \lor \exists_x Px$}
\RightLabel{$\lor$E}
\TrinaryInfC{$\forall_x \neg Px \lor \exists_x Px$}
\end{prooftree}$$

□

Proposition 15. *GLPO* $\supset$ *LEM*

Proof.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{GLPO}
\UnaryInfC{$\forall_x \neg A \lor \exists_x A$}
\AxiomC{}
\UnaryInfC{$\forall_x \neg A$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg A$}
\RightLabel{$\lor$I}
\UnaryInfC{$A \lor \neg A$}
\AxiomC{}
\UnaryInfC{$\exists_x A$}
\AxiomC{}
\UnaryInfC{A}
\RightLabel{$\exists$E}
\BinaryInfC{A}
\RightLabel{$\lor$I}
\UnaryInfC{$A \lor \neg A$}
\RightLabel{$\lor$E}
\TrinaryInfC{$A \lor \neg A$}
\end{prooftree}$$

□

Proposition 16. *DNS*$_\forall$ $\supset$ *WGMP*

Proof.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{DNSU}
\UnaryInfC{$\forall_x \neg\neg Px \to \neg\neg\forall_x Px$}
\AxiomC{}
\UnaryInfC{$\neg\exists_x \neg Px$}
\AxiomC{}
\UnaryInfC{$\neg Px$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x \neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg Px$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall_x \neg\neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg\neg\forall_x Px$}
\AxiomC{}
\UnaryInfC{$\neg\forall_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\exists_x \neg Px$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\forall_x Px \to \neg\neg\exists_x \neg Px$}
\end{prooftree}$$

□

Proposition 17. *WGMP* $\supset$ *DNS*$_\forall$

Proof.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{WGMP}
\UnaryInfC{$\neg\forall_x Px \to \neg\neg\exists_x\neg Px$}
\AxiomC{}
\UnaryInfC{$\neg\forall_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg\neg\exists_x\neg Px$}
\AxiomC{}
\UnaryInfC{$\exists_x\neg Px$}
\AxiomC{}
\UnaryInfC{$\forall_x\neg\neg Px$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg\neg Px$}
\AxiomC{}
\UnaryInfC{$\neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\exists$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\exists_x\neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\forall_x Px$}
\RightLabel{$\to$I}
\UnaryInfC{$\forall_x\neg\neg Px \to \neg\neg\forall_x Px$}
\end{prooftree}$$

$\square$

Proposition 18. *DP*(P) *is equivalent to* $\exists_y\forall_x\,(Py \to Px)$

Proof. ($\Longrightarrow$)

$$\begin{prooftree}
\AxiomC{$\exists_y\,(Py \to \forall_x Px)$}
\AxiomC{}
\UnaryInfC{$Py \to \forall_x Px$}
\AxiomC{}
\UnaryInfC{Py}
\RightLabel{$\to$E}
\BinaryInfC{$\forall_x Px$}
\RightLabel{$\forall$E}
\UnaryInfC{Px}
\RightLabel{$\to$I}
\UnaryInfC{$Py \to Px$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall_x\,(Py \to Px)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_y\forall_x\,(Py \to Px)$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists_y\forall_x\,(Py \to Px)$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$\exists_y\forall_x\,(Py \to Px)$}
\AxiomC{}
\UnaryInfC{$\forall_x\,(Py \to Px)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Py \to Px$}
\AxiomC{}
\UnaryInfC{Py}
\RightLabel{$\to$E}
\BinaryInfC{Px}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall_x Px$}
\RightLabel{$\to$I}
\UnaryInfC{$Py \to \forall_x Px$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_y\,(Py \to \forall_x Px)$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists_y\,(Py \to \forall_x Px)$}
\end{prooftree}$$

$\square$

Proposition 19. *$H\varepsilon(P)$ is equivalent to $\exists_y \forall_x (Px \to Py)$*

Proof. ($\Longrightarrow$)

$$
\begin{prooftree}
\AxiomC{$\exists_y (\exists_x Px \to Py)$}
\AxiomC{}
\UnaryInfC{$\exists_x Px \to Py$}
\AxiomC{}
\UnaryInfC{Px}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{Py}
\RightLabel{$\to$I}
\UnaryInfC{$Px \to Py$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall_x (Px \to Py)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_y \forall_x (Px \to Py)$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists_y \forall_x (Px \to Py)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\exists_y \forall_x (Px \to Py)$}
\AxiomC{}
\UnaryInfC{$\exists_x Px$}
\AxiomC{}
\UnaryInfC{$\forall_x (Px \to Py)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Px \to Py$}
\AxiomC{}
\UnaryInfC{Px}
\RightLabel{$\to$E}
\BinaryInfC{Py}
\RightLabel{$\exists$E}
\BinaryInfC{Py}
\RightLabel{$\to$I}
\UnaryInfC{$\exists_x Px \to Py$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_y (\exists_x Px \to Py)$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists_y (\exists_x Px \to Py)$}
\end{prooftree}
$$

□

Proposition 20. *DNE, LEM, EFQ $\supset$ DP*

Proof. First

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DNE}
\UnaryInfC{$\neg\neg\exists_x \neg Px \to \exists_x \neg Px$}
\AxiomC{$\neg\forall_x Px$}
\AxiomC{}
\RightLabel{DNE}
\UnaryInfC{$\neg\neg Px \to Px$}
\AxiomC{}
\UnaryInfC{$\neg\exists_x \neg Px$}
\AxiomC{}
\UnaryInfC{$\neg Px$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x \neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{Px}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\exists_x \neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\exists_x \neg Px$}
\end{prooftree}
$$

Now,

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{LEM}
\UnaryInfC{$\forall_x Px \lor \neg\forall_x Px$}
\AxiomC{}
\UnaryInfC{$\forall_x Px$}
\RightLabel{$\to$I}
\UnaryInfC{$Py \to \forall_x Px$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_y \, (Py \to \forall_x Px)$}
\AxiomC{}
\UnaryInfC{$\neg\forall_x Px \vdash_{DNE,LEM,EFQ} \exists_x \neg Px$}
\AxiomC{}
\RightLabel{EFQ}
\UnaryInfC{$\bot \to \forall_x Px$}
\AxiomC{}
\UnaryInfC{$\neg Px$}
\AxiomC{}
\UnaryInfC{Px}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$E}
\BinaryInfC{$\forall_x Px$}
\RightLabel{$\to$I}
\UnaryInfC{$Px \to \forall_x Px$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_y \, (Py \to \forall_x Px)$}
\RightLabel{$\exists$E}
\BinaryInfC{$\exists_y \, (Py \to \forall_x Px)$}
\RightLabel{$\vee$E}
\TrinaryInfC{$\exists_y \, (Py \to \forall_x Px)$}
\end{prooftree}
$$

□

Proposition 21. *LEM* $\supset$ *WLEM*

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{LEM}
\UnaryInfC{$\neg A \lor \neg\neg A$}
\end{prooftree}
$$

□

Proposition 22. *GMP* $\supset$ *WGMP*

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\exists_x\neg Px}\qquad\dfrac{\dfrac{}{\neg\forall_x Px\to\exists_x\neg Px}\,\text{GMP}\qquad\dfrac{}{\neg\forall_x Px}}{\exists_x\neg Px}\,{\to}\text{E}}{\bot}\,{\to}\text{E}}{\neg\neg\exists_x\neg Px}\,{\to}\text{I}}{\neg\forall_x Px\to\neg\neg\exists_x\neg Px}\,{\to}\text{I}$$

$\square$

Proposition 23. *DGP* $\supset$ *WLEM*

Proof.

$$\dfrac{\dfrac{}{(A\to\neg A)\vee(\neg A\to A)}\,\text{DGP}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\to\neg A}\quad\dfrac{}{A}}{\neg A}\,{\to}\text{E}\qquad\dfrac{}{A}}{\bot}\,{\to}\text{E}}{\neg A}\,{\to}\text{I}}{\neg A\vee\neg\neg A}\,{\vee}\text{I}\qquad\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\qquad\dfrac{\dfrac{}{\neg A\to A}\quad\dfrac{}{\neg A}}{A}\,{\to}\text{E}}{\bot}\,{\to}\text{E}}{\neg\neg A}\,{\to}\text{I}}{\neg A\vee\neg\neg A}\,{\vee}\text{I}}{\neg A\vee\neg\neg A}\,{\vee}\text{E}$$

$\square$

Proposition 24. *GLPO*$'$ $\supset$ *LEM*

Proof.

$$\dfrac{\dfrac{}{\forall_x A\vee\exists_x\neg A}\,\text{GLPOA}\qquad\dfrac{\dfrac{\dfrac{}{\forall_x A}}{A}\,\forall\text{E}}{A\vee\neg A}\,{\vee}\text{I}\qquad\dfrac{\dfrac{\dfrac{}{\exists_x\neg A}\quad\dfrac{}{\neg A}}{\neg A}\,\exists\text{E}}{A\vee\neg A}\,{\vee}\text{I}}{A\vee\neg A}\,{\vee}\text{E}$$

$\square$

Proposition 25. *GLPO*$'$ $\supset$ *GMP*

Proof.

$$\dfrac{\dfrac{\dfrac{}{\forall_x Px\vee\exists_x\neg Px}\,\text{GLPOA}\qquad\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\forall_x Px}\quad\dfrac{}{\forall_x Px}}{\bot}\,{\to}\text{E}}{\neg Px}\,{\to}\text{I}}{\exists_x\neg Px}\,\exists\text{I}\qquad\dfrac{}{\exists_x\neg Px}}{\exists_x\neg Px}\,{\vee}\text{E}}{\neg\forall_x Px\to\exists_x\neg Px}\,{\to}\text{I}$$

$\square$

Proposition 26. *DP* $\supset$ *UD*

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DP}
\UnaryInfC{$\exists_y \, (Py \to \forall_x Px)$}
\AxiomC{}
\UnaryInfC{$\forall_x \, (Px \lor \exists_x A)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Py \lor \exists_x A$}
\AxiomC{}
\UnaryInfC{$Py \to \forall_x Px$}
\AxiomC{}
\UnaryInfC{Py}
\RightLabel{$\to$E}
\BinaryInfC{$\forall_x Px$}
\RightLabel{$\lor$I}
\UnaryInfC{$\forall_x Px \lor \exists_x A$}
\AxiomC{}
\UnaryInfC{$\exists_x A$}
\RightLabel{$\lor$I}
\UnaryInfC{$\forall_x Px \lor \exists_x A$}
\RightLabel{$\lor$E}
\TrinaryInfC{$\forall_x Px \lor \exists_x A$}
\RightLabel{$\exists$E}
\BinaryInfC{$\forall_x Px \lor \exists_x A$}
\RightLabel{$\to$I}
\UnaryInfC{$\forall_x \, (Px \lor \exists_x A) \to (\forall_x Px \lor \exists_x A)$}
\end{prooftree}
$$

□

Proposition 27. *DP* $\supset$ *GMP*

Proof.

$$
\begin{prooftree}
\AxiomC{}\RightLabel{DP}\UnaryInfC{$\exists_y\,(Py\to\forall_x Px)$}
\AxiomC{}\UnaryInfC{$\neg\forall_x Px$}
\AxiomC{}\UnaryInfC{$Py\to\forall_x Px$}
\AxiomC{}\UnaryInfC{Py}
\RightLabel{$\to$E}\BinaryInfC{$\forall_x Px$}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg Py$}
\RightLabel{$\exists$I}\UnaryInfC{$\exists_x\neg Px$}
\RightLabel{$\exists$E}\BinaryInfC{$\exists_x\neg Px$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\forall_x Px\to\exists_x\neg Px$}
\end{prooftree}
$$

$\square$

Proposition 28. *H*ε $\supset$ *DNS*$_\exists$

Proof.

$$
\begin{prooftree}
\AxiomC{}\RightLabel{HE}\UnaryInfC{$\exists_y\,(\exists_x Px\to Py)$}
\AxiomC{}\UnaryInfC{$\neg\neg\exists_x Px$}
\AxiomC{}\UnaryInfC{$\neg Py$}
\AxiomC{}\UnaryInfC{$\exists_x Px\to Py$}
\AxiomC{}\UnaryInfC{$\exists_x Px$}
\RightLabel{$\to$E}\BinaryInfC{Py}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\exists_x Px$}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg Py$}
\RightLabel{$\exists$I}\UnaryInfC{$\exists_x\neg\neg Px$}
\RightLabel{$\exists$E}\BinaryInfC{$\exists_x\neg\neg Px$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg\exists_x Px\to\exists_x\neg\neg Px$}
\end{prooftree}
$$

$\square$

Proposition 29. *GLPO* $\supset$ *DNS*$_\exists$

Proof.

$$
\begin{prooftree}
\AxiomC{}\RightLabel{GLPO}\UnaryInfC{$\forall_x\neg Px\vee\exists_x Px$}
\AxiomC{}\UnaryInfC{$\neg\neg\exists_x Px$}
\AxiomC{}\UnaryInfC{$\exists_x Px$}
\AxiomC{}\UnaryInfC{$\forall_x\neg Px$}
\RightLabel{$\forall$E}\UnaryInfC{$\neg Px$}
\AxiomC{}\UnaryInfC{Px}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\exists$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\exists_x Px$}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg Px$}
\RightLabel{$\exists$I}\UnaryInfC{$\exists_x\neg\neg Px$}
\AxiomC{}\UnaryInfC{$\exists_x Px$}
\AxiomC{}\UnaryInfC{$\neg Px$}
\AxiomC{}\UnaryInfC{Px}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg Px$}
\RightLabel{$\exists$I}\UnaryInfC{$\exists_x\neg\neg Px$}
\RightLabel{$\exists$E}\BinaryInfC{$\exists_x\neg\neg Px$}
\RightLabel{$\vee$E}\TrinaryInfC{$\exists_x\neg\neg Px$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg\exists_x Px\to\exists_x\neg\neg Px$}
\end{prooftree}
$$

$\square$

Proposition 30. $GMP \supset DNS_{\exists}$

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{GMP}
\UnaryInfC{$\neg\forall_x\neg Px \to \exists_x\neg\neg Px$}
\AxiomC{}
\UnaryInfC{$\neg\neg\exists_x Px$}
\AxiomC{}
\UnaryInfC{$\exists_x Px$}
\AxiomC{}
\UnaryInfC{$\forall_x\neg Px$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Px$}
\AxiomC{}
\UnaryInfC{Px}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\exists$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\exists_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\forall_x\neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\exists_x\neg\neg Px$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\exists_x Px \to \exists_x\neg\neg Px$}
\end{prooftree}
$$

□

Proposition 31. $GLPO' \supset WGMP$

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{GLPOA}
\UnaryInfC{$\forall_x Px \vee \exists_x\neg Px$}
\AxiomC{}
\UnaryInfC{$\neg\forall_x Px$}
\AxiomC{}
\UnaryInfC{$\forall_x Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\exists_x\neg Px$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\forall_x Px \to \neg\neg\exists_x\neg Px$}
\AxiomC{}
\UnaryInfC{$\neg\exists_x\neg Px$}
\AxiomC{}
\UnaryInfC{$\exists_x\neg Px$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\exists_x\neg Px$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\forall_x Px \to \neg\neg\exists_x\neg Px$}
\RightLabel{$\vee$E}
\TrinaryInfC{$\neg\forall_x Px \to \neg\neg\exists_x\neg Px$}
\end{prooftree}
$$

□

Proposition 32. $DP, EFQ, TT \supset DGP$

Proof. Lemma 1:

$$
\begin{prooftree}
\AxiomC{$((Dy \to A) \land (\neg Dy \to B)) \to \forall_x ((Dx \to A) \land (\neg Dx \to B))$}
\AxiomC{}
\UnaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$Dy \to A$}
\AxiomC{}
\RightLabel{EFQ}
\UnaryInfC{$\bot \to B$}
\AxiomC{}
\UnaryInfC{$\neg Dy$}
\AxiomC{Dy}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$E}
\BinaryInfC{B}
\RightLabel{$\to$I}
\UnaryInfC{$\neg Dy \to B$}
\RightLabel{$\land$I}
\BinaryInfC{$(Dy \to A) \land (\neg Dy \to B)$}
\RightLabel{$\to$E}
\BinaryInfC{$\forall_x ((Dx \to A) \land (\neg Dx \to B))$}
\RightLabel{$\forall$E}
\UnaryInfC{$(D1 \to A) \land (\neg D1 \to B)$}
\AxiomC{}
\UnaryInfC{$\neg D1 \to B$}
\AxiomC{}
\RightLabel{DO}
\UnaryInfC{$\neg D1$}
\RightLabel{$\to$E}
\BinaryInfC{B}
\RightLabel{$\land$E}
\BinaryInfC{B}
\RightLabel{$\to$I}
\UnaryInfC{$A \to B$}
\RightLabel{$\lor$I}
\UnaryInfC{$(A \to B) \lor (B \to A)$}
\end{prooftree}
$$

Lemma 2:

$$
\begin{prooftree}
\AxiomC{$((Dy \to A) \land (\neg Dy \to B)) \to \forall_x ((Dx \to A) \land (\neg Dx \to B))$}
\AxiomC{}
\RightLabel{EFQ}
\UnaryInfC{$\bot \to A$}
\AxiomC{$\neg Dy$}
\AxiomC{}
\UnaryInfC{Dy}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$E}
\BinaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$Dy \to A$}
\AxiomC{}
\UnaryInfC{B}
\RightLabel{$\to$I}
\UnaryInfC{$\neg Dy \to B$}
\RightLabel{$\land$I}
\BinaryInfC{$(Dy \to A) \land (\neg Dy \to B)$}
\RightLabel{$\to$E}
\BinaryInfC{$\forall_x ((Dx \to A) \land (\neg Dx \to B))$}
\RightLabel{$\forall$E}
\UnaryInfC{$(D0 \to A) \land (\neg D0 \to B)$}
\AxiomC{}
\UnaryInfC{$D0 \to A$}
\AxiomC{}
\RightLabel{DZ}
\UnaryInfC{$D0$}
\RightLabel{$\to$E}
\BinaryInfC{A}
\RightLabel{$\land$E}
\BinaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$B \to A$}
\RightLabel{$\lor$I}
\UnaryInfC{$(A \to B) \lor (B \to A)$}
\end{prooftree}
$$

Now,

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DP}
\UnaryInfC{$\exists_y (((Dy \to A) \land (\neg Dy \to B)) \to \forall_x ((Dx \to A) \land (\neg Dx \to B)))$}
\AxiomC{}
\RightLabel{DX}
\UnaryInfC{$\forall_x (Dx \lor \neg Dx)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Dy \lor \neg Dy$}
\AxiomC{}
\UnaryInfC{Lemma 1}
\AxiomC{}
\UnaryInfC{Lemma 2}
\RightLabel{$\lor$E}
\TrinaryInfC{$(A \to B) \lor (B \to A)$}
\RightLabel{$\exists$E}
\BinaryInfC{$(A \to B) \lor (B \to A)$}
\end{prooftree}
$$

□

Proposition 33. *DP, TT* $\supset$ *WLEM*

Proof. Lemma 1:

$$
\begin{prooftree}
\AxiomC{$((Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)) \to \forall_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\AxiomC{}
\UnaryInfC{$\neg A$}
\AxiomC{}
\UnaryInfC{A}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$Dy \to \neg\neg A$}
\AxiomC{}
\UnaryInfC{$\neg Dy$}
\AxiomC{Dy}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg Dy \to \neg A$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)$}
\RightLabel{$\to$E}
\BinaryInfC{$\forall_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\RightLabel{$\forall$E}
\UnaryInfC{$(D1 \to \neg\neg A) \wedge (\neg D1 \to \neg A)$}
\AxiomC{}
\UnaryInfC{$\neg D1 \to \neg A$}
\AxiomC{}
\RightLabel{DO}
\UnaryInfC{$\neg D1$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg A$}
\AxiomC{}
\UnaryInfC{A}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\wedge$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg A$}
\RightLabel{$\vee$I}
\UnaryInfC{$\neg A \vee \neg\neg A$}
\end{prooftree}
$$

Lemma 2:

$$
\begin{prooftree}
\AxiomC{$((Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)) \to \forall_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\AxiomC{$\neg Dy$}
\AxiomC{}
\UnaryInfC{Dy}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$Dy \to \neg\neg A$}
\AxiomC{}
\UnaryInfC{$\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg Dy \to \neg A$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)$}
\RightLabel{$\to$E}
\BinaryInfC{$\forall_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\RightLabel{$\forall$E}
\UnaryInfC{$(D0 \to \neg\neg A) \wedge (\neg D0 \to \neg A)$}
\AxiomC{}
\UnaryInfC{$D0 \to \neg\neg A$}
\AxiomC{}
\RightLabel{DZ}
\UnaryInfC{$D0$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg\neg A$}
\AxiomC{}
\UnaryInfC{$\neg A$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\wedge$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\vee$I}
\UnaryInfC{$\neg A \vee \neg\neg A$}
\end{prooftree}
$$

Now,

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DP}
\UnaryInfC{$\exists_y (((Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)) \to \forall_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)))$}
\AxiomC{}
\RightLabel{DX}
\UnaryInfC{$\forall_x (Dx \vee \neg Dx)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Dy \vee \neg Dy$}
\AxiomC{}
\UnaryInfC{Lemma 1}
\AxiomC{}
\UnaryInfC{Lemma 2}
\RightLabel{$\vee$E}
\TrinaryInfC{$\neg A \vee \neg\neg A$}
\RightLabel{$\exists$E}
\BinaryInfC{$\neg A \vee \neg\neg A$}
\end{prooftree}
$$

$\square$

Proposition 34. $H\varepsilon, EFQ, TT \supset DGP$

Proof. Lemma 1:

$$
\begin{prooftree}
\AxiomC{$\exists_x\,((Dx \to A) \wedge (\neg Dx \to B)) \to ((Dy \to A) \wedge (\neg Dy \to B))$}
\AxiomC{}
\RightLabel{EFQ}
\UnaryInfC{$\bot \to A$}
\AxiomC{}
\RightLabel{DO}
\UnaryInfC{$\neg D1$}
\AxiomC{}
\UnaryInfC{$D1$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$E}
\BinaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$D1 \to A$}
\AxiomC{}
\UnaryInfC{B}
\RightLabel{$\to$I}
\UnaryInfC{$\neg D1 \to B$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(D1 \to A) \wedge (\neg D1 \to B)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x\,((Dx \to A) \wedge (\neg Dx \to B))$}
\RightLabel{$\to$E}
\BinaryInfC{$(Dy \to A) \wedge (\neg Dy \to B)$}
\AxiomC{}
\UnaryInfC{$Dy \to A$}
\AxiomC{Dy}
\RightLabel{$\to$E}
\BinaryInfC{A}
\RightLabel{$\wedge$E}
\BinaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$B \to A$}
\RightLabel{$\vee$I}
\UnaryInfC{$(A \to B) \vee (B \to A)$}
\end{prooftree}
$$

Lemma 2:

$$
\begin{prooftree}
\AxiomC{$\exists_x\,((Dx \to A) \wedge (\neg Dx \to B)) \to ((Dy \to A) \wedge (\neg Dy \to B))$}
\AxiomC{}
\UnaryInfC{A}
\RightLabel{$\to$I}
\UnaryInfC{$D0 \to A$}
\AxiomC{}
\RightLabel{EFQ}
\UnaryInfC{$\bot \to B$}
\AxiomC{}
\UnaryInfC{$\neg D0$}
\AxiomC{}
\RightLabel{DZ}
\UnaryInfC{$D0$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$E}
\BinaryInfC{B}
\RightLabel{$\to$I}
\UnaryInfC{$\neg D0 \to B$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(D0 \to A) \wedge (\neg D0 \to B)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x\,((Dx \to A) \wedge (\neg Dx \to B))$}
\RightLabel{$\to$E}
\BinaryInfC{$(Dy \to A) \wedge (\neg Dy \to B)$}
\AxiomC{}
\UnaryInfC{$\neg Dy \to B$}
\AxiomC{$\neg Dy$}
\RightLabel{$\to$E}
\BinaryInfC{B}
\RightLabel{$\wedge$E}
\BinaryInfC{B}
\RightLabel{$\to$I}
\UnaryInfC{$A \to B$}
\RightLabel{$\vee$I}
\UnaryInfC{$(A \to B) \vee (B \to A)$}
\end{prooftree}
$$

Now,

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{HE}
\UnaryInfC{$\exists_y\,(\exists_x\,((Dx \to A) \wedge (\neg Dx \to B)) \to ((Dy \to A) \wedge (\neg Dy \to B)))$}
\AxiomC{}
\RightLabel{DX}
\UnaryInfC{$\forall_x\,(Dx \vee \neg Dx)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Dy \vee \neg Dy$}
\AxiomC{}
\UnaryInfC{Lemma 1}
\AxiomC{}
\UnaryInfC{Lemma 2}
\RightLabel{$\vee$E}
\TrinaryInfC{$(A \to B) \vee (B \to A)$}
\RightLabel{$\exists$E}
\BinaryInfC{$(A \to B) \vee (B \to A)$}
\end{prooftree}
$$

$\square$

Proposition 35. $H\varepsilon, TT \supset WLEM$

Proof. Lemma 1:

$$
\AxiomC{$\exists_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)) \to ((Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A))$}
\AxiomC{}\RightLabel{DO}\UnaryInfC{$\neg D1$}
\AxiomC{}\UnaryInfC{$D1$}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg A$}
\RightLabel{$\to$I}\UnaryInfC{$D1 \to \neg\neg A$}
\AxiomC{}\UnaryInfC{$\neg A$}
\RightLabel{$\to$I}\UnaryInfC{$\neg D1 \to \neg A$}
\RightLabel{$\wedge$I}\BinaryInfC{$(D1 \to \neg\neg A) \wedge (\neg D1 \to \neg A)$}
\RightLabel{$\exists$I}\UnaryInfC{$\exists_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\RightLabel{$\to$E}\BinaryInfC{$(Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)$}
\AxiomC{}\UnaryInfC{$Dy \to \neg\neg A$}
\AxiomC{Dy}
\RightLabel{$\to$E}\BinaryInfC{$\neg\neg A$}
\AxiomC{}\UnaryInfC{$\neg A$}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\wedge$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg A$}
\RightLabel{$\vee$I}\UnaryInfC{$\neg A \vee \neg\neg A$}
\DisplayProof
$$

Lemma 2:

$$
\AxiomC{$\exists_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)) \to ((Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A))$}
\AxiomC{}\UnaryInfC{$\neg A$}
\AxiomC{}\UnaryInfC{A}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg\neg A$}
\RightLabel{$\to$I}\UnaryInfC{$D0 \to \neg\neg A$}
\AxiomC{}\UnaryInfC{$\neg D0$}
\AxiomC{}\RightLabel{DZ}\UnaryInfC{$D0$}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg A$}
\RightLabel{$\to$I}\UnaryInfC{$\neg D0 \to \neg A$}
\RightLabel{$\wedge$I}\BinaryInfC{$(D0 \to \neg\neg A) \wedge (\neg D0 \to \neg A)$}
\RightLabel{$\exists$I}\UnaryInfC{$\exists_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\RightLabel{$\to$E}\BinaryInfC{$(Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)$}
\AxiomC{}\UnaryInfC{$\neg Dy \to \neg A$}
\AxiomC{$\neg Dy$}
\RightLabel{$\to$E}\BinaryInfC{$\neg A$}
\AxiomC{}\UnaryInfC{A}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\wedge$E}\BinaryInfC{$\bot$}
\RightLabel{$\to$I}\UnaryInfC{$\neg A$}
\RightLabel{$\vee$I}\UnaryInfC{$\neg A \vee \neg\neg A$}
\DisplayProof
$$

Now,

$$
\AxiomC{}\RightLabel{HE}\UnaryInfC{$\exists_y (\exists_x ((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)) \to ((Dy \to \neg\neg A) \wedge (\neg Dy \to \neg A)))$}
\AxiomC{}\RightLabel{DX}\UnaryInfC{$\forall_x (Dx \vee \neg Dx)$}
\RightLabel{$\forall$E}\UnaryInfC{$Dy \vee \neg Dy$}
\AxiomC{}\UnaryInfC{Lemma 1}
\AxiomC{}\UnaryInfC{Lemma 2}
\RightLabel{$\vee$E}\TrinaryInfC{$\neg A \vee \neg\neg A$}
\RightLabel{$\exists$E}\BinaryInfC{$\neg A \vee \neg\neg A$}
\DisplayProof
$$

□

Proposition 36. *GMP, TT* $\supset$ *WLEM*

Proof. Lemma 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall_x\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right)}}{(D0 \to \neg\neg A) \land (\neg D0 \to \neg A)}\,\forall\mathrm{E} \qquad \dfrac{\dfrac{}{D0 \to \neg\neg A} \quad \dfrac{}{D0}\,\mathrm{DZ}}{\neg\neg A}\,{\to}\mathrm{E}}{\neg\neg A}\,\land\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{}{\forall_x\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right)}}{(D1 \to \neg\neg A) \land (\neg D1 \to \neg A)}\,\forall\mathrm{E} \qquad \dfrac{\dfrac{}{\neg D1 \to \neg A} \quad \dfrac{}{\neg D1}\,\mathrm{DO}}{\neg A}\,{\to}\mathrm{E}}{\neg A}\,\land\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\neg\forall_x\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right)}\,{\to}\mathrm{I}$$

Lemma 2:

$$\dfrac{\dfrac{\dfrac{\neg\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A} \quad \dfrac{}{A}}{\bot}\,{\to}\mathrm{E}}{\neg\neg A}\,{\to}\mathrm{I}}{Dx \to \neg\neg A}\,{\to}\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg Dx} \quad Dx}{\bot}\,{\to}\mathrm{E}}{\neg A}\,{\to}\mathrm{I}}{\neg Dx \to \neg A}\,{\to}\mathrm{I}}{(Dx \to \neg\neg A) \land (\neg Dx \to \neg A)}\,\land\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg A}\,{\to}\mathrm{I}}{\neg A \lor \neg\neg A}\,\lor\mathrm{I}$$

Lemma 3:

$$\dfrac{\dfrac{\dfrac{\neg\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg Dx \quad \dfrac{}{Dx}}{\bot}\,{\to}\mathrm{E}}{\neg\neg A}\,{\to}\mathrm{I}}{Dx \to \neg\neg A}\,{\to}\mathrm{I} \qquad \dfrac{\dfrac{}{\neg A}}{\neg Dx \to \neg A}\,{\to}\mathrm{I}}{(Dx \to \neg\neg A) \land (\neg Dx \to \neg A)}\,\land\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\neg A}\,{\to}\mathrm{I}}{\neg A \lor \neg\neg A}\,\lor\mathrm{I}$$

$$\dfrac{\dfrac{\dfrac{}{\neg\forall_x\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right) \to \exists_x\neg\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right)}\,\mathrm{GMP} \qquad \dfrac{}{\text{Lemma 1}}}{\exists_x\neg\left((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)\right)}\,{\to}\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{}{\forall_x(Dx \lor \neg Dx)}\,\mathrm{DX}}{Dx \lor \neg Dx}\,\forall\mathrm{E} \qquad \dfrac{}{\text{Lemma 2}} \qquad \dfrac{}{\text{Lemma 3}}}{\neg A \lor \neg\neg A}\,\lor\mathrm{E}}{\neg A \lor \neg\neg A}\,\exists\mathrm{E}$$

$\square$

Proposition 37. *$DP, LEM \supset GLPO'$*

Proof.

$$\dfrac{\dfrac{}{\exists_y\,(Py \to \forall_x Px)}\,\mathrm{DP} \qquad \dfrac{\dfrac{}{Py \vee \neg Py}\,\mathrm{LEM} \qquad \dfrac{\dfrac{\dfrac{}{Py \to \forall_x Px} \qquad \dfrac{}{Py}}{\forall_x Px}\,{\to}\mathrm{E}}{\forall_x Px \vee \exists_x \neg Px}\,{\vee}\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{}{\neg Py}}{\exists_x \neg Px}\,\exists\mathrm{I}}{\forall_x Px \vee \exists_x \neg Px}\,{\vee}\mathrm{I}}{\forall_x Px \vee \exists_x \neg Px}\,{\vee}\mathrm{E}}{\forall_x Px \vee \exists_x \neg Px}\,\exists\mathrm{E}$$

□

Proposition 38. $DNS_{\exists}, TT \supset WLEM$

Proof. Lemma 1:

$$
\begin{prooftree}
\AxiomC{$\neg\exists_x\,((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\AxiomC{}
\UnaryInfC{$(D0 \to \neg\neg A) \wedge (\neg D0 \to \neg A)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x\,((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg((D0 \to \neg\neg A) \wedge (\neg D0 \to \neg A))$}
\AxiomC{}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$D0 \to \neg\neg A$}
\AxiomC{}
\UnaryInfC{$\neg D0$}
\AxiomC{}
\RightLabel{DZ}
\UnaryInfC{$D0$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg D0 \to \neg A$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(D0 \to \neg\neg A) \wedge (\neg D0 \to \neg A)$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\neg A$}
\end{prooftree}
$$

Lemma 2:

$$
\begin{prooftree}
\AxiomC{$\neg\exists_x\,((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\AxiomC{}
\UnaryInfC{$(D1 \to \neg\neg A) \wedge (\neg D1 \to \neg A)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists_x\,((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg((D1 \to \neg\neg A) \wedge (\neg D1 \to \neg A))$}
\AxiomC{}
\RightLabel{DO}
\UnaryInfC{$\neg D1$}
\AxiomC{}
\UnaryInfC{$D1$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$D1 \to \neg\neg A$}
\AxiomC{}
\UnaryInfC{$\neg A$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg D1 \to \neg A$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(D1 \to \neg\neg A) \wedge (\neg D1 \to \neg A)$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A$}
\end{prooftree}
$$

Lemma 3:

$$
\begin{prooftree}
\AxiomC{$\neg\neg((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\AxiomC{}
\UnaryInfC{$(Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A)$}
\AxiomC{}
\UnaryInfC{$Dx \to \neg\neg A$}
\AxiomC{Dx}
\RightLabel{$\to$E}
\BinaryInfC{$\neg\neg A$}
\AxiomC{}
\UnaryInfC{$\neg A$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\wedge$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg((Dx \to \neg\neg A) \wedge (\neg Dx \to \neg A))$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\vee$I}
\UnaryInfC{$\neg A \vee \neg\neg A$}
\end{prooftree}
$$

Lemma 4:

$$
\dfrac{\neg\neg((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)) \qquad \dfrac{\dfrac{\overline{(Dx \to \neg\neg A) \land (\neg Dx \to \neg A)} \qquad \dfrac{\dfrac{\dfrac{\overline{\neg A} \qquad \overline{A}}{\bot}\,{\to}\mathrm{E}}{\neg\neg A}\,{\to}\mathrm{I} \qquad \dfrac{\overline{\neg Dx \to \neg A} \qquad \neg Dx}{\neg A}\,{\to}\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\bot}\,{\land}\mathrm{E}}{\neg((Dx \to \neg\neg A) \land (\neg Dx \to \neg A))}\,{\to}\mathrm{I}}{\dfrac{\dfrac{\bot}{\neg A}\,{\to}\mathrm{I}}{\neg A \lor \neg\neg A}\,{\lor}\mathrm{I}}\,{\to}\mathrm{E}
$$

Where $\Phi := \neg\neg\exists_x\,((Dx \to \neg\neg A) \land (\neg Dx \to \neg A)) \to \exists_x \neg\neg((Dx \to \neg\neg A) \land (\neg Dx \to \neg A))$,

$$
\dfrac{\dfrac{\overline{\Phi}\;\mathrm{DNSE} \qquad \dfrac{\dfrac{\overline{\text{Lemma 1}} \qquad \overline{\text{Lemma 2}}}{\bot}\,{\to}\mathrm{E}}{\neg\neg\exists_x\,((Dx \to \neg\neg A) \land (\neg Dx \to \neg A))}\,{\to}\mathrm{I}}{\exists_x \neg\neg((Dx \to \neg\neg A) \land (\neg Dx \to \neg A))}\,{\to}\mathrm{E} \qquad \dfrac{\dfrac{\overline{\forall_x\,(Dx \lor \neg Dx)}\;\mathrm{DX}}{Dx \lor \neg Dx}\,{\forall}\mathrm{E} \qquad \overline{\text{Lemma 3}} \qquad \overline{\text{Lemma 4}}}{\neg A \lor \neg\neg A}\,{\lor}\mathrm{E}}{\neg A \lor \neg\neg A}\,{\exists}\mathrm{E}
$$

$\square$