ON MAXIMUM-SIZED k–REGULAR MATROIDS

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Abstract. Let k be an integer exceeding one. The class of k–regular matroids is a generalization of the classes of regular and near-regular matroids. A simple rank–r regular matroid has the maximum number of points if and only if it is isomorphic to $M(K_{r+1})$, the cycle matroid of the complete graph on $r+1$ vertices. A simple rank–r near-regular matroid has the maximum number of points if and only if it is isomorphic to the simplification of $\overline{T}_{M(K_3)}(M(K_{r+2}))$, that is, the simplification of the matroid obtained, geometrically, by freely adding a point to a 3–point line of $M(K_{r+2})$ and then contracting this point. This paper determines the maximum number of points that a simple rank–r k–regular matroid can have and determines all such matroids having this number. With one exception, there is exactly one such matroid. This matroid is isomorphic to the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$, that is, the simplification of the matroid obtained, geometrically, by freely adding k independent points to a flat of $M(K_{r+k+1})$ isomorphic to $M(K_{k+2})$ and then contracting each of these points.

1. Introduction

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be k algebraically independent transcendentals over the rationals $\mathbf{Q}$. A matroid is *k–regular* if it can be represented by a matrix over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of which all subdeterminants are products of positive and negative powers of differences of pairs of elements in $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. The class of k–regular matroids is introduced in [10]. For $k \geq 2$, this class is a generalization of the classes of regular and near-regular (see [16, 17]) matroids. If $k = 0$ or $k = 1$, then the class of k–regular matroids is exactly the class of regular or near-regular matroids, respectively. Some of the attractive properties enjoyed by regular and near-regular matroids are also enjoyed by the class of k–regular matroids in general. For example, it follows from results in [11] (see also [9]) that the class of k–regular matroids is closed under standard matroid operations such as the taking of duals, minors, direct sums, and 2-sums. For readers familiar with the notion of a partial field [11, 9], the class of k–regular matroids can be defined as a class of matroids representable over a certain partial field. Although it should be noted that the study of partial fields strongly motivates this paper, the partial field framework is not used.

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A simple rank–r matroid is *maximum sized* in a class if it has the maximum number of points amongst all simple rank–r matroids in the class. This paper determines, for all r and all k, the maximum size of a rank–r k–regular matroid and determines all such matroids having this size. It turns out, with one exception, that there is a single maximum-sized rank–r k–regular matroid. Geometrically, such a maximum-sized matroid is obtained by freely adding k independent points to a flat of $M(K_{r+k+1})$ which is isomorphic to $M(K_{k+2})$, contracting each of these points, and simplifying the resulting matroid. Readers familiar with the matroid operation of complete principal truncation will recognize that the matroid obtained by this geometric construction is, in fact, the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$. This result generalizes the results for regular and near-regular matroids. It follows from a result of Heller [3] that a simple rank–r regular matroid is maximum sized if and only if it is isomorphic to $M(K_{r+1})$, the cycle matroid of the complete graph on $r+1$ vertices. Oxley, Vertigan and Whittle show [8, Corollary 2.2] that a simple rank–r near-regular matroid is maximum sized if and only if it is isomorphic to the matroid obtained, geometrically, by freely adding a point to a flat of $M(K_{r+2})$ isomorphic to $M(K_3)$, contracting this point, and simplifying the resulting matroid. This matroid is isomorphic to the simplification of $\overline{T}_{M(K_3)}(M(K_{r+2}))$.

The class of regular matroids is the class of matroids representable over all fields [13]. The class of near-regular matroids is the class of matroids representable over all fields except perhaps $GF(2)$ [17, Theorem 1.4]. For the class of k–regular matroids we have the following property. If a matroid is k–regular, then it is representable over all fields whose size is at least $k+2$ [10, Proposition 3.1]. The converse, however, is not true. The matroid $U_{3,6}$, which is representable over every field of size at least four [7, p. 504], is not 2–regular [10]. Furthermore, using the results of [10], it is straightforward to check that, for all k, the matroid $U_{4,8}$, which is representable over every field of size at least seven [7, Table 6.1], is not k–regular. Nevertheless, for a prime power q, there is evidence that the class of $(q-2)$–regular matroids will turn out to be fundamental in the study of matroids representable over $GF(q)$ and other fields.

It is interesting to compare the results of this paper with other characterizations of maximum-sized members of a class of matroids representable over a partial field. The class of $\sqrt[6]{1}$–matroids is the class of matroids representable over $GF(3)$ and $GF(4)$ [17, Theorem 1.2]. With a single exception, the maximum-sized rank–r $\sqrt[6]{1}$–matroid is isomorphic to the maximum-sized rank–r near-regular matroid [8, Theorem 2.1]. The class of dyadic matroids is the class of matroids representable over $GF(3)$ and the rationals [16, Theorem 7.1]. It follows from Kung [4], and Kung and Oxley [6] that a simple rank–r dyadic matroid is maximum sized if and only if it is isomorphic to the ternary Dowling geometry $Q_r(GF(3)^*)$. For each of these classes, if $r>3$, then there is a single maximum-sized rank–r matroid in the class. Moreover, in this case, the maximum-sized rank–r matroid in this class is a modular hyperplane of the maximum-sized rank–$(r+1)$ matroid of the class. It follows that these maximum-sized matroids share the very attractive structural property of being supersolvable. For these maximum-sized members of the class of k–regular matroids we will discuss this property further in the next section.

This paper has a similar organization to that of Oxley, Vertigan and Whittle's paper [8]. Indeed some of the results of [8] with appropriate modifications generalize straightforwardly. Section 2 details some of the properties of the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$ and states the main result of the paper. In Section 3 we prove some structure results for the class of k–regular matroids which will be used to prove the main result of the paper in Section 4.

We shall assume familiarity with the elements of matroid theory as set forth in [7]. In particular, we assume familiarity with matroid representation theory (see [7, Chapter 6]). Notation and terminology will follow that of [7] with two exceptions. We denote the simple matroid that is canonically associated with a matroid M by $\mathrm{si}(M)$. Secondly, since we are only concerned with simple matroids, we adopt the convention that, for an integer n with $n \geq 2$, an *n–point line* will mean a line that is isomorphic to $U_{2,n}$.

2. The Main Result

We begin this section by restating the definition of a k–regular matroid. Having done this we give a representation for the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$ and discuss some of the special properties of this matroid. The section ends by stating the main result.

Let $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ denote the field obtained by extending the rationals by the algebraically independent transcendentals $\alpha_1, \alpha_2, \ldots, \alpha_k$. If a matrix over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ has the property that all non-zero subdeterminants are in

$$\mathcal{A}_k = \{\pm \prod_{i=1}^{k} \alpha_i^{l_i} \prod_{i=1}^{k} (\alpha_i - 1)^{m_i} \prod_{1 \leq i < j \leq k} (\alpha_i - \alpha_j)^{n_{i,j}} : l_i, m_i, n_{i,j} \in Z\},$$

then this matrix is *k–unimodular*. A *k–regular* matroid is one that can be represented by a k–unimodular matrix. A matroid is *ω–regular* if, for some non-negative integer k, it is k–regular. As stated in the introduction, the classes of 0– and 1–regular matroids are the classes of regular and near-regular matroids, respectively.

For all $r \geq 2$, let D_r denote the $r \times \binom{r}{2}$ matrix whose columns consist of all r–tuples with exactly two non-zero entries, the first equal to 1 and the second equal to -1. For all $r \geq 3$ and all $k \geq 0$, let A_r^k denote the matrix

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 0\cdots 0 & 1\cdots 1 & \alpha_1\cdots\alpha_1 & \alpha_2\cdots\alpha_2 & \cdots & \alpha_k\cdots\alpha_k & 0\cdots 0 \\
\hline
0 & & & & & & & \\
\vdots & I_{r-1} & I_{r-1} & I_{r-1} & I_{r-1} & \cdots & I_{r-1} & D_{r-1} \\
0 & & & & & & &
\end{array}\right]$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Let $A_1^k = \left[\ 1\ \right]$ and let A_2^k be the matrix

$$\begin{bmatrix} 1 & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

The proof of [8, Lemma 3.1] generalizes straightforwardly to give a proof of the following result.

Lemma 2.1. *For all r and all k, the matrix A_r^k is k–unimodular.*

It follows from Lemma 2.1 that, for all r and all k, $M[A_r^k]$ is k–regular. Except for the single case $r = 3$ and $k = 2$, it turns out that $M[A_r^k]$ is the maximum-sized rank–r k–regular matroid.

Recall that, geometrically, for a flat F of a matroid M of positive rank, the *principal truncation* $T_F(M)$ is obtained by freely placing a point on F and then contracting this point. Geometrically, the *complete principal truncation* $\overline{T}_F(M)$ is obtained by freely placing $r(F) - 1$ independent points on F and then contracting each of these points. For precise definitions and properties of these matroid operations the reader is referred to Section 7.4 of Brylawski's paper in [14]. We now show that $M[A_r^k]$ is isomorphic to the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$. We start by first stating a result [15, Proposition 4.1.7] of Whittle.

2.2. *Let F_1 and F_2 be flats of a matroid M such that $r(F_2) > r(F_1) > 0$ and $F_1 \subseteq F_2$. Then $\overline{T}_{F_2}(\overline{T}_{F_1}(M)) = \overline{T}_{F_2}(M)$.*

Let $M(K_3), M(K_4), \ldots, M(K_{k+2})$ be fixed restrictions of $M(K_{r+k+1})$ such that $K_3, K_4, \ldots, K_{k+2}$ is a chain of cliques in K_{r+k+1}. Applying Whittle's result repeatedly to this chain of flats of $M(K_{r+k+1})$ beginning with $M(K_{k+1})$ and $M(K_{k+2})$, we get that

$$\overline{T}_{M(K_{k+2})}(M(K_{r+k+1})) = \overline{T}_{M(K_{k+2})}(\overline{T}_{M(K_{k+1})}(\cdots(\overline{T}_{M(K_3)}(M(K_{r+k+1})))\cdots)).$$

It is now easily seen that, geometrically, the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$ is obtained from $M(K_{r+k+1})$ by taking k concurrent 3–point lines and adding a point freely to each of these 3–point lines, contracting the added points and simplifying the resulting matroid. We use this equivalence to show that $M[A_r^k]$ is isomorphic to the simplification of $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$. Take a totally unimodular representation of $M(K_{r+k+1})$ of the form $[I_{r+k}|D_{r+k}]$. Adjoin the matrix

$$\begin{bmatrix} -\alpha_1 & -\alpha_2 & & -\alpha_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ & \vdots & \ddots & \\ 0 & 0 & & 1 \\ & \vdots & & \\ 0 & 0 & & 0 \end{bmatrix}$$

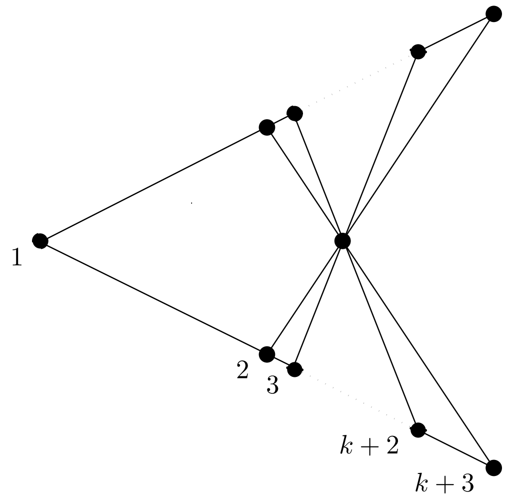

FIGURE 1. The matroid T_3^k.

to this representation. Each column corresponds to placing a point freely on a 3–point line of $M(K_{r+k+1})$. Moreover, each of the k 3–point lines to which a point has been freely added contains the point which corresponds to the first column of $[I_{r+k}|D_{r+k}]$. One can now obtain the specified representation for $M[A_r^k]$ in the following way. For each column of the adjoined matrix, first transform the column into a unit vector by pivoting on the second non-zero entry and then delete this column along with the row containing this entry. This corresponds to contracting each of the added points. By deleting certain columns of the resulting matrix, corresponding to simplifying the matroid obtained from these contractions, we can then obtain A_r^k by simply multiplying some rows and columns by -1.

To ease notation we define, for $r \geq 1$, T_r^k to be the simplification of the matroid $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$. Hence $T_1^k \cong U_{1,1}$ and $T_2^k \cong U_{2,k+3}$. A geometric representation of T_3^k is shown in Figure 1. If $k = 0$, then $T_r^0 \cong M(K_{r+1})$, the maximum-sized rank–r regular matroid. Furthermore, if $k = 1$, then $T_r^1 \cong T_r$, the maximum-sized rank–r near-regular matroid [8, Corollary 2.2].

Recall that a flat F of a matroid M is *modular* if, for every flat F' of M,

$$r(F) + r(F') = r(F \cup F') + r(F \cap F').$$

Furthermore, if there is a set of modular flats $\{F_0, F_1, \ldots, F_r\}$ of M such that, for $i \in \{0, 1, \ldots, r\}$, $r(F_i) = i$ and, for $i \in \{1, 2, \ldots, r\}$, $F_{i-1} \subseteq F_i$, then M is said to be *supersolvable* and $\{F_0, F_1, \ldots, F_r\}$ is called a *saturated chain of modular flats* of M. Now the matroid $M(K_{r+k+1})$ is supersolvable, where the saturated chain of modular flats is $\{M(K_1), M(K_2), \ldots, M(K_{r+k+1})\}$. Therefore, by [15, Corollary 4.1.9], $\overline{T}_{M(K_{k+2})}(M(K_{r+k+1}))$ is also supersolvable. Hence the simplification of this matroid, that is, T_r^k is supersolvable. Moreover, defining T_0^k to be $U_{0,0}$ for all k, its saturated chain of flats is $\{T_0^k, T_1^k, T_2^k, \ldots, T_r^k\}$ and so, for $i \in \{1, 2, \ldots, r\}$, T_{i-1}^k is a modular hyperplane of T_i^k. Thus, in general, the maximum-sized members of the class of k–regular matroids share the same attractive property of being

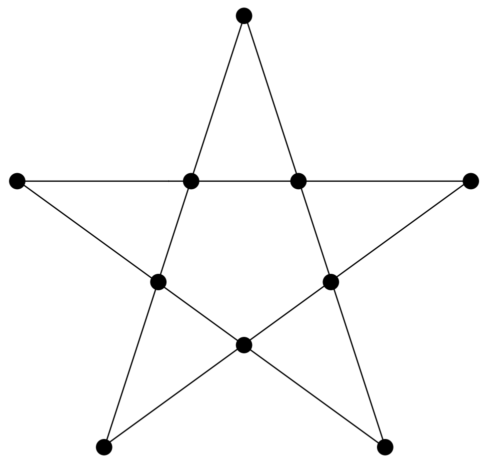

FIGURE 2. The matroid S_{10}.

supersolvable as the maximum-sized members of the classes of near-regular, dyadic, and $\sqrt[6]{1}$–matroids.

At last we state the main result, Theorem 2.3. A geometric representation for the matroid S_{10} appearing in Theorem 2.3 is shown in Figure 2. By [9], S_{10} is 2–regular and therefore, as S_{10} has a $U_{2,5}$–minor, it follows that S_{10} is k–regular if and only if $k \geq 2$.

Theorem 2.3. *Let M be a simple k–regular matroid having rank r. Then*

$$|E(M)| \leq \binom{r+k+1}{2} - \frac{k}{2}(k+3).$$

Moreover, for $r \neq 3$ or $k \neq 2$, T_r^k is the unique simple rank–r k–regular matroid whose ground set has cardinality equal to this bound. For $r = 3$ and $k = 2$, T_3^2 and S_{10} are the only simple matroids whose ground sets have cardinality equal to this bound.

The main difficulty in proving Theorem 2.3, which generalizes the corresponding results for the classes of regular and near-regular matroids, is the emergence of S_{10} when $k \geq 2$. Much of the argument is devoted to resolving this difficulty.

3. Some Structural Properties

In this section we obtain a number of structural properties of ω–regular matroids that will be needed in the proof of Theorem 2.3. We begin by showing that all k–unimodular representations of $U_{2,k+3}$ are equivalent.

Let A_1 and A_2 be two matrix representations of a matroid M over a field $\mathbf{F}$. Recall that A_1 and A_2 are equivalent representations of M if A_2 can be obtained from A_1 by a sequence of the following operations: pivoting on a non-zero entry;

interchanging two rows; interchanging two columns (along with their labels); multiplying a row or column by a non-zero scalar of $\mathbf{F}$; and applying an automorphism of $\mathbf{F}$ to the entries of A_1.

Let n be a non-negative integer and let $\mathbf{F}$ be a field. Let $a_1, a_2, \dots, a_n$ be distinct elements of $\mathbf{F} - \{0, 1\}$. We call an $\mathbf{F}$–representation of $U_{2,n+3}$ in the form

$$\begin{bmatrix} 1 & 0 & 1 & a_1 & a_2 & \cdots & a_n \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

a *standard* representation of $U_{2,n+3}$ over $\mathbf{F}$. Note that this slightly strengthens the usual definition of a representation being in standard form (see [7, p. 81]). Let A be the matrix

$$\begin{bmatrix} 1 & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$. By Lemma 2.1, A is a standard k–unimodular representation for $U_{2,k+3}$. Consider automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$. If $k = 0$, then the field is the rationals, which has no non-trivial automorphisms. For $k = 1$, the field is $\mathbf{Q}(\alpha_1)$, in which all non-trivial automorphisms are known (see [2, Proposition 2.3]). If $k \geq 2$, then it appears that the complete set of automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$ is not known (see [2, Section 5.2]). However, [10, Theorem 7] determines exactly when an automorphism φ of $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$ has the property that the matrix

$$\begin{bmatrix} 1 & 0 & 1 & \varphi(\alpha_1) & \varphi(\alpha_2) & \cdots & \varphi(\alpha_k) \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$ is also a standard k–unimodular representation of $U_{2,k+3}$. Using this theorem in combination with [10, Theorem 5 and Lemma 6], it is easily seen that if a matrix over $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$ is a standard k–unimodular representation of $U_{2,k+3}$, then we can obtain this representation by applying one of the automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$ mentioned above to the entries of A. Combining this with the fact that the set of all automorphisms of $\mathbf{Q}(\alpha_1, \alpha_2, \dots, \alpha_k)$ is a group under function composition, we deduce Lemma 3.1.

Lemma 3.1. *All k–unimodular representations of $U_{2,k+3}$ are equivalent.*

A matroid M is *strictly* k–regular if M is k–regular but not $(k-1)$–regular. Using Lemma 3.1 and the results of [10] again, it is straightforward to deduce the following corollary.

Corollary 3.2. *The matroid $U_{2,k+3}$ is strictly k–regular.*

Having established Lemma 3.1, it is not much more difficult, using the same results that proved Lemma 3.1, to realize Corollary 3.3.

Corollary 3.3. *Let n be a non-negative integer. Then, for all $k \geq n$, all k–unimodular representations of $U_{2,n+3}$ are equivalent.*

With Corollary 3.3 in hand we can now easily determine the k–regularity of matroids of small rank. The next two results are obtained by using the last corollary in conjunction with [10, Theorem 5 and Lemma 6].

Lemma 3.4. *The matroid $U_{3,k+3}$ is strictly k–regular.*

Lemma 3.5. *Let M be a simple rank–3 matroid with $|E(M)| = 7$. Then M is not ω–regular if and only if M is isomorphic to a matroid that can be obtained from the Fano matroid by relaxing up to six lines.*

We remark that all rank–3 matroids whose ground sets have size at most six are ω–regular. Now it immediately follows from Corollary 3.2 that $U_{2,k+3}$ is the maximum-sized rank–2 k–regular matroid. Furthermore, a routine check using Lemma 3.5 shows that S_{10} is a maximal ω–regular matroid of rank 3, that is, no rank–3 ω–regular matroid is a single-element extension of S_{10}.

Lemma 3.6. *Let M be a simple rank–3 k–regular matroid.*

(i) *If $k < 2$, then M is a restriction of T_3^k.*
(ii) *If $k = 2$, then M is a restriction of T_3^2 or S_{10}.*
(iii) *If $k > 2$, then M is a restriction of $U_{3,k+3}$, T_3^k, or S_{10}.*

Proof. The proof is a series of routine case checks which repeatedly use Lemma 3.5. Let M be an ω–regular matroid of rank 3. If M is regular, then M is a restriction of $M(K_4)$, which is isomorphic to T_3^0. If M is near-regular, then, by [8, Lemma 4.1], M is a restriction of T_3^1. Therefore assume that $k \geq 2$ and M is not near-regular.

Assume that $k = 2$. Using the fact that every rank–3 near-regular matroid is a restriction of T_3^1, it is easily seen that M has a minor isomorphic to either $U_{2,5}$ or $U_{3,5}$. Since the matroid obtained by placing a point on the intersection of two lines of $U_{3,5}$ is the only 2–regular single-element extension of $U_{3,5}$ and the only 2–regular single-element coextension of $U_{2,5}$, M has this matroid as a restriction. The rest of the proof for $k = 2$ is a straightforward case analysis based on this fact, Lemma 3.5, and the fact that, as P_6 is not $GF(4)$–representable, P_6 is not 2–regular.

Now assume that $k \geq 3$ and M is not 2–regular. Considering single-element extensions and coextensions of $U_{2,5}$, and single-element extensions of $U_{3,5}$, we get that M has, as a minor, one of the matroids $U_{2,6}$, $U_{3,6}$, or the matroid obtained by freely placing a point on a line of $U_{3,5}$. Following Oxley [7, p. 71], we call the last of these matroids P_6. Suppose that M has a $U_{3,6}$–minor. By Lemma 3.4, $U_{3,k+3}$ is strictly k–regular. Moreover, it is easily seen using Lemma 3.5 that the only single-element extension of $U_{3,k+3}$ that is ω–regular is $U_{3,k+4}$. Combining these two results, it follows that M is a restriction of $U_{3,k+3}$. Suppose that M has either a $U_{2,6}$– or P_6–minor, but no $U_{3,6}$–minor. We may assume that M is 3–connected, for otherwise M is a restriction of T_3^k. A routine check, considering single-element coextensions of rank–2 simple matroids with at least six points, now shows that if M has a $U_{2,6}$–minor, then it has a P_6–minor. So assume that this is indeed the case. By Lemma 3.5 again, every single-element extension of P_6 places a point on a line of P_6. Geometrically, this means that, every point of M, except exactly one, can be covered by two lines. The result follows routinely from this observation. □

A *long line* of a matroid is a line that contains at least three points. Let P_{2k+5} denote the matroid obtained from T_3^k by deleting a point that is on two $(k+3)$–point lines. In particular, if $k = 1$, then we get the matroid P_7. We note that, for $k \geq 1$, this point is unique. Furthermore, call the point of P_{2k+5} that is on $k+2$ 3–point lines its *tip*. We observe that if a point of a rank–3 ω–regular matroid is on at least three long lines, then, for some k, this matroid is a restriction of T_3^k.

Lemma 3.7. *If a rank–4 matroid M has four concurrent long lines no three of which are coplanar, then M is not an ω–regular matroid.*

Proof. Assume that M is ω–regular. Let p be the point of concurrency of four long lines, L_w, L_x, L_y, and L_z, no three of which are coplanar. Furthermore, let S be the union of these lines and, for all $i \in \{w, x, y, z\}$, let i_1 and i_2 be points of $L_i - p$. Consider $M|S$. If $q \in S - p$, then, by Lemma 3.5, $\mathrm{si}((M|S)/q) \cong P_7$. Therefore q is in exactly two 4–circuits that are not forced by q being on one of the four long lines and whose intersection is q. Thus we may assume without loss of generality that both $\{w_1, x_1, y_1, z_1\}$ and $\{w_1, x_2, y_2, z_2\}$ are 4–circuits of $M|S$. It now follows by the same reasoning that one of $\{z_1, w_2, x_2, y_2\}$, $\{z_1, w_2, x_1, y_2\}$, and $\{z_1, w_2, x_2, y_1\}$ is a 4–circuit of $M|S$. If $\{z_1, w_2, x_2, y_2\}$ is a 4–circuit of $M|S$, then y_2, as well as p, is on at least three 3–point lines in $\mathrm{si}((M|S)/x_2)$. This contradicts Lemma 3.5 and so $\{z_1, w_2, x_2, y_2\}$ is not a 4–circuit of $M|S$. Similarly, neither $\{z_1, w_2, x_1, y_2\}$ nor $\{z_1, w_2, x_2, y_1\}$ is a 4–circuit of $M|S$. This completes the proof of Lemma 3.7. □

The next two lemmas are obtained from the statements of [8, Lemmas 4.4 and 4.5] by replacing "$\sqrt[6]{1}$–matroid" with "ω–regular matroid". Moreover, for both these lemmas, the arguments used for [8, Lemmas 4.4 and 4.5] work when applied to ω–regular matroids instead of $\sqrt[6]{1}$–matroids.

Lemma 3.8. *Let M be a 3–connected ω–regular matroid. Then M does not have as a restriction the parallel connection of P_7 and $U_{2,4}$ in which the basepoint of the parallel connection is the tip of P_7.*

Lemma 3.9. *Let M be a 3–connected ω–regular matroid. Suppose that X and Y are subsets of $E(M)$ such that $M|X \cong P_7 \cong M|Y$ and $r(X \cup Y) \geq 4$. Then the tip of $M|Y$ is not in X.*

Lemma 3.10. *Let M be a 3–connected k–regular matroid of rank r. If $p \in E(M)$, then p is on at most $r + k - 1$ long lines. Moreover, for $i \in \{1, 2, \ldots, k\}$, if the point p is on exactly $r + i - 1$ long lines, then all long lines through p have exactly three points.*

Proof. Assume that p is on at least r long lines. Let S be the union of the long lines through p. Consider $M|S$. It follows by Lemmas 3.7 and 3.9 and the fact that p is on at least r long lines that exactly one plane P of $M|S$ spanned by two long lines through p contains more than two long lines. By Lemma 3.6, each of the long lines on P has size three. Moreover, by Lemma 3.6 again, there are at most $k+2$ long lines on P and so p is on at most $r+k-1$ long lines. Since M is 3–connected, it follows by Lemma 3.8 that all of the long lines not on P also have size three and the lemma is proved. □

For the last two lemmas of this section we first need some definitions. Both of these lemmas are essential in dealing with the difficulty caused by S_{10} being ω–regular. Firstly, since all single-element deletions of S_{10} are isomorphic, we denote such a matroid by $S_{10}-e$. A *ring* R of n long lines is a matroid with points $x_1, x_2, \ldots, x_n$ such that each of $\mathrm{cl}(\{x_1,x_2\}), \mathrm{cl}(\{x_2,x_3\}), \ldots, \mathrm{cl}(\{x_n,x_1\})$ is a long line of R and the ground set of R, $E(R)$, is the union of these n long lines (see [5, p. 39]). We call the points $x_1, x_2, \ldots, x_n$ the *joints* of R. If a ring R consists of r long lines and has rank r, then we say that R is a *standard* ring of rank r. Note that if each of the long lines in a standard ring R consists of three points, then R is isomorphic to either the rank–r whirl or the rank–r wheel. Let M be a rank–r standard ring with long lines $L_1, L_2, \ldots, L_r$ and x_1 be the joint of M that is on L_1 and L_r. Let M' be the matroid obtained from M by deleting all non-joint elements of L_r. A matroid N that is obtained from M' by adjoining a long line L'_r through x_r such that $r(N\backslash L_1) = r(N\backslash L'_r) = r(M)$ and $L_1 \cap L'_r$ is empty is called an *open ring* of rank r.

Lemma 3.11. *Let $r \geq 4$ and let M be a standard ring consisting of r long lines each of which has size at least four. Then M is not ω–regular.*

Proof. By contracting and deleting non-joint points of M, we can obtain a rank–4 minor N of M isomorphic to a rank–4 standard ring consisting of 4–point lines. Hence it suffices to prove that N is not ω–regular.

Assume that N is ω–regular. Let x_1, x_2, x_3, and x_4 be the joints of N and let $L_1 = \{x_1,u_1,v_1,x_2\}$, $L_2 = \{x_2,u_2,v_2,x_3\}$, $L_3 = \{x_3,u_3,v_3,x_4\}$, and $L_4 = \{x_4,u_4,v_4,x_1\}$ be the 4–point lines of N. As N is ω–regular, it follows by Lemma 3.6 that $\mathrm{si}(N/u_1)$ is isomorphic to $S_{10}-e$. Thus, without loss of generality, we may assume that $C_1 = \{u_1,u_2,u_3,u_4\}$ and $C_2 = \{u_1,v_2,v_3,v_4\}$ are both 4–circuits of N. Similarly, $\mathrm{si}(N/v_1)$ is isomorphic to $S_{10}-e$ and therefore v_1 must be an element of a 4–circuit C_3 that contains exactly one non-joint point from each of the 4–point lines of N. It follows that either $|C_1 \cap C_3|$ or $|C_2 \cap C_3|$ is equal to two. Say $|C_1 \cap C_3| = 2$. Then, by contracting an element of $C_1 \cap C_3$ from N, we obtain a rank–3 minor of N having three concurrent long lines one of which has four points; a contradiction to Lemma 3.6. Similarly, if $|C_2 \cap C_3| = 2$, we obtain a contradiction. This completes the proof of the lemma. □

Lemma 3.12. *Let $r \geq 3$ and let M be a rank–r open ring consisting of r long lines each of which has size at least four. Then M is not ω–regular.*

Proof. By deleting non-joint elements if necessary we may assume that each of the r long lines has exactly four points. We argue by induction on r. The result is clear for $r = 3$. For $r = 4$ we have

3.12.1. *Let M be a rank–4 open ring consisting of 4–point lines. Then M is not ω–regular.*

Proof. Assume that M is ω–regular. Let $L_1 = \{x_1, u_1, v_1, x_2\}$, $L_2 = \{x_2, u_2, v_2, x_3\}$, $L_3 = \{x_3, u_3, v_3, x_4\}$, and $L_4 = \{x_4, u_4, v_4, x_5\}$ be the 4–point lines of M. Now at least two elements of $\{u_4, v_4, x_5\}$ are not in the closure of $L_1 \cup L_2$. Without loss of generality we may assume that u_4 and v_4 are two such elements. If u_4 is in no 3–circuits of M other than those contained in L_4, then, by Lemma 3.6, M/u_4 is not ω–regular. Therefore $\{u_4, y, z\}$ is a 3–circuit of M such that $y \in \{x_1, u_1, v_1\}$ and $z \in \{u_3, v_3\}$. It is easily seen that we may assume $\{u_4, x_1, u_3\}$ is a 3–circuit of M. Moreover, this is the only such circuit containing u_4. It now follows by the same reasoning that $\{v_4, x_1, v_3\}$ must also be a 3–circuit of M. Since u_2 can be in at most one 3–circuit that contains either u_1 or v_1, it follows that, in $\text{si}(M/u_2)$, the point x_1 is the point of concurrency of three long lines one of which contains four points. By Lemma 3.6, $\text{si}(M/u_2)$ is not ω–regular and the proof is completed. □

Let M be a rank–r open ring consisting of r 4–point lines, where $r \geq 5$, and assume that the lemma holds for all smaller ranks. Let L be a 4–point line of M that contains exactly one joint. Let u be a non-joint point on L. Consider $\text{si}(M/u)$. Using the proof of the rank–4 case if need be, it is easily checked that $\text{si}(M/u)$ consists of $r-1$ long lines each of size four except perhaps one which has size five. Moreover, either $\text{si}(M/u)$ is a rank–$(r-1)$ open ring or a rank–$(r-1)$ standard ring. If $\text{si}(M/u)$ is an open ring of rank $r-1$, then, by the induction assumption, $\text{si}(M/u)$, and hence M, is not ω–regular. If $\text{si}(M/u)$ is a standard ring of rank $r-1$, then, as $r-1 \geq 4$, it follows by Lemma 3.11 that $\text{si}(M/u)$ is not ω–regular. □

4. Proof of Theorem 2.3

In this section we prove Theorem 2.3. The proof consists of a sequence of lemmas and has the same outline as the proof of [8, Theorem 2.1]. Indeed, the proofs of some lemmas are very similar to the proofs of particular lemmas used in proving [8, Theorem 2.1]. Where this is the case, the proof of the lemma is omitted and an appropriate remark is made preceding the statement of this lemma.

Proof of Theorem 2.3. The proof is by induction on r to simultaneously prove the bound and a characterization of the matroids whose ground sets have cardinality equal to this bound. If $k = 0$, then the result follows from [3]. If $k = 1$, then, by [8, Corollary 2.2], the theorem is proved. For $r = 2$, the result follows from Corollary 3.2. Moreover, by Lemma 3.6, the result is proved for $r = 3$.

Let M be a maximum-sized k–regular matroid of rank r, where $k \geq 2$ and $r \geq 4$, and assume that the theorem holds for all smaller ranks. Then

$$|E(M)| \geq |E(T_r^k)| = \binom{r+k+1}{2} - \frac{k}{2}(k+3). \tag{4.1}$$

Lemma 4.1. *M is 3–connected.*

Proof. The argument that M does not have a 1–separation is similar to the argument that M has no 2–separation. We present only the latter. Assume that M has a 2–separation $\{X_1, X_2\}$. Let $r_1 = r(X_1)$ and $r_2 = r(X_2)$. Then, by the induction assumption and since $r_1 + r_2 - 1 = r(M)$,

$$(4.2) \quad |E(M)| \leq \binom{r_1+k+1}{2} - \frac{k}{2}(k+3) + \binom{r_2+k+1}{2} - \frac{k}{2}(k+3).$$

Furthermore, by (4.1),

$$(4.3) \qquad |E(M)| \geq \binom{(r_1+r_2-1)+k+1}{2} - \frac{k}{2}(k+3).$$

Combining (4.2) and (4.3) we get

$$(r_1-1)(r_2-1) \leq 1.$$

This last inequality only holds when $r_1 = r_2 = 2$, that is, when $r = 3$. Since $r \geq 4$, the lemma is proved. □

Recall that, for a positive integer n, a matroid M is *vertically n–separated* if there is a partition $\{X_1, X_2\}$ of $E(M)$ with the properties that $\min\{r(X_1), r(X_2)\} \geq n$ and $r(X_1) + r(X_2) - r(M) \leq n - 1$. A matroid M is vertically 4–connected if, for all $n < 4$, it has no vertical n–separation.

Lemma 4.2. *M is vertically 4–connected.*

Proof. Since M is 3–connected, M has no vertical 1– or 2–separations. Therefore suppose that M has a vertical 3–separation $\{X_1, X_2\}$. Let $r_1 = r(X_1)$. Let $p \in E(M) - \text{cl}(X_2)$ and consider the long lines through p. Note that all such lines must lie in $\text{cl}(X_1)$.

We first show that p is on at most $r_1 - 1$ long lines. Suppose, to the contrary, that p is on at least r_1 long lines. Since M is 3–connected, for each e in $E(M) - \text{cl}(X_1)$, either $\text{co}(M\backslash e)$ or $\text{si}(M/e)$ is 3–connected [1] (see also [7, Proposition 8.4.6]). It follows by repeated application of this result that we can obtain a 3–connected k–regular minor N of M with the properties that $N|X_1 = M|X_1$ and $r(N) = r_1$. As all long lines through p are in the closure of X_1 in M, we deduce that p is on at least r_1 long lines in N. Therefore, by Lemma 3.10, p is on at most $r_1 + k - 1$ long lines in N each of which has exactly three points. This means that, in M, the point p is on at most $r_1 + k - 1$ long lines each of which has exactly three points. Therefore

$$|E(M)| \leq 1 + (r_1 + k - 1) + |E(\text{si}(M/p))|,$$

that is,

$$|E(\text{si}(M/p))| \geq |E(M)| - (1 + (r_1 + k - 1)).$$

By the induction assumption,

$$|E(\text{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3).$$

Combining the last two inequalities with (4.1), we obtain a contradiction. Hence p is on at most $r_1 - 1$ long lines. Assume that p is on at most one long line of size at least four. Then, as this line has at most $k+3$ points and p is on at most r_1-2 3–point lines,

$$|E(\mathrm{si}(M/p))| \geq |E(M)| - (1 + (k+1) + (r_1 - 2)).$$

Again, by the induction assumption,

$$|E(\mathrm{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3).$$

Combining the last two inequalities with (4.1), we get another contradiction. It now follows that every element of $E(M) - \mathrm{cl}(X_2)$ is on at least two lines of size at least four.

We next show that if p is on two 4–point lines, then p is on at least one other line of size at least four. Suppose not. Then, as p is on exactly two lines of size four and at most r_1-3 long lines of size three,

$$|E(\mathrm{si}(M/p))| \geq |E(M)| - (1 + 4 + (r_1 - 3)).$$

Therefore, by (4.1),

$$|E(\mathrm{si}(M/p))| \geq \frac{1}{2}(r^2 + (2k+1)r - 2k) - (1 + 4 + (r_1 - 3)).$$

By the induction assumption,

$$|E(\mathrm{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3).$$

Combining the last two inequalities we obtain $r+k \leq r_1+2$. Since $k \geq 2$, we have a contradiction. Thus if p is on two 4-point lines, then p is on at least one other line of size at least four.

We complete the proof of Lemma 4.2 by first constructing a restriction N of $M|\mathrm{cl}(X_1)$ with the following properties: N is isomorphic to a rank–r_1 standard ring with the non-joint elements of exactly one long line deleted and each of the remaining r_1-1 long lines has size at least four. Having obtained N, we use it to show that $M|\mathrm{cl}(X_1)$ has a restriction of rank r_1 isomorphic to either a standard or open ring in which each of the r_1 long lines has size at least four. In the following construction we repeatedly use the fact that every element of $E(M) - \mathrm{cl}(X_2)$ is on at least two long lines of size at least four. Start by choosing a point x_1 of $E(M) - \mathrm{cl}(X_2)$. Choose a line L_1 through x_1 of size at least four, and a point x_2 on L_1 distinct from x_1 and not in the closure of X_2. Repeat this process for x_2 to obtain a line L_2 of size at least four and a point x_3 not in the closure of X_2. Both L_1 and L_2 are long lines of N. We now show that there is a line, L_3 say, of size at least four through x_3 such that $L_3 \notin \mathrm{cl}(L_1 \cup L_2)$. Suppose, to the contrary, that this is not the case. Then there is a line L_3' of size at least four with the property that $L_3' \in \mathrm{cl}(L_1 \cup L_2)$. If one of L_1, L_2, and L_3' is a line of size at least five, then, by Lemma 3.6, M is not ω–regular. Therefore each of L_1, L_2, and L_3' must have exactly four points. Since x_3 is on two lines of size exactly four, x_3 is on a line of size at least four other than L_2 and L_3'. Moreover, by Lemma 3.6, this line is not contained in $\mathrm{cl}(L_1 \cup L_2)$; a contradiction. We choose L_3 to be a long line of N. Repeat this construction for L_3 to obtain a point x_4, that is not in the closure of X_2, and a line L_4 of size at least four through x_4 such that $r(L_2 \cup L_3 \cup L_4) \geq 4$. If $r(L_1 \cup L_2 \cup L_3 \cup L_4) = 4$, then, by Lemmas 3.11 and 3.12,

M is not ω–regular. Therefore $r(L_1 \cup L_2 \cup L_3 \cup L_4) = 5$. Continuing in this way we eventually obtain the restriction N of $M|\mathrm{cl}(X_1)$ that has rank r_1 and consists of $r_1 - 1$ long lines each of which has at least four points. Let $L_1, L_2, \ldots, L_{r_1-1}$ be the long lines of N, and x_{r_1} be a point on L_{r_1-1} such that x_{r_1} is not on L_{r_1-2} and is not in $\mathrm{cl}(X_2)$. As before, choose a line L_{r_1} of size at least four through x_{r_1} such that $r(L_{r_1-2} \cup L_{r_1-1} \cup L_{r_1}) = 4$. It follows that $M|\mathrm{cl}(X_1)$, and hence M, has a restriction containing L_{r_1-2}, L_{r_1-1}, and L_{r_1} that is isomorphic to either a standard or open ring of rank at least four. In both cases each of the ring's long lines has at least four points and therefore by Lemmas 3.11 and 3.12 this restriction, and hence M, is not ω–regular. We conclude that M is vertically 4–connected. □

Lemma 4.3. *Suppose $p \in E(M)$ and p is on at least r long lines. Then p is on exactly $r + k - 1$ long lines. Moreover, each of the $r + k - 1$ long lines has exactly three points.*

Proof. By Lemma 3.10, p is on at most $r+k-1$ long lines each of which has exactly three points. Therefore

$$|E(M)| \leq 1 + (r + k - 1) + |E(\mathrm{si}(M/p))|. \tag{4.4}$$

By the induction assumption,

$$|E(\mathrm{si}(M/p))| \leq \binom{r+k}{2} - \frac{k}{2}(k+3) \tag{4.5}$$

and so

$$|E(M)| \leq \binom{r+k+1}{2} - \frac{k}{2}(k+3).$$

Hence, by (4.1), equality holds in (4.4) and (4.5). Thus if p is on at least r long lines, then p is on exactly $r + k - 1$ long lines each of which has exactly three points. □

Lemma 4.4. *Let $p \in E(M)$. Let S be the union of the long lines through p and let $e \in \mathrm{cl}(S)$. If either*

(i) *$M|S$ is a union of three point lines in which P_{2k+5} is a restriction; or*
(ii) *p is on a long line containing at least four points;*

then e is on a plane spanned by two long lines through p.

Proof. Assume, to the contrary, that e is not in a plane spanned by two long lines through p. Say $M|S$ satisfies (i) in the statement of the lemma. Then it follows from the proof of Lemma 3.10 that p is on $r(S) + k - 1$ 3–point lines. Therefore, in $\mathrm{si}(M/e)$, p is on $r(S) + k - 1$ 3–point lines and $\mathrm{si}(M/e)|S$ has rank $r(S) - 1$. Since M is vertically 4–connected, $\mathrm{si}(M/e)$ is 3–connected and therefore we contradict Lemma 3.10. This completes the proof of (i). If p is on a 4–point line, then, by Lemma 3.10, p is on $r(S) - 1$ long lines. Using an argument similar to that which proved (i) we again obtain a contradiction and so the lemma is proved. □

Corollary 4.5. *Let $p \in E(M)$ and suppose that p is on a line L of size at least four. If M restricted to the long lines through p has rank r, then all long lines through points on L lie on a plane spanned by L and a long line through p.*

Proof. Let x be a point, other than p, on L. Let L_x be a long line through x, and let y and z be two other points on L_x. Since M restricted to the long lines through p has rank r, it follows by Lemma 4.4 that y must lie on a plane spanned by two long lines through p. To prove the corollary, it suffices to show that y lies on a plane spanned by L and one other long line through p. Suppose, to the contrary, that this is not the case. Then y does not lie on a long line through p. Let L' and L'' be the unique pair of long lines though p such that y lies in the span of L' and L''. Let S be the union of the lines L, L_x, L', and L''. In $M|S$, the point z does not lie on a plane spanned by two long lines through p. Therefore $(M|S)/z$ is a rank–3 minor of M with three concurrent long lines one of which has at least four points. This contradiction to Lemma 3.6 completes the proof of Corollary 4.5. □

Lemma 4.6. *If $p \in E(M)$ and p is on at least two long lines each of which has at least four points, then M/p is regular.*

Proof. Let L_1 and L_2 be two such lines through p and assume that M/p is non-regular. Then M/p has a minor isomorphic to one of the matroids $U_{2,4}$, F_7, and F_7^* [13]. Since neither F_7 nor F_7^* is ω–regular, M/p must have a minor isomorphic to $U_{2,4}$. Since M is vertically 4–connected, $\mathrm{si}(M/p)$ is 3–connected. Let x_1 and x_2 be the points in $\mathrm{si}(M/p)$ corresponding to L_1 and L_2 in M, respectively. Then, as M/p has a $U_{2,4}$–minor, $\mathrm{si}(M/p)$ has a $U_{2,4}$–minor whose ground set contains x_1 and x_2 (Seymour [12], see also [7, Proposition 11.3.8]). Therefore M has a rank–3 minor that contains the two lines L_1 and L_2, and two points neither of which is on L_1 or L_2. If either $|L_1| \geq 5$ or $|L_2| \geq 5$, then, by Lemma 3.6, M is not ω-regular. Therefore we may assume that both L_1 and L_2 have size four.

Let $q \in E(M)$. The next three results establish that q is on at least two 4–point lines if $k = 2$ and on at least three 4–point lines if $k \geq 3$.

4.6.1. *No line through q has more than four points.*

Proof. Assume that q is on a line L containing at least five points. Then, by Lemma 3.10, q is on at most $r-1$ long lines. Suppose that q is on a line, other than L, which has size at least four. Since q is on a line containing at least five points, q and p are distinct and so M/q contains a 4–point line. Therefore M/q is non-binary. Since $\mathrm{si}(M/q)$ is 3–connected, we can argue as before to obtain a contradiction. Therefore, other than L, all long lines through q have size three. Thus, as q is on at most $r-2$ 3–point lines,

$$|E(M)| \leq 1 + (k+1) + (r-2) + |E(\mathrm{si}(M/q))|. \tag{4.6}$$

By (4.1),

$$|E(M)| \geq \binom{r+k+1}{2} - \frac{k}{2}(k+3). \tag{4.7}$$

Combining (4.6) and (4.7) we deduce that equality holds in (4.6). Thus q is on exactly one $(k+3)$–point line and exactly $r-2$ 3–point lines. By the same reasoning, each point of L is on exactly $r-2$ 3–point lines.

By Lemmas 3.7 and 3.8, M restricted to the long lines through some point on L has rank r. Since $|L| \geq 4$, it follows by Corollary 4.5 that every plane spanned by L

and a 3–point line through q contains exactly one 3–point line that passes through each point on L. By considering such a plane of M, we obtain a contradiction to Lemma 3.6. We conclude that no line through q has more than four points. □

The next result is obtained by combining the last result with the fact that if q is on a 4–point line, then q is on at most $r-1$ long lines.

4.6.2. *Suppose that q is on a 4–point line. Then q is on at least k 4–point lines.*

4.6.3. *q is on at least one 4–point line.*

Proof. Suppose that every long line through q has exactly three points. Then, from the proof of Lemma 4.3, q is on exactly $r+k-1$ 3–point lines. Let S be the union of the long lines through q. Using Lemma 3.6 and the fact that M has no 5–point line restriction, it is easily seen that in $M|S$ there are at most four 3–point lines in a plane. Therefore, by Lemmas 3.7 and 3.9, $r(M|S) = r(M) + k - 2$. If $k > 2$, then we have a contradiction. So assume that $k = 2$. Then q is on $r+1$ 3–point lines and $r(M|S) = r(M)$. Therefore, by Lemmas 3.7 and 3.9, $M|S$ has a restriction isomorphic to P_9 in which q is the tip. Let L_3 be a 3–point line through q in this restriction. Let x_1 be a point of $L_3 - q$. Then x_1 is on a 4–point line L_4 of this restriction. By (4.6.2), x_1 is on at least one other 4–point line L'_4. By Lemma 3.6, L'_4 does not lie on the plane of M spanned by the four coplanar 3–point lines through q. Using the fact that $r(M|S) = r(M)$, it is straightforward to deduce, by Lemma 4.4 and an argument similar to the proof of Corollary 4.5, that L'_4 lies on a plane spanned by L_3 and a 3–point line, L'_3 say, through q that is not in the closure of the restriction isomorphic to P_9. Let x_2 be a point on L'_3 that is on neither L_3 nor L'_4. By contracting x_2 we obtain a rank–3 minor of M with four concurrent long lines one of which has four points; a contradiction. Hence every element of M is on at least one 4–point line. □

Like Lemma 4.2, the proof of Lemma 4.6 is completed by showing that M has a restriction isomorphic to either a standard or open ring of rank at least four in which each of the ring's long lines has four points and thereby obtaining a contradiction to Lemmas 3.11 and 3.12. For $k \geq 3$, the argument that M has such a restriction is similar to, but simpler than, the analogous argument used in the proof of Lemma 4.2. We omit the straightforward details and remark that the proof relies on the fact that every member of $E(M)$ is on at least three 4–point lines. To prove the result for $k = 2$, however, we first require an additional result.

4.6.4. *If M has a restriction isomorphic to S_{10}, then, for every 4–point line of this restriction, there is a pair of points with the property that each point is on at least three 4–point lines.*

Proof. Suppose that M has a restriction isomorphic to S_{10} and let L be a 4–point line of this restriction. Suppose, to the contrary, that there are three points x, y, and z on L that are each on exactly two 4–point lines. Then, using (4.1), it is routine to deduce that each of x, y, and z is on exactly $r-3$ 3–point lines. By Lemmas 3.7 and 3.8, M restricted to the long lines through any one of x, y, and z has rank r. Therefore, as L is a 4–point line, it follows by Corollary 4.5 that every plane spanned by L and a 3–point line through x contains exactly one 3–point line that passes through each of y and z. Since $r \geq 4$, there exists such a plane.

Let w denote the fourth point on L. Then, using (4.1) again, we deduce that, besides the two 4–point lines of the S_{10}–restriction, w is on one other long line. Furthermore, by Lemma 3.6 and Corollary 4.5 such a line must lie in a plane, P say, spanned by L and a 3–point line through x. Consider the plane P. Since each of x, y, and z is on exactly two 4–point lines, it is easily checked by Lemma 3.6 that P is a restriction of T_3^2. A further check now shows that P has a restriction isomorphic to P_7. By (4.6.3), the tip of this P_7–restriction is on a 4–point line. Moreover, by Lemma 3.6, this 4–point line is not in the closure of P in M. It now follows by Lemma 3.8 that M is not ω–regular. This contradiction completes the proof of (4.6.4). □

As mentioned above, the proof of Lemma 4.6, for $k = 2$, is completed by showing that M has a restriction isomorphic to either a standard or open ring of rank at least four in which each of the ring's long lines has four points. As in the proof of Lemma 4.2, we do this by first constructing a restriction N of M that is isomorphic to a rank–r standard ring with the non-joint elements of exactly one long line deleted and in which each of the remaining $r - 1$ long lines has size exactly four. The construction of N and the obtaining of the desired restriction is similar to that in the proof of Lemma 4.2, but with one important difference. We highlight this difference with the first few steps in the construction of N and leave the remaining straightforward details to the reader.

Start by choosing a point x'_1 of $E(M)$. Choose a line L'_1 through x'_1 of size four and a point x'_2 on L'_1 distinct from x'_1. Now choose a 4–point line L'_2 through x'_2 that is distinct from L'_1. Unlike the construction in the proof of Lemma 4.2, we cannot arbitrarily choose the third joint element of N. However, (4.6.4) determines such a point for us. This is done in the following way. Suppose that there is no point on L'_2, distinct from x'_2, that is on a 4–point line which is not in $\mathrm{cl}(L'_1 \cup L'_2)$. Then, as every point of L'_2 is on at least two 4–point lines, it follows by Lemma 3.6 that M has a restriction isomorphic to S_{10} that is spanned by the union of L'_1 and L'_2. Combining (4.6.4) with Lemma 3.6 we obtain a contradiction. Hence there is a point on L'_2, distinct from x'_2, that is on a 4–point line which is not in $\mathrm{cl}(L'_1 \cup L'_2)$. Label this point and 4–point line x'_3 and L'_3, respectively. The completion of the construction of N is the same as that in the proof of Lemma 4.2, but with the obvious exception. Having obtained N, the proof of Lemma 4.6 for $k = 2$ is concluded in the same way that Lemma 4.2 was concluded. □

The proof of the next result, which confirms the bound on $|E(M)|$, is similar to the proof of [8, Lemma 5.5]. We omit the details here and just remark that parts (ii) and (iii) of Lemma 4.7, respectively, are established by considering the cases of a point p of M being on

(a) at most one long line of size at least four; and
(b) at least two long lines of size at least four.

Lemma 4.7. $|E(M)| = \binom{r+k+1}{2} - \frac{k}{2}(k+3)$. *Moreover, every point* p *of* M *satisfies one of the following:*

(i) p *is on exactly* $r+k-1$ *long lines each of which has exactly three points, and* p *is the tip of a unique* P_{2k+5}*–restriction of* M*;*
(ii) p *is on exactly* $r-1$ *long lines, one of which has exactly* $k+3$ *points and* $r-2$ *of which have exactly three points;*
(iii) p *is on exactly* $r-1$ *long lines, each of which has exactly* $k+3$ *points, and* $\mathrm{si}(M/p) \cong M(K_r)$.

The three possibilities for a point p of M generalize those for the near-regular case in [8, Lemma 5.5]. Therefore, as in [8], we shall say that p is of type (i), (ii), or (iii) depending on which of (i)–(iii) of Lemma 4.7 p satisfies.

The next result is needed for Lemma 4.9.

Corollary 4.8. *If* M *is a maximum-sized* 2*–regular matroid, then* M *has no point* p *for which* $\mathrm{si}(M/p) \cong S_{10}$.

Proof. Suppose that M has such a point p. Then $r(M) = 4$ and so, by Lemma 4.7, the union of the long lines through p has rank 4. Therefore, by Lemma 4.4, every element of $E(M)$ is on a plane spanned by two long lines through p. Say p is of type (ii). Then $\mathrm{si}(M/p)$ has at most three long lines in which each line contains at least four points. Each of these lines corresponds to one of the three planes spanned by two long lines through p in M. Since S_{10} has five 4–point lines, we have a contradiction. Therefore assume that p is of type (i). Then M has a P_9–restriction in which p is the tip. Moreover, as every element of M is of type (i), (ii), or (iii), every point of this P_9–restriction, other than p, is on a 5–point line of M. Hence $\mathrm{si}(M/p)$ has a 5–point line restriction and so it is not isomorphic to S_{10}. □

The proof of Lemma 4.9 is a routine modification of the proof of [8, Lemma 5.6]. We note that Corollary 4.8 plays the role of [8, Lemma 5.4] in this modification and omit the details of the proof.

Lemma 4.9. M *has a point of type (i) or (iii).*

Lemma 4.10. M *has a point of type (iii).*

Proof. Assume that every point of M is of type (i) or (ii). By Lemma 4.9, M has a point p of type (i). Let N be the P_{2k+5}–restriction of M having p as its tip. Let L be a 3–point line of N and let $L = \{p, x_1, x_2\}$. Since $k \geq 2$, x_1 and x_2 are on long lines L_1 and L_2, respectively, of N in which both contain at least four points and therefore both x_1 and x_2 must be of type (ii). Thus both L_1 and L_2 are of size $k+3$, so, by Lemma 3.6, M has a rank–3 restriction isomorphic to T_3^k. But then M has a point that is on two long lines of size $k+3$ and the fact that M has no point of type (iii) is contradicted. □

Corollary 4.11. M *has a unique point* p_o *of type (iii).*

Proof. By Lemma 4.10, M has a point p_o of type (iii). By Lemma 4.6, M/p_o is regular. Therefore every $(k+3)$–point line of M meets p_o and so p_o is the only point of type (iii). □

The next result follows from Lemma 4.4.

Lemma 4.12. *Every element of M is on a plane spanned by two $(k+3)$–point lines through p_o.*

We are now able to determine, for $k \geq 2$, the maximum-sized rank–r k–regular matroids.

Lemma 4.13. $M \cong T_r^k$.

Proof. By the last lemma, every point of M is on a plane spanned by two $(k+3)$–point lines through p_o. By Lemma 3.6, this plane is a restriction of T_3^k and so it has at most one additional point. Since p_o is of type (iii), M has $\binom{r-1}{2}$ such planes. Therefore

$$|E(M)| \leq 1 + (k+2)(r-1) + \binom{r-1}{2}. \tag{4.8}$$

Since $|E(M)| = \binom{r+k+1}{2} - \frac{k}{2}(k+3)$, which is equal to the right-hand side of (4.8), it follows that every plane that contains two $(k+3)$–point lines through p_o contains exactly one additional point and is therefore isomorphic to T_3^k.

We complete the proof of the lemma, and Theorem 2.3, by obtaining a k–regular representation for M. It will turn out that the representation obtained is a k–regular representation for T_r^k and in the same form as the one shown in Section 2. Label the $(k+3)$–point lines of M through p_o by $L_1, L_2, \ldots, L_{r-1}$ and, for each $i < j$, let w_{ij} be the unique point of M in $\mathrm{cl}(L_i \cup L_j) - (L_i \cup L_j)$. Label the points of $L_1 - p_o$ arbitrarily by $x_1^1, x_1^2, \ldots, x_1^{k+2}$. Then, for each $i \in \{2, 3, \ldots, r-1\}$, let $x_i^1, x_i^2, \ldots, x_i^{k+2}$ be the points of intersection of L_i with $\mathrm{cl}(\{x_1^1, w_{1i}\}), \mathrm{cl}(\{x_1^2, w_{1i}\}), \ldots, \mathrm{cl}(\{x_1^{k+2}, w_{1i}\})$, respectively. A basis for M is $B = \{p_o, x_1^1, x_2^1, \ldots, x_{r-1}^1\}$. As M is a k–regular matroid, there is a k–unimodular matrix X representing M. We will partition X into $k+4$ parts and label the columns of X in the following way. The first and second partition of X will correspond to p_o and $B - p_o$, respectively. For $l \in \{3, 4, \ldots, k+3\}$, we will label the l–th partition's columns by $x_1^{l-1}, x_2^{l-1}, \ldots, x_{r-1}^{l-1}$. In other words, the elements of $E(M)$ corresponding to the columns of the l–th partition are those elements which share a 3–point line with x_1^{l-1}. The last partition consists of columns whose corresponding elements of $E(M)$ have the form w_{ij}. Since, for each $i \in \{2, 3, \ldots, r-1\}$, $\{x_1^1, w_{1i}, x_i^1\}$ is a 3–circuit, we deduce that the first entry in each of the columns labelled $w_{12}, w_{13}, \ldots, w_{1(r-1)}$ is zero. We may assume that X is as shown in Figures 3 and 4. In the first matrix, the entries $a_1^2, \ldots, a_{r-1}^2, a_1^3, \ldots, a_{r-1}^{k+2}$ are non-zero. In the second matrix, the entries $b_2, b_3, \ldots, b_{r-1}$ and $d_{23}, d_{24}, \ldots, d_{(r-2)(r-1)}$ are all non-zero, but the entries $c_{23}, c_{24}, \ldots, c_{(r-2)(r-1)}$ may be zero. Whether each of the entries $c_{23}, c_{24}, \ldots, c_{(r-2)(r-1)}$ is zero or not, depends on $\{w_{ij}, x_i^1, x_j^1\}$ being a 3–circuit.

$$
\begin{array}{c|c|c|c|c}
p_0 & x_1^1 \ \cdots \ x_{r-1}^1 & x_1^2 \ \cdots \ x_{r-1}^2 & \cdots & x_1^{k+2} \ \cdots \ x_{r-1}^{k+2} \\
1 & 0 \ \cdots \ 0 & a_1^2 \ \cdots \ a_{r-1}^2 & & a_1^{k+2} \ \cdots \ a_{r-1}^{k+2} \\
\hline
0 & & & & \\
0 & & & & \\
\vdots & I_{r-1} & I_{r-1} & \cdots & I_{r-1} \\
0 & & & & \\
0 & & & &
\end{array}
$$

FIGURE 3. The first $k+3$ partitions of X.

$$
\begin{array}{cccccccc}
w_{12} & w_{13} & \cdots & w_{1(r-1)} & w_{23} & w_{24} & \cdots & w_{(r-2)(r-1)} \\
0 & 0 & \cdots & 0 & c_{23} & c_{24} & \cdots & c_{(r-2)(r-1)} \\
\hline
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
b_2 & & & & 1 & & & \\
 & b_3 & & & d_{23} & 1 & & \\
 & & \ddots & & & & \ddots & \\
 & & & & & & & 1 \\
 & & & b_{r-1} & & & & d_{(r-2)(r-1)}
\end{array}
$$

FIGURE 4. The last partition of X.

We now determine the unknown entries of X. By scaling the first row and first column, we may assume that $a_1^2 = 1$. Furthermore, by scaling rows $3, 4, \ldots, r$ and then those columns whose entries were affected by this row scaling, we may also assume that $b_2 = b_3 = \cdots = b_{r-1} = -1$. As $\{x_1^{l-1}, w_{1i}, x_i^{l-1}\}$ is a long line of M, it now follows that, for each l in $\{3, 4, \ldots, k+3\}$, $a_1^{l-1} = a_i^{l-1}$, for all i in $\{2, 3, \ldots, r-1\}$. Moreover, for all l in $\{3, 4, \ldots, k+3\}$, the elements a_1^{l-1} are all distinct.

Next we determine $d_{23}, d_{24}, \ldots, d_{(r-2)(r-1)}$. Let S be the union of L_1 and two other $(k+3)$–point lines of M through p_o. Consider the restriction of $\mathrm{si}(M/p_o)$ to those elements of $E(M)$ in the closure of S. Then, as $\mathrm{si}(M/p_o)$ is regular, this restriction of $\mathrm{si}(M/p_o)$ must be isomorphic to $M(K_4)$. It immediately follows that for all i and j in $\{2, 3, \ldots, r-1\}$ with $i < j$, the matrix

$$
\begin{bmatrix} 1 & 1 & 0 \\ b_i & 0 & 1 \\ 0 & b_j & d_{ij} \end{bmatrix}
$$

has zero determinant. Since $b_i = b_j = -1$, $d_{ij} = -1$.

Now we show that $c_{ij} = 0$ for all i and j in $\{2, 3, \ldots, r-1\}$ with $i < j$. Consider $M|\mathrm{cl}(L_i \cup L_j)$. Recall that this matroid is isomorphic to T_3^k. If, for some i and j in$\{2, 3, \ldots, r-1\}$, the elements x_i^{l-1}, w_{ij}, and x_j^{l-1} are all on the same long line, then $c_{ij} = 0$. So assume that this is not the case. Then, as $M|\mathrm{cl}(L_i \cup L_j) \cong T_3^k$, there exists distinct elements m and n of $\{1, 2, \ldots, k+2\}$ such that $\{x_i^1, x_j^m, w_{ij}\}$ and $\{x_i^2, x_j^n, w_{ij}\}$, where $m \neq 1$ and $n \neq 2$, are both lines of M. This implies that the submatrices

$$\begin{array}{ccc} x_i^1 & x_j^m & w_{ij} \end{array} \qquad \begin{array}{ccc} x_i^2 & x_j^n & w_{ij} \end{array}$$
$$\begin{bmatrix} 0 & a_j^m & c_{ij} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & a_j^n & c_{ij} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

of X both have zero determinant. Thus $c_{ij} = -a_j^m$ and $c_{ij} = 1 - a_j^n$, and so $-a_j^m = 1 - a_j^n$. If $m = 2$, then $a_j^m = 1$ and therefore $a_j^n = 2$ which is not in $\mathcal{A}_k$. Hence a_j^m and a_j^n are both elements of $\mathcal{A}_k - \{1\}$. Since X is a k–unimodular matrix, $a_j^m - 1$ and $a_j^n - 1$ are also in $\mathcal{A}_k$. One now readily checks using [10, Lemma 6] that no choice of a_j^m and a_j^n satisfy $a_j^n - a_j^m = 1$. We conclude that, for all i and j, $c_{ij} = 0$ and therefore $M \cong T_r^k$. Hence Lemma 4.13 and, in particular, Theorem 2.3 is proved. □

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