What the applicability of mathematics says about its philosophy

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Introduction

We use mathematics to understand the world. This fact lies behind all of modern science and technology. Mathematics is the tool used by physicists, engineers, biologists, neuroscientists, chemists, astrophysicists and applied mathematicians to investigate, explain, and manipulate the world around us. The importance of mathematics to science cannot be overstated. It is the daily and ubiquitous tool of millions of scientists and engineers throughout the world and in all areas of science. The undeniable power of mathematics not only to predict but also to explain phenomena is what physics Nobel laureate Eugene Wigner dubbed the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960).

Yet the success of mathematics in explaining the world belies a great mystery: why is that possible? Why are our abstract thought and our manipulation of symbols able to successfully explain the workings of distant stars, the patterns of stripes on a tiger, and the weirdest behaviour of the smallest units of matter? Why is applying mathematics to the real world even possible?

This is a question in *the philosophy of mathematics*. The traditional approach to answering it is to first decide (hopefully on rational grounds) what to believe about the nature of mathematics and its objects of study, and then to explore what this philosophical standpoint says about the applicability of mathematics to the world. In this chapter, I take a different approach.

I take as given *the existence of applied mathematics*. On this foundational axiom, I ask the question *"what does the existence of applied mathematics say about the philosophy of mathematics?"* In this way, we treat the existence of applied mathematics as a lens through which to examine competing claims about the nature of mathematics. What then do we mean by the existence of applied mathematics, by the philosophy of mathematics, and what are the claims on the nature of mathematics?

Applied mathematics

It is not easy to define applied mathematics. The authoritative *Princeton Companion to Applied Mathematics* (Higham, 2015) sidesteps this difficulty by instead describing what applied mathematics is based on what applied mathematicians do. This is a strategy, the *Companion* argues (Higham, 2015a, p.1), with some distinguished historical precedent (for example, Courant & Robbins, 1941).

In this chapter I borrow a concise definition of applied mathematics from mathematician Garrett Birkhoff (1911-1996), who took inspiration from physicist Lord Rayleigh (1842-1919): “mathematics becomes ‘applied’ when it is used to solve real-world problems” (quoted in Higham, 2015a, p.1). The breadth of this definition, which includes “everything from counting change to climate change” (Wilson, 2014, p.176), is important. It means that we can use the shorthand “applied mathematics” for any application of mathematics to understanding the real world, and the name “applied mathematician” for any person doing so. This usage of “applied mathematics” and “applied mathematician” means we avoid any confusion over how a particular example or person might be categorised according to contingent academic disciplines in the workplace.

For our purposes, then, applied mathematics, is simply *mathematics* which is *applied*. An applied mathematician is anyone who applies mathematics.

In a book like this we can take it for granted that the existence of applied mathematics is undisputed. Its chapters present case after case of the overwhelming success and importance of the application of mathematics to the world around us. Applied mathematics not only predicts the outcome of experiment, it also provides understanding and explanation of the forces, fields, and principles at work. Indeed, “Mathematics … has become the definition of explanation in the physical sciences.” (Barrow, 2000). This is what I mean by *the existence of applied mathematics*, a useful phrase which I will abbreviate to TEAM.

Here I take mathematics, science, and technology seriously, in that I believe they have something important and objective to say about the world. While there are cultural and social concerns with the institutional forms of transmission of mathematics, I firmly reject the “woefully inadequate explanation” (Barrow, 2000) that mathematics is merely a social construct. This postmodern fallacy has been hilariously exposed by Sokal (1996, 2008) and others. As a mathematician and scientist, I also reject the notion, fashionable among some famous physicists, that philosophy has nothing useful to say about science; see for example Weinberg (1992), or Krauss (2012). This chapter is evidence against that view.

The four schools of the philosophy of mathematics

What is mathematics? What is the status of the objects it studies? How can we obtain reliable knowledge of them? These are the general types of questions which animate and define the philosophy of mathematics, and on which we will focus below. If you think this sounds vague, I agree with you. In *Philosophy of Mathematics: selected readings* (Benacerraf & Putnam, 1983) compiled by the highly influential philosophers of mathematics Paul Benacerraf and Hilary Putnam, the editors write in their first sentence “It would be difficult to say just what comprises the philosophy of mathematics”.

But we have to talk about something, so in what follows I present some of the main ideas from the long history of this vaguely-defined area of philosophy. This is not an exhaustive study of all of the schools of the philosophy of mathematics, neither will we see all of the main areas of study. Those in the know might find it shocking that I do not mention Descartes, Locke, Berkeley, or Wittgenstein, and spend scant time on Kant and Hume. Their ideas fill these pages through their influence on their contemporaries and those who came after them and on whose ideas I focus. And while I try to present some historical development, this can only ever be cursory in a single chapter covering over 2,500 years from Pythagoras to the present. I am painfully aware of the Western bias in my presentation, with no mention of the great Indian, Chinese, and Arabic traditions. I hope that you are intrigued enough to follow the references. If you are eager to start right now, then Bostock (2009) gives a highly readable and comprehensive introduction, Benacerraf & Putnam (1983) contains selected key papers and readings, Horsten (2016) is an excellent starting point for an educational internet journey, and Mancosu (2008) is a survey of the modern perspective. But I hope you will read this chapter first.

The chapter divides the philosophy of mathematics into four schools, each of which has its own section. This division is broadly accepted and historically relevant, but not without controversy. I have also tried to present the arguments of smaller subschools of the philosophy of mathematics. Sometimes this has required discussing a subschool when a theme arises, even if historically it does not belong in that section. I hope that historians of the philosophy of mathematics, and the philosophers themselves, will forgive me.

Mostly I have tried to avoid jargon, but there are some important concepts that I have tried to develop as they arise. However, there are two words needed from the start: *ontology* and *epistemology*. Ontology concerns the nature of being. In terms of mathematics: what do we mean when we say that a mathematical object exists? Are mathematical objects pure and outside of space and time, as the platonist insists, or are they purely mental, as the intuitionist would argue, or the fairy tales of the fictionalist? Epistemology concerns the nature of knowledge, how we can come to have it, and what justifies our belief in it. Speaking loosely, we can say that if ontology is concerns the nature of what we know, then epistemology concerns how we know it.

The lens

I focus on what TEAM says about the philosophy of mathematics. It is important to distinguish this concern with what the applicability of mathematics says about the nature of mathematics from a concern (even a philosophical one) with the nature of the work done in applying mathematics. This latter question focusses on the praxis of applying mathematics: how applied mathematicians choose which problems to work on, how they turn a real-world problem into a mathematical one, what their aesthetic is, how they choose a solution method, how they communicate their work, and related questions. See for example Davis & Hersh (1981), Ruelle (2007), Mancosu (2008), and Higham (2015).

In training our TEAM lens on the four main schools of the philosophy of mathematics, we bring into focus some aspects of old questions. This is complementary to a more modern focus on the so-called "philosophy of real mathematics" (Barrow-Green & Siegmund-Schultze, 2015, p.58).

This "new wave" as outlined in the introduction to Mancosu (2008), currently avoids the daunting ontological question of why mathematics is applicable, and focusses instead on expanding the epistemological objects of study to include "fruitfulness, evidence, visualisation, diagrammatic reasoning, understanding, explanation" (Mancosu, 2008, p.1) and more besides. These everyday epistemological issues raised by working with mathematics are used to refine what is meant by applied mathematics, to study how applied mathematics and its objects of study relate to the rest of mathematics, and what mathematical value there is in applied mathematics. Indeed, Pincock (2009, p. 184) states "a strong case can be made that significant epistemic, semantic and metaphysical consequences result from reflecting on applied mathematics". The interested reader is referred to the excellent overviews collected in Mancosu (2008) and Bueno & Linnebo (2009).

I take TEAM as axiomatic in order to examine the claims of various schools of the philosophy of mathematics. This is distinct from those like Quine (1948) and Putnam (1971) who take TEAM as axiomatic in order to provide a justification for "faith" in mathematics. As outlined by Bostock (2009, pp. 275 ff), the Quine/Putnam position is that mathematics is similar to the physical sciences in the sense that both postulate the existence of objects which are not directly perceptible with human senses. In the case of mathematics, this includes the integers, while for the physical sciences, this includes atoms, to take an example in each field. The Quine/Putnam position is that mathematics as well as the physical sciences should be exposed to the "tribunal of experience". In particular, since our atomic theory leads to predictions which conform to our experience, we should accept the existence of atoms as real. Crucially, claim Quine and Putnam, since all our physical theories are mathematical in nature, and since those theories work, we must accept the existence of the mathematical entities on which those theories depend as also being real. The Quine/Putnam *indispensability argument* is that we must believe that mathematical objects exist because mathematics works. We will return to the indispensability argument, but I reiterate that we will use TEAM as an axiom for examining competing claims on the nature of mathematics, rather than using TEAM as an axiom for a new claim on the nature of mathematics.

The remainder of the chapter is structured as follows. We will examine each of the four schools in turn, introducing their main ideas, explaining their ontology and epistemology, and giving a brief overview of their history and structure. Within each school's section, we will use the TEAM lens to bring into focus the challenges faced by the school's followers as they attempt to explain the applicability of mathematics. We end with a discussion and conclusion.

Platonism

The platonist believes that mathematical objects are real and exist independently of humans in the same way that stars exist independently of us. Stars burn in all ignorance of us, and while their properties are discoverable by humans, they are independent of us. The same is true, says the platonist, of the existence and properties of numbers, and of all mathematical objects. Thus the platonist mathematician believes that we discover mathematics, rather than invent it.

The platonist position is that all abstract objects are real. An "abstract object" is one which is both entirely nonphysical and entirely nonmental. The triangle formed by the three beams over my head is an entirely physical object. When I hold it in my mind, and as you now attempt to picture it in yours, we have a mental object which is drawn from our experiences of the physical. But this mental object is still not yet a platonic object. For the platonist there exists in a third “realm” apart from the physical and mental ones the perfect, ideal form of a triangle, of which the imperfect triangles in our minds, and the still less imperfect ones in our physical world, are merely poor approximations.

The platonist does not believe that mathematical objects are drawn off or abstracted from the physical world; rather, that they exist in a realm of perfect, idealised forms outside of space and time. But what does “existence” mean in this statement? Existence usually refers to an object embedded in time and space, yet these platonic forms are taken to be outside of time and space. Their existence is of a different type to all other forms of existence of which we know. We can say that as I type this my laptop rests on an oak table in New Zealand early in 2017. We can say that our sun will be in the Milky Way galaxy next year, and that Ceaser lay bleeding in Rome two millennia ago. The verbs “rest”, “be”, “bleed” in these statements are fancy ways of saying “is”, and the locations and times in each example are not two pieces of information but one: a single point in the fabric of spacetime which Albert Einstein (1879-1955) wove for us a century ago. By contrast, platonic objects “are” in a “place” outside of spacetime.

Platonism is the oldest of our four schools, and for many mathematicians in history this perspective was taken to be natural and obvious – and this remains true for the typical mathematician or scientist today (Bostock, 2009, p. 263). There is some evidence that Plato (427-347 BCE) held this view (Cooper, 1997), possibly swayed towards the life of the mind and away from the life of the engaged citizen philosopher after his great mentor Socrates was condemned to death. Plato presented his theory of forms in his *Phaedo*, and developed it in his *Republic* (Cooper, 1997), with its enduring image of a shackled humanity deluded by shadows cast by ideal forms on a cave wall. It is much less clear that the platonism we are discussing here was a view held by Plato, since in later life Plato saw mathematical forms as being intermediaries between ideal forms and perceptible objects in our world (Bostock, 2009, p. 16). For this reason I do not capitalise the word platonism.

Mathematical platonism is the position that mathematical objects have a reality or existence independent not only of space and time but also of the human mind. Within this statement are the three claims that (1) mathematical objects exist, (2) they are abstract (they sit outside of spacetime), and (3) they are independent of humans or other intelligent agents (Linnebo, 2013). All three claims have been challenged by various schools, but the claim of independence sets platonism apart from the other schools, as we shall see. For the platonist, the concept of number, the concept of a group, the notion of infinity – all of these would exist without humans, and even, remarkably, without the physical universe. The platonist ontology is that mathematical objects are real, the realest things that exist.

But how can we know about them? Even mathematicians are physical beings containing mental processes and which are embedded in space and time, so how can they access this platonic realm, which sits outside of spacetime? The only platonist answer to this epistemological problem is that we know about these abstract objects *a priori* – that is, that they are innate, and independent of sensory evidence.

This is surely an unsatisfactory answer. To say that we know something *a priori* is merely to rename the fact that we do not know how we know it. It is dodging the issue – begging the question. If the innateness claim is taken to its extreme, the idea that every abstract concept that humanity might ever uncover is somehow hardwired from birth into a finite brain of finite storage capacity seems questionable to say the least. And where is the information encoded which is uploaded into the developing foetal brain? DNA has a finite, if colossal, storage capacity (Extance, 2016).

The other option is that (at least) the human mind somehow has the capacity to access the platonic realm. But how can a physical, mental being access a realm outside of those two realms? Plato himself saw this epistemological problem as a grave issue, and in his later life he moved away from the viewpoint which bears his name, as we saw above.

This *problem of epistemological access* was precisely formulated by Benacerraf (1973). By breaking the problem into its constituent assumptions and deductions, Benacerraf gave philosophers of mathematics more precise targets at which to aim. There have been many responses to this challenge, as we shall see. But as summarised in Horsten (2016), the fundamental problem of a how a “flesh and blood” mathematician can access the platonic realm “is remarkably robust under variation of epistemological theory” – that is, “[t]he platonist therefore owes us a plausible account of how we (physically embodied humans) are able” to access the platonic realm.

Such an account is elusive, although attempts are being made; see (Balauger, 2016, section 5) for an excellent summary. It is worth noting here that even ardent platonists such as Kurt Gödel (1906-1978) failed to avoid dodging the issue. Gödel is a central figure in the philosophy of mathematics. As we shall see, he was a platonist, who destroyed both logicism and formalism, and shackled the consistency of intuitionistic arithmetic to that of classical arithmetic (Ferreirós, 2008, p. 151), where *consistency* means that contradictions cannot be derived. But returning to the issue of epistemological access, we see for example, in Gödel (1947, pp. 483-4) how he skips over it by stating “axioms force themselves on us as being true. I don’t see why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception.” But how do we come by such intuitions? Whether they are innate (following the great Immanuel Kant (1724-1804)) or acquired (following the equally great David Hume (1711-1776)) there remains the question of how mental events correlated with physical brains localised in spacetime are able to have them.

Platonism is a kind of *realism*. The realist believes that mathematical objects exist, and do so independently of the human mind. Gödel was certainly a platonic realist (Bostock, 2009, p. 261). There are, however, non-platonic forms of realism, and the Quine/Putnam position outlined in the Introduction is one example. Quine and Putnam argue that mathematics is real because it underpins our physical theories – since they work, mathematics must be true. By “work” here I mean precisely what I meant when I defined applied mathematics in the Introduction, and the breadth of that definition is important. Since it really does cover everything from counting change to climate change, it is not just the use of mathematics in highfalutin scientific domains such as climate modelling or fundamental particle physics, but also includes the utility of basic arithmetic for counting sheep.

Platonism under the lens

Under even the closest scrutiny beneath the TEAM lens, the ontology of platonism remains as pure and perfect as its own ideal forms. Since the platonist believes that the physical world is an imperfect shadow of a realm of perfect ideal objects, and since in this worldview mathematics is itself a very sharp shadow cast by a more ideal form, it is no surprise that our mathematics becomes applicable to the physical world. This is not *evidence for* platonism, but the TEAM lens does not reveal any *evidence against* platonism based on its ontology.

However, as we have seen, cracks appear when we examine the epistemology of platonism – that is, when we ask how we are able to have knowledge of the platonic realm of ideal forms. The problem of epistemological access is such a serious one that it has prompted a rejection of platonism altogether, which we consider in the following three sections. Another approach has been to recast platonism in forms which avoids the epistemic access problem.

One example is *plenitudinous platonism*; see Balauger (1998) and Linsky & Zalta (1995, 2006) for two different versions. The central idea is that any mathematical objects which *can* exist, *do* exist. Summarising how this approach may solve the problem of epistemological access, Linnebo (2013) says "If every consistent mathematical theory is true of some universe of mathematical objects, then mathematical knowledge will, in some sense, be easy to obtain: provided that our mathematical theories are consistent, they are guaranteed to be true of some universe of mathematical objects."

While plenitudinous platonism may solve the epistemic access problem (though this remains controversial), it does not yet explain why mathematics is able to be applied to the real world. Both platonism and plenitudinous platonism fail to explain why any part of mathematics should explain the physical world. Simply because our mathematical objects (platonism), or objects in all forms of mathematics (plenitudinous platonism) have an independent existence does not in any way explain why they are applicable to the real world around us. Something further is required, some explanation of why the platonic realm entails the physical realm. This is what the TEAM lens brings sharply into focus for the platonist and plenitudinous platonist arguments.

An idea similar to plenitudinous platonism and which goes some way to addressing epistemic concerns was developed by the mathematical physicist Max Tegmark (b. 1967). In a series of papers beginning with Tegmark (2008), and explained in layperson terms in Tegmark (2014), Tegmark shows that platonic realism about physical objects implies a radical platonic realism about mathematical objects. Tegmark argues that the hypothesis that physical objects have an independent existence implies his Mathematical Universe Hypothesis (MUH): "our physical world is an abstract mathematical structure" (Tegmark, 2008, p. 101). He goes on to argue, echoing the plenitudinous platonists, that all mathematics which can exist does exist in some sense, that our physical world *is mathematics* (not simply *mathematical*), and that our minds are themselves self-aware substructures of this mathematical universe. In the MUH, our selves, our universe, and the various multiverses which our physical theories imply are subsets of this grand mathematical ensemble. The MUH addresses (though was not motivated by) the same epistemic concerns which motivated the plenitudinous platonists. Tegmark's ideas have spawned much debate, and in the grand tradition he has both defended and amended his hypothesis. It is heartening to see a mathematical physicist engaging with philosophers and mathematicians precisely around the issues of this chapter. For a starting point of objections and Tegmark's responses to them, see Wikipedia (2017).

As for Quine/Putnam realism, Bostock (2009, p. 278) observes that when considering objections to the Quinean position it is important to be careful about what is meant by science and the applications of mathematics. He argues that adopting the kind of broad definition of applied mathematics that I have taken for this chapter will undermine some of the objections to the Quine/Putnam theory, such as those in Parsons (1979/80) and Maddy (1990). However, surely we can conclude that the Quine/Putnam idea is attractive under the TEAM lens?

Not so, claims Bostock (2009, pp. 305-6). One problem is the tenuous nature of truth when it is defined in this quasi-instrumentalist and utilitarian way, when the only true mathematical things are those which currently support our physical theories. As the theories change, so does truth. Worse, it is possible to argue that fewer and fewer parts of classical mathematics are required for our scientific theories, leading, in the extreme, to the fictionalism of Hartry Field (b. 1946) in which absolutely no mathematical objects are necessary; see the discussion in the Formalism section. But even if a time-dependent notion of mathematical truth is accepted, Paseau (2007) observes that the Quine/Putnam theory leaves unspecified the ontological status of the objects it posits. Mathematical statements are true when they are useful, but the Quinean can only shrug when asked whether mathematical objects are platonic or have one of the other possible statuses given in the following sections.

A final comment concerns the issue of causal agency for Quine and Putnam. Their position argues that both quarks and real numbers are to be considered true in as much as they are required in our quantum mechanics. Yet the former is a name for something which has a causal role in the world, while the latter is the name for a temporarily useful fiction with no causal power.

Logicism

To the logicist, mathematics is logic in disguise. All of the varied fields of mathematics are simply the fecund outpourings produced when logic combines with interesting definitions (Bostock, 2009, p. 114). Mathematics equals logic plus definitions.

In this way, logicists seek to *reduce* mathematics to something else: logic. This idea can trace its lineage to Aristotle (384-322 BCE), who invented logic and tried to formulate his mathematical arguments in logical terms. Aristotle rejected Plato's insistence on a higher realm of ideal objects.

He did not reject abstraction, but saw it as a process of generalisation of examples in the world. To him, the concept of triangle generalised real-world triangles. While Plato believed that all Earthly triangles were poor shadows of an ideal triangle with an independent existence beyond space and time, Aristotle believed that the concept of a triangle was abstracted from our everyday experience of triangles in the world. All Aristotle's science and mathematics concerns these abstractions. His ontology is of generalised ideas in the human mind, and his epistemology is one of perception, even in mathematics. (Bostock, 2009, p. 16). Thus to Aristotle, and his *conceptualist* viewpoint just outlined, we invent rather than discover mathematics, which is why I described him as having *invented* logic.

Central to a reductionist view of mathematics is that it can be reduced to something more fundamental, that the definitions of mathematics are a type of name or shorthand for relationships between sets of the fundamental objects, and that the correspondence of those names with things in the real world is of little interest or relevance to mathematics. This type of reduction can be called *nominalism*, since it concerns names, and there are two types (Bostock, 2009, p. 262). One is logicism, which reduces mathematics to logic, and states that mathematics is a collection of names applied to logical objects. In this view, mathematics is a set of truths derived (or discovered) by the use of logic. It is worth noting that in this nominalist account, the mathematical objects have no independent existence. The second type of nominalism is the fictionalism of Hartry Field, which we discuss below in the section on Formalism.

The logicist ontology is that mathematical objects are merely logical ones in disguise. This ontology neatly explains why the varied fields of mathematics are connected: they lie in correspondence with one another because their objects of study are at root the same logical objects (or collections of them), but with a different overlay of definitions. Moreover, the central practice of mathematicians, the proving of theorems, follows well-defined and closely prescribed logical rules which themselves guarantee the validity and truth of the outcomes. No matter the definitions of the objects, when logical operations are correctly applied to logical objects (disguised as mathematical ones) the outcome will certainly be true.

In the logicist worldview mathematicians take disguised logical objects and perform logical operations on them. Because of this derivation of new results by a logical analysis of existing concepts, it is tempting to refer to these truths as *analytic*, and thereby to invoke Kant, and in particular to set up an opposition with Kant's *synthetic* truths derived from experience. But to use these words here might be misleading, since Kant himself argued for the synthetic nature of some, if not all, mathematical truth (Bostock, 2009, p. 50). To Kant, mathematical truths could not be wholly derived by the action of logic; some *a priori* "intuition" of the objects involved was required. In the context of logicism, an analytic truth means one which is derived by the action of logic on logical objects plus definitions. This is the usage employed by the key figure Gottlob Frege, as we shall see below.

To explore what it means to say that mathematics is logic plus definitions, we can ask: what is a number in the logicist worldview? Surely something so fundamental to mathematics, at the core of arithmetic, cannot be open to debate? Yet to the logicist, the idea of number is in some sense superfluous to the truths of arithmetic. Defining number in a mathematical way simply overlays mathematical definitions on logical objects. The overlay is done on multiple objects rather than single objects, since if the latter were true then the logicist worldview would be rather barren. Merely positing a one-to-one correspondence between mathematical objects and logical ones would be no more interesting than compiling a very accurate thesaurus. If I observe that every eggplant is an aubergine and that every aubergine is an eggplant, then I can merely use the two words interchangeably, and I have not learned anything new about eggplants. Or aubergines. Rather, in the logicist worldview, a mathematical definition is powerful because it encodes multiple logical objects and the relationships between them. The apparently simple task of defining number logically takes us from the budding of logicism in the garden of a man named Frege, through its flowering in the care of a man named Russell, to its wilting in the shadow cast by a man named Gödel.

The soil for Frege's garden was laid down by Richard Dedekind (1831-1916). Dedekind is known to undergraduate mathematicians for putting the real numbers on a solid basis. He defined them by means of "cuts": an irrational such as the square root of 2 cuts the rational numbers into two classes, or sets. One of these contains all of the rational numbers smaller than the square root of 2, while the other contains all of the rational numbers larger than the square root of 2. This gave Dedekind the hope that all of mathematics could be built on logic plus set theory, with sets conceived of as logical objects.

This dream was shared by Gottlob Frege (1848-1925), who is considered the founder of logicism. Bostock in his (2009, p. 115) says "Frege's first, and … greatest contribution … is that he invented modern logic." Extending Dedekind's ideas, Frege defined number in terms of classes of equinumerous classes. In this way, the number 2 is the name for all sets which have two elements. Although this smacks of circularity, it is formalised in a way which avoids it. However, Bertrand Russell (1872-1970) found a paradox nestled at the heart of logicism as conceived by Frege as a combination of set theory and logic. This is the famous Russell's paradox, which in words is the following. Consider a set which contains all sets which do not contain themselves as members. Does this set contain itself? If it does, then it does not, and if it does not, then it does.

A popular analogy is the following. Suppose there is a town in which every man either always shaves himself, or is always shaved by the barber. This seems to divide the men of the town into two neat classes; no man can be in both sets by definition. But what about the barber? If he is a man who always shaves himself then he cannot be, since he is also then a man shaved by the barber. And if he is a man who is always shaved by the barber, then he will always shave himself, which he cannot.

Thus even the definition of quite simple sets is problematic. The problem is surprisingly difficult to eliminate, leaving aside solutions such as a barber who does not shave or is a woman. So difficult, in fact, that Frege gave up on his own logicist dream. Russell did not. He developed with Alfred North Whitehead (1861-1947) a new theory of "types", which in essence are hierarchical sets. This "ramified" theory eliminated the type of paradoxes which bedevilled Frege's logicism. A set could no longer contain itself as a member. In the shaving story, it is as if the town now has a caste system, and a man can be shaved only by someone of a lower caste. Thus the barber can be shaved by someone of a lower caste, and can shave anyone in a higher caste, but no-one can shave themselves (the lowest caste grows beards).

Russell and Whitehead wrote the monumental *Principia Mathematica* (Russell & Whitehead, 1910) to bring Frege's dream to fruition through their ramified theory of typyes. The power of the mantle of meaning which mathematics places over logic is revealed by the fact that it takes 378 pages of dense argument in the *Principia* to prove (logically) that one plus one equals two.

But despite these Herculean efforts, the dream of reducing mathematics to logic died when Gödel rocked the mathematical world in 1931 with the publication of his two *incompleteness theorems* (Gödel, 1931; see also Smoryński, 1977). The first theorem is bad enough news: it says that any system which aims to formalise arithmetic must necessarily be incomplete. *Incomplete* means that the system must contain true statements which cannot be proved. And Gödel showed that this is true for *any* system which aimed to formalise arithmetic, and, worse, for any system which contained arithmetic. Thus Gödel's theorem not only destroyed the approach based on a combination of logic and ramified types developed by Russell and Whitehead, but all possible approaches. This was a profound and philosophically disturbing shock to mathematicians, who until that moment believed that all true statements must be provable. Mathematics has not been the same since.

Even worse was to come from Gödel's second incompleteness theorem: it is impossible to prove the consistency of arithmetic using only the methods of argument from within arithmetic. Thus to prove even the most basic of mathematical areas consistent, that is to show that contradictions can never be derived within it, requires stepping outside of that area. But then the new area of mathematics used to establish consistency of the first area would itself require external techniques in order to establish its consistency, and so on.

Gödel showed that any system which aims to formalise an area of mathematics contains unprovable true statements, and whose consistency can only be established by stepping outside of itself. Logicism (and not just logicism, as we shall see) seemed well and truly dead. But logicism lives on in modified forms; the idea of number as a powerful naming convention for a set of interconnected logical objects is closer to what is now called the neo-Fregean standpoint. The difference between Fregean logicism and neo-Fregean logicism revolves around "Hume's Principle", which we do not have the space to consider here; see for example (Bostock, 2009, pp. 266 ff). Moroever, Russell's theory of types is now considered the start of predicativism. Both neo-Fregean logicism and predicativism seek to avoid paradox while retaining logic as fundamental. These ideas have been developed for example by Bostock (1980); see also his (2009, section 5.3).

If in some sense all mathematics can be reduced to logic, what is the ontology of logic? The logicist rejects the realist idea that mathematical objects have an independent existence in a platonic realm of ideal forms, and substitutes logic as a foundation for mathematics. But this merely shifts the ontological question on to logic, and here we see a divergence in the history of logicist thought. Its founding father, Frege, was a realist of sorts, since he believed that logic and its objects had a platonic existence (Bostock, 2009, chapter 9). Although Russell's views were complex and evolved throughout his life, he also seemed to remain essentially a platonic realist when it came to mathematics. Other logicists choose to remain silent on ontology.

Logicism under the lens

What can the logicist say about the existence of applied mathematics? If at the heart of mathematics we find only logic, and if the familiar objects of mathematics are merely names under which hides a Rude Goldberg arrangement of logical objects, then why should mathematics have anything useful to say about the real world? The logicist is not allowed to answer that the universe is merely an embodiment of a higher platonic realm of logic. To do so makes them a platonist.

There does not seem to be much more to see of logicism under the TEAM lens. At its heart, there is either a dormant platonism in its classical form (which Gödel destroyed anyway), or an echoing ontological silence in the modern forms. Since these modern forms do not propose any ontology, it is hard to critique them via the existence of applied mathematics. However, even they seem to have an implied platonism at their heart, since the neo-Fregean adoption of Hume's principle brings with it a notion of infinity which is platonic in the extreme – see Bostock (2009, p. 270) for some of the controversy.

Perhaps one observation can be made using the TEAM lens. If even such a simple concept as number veils a hidden complexity of logical objects, maybe what mathematicians do is to select definitions which excel at encoding logical objects and their interrelations. Having done so, perhaps mathematics is then a process of selection and evolution. This *principle of fecundity* and an evolutionary perspective is sufficiently general that it may apply in a broad sense to other schools in the philosophy of mathematics. However, it has problems. For a start, what is the ontological status of the fecund objects upon which evolution acts? Secondly, there are epistemological problems with the claim (see for example Mohr, 1977) that minds with the best model of reality are those which are selected as fittest evolutionarily. It is not clear that the objects of the human mind need faithfully represent the objects of the physical universe. Mental maps of reality survive not because they are faithful to reality, but because of the advantage they conferred to our ancestors in their struggles to survive and to mate. Moreover, while concepts such as number and causality have obvious correlates in the real world, our modern theories of physics involve concepts which have no obvious correlates in the real world, such as complex analysis or the common-sense defying nature of quantum mechanics.

Formalism

The formalist holds a radical ontological perspective: mathematical objects have no real existence, they are merely symbols. The mathematician shuffles and recombines these meaningless symbols according to the dictates of systems of postulates. No meaning is ever to be ascribed to the symbols or the statements in which they appear, nor is any kind of interpretation of these symbols or statements ever to be done. Some formalists may be content to remain agnostic on whether meaning can ever be ascribed to mathematical symbols and statements, preferring simply to insist that no meaning is necessary, that the symbols and their interrelations suffice. Others, more radically still, insist that no meaning can ever be given to mathematical symbols and statements, and the systems in which they are used.

These symbols are manipulated within systems of postulates and rules, the *formal systems* which give formalists their moniker. The formalist is in theory able to study any formal system, but usually certain restrictions are placed on what counts as a postulate, and what is an allowable rule. One of the main criteria for a formal system is the concept of consistency which we have already encountered.

A formal system is consistent when its axioms and rules do not allow the deduction of a contradiction. In the early days of the formalist school, its leader, David Hilbert (1862-1943) believed that consistency implied existence (Bostock, 2009, p. 168). It is hard to discern what is meant by "existence" here, given the formalist insistence on the meaningless of mathematics – indeed, Hilbert himself seems somewhat agnostic on this point (Reid, 1996). However, I take it to mean that any statement derived from the axioms and rules has (at the very least) the same ontological existence as the axioms themselves. Thus while mathematics may be seen as one among many formal systems, and while each can be studied in the same way, if the axioms of mathematics are shown to have a more significant existence then so do all other mathematical objects.

It is impossible to talk about formalism without talking about Hilbert. The school probably would not exist without him. Hilbert was a towering figure of 19^{th} and 20^{th} Century mathematics, and his name is attached to several important concepts and theories (Reid, 1996). He is also famous for listing 23 open problems in mathematics in the published form of his address to the International Congress of Mathematicians in Paris in 1900 (Hilbert, 1902). Many of Hilbert's problems are still unanswered and remain the focus of research today. Hilbert in 1920 began his so-called program to show that mathematics is a consistent formal system. As we have seen, Gödel would show a decade later that this is impossible.

Hilbert was already on the formalist track when in 1899 he published his *Grundlagen der Geometrie* (*The Foundations of Geometry*) (Hilbert, 1899), in which he formulated axioms of Euclidean geometry and showed their consistency. Hilbert is not the only mathematician to axiomatize Euclid's geometry. The idea is to eliminate geometrical intuition from geometry and to replace that intuition with definitions and axioms about objects bearing geometrical names. From those postulates can be derived all the theorems of Euclid's geometry, but crucially and as a direct result of the formulation of geometry as a formal system, those theorems need no longer be taken as referring to geometrical objects in the real world. In fact, they need not even be taken as referring to any kind of abstract geometry, neither to the platonist's ideal geometry, not to the aristotelian's geometry generalised from the real world. Although the postulates use words such as "line" and "point", these objects are only defined by the formal system, and are not supposed to be taken as referring to our everyday notion of lines and points. The words could just as easily be replaced by "lavender" and "porpoise" – but again, without any sense that there is any correspondence with lavender or porpoises in the real world. This is the start of the formalist dream.

It was no great surprise when Hilbert showed in his *Foundations of Geometry* that Euclidean geometry was consistent. At the time, the only area of mathematics over which there was any doubt as to its consistency was Georg Cantor's (1845-1918) theory of infinite numbers (Bostock, 2009, p.168). To introduce this theory, we first need to consider the notion of countability.

A finite set is *countable* if it can be placed in one-to-one correspondence with a subset of the natural numbers. This is a formal definition of what it means to count the objects in the set. Counting means assigning each object a unique number, which puts them in a one-to-one correspondence with a subset of the natural numbers, say the subset of numbers from 1 to 10 if there are 10 objects in the set. If the set is infinite, we call it countable if it can be placed in one-to-one correspondence with all of the natural numbers (not just a subset). (Some authors reserve countable for finite sets and call countable infinite sets *enumerable*.)

The concept of countability puts infinity within our grasp. If the elements in an infinite set can be paired with the counting numbers, then an incremental counting-type algorithmic process can be set up to "access" everything in the set. For every element in the set there is a unique positive whole number, and for every positive whole number there is a unique object. However, this immediately leads to apparent paradoxes. For example, the even natural numbers can be paired in an obvious way with the natural numbers, and are thus countable. This means that the size of the set of even natural numbers is the same as the size of the set of all natural numbers, despite the fact that the latter contains all of the former!

Cantor asked whether the set of all numbers is countable. This set of real numbers contains not just the natural numbers, but all integers, all rational numbers, and all irrational numbers. He assumed first that the reals are countable, in which case, by definition they can be listed alongside the natural numbers. The next step was Cantor's stroke of genius. He considered a real number whose decimal expansion differs from the first real number on the list in the first decimal place, from the second real number in the second decimal place, and so on for every decimal place. This number is therefore different from every number on the list, and so it is not on the list. Yet it is a real number, and so if the assumption of the countability of the reals were correct it is on the list. This contradiction implies that the assumption of countability was wrong, and Cantor concluded that the reals are uncountable. Stunningly, this means that there is a "bigger size" of infinity than the size of the set of natural numbers. Moreover, Cantor showed that there is an infinite succession of sizes of infinities, each bigger than the last, and he constructed a beautiful theory of these infinite numbers. Within this theory, his famous continuum hypothesis is that the second smallest size of infinity is the size of the set of real numbers (Bagaria, 2008)

Hilbert so loved Cantor's theory that he desired that "[n]o one shall drive us out of the paradise which Cantor has created" (Hilbert, 1926, p. 170), and so he was desperate to prove its consistency. He never did so, and Gödel incompleteness theorems showed its impossibility before Hilbert had even finished shoring up the foundations of arithmetic. As Hilbert waded through the mud he found in the formalist foundations, he repeatedly encountered the notion of infinity. Although he hoped to construct an edifice which up to Cantor's theory, Hilbert did not want infinity in the formalist foundations on which he built. Hilbert could not prove the consistency of arithmetic based on a finitary formal system. This insistence that as a finite human in an apparently finite world we should use only "finitary" definitions and methods will recur in our final school of mathematical philosophy, intuitionism, to which Hilbert ironically was bitterly opposed.

The death blow for Hilbert and the formalist's dream came with Gödel's incompleteness theorems, as described in the Logicism section above. These theorems not only destroyed the logicist dream of a mathematics founded on (and in some sense no more than) logic, but simultaneously destroyed Hilbert's formalism. This is because the theorems showed that any formal system sophisticated enough to contain simple arithmetic would necessarily contain unprovable true statements, and whose consistency required an external system. There was no way out, and formalism was dead.

Consequently, it is unlikely that anyone would call themselves a formalist today (Bostock, 2009, p. 195). The idea which died is that formal systems are primary in the sense that they are the object of study, and that any application of them to an area of mathematics is essentially meaningless. But formalism evolved and survived in the same way that dinosaurs both died out and are alive in the birds we see around us. One surviving form is *structuralism*. The idea behind it, as advanced by Dedekind (1888) and Benacerraf (1965) is that the common structures of particular areas of mathematics are the object of study; they are primary. Like the formalist, the structuralist believes that applications of the structures are secondary, and that it is the structures themselves which must be studied. For example, the natural numbers can be taken to be an example of a *progression*: a non-empty set of objects each of which has a successor, as formalised in Peano's Axioms (Gowers, 2008, pp. 258-9). Because natural numbers are an example of a progression, they are less interesting to the structuralist than the progression structure they model.

The idea of structure being fundamental seems to be attractive to some physicists, even if they do not necessarily acknowledge structuralism. Writing popular accounts of the power of mathematics in the physical sciences, people like John Barrow, David Deutsch, and Ian Stewart argue for the primacy of pattern or structure. For example, Deutsch (b. 1953), a mathematical physicist, argues that the human brain both embodies the mathematical relationships and causal structure of physical objects such as quasars, and that this embodiment becomes more accurate over time. This happens because our study of these objects aligns the structure of our brains with the structure of the objects themselves, with mathematics as the encoding language of structure (Deutsch, 2011). What is the ontology of such structures? The question is somewhat avoided by structuralists, but in essence they must claim either a platonic existence for them, or one of the other positions detailed here. Thus any claims of the structuralist are subject to some of the same ontological and epistemological objections as the other schools herein.

Finally here, we consider not a variant of logicism but a subschool which has in common with logicism the denial of any meaning in the objects of mathematics. In the Logicism section I said that logicism could be considered to be one form of nominalism. Another is given in Field (1980); see also Bostock (1979). By this account, mathematics is a “fairytale world which has no genuine reality” (Bostock, 2009, p.262). In this fairytale wold, numbers (and other mathematical concepts) are powerful names for a collection of underlying objects and structures. These names allow us to use, say, arithmetic rather than logic or set theory in our deductions. This use of arithmetic as a set of names and rules is conservative in the sense that we cannot prove anything in arithmetic that could not be proved by stripping away the arithmetical names and working with a more fundamental structure (such as logic). Thus the names are useful but not required, and no meaning is given to them. Moreover, even if it is a useful fiction to treat them as real, the things to which the names seem to point have no independent existence; they may be abstractions of some kind, but they are not real in the sense of having an independent platonic existence.

Of course, we sometimes choose names which correspond to things in the real world. We know about numbers when we count shirt buttons, which is a kind of instrumentalist view of the existence of numbers. Thus arithmetic can be taken to be about the countable things we encounter in the world, whose ontological status is either left vague or has a minimalist instrumentalist view. Any correctly derived arithmetical statements are true both of numbers as fictions and of real-world numbers. Arithmetical deductions which go beyond what can be encountered in the world are true, but only in some fictional sense.

Formalism under the lens

If mathematics is a game, why should it tell us anything about the world? To the pure formalist, mathematical objects have no “real” existence, and to do mathematics is simply to explore a formal system or systems. But no particular formal system should be privileged over any other – some may be more interesting than others, for sure, but none of them is taken to have any special ontological status. Why, then, does mathematics help to explain the world?

The only way out of this conundrum seems to be to take Hilbert’s less hard line view in which mathematical objects have a special ontological status, and that the formal system or systems at the foundations of mathematics are therefore more special than others. Although this does fix one problem, it creates another: what does it mean for mathematical objects to have special ontological status? What is that ontological status? The options are presumably those held by one of the other schools of the philosophy of mathematics and therefore subject to the same criticisms under the TEAM lens (amongst others).

Putting those criticisms to one side, and playing devil’s advocate, I could point out that some games do teach us about the world. For example, in 1970 Martin Gardner introduced the world to John Conway’s “Game of Life” (Gardner, 1970). Since that time, this simple game has become a field of study both in its own right and as a model for processes in biology, economics, physics, and computer science, as revealed by a quick search of Google Scholar. But although some features of the Game of Life are emergent and therefore could not be predicted, the simple rules of the game were chosen in order to mimic those of simple real-world systems. If we wish to claim that this is comparable to the far more complex game of mathematics mimicking the real world, then we would have to assert that the rules of mathematics were chosen in order to mimic those in the real world. Once again, we are forced to abandon the ontology of pure formalism, at least.

Other problems are visible under the TEAM lens. While it is easy to accept that, say, the rules of arithmetic have been chosen because they mimic real-world counting, it is harder to explain the important role that, say, complex analysis or Hilbert spaces play in our best theories of the universe. In geometry, it is "natural" to consider flat Euclidean geometry, and so the non-Euclidean geometry which arose in the last half of the 19th century was viewed initially with distaste and seen as something of a pointless game. Yet Einstein has taught us that our universe is non-Euclidean. How, then, are we to know which of our formal systems have special ontological status? Only those which are later shown to correspond to some aspect of the real world? But this is surely a poor ontological status which seems predicated both on time and on our ignorance. What if when our theories change we need an area of mathematics and so it becomes "real" – but then later find we no longer need it, at which it returns to being unreal? It seems that this is indistinguishable from the Quine/Putnam indispensability argument, and so arguments against that position are also valid here.

The structuralist might choose to argue that the structures of mathematics are chosen because they mimic some aspect of the real world. But does this not give a privileged ontological status to the real world, and the structures within it? What is their ontological status? At this point, the structuralist has passed the buck. The fictionalist seems Quinean when examined under the TEAM lens, for the only way to distinguish between the real and the fictional is to expose a truth to the crucible of the real world. The other option is to admit a platonic existence at the heart of your fictionalist worldview, as Field himself did when he sought to remove it in Field (1992).

Intuitionism

Intuitionism was the first and remains the largest "constructivist" schools of mathematics (Chabert, 2008). Most of what I say in this section can be taken to be true of the other constructivist schools, which include (i) finitism, (ii) the Russian recursive mathematics of Shanin and Markov, (iii) Bishop's constructive analysis, and (iv) constructive set theory. It is always a pleasure to note that intuitionists claim constructivism as a subschool and constructivists claim intuitionists likewise, but I will mostly use the word "intuitionism" as an umbrella term in this section, and look forward to the deluge it provokes from constructivists.

The defining characteristic of intuitionism is that existence requires *construction*. The perspective of intuitionists, for example in Bridges (1999), is that believing that existence requires construction forces upon the mathematician the requirement to use a different logic. This logic is the intuitionistic logic which has at its heart a rejection of the *law of excluded middle* and a rejection of the *axiom of choice*. I will explain each of these points below. It is worth noting that, as in every area we discuss herein, the argument for intuitionism has at least two sides. For every Bridges arguing that construction implies intuitionistic logic, there is a Dummett arguing that this is untrue (see his 1977, and Bostock, 2009, pp 215 ff). But we continue, since all schools presented herein have adherents arguing their corner and antagonists arguing them into one.

All mathematicians distinguish between an *existence proof* and a *construction proof*. The former merely establishes whether a statement is true or not. A construction proof, by contrast, gives steps which construct the properties of the object in question, and so gives in addition to a proof of truth some insight as to why. In the case in which the statement is not true, an actual counterexample is constructed. I now try to put a little flesh on these bones.

A common question in mathematics concerns the *existence* of a mathematical object. This is not the metaphysical notion of existence central to this chapter. When a mathematician asks whether a mathematical object exists, she is not worried about whether scientific methods can show it to be a real, physical thing in the world, nor is she usually bothered with the ontological status of that object. Instead, she is interested in whether the object exists in a mathematical sense.

For the majority of mathematicians, existence proofs suffice, even if construction proofs provide more information. Not so the intuitionists, who believe that existence is shown only when the object has been constructed. Construction here has a specific meaning, and once again this has nothing to do with building an object in the real world. Rather it has to do with providing a proof of a statement from which, at least in principle, an algorithm could be extracted which would compute the object in question, and any of its properties. Only when a constructive proof has been found is the object said to exist. For the intuitionist, “existence” means “construction”.

For a real-world analogy, we can turn the weather. When I look up the weather records for my home town of Christchurch, New Zealand, I can see that in 2016 the maximum recorded temperature was 34^{o}C on 27th February, and the minimum recorded temperature was -5^{o}C on 11th August (WolframAlpha, 2017). This means that with confidence I can claim that there was a moment between 27th February and 11th August when the temperature was precisely 0^{o}C. My assertion rests on two points: that for this time range the temperature starts at a positive value (34) and ends on a negative one (-5), and that temperature cannot instantaneously change. From these two observations, I know that there must exist a time, however short, when the thermometer read 0^{o}, since it is impossible to go smoothly from 34 down to -5 without passing through 0. Of course, there were probably many such times, but the mathematician’s interest in uniqueness is not our concern here, only existence. In our temperature analogy we have demonstrated the existence of a time at which the temperature was 0^{o} in a way which would satisfy most mathematicians.

But the intuitionist weather-watcher would not be satisfied. She wants something more: she wants an actual moment at which the thermometer read 0. In our analogy, this means going through the weather station data until such a time is found. That is a “constructive” proof of the existence of a 0^{o} time.

Our analogy has flaws, as all do. It could give the impression that intuitionistic mathematics is about data-sifting; this is untrue. Intuitionistic mathematics is mathematics, but with tighter constraints on what can be used in the logical arguments called proofs which establish truths. Indeed, Bridges argues in his (1999) that the intuitionistic mathematician is free to work with whatever mathematical objects she so desires. Another flaw is that although the analogy illustrates the difference between existence and construction, it does not have an analogy for intuitionistic logic.

I said above that intuitionistic logic has two features which distinguish it from classical logic, and both features involve a rejection. The first of these is a rejection of the law of excluded middle (LEM). For most mathematicians, something either is, or is not. A number is either rational, or irrational. It cannot be both; it is either. But the intuitionist will not say it is one or the other until it has been constructed. A classical mathematician may present the following argument. Object X can either have property P or not. If we assume for the sake of argument that it has property P, we can investigate the consequences of our assumption. Suppose that when we do that, we uncover a contradiction, an absurdity. Then (assuming we have done everything correctly) the only problem was our assumption that object X had property P. Thus it cannot have property P. This is the commonly used proof technique called *proof by contradiction*, and we saw an example of it above when we presented Cantor's diagonal argument.

Such a proof would not be considered valid in intuitionistic logic. The reason that it is invalid is that X has not been shown to have a particular property or not, but simply that by assuming the converse a contradiction has been found. At issue is not the assumption of whether or not X has property P. If the objects of study of which X is an example are such that they must either have property P or not, then it would be absurd to argue that they have neither, or, somehow, a superposition of both. The intuitionist does not argue this. Rather, the idea is of a radical redefinition of truth. To the intuitionist mathematician, a statement is true only when a constructive proof without recourse to the LEM has been given. A statement is false precisely when a counterexample has been given. Since truth now has this specific meaning, a statement is neither true nor false until such a constructive proof is furnished.

Although the truth of a statement becomes time-dependent, it is not the same time-dependency as in the Quine/Putnam indispensability argument. There, something is real only for as long as it is necessary for a successful theory of the real world; the status of mathematical objects are forever conditional. For the intuitionist, on the other hand, truth is defined to mean proof by construction. Thus an object is neither real nor not real until it is constructed, at which point it becomes and forever remains real (or becomes and remains forever not real when a counterexample is constructed).

To object that surely, say, the statement “the trillionth decimal digit of pi is zero” has been true or false since the dawn of time is to confuse the platonic notion of truth with the intuitionist one. The point is that although the trillionth decimal digit of pi has a value entirely independent of the free will of humans, that it is indeed dictated by something deeper than whatever human whimsy may want it to be, until its value is actually calculated the statement has no (intuitionist) truth value associated with it.

Although for the intuitionist mathematical objects have properties which can be rigorously defined or derived, they nevertheless have the ontological status of being purely mental objects. In this way, intuitionism is a form of the conceptualism which harks back to Aristotle (Bostock, 2009, p. 44). By making mathematics mental, intuitionists avoid problems of epistemic access, since naturally we can access the objects of our own minds. There is an ontological issue associated with insisting that mathematical objects are purely mental. We must ask why they have properties independent of the individual mind which explores or creates them. Thus an obvious objection to this conceptualism is that these objects must rely on some deeper structure that at the very least is shared by other human minds. But that suggests that there is something more fundamental than the mathematics itself – and the intuitionist certainly cannot claim that something like logic, language, "structure", or a platonic realm of ideas is more fundamental.

Indeed, the founder of intuitionism, Luitzen Egbertus Jan Brouwer (1881-1966), echoing Kant and in agreement with the mathematicians Felix Klein (1849-1925) and Henri Poincaré (1854-1912), believed that the basic axioms of mathematics are intuited. In this he meant that they were known to our minds, but not that our intuition reveals anything which exists outside of the mind. He went further, claiming a stark independence of mathematics from both language and logic. If there was any relation there, it was that logic and language rested on mathematics, rather than the other way around. This was revolutionary, and put Brouwer directly in harm's way. His point of view, given in Brouwer (1907), was directly contrary to both logicism and to Hilbert's program of formalism as it developed in the 1920s. Hilbert's program was popular and Hilbert himself was powerful. Brouwer apparently did nothing other than disagree with Hilbert, yet Hilbert had Brouwer removed from the editorial board of the prestigious journal *Mathematische Annalen*, and sought to discredit him at every turn (van Dalen, 2008, p.800).

Having discussed construction, the law of excluded middle, and the redefinition of "truth", I now consider the other idea which intuitionists reject, the axiom of choice. Stated in words, it says that we can always select an element from each of a family of sets. This is uncontroversial for a finite family of finite sets, but becomes controversial otherwise, because an infinite number of choices can be made. For most mathematicians this is not a problem; to put it crudely, the fact that there are an infinite number of choices which can be made guarantees that one can be made. For an intuitionist, the mere fact that a choice can be made is not enough: the choice must be specified in order to count as a construction. Yet when a classical mathematician invokes the axiom of choice it is usually for very general cases in which specificity is impossible.

To make this point clearer, suppose we have a countable number of sets, each of which is countable. Now suppose that we wish to form a superset containing all of the elements in all of the sets and to ask whether that new set is itself countable. This is easy for a classical mathematician. For each set, she first lists the elements, which we know can be done because every set in the family is countable. Then she runs the lists together in turn, and hey presto, the superset is listed out, and therefore countable. There is no "problem" with this proof for most mathematicians, but the intuitionist asks: how did she choose the ordering for each set, and for the family of sets? There is an infinite number of choices in each case, so the choice function is unspecified. The proof uses (in quite a disguised way) the axiom of choice. Whenever the axiom of choice is used, the proof is non-constructive.

Uncountable infinity is the heart of the rejections which define intuitionism. To be clear, if the axiom of choice is invoked either in a finite context or in one which is countable, then a choice function can be defined and the intuitionist is happy. The problem is in the uncountable case. Likewise, the law of excluded middle is connected with the notion of infinity; recently Bridges has argued that the continuum hypothesis implies LEM (Bridges, 2016). Only potential infinities, namely those accessible through enumeration or by an algorithmic process are acceptable to the intuitionist.

But to return to our starting point that intuitionistic mathematics is mathematics done with intuitionistic logic, we note that it is sometimes possible to construct intuitionistic theories of mathematical objects which in classical mathematics require uncountable infinities. For example, Brouwer introduced the notion of *choice sequences* to create a theory of the continuum (that is, the real number line) which was apparently out of reach to intuitionists (Brouwer, 1981). Brouwer never defined choice sequences carefully enough to avoid problems, but Bishop's constructive mathematics (Bishop, 1967, Bishop & Bridges 1985) does contain an apparently sound theory of the reals which avoids uncountable infinities. This is an example of how something which in classical mathematics requires uncountable infinities can be given an intuitionistic theory which only uses countable processes.

Intuitionism under the lens

Intuitionism has never been popular with mathematicians, and few applied mathematicians insist on a constructive approach to their work. But is it possible to argue that intuitionistic logic's insistence on countability, apparently so true of our physical universe, is the reason for the success of mathematics in modelling the world?

Does the universe only appear to rely on countability, and so are there unavoidable instances of uncountable infinities, both in our theories of the world and in the universe itself? Since infinity is implied in our best theories of the very big and the very small, it is no wonder that when intuitionism is under the TEAM lens what comes into focus is quantum mechanics (QM) and general relativity (GR).

It may seem that on a large scale our universe is a finite (though huge) thing containing a finite number (though huge) of discrete things. But we do not know that to be a fact. At the other end of the scale, quantum mechanics suggests that the structure of spacetime is granular at the very smallest of time and length scales. However, that prediction has not yet been verified. It may be the result of our most successful and accurate theory of science, but we do not know it to be true. Could the universe be infinite in extent? Might spacetime be a continuum?

Continuous spacetime does not necessarily cause a fatal problem for intuitionism since Bishop's constructive mathematics has an intuitionistic theory of continua. A potentially deeper argument, given by Hellman (1993, 1997), that intuitionism must be wrong because QM requires a theory of unbounded operators which seems to defy intuitionism, has been refuted by Bridges (1995, 1999) on the grounds that such a theory is possible with an intuitionistic approach. These Hellman-type arguments have also been refuted in the context of GR: see Billinge's (2000) response to Hellman

(1998). However, what of mathematical objects essential to our theories of the universe but for which no intuitionistic theory has yet been found? Does their necessity destroy intuitionism? Billinge (2000) says no, when she powerfully argues that just because we have not yet found a constructive proof of something does not mean that it cannot ever be found.

The intuitionist's belief that the objects of mathematics are purely mental avoids the plantonist's problem of epistemological access. But the TEAM lens shows us a deeper ontological problem: if the objects of mathematics are purely mental, why should they ever have any correspondence with the real world? Why should mathematics ever be useful?

Discussion and conclusion

The platonist see mathematics as eternal and changeless, existing outside of spacetime. But how do we access such an ideal realm? How does this ideal realm cast the physical "shadows" in our world which mathematics explains? The logicist reduces mathematics to logic in disguise. But why should logic explain the world? Does logic have a platonic existence? The formalist is the ultimate reductionist, claiming that mathematics is naught but a game, a meaningless shuffling of semantically empty symbols. But why should the game of mathematics be able to explain the world? Why that game and not another? Finally, conceptualism returns with the intuitionists, who believe that only construction means truth. But while intuitionistic logic and an insistence on construction are not at odds with our best theories of the universe (our best applied mathematics), the intuitionist believes that all mathematical objects are mental constructions. Why should such mental constructions explain the world?

This last point is subtle, and slippery. Of course we expect that any idea which explains the world will be in our minds; that is where we experience ideas. The issue concerns how an idea can come to mimic and explain the outside world. This is a debate with a long history. In the middle stand two figures directly opposed to one another. Kant believed that our minds are primary, and thus that our applied mathematics works not because our minds come to mirror reality, but because reality must conform to the mind in order to be perceptible and comprehensible to us. By contrast, Hume was an empiricist, naturalist, and sceptic, who believed that our concepts came from experience of an independently-existing natural world, without imposing an ontology on that world. At the far end of the chronology is Plato, who believed that our mental realm can access a world of forms which projects the physical world. This raises more questions than it answers. Nevertheless, it seems to be the perspective of many theoretical physicists today, perhaps without considering its epistemic problems. The modern structuralist, by contrast, might argue that structure is fundamental, and so our mental world can be structured to mimic the external world. We have already observed in the Formalism section that such a perspective seems to pass the buck on the ontological status of structure. This structuralist approach seems attractive to physicists such as Deutsch, who we encountered in our discussion of structuralism above, and who otherwise seems to be a realist in his worldview.

When physicists make pronouncements about mathematics they are usually motivated not by concern about what mathematics is or what its foundations are, but only by what sort of mathematics should or can be taken to be the foundation of *physics*. For example, the Nobel laureate in Physics Gerard ’t Hooft (b. 1946) wants only finiteness in his theories of quantum mechanics (Musser, 2013). It is not completely clear what he means by this, but it seems to be a kind of countability, since he mentions basing theory on the integers or finite sets (though the former is countably infinite). ‘t Hooft seems to be directly motivated by the granular discreteness of spacetime at the Planck scale predicted by QM. It would be wrong to suggest that he is rejecting classical mathematics and a platonic ontology in favour of, say, neo-Fregean logic, intuitionism, or a Hilbertean finitism, when he is only restricting himself to finite methods and objects for the mathematics of QM. He says nothing about the ontological status of other mathematics. Likewise, the physicist Lee Smolin (b. 1955) claims in his (2000) that topos theory is “required” for cosmology, and topos theory itself requires constructive set theory, a form of intuitionism. Once again, this is not a statement of ontological intent for the whole of mathematics, just for what mathematics can be applied to physics. In both cases, the question of epistemology is left open, as is the ontological status of the objects being studied. However, without acknowledging the philosophical position they adopt it is possible that applied mathematicians such as these physicists are overlooking some difficulties, especially when combining ideas from different philosophical schools. This seems particularly acute when the physical objects are considered real but the mathematics used to model them is considered to be entirely mental. Note that neither of these physicists claim that the mathematics which helps them is the only mathematics which is true; there is no evidence that they adhere to the Quine/Putnam indispensability argument.

The TEAM lens reveals other issues which we have not discussed above. For example, it is one thing to say that applied mathematics is possible, but we could also ask why *we* are able to do it. Why is the mathematics which seems to do so well at explaining the world accessible to our minds? We can imagine a universe in which rational, intelligent beings existed who were incapable of developing sufficiently advanced mathematics to understand that universe even though it were capable of being comprehended mathematically.

Also, what about beauty, or the role of aesthetics? This is a commonly-observed inspiration for both mathematicians and those who apply mathematics. The mathematician GH Hardy (1877-1947) said of mathematics “Beauty is the first test: there is no permanent place in the world for ugly mathematics” (Hardy, 1940). Einstein is quoted in Farmelo (2002, p. xii) as saying “the only physical theories that we are willing to accept are the beautiful ones”, while physicist colleague Hermann Weyl (1885-1955) said “My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful” (quoted in Stewart, 2007, p. 278). But why should an aesthetic of mathematics help create new mathematics, and new applied mathematics? Are we simply wrong about beauty, especially when we use it as a selection criterion, and that ugly theories could better explain the world, and even be more fecund mathematically? Perhaps we have been misled by mathematics because we are in the early days of science; are we even wrong about the power of mathematics to explain the world?

Another question we have overlooked as we peered down the TEAM lens concerns the meaning of deductive steps in an applied mathematical argument. More specifically, if I have a mathematical model of a physical process which I then analyse mathematically to arrive at a physically-verifiable result, need each of the intermediate logical steps also have physical meaning? This question has been considered by Nancy Cartwright, among others; see for example her (1984), in which she says “derivations do not provide maps of causal processes. A derivation may start with the basic equations that govern a phenomenon. It may be highly accurate, and extremely realistic. Yet it may not pass through the causes.” This question, and the others raised above, deserve more attention.

Conclusion

We do not know the ontological or epistemological status of mathematical objects. We do not know why mathematics can be applied to the world around us. Though it was too much to hope that the TEAM lens would itself provide an *experimentum crucis* which would eliminate all but one philosophy of mathematics and therefore resolve a millennia-old debate, the TEAM lens has brought into focus the questions which must be clearly addressed when defending a particular philosophical standpoint.

I have attempted to summarise the systems of ideas which constitute these standpoints in four broad schools. Despite presenting them as separate, they are united in their concern with the ontological and epistemological questions, and in their focus on key ideas: what is number, what is a set, what is a proof, what is infinity, and more besides. As we saw, one person who has united them in a stunningly destructive way was Kurt Gödel.

Another figure may pull some of these strands together. Max Tegmark introduced the radically realist Mathematical Universe Hypothesis, which earns him a capital P on Platonist if anyone ever deserves it. The MUH is a tentative, new, and controversial idea, and my positive view of it may not be representative. But I do think it takes seriously these philosophical questions and that it represents an important attempt to think clearly about them, and possibly to unite some of the schools. For example, structuralists and fictionalists might observe that in the MUH all mathematical objects exist and all things which exist are mathematical, and so there is no need for any particular structure or fiction to be privileged. Even the debate between Kantian innateness and Humean empiricism may be erased: if the mind is a self-aware substructure of the mathematical universe, then there is no epistemic gap between the mind and the world. For plantoists, the problem of epistemological access may be solved because the MUH is more than plenitudinous platonism, which addressed the epistemic concerns. However, it also potentially fixes the platonist ontological issues and leaves us with an inspiring thought: if everything is mathematics and mathematics is everything, then there is only one realm. We are self-aware substructures of mathematics.

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