The Philosophy of Applied Mathematics

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Pity the applied mathematician. Not only does she have to suffer the perennial people-repelling problem which all mathematicians have experienced at parties, but in addition she faces a general ignorance of the importance of mathematics in understanding the natural world and developing new technologies. I told a guest at a recent party that I use mathematics to try to understand migraines; she thought I meant that I ask migraine sufferers to do mental arithmetic to alleviate their pain. What does such a misconception reveal? There is the surface fact of a continuing need to publicly discuss the work of mathematicians. But at a deeper level, this response reveals a stunning fact we often overlook: the world can be understood mathematically. Moreover, the misconception reminds us that this fact is not obvious. I want to discuss the question of the applicability of mathematics to an understanding of the world. To do so requires reviewing the long history of the general philosophy of mathematics - what I will loosely here call “metamath” - before discussing our Big Question: “why can math be used to describe the world?”, or to extend it more provocatively, “why is applied math even possible?”

Before we go any further, we should be clear on what we mean by applied mathematics. I will borrow a definition given by an important applied mathematician of the 20th and 21st centuries, Tim Pedley, the GI Taylor Professor of Fluid Mechanics at the University of Cambridge. In his Presidential Address to the Institute of Mathematics and its Applications in 2004, he said “Applying mathematics means using a mathematical technique to derive an answer to a question posed from outside mathematics.” This definition is deliberately broad - including everything from counting change to climate change - and the very possibility of such breadth of definition is part of the mystery we are discussing.

Pedley’s definition reaffirms what the phrase “applied mathematics” tells us of its subject: it is *mathematics* which is *applied*. As such, the question of the applicability of mathematics is arguably more important than the questions typically addressed in metamath for the following reasons. Firstly, because applied mathematics is *mathematics*, it raises all the same issues as those traditionally arising in metamath. Secondly, being *applied*, it raises some of the issues addressed in the philosophy of science. I suspect that the case could be made for our big question being in fact *the* Big Question in the philosophy of science and mathematics - and perhaps beyond, as we will see later. However, let us now turn to the history of metamath: what has been said about mathematics, its nature and its applicability?

The long history of mathematics generally lacks a pure/applied distinction, yet in the “modern” era of mathematics over, say, the last two centuries, there has been an almost exclusive focus on a philosophy of pure mathematics, on metamath. In particular, emphasis has been given to the so-called “foundations” of mathematics - what is it that gives mathematical statements truth? Metamathematicians interested in foundations are commonly grouped into four camps.

Formalists such as David Hilbert view mathematics as being founded on a combination of sets and logic, and to some extent view the process of doing mathematics as an essentially meaningless shuffling of symbols according to certain prescribed rules. *Logicists* see mathematics as being an extension of logic. The arch-logicists Bertrand Russell and Alfred North Whitehead famously took hundreds of pages to prove (logically) that two plus two equals four. *Intuitionists* are exemplified by LEJ Brouwer, a man about whom it has been said that “he wouldn’t believe that it was raining or not until he looked out of the window” (Knuth). This quote satirises one of the central intuitionist ideas, the rejection of the law of the excluded middle, which says that a statement is either true or false: intuitionists believe in the existence of neuter statements, which have been neither proved nor disproved. Moreover, they admit only enumerable operations into their proofs, since they believe that mathematics is entirely a product of the human mind, which they postulate to be capable of grasping infinity only as an extension of an algorithmic or pseudo-mechanical one-two-three kind of process, of the kind we experience in the physical world. Finally, *Platonists*, members of the oldest of the four camps, believe in an independent reality or existence of numbers and the other objects of mathematics. For a platonist such as Kurt Godel, mathematics exists without the human mind, possibly without the physical universe, but there is a mysterious link between the mental world of humans and the platonic realm of mathematics.

No undisputed answer has been given to which of these four alternatives - if any - serves as the foundations of mathematics. While it might seem like such rarefied discussions have nothing to do with the question of applicability, it has even been argued that this uncertainty over foundations has influenced the very practice of applying mathematics. In “The loss of certainty”, Morris Kline wrote in 1980 that “The crises and conflicts over what sound mathematics is have also discouraged the application of mathematical methodology to many areas of our culture such as philosophy, political science, ethics, and aesthetics . . . The Age of Reason is gone.” Thankfully, mathematics is now beginning to be applied to these areas, but we have learned the important historical lesson that there is to the choice of applications of mathematics a sociological dimension sensitive to metamathematical problems.

However, the logical next step for the metamathematician who bothers to think about the applicability of mathematics, would be to ask what each of the four foundational views has to say about our big question. Accessible discussions along this line have been written by a number of mathematicians and scientists, such as Roger Penrose in the book “The Road to Reality”, or Paul Davies in his book “The Mind of God”.

I would like to take a different path here by reversing the “logical” next step: I want to ask “what does the applicability of mathematics have to say about the foundations of mathematics?” In asking this question I take for granted that there is no serious disagreement about whether mathematics is applicable: the entire edifice of modern science and technology, depending heavily as it does on the mathematization of nature, bears witness to this fact.

But what can a formalist say to explain the applicability of mathematics? If mathematics really is nothing other than the shuffling of mathematical symbols in the world's longest running and most multiplayer game, then why should it describe the world? What privileges the game of math to describe the world rather than any other game? Remember, the formalist must answer from within the formalist worldview, so no Plato-like appeals to a deeper meaning of math or hidden connection to the physical world is allowed. For similar reasons, the logicists are left floundering, for if they say "well, perhaps the universe is an embodiment of logic" then they are tacitly assuming the existence of a Platonic realm of logic which can then be (but need not necessarily be) embodied. Thus for both formalists and logicists the very existence of applicable mathematics poses a problem apparently fatal to their position.

Neither logicism nor formalism is widely "believed" any more, despite the cliche that mathematicians are platonists during the week and formalists at the weekend. Both perspectives fell out of favour for reasons other than the potentially fatal one about the applicability of mathematics, reasons largely connected with the work of Godel, Skolem, and others. But the third proposed "foundation", intuitionism, never really garnered much support in the first place. To this day, it is muttered about in dark tones by most working mathematicians, if it is considered at all. What is seen as a highly restricted toolkit for proofs and a bizarre notion of limbo in which a statement is neither true nor false until a proof has been constructed one way or the other, lower the attraction of this viewpoint to most mathematicians. However, the central idea of the enumerable nature of processes in the universe and certainly as it is perceived by the human mind, appears to be deduced from reality. In this way, perhaps intuitionism is derived from reality, from the apparently at-most-countably infinite physical world. It appears that intuitionism offers a neat answer to the question of the applicability of mathematics: it is applicable because it is derived from the world.

However, this answer may fall apart on closer inspection. For a start, there is much in modern mathematical physics which requires notions of infinity beyond the enumerable, and therefore possibly forever beyond the explicatory power of intuitionistic mathematics. Even one of the modern ideas which could benefit from the finitist logic of the inuitionists fails to be truly intuitionistic and seems to sneak in some platonic ideas. Thus the "digital physics" community, whose motto is Ed Fredkin's "It from Bit", involves the "bit" of information theory, seemingly positing a platonic existence to information from which the physical world is derived. But more fundamentally, intuitionism has no answer to the question of why non-intuitionistic mathematics is applicable. It may well be that a non-intuitionistic mathematical theorem is only applicable to the natural world when an intuitionistic proof of the same theorem also exists, but this has not been established. Moreover, it is not clear that the objects of the human mind need faithfully represent the objects of the physical universe. Mental representations have been selected for over evolutionary time not for their fidelity but for the advantage they gave our forebears in their struggles to survive and to mate.

Formalism and logicism have failed to answer our big question. The jury is out on whether inuitionism might do so, but huge conceptual difficulties remain. What, then, of Platonism?

The platonist believes that the physical world is an imperfect shadow of a realm of mathematical objects (and possibly of notions like “truth” and “beauty” as well). The physical world emerges, somehow, from this platonic realm, is rooted in it, and therefore objects and relationships between objects in the world shadow those in the platonic realm. The fact that the world is described by mathematics then ceases to be a mystery as it has become an axiom: the world is rooted in a mathematical realm. But even greater problems then arise: why should the physical realm emerge from and be rooted in the platonic realm? Why should the mental realm emerge from the physical? Why should the mental realm have any direct connection with the platonic? And in what way do any of these questions differ from those surrounding ancient myths of the emergence of the world from the slain bodies of gods or titans, the Buddha-nature of all natural objects, or the Abrahamic notion that we are “created in the image of God”? Indeed, the belief that we live in a divine universe and partake in a study of the divine mind by studying mathematics and science has arguably been the longest-running motivation for rational thought, from Pythagoras in ancient times, through Newton, “the last of the magicians”, and to many scientists today. “God”, in this sense, seems to be neither an object in the space-time world, nor the sum total of objects in that physical world, not yet an element in the Platonic world, but rather something closer to the entirety of the Platonic realm. In this way, many of the difficulties outlined above which a Platonist faces in explaining how the Platonic world is embodied by the physical world, accessed by the mental world, and interacts mutually with both, are identical with those faced by theologians of the Judeo-Christian world - and possibly of other religious or quasi-religious systems.

So that secular icon Galileo believed that the “book of the universe” was written in the “language” of mathematics - a platonic statement begging an answer (if not the question) if ever there was one. Even non-religious mathematical scientists today regularly report feelings of awe and wonder at their explorations of what feels like a Platonic realm. Paul Davies goes further in “The Mind of God”, and highlights the two-way nature of this motivation. Not only may a mathematician be driven to understand mathematics in a bid to glimpse the mind of God (a non-personal God like that of Spinoza or Einstein), but our very ability to access this “key to the universe” is “too intimate”, suggesting that “the existence of mind in some organism on some planet in the universe is surely a fact of fundamental significance . . . We are truly meant to be here.”

In fact, a point rarely mentioned is that the hypothesis of the mathematical structure and physical nature of the universe, and our mental access to study both, as being somehow a part of and upheld by the mind, being, and body of a “god” is a considerably tidier answer to the questions of the foundation of mathematics and its applicability than those described above. Such a hypothesis, though rarely called such, has been found in a wide variety of religious, cultural, and scientific systems of the past several millenia. It is not natural, however, for a philosopher or scientist to wholeheartedly embrace such a view (even if they may wish to) since it tends to encourage the preservation of mystery rather than the drawing back of the obscuring veil.

Roger Penrose has most lucidly illustrated some of this mystery with a “three-worlds” diagram. The platonic, physical, and mental worlds are the three in question, and he sketches them as spheres arranged in a triangle. A cone then connects the platonic world with the physical: in its most general form, the diagram shows the narrow end of the cone penetrating the platonic world and the wider part penetrating some of the physical world. This is to show that (at least some of) the physical world is embedded in at least some of the platonic world. A similar cone connects the physical to the mental world: (at least some of) the mental world is embedded in the physical world. Finally, and most mysteriously, the triangle is completed by a cone from the mental to the platonic world: (at least some of) the platonic world is embedded in the mental world. Each cone, each world, remains a mystery.

We seem to have reached the rather depressing impasse in which none of the four proposed foundations of mathematics can cope unambiguously with the question of the applicability of mathematics. But I want you to finish this essay instead with the feeling that this is incredibly good news! The teasing out of the nuances of the Big Question - why does applied mathematics exist? - is a future project which could yet yield deep insight into the nature of mathematics, the physical universe, and our place within both systems as embodied, meaning-making, pattern-finding systems.