The circle space of a spherical circle plane

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Abstract

We show that the circle space of a spherical circle plane is a punctured projective 3-space. The main ingredient is a partial solution of the problem of Apollonius on common touching circles.

1 Introduction

In 1974, K. Strambach [18] announced the result that all compact connected Möbius planes have the same circle space, up to homeomorphism. He gives strong arguments to support this statement, but some details in his construction of a homeomorphism are not consistent. We add some further ideas sufficient to make the proof work. In fact, our proof becomes simpler and does not use the axiom of touching, hence we prove the result for spherical circle planes:

Theorem 1.1. *The circle space of every spherical circle plane is homeomorphic to real projective 3-space minus a point.*

In the case of the classical Möbius plane, formed by the unit sphere $\mathbb{S}_2$ in $\mathbb{R}^3$ with its infinite plane sections as circles, the statement of this theorem is easily verified. Indeed, under the polarity σ of $P_3\mathbb{R}$ having $\mathbb{S}_2$ as its set of absolute points, the planes meeting $\mathbb{S}_2$ in infinitely many points correspond to the points of the projective space outside the sphere. The set of these points is homeomorphic to the punctured projective space. (A different argument for the classical circle space can be found in [10, p. 153].) Therefore, in order to prove the theorem, it remains to show that all spherical circle planes have the same circle space, up to homeomorphism.

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A. Lightfoot [9] considered embeddable spherical circle planes and obtained the result of Theorem 1.1 in this special case, see [9, Theorem 4.3.7]. In his proof he makes essential use of the fact that such circle planes sit in $\mathbb{R}^3$ and that circles come from affine planes of $\mathbb{R}^3$.

2 Definition and basic properties

Definition 2.1. A *spherical circle plane* $\mathcal{C} = (S, \mathcal{K})$ consists of its point set $S = \mathbb{S}_2$, the 2-sphere, and its set $\mathcal{K}$ of circles. Each circle $K \in \mathcal{K}$ is a subset of S homeomorphic to the unit circle $\mathbb{S}_1$, and for any three distinct points $x, y, z \in S$, there is a unique circle $K(x, y, z)$ joining them. Two distinct circles K, L are said to *touch* in a point x if $\{x\} = K \cap L$. We also say that L touches itself in any of its points.

A *spherical Möbius plane* is a spherical circle plane satisfying the *axiom of touching:* Given a circle K and two points x, y such that $x \notin K$ and $y \in K$, there is a unique circle $L = T(x, K, y)$ containing x and touching K in the point y.

The geometric operations $K(-,-,-)$ and $T(-,-,-)$ introduced above are automatically continuous, see [15, 2.2] and [19, Satz 7.1]. Also the operation $\cap$ that sends a pair (K, L) of distinct circles with nonvoid intersection to $K \cap L$ is continuous when we consider $K \cap L$ as an element of the symmetric square $S * S = S \times S/\sim$, where $\sim$ is the equivalence relation defined by $(x, y) \sim (y, x)$. To make this work, we identify the singleton x with the pair (x, x).

Examples of spherical circle planes abound. A special class of examples are G. Ewald's ovoidal (or embeddable) planes [5]. Their point sets are strictly convex closed surfaces $S \subseteq \mathbb{R}^3$, and circles are the plane sections of S having more than one point. The plane is a Möbius plane if and only if S is differentiable by [5, Satz 2]. Examples of a different kind are given by G. Ewald in [6]. See [10, Section 3.3] for more examples. Highly homogeneous spherical circle planes have been classified by K. Strambach [16], [17].

Definition 2.2. A spherical circle plane $\mathcal{C}$ may be derived at any point p. The *derived plane* $\mathcal{C}_p$ has point set $S_p := S \setminus \{p\}$ and the set of lines $\mathcal{K}_p := \{K \setminus \{p\} \mid p \in K \in \mathcal{K}\}$.

Thus, $\mathcal{C}_p$ is an $\mathbb{R}^2$-plane in the sense of [13, Section 31]. This means that the point set S_p is homeomorphic to $\mathbb{R}^2$ and each line $L \in \mathcal{K}_p$ is a closed subset of S_p homeomorphic to $\mathbb{R}$, and that any two distinct points in S_p are joined by a unique line. According to Skornyakov's theorem, see [13, 31.22], the line space of an $\mathbb{R}^2$-plane admits a unique topology making the plane a stable plane, that is, rendering the set of pairs of intersecting lines open and the geometric operations of join and intersection continuous. By uniqueness, this topology coincides with the topology induced from $\mathcal{K}$. Further properties of $\mathbb{R}^2$-planes may be found in [13, Section 31]. The circle plane $\mathcal{C}$ is a Möbius plane if and only if all derived planes are affine planes.

Suppose that K and L are two circles intersecting in two points x, y. Then y is the intersection point of the two lines $K \setminus \{x\}$ and $L \setminus \{x\}$ in the derived plane $\mathcal{C}_x$. Therefore, we know from [13, 31.5b] that the intersection is *transversal,* that is, in some neighborhood of y, the traces of K and L are embedded like the Cartesian factors of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. On the other hand, if two circles K, L touch in the point x, then $L \setminus \{x\}$ is contained in one of the two connected components of $S \setminus K$. This implies that K and L intersect non-transversally in this case. Thus we have proved:

Proposition 2.3. *Two circles K and L in a spherical circle plane touch in a common point x if and only if their intersection at x is not transversal.* ■

Corollary 2.4. *The pairs of touching circles form a closed subset of* $\mathcal{K} \times \mathcal{K}$.

Proof. The set of pairs of circles intersecting in two points is open by Proposition 2.3. Since also the set of pairs of disjoint circles is open, the statement follows. ■

Let K be a circle and x, y be two points not on K. If x and y belong to different connected components of $S \setminus K$, then no circle containing x and y can touch K. However, if x and y belong to the same component, then the set $\mathcal{K}_{x,y}$ of all circles passing through both points is a line pencil in the derived plane $\mathcal{C}_x$, hence it is homeomorphic to the circle $\mathbb{S}_1$, see [13, 31.17]. Given a point p in K, there is a unique circle L_p in this set containing p, and it contains a unique further point $f(p) \in K$; if L_p happens to touch K, then we interpret this as saying that $f(p) = p$. The map $f\colon K \to K$ is an involution, i.e., $f \circ f = \mathrm{id}$ but $f \neq \mathrm{id}$. By continuity of geometric operations, f is continuous, and hence conjugate to one of the two linear involutions on $\mathbb{S}_1$. If f were conjugate to the antipodal map of $\mathbb{S}_1$, then every pair of circles in $\mathcal{K}_{x,y}$ would have a common point in the complementary component of K not containing x and y, a contradiction. Thus f has exactly two fixed points, corresponding to exactly two circles in $\mathcal{K}_{x,y}$ touching K. By these considerations, taken essentially from [3], we have proved the following result. It was first obtained by K. Strambach [15, 3.13], but the method we used here prepares a similar argument that we shall need in Section 3.

Proposition 2.5. *Let K be a circle in a spherical circle plane* $(S, \mathcal{K})$ *and let x, y be two points not on K. If x and y belong to different components of* $S \setminus K$*, then there is no circle containing these points and touching K.*

If, on the other hand, x and y belong to the same component, then there are exactly two circles touching K and containing x and y. This pair of touching circles depends continuously on (K, x, y) *by Corollary 2.4.* ■

In the last proposition, fix the point x and move y to approach a point on K. In view of Corollary 2.4, this shows that every point on K lies on a touching circle that contains x. In fact, we can show more than that:

Proposition 2.6. *For a point p on a circle K and a point* $x \notin K$*, the set of all circles in* $\mathcal{K}_x$ *touching K in the point p is homeomorphic to a closed interval, which may be reduced to a point.*

Proof. The map $g : K \setminus \{p\} \to \mathcal{K}_{x,p}$ defined by $g(y) = K(y, x, p)$ is continuous and injective, hence its image is an open interval in the space $\mathcal{K}_{x,p}$, which is homeomorphic to the circle $\mathbb{S}_1$. The proposition follows; see also Figure 4 and the example at the end of Section 3. ■

In a spherical Möbius plane, the following fact is rather obvious.

Proposition 2.7. *Let K be a circle and x be a point not on K. Then the set $\mathcal{T}(x,K)$ of all circles in $\mathcal{K}_x$ touching K is homeomorphic to* $\mathbb{S}_1$.

Proof. The point x has a neighborhood basis consisting of complementary components of circles. In particular, there is a circle L separating K from x. Every tangent circle $T \in \mathcal{T}(x,K)$ intersects L in two distinct points. We use this in order to construct a homeomorphism $f\colon L \to \mathcal{T}(x,K)$. Given a point $y \in L$, Proposition 2.5 asserts that there are exactly two tangents in $\mathcal{T}(x,K) \cap \mathcal{K}_y$. They meet L in a total of three points y, a, b; see the diagram on the left of Figure 1. Choose an orientation on L and let $g(y) \in \{a, b\}$ be the point following y in this orientation. Let $f(y) \in \mathcal{T}(x,K)$ be the circle joining x, y and $g(y)$. Then f is a continuous map $L \to \mathcal{T}(x,K)$, and we aim to show that f is bijective.

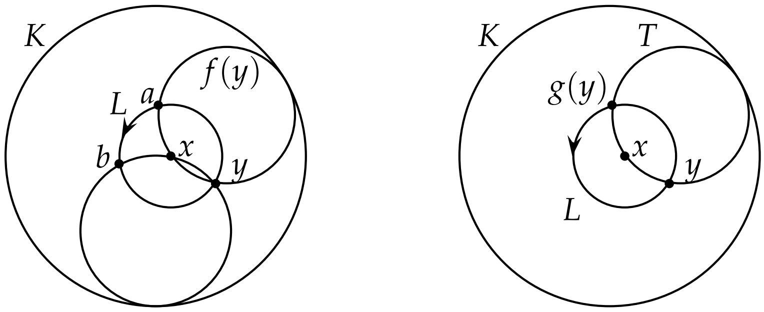

Figure 1: The topology of $\mathcal{T}(x,K)$

Every tangent $T \in \mathcal{T}(x,K)$ separates S into two disks, one of which contains $K \setminus T$. The boundary of the other disk intersects L in an interval I bounded by the two intersection points of T and L. If y denotes the first of these points of intersection in the order of I induced by the orientation of L, then $g(y)$ is the other end point of I, and hence $T \mapsto y$ is an inverse map for f. ■

3 The Apollonius problem

This problem asks for the number of common touching circles of some given configuration of three points and/or circles, where touching a point is interpreted as containment. A simple instance of this problem is answered by Proposition 2.5. In spherical Möbius planes, the problem has been studied extensively by H. Groh [7]. We need to generalize his result 5.4 to spherical circle planes. A more sophisticated approach to the Apollonius problem in the Möbius case is given by A.E. Schroth in [14, Chapter 7]; he uses an embedding of a Möbius plane into a generalized quadrangle and obtains a complete result.

Before we state our result, note that two disjoint circles K, L separate the point set S into three connected components, two disks bounded by a single circle (K or L), and one annulus bounded by K and L together.

Theorem 3.1. *In a spherical circle plane $(S, \mathcal{K})$, let K and L be disjoint circles, and consider a point x in the annulus bounded by $K \cup L$. Then $\mathcal{K}_x$ contains exactly four circles touching both K and L. Exactly two of these separate K from L, that is, with the exception of the points of touching, K and L lie in different complementary components with respect to these circles.*

Proof. 1) The proof will follow the same pattern as that of Proposition 2.5, which is why we included the latter. Most of the time, we work in the derived plane $\mathcal{C}_x$ with point set $E = S \setminus \{x\}$, and we use the terms 'line' and 'tangent' to designate elements of $\mathcal{K}_x$ and of $\mathcal{T}(x, K)$ or $\mathcal{T}(x, L)$, respectively. By our hypothesis, K is contained in the unbounded component of $E \setminus L$ and vice versa.

Let T be a line touching K in a point p, and let G be a secant of K containing p. Then G separates E into two open half planes, and splits T into two rays contained in different half planes. At the same time, K is split into two half circles K_1 and K_2. Fix an orientation of K and let K_1 be the half circle beginning at the point p. Now let T_1 denote the 'forward' ray contained in the same half plane which contains K_1. The other ray T_2 will be referred to as the backward ray of T. This choice of T_1 depends on the orientation of K, but not on the choice of a secant. Now consider the situation of Proposition 2.5 with the same notation as used there, and look at the two tangents of K passing through a point y in the unbounded domain of $E \setminus K$. We observe that y belongs to the forward ray of exactly one of these tangents, and to the backward ray of the other, see Figure 2.

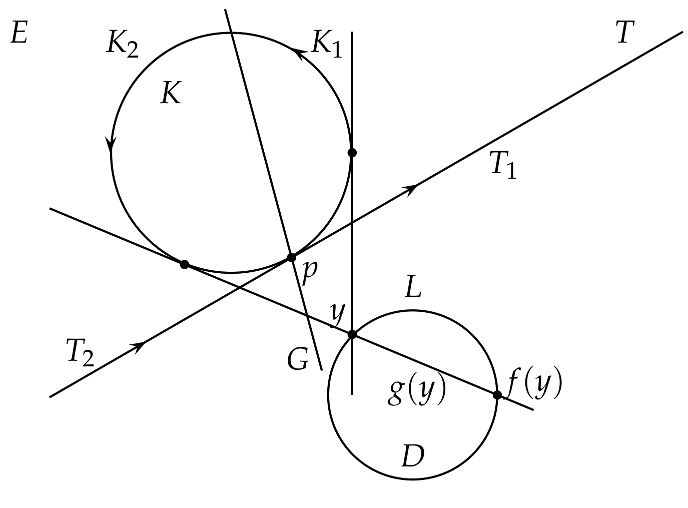

Figure 2: Forward and backward rays

2) Let $g(y) \in \mathcal{T}(x, K)$ be the tangent whose forward ray contains y. Then g is a continuous map, and we obtain a continuous involution $f \colon L \to L$ by sending $y \in L$ to the second intersection point of $g(y)$ and L, or to y itself if $g(y)$ happens to touch L. Clearly, f is not the identity, because Proposition 2.5 shows that there are forward rays containing a point in the bounded complementary domain D of L in E. The involution f cannot be fixed point free either, for in this case any two forward rays properly intersecting L would have to intersect in D, producing a point which lies on two forward tangent rays, a contradiction to step (1). This

shows that f must have precisely two fixed points, that is, exactly two forward rays touch L. In the same way, we obtain exactly two touching backward rays.

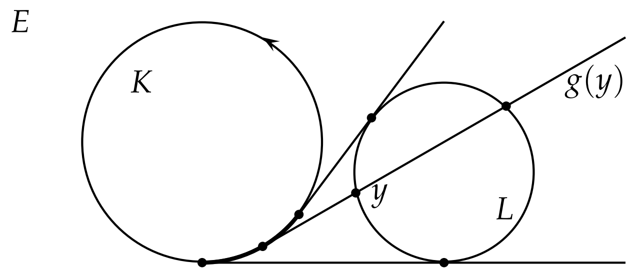

Figure 3: Region bounded by forward rays

3) Suppose that $g(y)$ is a secant of L. Then y is a transversal intersection point of $g(y)$ and L. It follows that the forward ray of $g(y)$ intersects L twice, because it enters the bounded component D and is closed in E. The result of step (2) shows that the image of the map g is an interval in $\mathcal{T}(x, K)$ whose end points are forward tangents of L, while the rest of the interval consists of secants. The circle L is therefore contained in a region bounded by the two forward rays tangent to L together with a (possibly degenerate) interval of K joining the two points of touching; compare Figure 3. The first of the end points of this interval is the touching point of the first tangent ray. These facts show that the first forward tangent does not separate the circles K and L while the second one does. The same applies to backward tangents, which completes the proof. ■

If we move x to approach a point p on K, then, in view of Corollary 2.4, the touching circles provided by Theorem 3.1 will accumulate at circles touching K in p. Hence, we obtain the following corollary.

Corollary 3.2. *In a spherical circle plane $(S, \mathcal{K})$, let K and L be disjoint circles, and consider a point x on K. Then $\mathcal{K}_x$ contains at least two circles touching both K and L. One of them separates K from L and some other one does not.*

Here, separating K from L means for a circle T that each of the closed half planes defined by T contains one of the circles K, L.

In Möbius planes there are exactly two circles in $\mathcal{K}_x$ that touch both K and L. However, in spherical circle planes we may have two closed intervals of such touching circles in $\mathcal{K}_x$, one consisting of separating circles and the other of non-separating ones; we do not know whether even more involved situations may occur. In fact, the set of circles that touch K in x and L in a fixed point y may even be homeomorphic to a closed interval. A simple example for this behaviour can be found by considering the strictly convex closed surface $S \subset \mathbb{R}^3$ which is the union of a closed hemisphere of the unit 2-sphere $\mathbb{S}_2$ and a fitting cap of a larger sphere. More precisely, let

$$S = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1, u \geq 0\} \cup \{(u, v, w) \in \mathbb{R}^3 \mid u^2 - 2ru + v^2 + w^2 = 1, u \leq 0\}$$

where $r \geq 0$. In case $r = 0$ we obtain the unit 2-sphere $\mathbb{S}_2$, but for $r > 0$ the surface S is no longer differentiable at points whose u-coordinate is 0. We use stereographic projection π from the north pole $n = (0, 0, 1)$ onto the uv-plane. $\pi(S \setminus \{n\})$ consists of the open half-plane $H = \{(u, v) \in \mathbb{R}^2 \mid u > -\frac{1}{r}\}$. Any circle through n will appear as a straight line in H, and two such circles K_1 and K_2 touch in n if and only if $\pi(K_1)$ and $\pi(K_2)$ are disjoint, that is, the underlying Euclidean lines are parallel or intersect in a point outside H. The entire spherical circle plane can be represented in H; straight lines have to be extended by the point n at infinity.

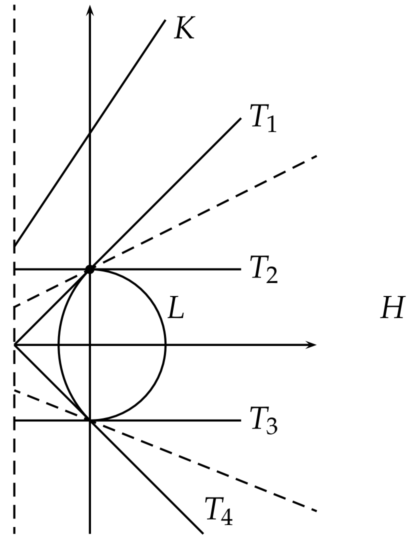

Figure 4: Circles touching K and L

The intersection of S with the uv-plane is the circle

$$L = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 = 1, v \geq 0\} \cup \{(u, v) \in \mathbb{R}^2 \mid u^2 - 2ru + v^2 = 1, u \leq 0\},$$

which is shown in Figure 4. This set has a 'left' tangent T_1 in $(0, 1)$ and a 'right' tangent T_2 in the same point. If K is a line as in Figure 4, that is, above T_1 and T_2 and not intersecting these tangents in H, any straight line between T_1 and T_2 touches K in n and L in $(0, 1)$, and separates K from L. One such line is shown in Figure 4 as a dashed line. One similarly has tangents T_3 and T_4 to L at $(0, -1)$; any straight line between them touches K in n and L in $(0, -1)$, but does not separate K from L. Of course, there are further tangents touching L in points $\neq (0, \pm 1)$ and still disjoint from K except for n.

4 Flocks

Definition 4.1. Let $n, s \in S$ be two points, thought of as the north pole and the south pole. A *flock* with *carrier* $\{n, s\}$ is a set $\mathcal{F} \subseteq \mathcal{K}$ of mutually disjoint circles covering $S \setminus \{n, s\}$.

When working with flocks, it is convenient to use the *extended circle space* $\bar{\mathcal{K}} := \mathcal{K} \cup S$ with the following topology. Open sets in $\mathcal{K}$ are open in $\bar{\mathcal{K}}$, and a neighborhood of $x \in S$ consists of a neighborhood U of x in S together with all circles contained in U.

Proposition 4.2. *In the extended circle space $\bar{\mathcal{K}}$, the closure $\bar{\mathcal{F}} = \mathcal{F} \cup \{n, s\}$ of a flock with carrier $\{n, s\}$ is homeomorphic to the interval $[-1, 1]$ of real numbers, with n and s corresponding to the end points. A homeomorphism is given by intersecting flock circles with a meridian M, i.e., with a semicircle joining n and s.*

Proof. The map $f\colon M \to \bar{\mathcal{F}}$ sending a point to the unique flock circle containing it (and sending n and s to themselves) is clearly bijective. Next, we prove continuity of f. Suppose that $x_k \to x$ in $M \setminus \{n, s\}$. We have to exclude the possibility that (some subsequence of) $f(x_k)$ converges to some circle K other than $f(x)$. We have $x \in K$, and if $f(x)$ intersects K transversally at x, then $f(x_n)$ intersects $f(x)$ for large k, a contradiction. There remains the case that K touches $f(x)$ at the point x. By the same reason as before, K and all circles $f(x_k)$ must be contained in the same disk bounded by $f(x)$. But then the region bounded by K together with $f(x)$ cannot contain points of any flock circle. Now suppose that $x_k \to n$ and that $f(x_k)$ accumulates at some circle K. Then $n \in K$ and K must intersect some flock circle transversally, a contradiction. Thus, we have proved that f is continuous. By compactness, f is a homeomorphism. ∎

Flocks of circles are an important tool for the introduction of coordinates in the circle space. Their existence has been proved for Möbius planes by K. Strambach [18], see also [11] and [12]. Our proof for spherical circle planes will resemble the existing proofs. The main ingredient is the following consequence of our previous results.

Proposition 4.3. *Let K and L be disjoint circles and consider a point x in the annulus $A \subset S$ bounded by $K \cup L$. There exists a circle $M \subset A$ containing x and separating K from L.*

Proof. As in the proof of Theorem 3.1, we work in the derived plane $\mathcal{C}_x$. The two separating common tangents T_1, T_2 of K and L intersect in a point p, and in the line pencil of p they define two open intervals. One of them consists of common secants of K and L, and the lines in the other interval are disjoint from both circles and separate them. ∎

Theorem 4.4. *For every pair of points $n, s \in S$ there exists a flock $\mathcal{F}$ with carrier $\{n, s\}$.*

Proof. Let $\{U_k \mid k \in \mathbb{N}\}$ be a countable basis for the topology of S. Inductively we define F_k to be a circle meeting U_k and disjoint from $F_1, \ldots, F_{k-1}$. Such a circle F_k can be found using Proposition 4.3 if U_k contains points that lie between two previously chosen circles. If this is not the case, choose $x \in U_k$. If none of the circles $F_1, \ldots, F_{k-1}$ separates x from n, choose an auxiliary circle separating x from n and apply 4.3 to this circle and the previously chosen circle closest to x. Proceed similarly with n replaced by s. There results a partial flock $\mathcal{P} = \{F_k \mid k \in \mathbb{N}\}$ whose union is dense in S.

Now let $\mathcal{F}$ be the closure of $\mathcal{P}$ in $\mathcal{K}$. Then $\mathcal{F}$ covers S. If two circles $K, L \in \mathcal{F}$ intersect, then some approximating circles F_k and F_l also intersect, by arguments as in the proof of Proposition 4.2. The contradiction shows that $\mathcal{F}$ is a flock. ∎

5 The circle space

The main purpose of this section is to prove Theorem 1.1, using the tools presented in the preceding sections. In particular, we make use of a flock $\mathcal{F}$ with carrier $\{n, s\}$, where n and s are arbitrary distinct points. Our first aim is to introduce a kind of polar coordinates on the point set. A point $p \in S$ is specified by its latitude and longitude. The latitude a of p is determined by the flock circle $F_a \in \mathcal{F}$ containing p and takes values in the interval $[-1, 1] \approx \bar{\mathcal{F}}$, the values -1 and 1 pertaining to s and n, respectively. For the longitude, we use the pencil $\mathcal{K}_{n,s}$ of *longitudes*. Being a line pencil in the derived plane $\mathcal{C}_n$, this pencil is a topological circle. Its twofold covering space $\tilde{\mathcal{K}}_{n,s}$ may be thought of as the space of *meridians* (semicircles joining n to s) and is again a topological circle, parametrized by the interval $[0, 2\pi]$ subject to the identification $0 = 2\pi$. The longitude φ of p is the parameter of the meridian containing p and is arbitrary for $p \in \{n, s\}$. We write (a, φ) for the point with coordinates a, φ.

Introducing coordinates for circles is much more difficult and cannot be done using a single chart from the start. We split the space $\mathcal{K}$ into two subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ which are not disjoint. Both contain the circles passing through n or s, and those containing s receive different coordinates when regarded as members of $\mathcal{K}_1$ or $\mathcal{K}_2$. The longitudes occur twice in $\mathcal{K}_1$ and twice in $\mathcal{K}_2$, carrying two sets of coordinates in each chart. The circles containing n also belong to the intersection of the two charts, but their coordinates agree, so the charts may be glued together, resulting in a closed ball $\mathbb{D}_3$, as we shall see. After gluing, all coordinates are unique, except for circles containing s, which have two distinct sets of coordinates. Thus it will turn out that the space $\mathcal{K}$ is a quotient of $\mathbb{D}_3$ modulo an identification given by the orbits of a fixed point free involutory homeomorphism of its boundary sphere. To be precise, one of the orbits of the involution is missing. Now we use a well-known theorem by Brouwer and Kerékjártó [2], [8], see also [4] and [1] for completions of the proof. It asserts that every fixed point free involution of $\mathbb{S}_2$ is conjugate to the antipodal map $x \mapsto -x$. The conjugating homeomorphism may be radially extended over all of $\mathbb{D}_3$, so this shows that the resulting quotient space of $\mathbb{D}_3$ is homeomorphic to the real projective 3-space $P_3\mathbb{R}$. Due to the single missing orbit, we obtain that the circle space is the punctured projective space.

Before we define the sets $\mathcal{K}_i$, let us first describe how the coordinates will be defined. For every circle K, there is a continuous mapping $K \to \bar{\mathcal{F}}$ sending a point in K to the unique flock circle containing it; if applicable, n and s are mapped to themselves. This gives us a compact subinterval of $\bar{\mathcal{F}} \approx [-1, 1]$, and we let the maximum a and the minimum b of this subinterval be coordinates of K. Note that by Proposition 2.3 the flock circles F_a and F_b touch K. The map $K \mapsto \{a, b\}$ is continuous, and we have $a = b$ if and only if $K \in \mathcal{F}$. In that case, the circle is already determined by its coordinates.

If $a \neq b$, we need a third coordinate for our circle K. We shall use the two intersection points of K with the middle circle F_m, where $m = \frac{1}{2}(a + b)$. The fact that there are always two intersection points causes the trouble, forcing us to make a choice. What we do is to divide $\mathcal{K}$ into $\mathcal{K}_1$ and $\mathcal{K}_2$ in such a way that it is possible to choose an orientation coherently for all circles in a given set $\mathcal{K}_i$, and then we generally choose an intersection point depending on this orientation. Specifying an orientation of K amounts to choosing a connected component U of $S \setminus K$, and the preferred intersection point in $K \cap F_m$ will be the initial point of the closed interval $F_m \cap \overline{U}$, in the sense of the orientation of meridians given by the angle parameter. The coordinate set of K will be (a, b, φ), where φ is the angle coordinate of the preferred intersection point.

Given the coordinates (a, b, φ) of K, we can determine the preferred intersection point $x = (m, \varphi)$ of K and F_m. According to Theorem 3.1, there are exactly four circles containing x and touching F_a and F_b, so K must be one of these. Exactly two of the four circles separate the two flock circles and, hence, the poles n and s. This fact, together with suitable orientation conventions, will allow us to distinguish between the four circles determined by (a, b, φ).

Here then is the definition of $\mathcal{K}_1$. This set consists of all circles that either separate the poles n and s or contain one of them. The orientation of $K \in \mathcal{K}_1$ is given by the complementary component U containing s, unless $s \in K$, in which case U is the component *not* containing n. This does not determine the orientation of longitudes, and in fact, each longitude is counted as two elements of $\mathcal{K}_1$, with opposite orientations.

Next, we define the set $\mathcal{K}_2$. It contains the circles that do not separate n and s, including those that contain one or two poles. The orientation convention differs from that of $\mathcal{K}_1$. The component U is defined to be the one containing s, if $s \notin K$; if $s \in K$, then U is the component containing n. Longitudes are given two orientations like in $\mathcal{K}_1$.

The orientation conventions for $\mathcal{K}_1$ and $\mathcal{K}_2$ match for all circles except those passing through s; for the latter, we have made opposite choices.

Now we start visualizing the coordinate patches. At the beginning, we fix the angle coordinate. By combining the contributions from $\mathcal{K}_1$ and $\mathcal{K}_2$, we shall obtain a half disk H lacking the end points of the diameter that defines it. The remainder of the diameter represents the flock circles, which need no angle coordinate. To be more explicit, we want H to be the part of the unit disk in the (u, v)-plane defined by $u \geq 0$, with the points $(0, \pm 1)$ omitted. Hence we may later use φ in order to describe the body of revolution obtained by rotating H about the v-axis.

For fixed φ, we represent $\mathcal{K}_1$ as the triangle in the (a, b)-plane defined by $-1 \leq b \leq a \leq 1$. Its vertices are $(-1, -1)$, $(1, 1)$, and $(1, -1)$, see Figure 5. The first two vertices do not represent circles, and $(1, -1)$ represents a longitude. (Note that, since φ parametrizes meridians, every longitude will occur twice). The flock corresponds to the side $a = b$ of the triangle, and the other two sides $a = 1$ and $b = -1$ consist of the circles passing through n and s, respectively.

The triangle will be mapped homeomorphically to the part of H below and on the line $u + v = 1$, with the side $a = b$ going to the diameter, the side $b = -1$ going to the quarter circle from $(0, -1)$ to $(1, 0)$ with the orientation indicated in the figure, and the side $a = 1$ going to the segment defined by $u + v = 1$.

For $\mathcal{K}_2$, we picture the coordinates in $\mathbb{R}^2$ as $(-b, -a)$, in order to facilitate the subsequent mapping into H. Thus, our triangle looks the same as before, but now the side $-a = -b$ is not there, the side $-b = 1$ represents circles containing s, the side $-a = -1$ represents circles containing n, and the vertex $(-b, -a) = (1, -1)$

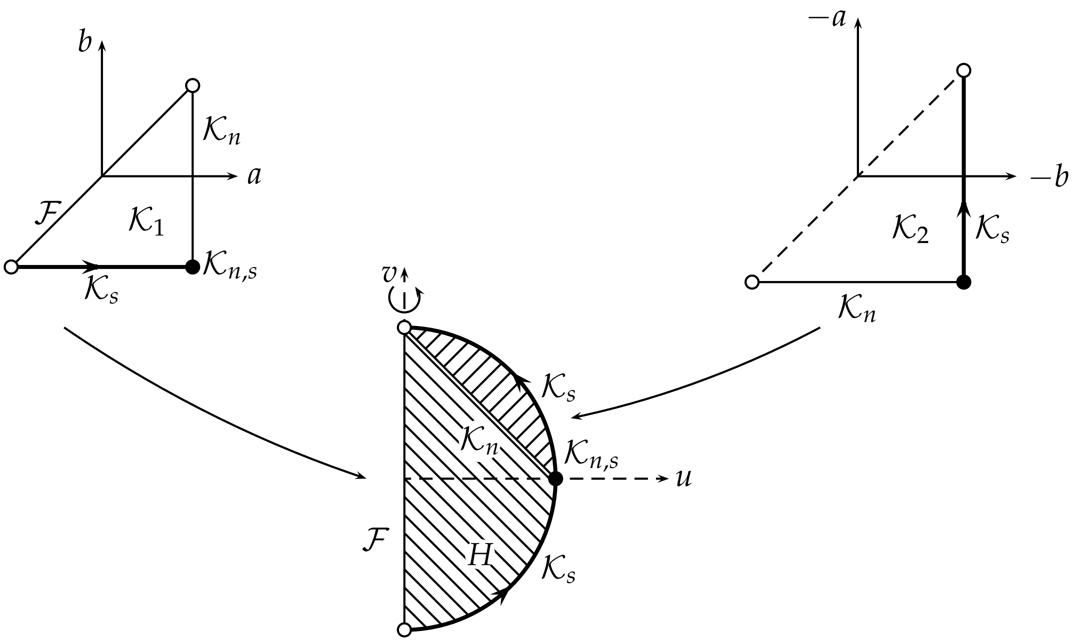

Figure 5: Gluing together the coordinate patches

represents a longitude. This triangle is mapped to the part of H on and above the segment $u + v = 1$, and we can insist that this matches the previous homeomorphism coming from $\mathcal{K}_1$ in the case of circles passing through n (the angle coordinates for these circles do match). The missing side $-a = -b$ is compressed to the missing boundary point $(0, 1)$ of H, and the side $-b = 1$ of the triangle is sent to the quarter circle joining $(1, 0)$ to $(0, 1)$, with the orientation indicated in the figure.

We leave it to the reader to check that this representation has all the features announced above. We only mention some of the trickier points. For instance, the circles containing s but not n occur twice on the boundary of the ball obtained by rotating H, with distinct values of φ, and with different signs of v. The longitudes occur twice on the equator $v = 0$, with distinct values of φ. Every point z in the ball $\mathbb{D}_3$ except those with $v = \pm 1$ represents a unique circle, which depends continuously on z. Circles passing through s occur twice as images, other circles occur exactly once. The former implies that one has a continuous fixed point free involution on $\mathbb{S}_2 \setminus \{(0, 0, \pm 1)\}$, and this involution extends to a continuous fixed point free involution over all of $\mathbb{S}_2$ interchanging the two points $(0, 0, \pm 1)$.

By passing to the corresponding quotient of $\mathbb{D}_3$, we obtain a continuous bijection onto $\mathcal{K}$. Now both the quotient and $\mathcal{K}$ are 3-dimensional topological manifolds (for $\mathcal{K}$, see [15, 2.9]), hence a continuous bijection is a homeomorphism by domain invariance. This ends the proof of Theorem 1.1. ∎

The homeomorphism obtained here can be further extended to one from the extended circle space.

Theorem 5.1. *The extended circle space $\bar{\mathcal{K}}$ of every spherical circle plane is homeomorphic to $P_3\mathbb{R} \setminus X$, where X is an open ball. This is a 3-dimensional topological manifold with boundary with the point set S forming the boundary.*

Proof. From the coordinates (a_k, b_k, φ_k) it is possible to recognize whether or not a sequence of circles K_k converges to a given point $p = (a, \varphi)$. If $p = n$ or s, then convergence to p just means that $b_k \to 1$ or $a_k \to -1$, respectively. Convergence to any other point point p means that both a_k and b_k converge to a and $\varphi_k \to \varphi$. Note that this convergence can only occur for circles in $\mathcal{K}_2$ (ie, circles that do not separate n and s). This implies that, given two spherical circle planes, we can extend the homeomorphism of their circle spaces constructed in the present section. Then the homeomorphism type of $\bar{\mathcal{K}}$ can be determined in the case of the classical Möbius plane. That the classical Möbius plane has $P_3\mathbb{R} \setminus X$ as its extended circle space may be seen by extending the argument given for $\mathcal{K}$ in Section 1.

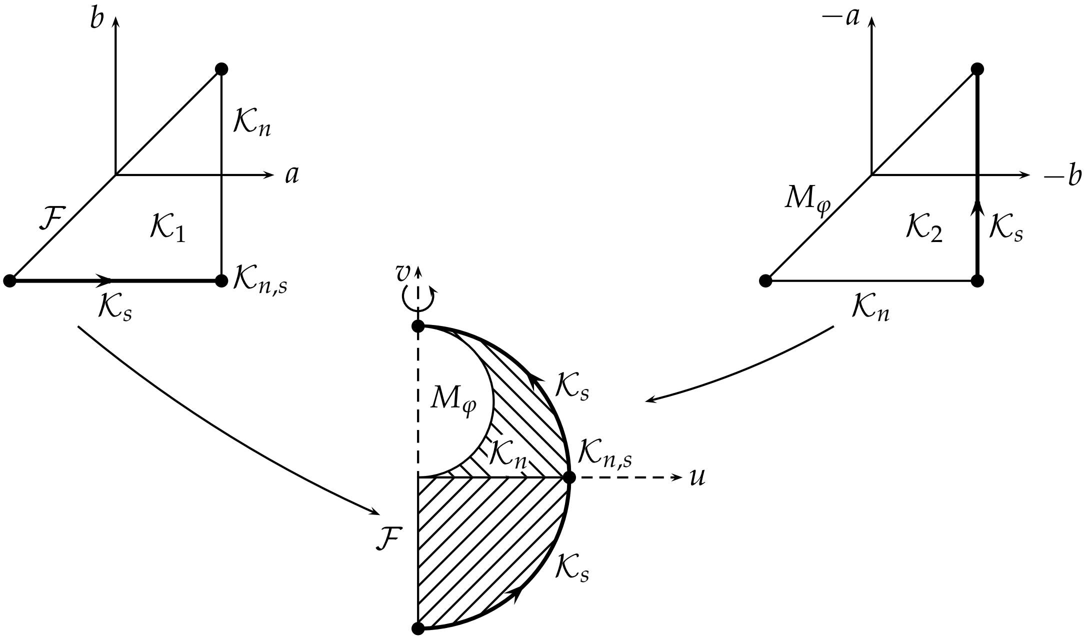

Figure 6: Obtaining the extended circle space

We can also modify the gluing process used before, see Figure 6. We are keeping the coordinates for $\mathcal{K}_1$ and $\mathcal{K}_2$. However, for fixed φ, now all three vertices of the triangles are there and also the side $-a = -b$ in the coordinate triangle for $\mathcal{K}_2$ is there, the latter representing the points on the meridian M_φ of all points that have angle φ. For $\mathcal{K}_1$ the triangle is mapped homeomorphically to the quarter circle below the u-axis. The triangle for $\mathcal{K}_2$ is mapped to the quarter circle above the u-axis minus a semi-circle of radius $\frac{1}{2}$ with centre at $(0, \frac{1}{2})$, where the side $-a = -b$ is taken to the boundary of this semi-circle. Rotation about the v-axis yields the ball $\mathbb{D}_3$ from which a smaller open ball is missing. The identifications on the boundary of $\mathbb{D}_3$ are the same as before so that we again obtain a projective space $P_3\mathbb{R}$, but now minus an open ball. ∎

6 The flag space

It would be desirable to prove a similar theorem about the *flag space*, consisting of all pairs $(p, K) \in S \times \mathcal{K}$ such that $p \in K$. A. Lightfoot [9, Theorem 5.1.1] showed that the flag space of a spherical circle plane is a 4-dimensional topological manifold. Furthermore, in [9, Theorem 5.2.1] he proved that the flag space of an embeddable spherical circle plane is homeomorphic to the flag space of the classical Möbius plane and the latter is determined in Theorem 5.2.4. Combining these two results one has the following.

Proposition 6.1. *The flag space of an embeddable spherical circle plane is homeomorphic to a subset of* $\mathbb{S}_2 \times P_2\mathbb{R}$*, namely, to the complement of the set* $\{(x, \pm x) \mid x \in \mathbb{S}_2\}$.

The flag space of the classical Möbius plane can also be obtained by taking the tangent bundle of $\mathbb{S}_2$, passing to the projective closure in each fiber, and deleting the zero section. In order to see the latter we take up ideas from the introduction. Let σ be the polarity of $P_3\mathbb{R}$ having the sphere $S = \mathbb{S}_2$ as its set of absolute points, and let $E = P_3\mathbb{R} \setminus \mathbb{D}_3$ be the set of exterior points with respect to S. Then the circles are the sets $q^\sigma \cap S$ for $q \in E$. By the properties of a polarity, a point $p \in S$ belongs to a circle $K = q^\sigma \cap S$ if and only if $q \in p^\sigma \cap E$. Now for $p \in S$, the plane p^σ is the tangent plane T_pS, extended to infinity, that is, the projective closure $\overline{T_pS}$ of the tangent plane in the analytic sense. Here, $\overline{T_pS}$ should be considered as a projective subspace of $P_3\mathbb{R}$, and we have $\overline{T_pS} \cap E = \overline{T_pS} \setminus \{p\}$. Hence the pairs (p, q) corresponding to flags are indeed the elements of the projectively closed tangent bundle minus its zero section. The projective tangent bundle is trivial, because all fibers can simultaneously be projected to the plane 0^σ at infinity by central projection from the origin. However, the zero section is not a factor of the product $S \times P_2\mathbb{R}$ but rather equals the set $\{(x, \pm x) \mid x \in S\}$.

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