On the classification of toroidal circle planes

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Abstract

We consider the problem of classifying toroidal circle planes with respect to the dimension of their automorphism groups. With tools from topology, we prove that these groups are Lie groups of dimension at most 6. From the results on flat Minkowski planes by Schenkel, we classify planes whose automorphism group has dimension at least 4.

In the case of dimension 3, we propose a framework for the full classification based on all possible geometric invariants of the automorphism group. When the group fixes exactly one point, we characterise two cases completely with a new family of planes called (modified) strongly hyperbolic planes and the family constructed by Artzy and Groh. Using these results, we determine the automorphism group of the planes constructed by Polster.

To my grandparents

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List of notations

Plane

$\mathbb{T}$	toroidal circle plane
$\mathcal{P}$	point set
$\mathcal{C}$	circle set
$\mathcal{C}^-$	negative half of the circle set
$\mathcal{G}^\pm$	the set of $(\pm)$-parallel classes
$[p]_\pm$	$(\pm)$-parallel class going through the point p
$\mathcal{M}$	flat Minkowski plane
$\mathcal{M}_C$	classical flat Minkowski plane
$\mathcal{M}_M(m)$	modified classical flat Minkowski plane
$\mathcal{M}_{AG}(f,g)$	Artzy-Groh plane
$\mathcal{M}_{GH}(r_1,s_1;r_2,s_2)$	generalised Hartmann plane
$\mathcal{M}_{MS}(m^-,f_1,f_2;m^+,f_3,f_4)$	modified strongly hyperbolic plane
$\mathcal{M}_{SH}(f_i)$	strongly hyperbolic plane
$\mathcal{M}(f,g)$	swapping half plane
$\mathbb{T}_P(f,g)$	Polster plane

Topology

$C(\mathcal{X})$	space of continuous maps from $\mathcal{X}$ to itself
$\mathcal{H}(\mathcal{X})$	space of all non-empty compact subsets of $\mathcal{X}$
$\mathcal{P}^{1,2}$	subspace of $\mathcal{H}(\mathcal{P})$ consisting of pairs of points
$\mathcal{P}^3$	product space $\mathcal{P} \times \mathcal{P} \times \mathcal{P}$
$\widetilde{\mathcal{P}^3}$	subspace of $\mathcal{P}^3$ of triples of pairwise nonparallel points
$\mathcal{C}^2$	product space $\mathcal{C} \times \mathcal{C}$
$\mathcal{C}^{2*}$	subspace of $\mathcal{C}^2$ consisting of pairs of intersecting circles
$\mathcal{C}^{1*}$	subspace of $\mathcal{C}^{2*}$ consisting of pairs of touching circles
$\mathbf{d}$	maximum metric on $\mathcal{P}$
$\mathbf{e}$	supremum metric on $C(\mathcal{P})$
$\partial\mathcal{R}$	boundary of $\mathcal{R}$

Group

$\mathrm{Aut}(\mathbb{T})$	the automorphism group of $\mathbb{T}$
Σ	the connected component of $\mathrm{Aut}(\mathbb{T})$
$T^{\pm}$	the kernels of $\mathrm{Aut}(\mathbb{T})$
$\Delta^{\pm}$	the kernel of the action of a specified connected group on $\mathcal{G}^{\pm}$
$Z(G)$	centre of a group G
Φ_d	group of maps $\{(x, y) \mapsto (ax + b, a^d y + c) \mid a > 0, b, c \in \mathbb{R}\}$
Φ_∞	group of maps $\{(x, y) \mapsto (x + b, ay + c) \mid a > 0, b, c \in \mathbb{R}\}$
Φ	group of maps $\{(x, y) \mapsto (rx + a, sy + b) \mid r, s > 0, a, b \in \mathbb{R}\}$
L_2	group of maps $\{x \mapsto ax + b \mid a > 0, b \in \mathbb{R}\}$

Miscellany

IVT	Intermediate Value Theorem
MVT	Mean Value Theorem
$f_{r,s}$	semi-multiplicative homeomorphism of $\mathbb{S}^1$
f'_-	left derivative of f
$f(x^-)$	the left-sided limit of f at x
id	identity map

Chapter 1

Introduction

The geometry of the Euclidean plane is well known. In the modern language of incidence geometry, the Euclidean plane $\mathscr{E} = (\mathcal{P}, \mathcal{L})$ consists of a *point set* $\mathcal{P}$, and a *line set* $\mathcal{L}$. Here the point set $\mathcal{P}$ is the set of *points* of the 2-dimensional Euclidean space $\mathbb{R}^2$. If we identify $\mathbb{R}^2$ with the Cartesian coordinate system, a *line* in the line set $\mathcal{L}$ can be described as a set of points (x, y) satisfying the equation

$$ax + by + c = 0,$$

for some $a, b, c \in \mathbb{R}, a^2 + b^2 \neq 0$. Furthermore, the Euclidean plane satisfies the two following incidence axioms.

A1: Two distinct points belong to a unique line.

A2: Given a line L and a point p not on L, there is a unique line L' going through p and not intersecting L.

Similar to the way we describe the Euclidean plane, by *a geometry* we mean a pair $(\mathcal{P}, \mathcal{L})$ satisfying some pre-defined axioms. In this thesis, we specialise in *geometries on surfaces*, which are geometries whose lines 'look like' (are homeomorphic to) the real line $\mathbb{R}$ or the circle $\mathbb{S}^1$. A geometry, with points in $\mathbb{R}^2$ and lines homeomorphic to $\mathbb{R}$, that satisfies Axioms A1 and A2 is called a *flat affine plane*. An example of a non-Euclidean flat affine plane is a *Moulton plane* $\mathscr{M}_k$, cf. [Mou02]. The line set of a Moulton plane $\mathscr{M}_k$ is the same as that of the Euclidean plane, except that each Euclidean line

$$L = \{(x, sx + t) \mid x \in \mathbb{R}\}$$

with positive slope s is replaced by a set of points

$$L' = \{(x, sx + t) \mid x \geqslant 0\} \cup \{(x, ksx + t) \mid x \leqslant 0\},$$

for a fixed $k > 0$.

There was a long debate whether Axiom A2 can be inferred from Axiom A1. To answer this question, a theory of non-Euclidean geometries had to be developed. This study reached a peak in the 19th century with the likes of Gauss, Bolyai and Lobachevsky. Geometries not satisfying Axiom A2 were constructed; some examples are the *real hyperbolic plane* and the *real cylinder plane*, which are the restriction of the Euclidean plane to the open unit disc and the open upper half-plane, respectively. In today's terminology, we call such a geometry, with the point set $\mathbb{R}^2$ and lines homeomorphic to $\mathbb{R}$, satisfying Axiom A1 simply an *$\mathbb{R}^2$-plane*.

The emergence of various non-Euclidean geometries inevitably led to the desire of classifying these objects. In 1872, Felix Klein proposed the Erlangen program with the aim to characterise geometries according to their geometric invariants (cf. [Kle93]). The program encourages the use of methods from projective geometry and group theory as long as the 'symmetries' of the object are rich enough. Following this point of view, in the late 20th century, Salzmann and his school completed a classification of flat affine planes, of their close relatives 2-dimensional compact projective planes, and of $\mathbb{R}^2$-planes. A summary of the work can be found in [Sal+95].

In 1973, Benz [Ben73] introduced a system of axioms for geometries on point sets other than $\mathbb{R}^2$. In dimension 2, these are the sphere $\mathbb{S}^2$, the torus $\mathbb{S}^1 \times \mathbb{S}^1$ and the cylinder $\mathbb{S}^1 \times \mathbb{R}$. On each of these point sets, there are two axioms, namely Axiom of Joining and Axiom of Touching, playing a similar role as Axioms A1 and A2 on the point set $\mathbb{R}^2$. We call geometries satisfying Axiom of Joining and Axiom of Touching on these three new point sets *flat Möbius planes*, *flat Minkowski planes* and *flat Laguerre planes*, respectively. A more general way to indicate a plane of these types is to call it a *flat Benz plane*.

On the point set $\mathbb{R}^2$, we can think of a flat affine plane as an $\mathbb{R}^2$-plane satisfying the additional Axiom A2. Analogously, there are geometries similar to flat Benz planes but not satisfying the corresponding Axiom of Touching. These geometries are called *spherical circle planes*, *toroidal circle planes* and *cylindrical circle planes*, respectively.

In the current literature, research on flat Benz planes was initiated by Wölk [Wol66], Schenkel [Sch80], and Groh [Gro68], [Gro70]. Spherical circle planes have also been studied in a series of papers by Strambach (cf. [Str70]). Yet toroidal circle planes and cylindrical circle planes are to be explored in details.

Our goal for this project is to carry out an investigation of toroidal circle planes with respect to their symmetries.

1.1 Research objectives and outcomes

The symmetries of a geometry are encoded in its *automorphism group*. In the case of flat Minkowski planes, Schenkel [Sch80] showed that their automorphism groups are Lie groups (with respect to the compact-open topology) of dimension at most 6. This result strongly suggests that the automorphism group of toroidal circle planes follows this pattern, in view of the fact that flat Minkowski planes are toroidal circle planes.

In the case of $\mathbb{R}^2$-planes, Salzmann [Sal67a] showed that their automorphism groups are also Lie groups with dimension at most 6. The proof of this result makes use of the continuity of geometric operations. The topology on the point set can be assumed to be the standard topology. In contrast, since lines are merely sets of points, there are various candidates for the topology on the line set. Hence it is non-trivial to describe a topology for the line set such that geometric operations are continuous.

To verify that automorphism groups of flat Minkowski planes are Lie groups, Schenkel applied a technique relying on the Axiom of Touching. This means we cannot apply the same technique for toroidal circle planes. Fortunately, the theories of $\mathbb{R}^2$-planes and toroidal circle planes are inextricably intertwined, as *derived planes* of toroidal circle planes are $\mathbb{R}^2$-planes (cf. Section 2.2). The line set of such derived $\mathbb{R}^2$-planes originates from the *circle set* of toroidal circle planes. To adapt the proof in the case of $\mathbb{R}^2$-planes for toroidal circle planes, our first task is to develop a suitable topology on the circle set.

Most known results on the automorphism group of flat Minkowski planes come from Lie theory and are independent from the Axiom of Touching (cf. Theorems 2.3.4, 2.3.3 and 2.3.5). If automorphism groups of toroidal circle planes are indeed Lie groups, then we can generalise these results.

Although there are examples of flat Minkowski planes with 3-dimensional group of automorphisms, a full classification of these planes (and of toroidal circle planes in general) has not been carried out and is out of scope for this project. As a first step towards this direction, we aim to determine the action of such a group. We also aim to construct additional planes of this type and classify them under suitable restrictions.

Briefly, our research outcomes are the following.

(i) Establishment of a natural topology on the circle set of toroidal circle planes on which geometric operations are continuous.

(ii) Proof of the conjecture that the automorphism group of a toroidal circle plane is a Lie group with dimension at most 6.

(iii) Description of all toroidal circle planes whose automorphism group is at least 4-dimensional, or one of whose kernels is 3-dimensional.

(iv) Description of the action of 3-dimensional connected groups of automorphisms on toroidal circle planes.

(v) New examples of flat Minkowski planes whose automorphism group is at least 2-dimensional.

(vi) Characterisations of two families of flat Minkowski planes with 3-dimensional groups of automorphisms.

(vii) Description of the automorphism groups of a family of toroidal circle planes that are not flat Minkowski planes called Polster planes.

1.2 Outline of thesis

The thesis is organised as follows.

Chapter 2 contains background information about toroidal circle planes and $\mathbb{R}^2$-planes. This includes definitions and examples, as well as mathematical jargons which are used throughout the thesis. Concepts such as derived planes, automorphism groups and Klein-Kroll types are also introduced.

In Chapter 3, we examine toroidal circle planes from a topological point of view. We first define appropriate geometric operations on the point set and the circle set. We then establish a topology on the circle set and show that geometric operations are continuous with respect to this topology. The results contained in this chapter will be used mainly to support the arguments in Chapter 4.

In Chapter 4, we show that automorphism groups of toroidal circle planes are Lie groups with dimension at most 6. The structure of this chapter is linear: we first recall what we need from the literature, then prove the result in a series of lemmas.

Chapter 5 is about the structure of the automorphism group with dimension at least 3. Here we extend some results on flat Minkowski planes to toroidal circle planes. We then consider almost simple groups of automorphisms, since they play a prominent role in our analysis of possible automorphism groups with dimension 3. For clarity purposes, some technical details used in this chapter are summarised in Appendix A.

Chapter 6 is devoted to the construction of a family of flat Minkowski planes with 3-dimensional groups of automorphisms. The verification of axioms relies heavily on results about real functions, which are listed in Appendix B. This chapter includes a discussion of isomorphism classes of a subfamily of these planes, and some examples.

In Chapter 7, we characterise two families of flat Minkowski planes with 3-dimensional groups of automorphisms, one of which is from Chapter 6. We also use these results to describe the automorphism group of a family of proper toroidal circle planes that are not flat Minkowski planes.

The thesis concludes with a summary of contributions and discussion of future work in Chapter 8.

Chapter 2

Background

This chapter contains background on toroidal circle planes. In Section 2.1, we introduce the definitions and some examples of toroidal circle planes and flat Minkowski planes. In Section 2.2, we define derived planes and consider some examples. In Section 2.3, we discuss automorphism groups of toroidal circle planes and their derived planes. Section 2.4 contains results on the Klein-Kroll types of flat Minkowski planes.

2.1 Definitions and examples

In this section we define toroidal circle planes and flat Minkowski planes. We also consider some families of toroidal circle planes and discuss the methods behind these examples along the way. These methods will be used in Chapter 6.

Definition 2.1.1. A *toroidal circle plane* is a geometry $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$, whose

> point set $\mathcal{P}$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$,
>
> circles (elements of $\mathcal{C}$) are graphs of homeomorphisms of $\mathbb{S}^1$,
>
> $(+)$-parallel classes (elements of $\mathcal{G}^+$) are the verticals $\{x_0\} \times \mathbb{S}^1$,
>
> $(-)$-parallel classes (elements of $\mathcal{G}^-$) are the horizontals $\mathbb{S}^1 \times \{y_0\}$,

where $x_0, y_0 \in \mathbb{S}^1$. Furthermore, a toroidal circle plane satisfies the following

> *Axiom of Joining*: three pairwise nonparallel points (no two of which are on the same parallel class) can be joined by a unique circle.

We denote the $(\pm)$-parallel class containing a point p by $[p]_\pm$. When two points p, q are on the same $(\pm)$-parallel class, we say they are $(\pm)$-*parallel* and denote this by $p \parallel_{\pm} q$. Two points p, q are *parallel* if they are $(+)$-parallel or $(-)$-parallel, and we denote $p \parallel q$.

Definition 2.1.2. A toroidal circle plane is a *flat Minkowski plane* if it also satisfies the following

> *Axiom of Touching*: for each circle C and any two nonparallel points p, q with $p \in C$ and $q \notin C$, there is exactly one circle D that contains both points p, q and intersects C only at the point p.

In general, for two distinct sets of points C and D (not necessarily circles of a toroidal circle planes), we say *C touches D (combinatorially)* at p if $C \cap D = \{p\}$. If C and D are circles of a toroidal circle plane and C touches D, then *C touches D topologically*. This will be discussed more thoroughly in Lemma 3.3.5.

A toroidal circle plane is in *standard representation* if the set $\{(x, x) \mid x \in \mathbb{S}^1\}$ is one of its circles. Up to isomorphisms (cf. Section 2.3), every toroidal circle plane can be described in standard representation (cf. [PS01] Subsection 4.2.3). All toroidal circle planes presented in this thesis are in standard representation. We note that it is sufficient to describe a toroidal circle plane in standard representation by describing its circle set. All examples of toroidal circle planes in this section can be found in [PS01] Section 4.3. Additional explanations can be found in [Pol98a] Chapter 21. We take this opportunity to introduce some notations.

Toroidal circle planes arise from the geometry of plane sections of the standard non-degenerated ruled quadric

$$\mathcal{Q} = \{(x_0 : x_l : x_2 : x_3) \in \mathbb{P}_3(\mathbb{R}) \mid x_0x_3 + x_1x_2 = 0\}$$

in the real 3-dimensional projective space $\mathbb{P}_3(\mathbb{R})$. This prototype model is called the *classical flat Minkowski plane*. As stated by Delandtsheer [Del95], the classical flat Minkowski plane is the 2-dimensional analogue of the 4-dimensional Minkowski spacetime of special relativity. In incidence geometry, it plays the same role in the theory of toroidal circle planes as that of the Euclidean plane in the theory of $\mathbb{R}^2$-planes.

Different models of the classical flat Minkowski plane are described in [PS01] Section 4.1. The following description is in standard representation (cf. [PS01] Subsection 4.1.2).

Example 2.1.3 (Classical flat Minkowski plane $\mathcal{M}_C$)**.** The circle set $\mathcal{C}_C$ of the *classical flat Minkowski plane $\mathcal{M}_C$* consists of sets of the form

$$\{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\},$$

where $s, t \in \mathbb{R}$, $s \neq 0$, and

$$\{(x, y) \in \mathbb{R}^2 \mid (x - b)(y - c) = a\} \cup \{(\infty, c), (b, \infty)\},$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$.

In the description above, we note that sets of the first form are non-horizontal and non-vertical Euclidean lines extended by the point (∞, ∞). Sets of the second form are Euclidean hyperbolas extended by two infinite points.

Let $\mathbb{T}$ be a toroidal circle plane with circle set $\mathcal{C}$. Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sets of all circles that are graphs of orientation-preserving and orientation-reversing homeomorphisms $\mathbb{S}^1 \to \mathbb{S}^1$, respectively. Then $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. We call $\mathcal{C}^+$ the *positive half* and $\mathcal{C}^-$ the *negative half* of $\mathbb{T}$. These two halves are independent of each other, that is, we can combine halves from different toroidal circle planes to obtain another toroidal circle plane. This method can even be specialised to flat Minkowski planes, and is summarised as follows.

Theorem 2.1.4 (cf. [PS01] Theorem 4.3.1)**.** *For $i = 1, 2$, let $\mathbb{T}_i$ be a toroidal circle plane with circle set $\mathcal{C}_i$. Let $\mathcal{C} = \mathcal{C}_1^+ \cup \mathcal{C}_2^-$. Then $\mathcal{C}$ is the circle set of a toroidal circle plane $\mathbb{T}$. If both $\mathbb{T}_1$ and $\mathbb{T}_2$ are flat Minkowski planes, so is $\mathbb{T}$.*

Based on this property of the circle set, the classical flat Minkowski plane can be modified in the following way. Denote $\mathrm{PGL}(2, \mathbb{R})$ by Ξ and $\mathrm{PSL}(2, \mathbb{R})$ by Λ. The group Λ is a normal subgroup of index 2 in Ξ. In the standard action of Ξ on $\mathbb{S}^1$ as fractional linear transformations, Λ consists of orientation-preserving homeomorphisms of $\mathbb{S}^1$ where as $\Xi \backslash \Lambda$ consists of orientation-reversing homeomorphisms of $\mathbb{S}^1$. The circles of the classical flat Minkowski plane are the graphs of the homeomorphisms in Ξ. More specifically, the two halves of its circle set are obtained from Λ and $\Xi \backslash \Lambda$. When we replace the set $\Xi \backslash \Lambda$ with $g^{-1}(\Xi \backslash \Lambda) f$ for suitable functions f and g, we obtain non-classical flat Minkowski planes called *swapping half planes*. These planes were first introduced by Schenkel [Sch80] and then generalised by Steinke [Ste94].

Example 2.1.5 (Swapping half plane $\mathcal{M}(f, g)$)**.** Let f and g be two orientation-preserving homeomorphisms of $\mathbb{S}^1$. Denote $\mathrm{PGL}(2, \mathbb{R})$ by Ξ and $\mathrm{PSL}(2, \mathbb{R})$ by Λ. The circle set $\mathcal{C}(f, g)$ of a *swapping half plane* $\mathcal{M}(f, g)$ consists of sets of the form

$$\{(x, \gamma(x)) \mid x \in \mathbb{S}^1\},$$

where $\gamma \in \Lambda \cup g^{-1}(\Xi \backslash \Lambda) f$.

To describe the construction of the next example, we need some notations. Explicitly, the negative half of the classical flat Minkowski plane consists of sets of the form

$$\{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\},$$

where $s, t \in \mathbb{R}$, $s < 0$, and

$$\{(x, y) \in \mathbb{R}^2 \mid (x - b)(y - c) = a\} \cup \{(\infty, c), (b, \infty)\},$$

where $a, b, c \in \mathbb{R}$, $a > 0$.

In the description above, let E and H be the set of circles having the former and latter form, respectively.

Let C be a circle in H. The restriction of C to $\mathbb{R}^2$ has two connected components, one is the graph of a convex function and the other is the graph of a concave function. We call these two components the *convex branch* and the *concave branch* of C, respectively. In particular, a circle in H consists of a convex branch, a concave branch, and two points not in $\mathbb{R}^2$.

Let H_0 be the set of circles in H going through the points $(0, \infty), (\infty, 0)$. Then every circle in H is a translation of a circle in H_0.

Let Q_1 be the set of convex branches of circles in H_0 and Q_2 be the set of concave branches of circles in H_0. We note that Q_1 partitions the first quadrant and Q_2 partitions the third quadrant.

A second method to modify the negative half of the classical flat Minkowski plane is the following. For each circle C in H_0, we replace its concave branch by an element V in Q_2. The choice of this replacement is determined by a pre-defined function m^-. The result of this replacement is that C becomes a set C', consisting of the convex branch of C, the set V, and two points $(0, \infty), (\infty, 0)$. We denote the collection of all such C' by $H_0(m^-)$. Let $H(m^-)$ be the set of all translations of elements in $H_0(m^-)$. Then $E \cup H(m^-)$ is the negative half of a flat Minkowski plane.

Similarly, the positive half of the classical flat Minkowski plane can be modified by a function m^+. When we modify both halves of the classical flat Minkowski plane this way, we obtain a *modified classical flat Minkowski plane* $\mathcal{M}_M(m)$. Here m is a function determined by m^- and m^+.

This construction was introduced by Steinke [Ste85]. More insights into this method are provided by Polster [Pol96].

Example 2.1.6 (Modified classical flat Minkowksi plane $\mathcal{M}_M(m)$)**.** Let $m : \mathbb{R} \to \mathbb{R}$ be an orientation-preserving homeomorphism of $\mathbb{R}$ that fixes 0. The circle set $\mathcal{C}_M(m)$ of a *modified classical flat Minkowski plane* $\mathcal{M}_M(m)$ consists of sets of the form

$$\{(x, sx+t) \mid x \in \mathbb{R}\} \cup \{(\infty,\infty)\},$$

where $s, t \in \mathbb{R}$, $s \neq 0$, and

$$\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; (x-b)(y-c) = \begin{cases} a & \text{for } x > b, \\ m(a) & \text{for } x < b, \end{cases} \right\} \cup \{(\infty, c), (b, \infty)\},$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$.

Perhaps the first group-theoretic construction of flat Minkowski planes is the one introduced by Artzy and Groh [AG86]. We first start with two 'nice' homeomorphisms f and g of $\mathbb{R}\backslash\{0\}$ and extend them to homeomorphisms of $\mathbb{S}^1$ in the canonical way. We then generate most of the circle set with the images of these two homeomorphisms under the group

$$\Phi_1 = \{(x,y) \mapsto (ax+b, ay+c) \mid a, b, c \in \mathbb{R}, a > 0\}.$$

We complete the circle set by including the set of non-horizontal and non-vertical Euclidean lines extended by the point (∞,∞). This set is also invariant under the group Φ_1. The advantage of this construction is that we know before-hand the group Φ_1 is a group of *automorphisms* of such a plane. Automorphisms will be discussed separately in Section 2.3. In the meantime, we can describe the circle set of an Artzy-Groh plane explicitly as follows.

Example 2.1.7 (Artzy-Groh plane $\mathcal{M}_{AG}(f,g)$)**.** Let $f, g : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ be two homeomorphisms satisfying the following conditions.

(i) The functions f and g are differentiable.

(ii) The restrictions of f' and g' to both $\mathbb{R}^+$ and $\mathbb{R}^-$ are strictly monotonic.

(iii) The restrictions of f and g to both $\mathbb{R}^+$ and $\mathbb{R}^-$ have the x-axis and the y-axis as asymptotes.

(iv) $f'(x) < 0$ and $g'(x) > 0$ for all $x \in \mathbb{R}\backslash\{0\}$.

For $a > 0, b, c \in \mathbb{R}$, let $f_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$f_{a,b,c}(x) = af\left(\frac{x+b}{a}\right) + c.$$

For $a < 0, b, c \in \mathbb{R}$, let $g_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$g_{a,b,c}(x) = ag\left(\frac{x+b}{|a|}\right) + c.$$

The circle set $\mathcal{C}_{AG}(f,g)$ of an *Artzy-Groh plane* $\mathcal{M}_{AG}(f,g)$ consists of sets of the form

$$\{(x, sx+t) \mid x \in \mathbb{R}\} \cup \{(\infty,\infty)\},$$

where $s,t \in \mathbb{R}$, $s \neq 0$, sets of the form

$$\{(x, f_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty,c),(-b,\infty)\},$$

where $a,b,c \in \mathbb{R}$, $a > 0$, and sets of the form

$$\{(x, g_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty,c),(-b,\infty)\},$$

where $a,b,c \in \mathbb{R}$, $a < 0$.

It is worth noting that the negative half of an Artzy-Groh plane $\mathcal{M}_{AG}(f,g)$ consists of images of the graph of f under Φ_1 and proper (non-horizontal and non-vertical) Euclidean lines with negative slope, extended suitably to homeomorphisms of $\mathbb{S}^1$.

The classical flat Minkowski plane is the Artzy-Groh plane $\mathcal{M}_{AG}(x^{-1},x^{-1})$. In the next example, we consider another special type of Artzy-Groh planes. For $r,s > 0$, let $f_{r,s}$ be the orientation-preserving *semi-multiplicative homeomorphism* of $\mathbb{S}^1$ defined by

$$f_{r,s}(x) = \begin{cases} x^r & \text{for } x \geqslant 0, \\ -s|x|^r & \text{for } x < 0, \\ \infty & \text{for } x = \infty. \end{cases}$$

A homeomorphism f of $\mathbb{S}^1$ is *inversely semi-multiplicative* if it has the form $f(x) = f_{r,s}^{-1}(x)$. The Artzy-Groh plane $\mathcal{M}_{AG}(f,g)$ where f and g are inversely semi-multiplicative is called a *generalised Hartmann plane.*

Example 2.1.8 (Generalised Hartmann plane $\mathcal{M}_{GH}(r_1,s_1;r_2,s_2)$)**.** Let $r_1,s_1,r_2,s_2 > 0$. A *generalised Hartmann plane* $\mathcal{M}_{GH}(r_1,s_1;r_2,s_2)$ is the plane $\mathcal{M}_{AG}(f^{-1}_{r_1,s_1}, g^{-1}_{r_2,s_2})$.

Original *Hartmann planes* were constructed by Hartmann [Har81] as planes of the form $\mathcal{M}_{GH}(r_1,1;r_2,1)$. Generalised Hartmann planes were introduced by Schenkel [Sch80]. The plane $\mathcal{M}_{GH}(1,1;1,1)$ is none other than the classical flat Minkowski plane.

The last example in this section is a proper toroidal circle plane that is not a flat Minkowski plane. This plane was introduced by Polster [Pol98b] originally in the extended Cartesian coordinate system rotated by 45°. An intuitive explanation of this construction can be found in [PS01] Subsection 4.3.7. The description below is in the standard coordinate system and is equivalent to the description in [Pol98b].

Example 2.1.9 (Polster plane $\mathbb{T}_P(f,g)$)**.** Let f, g be functions satisfying the four conditions in Example 2.1.7, with f satisfying the additional property $f = f^{-1}$.

For $a > 0, b, c \in \mathbb{R}$, let $f_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$f_{a,b,c}(x) = \begin{cases} af\left(\dfrac{x+b+1}{a}\right) + c & \text{for } x \geqslant x^*, \\ af\left(\dfrac{x+b}{a}\right) + c - 1 & \text{for } -b < x \leqslant x^*, \\ af\left(\dfrac{x+b}{a}\right) + c & \text{for } x < -b, \end{cases}$$

where $x^* \in (-b, +\infty)$ satisfies

$$af\left(\frac{x^*+b+1}{a}\right) = x^* + b.$$

For $a < 0, b, c \in \mathbb{R}$, let $g_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$g_{a,b,c}(x) = ag\left(\frac{x+b}{|a|}\right) + c.$$

The circle set $\mathcal{C}_P(f,g)$ of a *Polster plane* $\mathbb{T}_P(f,g)$ consists of sets of the form

$$\{(x, sx+t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\},$$

where $s, t \in \mathbb{R}$, $s \neq 0$, sets of the form

$$\{(x, f_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty, c), (-b, \infty)\},$$

where $a, b, c \in \mathbb{R}$, $a > 0$, and sets of the form

$$\{(x, g_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty, c), (-b, \infty)\},$$

where $a, b, c \in \mathbb{R}$, $a < 0$.

2.2 Derived planes

Studying toroidal circle planes is sometimes hard, globally. In such moments, one can take a more local point of view via *derived planes*. Informally speaking, a derived plane contains local information of the toroidal circle plane at a point.

Definition 2.2.1. The *derived plane* $\mathbb{T}_p$ *of* $\mathbb{T}$ *at the point* p is the incidence geometry whose point set is $\mathcal{P}\backslash([p]_+ \cup [p]_-)$ and whose lines are $(\mathcal{G}^+ \cup \mathcal{G}^-)\backslash\{[p]_+, [p]_-\}$ plus all circles of $\mathbb{T}$ going through p.

The derived plane $\mathbb{T}_p$ is an $\mathbb{R}^2$-plane for every point $p \in \mathcal{P}$ and even a flat affine plane in the case $\mathbb{T}$ is a flat Minkowski plane. The converse also holds in the following sense (cf. [PS01] Theorem 4.2.1).

Theorem 2.2.2 (Characterisation via Derived Planes)**.** *Let $\mathcal{M}$ be a geometry whose point set is the torus $\mathbb{S}^1 \times \mathbb{S}^1$ equipped with two nontrivial parallelisms the parallel classes of which are the horizontals and verticals on the torus. Then $\mathcal{M}$ is a toroidal circle plane or a flat Minkowski plane if and only if all its derived planes are $\mathbb{R}^2$-planes or flat affine planes, respectively.*

For convenience, definitions of $\mathbb{R}^2$-planes and flat affine planes are included here. The following definition of a $\mathbb{R}^2$-plane is adapted from [Sal+95] Definition 31.1.

Definition 2.2.3. Let $\mathcal{L}$ be a system of closed subsets in the topological space $\mathbb{R}^2$ which are homeomorphic to $\mathbb{R}$. The elements of $\mathbb{R}^2$ are called *points*, and the elements of $\mathcal{L}$ are called *lines*. We say that $(\mathbb{R}^2, \mathcal{L})$ is an $\mathbb{R}^2$*-plane* if the following axiom holds.

> *A1*: Two distinct points p, q are contained in exactly one line $L \in \mathcal{L}$.

The joining line L is denoted by $L = pq$. The *closed interval* $[p, q]$ is the intersection of all connected subsets of the line pq that contain p and q; *open intervals* are defined by $(p, q) := [p, q] \backslash \{p, q\}$; *half-open intervals* are defined by $[p, q) = [p, q] \backslash \{q\}$ and $(p, q] = [p, q] \backslash \{p\}$.

Definition 2.2.4. An $\mathbb{R}^2$-plane is a *flat affine plane* if it also satisfies the following axiom.

> *A2*: Given a line L and a point p not on L, there is a unique line L' going through p and not intersecting L.

The prototype example of an $\mathbb{R}^2$-plane is none other than the Euclidean plane. It is a flat affine plane and is sometimes referred to as the *classical flat affine plane*. In [Sal+95] Subsection 31.2, it is also called the *real affine plane*.

Example 2.2.5 (Classical flat affine plane, real affine plane, Euclidean plane $\mathscr{E}$)**.** The line set of the *Euclidean plane* consists of sets of points (x, y) satisfying the equation

$$ax + by + c = 0,$$

where $a, b, c \in \mathbb{R}, a^2 + b^2 \neq 0$. An element in the line set of the Euclidean plane is called a *Euclidean line*.

A family of flat affine planes with a 'rich' collection of symmetries are that of *radial Moulton planes*. The following description of a radial Moulton plane is taken from [Sal+95] Subsections 34.1 and 34.2.

Example 2.2.6 (Radial Moulton plane $\mathscr{M}(s)$)**.** Fix a real number $s \geqslant 0$ and define the function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (0, \infty)$ by

$$f(\varphi) = \frac{e^{s\varphi}}{\cos\varphi}.$$

The line set of a *radial Moulton plane* $\mathscr{M}(s)$ is the set

$$\mathcal{L} = \{cL \mid c \in \mathbb{C}^\times\} \cup \left\{\mathbb{R}e^{i\varphi} \mid -\frac{\pi}{2} < \varphi \leqslant \frac{\pi}{2}\right\},$$

where $L = \left\{f(\varphi)e^{i\varphi} \mid -\frac{\pi}{2} < \varphi \leqslant \frac{\pi}{2}\right\}$.

An alternative description can be found in [PS01] Subsection 2.7.3. Under projective completion, the Moulton plane $\mathscr{M}_k$ in the Introduction chapter is 'the same as' (isomorphic to) a radial Moulton plane $\mathscr{M}(s)$ when $k = e^{2\pi s}$. A proof of this fact can be found in [Sal+95] Subsection 34.2. In particular, $\mathscr{M}(0)$ is the Euclidean plane.

We can obtain another model of the Euclidean plane by deforming it via the homeomorphism

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, y + x^2).$$

Under this deformation the verticals stay unchanged and the nonvertical line that is the graph of the affine function

$$\mathbb{R} \to \mathbb{R} : x \mapsto bx + c$$

becomes the parabola given by the quadratic function

$$\mathbb{R} \to \mathbb{R} : x \mapsto x^2 + bx + c.$$

In other words, we can construct the line set of this model by keeping all vertical lines and replacing all nonvertical lines by the parabola

$$\{(x, x^2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$

and its images under the group $\mathbb{R}^2$ of Euclidean translations

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + a, y + b),$$

where $a, b \in \mathbb{R}$.

When we generalise this construction by replacing the parabola with the graph of a 'nice' function f, we obtain a family of flat affine planes called *shift planes*. More information on shift planes and properties of f can be found in [Sal+95] Subsection 31.25c or [PS01] Subsection 2.7.4. In the following example, we consider a special class of shift planes, called *skew parabola planes.*

Example 2.2.7 (Skew parabola plane $\mathscr{P}_{c,d}$)**.** For $c, d \in \mathbb{R}$ with $c > 0$ and $d > 1$, let

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x^d & \text{for } x \geqslant 0, \\ c|x|^d & \text{for } x \leqslant 0. \end{cases}$$

The line set of a *skew parabola plane* $\mathscr{P}_{c,d}$ consists of vertical lines, the graph of f and its images under the group $\mathbb{R}^2$ of Euclidean translations

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + a, y + b),$$

where $a, b \in \mathbb{R}$.

A thorough summary of skew parabola planes can be found in [Sal+95] Section 36 or [PS01] Subsection 2.7.4.

In the remaining examples of this section, we consider proper (non-affine) $\mathbb{R}^2$-planes. The following two examples are simply the restrictions of the Euclidean plane to subsets of the point set $\mathbb{R}^2$.

Example 2.2.8 (Real hyperbolic plane $\mathscr{H}(\mathbb{R})$)**.** The *real hyperbolic plane* $\mathscr{H}(\mathbb{R})$ is the restriction of the Euclidean plane to the open unit disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.

Example 2.2.9 (Real cylinder plane, real half plane $\mathscr{C}(\mathbb{R})$)**.** The *real cylinder plane* $\mathscr{C}(\mathbb{R})$ is the restriction of the Euclidean plane to the open set $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$.

In 1968, Strambach [Str68] began the program to classify $\mathbb{R}^2$-planes with 3-dimensional automorphism group. The completion of this classification required the notion of arc planes, which was introduced and developed by Groh [Gro76], [Gro79], [Gro82b] and [Gro82a]. There are two types of arc planes: type $\mathbb{R}^2$ and type L_2. For our purpose, we only consider arc planes of type $\mathbb{R}^2$.

Let G be the group $\mathbb{R}^2$ consisting of all Euclidean translations

$$t_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + a, y + b),$$

where $a, b \in \mathbb{R}$.

Let $\mathscr{E} = (\mathbb{R}^2, \mathcal{L}_e)$ be the Euclidean plane. We define the *slope* $s \in \mathbb{R} \cup \{\infty\}$ of a Euclidean line in the usual way. The *slope supply* of a set of points in $\mathbb{R}^2$ is the set of all slopes of secant lines of the set. Two sets of points are *slope disjoint* if their slope supplies do not have any elements in common. A set of points is called *projectable* if there is a parallel class of Euclidean lines such that every one of these lines intersects the set in at most one point and the set itself is not contained in any strip bounded by two of these parallel lines.

An *arc* is a set of points in $\mathbb{R}^2$ that has at most 2 intersections with every Euclidean line. An arc is called a *topological arc* if it is closed and homeomorphic to $\mathbb{R}$.

Definition 2.2.10 (Arc plane of type $\mathbb{R}^2$ $\mathscr{A}(A)$)**.** Let A be a collection of sets of points in $\mathbb{R}^2$ such that the following holds.

(i) All elements of A are topological arcs in $\mathscr{E}$.

(ii) None of the elements of A has a slope supply equal to $\mathbb{R} \cup \{\infty\}$.

(iii) Any two distinct elements of A are slope disjoint.

(iv) All elements of A are projectable.

Let S be the set of all Euclidean lines that have a slope that is not contained in the slope supply of any arc in A. Let $\mathcal{L}$ consist of all elements of S and all images of the elements of A under the group G. The geometry $\mathscr{A}(A) = (\mathbb{R}^2, \mathcal{L})$ is called the *arc plane (of type* $\mathbb{R}^2$*)* generated by A.

The following two examples are arc planes of type $\mathbb{R}^2$. They can be found in [Gro82b] or [PS01] Subsection 2.7.5.

Example 2.2.11 (Hyperbolic arc planes)**.** A *hyperbolic arc plane of type 1* is the arc plane $\mathscr{A}(x^s)$ generated by the graph of one of the functions

$$\mathbb{R}^+ \to \mathbb{R} : x \mapsto x^s,$$

where $s \leqslant -1$. A *hyperbolic arc plane of type 2* is the arc plane $\mathscr{A}(x^s, rx^s)$ generated by the graph of the two functions

$$\mathbb{R}^+ \to \mathbb{R} : x \mapsto x^s,$$

and

$$\mathbb{R}^+ \to \mathbb{R} : x \mapsto rx^s,$$

where $r \leqslant -1$ and $s < 0$.

Example 2.2.12 (Exponential arc planes)**.** An *exponential arc plane of type 1* is the plane $\mathscr{A}(e^s)$ generated by the graph of the exponential function

$$\mathbb{R}^+ \to \mathbb{R} : x \mapsto e^x.$$

An *exponential arc plane of type 2* is the arc plane $\mathscr{A}(e^x, -\mathrm{sgn}(s)e^{sx})$ generated by the graph of the exponential function and the graph of one of the functions

$$\mathbb{R}^+ \to \mathbb{R} : x \mapsto -\mathrm{sgn}(s)e^{sx},$$

where $|s| \geqslant 1$.

2.3 The automorphism group

In this section we recall some facts about the symmetries of toroidal circle planes and their derived $\mathbb{R}^2$-planes. Most of the material here can be found in [PS01] Section 4.4 and Section 2.6, respectively.

An *isomorphism between two toroidal circle planes* is a bijection between the point sets that maps circles to circles, and induces a bijection between the circle sets. In particular, parallel classes are mapped to parallel classes. There are two kinds of isomorphisms between two toroidal circle planes $\mathbb{T}_1$ and $\mathbb{T}_2$. If $(+)$-parallel classes are always taken to $(+)$-parallel classes, an isomorphism must have the form

$$(x, y) \mapsto (\alpha(x), \beta(y)),$$

where α and β are permutations of $\mathbb{S}^1$. In the other kind of isomorphism $(+)$-parallel classes of $\mathbb{T}_1$ are mapped to $(-)$-parallel classes of $\mathbb{T}_2$ and such an isomorphism must have the form

$$(x, y) \mapsto (\alpha(y), \beta(x)),$$

where α and β are permutations of $\mathbb{S}^1$.

An *automorphism of a toroidal circle plane* $\mathbb{T}$ is an isomorphism from $\mathbb{T}$ to itself. With respect to composition, the set of all automorphisms of a toroidal circle plane is an abstract group. We denote this group by $\mathrm{Aut}(\mathbb{T})$. Every automorphism of a toroidal circle plane is continuous and thus a homeomorphism of the torus, cf. [PS01] Theorem 4.4.1.

In the case of flat Minkowski planes, Schenkel [Sch80] obtained the following characterisation of $\mathrm{Aut}(\mathbb{T})$.

Theorem 2.3.1. *The automorphism group of a flat Minkowski plane is a Lie group with respect to the compact-open topology of dimension at most 6.*

For ease of communication, we say a toroidal circle plane has *group dimension* n if its automorphism group has dimension n.

The automorphism group $\mathrm{Aut}(\mathbb{T})$ has two distinguished normal subgroups, the *kernels* T^+ and T^- of the action of $\mathrm{Aut}(\mathbb{T})$ on the set of parallel classes $\mathcal{G}^+$ and $\mathcal{G}^-$, respectively. In other words, the kernel $T^\pm$ of $\mathbb{T}$ consists of all automorphisms of $\mathbb{T}$ that fix every $(\pm)$-parallel class.

If $\mathbb{T}$ is a flat Minkowski plane, then each of these kernels $T^\pm$ is restricted to the following.

Theorem 2.3.2. *The connected component of a kernel $T^\pm$ of a flat Minkowski plane is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, L_2, $\mathrm{SO}(2,\mathbb{R})$, $\mathbb{R}$, or the trivial group $\{id\}$.*

In Theorem 2.3.2, we denote

$$\mathrm{L}_2 = \{x \mapsto ax + b \mid a > 0, b \in \mathbb{R}\}.$$

Up to isomorphisms, L_2 is the only non-commutative, 2-dimensional, connected Lie group (cf. [Sal+95] Proposition 94.40).

In [Sch80], Schenkel also classified flat Minkowski planes whose automorphism group is at least 4 dimensional, or one of whose kernels is 3 dimensional. These results are summarised in the following three theorems.

Theorem 2.3.3. *Let $\mathbb{T}$ be a flat Minkowski plane. If one of the kernels of $\mathbb{T}$ is 3-dimensional, then $\mathbb{T}$ is isomorphic to a plane $\mathcal{M}(f, id)$, where f is an orientation-preserving homeomorphism of $\mathbb{S}^1$.*

Theorem 2.3.4. *A flat Minkowski plane is isomorphic to the classical flat Minkowski plane if and only if $\mathrm{Aut}(\mathbb{T})$ has dimension at least 5.*

Theorem 2.3.5. *Let $\mathbb{T}$ be a flat Minkowski plane. If $\mathrm{Aut}(\mathbb{T})$ has dimension 4, then $\mathbb{T}$ is isomorphic to one of the following planes.*

(i) A nonclassical swapping half plane $\mathcal{M}(f, id)$, where f is a semi-multiplicative homeomorphism of the form $f_{d,s}$, $(d, s) \neq (1, 1)$. This plane admits the 4-dimensional group of automorphisms

$$\{(x, y) \mapsto (rx, \delta(y)) \mid r \in \mathbb{R}^+, \delta \in \mathrm{PSL}(2,\mathbb{R})\}.$$

(ii) A nonclassical generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$, $r_1, s_1, r_2, s_2 \in \mathbb{R}^+$, $(r_1, s_1, r_2, s_2) \neq (1, 1, 1, 1)$. This plane admits the 4-dimensional group of automor-

phisms

$$\{(x,y) \mapsto (rx+a, sy+b) \mid a,b,r,s \in \mathbb{R}, r,s > 0\}.$$

The proofs of Theorems 2.3.2, 2.3.3, 2.3.4, 2.3.5 can be found in [PS01] Proposition 4.4.9, Theorems 4.4.10, 4.4.12, 4.4.15, respectively.

If an automorphism of a toroidal circle plane $\mathbb{T}$ fixes a point p, then it induces an automorphism of the derived $\mathbb{R}^2$-plane $\mathbb{T}_p$, that is, a bijection on the point set $\mathbb{R}^2$ which also maps lines to lines. This simple fact often leads us to the strategy of first finding a subgroup G of $\mathrm{Aut}(\mathbb{T})$ that fixes a point, then using the results of $\mathbb{R}^2$-planes to describe the structure of G.

It is therefore necessary to invest time in getting a grasp of the structure of automorphism groups of flat affine planes and $\mathbb{R}^2$-planes.

The automorphism group of these geometries is a Lie group (cf. [Sal+95] Theorem 32.21). Flat affine planes and $\mathbb{R}^2$-planes with group dimension at least 3 have been classified, cf. [PS01] Theorem 2.6.3 for flat affine planes and Theorem 2.6.4 for proper (non-affine) $\mathbb{R}^2$-planes, and references therein. For our purposes, we only list planes with group dimension at least 4.

Theorem 2.3.6 (Classification $\mathbb{R}^2$-plane with group dimension at least 4)**.** *Let $\mathcal{R}$ be an $\mathbb{R}^2$-plane.*

(i) The maximal possible group dimension of $\mathcal{R}$ is that of the classical flat affine plane. Its group dimension is 6.

(ii) The plane $\mathcal{R}$ has group dimension at least 5 if and only if it is classical.

(iii) The plane $\mathcal{R}$ has group dimension 4 if and only if it is isomorphic to one of the following planes.

(i) A nonclassical radial Moulton plane $\mathscr{M}(s)$.

(ii) The real cylinder plane $\mathscr{C}(\mathbb{R})$.

$\mathbb{R}^2$-planes with 3-dimensional point-transitive automorphism group have been classified by Groh. For our purposes, the following is adapted from [Gro82b] Main Theorem 2.6.

Theorem 2.3.7. *Let $\mathcal{R}$ be an $\mathbb{R}^2$-plane. The automorphism group of $\mathcal{R}$ is point-transitive and at least 3-dimensional if and only if $\mathcal{R}$ is isomorphic to one of the following planes.*

(i) The real hyperbolic plane $\mathscr{H}(\mathbb{R})$.

(ii) A hyperbolic arc plane.

(iii) An exponential arc plane.

(iv) A skew parabola plane.

$\mathbb{R}^2$-planes with 3-dimensional automorphism group fixing precisely a line have also been classified. Stating this classification requires the notion of *pasted planes*, which is not needed in this thesis. Here we only describe the connected component of the automorphism group of such planes abstractly. The full classification can be found in [GLP83] Main Theorem 7.5. References for pasted planes are [Gro81] and [PS01] Subsection 2.7.11. For $d \in \mathbb{R}$, we denote

$$\Phi_d = \{(x, y) \mapsto (ax + b, a^d y + c) \mid a > 0, b, c \in \mathbb{R}\}.$$

Theorem 2.3.8. *Let Σ be a closed connected 3-dimensional group of automorphisms of an $\mathbb{R}^2$-plane $\mathcal{R}$. If Σ fixes precisely one line of $\mathcal{R}$, then either $\Sigma \cong \Phi_d$ for some $d < 0$, or $\Sigma \cong \mathbb{R} \times \mathrm{L}_2$.*

2.4 The Klein-Kroll types of flat Minkowski planes

The material in this section is used in Subsection 6.5.3. Main references for this section are [KK89], [Kle92], [Ste07] and [PS01] Section 4.5.

In Section 2.3 we have seen that flat Minkowski planes (and toroidal circle planes in general) can be classified with respect to the dimension of their automorphism groups. An alternative way to classify flat Minkowski planes is via *central automorphisms*, which are automorphisms that fix at least one point and induce central collineations in the derived plane at that fixed point. In the case of the more general abstract Minkowski planes, a classification with respect to central automorphisms was first carried out by Klein and Kroll [KK89] and Klein [Kle92]. For a Minkowski plane $\mathcal{M}$, there are three types of central automorphisms: q-translations, G-translations, and (p, q)-homotheties.

Let q be a point of $\mathcal{M}$. A *q-translation* is an automorphism of $\mathcal{M}$ that is either the identity or fixes precisely the point q and induces a translation on the derived affine plane $\mathcal{M}_q$. Let C be a circle passing through q. Let $B(q, C)$ denote *the touching pencil with support* q, that is, $B(q, C)$ consists of all circles that touch the circle C at the point q. A $(q, B(q, C))$*-translation* of $\mathcal{M}$ is a q-translation that fixes C (and thus each circle in $B(p, C)$) globally. A group of $(q, B(q, C))$-translations of $\mathcal{M}$ is called $(q, B(q, C))$-*transitive*, if it acts transitively on $C \backslash \{q\}$. A group of q-translations is called q-transitive, if it acts transitively on $\mathcal{P}$ $([q]_+ \cup [q]_-)$. We say that the automorphism group $\mathrm{Aut}(\mathcal{M})$ is $(q, B(q, C))$-transitive or q-transitive if $\mathrm{Aut}(\mathcal{M})$ contains a $(q, B(q, C))$-transitive subgroup of (q,B(q,C))-translations or a q-transitive subgroup of q-translations, respectively. With respect to q-translations, there are seven types of Minkowski planes.

If $\mathcal{T}$ denotes the set of all points q for which $\mathrm{Aut}(\mathcal{M})$ is $(q, B(q, C))$-transitive for some touching pencil $B(q, C)$ with support q, then exactly one of the following statements is valid.

I. $\mathcal{T} = \varnothing$.

II. There is a point q such that $\mathcal{T} = \{q\}$ and there is exactly one touching pencil with support q such that $\mathcal{M}$ is $(q, B(q, C))$-transitive.

III. There is a point q such that $\mathcal{T} = \{q\}$ and $\mathcal{M}$ is q-transitive.

IV. $\mathcal{T}$ consists of the points on a circle.

V. $\mathcal{T}$ consists of the points on a parallel class.

VI. $\mathcal{T} = \mathcal{P}$ and for each point q there is exactly one touching pencil $B(q, C)$ with support q such that $\mathcal{T}$ is $(q, B(q, C))$-transitive.

VII. $\mathcal{T} = \mathcal{P}$ and $\mathcal{M}$ is q-transitive for every point $q \in \mathcal{P}$.

Let G be a parallel class of $\mathcal{M}$. A *G-translation* of $\mathcal{M}$ is an automorphism of $\mathcal{M}$ that is either the identity or fixes precisely the points of G. A group of G-translations of $\mathcal{M}$ is called *G-transitive*, if it acts transitively on each parallel class H of type opposite the type of G without the point of intersection with G. We say that the automorphism group $\mathrm{Aut}(\mathcal{M})$ is G-transitive if it contains a G-transitive subgroup of G-translations. With respect to G-translations, there are six types of Minkowski planes.

If $\mathcal{Z}$ denotes the set of all parallel classes G for which $\mathrm{Aut}(\mathcal{M})$ is G-transitive, then exactly one of the following statements is valid.

A. $\mathcal{Z} = \varnothing$.

B. $|\mathcal{Z}| = 1$.

C. There is a point p such that $\mathcal{Z} = \{[p]_+, [p]_-\}$.

D. $\mathcal{Z}$ consists of all $(+)$-parallel classes or of all $(-)$-parallel classes.

E. $\mathcal{Z}$ consists of all $(+)$-parallel classes plus one $(-)$-parallel class or of all $(-)$-parallel classes plus one $(+)$-parallel class.

F. $\mathcal{Z}$ consists of all $(+)$- and all $(-)$-parallel classes.

Let p and q be two non-parallel points of $\mathcal{M}$. A (p, q)-*homothety* is an automorphism of $\mathcal{M}$ that fixes two non-parallel points p and q and induces a homothety with centre q on the derived plane $\mathcal{M}_p$. A group of (p, q)-homotheties is called (p, q)-*transitive* if it acts transitively on each circle through p and q minus the two points p and q. We say that the automorphism group $\mathrm{Aut}(\mathcal{M})$ is (p, q)-transitive if contains a (p, q)-transitive subgroup of (p, q)-homotheties. With respect to (p, q)-homotheties, there are 23 types of Minkowski planes. These types are numbered from 1 to 23. A full description of these types can be found in Klein [Kle92]. In the following we only list types with planes that appear in this thesis (particularly Theorems 2.4.2 and 6.5.9).

Let $\mathcal{H}$ be the set of all unordered pairs of points $\{p, q\}$ for which the Minkowski plane $\mathcal{M}$ is (p, q)-transitive. Then $\mathcal{M}$ is of type i with respect to (p, q)-homotheties if it satisfies the following statement i.

1. $\mathcal{H} = \varnothing$.

18. There is a point p such that $\mathcal{H} = \{\{p, q\} \mid q \nparallel p\}$.

19. There is a point p such that

$$\mathcal{H} = \{\{p, q\} \mid q \nparallel p\} \cup \{\{r, s\} \mid r \in [p]_+ \backslash \{p\}, s \in [p]_- \backslash \{p\}\}.$$

23. $\mathcal{H}$ consists of all pairs of non-parallel points.

The automorphism group $\mathrm{Aut}(\mathcal{M})$ can admit different types of central automorphisms. We call $\mathcal{M}$ has Klein-Kroll type $X.Y.Z$ if its automorphism group is of type X with respect to q-translations, is of type Y with respect to G-translations, and is of type Z with respect to (p, q)-homotheties.

In the case of flat Minkowski planes, possible Klein-Kroll types were determined by Steinke [Ste07] as follows.

Theorem 2.4.1. *A flat Minkowski plane has Klein-Kroll type*

I. A.1, A.2, A.3, B.1, B.10, B.11, D.1,
II. A.1, A.15,
III. C.1, C.18, C.19,
IV. A.1, *or*
VII. F.23.

For each of these 14 types, except type II.A.15, examples are given in [Ste07].

In the same paper, Steinke also characterised some families of flat Minkowski planes. The following result is adapted from [Ste07] Proposition 5.9. By a *normalised* function f we mean f satisfies $f(1) = 1$.

Theorem 2.4.2. *A flat Minkowski plane of Klein-Kroll type*

VII.F.23 *is isomorphic to the classical flat Minkowski plane;*

III.C.19 *is isomorphic to a proper Hartmann plane* $\mathcal{M}_{GH}(r, 1; r, 1), r \neq 1$*;*

III.C.18 *is isomorphic to an Artzy-Groh plane* $\mathcal{M}_{AG}(f, g)$ *where* f *and* g *are normalised and odd except when* $f = g$ *is inversely semi-multiplicative.*

Chapter 3

Toroidal circle planes as topological geometries

In the literature (cf. [Sal67b], [Sal+95] Chapter 3), topological geometries are incidence geometries endowed with topologies such that geometric operations are continuous. Particularly to our setting, we will define topological toroidal circle planes in Definition 3.4.1.

In Section 3.1, we define the relevant geometric operations on toroidal circle planes. We will also introduce notations and the metric we use on the point set. In Section 3.2, we define different topologies on the circle set and show that these topologies coincide with a unique topology called the *topology* H (cf. Definition 3.2.1 and Theorem 3.2.7).

In Section 3.3, we study some properties of geometric operations (defined in Section 3.1) with respect to the topology H defined in Section 3.2. Based on these properties, in Section 3.4, we show that a toroidal circle plane is topological if and only if its circle set is equipped with the topology H. We will also prove that the circle set of a topological toroidal circle plane is homeomorphic to $\mathrm{PGL}(2, \mathbb{R})$.

3.1 Preliminaries

Naturally, the point set $\mathcal{P}$ of a toroidal circle plane $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$ is equipped with the usual Euclidean metric. However, in this chapter, we use the maximum metric $\mathbf{d}$ for the point set instead, which is equivalent to the Euclidean metric. The metric $\mathbf{d}$ between two points $p := (x_1, x_2), q := (y_1, y_2)$ is defined as

$$\mathbf{d}(p, q) := \max(|x_1 - x_2|, |y_1 - y_2|).$$

For ease of communication, we introduce a few notations for subspaces. First, let $\mathcal{H}(X)$ be the set of all compact subsets of a space X. The set $\mathcal{H}(X)$ is a metric space if we equip it with the Hausdorff distance, which will be covered in the next section.

Let $\widetilde{\mathcal{P}^3}$ be the subspace of the product space $\mathcal{P}^3$ consisting of all triples of pairwise nonparallel points. Let $\mathcal{P}^{1,2} = \{\{x, y\} \mid x, y \in \mathcal{P}\}$ be a subspace of $\mathcal{H}(\mathcal{P} \times \mathcal{P})$.

Let $\mathcal{C}^{2*}$ be the subspace of the product space $\mathcal{C}^2$ which consists of all pairs of distinct circles that have non-empty intersection. Let $C^{1*} \subset C^{2*}$ be the subspace of pairs of touching circles.

There are five geometric operations on toroidal circle planes defined as follows.

Definition 3.1.1 (Geometric operations)**.** Let $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$ be a toroidal circle plane.

(i) *Joining* $\alpha : \widetilde{\mathcal{P}^3} \to \mathcal{C}$ is defined by $\alpha(x, y, z)$ being the unique circle going through three pairwise nonparallel points x, y, z.

(ii) *Parallel Intersection* $\pi : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is defined as $\pi : (x, y) \mapsto [x]_+ \cap [y]_-$.

(iii) *Parallel Projection* $\pi^{+(-)} : \mathcal{P} \times \mathcal{C} \to P$ is defined as $\pi^{+(-)} : (x, C) \mapsto [x]_{+(-)} \cap C$.

(iv) *Intersection* $\gamma : \mathcal{C}^{2*} \to \mathcal{P}^{1,2}$ is defined as $\gamma(C, D) = C \cap D$.

(v) *Touching* $\beta : C^{1*} \to \mathcal{P}$ is defined as $\beta(C, D) = C \cap D$.

The geometric operations on the derived $\mathbb{R}^2$-plane $\mathbb{T}_p$ at a point p are inherited from those on $\mathbb{T}$.

3.2 The topology on the circle space

Recall that $\mathcal{H}(\mathcal{P})$ is the set of all compact subsets of $\mathcal{P}$. In view of circles as compact subsets of $\mathcal{P}$, a candidate for a topology on the circle set is the one induced from the Hausdorff metric on $\mathcal{H}(\mathcal{P})$, which is described as follows.

For every pair of circles $C, D \in \mathcal{C}$, we define their *Hausdorff distance* by

$$\mathbf{h}(C, D) = \max\{\sup_{y \in D} \inf_{x \in C} \mathbf{d}(x, y), \sup_{x \in C} \inf_{y \in D} \mathbf{d}(x, y)\}.$$

Since each circle is compact, $\mathbf{h}$ is a metric on $\mathcal{C}$. We then have

Definition 3.2.1. The *topology* H is the topology on $\mathcal{C}$ induced by the Hausdorff metric $\mathbf{h}$.

In the case of $\mathbb{R}^2$-planes, the topology H (defined appropriately on the line set of the $\mathbb{R}^2$-plane) is the unique topology such that the geometric operations are continuous (cf. [Sal+95] Theorems 31.19 and 31.22). To adapt the proofs from the case of $\mathbb{R}^2$-planes to our situation, we first describe other topologies on the circle set $\mathcal{C}$, all of which will be used in the next section. The main result of this section is Theorem 3.2.7.

Definition 3.2.2. The *open join topology* OJ is the topology whose basis consists of sets of the form

$$\alpha(O_1, O_2, O_3) := \{\alpha(p_1, p_2, p_3) \mid p_i \in O_i\},$$

where O_1, O_2, O_3 are nonempty open subsets of $\mathcal{P}$ such that $p_i \in O_i$, for $i = 1, 2, 3$, are pairwise nonparallel.

Definition 3.2.3. The *open meet topology* OM is the topology whose subbasis consists of sets of the form

$$M_O := \{C \in \mathcal{C} \mid C \cap O \neq \varnothing\},$$

where O is a nonempty open set of $\mathcal{P}$.

Definition 3.2.4. The *interval join topology* IJ is the topology whose subbasis consists of sets of the form

$$\alpha(I_1, I_2, I_3) := \{\alpha(p_1, p_2, p_3) \mid p_i \in I_i\},$$

where I_1, I_2, I_3 are open intervals on distinct (+)-parallel classes.

Definition 3.2.5. The *final topology* F is the finest topology such that Joining α is continuous.

Definition 3.2.6. The *parallel topology* $P^{+(-)}$ is the topology induced from the metric $\mathbf{p}^{+(-)}$ defined by

$$\mathbf{p}^{+(-)}(C_1, C_2) = \sup\{\mathbf{d}(p_1, p_2) : p_1 \parallel_{+(-)} p_2; p_i \in C_i, i = 1, 2\}.$$

For the subsequent proof, it makes no difference whether we work with P^+ or P^-. Therefore our arguments will show that both these topologies equal other topologies under consideration. From now on we simply write P to denote any fixed choice of either P^+ or P^-.

The following theorem is particularly convenient when we consider the properties of geometric operations in Section 3.3.

Theorem 3.2.7. *The topologies OJ, OM, IJ, H, F, P coincide.*

The proof of the theorem is presented in a series of lemmas, as we show $OJ \subseteq OM \subseteq IJ \subseteq H = P \subseteq F \subseteq OJ$.

Lemma 3.2.8 ($OJ \subseteq OM$)**.** *Every OJ-open set is OM-open.*

Proof. Let O_1, O_2, O_3 be nonempty open sets. Then the set $\alpha(O_1, O_2, O_3)$ is precisely the intersection of M_{O_1}, M_{O_2} and M_{O_3}. This verifies the claim. □

Lemma 3.2.9 ($OM \subseteq IJ$)**.** *For each $C \in \mathcal{C}$ and each subbasis element M_O of OM containing C, there is a subbasis element B of IJ such that $C \in B \subset M_O$.*

Proof. Let M_O be a subbasis element of OM and let C be a circle in M_O. Also, let p be a point in $C \cap O$. Then there is an open interval I on $[p]_+$ contained in O. Therefore $B := \alpha(I, P_1, P_2) \subseteq M_O$, where P_1, P_2 are distinct $(+)$-parallel classes different from $[p]_+$. This implies $OM \subseteq IJ$. □

Lemma 3.2.10 ($IJ \subseteq H$)**.** *For each circle $C \in \mathcal{C}$ and each circle set of the form*

$$B := \bigcap_{i=1}^{n} \alpha(I_{i_1}, I_{i_2}, I_{i_3}),$$

where $C \in B$, there is a basis element B' of H such that $C \in B' \subset B$.

Proof. Let $C \in B$ be a circle and $c_{i_k} := C \cap I_{i_k}$, for $1 \leqslant i \leqslant n, 1 \leqslant k \leqslant 3$. Then there are open intervals $(c_{i_k} - \varepsilon_{i_k}, c_{i_k} + \varepsilon_{i_k})$ contained in each I_i. If we let $\varepsilon_i := \min\{\varepsilon_{i_1}, \varepsilon_{i_2}, \varepsilon_{i_3}\}$ and $\varepsilon := \min_i \varepsilon_i$, then $B' := B_{\mathbf{h}}(C, \varepsilon)$ is contained in B. □

Let $\liminf_n C_n$ be the set of all limits of convergent sequences (p_n) with $p_n \in C_n$ and $\limsup_n C_n$ be the set of all accumulation points of sequences (q_n) with $q_n \in C_n$. A sequence $(C_n) \in \mathcal{C}$ converges to C with respect to the Hausdorff metric if and only if $\liminf_n C_n = C = \limsup_n C_n$.

The following is adapted from [Sch80] Satz 4.4.

Lemma 3.2.11 ($H \subseteq F$)**.** *If the circle set $\mathcal{C}$ is equipped with the topology H, then Joining α is sequentially continuous.*

Proof. For an illutration, compare Figure 3.1.

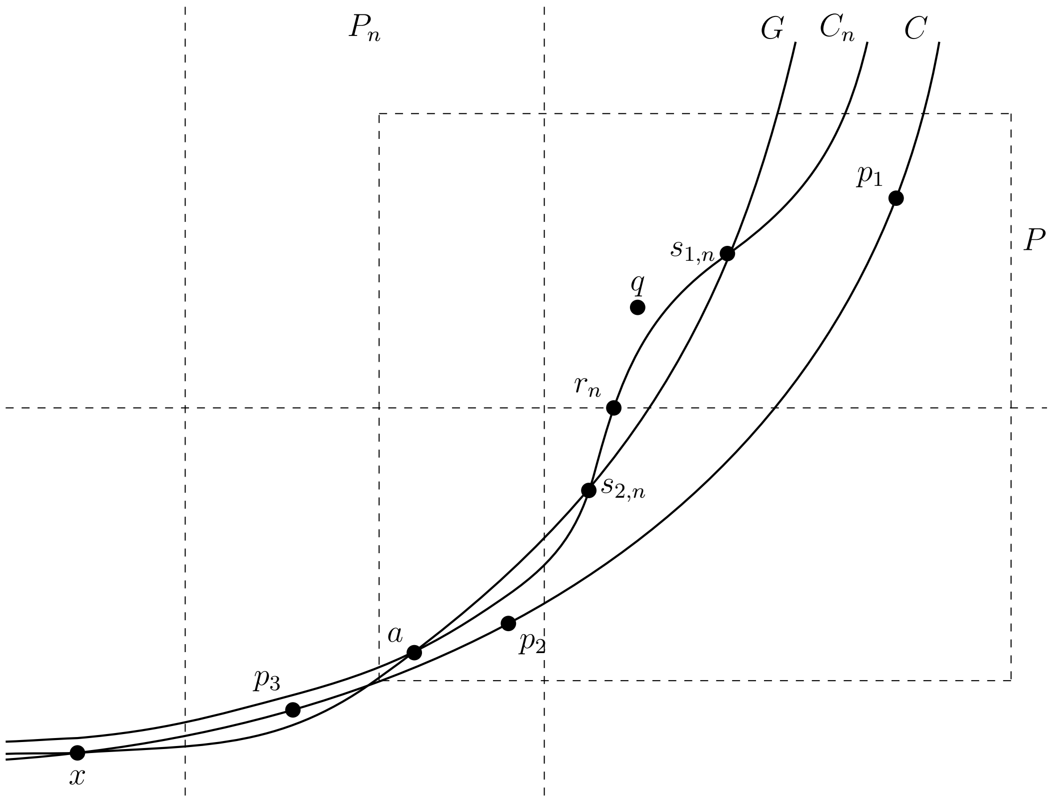

Figure 3.1

Let $(p_{i,n})$ be sequences convergent to p_i, which are pairwise nonparallel, for $i = 1, 2, 3$. Let $C_n := \alpha(p_{1,n}, p_{2,n}, p_{3,n})$ and $C := \alpha(p_1, p_2, p_3)$. We show $(C_n) \to C$ with respect to the Hausdorff metric, that is, $\liminf_n C_n = C = \limsup_n C_n$.

It is clear that $\liminf_n C_n \subseteq \limsup_n C_n$. It is then sufficient to show

$$\limsup_n C_n \subseteq C \subseteq \liminf_n C_n.$$

1) $\limsup_n C_n \subseteq C$. We assume that there is a sequence of points $q_n \in C_n$ which has an accumulation point $q \notin C$. Without loss of generality, assume q_n lies on C_n between $p_{1,n}, p_{2,n}$ and the point p_2 is between p_1 and p_3 on C.

Since each circle intersects each parallel class only once, because of connectivity, q must belong to the intersection of the components $\mathcal{P}\backslash([p_1]_+ \cup [p_2]_+)$ and $\mathcal{P}\backslash([p_1]_- \cup [p_2]_-)$ not containing p_3.

Let x be a point on C which is separated from p_2 by p_1 and p_3. In $\mathbb{T}_x$, let G be a straight line that separates p_1, p_2 from q, p_3.

Let P be the intersection of a closed $(+)$-parallel and a $(-)$-parallel strip, both of which contain neither x nor p_3, but contain neighbourhoods of p_1, p_2, and q. We then have

(i) G separates p_1, p_2 from q in P;

(ii) For n sufficiently large, the restriction of C_n in P is connected because the circle C_n meets each parallel class exactly once, otherwise there are no points in the vicinity of p_3.

Therefore, for n sufficiently large, each circle C_n intersects G at $s_{i,n}$, for $i = 1, 2$, such that $s_{i,n}$ is between $p_{i,n}$ and q_n.

Let r_n be a point on C_n that separates $s_{1,n}$ and $s_{2,n}$ in P. Then the intersection of $[r_n]_-$ and G is also in P. We then choose closed $(+)$-parallel strips P_n so that $p_{2,n}$ and $p_{3,n}$ lie inside of P_n, but P_n does not contain any of the points $p_{1,n}, s_{i,n}, q_n$ or x. This is possible since $\mathbb{S}_1$ is dense.

Let R be the region bounded by G, $[r_n]_-$ and P_n. The boundary ∂R of R is a Jordan curve which cuts C_n in at least two points a, b. Since $[r_n]_-$ and C_n has no intersection within P_n and $p_{3,n}$ lies outside of the region yet inside P_n, one of the intersection points, say a, must lies on G. However, this is a contradiction to the uniqueness of the circle G, since $s_{1,n}$ and $s_{2,n}$ are also on both G and C_n.

2) $C \subseteq \liminf_n C_n$. Let $p \in C$ and let $p_n := \pi^-(p, C_n)$. From the previous part of the proof, $(p_n) \to \pi^-(p, C) = p$. Hence $p \in \liminf_n C_n$. $\square$

Lemma 3.2.12 ($F \subseteq OJ$)**.** *Every F-open set is OJ-open.*

Proof. Let O be an F-open set. Since Joining α is continuous with respect to F, the preimage $\alpha^{-1}(O)$ is open in the product topology of $\widetilde{\mathcal{P}^3}$. Then $\alpha^{-1}(O)$ is the union of sets of the form (O_1, O_2, O_3), where O_1, O_2, O_3 are open subsets of $\mathcal{P}$ such that $p_i \in O_i$, for $i = 1, 2, 3$, are pairwise nonparallel. The set O is then the union of the corresponding sets $\alpha(O_1, O_2, O_3)$, which implies O is open in the topology OJ. $\square$

The following is also adapted from [Sch80] Satz 3.4. It is clear that $H \subseteq P$. We now show the converse.

Lemma 3.2.13 ($P \subseteq H$)**.** *For every $C \in \mathcal{C}$ and every basis element B of P containing C, there is a basis element B' of H such that $C \in B' \subset B$.*

Proof. Let $C \in \mathcal{C}$ be a circle and $B_{\mathbf{p}}(C, \varepsilon)$ an ε-neighbourhood of C with respect to the metric $\mathbf{p}$. Because of the continuity of $\pi^{+(-)}$ with respect to H, there exists at each point $p \in C$ an open neighbourhood (U_p, V_p) of (p, C) with $U_p \subseteq U_{\varepsilon/2}(p)$ and $\pi^{+(-)}(U_p, V_p) \subseteq U_{\varepsilon/2}(p)$, where $U_{\varepsilon/2}(p)$ is the open ball of p radius $\varepsilon/2$ with respect to d. The union of all

U_p, with $p \in C$, forms an open cover of C. Since C is compact, there exists a finite cover $\bigcup_{i=1}^{n} U_{p_i}$.

We claim that $V := \bigcap_{i=1}^{n} V_{p_i}$ is a neighbourhood of C and $V \subseteq B_{\mathbf{p}}(C, \varepsilon)$. Indeed, let $C' \in V$ and $p' \in C'$, then we have $p := \pi^{+(-)}(p', C) \in U_{p_i}$ for some i; $1 \leqslant i \leqslant n$. Therefore $d(p, p_i) < \varepsilon/2$. On the other hand, because $C' \in V_{p_i}$ it follows that $d(p', p_i) = d(\pi^{+(-)}(p, C'), p_i) < \varepsilon/2$ also $d(p', p) < \varepsilon$. □

Theorem 3.2.7 now follows from Lemmas 3.2.8, 3.2.9, 3.2.10, 3.2.11, 3.2.12 and 3.2.13.

3.3 Properties of geometric operations

In this section, we study topological properties of geometric operations (cf. Definition 3.1.1) with respect to the topology H defined at the beginning of Section 3.2.

We have already established the properties of Joining α from the previous section.

Lemma 3.3.1. *Joining α is open and continuous with respect to the topology H.*

Proof. By definition, Joining α is open with respect to the topology OJ and is continuous with respect to the topology F. The claim now follows from Theorem 3.2.7. □

For the continuity of operations $\pi^{+(-)}$ and π, we need a general result in metric spaces as a helping lemma. It is mentioned in [Gro70] 3.10 but without a proof. We proceed by verifying this lemma.

Lemma 3.3.2. *In a compact metric space (X,d) let A, B be two compact sets and let U be an open set such that $A \cap B \subset U$. Then there are neighbourhoods V of A and W of B with respect to the Hausdorff metric induced by d such that for all closed subsets $A' \in V$, $B' \in W$ one has $A' \cap B' \subset U$.*

Proof. Assume by way of contradiction that there are no neighbourhoods V of A and W of B such that if $A' \in V$ and $B' \in W$, then $A' \cap B' \subset U$. Then there are closed sets A_n and B_n such that $\mathbf{h}(A_n, A), \mathbf{h}(B_n, B) < 1/n$ and $A_n \cap B_n \not\subset U$. Thus there is $c_n \in A_n \cap B_n$ such that $c_n \notin U$. Since X is compact, the sequence (c_n) has an accumulation point c, and $c \notin U$ since U is open. However, $\mathbf{h}(A_n, A) < 1/n$ and $c_n \in A_n$ implies that $\mathbf{d}(c_n, A) < 1/n$ and therefore $\mathbf{d}(c, A) = 0$. Hence $c \in A$ because A is closed. One similarly obtains that $c \in B$. But then $c \in A \cap B \subset U$, a contradiction. □

It is then straightforward to verify that

Lemma 3.3.3. *Parallel Projection $\pi^{+(-)}$ and Parallel Intersection π are continuous with respect to the topology H.*

We proceed with another property of the operation $\pi^{+(-)}$.

Lemma 3.3.4. *Parallel Projection $\pi^{+(-)}$ is open with respect to the topology H.*

Proof. We show that Parallel Projection $\pi^{+(-)}$ is open with respect to the topology OM. For an illustration, compare Figure 3.2.

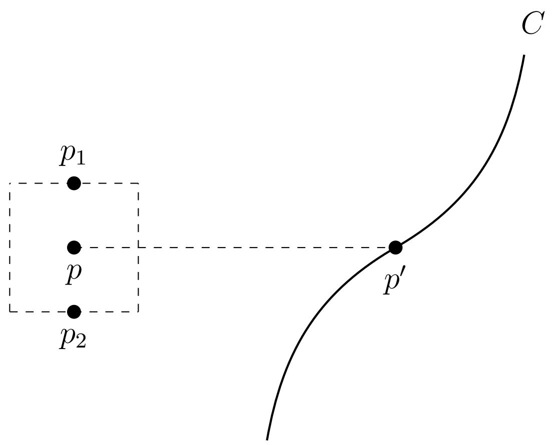

Figure 3.2

It is sufficient to show that π^- is open. Fix (p, C) and let $p' := \pi^-(p, C)$. Let $O := B_{\mathbf{d}}(p, \varepsilon_1), O' := B_{\mathbf{d}}(p', \varepsilon_2)$. Then $(O, M_{O'})$ is a basis element of the product topology on $\mathcal{P} \times \mathcal{C}$ and it contains (p, C).

Let ∂O be the boundary of O and let $\partial O \cap [p]_+$ be the points p_1 and p_2. Then $\pi^-(O, M_{O'})$ is the $(-)$-parallel strip that contains p and whose boundary is $[p_1]_- \cup [p_2]_-$, which is open. $\square$

We now turn our attention to the operation γ. Let C_1, C_2 be two circles that have the point p in common. We say that C_1 *touches* C_2 *topologically* at the point p if there are a neighbourhood U of p and a homeomorphism between U and $\mathbb{R}^2$ that maps $C_1 \cap U$ to the x-axis and $(C_2 \cap U) \backslash \{p\}$ to a subset of the upper half-plane. We say that C_1 *intersects* C_2 *transversally at* p if there are a neighbourhood U of p and a homeomorphism between U and $\mathbb{R}^2$ that maps $C_1 \cap U$ to the x-axis and $C_2 \cap U$ to the y-axis (cf. [PS01] p. 22). We have

Lemma 3.3.5 (cf. [PS01] Theorem 4.2.2)**.** *Let $C_1, C_2 \in \mathcal{C}$ be two circles of a toroidal circle plane. Then $|C_1 \cap C_2| \leqslant 2$.*

(i) If $|C_1 \cap C_2| = 1$, then the circles touch topologically.

(ii) If $|C_1 \cap C_2| = 2$, then the circles intersect transversally at the intersection points.

We recall that if $|C_1 \cap C_2| = 1$, then we say C_1 touches C_2 (combinatorially). This means, if C_1 and C_2 are circles of a toroidal circle plane and C_1 touches C_2, then C_1 touches C_2 topologically.

In the case of transversal intersection, the technique used in the proof of continuity of parallel projection is not sufficient to ensure that the two sequences of intersections converge to proper limits. The following example demonstrates that Lemma 3.3.2 does not apply for the continuity of γ. Recall that $\mathcal{C}^{2*}$ is the subspace of $\mathcal{C}^2$ which consists of all pairs of distinct circles that have non-empty intersection, and $\mathcal{P}^{1,2} = \{\{x, y\} \mid x, y \in \mathcal{P}\}$ a subspace of $\mathcal{H}(\mathcal{P}^2)$. The topology on both $\mathcal{C}^{2*}$ and $\mathcal{P}^{1,2}$ is the induced topology H.

Example 3.3.6. Close circles in $\mathcal{C}^{2*}$ need not have close intersections in $\mathcal{P}^{1,2}$.

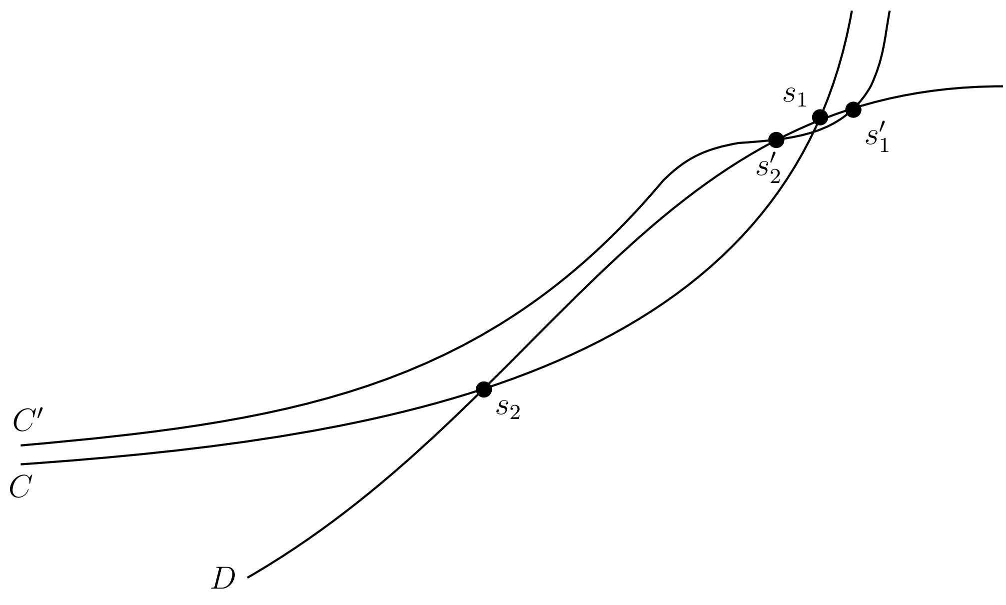

Figure 3.3

As illustrated in Figure 3.3, we have two circles C and D intersecting at s_1 and s_2. The circle C' is close to C in the Hausdorff metric but intersects D at s_1', s_2', which are in the neighbourhood of s_1 only.

This leads to the desire of constructing a neighbourhood of the pairs C and D whose members still have intersections close to $\{s_1, s_2\}$ in the Hausdorff sense.

Lemma 3.3.7. *Let $\gamma(C, D) = \{s_1, s_2\}$ and $s_1 \neq s_2$, that is, C and D intersect transversally. Let $O := B_{\mathbf{d}}(s_1, \varepsilon)$. Then there exists $\delta > 0$ such that if $C' \in B_{\mathbf{p}}(C, \delta)$ and $D' \in B_{\mathbf{p}}(D, \delta)$, then $(C' \cap O) \cap (D' \cap O) \neq \varnothing$.*

Proof. For an illustration, compare Figure 3.4.

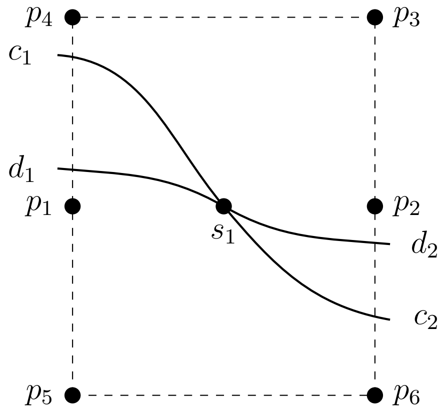

Figure 3.4

Without loss of generality, we restrict O to be a rectangle inside $B_{\mathbf{d}}(s_1, \varepsilon)$ which does not contain s_2. Furthermore, for $i = 1, 2$, let $p_i = \partial O \cap [s_1]_-$, and let c_i, d_i be the intersections of ∂O with C and D, respectively. We assume $c_i, d_i \in [p_i]_+$.

Let x be a point that does not lie on any circle of $B_{\mathbf{p}}(C, \varepsilon)$ or $B_{\mathbf{p}}(D, \varepsilon)$. In the derived plane $\mathbb{T}_x$, we coordinatize $s_1 = (0, 0)$ and the rectangle O whose sides of length $2\epsilon_1, 2\epsilon_2$ by letting $p_3 = (\varepsilon_1, \varepsilon_2), p_4 = (-\varepsilon_1, \varepsilon_2), p_5 = (-\varepsilon_1, -\varepsilon_2), p_6 = (\varepsilon_1, -\varepsilon_2)$. It follows that $p_1 = (-\varepsilon_1, 0), p_2 = (\varepsilon_1, 0)$.

Let $f_K : \mathbb{R} \to \mathbb{R}$ be the homeomorphism whose graph is the restriction of circle K on $\mathbb{T}_x$. Then $c_i = f_C(p_i)$ and $d_i = f_D(p_i)$.

By Lemma 3.3.5, we then assume $f_C(p_1) - f_D(p_1) > 0$ and $f_C(p_2) - f_D(p_2) < 0$.

Let

$$\delta = \min \left\{ \frac{f_C(p_1) - f_D(p_1)}{2}, \frac{f_D(p_2) - f_C(p_2)}{2}, \min_{\substack{i=1,2 \\ j=3,4,5,6}} \mathbf{d}(c_i, p_j), \min_{\substack{i=1,2 \\ j=3,4,5,6}} \mathbf{d}(d_i, p_j) \right\}.$$

The last two conditions ensure that for $C' \in B_{\mathbf{p}}(C, \delta)$ and $D' \in B_{\mathbf{p}}(D, \delta)$, we still have $f_{C'}(p_1), f_{D'}(p_1) \in \partial O$. From the first condition, we have

$$|f_{C'}(p_1) - f_C(p_1)| < \frac{f_C(p_1) - f_D(p_1)}{2},$$

and

$$|f_{D'}(p_1) - f_D(p_1)| < \frac{f_C(p_1) - f_D(p_1)}{2}.$$

Thus

$$f_{C'}(p_1) - f_{D'}(p_1) > f_C(p_1) - \frac{f_C(p_1) - f_D(p_1)}{2} - f_D(p_1) - \frac{f_C(p_1) - f_D(p_1)}{2} = 0.$$

In a similar fashion, for p_2 we obtain

$$f_{C'}(p_2) - f_{D'}(p_2) < 0.$$

Since $f_{C'}$ and $f_{D'}$ are continuous, so is the difference $f_{C'} - f_{D'}$. By the IVT, there exists $p \in (p_1, p_2) \subset O$ such that $f_{C'}(p) - f_{D'}(p) = 0$, which shows that C' intersects D' at least once inside O. □

We now verify

Lemma 3.3.8. *Intersection γ is continuous with respect to the topology H.*

Proof. We show that Intersection γ is continuous with respect to the topology P. Let (C_1, C_2) be a pair of circles in $\mathcal{C}^{2*}$. We consider 2 cases.

Case 1: $|C_1 \cap C_2| = 1$. This case is a direct consequence of Lemma 3.3.2.

Case 2: $|C_1 \cap C_2| = 2$. Let O_i be **d**-open ε-square containing s_i, for $i = 1, 2$. From Lemma 3.3.7, there are $\delta_1, \delta_2 > 0$ such that if $C' \in B_{\mathbf{p}}(C, \delta_i)$ and $D' \in B_{\mathbf{p}}(D, \delta_i)$, then $(C' \cap O_i) \cap (D' \cap O_i) \neq \varnothing$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $B_{\mathbf{p}}(C, \delta)$ and $B_{\mathbf{p}}(D, \delta)$ are still neighbourhoods of C and D respectively and they preserve the property above locally at each O_i. This proves the continuity of γ. □

Since $\mathcal{P}$ is homeomorphic to a closed subset of $\mathcal{P}^{1,2}$, we also obtain the following as a corollary of Lemma 3.3.8.

Lemma 3.3.9. *Touching β is continuous with respect to the topology H.*

3.4 Toroidal circle planes are topological geometries

Definition 2.1.1 of toroidal circle planes does not refer to any topology on the circle set. During the last two sections, we examined the topology H on the circle set and how it interacts with the geometric operations. In this section, we justify that the topology H is the only topology on the circle set that turns toroidal circle planes into topological geometries.

Definition 3.4.1. A toroidal circle plane $\mathbb{T}$ is *topological* if its circle set carries a topology such that the geometric operations $\alpha, \pi, \pi^{+(-)}, \gamma, \beta$ are continuous on their domains of definition.

We have verified that the geometric operations $\alpha, \pi, \pi^{+(-)}, \gamma, \beta$ are continuous with respect to the topology H (cf. Lemmas 3.3.1, 3.3.3, 3.3.8, and 3.3.9). Thus, we have

Lemma 3.4.2. *If $\mathcal{C}$ is equipped with the topology H, then $\mathbb{T}$ is topological.*

A particularly interesting question is whether the converse direction holds true. We show that the answer to this question is affirmative in the following lemmas.

Let (x_1, x_2, x_3) be a fixed triple in $\widetilde{\mathcal{P}^3}$ and let

$$([x_1]_+, [x_2]_+, [x_3]_+) = \{(x_1', x_2', x_3') \mid x_i' \parallel_+ x_i\}.$$

Let $\widetilde{[\mathcal{P}]^3} = \widetilde{\mathcal{P}^3} \cap ([x_1]_+, [x_2]_+, [x_3]_+)$. We equip $\widetilde{[\mathcal{P}]^3}$ with the topology inherited from $\widetilde{\mathcal{P}^3}$.

Lemma 3.4.3. *If $\mathbb{T}$ is topological, then the restriction $\widetilde{\alpha}$: $\widetilde{[\mathcal{P}]^3} \to \mathcal{C}$ is a homeomorphism.*

Proof. It is clear that $\widetilde{\alpha}$ is bijective from the Axiom of Joining and the construction of $\widetilde{[\mathcal{P}]^3}$. From Definition 3.4.1, we readily have that $\widetilde{\alpha}$ is continuous with respect to the subspace topology of $\widetilde{\mathcal{P}^3}$.

The projection of $\widetilde{\alpha}^{-1}$ onto each of $[x_i]_+$ is precisely $\pi^-(C, [x_i]_+)$, which is continuous; therefore $\widetilde{\alpha}^{-1}$ is also continuous. $\square$

Hence, on $\mathcal{C}$, the topology IJ is the coarsest topology such that $\widetilde{\alpha}$ is open, and, by definition, the topology F is the finest topology such that $\widetilde{\alpha}$ is continuous. By Theorem 3.2.7, we have the following.

Lemma 3.4.4. *The topology H is the unique topology that renders the circle space $\mathcal{C}$ homeomorphic to $\widetilde{[\mathcal{P}]^3}$.*

From Lemmas 3.4.2, 3.4.3 and 3.4.4, we obtain the main result of this chapter.

Theorem 3.4.5. *Let $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}_+, \mathcal{G}_-)$ be a toroidal circle plane. Then $\mathbb{T}$ is topological if and only if $\mathcal{C}$ is equipped with the topology H.*

More can be said about the circle set of a topological toroidal circle plane. We have the following.

Lemma 3.4.6. *If a toroidal circle plane* $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}_+, \mathcal{G}_-)$ *is topological, then its circle set* $\mathcal{C}$ *is homeomorphic to* $\mathrm{PGL}(2, \mathbb{R})$.

Proof. By Lemma 3.4.3, $\mathcal{C}$ is homeomorphic to $\widetilde{[\mathcal{P}]^3}$. To prove the lemma, we prove that the evaluation mapping $e : \mathrm{PGL}(2, \mathbb{R}) \to \widetilde{[\mathcal{P}]^3}$ defined as $e : \varphi \mapsto (\varphi(x_1), \varphi(x_2), \varphi(x_3))$ is a homeomorphism.

Every triple of points (x_1', x_2', x_3') in $\widetilde{[\mathcal{P}]^3}$ can also be uniquely determined by the corresponding triple $(x_1', [x_1]_+ \cap [x_2']_-, [x_1]_+ \cap [x_3']_-)$ on the parallel class $[x_1]_+$. Since $\mathrm{PGL}(2, \mathbb{R})$ acts sharply 3-transitive on $\mathbb{S}_1$ and thus on $[x_1]_+$, it follows that e is bijective.

Since the topology of $\mathrm{PGL}(2, \mathbb{R})$ is the compact-open topology, the evaluation mapping is automatically continuous, cf. [Mun75] p. 287.

Finally, e is open by [Sal+95] Proposition 96.8. □

Chapter 4

On the automorphism group of a toroidal circle plane

Our main objective in this chapter is to prove the following theorem.

Theorem 4.0.1. *The automorphism group of a toroidal circle plane is a Lie group with respect to the compact-open topology and has dimension at most 6.*

We will assume the circle space $\mathcal{C}$ is equipped with the topology H (cf. Section 3.2) so that the continuity of geometric operations will be available as our primary machinery of use (cf. Theorem 3.4.5). Additional preliminaries will be introduced in Section 4.1. We then discuss the topological setting of the automorphism group in Section 4.2. Finally, Section 4.3 is where the proof of Theorem 4.0.1 is presented.

4.1 Preliminaries

The first result we need from the literature is concerned with the collinearity of points on derived $\mathbb{R}^2$-planes.

Lemma 4.1.1 (cf. [Sal+95] Proposition 31.12)**.** *In $\mathbb{R}^2$-planes, collinearity and the order of (collinear) point triples are preserved under limits, in the following sense:*

(a) If the point sequences $(a_n), (b_n), (c_n)$ have mutually distinct limits a, b, c and if a_n, b_n, c_n are collinear for infinitely many $n \in \mathbb{N}$, then a, b, c are collinear as well.

(b) If, in addition, $b_n \in (a_n, c_n)$ for infinitely many $i \in \mathbb{N}$, then $b \in (a, c)$.

The next lemma is an application of Hilbert's Fifth problem and special properties of surfaces. It implies that, to show the automorphism group $\mathrm{Aut}(\mathbb{T})$ is a Lie group, we have to show it is locally compact.

Lemma 4.1.2 (cf. [PS01] Theorem A.2.3.5, also cf. Lemma A.3.2)**.** *If G is a locally compact effective transformation group on a surface, then G is a Lie group.*

Toroidal circle planes satisfy the K4 coherence condition for flat Minkowski planes introduced by Schenkel [Sch80] 3.8. In particular, we have the following property which is a special case of the aforementioned K4 condition.

Lemma 4.1.3 (K4 coherence condition, special case)**.** *Let $(C_n) \in \mathcal{C}$ be a sequence of circles and $(p_{i,n}) \to p_i, i = 1, 2, 3$ be three converging sequences of points such that $p_{i,n} \in C_n$, $p_1 \parallel_+ p_2$ and $p_3 \notin [p_1]_+$.*

Suppose $(p_n) \to p \subseteq \mathcal{P}$ such that p is not parallel to any p_i. Then $(\pi^-(p_n, C_n)) \to \pi(p_1, p)$ and $(\pi^+(p_n, C_n)) \to \pi(p, p_3)$.

Proof. For an illustration, compare Figure 4.1. We show that $\pi^-(p_n, C_n) \to \pi(p_1, p)$, the remaining case is analogous.

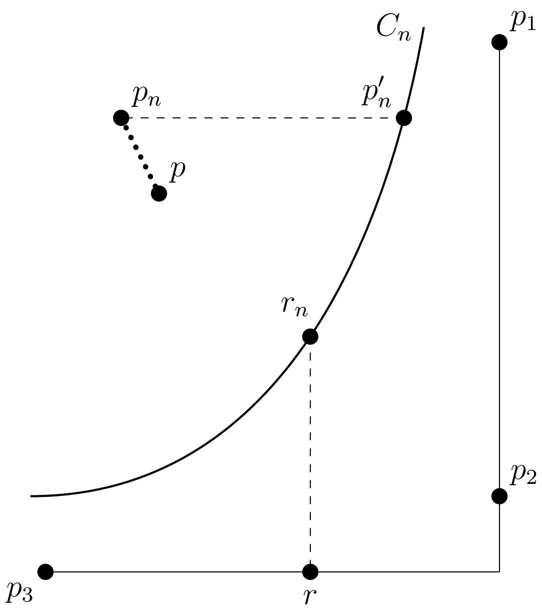

Figure 4.1

For convenience, let $p'_n := \pi^-(p_n, C_n)$. Since $p_1 \parallel_+ p_2$, (C_n) does not converge in $\mathcal{C}$.

As the point set $\mathcal{P}$ is compact, we may assume that (p'_n) converges to p' and aim to show $p' = \pi(p_1, p)$. Since the mapping π^- is continuous, it follows that $p' \in [p]_-$. We now show that $p' \in [p_1]_+$.

If $p' \notin [p_3]_+ \cup [p_1]_+$, then three points p_1, p_3, p' are pairwise nonparallel. The continuity of joining α then implies (C_n) converges in $\mathcal{C}$, a contradiction. Otherwise, suppose $p' \in [p_3]_+$. Let r be a point on $[p_3]_-$ and $r \notin \{p_3, \pi(p_1, p_3)\}$. Also, let $r_n := \pi^+(r, C_n)$. Since the point set is compact, the sequence (r_n) must have an accumulation point $r' \in [r]_+$. We now show that there is no such choice of r'.

If $r' \notin [p_i]_-$ for $i = 1, 2, 3$, then r', p_1 and p_3 are pairwise nonparallel. Otherwise, if $r' \in [p_i]_-$, then r', p_j, p_3 are pairwise nonparallel, for $i, j = 1, 2$, $i \neq j$. If $r' \in [p_3]_-$, then r', p' and p_1 are pairwise nonparallel. In any case, we obtain a contradiction.

Therefore $p' \in [p_1]_+$ and this verifies the claim. □

Let $\mathbb{T}$ be a toroidal circle plane with the automorphism group $\mathrm{Aut}(\mathbb{T})$. Let Σ be the subgroup of $\mathrm{Aut}(\mathbb{T})$ consisting of all automorphisms that leave the sets $\mathcal{G}^{\pm}$ invariant and preserve the orientation of parallel classes. From the proof of [PS01] Lemma 4.4.2, we have the following result for flat Minkowski planes, which still holds for toroidal circle planes.

Lemma 4.1.4. *In Σ, the identity is the only automorphism that fixes three pairwise nonparallel points.*

4.2 The topology of the automorphism group

Let $C(\mathcal{P})$ be the space of continuous mappings from $\mathcal{P}$ to itself. As each automorphism is an element of $C(\mathcal{P})$, it is natural to equip the automorphism group with the compact-open topology. Let

$$(A, B) = \{\sigma \in C(\mathcal{P}) \mid \sigma(A) \subset B\},$$

where $A \subset \mathcal{P}$ is compact and $B \subset \mathcal{P}$ is open. The collection of all sets of the form (A, B) is a subbasis for the compact-open topology.

Since $\mathcal{P}$ is a compact metric space, the compact-open topology is equivalent to the topology of uniform convergence, cf. [Mun75] p. 283 and p. 286. Thus we have the following.

Lemma 4.2.1. *Convergence in Σ is equivalent to uniform convergence on the point set $\mathcal{P}$.*

Furthermore, since $\mathcal{P}$ is a metric space, the compact-open topology is metrisable and a metric is given by

$$\mathbf{e}(\sigma, \tau) = \sup\{\mathbf{d}(\sigma(x), \tau(x)) \mid x \in \mathcal{P}\}.$$

4.3 The automorphism group is a Lie group

We recall from Section 4.1 that Σ is the subgroup of $\mathrm{Aut}(\mathbb{T})$ consisting of all automorphisms that leave the sets $\mathcal{G}^{\pm}$ invariant and preserve the orientation of parallel classes.

In this section, we prove Theorem 4.0.1. By Lemma 4.1.2, it is sufficient to prove that the subgroup Σ is locally compact. Our strategy is to prove that Σ is homeomorphic to a closed subset N of $\widetilde{\mathcal{P}^3}$. Since $\widetilde{\mathcal{P}^3}$ is locally compact, Σ will then be locally compact.

Let $d = (d_1, d_2, d_3) \in \widetilde{\mathcal{P}^3}$ be a triple of pairwise nonparallel points. Also, let $d_4 := \pi(d_2, d_3), d_5 := \pi(d_3, d_2)$. Let N be the subset of $\widetilde{\mathcal{P}^3}$ defined as

$$N = \{\sigma(d) := (\sigma(d_1), \sigma(d_2), \sigma(d_3)) \mid \sigma \in \Sigma\}.$$

Lemma 4.3.1. *Let $\omega : \Sigma \to N$ be a mapping defined as $\omega : \sigma \mapsto \sigma(d)$. Then ω is a continuous bijection.*

Proof. By construction, ω is surjective. We next show that ω is injective. Suppose there are $\sigma_1, \sigma_2 \in \Sigma$ such that $\sigma_1(d) = \sigma_2(d)$. Since automorphisms are invertible, we have $\sigma_2^{-1}\sigma_1(d) = d$. By Lemma 4.1.4, it follows that $\sigma_2^{-1}\sigma_1 = \mathrm{id}$, and so $\sigma_1 = \sigma_2$.

To show that ω is continuous, let U be an open set containing $\sigma(d)$ in N. Since the point set is compact and σ is continuous, there is a compact neighbourhood K of d such that $\sigma(K) \subseteq U$. Let M be the set of automorphisms that map K into U. Then M is an open set of Σ containing σ such that $\omega(M) \subseteq U$. Hence ω is continuous. □

For the proof of the main theorem, the following technical lemma is required. We maintain the notations at the beginning of this section.

Lemma 4.3.2. *In the derived plane $\mathbb{T}_{d_1}$, by means of joining, intersecting, parallel intersecting and parallel projecting, the two points d_2, d_3 generate a dense subset $\mathcal{D}$.*

Proof. It suffices to show that $\mathcal{D}$ is dense on the parallel class $[d_2]_-$.

We first show $\mathcal{D}$ is dense in $[d_2, d_5]$. Suppose for a contradiction that there is an open interval $(a, b) \in [d_2, d_5] \backslash \overline{\mathcal{D}}$. Without loss of generality, suppose $a \in (d_2, b)$, compare Figure 4.2. We aim to construct a point $x \in (a, b) \cap \mathcal{D}$ by means of geometric operations.

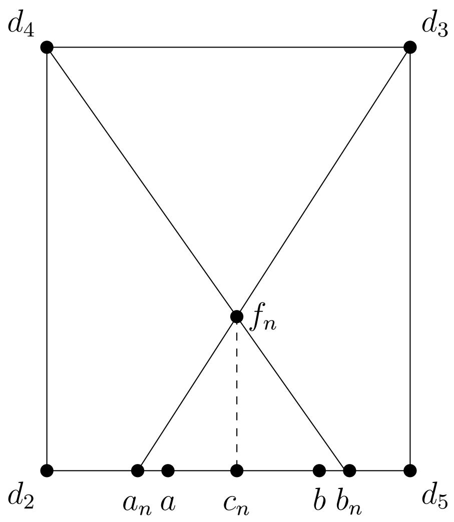

Figure 4.2

Let $(a_n) \in \mathcal{D} \cap [d_2, a)$ and $(b_n) \in \mathcal{D} \cap (b, d_5]$ be sequences convergent to a and b respectively. For each n, let R_n be the rectangle formed by $[a_n]_+, [b_n]_+, d_2d_5, d_4d_3$. Furthermore, let $f_n := a_nd_3 \cap d_4b_n$ and let $c_n := \pi(f_n, d_2)$. Since f_n is in the interior of R_n, we must have $c_n \in (a_n, b_n)$. Since the geometric operations are continuous, the limit $c := \lim_n c_n$ exists. We readily have $c \in [a, b]$.

We prove that $c \neq a$ by contradiction. If $f := \lim_n f_n$, then $f = ad_3 \cap d_4b \neq a$. On the other hand, since π^+ is continuous, we have $f \parallel_+ c$. We then consider the sequence (C_n) of circles $C_n := \alpha(d_3, f_n, a_n)$ and the constant sequence (d_1) on $\mathbb{T}$. By Lemma 4.1.3, the projection sequence $(\pi^-(d_1, C_n))$ converges to $\pi(a, d_1) \neq d_1$, which contradicts the continuity of joining. Similarly, $c \neq b$ and so $c \in (a, b)$. Then, for sufficiently large n, there is a point $x := c_n$ that belongs to (a, b), which contradicts the assumption $(a, b) \cap \mathcal{D} = \varnothing$. Therefore $\mathcal{D}$ is dense in $[d_2, d_5]$.

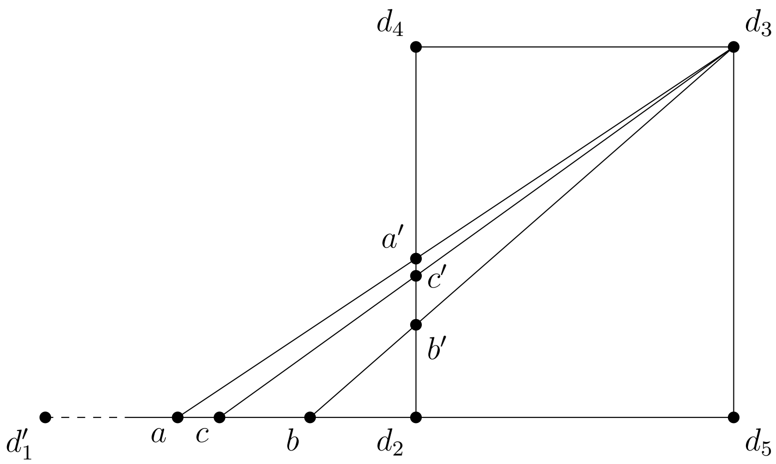

Figure 4.3

Let $d_1' := \pi(d_1, d_2)$. We now show that $\mathcal{D}$ is dense in the interval (d_1', d_2) not containing d_5. For an open interval $(a,b) \subseteq (d_1', d_2)$, let $a' := ad_3 \cap d_2d_4$ and $b' := bd_3 \cap d_2d_4$, compare Figure 4.3. As (a,b) and d_3 are separated by d_2d_4, the open interval (a', b') is contained in (d_2, d_4). Analogous to the case (d_2, d_5), the set $\mathcal{D}$ is dense in the interval (d_2, d_4) and so there exists a point $c' \in \mathcal{D} \cap (a', b')$. Then $c := c'd_3 \cap d_2d_5 \in (a,b)$. This shows $\mathcal{D}$ is dense in the interval (d_1', d_2) not containing d_5.

Consequently, $\mathcal{D}$ is dense in the remaining interval and hence in $[d_2]_-$. $\square$

We are now ready to show that N is a closed subspace of $\widetilde{\mathcal{P}^3}$. In Σ, let (σ_n) be a sequence of automorphisms that converges on the points d_i, and let $e_i := \lim_n \sigma_n(d_i)$, which we assume to be also pairwise nonparallel, for $i = 1, 2, 3$. With the aid of Lemma 4.2.1, we aim to show that (σ_n) converges uniformly on $\mathcal{P}$ to an automorphism σ with $\sigma(d_i) = e_i$.

To this end, we shall consider the derived planes $\mathbb{T}_{d_1}$ and $\mathbb{T}_{e_1}$, which are $\mathbb{R}^2$-planes. By Lemma 4.3.2, the points d_2, d_3 generate a dense set $\mathcal{D}$. Also, let $\mathcal{D}'$ be the dense set generated by e_2 and e_3 in the derived plane $\mathbb{T}_{e_1}$. We first define the limit automorphism σ between these dense sets and then extend it to the point set $\mathcal{P}$.

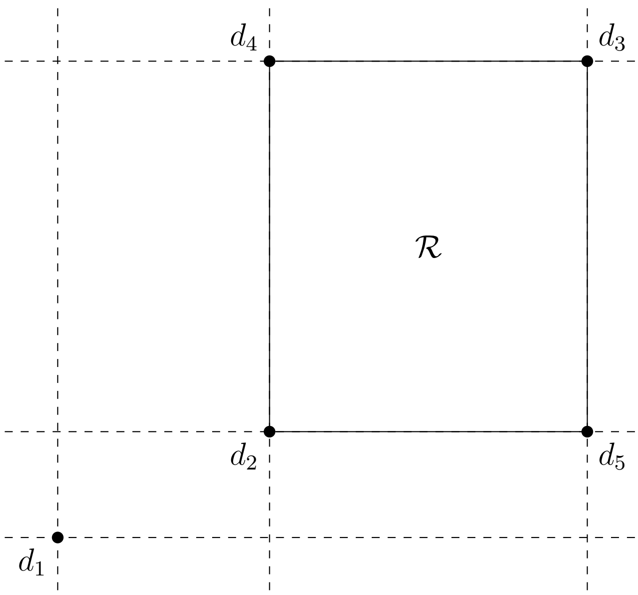

Figure 4.4

In particular, define $\sigma : \mathcal{D} \to \mathcal{D}'$ by $\sigma : x \mapsto \lim_n \sigma_n(x)$. The limit exists because of the continuity of geometric operations. Let $\mathcal{R}$ be the rectangle formed by d_2, d_3, d_4, d_5 and $\mathcal{R}'$ be the rectangle formed by e_2, e_3, e_4, e_5, where $e_i := \lim_n \sigma_n(d_i)$, for $i = 4, 5$. The parallel classes of d_1, d_2, d_3 partition the point set $\mathcal{P}$ into nine rectangles as depicted in the following Figure 4.4.

In the following steps, we show that (σ_n) converges uniformly to the extension of σ on $\mathcal{R}$.

By taking appropriate derived planes and obtaining uniform convergence on the remaining rectangles, we then obtain uniform convergence of (σ_n) on $\mathcal{P}$.

Lemma 4.3.3. *If a point $p \in \mathcal{D}$ is in the interior of the rectangle $\mathcal{R}$, then $q := \sigma(p)$ is in the interior of the rectangle $\mathcal{R}'$.*

Proof. For an illustration, compare Figure 4.5.

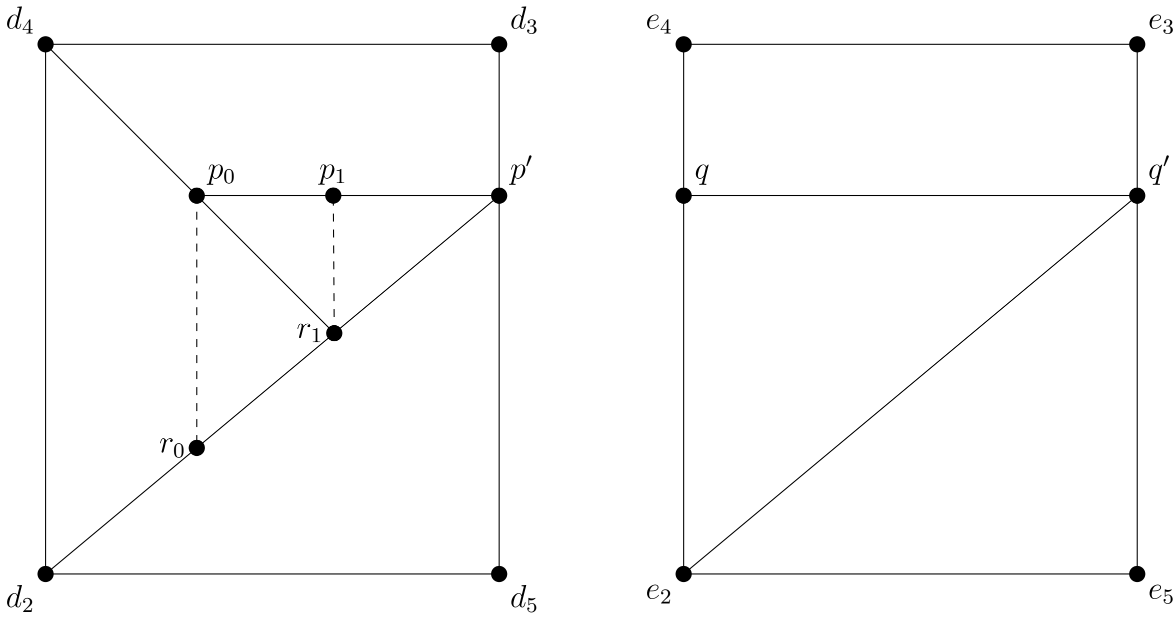

Figure 4.5

Since R' is compact, $q \in R'$. Without loss of generality, suppose for a contradiction that $q \in [e_2, e_4)$. In the sequel, we define $p_0 := p$. For inductive purposes, let $p' := \pi(d_3, p_0)$, and $r_0 := [p_0]_+ \cap d_2p'$. Since $[p_0]_+$ is disjoint from d_2d_4 and p_0 is inside the triangle d_4d_2p', it follows that $r_0 \in (d_2, p')$.

Let $r_1 := d_4p_0 \cap d_2p'$. We observe that p_0 must be inside the triangle d_4r_0p' as d_4 and p' are separated by $[p_0]_+$, and likewise d_4, r_0 are separated by $[p_0]_-$. Hence $r_1 \in (r_0, p')$. Since parallel projections preserve betweenness of points, if we let $p_1 := \pi(r_1, p)$, then $p_1 \in (p_0, p')$. This implies p_1 is inside the triangle d_4r_1p'.

We then construct the sequences (p_n) and (r_n) inductively by letting $r_n := d_4p_{n-1} \cap d_2p'$, and $p_n := \pi(r_n, p)$, for $n \geqslant 2$. It readily follows that $p_{n+1} \in (p_n, p')$, for $n \geqslant 1$.

Suppose that the sequence (r_n) converges to $r' \in (d_2, p')$. Then (p_n) converges to $p'' := \pi(r', p)$. We now consider the circles $C_n := \alpha(d_4, p_n, r_{n+1})$ and the constant sequence (d_1) on $\mathbb{T}$. By Lemma 4.1.3, the sequence of parallel projections $(\pi^-(d_1, C_n))$ converges to $\pi(r', d_1) \neq d_1$, a contradiction. Therefore both (r_n) and (p_n) converge to p'.

We now consider the images of these sequences under σ. Let $q' := \sigma(p') \equiv \pi(e_3, q)$. Initially, $\sigma(p_0) = q$ and $\sigma(r_0) = [q]_+ \cap e_2 q' = e_2$. From the inductive construction, we then obtain $\sigma(p_n) = q$ and $\sigma(r_n) = e_2$ for all n.

Let $d_6 := d_2 d_3 \cap d_4 d_5$ and let $d_6' := \pi(d_6, p)$. Then there exists $p_* \in (p_n)$ such that $p_* \in (d_6', p')$. On the other hand, $\sigma(p_*) \in e_2 e_4$ implies $\sigma(p_*) \notin (\sigma(d_6'), q')$, which contradicts Lemma 4.1.1. We conclude q is in the interior of the rectangle $\mathcal{R}'$. □

By repeating the same argument used in Lemma 4.3.3 for arbitrarily small rectangles, we then have the following.

Lemma 4.3.4. *The limit σ maps two nonparallel points to two nonparallel points.*

This is the key ingredient for proving the subsequent lemma.

Lemma 4.3.5. *The limit σ is a bijection between the dense subsets $\mathcal{D}$ and $\mathcal{D}'$.*

Proof. We observe that σ is surjective as each $y \in \mathcal{D}'$ is obtained from e_i in finitely many steps using geometric operations. The same operations applied to d_i yield $x \in \mathcal{D}$ such that $\sigma(x) = y$.

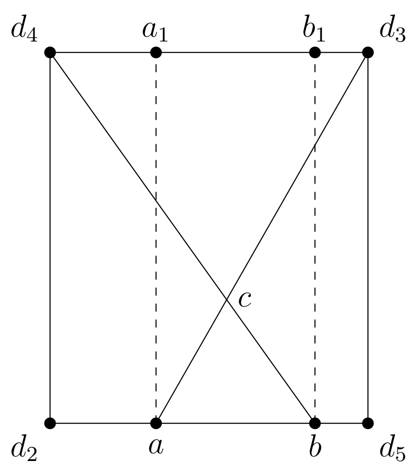

Figure 4.6

If two points a, b are nonparallel, then Lemma 4.3.4 implies $\sigma(a) \neq \sigma(b)$. It is then sufficient to show that σ is injective on $[d_2]_-$, particularly on (d_2, d_5). Suppose for a contradiction that there are two points $a, b \in \mathcal{D} \cap (d_2, d_5)$ such that $\sigma(a) = \sigma(b)$. Without loss of generality, further suppose that $a \in (d_2, b)$ and $\sigma(a) \notin \{e_2, e_5\}$, compare Figure 4.6. Let $a_1 := \pi(a, d_3)$ and $b_1 := \pi(b, d_3)$. Let $\mathcal{S}_1$ be the rectangle formed by d_2, b, b_1, d_4 and let $\mathcal{S}_2$ be the rectangle formed by a, d_5, d_3, a_1. Let $c = ad_3 \cap bd_4$. Then $c \in \mathcal{S}_1 \cap \mathcal{S}_2$, which implies $\sigma(c) \in \sigma(\mathcal{S}_1) \cap \sigma(\mathcal{S}_2) = (\sigma(a), \sigma(a_1))$. This contradicts what we have obtained in Lemma 4.3.3. $\square$

If two points are in the interior of an ε-rectangle, then under the mapping σ, their images are still in the interior of the image of that rectangle. Hence σ is open. Conversely, let $\mathcal{S}'$ be an ε-rectangle whose vertices belong to $\mathcal{D}'$. From Lemma 4.3.4 and the bijectivity of σ, there exists a δ-rectangle $\mathcal{S}$ whose vertices are in $\mathcal{D}$ and whose interior is mapped to the interior of $\mathcal{S}'$. This implies σ is continuous. We conclude the following.

Lemma 4.3.6. *The limit σ is a homeomorphism between $\mathcal{D}$ and $\mathcal{D}'$.*

To extend the map σ from the dense set $\mathcal{D}$ to the point set $\mathcal{P}$, we consider the convergence of the sequence (σ_n) at each point $x \in \mathcal{P}$.

Lemma 4.3.7. *The sequence (σ_n) converges pointwise on the point set $\mathcal{P}$.*

Proof. We first show that (σ_n) converges at each point x on (d_2, d_5) in the derived plane $\mathbb{T}_{d_1}$. The point set $\mathcal{P}$ is compact, and so we let x^* be an accumulation point of the sequence $(\sigma_n(x))$. Passing to subsequences, we assume $(\sigma_n(x))$ converges to x^* on $[e_2]_-$.

Let $(y_n) \in \mathcal{D} \cap [d_2, x)$ be a sequence convergent to x. From Lemma 4.3.5, we have defined $\sigma(y_n)$ for every n. We then assume the sequence $(\sigma(y_n))$ converges to a point y^*. Similarly, let $(z_n) \in \mathcal{D} \cap (x, d_5]$ be a sequence convergent to x such that $(\sigma(z_n)) \to z^*$.

As a consequence of Lemma 4.1.1, we have $x^* \in [y^*, z^*]$ in the derived plane $\mathbb{T}_{e_1}$. Suppose for a contradiction that $y^* \neq z^*$. Since $\mathcal{D}'$ is dense and σ is a bijection on $\mathcal{D}$, there exist two distinct points $u, v \in \mathcal{D}$ such that $\sigma(u), \sigma(v) \in (y^*, z^*)$. This implies $\sigma(u) \in (\sigma(y_n), \sigma(z_n))$ for sufficiently large n. From Lemma 4.3.4 and Lemma 4.3.6, it follows that $u \in (y_n, z_n)$ for sufficiently large n. The only point in the point set $\mathcal{P}$ satisfying this condition is x, and so $u = x$. The same argument implies that $v = x$, which then yields a contradiction. Therefore $x^* = y^* = z^*$ and so every accumulation point of $(\sigma_n(x))$ coincides with x^*.

Repeating the argument above for the interval on $[d_2]_-$ not containing x, we conclude that (σ_n) converges pointwise on $[d_2]_-$. The same result holds for $[d_2]_+$. This completes the proof. $\square$

We now prove the main theorem of this section.

Theorem 4.3.8. *Let $d \in \widetilde{\mathcal{P}^3}$ be fixed and define $\omega : \Sigma \to \widetilde{\mathcal{P}^3}$ by $\omega(\sigma) = \sigma(d)$. If $(\sigma_n) \in \Sigma$ is a sequence such that $(\omega(\sigma_n))$ converges in $\widetilde{\mathcal{P}^3}$, then (σ_n) converges in Σ. In particular,* *the image $N = \omega(\Sigma)$ is a closed subset of $\widetilde{\mathcal{P}^3}$, and the inverse map $\omega^{-1} : N \to \Sigma$ is continuous.*

Proof. Let $\sigma : \mathcal{P} \to \mathcal{P}$ be defined by $\sigma(x) = \lim_n \sigma_n(x)$. By Lemma 4.3.7, σ is well-defined. By Lemma 4.2.1, it is sufficient to show (σ_n) converges uniformly to σ on the rectangle $\mathcal{R}$.

Fix $\epsilon > 0$. For every $\xi \in \mathcal{R}$ let $\mathcal{R}_\xi$ be a rectangle whose vertices $\xi_1, \xi_2, \xi_3, \xi_4$ belong to the dense set $\mathcal{D}$ such that ξ is inside $\mathcal{R}_\xi$ and $\max_i \mathbf{d}(\sigma(\xi), \sigma(\xi_i)) \leqslant \epsilon/2$, for $i = 1, \ldots, 4$. This is facilitated by Lemma 4.3.6, which implies the continuity of σ on $\mathcal{P}$. The union of interior of all such rectangles is an open cover of $\mathcal{R}$ and thus has a finite subcover $\mathscr{F}$ with the set of finite vertices $\mathcal{D}_{\epsilon/2} \subset \mathcal{D}$. Also because (σ_n) converges pointwise, there exists n_0 such that if $n \geqslant n_0$ and $\xi \in \mathcal{D}_{\epsilon/2}$, then $\mathbf{d}(\sigma_n(\xi), \sigma(\xi)) \leqslant \epsilon/2$.

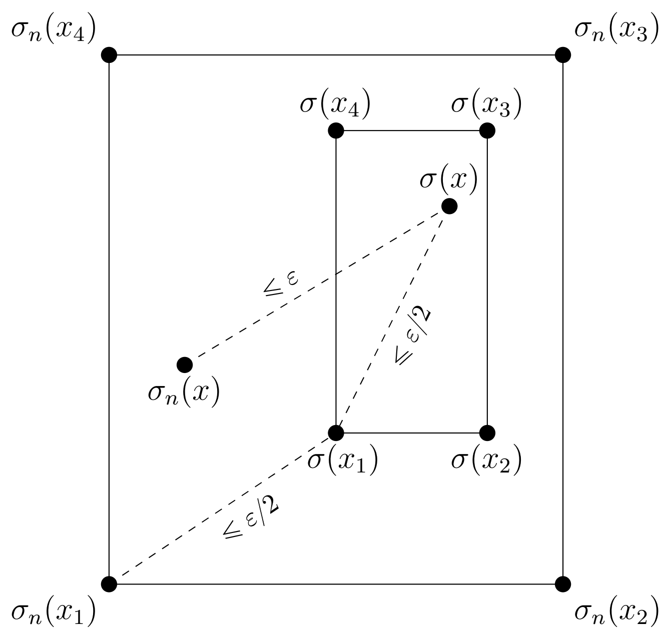

Figure 4.7

Let $x \in \mathcal{R}$ and let $\mathcal{R}_x$ be a rectangle from $\mathscr{F}$ that covers x, compare Figure 4.7. Then $\max_i \mathbf{d}(\sigma(x), \sigma(x_i)) \leqslant \epsilon/2$. For $n \geqslant n_0$, we have $\mathbf{d}(\sigma_n(x_i), \sigma(x_i)) \leqslant \epsilon/2$, which implies $\max_i \mathbf{d}(\sigma(x), \sigma_n(x_i)) \leqslant \epsilon$. Since $\sigma_n(x)$ is in the interior of the rectangle formed by $\sigma_n(x_i)$, for $i = 1, \ldots, 4$, we readily have $\mathbf{d}(\sigma_n(x), \sigma(x)) \leqslant \epsilon$. Therefore (σ_n) converges uniformly to σ on $\mathcal{R}$. $\square$

From Lemma 4.3.1 and Theorem 4.3.8, Σ is homeomorphic to N. Since $\widetilde{\mathcal{P}^3}$ is locally compact, N is locally compact as well. This leads us to the local compactness of Σ. By Lemma 4.1.2, we then have

Theorem 4.3.9. Σ *is a Lie group of dimension at most 6.*

Theorem 4.0.1 now follows from Theorem 4.3.9 and the fact that Σ is a closed normal subgroup of $\mathrm{Aut}(\mathbb{T})$ with index at most 8.

Chapter 5

On the classification with respect to group dimension

Having established Theorem 4.0.1, we now generalise some results on flat Minkowski planes to toroidal circle planes. In particular, we determine all toroidal circle planes whose automorphism group is at least 4-dimensional, or one of whose kernels is 3-dimensional. These are the results of Section 5.1.

We then consider almost simple groups of automorphisms in Section 5.2. From the list of almost simple Lie groups of low dimension (cf. Lemma A.2.5), we show that such a group of automorphisms is isomorphic to the $\mathrm{PSL}(2,\mathbb{R})$. We will also describe the action of these groups.

Lastly, based on results from Section 5.2, we determine all possible 3-dimensional connected groups of automorphisms of toroidal circle planes in Section 5.3.

5.1 Planes with group dimension at least 4

In this section, we generalise Theorems 2.3.2 2.3.3, 2.3.4, 2.3.5 for flat Minkowski planes to toroidal circle planes. Inspection of proofs shows that the first three theorems only depend on the fact that the automorphism group of a flat Minkowski plane is a Lie group (cf. Theorem 2.3.1). By Theorem 4.0.1, we readily have the corresponding results.

Theorem 5.1.1. *The connected component of a kernel $T^{\pm}$ of a toroidal circle plane is isomorphic to* $\mathrm{PSL}(2,\mathbb{R})$*,* L_2*,* $\mathrm{SO}(2,\mathbb{R})$*,* $\mathbb{R}$*, or the trivial group* $\{id\}$*.*

Theorem 5.1.2. *Let $\mathbb{T}$ be a toroidal circle plane such that the kernel T^+ is 3-dimensional. Then $\mathbb{T}$ is a flat Minkowski plane. In particular, $\mathbb{T}$ is isomorphic to a swapping half plane*

$\mathcal{M}(f, id)$, where f is an orientation-preserving homeomorphism of $\mathbb{S}^1$.

Theorem 5.1.3. *A toroidal circle plane is isomorphic to the classical flat Minkowski plane if and only if its full automorphism group has dimension at least 5.*

We now describe all possible toroidal circle planes with group dimension 4. In the case of flat Minkowski planes, the proof of Theorem 2.3.5 uses the fact that derived planes are flat affine planes. This is not true in the case of toroidal circle planes, where derived planes are only $\mathbb{R}^2$-planes, hence an extra argument is needed. We address this problem with a sketch of proof. The details follow in the same manner as in the proof of Theorem 2.3.5 (cf. [PS01] Theorem 4.4.15).

Theorem 5.1.4. *Let $\mathbb{T}$ be a toroidal circle plane. If the full automorphism group of $\mathbb{T}$ has dimension 4, then $\mathbb{T}$ is a flat Minkowski plane. In particular, $\mathbb{T}$ is isomorphic to one of the following planes.*

(i) A nonclassical swapping half plane $\mathcal{M}(f, id)$, where f is a semi-multiplicative homeomorphism of the form $f_{d,s}$, $(d, s) \neq (1, 1)$. This plane admits the 4-dimensional group of automorphisms

$$\{(x, y) \mapsto (rx, \delta(y)) \mid r \in \mathbb{R}^+, \delta \in \mathrm{PSL}(2, \mathbb{R})\}.$$

(ii) A nonclassical generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$, $r_1, s_1, r_2, s_2 \in \mathbb{R}^+$, $(r_1, s_1, r_2, s_2) \neq (1, 1, 1, 1)$. This plane admits the 4-dimensional group of automorphisms

$$\{(x, y) \mapsto (rx + a, sy + b) \mid a, b, r, s \in \mathbb{R}, r, s > 0\}.$$

Sketch of Proof. Let Σ be the connected component of $\mathrm{Aut}(\mathbb{T})$ and let $\Delta^{\pm}$ be the kernel of the action of Σ on $\mathcal{G}^{\pm}$.

If at least one of $\Sigma/\Delta^{\pm}$ is transitive on $\mathcal{G}^{\pm}$, then $\mathbb{T}$ is isomorphic to a plane $\mathcal{M}(f, id)$. Otherwise, Σ fixes a point p and acts transitively on $\mathcal{P}\backslash([p]_+ \cup [p]_-)$. Then Σ induces a 4-dimensional point-transitive group of automorphisms $\overline{\Sigma}$ on the derived plane $\mathbb{T}_p$. By [Sal67b] Theorem 4.12, $\mathbb{T}_p$ is isomorphic to either the Desarguesian plane or a real cylinder plane.

We show $\mathbb{T}_p$ is not isomorphic to a real cylinder plane. Suppose the contrary. Since $\overline{\Sigma}$ acts transitively on the two sets of lines derived from $\mathcal{G}^{\pm}$, from [Sal67b, p. 31], $\dim \Delta^{\pm} = 3$. But then $\dim \Delta^+\Delta^- = 6 > \dim \Sigma$, a contradiction.

Hence $\mathbb{T}_p$ is a Desarguesian plane. It follows that $\mathbb{T}$ is a generalised Hartmann plane. □

5.2 Almost simple groups of automorphisms

Let $\mathbb{T}$ be a toroidal circle plane with full automorphism group $\mathrm{Aut}(\mathbb{T})$. Denote Σ the connected component of $\mathrm{Aut}(\mathbb{T})$. Let S be an almost simple Lie subgroup of Σ. Let $\Delta^{\pm}$ be the kernel of the action of S on $\mathcal{G}^{\pm}$.

The overall strategy in this section is as follows. We first show that S is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. This implies S is isomorphic to one of the covering groups of $\mathrm{PSL}(2,\mathbb{R})$; in particular, the centre $Z(S)$ is cyclic. We then show that S is in fact isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, by showing the centre $Z(S)$ is trivial. Finally, we apply Brouwer's Theorem (cf. Lemma A.3.6) to determine all possible actions of S.

We start with the following observation.

Lemma 5.2.1. *If S contains a subgroup H isomorphic to $\mathrm{SO}(2,\mathbb{R})$, then S acts transitively on at least one of $\mathcal{G}^{\pm}$.*

Proof. Since $T^+ \cap T^- = \{id\}$, the subgroup H cannot be contained in both $T^{\pm}$. Without loss of generality, we assume $H \not\subseteq T^+$. This means H cannot act trivially, and therefore, acts transitively on $\mathcal{G}^+$ (cf. Lemma A.3.4). □

Lemma 5.2.2. *S is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.*

Proof. We first note that S cannot have dimension 6, because this implies $\mathbb{T}$ is the classical Minkowski plane and $\Sigma \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, which has no almost simple subgroups of dimension 6. From the list of almost simple groups (cf. Lemma A.2.5), S is locally isomorphic to either $\mathrm{SO}(3,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{R})$.

Suppose for a contradiction that S is locally isomorphic to $\mathrm{SO}(3,\mathbb{R})$. By Lemma A.2.6, S is isomorphic to either $\mathrm{SO}(3,\mathbb{R})$ or $\mathrm{SU}(2,\mathbb{C})$. This implies S is compact and contains a subgroup isomorphic to $\mathrm{SO}(2,\mathbb{R})$. By Lemma 5.2.1, we can assume S is transitive on $\mathcal{G}^+$. By Brouwer's Theorem, the factor group S/Δ^+ is isomorphic to either $\mathrm{SO}(2,\mathbb{R})$ or $\mathrm{PSL}^{(k)}(2,\mathbb{R})$. Since S is compact, S/Δ^+ is also compact, and so $S/\Delta^+ \cong \mathrm{SO}(2,\mathbb{R})$. But this implies $\dim \Delta^+ = 2$, which contradicts the assumption S is almost simple. □

Since S is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, it is isomorphic to either the universal covering group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ or a finite covering group $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$. In particular, the centre $Z(S)$ is cyclic. In the following three lemmas, we show that $Z(S)$ is trivial. Let κ be the generator of $Z(S)$.

Lemma 5.2.3. *Let p be a point. Then $\kappa(p) \parallel p$.*

Proof. Suppose for a contradiction that there exists a point p such that $\kappa(p) \nparallel p$.

1) We show $\dim S_{[p]_\pm} = 2$. By the dimension formula A.3.3, we have

$$3 = \dim S = \dim S_{[p]_+} + \dim S([p]_+).$$

Then $\dim S_{[p]_+}$ is either 3 or 2, since $\dim S([p]_+)$ is either 0 or 1. If $\dim S_{[p]_+} = 3$, then by Lemma A.2.1, $S = S_{[p]_+}$. This cannot be true however, since κ does not fix $[p]_+$. Hence $\dim S_{[p]_+} = 2$. Similarly, we have $\dim S_{[p]_-} = 2$.

2) We claim that $S_{[p]_\pm}$ fixes at least one point on $[p]_\pm$. If $S_{[p]_+}$ is transitive on $[p]_+$, then by Brouwer's Theorem, $S_{[p]_+}/\Delta^- \cong \mathrm{SO}(2, \mathbb{R})$. Since $\dim S_{[p]_+} = 2$, $\dim \Delta^- = 1$. But this contradicts the fact that S is almost simple. Hence $S_{[p]_+}$ is not transitive on $[p]_+$. By Lemma A.3.4, $S_{[p]_+}$ fixes at least one point on $[p]_+$. The same argument holds for $S_{[p]_-}$ on $[p]_-$.

3) We show that either $S_p = S_{[p]_-}$ or $S_p = S_{[p]_+}$. Following part 2), without loss of generality, let q be a point such that $[q]_\mp$ is fixed by $S_{[p]_\pm}$. Since $S_p \leqslant S_{[p]_+} \cap S_{[p]_-}$, $S_p = S_{p,q}$. We also have $S_p = S_{p,\kappa(p)}$, and so $S_p = S_{p,q,\kappa(p)}$. If q is nonparallel to p and $\kappa(p)$, then S_p is trivial and $\dim S_p = 0$. From the dimension formula, we get $\dim S(p) = \dim S - \dim S_p = 3$, which cannot be true. Hence either $q \parallel p$ or $q \parallel \kappa(p)$.

If $q \parallel_+ p$, then $[q]_+ = [p]_+$, so that $S_{[p]_-}$ fixes $[p]_+$. If $q \parallel_+ \kappa(p)$, then $S_{[p]_-}$ fixes $\kappa([p]_+)$, and since κ commutes with every element of S, we see that $S_{[p]_-}$ also fixes $[p]_+$. In particular, $S_p = S_{[p]_-}$.

In the cases $q \parallel_- p$ and $q \parallel_- \kappa(p)$, we obtain $S_p = S_{[p]_+}$ in a similar manner.

4) We show there exists $r \in S(p)$ such that $r \nparallel p$ and $r \nparallel \kappa(p)$. From part 1) and 3), $\dim S_p = 2$, so that $\dim S(p) = 1$. Also from part 3), we can assume $S_p = S_{[p]_-}$, and so $S_{\kappa(p)} = S_{\kappa([p]_-)}$.

Suppose for a contradiction that for every $\alpha \in S$, either $\alpha(p) \parallel p$ or $\alpha(p) \parallel \kappa(p)$. From the assumption $S_p = S_{[p]_-}$, if $\alpha(p) \parallel_- p$, then α fixes p. Similarly, since $S_{\kappa(p)} = S_{\kappa([p]_-)}$, if $\alpha(p) \parallel_- \kappa(p)$, then α fixes $\kappa(p)$. It follows that the orbit $S(p)$ consists of p, $\kappa(p)$, and the points $\alpha(p)$ such that $\alpha(p) \parallel_+ p$ or $\alpha(p) \parallel_+ \kappa(p)$. But this is impossible, since $S(p)$ is connected.

5) Let r be as in part 4). From the dimension formula, we have

$$\begin{aligned}\dim S &= \dim S_p + \dim S(p) \\ &= \dim S_{p,r} + \dim S_p(r) + \dim S(p) \\ &\leqslant \dim S_{p,r,\kappa(p)} + 1 + 1 = 2,\end{aligned}$$

a contradiction. This completes the proof. □

Lemma 5.2.4. *$Z(S)$ is contained in at least one of the kernels $T^{\pm}$.*

Proof. For an illustration, compare Figure 5.1.

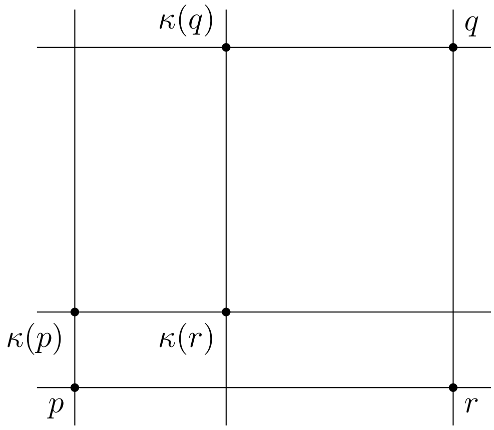

Figure 5.1

Suppose for a contradiction that there are two nonparallel points p, q such that $\kappa(p) \neq p, \kappa(q) \neq q$, $\kappa(p) \parallel_+ p$, $\kappa(q) \parallel_- q$. Let $r = [p]_- \cap [q]_+$. Then $\kappa(r) \parallel_- \kappa(p)$ and $\kappa(r) \parallel_+ \kappa(q)$. Since $\kappa(p) \neq p$ and $\kappa(q) \neq q$, we see that $\kappa(r)$ is nonparallel to r, contradicting Lemma 5.2.3. $\square$

Lemma 5.2.5. *$Z(S)$ is trivial.*

Proof. By Lemma 5.2.4, we can assume $Z(S) \leqslant T^+$. Since Δ^+ is a normal subgroup of S, the dimension of Δ^+ is either 0 or 3. If $\dim \Delta^+ = 3$, then by Lemma A.2.1, $S = \Delta^+$. On the other hand, by Lemma 5.1.1, $\Delta^+ \cong \mathrm{PSL}(2, \mathbb{R})$. It follows that $S \cong \mathrm{PSL}(2, \mathbb{R})$ and thus $Z(S)$ is trivial.

In the remainder of the proof, we deal with the case $\dim \Delta^+ = 0$. By Lemma A.2.3, $\Delta^+ \leqslant Z(S)$. Since $\Delta^+ = S \cap T^+$, we also have $Z(S) \leqslant \Delta^+$ and hence $Z(S) = \Delta^+$. Suppose for a contradiction that $\kappa \neq id$, that is, there exists a point p such that $\kappa(p) \neq p$ and $\kappa(p) \parallel_+ p$.

1) We claim that the action of S/Δ^+ on $\mathcal{G}^+$ is equivalent to the standard action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{S}^1$. We first note that $S/\Delta^+ = S/Z(S) \cong \mathrm{PSL}(2, \mathbb{R})$ so that it contains a subgroup H isomorphic to $\mathrm{SO}(2, \mathbb{R})$. By Lemma A.3.5, H acts either trivially or transitively on $\mathcal{G}^+$. If H acts trivially on $\mathcal{G}^+$, then $H \leqslant \Delta^+$, and since S is almost simple, $S = \Delta^+ = Z(S)$, which cannot be true because S is not commutative. Hence H and thus S/Δ^+ is transitive on $\mathcal{G}^+$. The claim now follows from Brouwer's Theorem.

2) We show that, without loss of generality, we can assume $\dim S(p) = 1$.

Since $\Delta^+ \cap \Delta^- = \{id\}$, $S \neq \Delta^-$. This implies S is transitive on either a subset $I \cong \mathbb{R}$ of $\mathcal{G}^-$ or $\mathcal{G}^-$ itself. We consider the first case. Let K^- be the kernel of the action of S on I. By Brouwer's Theorem, S/K^- is isomorphic and acts equivalently to $\mathrm{PS}\widetilde{\mathrm{L}(2,\mathbb{R}})$.

We identify the point set $\mathcal{P}$ as $\mathcal{G}^+ \times \mathcal{G}^-$. Let $\widetilde{\mathcal{P}} := \mathcal{G}^+ \times I$. Let π be the standard covering map from $\mathbb{R}$ to $\mathbb{S}^1$. We will abuse notation and let $\pi(I)$ be the image of I under the composition of π with a homeomorphism from I onto $\mathbb{R}$. Define $\widetilde{\pi} : \widetilde{\mathcal{P}} \to \mathcal{G}^+ \times \pi(I)$ by

$$\widetilde{\pi}((x, y)) = (x, \pi(y)).$$

The action of S on $\widetilde{\pi}(\widetilde{\mathcal{P}})$ is equivalent to the diagonal action of $\mathrm{PSL}(2, \mathbb{R})$ under a suitable coordinate system. Let Δ be the diagonal $\widetilde{\pi}(\widetilde{\mathcal{P}})$. Then $D := \widetilde{\pi}^{-1}(\Delta)$ is a 1-dimensional orbit of S on $\widetilde{\mathcal{P}}$. An illustration of the orbit D on the torus can be found in Figure 5.2.

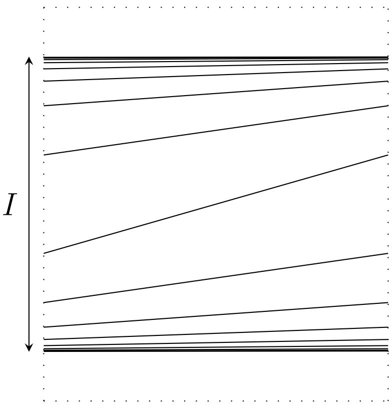

Figure 5.2

Let $p' \in D$. If κ fixes p', then κ fixes D pointwise and thus is trivial, which is a contradiction. Hence, without loss of generality, we can assume $p = p'$, which implies $S(p) = D$ and so $\dim S(p) = 1$.

In the case S is transitive on $\mathcal{G}^-$, the factor group S/Δ^- is isomorphic and acts equivalently to $\mathrm{PSL}^{(k)}(2, \mathbb{R})$, for some $k \in \mathbb{N}$, $k > 1$. Let π be the k-sheeted covering map from $\mathbb{S}^1$ to $\mathbb{S}^1$. Define $\widetilde{\pi} : \mathcal{P} \to \mathcal{G}^+ \times \pi(\mathcal{G}^-)$ by

$$\widetilde{\pi}((x, y)) = (x, \pi(y)).$$

The action of S on $\widetilde{\pi}(\widetilde{\mathcal{P}})$ is equivalent to the diagonal action of $\mathrm{PSL}(2, \mathbb{R})$ under a suitable coordinate system. Similar to the previous case, we can also assume $\dim S(p) = 1$.

3) We show that, if q is a point such that $q \nparallel p, q \nparallel \kappa(p)$, then the stabiliser $S_{p,q}$ is sharply transitive on each of $\mathcal{C}^{\pm}_{p,q}$. This comes from 1) and from the fact that there is a continuous bijection ϕ from $\mathcal{C}_{p,q}$ onto the set of parallel classes $\mathcal{G}^+ \backslash \{[p]_+, [q]_+\}$. If we identify $\mathcal{G}^+ \backslash \{[p]_+, [q]_+\}$ as points on $[\kappa(p)]_-$, then the bijection ϕ can be defined via the map

$$\phi : C \mapsto C \cap [\kappa(p)]_-.$$

Since ϕ is induced from the operation Parallel Projection, it is continuous. In particular, ϕ maps each of $\mathcal{C}^{\pm}_{p,q}$ onto a connected component of $\mathcal{G}^{+}\backslash\{[p]_+, [q]_+\}$.

4) Let V be the connected component of $\mathcal{G}^{-}\backslash\{[p]_-, [\kappa(p)]_-\}$ which does not contain $[\kappa^n(p)]_-$, $n \geqslant 2$. Define $U := \{q \in S(p) \mid [q]_- \in V\}$. By part 2), we have that $U \cong \mathbb{R}$. For every point $q \neq p$ on U, let $U(q,i)$, $i = 1, 2$, be the two connected components of $U\backslash\{q\}$. For an illustration, compare Figure 5.3. Then the stabiliser $S_{p,q}$ is sharply transitive on each of $U(q,i)$.

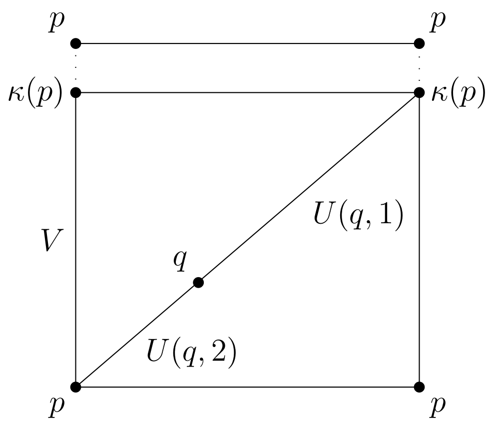

Figure 5.3

5) Maintain the notation from part 4). We show there exists a point $q \in U$ such that the following condition $(*)$ is satisfied.

$(*)$ There exist $r \in U(q,1), s \in U(q,2)$ such that both circles $\alpha(p,q,r)$ and $\alpha(p,q,s)$ belong to the same connected component of $\mathcal{C}_{p,q}$.

Suppose the contrary that for every point $q \in U$, if $r \in U(q,1)$, then $\alpha(p,q,r) \in \mathcal{C}^{+}_{p,q}$; and if $s \in U(q,2)$, then $\alpha(p,q,s) \in \mathcal{C}^{-}_{p,q}$. Fix $q \in U$, $r \in U(q,1), s \in U(q,2)$. Assume $q \in U(r,1)$ and let $t \in U(r,2)$. Then $\alpha(p,r,t) \in \mathcal{C}^{-}_{p,r}$. Since s,t are not in the same connected component of $U\backslash\{r\}$, we see that $\alpha(p,r,s) \in \mathcal{C}^{+}_{p,r}$. Now, since q,r are in the same connected component of $U\backslash\{s\}$, $\alpha(p,q,s) \in \mathcal{C}^{+}_{p,s}$. This implies $\alpha(p,q,s)$ is orientation-preserving, contradicting the assumption $\alpha(p,q,s) \in \mathcal{C}^{-}_{p,q}$.

The case $q \in U(r,2)$ is similar.

6) Let q be a point satisfying condition $(*)$ in part 5). Without loss of generality, let $r \in U(q,1), s \in U(q,2)$ such that $\alpha(p,q,r), \alpha(p,q,s) \in \mathcal{C}^{+}_{p,q}$. Define $A := \alpha(p,q,r), B := \alpha(p,q,s)$. For an illustration, compare Figure 5.4.

By part 3), there exists a unique $\sigma \in S_{p,q}$ such that $\sigma(B) = A$. This implies $\sigma(s) \in A$, and by part 4), $\sigma(s) \in U(q,2)$. Since $\sigma(s), q$ are in the same connected component of $U\backslash\{r\}$,

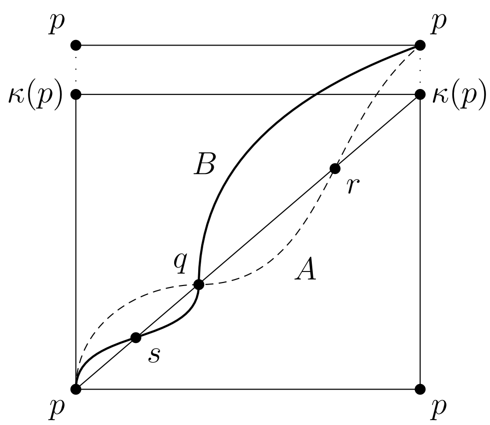

Figure 5.4

again by part 4), there exists a unique $\tau \in S_{p,r}$ such that $\tau(q) = \sigma(s)$. In other words, τ fixes $p, r \in A$ and maps $q \in A$ to $\sigma(s) \in A$, so τ must fix A. It then follows from part 3) that $\tau = id$. But this is impossible, because $\sigma(s) \neq q$. This completes the proof. □

Among the covering groups of $\mathrm{PSL}(2, \mathbb{R})$, only $\mathrm{PSL}(2, \mathbb{R})$ has trival centre. Hence $S \cong \mathrm{PSL}(2, \mathbb{R})$. Thus we have proved the following.

Theorem 5.2.6. *Let Σ be a 3-dimensional connected group of automorphisms of a toroidal circle plane. If Σ is almost simple, then $\Sigma \cong \mathrm{PSL}(2, \mathbb{R})$.*

As a corollary of Theorem 5.2.6, we have

Theorem 5.2.7. *Let Σ be a 3-dimensional connected group of automorphisms of a toroidal circle plane. Then Σ is either solvable or isomorphic to $\mathrm{PSL}(2, \mathbb{R})$.*

Proof. Let L be the Lie algebra of Σ. If L is not simple, then it contains an ideal I whose dimension is either 1 or 2. Hence I is solvable. The quotient L/I then has dimension 2 or 1 correspondingly, and thus also solvable. Therefore L is also solvable.

The claim now follows from Theorem 5.2.6. □

We conclude this section by discussing the action of the group $\mathrm{PSL}(2, \mathbb{R})$.

Theorem 5.2.8. *Let $\Sigma \cong \mathrm{PSL}(2, \mathbb{R})$. Then exactly one of the following occurs.*

(i) Σ fixes every $(+)$-parallel class.

(ii) Σ fixes every $(-)$-parallel class.

(iii) Σ acts diagonally on the point set. The diagonal D is a circle, and Σ fixes D.

Proof. Since $\Delta^\pm$ are normal subgroups of the simple group Σ, either $\Delta^\pm = \Sigma$ or $\Delta^\pm = \{id\}$. If $\Delta^\pm = \Sigma$, then Σ fixes every $(\pm)$-parallel class.

We now consider the case both $\Delta^\pm = \{id\}$. Since Σ contains a subgroup $H \cong \mathrm{SO}(2,\mathbb{R})$ which is not contained in $\Delta^\pm$, by Lemma A.3.5, H and thus Σ acts transitively on $\mathcal{G}^\pm$. Hence Σ acts equivalently to $\mathrm{PSL}(2,\mathbb{R})$ on both $\mathcal{G}^\pm$. It follows that Σ has two orbits: the diagonal $D \cong \mathbb{S}^1$, in a suitable coordinate system, and its complement $\mathcal{P}\backslash D$.

In the remainder of the proof we show that $D \in \mathcal{C}$. Fix $p \in D$. For any point $x \in D\backslash\{p\}$, we denote $D(x,i)$, $i = 1,2$ the two connected components of $D\backslash\{p,x\}$.

1) It is easy to check by direct calculations that for any point q on $D\backslash\{p\}$, the stabiliser $\Sigma_{p,q}$ is isomorphic to $\mathbb{R}$. The orbits of points in $\mathcal{P}\backslash\{[p],[q]\}$ under $\Sigma_{p,q}$ are 1-dimensional and partition $\mathcal{P}\backslash\{[p],[q]\}$. Under a suitable coordinate system with $p = (\infty,\infty)$ and $q = (0,0)$, these orbits can be represented as sets of the form $\{(x,ax) | x > 0\}$ or $\{(x,ax) | x < 0\}$, where each $a > 0$ determines such an orbit.

2) We claim that for every pair of distinct points $q, r \in D\backslash\{p\}$, there exists an orbit O, whose form is either $O := S_{p,q}(\xi)$ or $O := S_{p,r}(\xi)$, intersecting $C := \alpha(p,r,q)$ at least two times. Assume $r \in D(q,2)$. If C intersects D at an additional point v, then O is either $S_{p,q}(r)$ or $S_{p,r}(q)$. Otherwise, at least one intersection of C and D is not transversal. By changing the roles of p,q,r if necessary, we can assume this intersection is r. In this case we choose $O := S_{p,q}(\xi)$ for some ξ. For an illustration of these two cases, compare Figure 5.5.

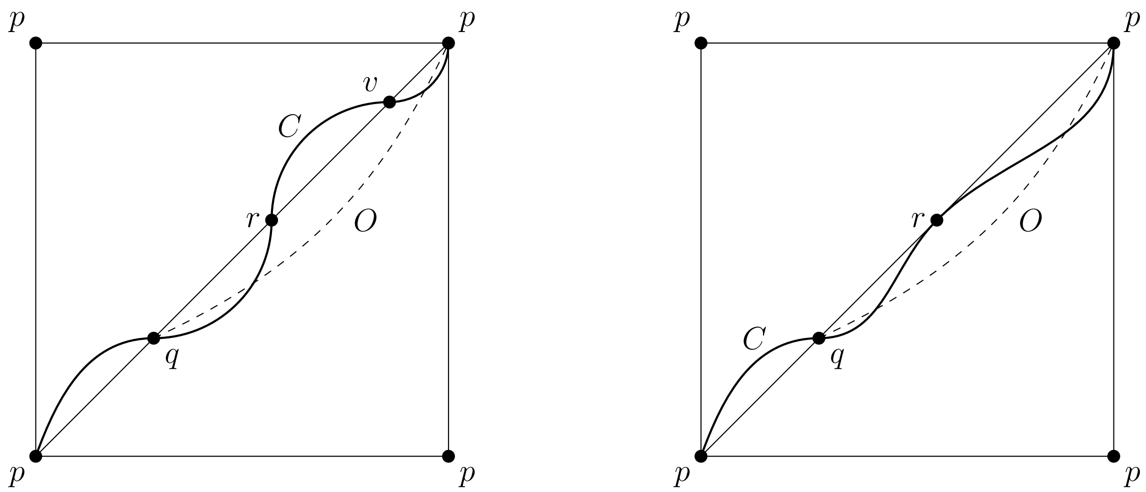

Figure 5.5

3) Maintain the set up from part 2). We show $D(q,2)$ intersects C infinitely many times. Assume $O \neq D(q,2)$, and let $s,t \in O \cap C$, $s \neq t$. Then there exists $\sigma \in \Sigma_{p,q}$ such that

and $\sigma(s) = t$. This implies $\sigma(C) = C$. But since $s \neq t$, $\sigma \neq id$, so that $\sigma(r) \neq r$ and $\sigma(r) \in D(q,2) \cap C$. This implies $|D(q,2) \cap C| \geqslant 2$. Hence, without loss of generality, we can assume $O = D(q,2)$ and $r, \sigma(r) \in O \cap C$. Since σ has infinite order, the points $\sigma^n(r)$ are distinct and belong to $O \cap C$.

4) By interchanging the role of q and $\sigma^n(r)$ for each n, from the arguments in part 2) and 3), the set $D \cap C$ is dense on D. From the compactness of D and C, we have $D \subseteq C$. As there is no proper subset of $\mathbb{S}^1$ homeomorphic to $\mathbb{S}^1$, it follows that $D = C$. This completes the proof. □

5.3 3-dimensional connected group of automorphisms

In this section we determine all possible 3-dimensional connected group of automorphisms of toroidal circle planes. This is the content of Theorem 5.3.3. Essentially, it is proved in Lemmas 5.3.1 and 5.3.2.

Lemma 5.3.1. *If Σ is a 3-dimensional connected group of automorphisms, then exactly one of the following occurs.*

(i) Σ fixes at least one point.

(ii) Σ fixes no points but fixes and acts transitively on at least one circle. In this case $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$.

(iii) Σ fixes no points but fixes and acts transitively on at least one parallel class. In this case either $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$ or $\Sigma \cong \mathrm{SO}(2,\mathbb{R}) \times \mathrm{L}_2$.

Proof. Depending on the transitivity of $\Sigma/\Delta^{\pm}$ on on $\mathcal{G}^{\pm}$, we have 3 cases.

Case 1: Neither of $\Sigma/\Delta^{\pm}$ is transitive on $\mathcal{G}^{\pm}$. By applying Lemma A.3.4, Σ fixes at least one point.

Case 2: Both $\Sigma/\Delta^{\pm}$ are transitive on $\mathcal{G}^{\pm}$. By Brouwer's Theorem, $\Sigma/\Delta^{\pm}$ is isomorphic and acts equivalently to either $\mathrm{SO}(2,\mathbb{R})$ or $\mathrm{PSL}^k(2,\mathbb{R})$. Furthermore, it cannot be the case both $\Sigma/\Delta^{\pm}$ are isomorphic to $\mathrm{SO}(2,\mathbb{R})$, otherwise $\dim \Delta^{\pm} = 2$, which in turn implies $\dim \Delta^+\Delta^- = 4 > \dim \Sigma$, a contradiction. If Σ/Δ^+ is isomorphic to $\mathrm{PSL}^k(2,\mathbb{R})$, then the dimension formula tells us that $\dim \Delta^+ = 0$ and so Σ is almost simple. By Theorem 5.2.6, $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$. From Lemma 5.2.8, it follows that Σ fixes and acts transitively on a circle.

Case 3: Σ/Δ^+ is transitive on $\mathcal{G}^+$, Σ/Δ^- is not transitive on $\mathcal{G}^-$, or vice versa. Here we only consider the stated instance, the other can be dealt with similarly. By Brouwer's Theorem, Σ/Δ^+ is isomorphic and acts equivalently to either $\mathrm{SO}(2,\mathbb{R})$ or $\mathrm{PSL}^k(2,\mathbb{R})$. We then have 2 subcases.

Subcase 3A: $\Sigma/\Delta^+ \cong \mathrm{SO}(2,\mathbb{R})$. If Σ/Δ^- acts trivially on $\mathcal{G}^-$, then $\Sigma = \Delta^-$, and by Lemma 5.1.1, $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$. Otherwise, by Lemma A.3.4, Σ/Δ^- acts transitively on an open subset $I \cong \mathbb{R}$ of $\mathcal{G}^-$. By Lemma A.1.1, Σ is isomorphic to a subgroup of $\Sigma/\Delta^+ \times \Sigma/\Delta^-$. From the dimension formula,

$$2 \leqslant \dim \Sigma/\Delta^- \leqslant 3.$$

From Brouwer's Theorem and Theorem 5.2.6, it follows that $\Sigma/\Delta^- \cong \mathrm{L}_2$. By Lemma A.2.1, $\Sigma \cong \Sigma/\Delta^+ \times \Sigma/\Delta^- = \mathrm{SO}(2,\mathbb{R}) \times \mathrm{L}_2$.

Subcase 3B: $\Sigma/\Delta^+ \cong \mathrm{PSL}^k(2,\mathbb{R})$. We have $\dim \Delta^+ = 0$ and so Σ is almost simple. By Theorem 5.2.6, $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$. By Theorem 5.2.8, Σ fixes every $(-)$-parallel class.

The proof is complete here. □

There are examples of toroidal circle planes for each statement in Lemma 5.3.1. An Artzy-Groh plane $\mathcal{M}_{AG}(f,g)$ in Example 2.1.7 admits the group Φ_1 as a 3-dimensional group of automorphisms fixing exactly one point. A swapping half plane $\mathcal{M}(f,g)$ in Example 2.1.5 has a kernel of dimension 3, and thus, admits a 3-dimensional group of automorphisms fixing no points but every parallel class (of one type). A family of flat Minkowski planes admitting 3-dimensional groups fixing no points but fixing and acting transitively on a circle was constructed by Steinke [Ste04].

More can be said about the first statement in Lemma 5.3.1. We have the following.

Lemma 5.3.2. *Let Σ be a 3-dimensional connected group of automorphisms fixing at least one point. Then exactly one of the following occurs.*

(i) Σ fixes exactly two parallel points. In this case either $\Sigma \cong \mathbb{R} \times \mathrm{L}_2$ or $\Sigma \cong \Phi_d$, for some $d < 0$.

(ii) Σ fixes exactly one point. In this case $\Sigma \cong \Phi_d$, for some $d \in \mathbb{R} \cup \{\infty\}$.

Proof. Under a suitable coordinate system, we can let the fixed point be $p = (\infty, \infty)$. There are 3 cases depending on the transitivity of $\Sigma/\Delta^\pm$ on $\mathcal{G}_\pm \backslash \{[p]_\pm\}$.

Case 1: Neither of $\Sigma/\Delta^\pm$ is transitive on $\mathcal{G}_\pm \backslash \{[p]_\pm\}$. This implies there is an additional fixed point q nonparallel to p. From the dimension formula, for a point r nonparallel to

p and q, we have

$$3 = \dim \Sigma_p = \dim \Sigma_{p,q} = \dim \Sigma_{p,q,r} + \dim \Sigma_{p,q}(r) \leqslant 2,$$

a contradiction. Thus, this case cannot occur.

Case 2: Σ/Δ^- is transitive on $\mathcal{G}^-\backslash\{[p]_-\}$, Σ/Δ^+ is not transitive on $\mathcal{G}^+\backslash\{[p]_+\}$, or vice versa. By Lemma A.3.4, Σ fixes an additional point $q \in [p]_-$. We show Σ fixes at most two points. Suppose the contrary that Σ fixes 3 points on $[p]_-$, thus 3 $(+)$-parallel classes. Then Δ^- fixes these 3 parallel classes pointwise and so must be trivial. By Brouwer's Theorem, $\Sigma \cong \widetilde{\mathrm{PSL}(2,\mathbb{R})}$, which contradicts Theorems 5.2.6 and 5.2.8.

Hence Σ fixes precisely two parallel points p and q. On the derived plane $\mathbb{T}_p$, Σ induces a group of automorphisms that fixes precisely a line. By [GLP83] Theorem 7.5B, Σ is isomorphic to either $\mathbb{R} \times \mathrm{L}_2$ or Φ_d, for some $d < 0$. The actions of these groups are described by

$$\{(x, y) \mapsto (ax, by + c) \mid a, b > 0, c \in \mathbb{R}\},$$

and

$$\{(x, y) \mapsto (a \operatorname{sgn}(x) \cdot |x|^c, c^d y + b) \mid a, c > 0, b \in \mathbb{R}\},$$

respectively.

Case 3: Both $\Sigma/\Delta^\pm$ are transitive on $\mathcal{G}^\pm\backslash\{[p]_\pm\}$. Brouwer's Theorem implies $\Sigma/\Delta^\pm$ is isomorphic to $\mathbb{R}$, L_2 or $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. From Theorems 5.2.6 and 5.2.8, we can rule out the case $\Sigma/\Delta^\pm \cong \widetilde{\mathrm{PSL}(2,\mathbb{R})}$. By Lemma A.1.1, Σ is isomorphic to a subgroup of $\Sigma/\Delta^+ \times \Sigma/\Delta^-$. From the dimension of Σ, it cannot be the case both $\Sigma/\Delta^\pm \cong \mathbb{R}$. This leads to 2 subcases.

Subcase 3A: $\Sigma/\Delta^+ \cong \mathbb{R}$ and $\Sigma/\Delta^- \cong \mathrm{L}_2$, or vice versa. Since $\dim \Sigma = \dim \Sigma/\Delta^+ \times \Sigma/\Delta^-$, by Lemma A.2.1, $\Sigma = \Sigma/\Delta^+ \times \Sigma/\Delta^- \cong \mathbb{R} \times \mathrm{L}_2$. By Brouwer's Theorem, the action of $\Sigma/\Delta^\pm$ is standard, and so the action of Σ is described by the maps

$$\{(x, y) \mapsto (x + b, ay + c) \mid a > 0, b, c \in \mathbb{R}\}.$$

In particular, $\Sigma \cong \Phi_\infty$.

When $\Sigma/\Delta^- \cong \mathbb{R}$ and $\Sigma/\Delta^+ \cong \mathrm{L}_2$, we obtain $\Sigma \cong \Phi_0$ in a similar fashion.

Subcase 3B: $\Sigma/\Delta^\pm \cong \mathrm{L}_2$. We show that $\Sigma \cong \Phi_d$ for some $d \in \mathbb{R} \cup \{\infty\}$.

Let $\overline{\mathcal{P}} := \mathcal{P}\backslash([p]_+ \cup [p]_-) \cong \mathbb{R}^2$. We consider the action of $\Delta^+\Delta^-$ on $\overline{\mathcal{P}}$. We have $\dim \Delta^+ = 1$. If Δ^+ fixes a parallel class $[q]_- \in \mathcal{G}^-\backslash\{[p]_-\}$, then, as a normal subgroup, it fixes the orbit of $[q]_-$ pointwise. But since Σ/Δ^- is transitive on $\mathcal{G}^-\backslash\{[p]_-\}$, Δ^+ is then trivial, which is impossible. Hence Δ^+ is transitive on $\mathcal{G}^-\backslash\{[p]_-\}$. With the same reasoning for Δ^-, we conclude $\Delta^\pm$ is isomorphic and acts equivalently to $\mathbb{R}$ on $\mathcal{G}^\mp\backslash\{[p]_\mp\}$.

Since $\Delta^+\Delta^- = \Delta^+ \times \Delta^-$, $\Delta^+\Delta^-$ is the translation group $\mathbb{R}^2$ (in suitable coordinates) and is sharply transitive on $\overline{\mathcal{P}}$.

Denote $o := (0,0)$. From the sharp transitivity of $\Delta^+\Delta^-$, for each $\sigma \in \Sigma$, there is a unique $\kappa \in \Delta^+\Delta^-$ such that $\kappa(\sigma(o)) = o$, which means $\kappa\sigma \in \Sigma_o$. This implies $\sigma \in \Delta^+\Delta^-\Sigma_o$ and so $\Sigma = \Delta^+\Delta^-\Sigma_0$. By direct calculation, for $\sigma \in \Sigma$ and $\tau_u \in \Delta^+\Delta^-$ a translation with vector u, we have $\sigma\tau_u\sigma^{-1} = \tau_{\sigma(u)} \in \Delta^+\Delta^-$. Hence, $\Delta^+\Delta^-$ is a normal subgroup of Σ. Also, $\Delta^+\Delta^- \cap \Sigma_o$ is trivial. Therefore $\Sigma = \Sigma_o \ltimes \Delta^+\Delta^-$.

From dimension of Σ and $\Delta^\pm$, $\dim \Sigma_o = 1$. By [Sal+95, Corollary 94.39], $\Sigma_o \cong \mathbb{R}$. The action of Σ_o is described by the maps

$$\{(x,y) \mapsto (x, ay) \mid a > 0\},$$

or

$$\{(x,y) \mapsto (ax, a^d y) \mid a > 0\}.$$

This shows that $\Sigma \cong \Phi_d$ for some $d \in \mathbb{R} \cup \{\infty\}$. The proof is complete here. $\square$

The results from Lemmas 5.3.1 and 5.3.2 are summarised in the following main theorem of this section.

Theorem 5.3.3. *If Σ is a 3-dimensional connected group of automorphisms, then exactly one of the following occurs.*

(i) Σ fixes exactly two parallel points. In this case either $\Sigma \cong \mathbb{R} \times \mathrm{L}_2$ or $\Sigma \cong \Phi_d$, for some $d < 0$.

(ii) Σ fixes exactly one point. In this case $\Sigma \cong \Phi_d$, for some $d \in \mathbb{R} \cup \{\infty\}$.

(iii) Σ fixes no points but fixes and acts transitively on at least one circle. In this case $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$.

(iv) Σ fixes no points but fixes and acts transitively on at least one parallel class. In this case either $\Sigma \cong \mathrm{PSL}(2,\mathbb{R})$ or $\Sigma \cong \mathrm{SO}(2,\mathbb{R}) \times \mathrm{L}_2$.

Chapter 6

Modified strongly hyperbolic planes

In this chapter we describe a family of flat Minkowski planes, called *modified strongly hyperbolic planes*. The organisation of this chapter is as follows.

In Section 6.1, we motivate and discuss the ideas behind the construction of modified strongly hyperbolic planes. In Section 6.2, we introduce notations that are used in this chapter. In particular, we define the notation $\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$ (cf. Definition 6.2.6). In Section 6.4, we prove

Theorem 6.0.1. *$\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$ is the circle set of a flat Minkowski plane.*

We call the plane with such a circle set a modified strongly hyperbolic plane (cf. Definition 6.2.7). Theorem 6.0.1 follows from Theorems 6.4.9, 6.4.12, 6.4.17, and 6.4.19, which are verifications of incidence axioms. To prove these theorems, we require some preliminary results, which are covered in Section 6.3.

In Section 6.5, we determine the isomorphism classes and the Klein-Kroll types of a subfamily of modified strongly hyperbolic planes, called *strongly hyperbolic planes* (cf. Definition 6.2.8). Some examples are then presented in Section 6.6.

6.1 Motivation and methods

The original motivation to construct modified strongly hyperbolic planes is to describe a family of toroidal circle planes that exhibit the group

$$\Phi_\infty = \{(x, y) \mapsto (x + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\}$$

as their groups of automorphisms. The construction has its roots in a family of Laguerre planes, called *Laguerre planes of translation type*, introduced by Löwen and Pfüller

[LP87a], [LP87b]. Albeit different incidence structures, these Laguerre planes also have the group Φ_∞ as their groups of automorphisms. Informally speaking, Löwen and Pfüller constructed these Laguerre planes as follows: they first describe a 'nice' function f, then they generate the circle set with the images of f under the group Φ_∞. The properties of f are chosen in a way such that the incidence axioms of Laguerre planes are satisfied. In essence, our construction of (the negative half of) modified strongly hyperbolic planes will follow the same strategy, and our choice of properties for the function f is adapted to fit in the incidence axioms of toroidal circle planes.

We recall from Theorem 2.1.4 that the circle set of a toroidal circle plane has two independent halves and we can combine halves from different planes to obtain a new plane. This means we can first start with a candidate for the negative half and verify the incidence axioms accordingly. Specifically, we define this candidate as a set $\mathcal{C}^-_{MS}(m, f_1, f_2)$ (cf. Definition 6.2.3), and prove

Theorem 6.1.1. *$\mathcal{C}^-_{MS}(m, f_1, f_2)$ is the negative half of a flat Minkowski plane.*

We then define a candidate $\mathcal{C}^+_{MS}(m, f_3, f_4)$ for the positive half as the image of the set $\mathcal{C}^-_{MS}(m, f_3, f_4)$ under the map

$$\varphi : \mathbb{S}^1 \to \mathbb{S}^1 : (x, y) \mapsto (-x, y).$$

As a consequence of Theorem 6.1.1, the union of $\mathcal{C}^-_{MS}(m^-, f_1, f_2)$ and $\mathcal{C}^+_{MS}(m^+, f_3, f_4)$ is then the circle set of a flat Minkowski plane. This union is precisely the circle set $\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$ stated in Theorem 6.0.1.

Therefore, to prove Theorem 6.0.1, it is sufficient to prove Theorem 6.1.1.

The method used in the construction of a modified classical flat Minkowski plane (cf. Example 2.1.6) will also be applied to construct the set $\mathcal{C}^-_{MS}(m, f_1, f_2)$. In particular, this method consists of two steps. First, we generate convex branches and concave branches of circles with images of two *strongly hyperbolic functions* f_1 and f_2 under the group Φ_∞, respectively. Then, we match convex branches with concave branches by a function m.

Besides f_1 and f_2, this method gives us another layer of customisation via the function m. This results in a larger family of planes (than that from following the strategy by Löwen and Pfüller). However, these planes do not necessarily admit Φ_∞ as their group of automorphisms. Consequently, determining isomorphism classes of these planes becomes rather tedious. This remains as future work. In Section 6.5 we will only determine isomorphism classes of (non-modified) strongly hyperbolic planes, which, by construction, admit Φ_∞ as their group of automorphisms.

6.2 Definitions

In this section, we define hyperbolic functions and strongly hyperbolic functions. We also define a (modified) strongly hyperbolic plane, and notations for its positive half, negative half and circle set.

Definition 6.2.1. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *hyperbolic* if it satisfies the following conditions.

(i) $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to +\infty} f(x) = 0$.

(ii) f is strictly convex.

A hyperbolic function is called a *strongly hyperbolic function* if it satisfies additional properties as follows. For clarity of Chapters 6 and 7, we define strongly hyperbolic functions independently from hyperbolic functions.

Definition 6.2.2. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *strongly hyperbolic function* if it satisfies the following conditions.

(i) $\lim_{x\to 0+} f(x) = +\infty$ and $\lim_{x\to +\infty} f(x) = 0$.

(ii) f is strictly convex.

(iii) For each $b \in \mathbb{R}$,
$$\lim_{x\to +\infty} \frac{f(x+b)}{f(x)} = 1.$$

(iv) f is differentiable.

(v) $\ln |f'(x)|$ is strictly convex.

A strongly hyperbolic function f is *normalised* if $f(1) = 1$.

In the construction of Laguerre planes by Löwen and Pfüller, the function f was assumed to be twice differentiable. As noted by the authors in [LP87a] Remark 2.6, this condition was proved to be unnecessary by Schellhammer [Sch81]. In the analog version above, we drop this condition.

With the definition of strongly hyperbolic functions at hand, we define the set $\mathcal{C}^-_{MS}(m, f_1, f_2)$ as follows.

Definition 6.2.3 (Notation $\mathcal{C}^-_{MS}(m, f_1, f_2)$)**.** Let f_1, f_2 be strongly hyperbolic functions. Let $m : \mathbb{R}^+ \to \mathbb{R}^+$ be an orientation-preserving homeomorphism.

For $a > 0, b, c \in \mathbb{R}$, let

$$f_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\} : x \mapsto \begin{cases} af_1(x+b)+c & \text{for } x > -b, \\ -m(a)f_2(-x-b)+c & \text{for } x < -b. \end{cases}$$

Let

$$\begin{aligned} \overline{f_{a,b,c}} &\coloneqq \{(x, f_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(-b, \infty), (\infty, c)\}, \\ F &\coloneqq \{\overline{f_{a,b,c}} \mid a > 0, b, c \in \mathbb{R}\}, \\ \overline{l_{s,t}} &\coloneqq \{(x, sx+t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\}, \\ L &\coloneqq \{\overline{l_{s,t}} \mid s, t \in \mathbb{R}, s < 0\}. \end{aligned}$$

Define $\mathcal{C}^-_{MS}(m, f_1, f_2) \coloneqq F \cup L$.

An element $\overline{f_{a,b,c}}$ in Definition 6.2.3 has two connected components. One component is the graph of a strictly convex function, and the other is the graph of a strictly concave function. For clarity, we state this observation as a definition.

Definition 6.2.4 (Convex/concave branch)**.** Let $\overline{f_{a,b,c}}$ be defined as in Definition 6.2.3. The set

$$\{(x, f_{a,b,c}(x)) \mid x > -b\}$$

is called *the convex branch of* $\overline{f_{a,b,c}}$, and the set

$$\{(x, f_{a,b,c}(x)) \mid x < -b\}$$

is called *the concave branch of* $\overline{f_{a,b,c}}$.

We now define the sets $\mathcal{C}^+_{MS}(m, f_1, f_2)$ and $\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$.

Definition 6.2.5 (Notation $\mathcal{C}^+_{MS}(m, f_1, f_2)$)**.** Let f_1, f_2 be strongly hyperbolic functions. Let $m : \mathbb{R}^+ \to \mathbb{R}^+$ be an orientation-preserving homeomorphism. Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by

$$\varphi : (x, y) \mapsto (-x, y).$$

Define $\mathcal{C}^+_{MS}(m, f_1, f_2) \coloneqq \varphi(\mathcal{C}^-_{MS}(m, f_1, f_2))$.

Definition 6.2.6 (Notation $\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$)**.** For $i = 1..4$, let f_i be a strongly hyperbolic function. Let m^+ and m^- be two orientation-preserving homeomorphisms of $\mathbb{R}^+$.

Define $\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4) := \mathcal{C}^-_{MS}(m^-, f_1, f_2) \cup \mathcal{C}^+_{MS}(m^+, f_3, f_4)$.

By Theorem 6.0.1 (which will be proved in Section 6.4), $\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$ is the circle set of a flat Minkowski plane. We then define the following.

Definition 6.2.7. A *modified strongly hyperbolic plane* $\mathcal{M}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$ is a flat Minkowski plane with the circle set $\mathcal{C}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$.

We say a plane $\mathcal{M}_{MS}(m^-, f_1, f_2; m^+, f_3, f_4)$ is *normalised* if f_1 and f_3 are normalised. When the functions m^- and m^+ are the identity map, we obtain a subfamily called strongly hyperbolic planes.

Definition 6.2.8. A *strongly hyperbolic plane* $\mathcal{M}_{SH}(f_1, f_2; f_3, f_4)$ is the modified strongly hyperbolic plane $\mathcal{C}_{MS}(id, f_1, f_2; id, f_3, f_4)$.

For ease of notation, we denote a strongly hyperbolic plane $\mathcal{M}_{SH}(f_1, f_2; f_3, f_4)$ by $\mathcal{M}_{SH}(f_i)$.

6.3 Preliminaries

In this section we obtain necessary results on strongly hyperbolic functions which are used to verify Theorem 6.1.1. In Subsection 6.3.1, we collect some properties of hyperbolic and strongly hyperbolic functions. We then use these results to study the relationship between strongly hyperbolic functions and their images under the group Φ_∞. This is done in Subsections 6.3.2 and 6.3.3.

6.3.1 Properties of hyperbolic and strongly hyperbolic functions

In this subsection, we prove some results on hyperbolic and strongly hyperbolic functions. Visually (and intuitively), these functions are orientation-reversing homeomorphisms of $\mathbb{R}^+$. We make this more precise in Lemma 6.3.1. We then consider the behaviour of a strongly hyperbolic function at the boundaries of its domain, in Lemmas 6.3.3, 6.3.4, and 6.3.5.

Lemma 6.3.1. *Let f be a hyperbolic function. Then f is strictly decreasing.*

Proof. 1) We first show $f(x) \neq f(y)$ for $x < y \in \mathbb{R}^+$. Suppose for a contradiction there are $x < y \in \mathbb{R}^+$ such that $f(x) = f(y)$. Since $\lim_{\xi \to +\infty} f(\xi) = 0$, there exists $z > y$ satisfying $f(z) < f(x)$. As $x < y < z$, we can rewrite

$$y = tx + (1-t)z,$$

for some $t \in (0,1)$. From strict convexity, we have

$$f(y) < tf(x) + (1-t)f(z).$$

The assumption $f(x) = f(y)$ then yields

$$(1-t)(f(z) - f(x)) > 0,$$

which implies $f(z) > f(x)$, a contradiction.

2) Let $x < y \in \mathbb{R}^+$. Suppose $f(x) < f(y)$. Since f is strictly convex, it is continuous. Also, since $\lim_{x \to 0^+} f(x) = +\infty$, by the IVT, there exists $x' \in (0,x)$ such that $f(x') = f(y)$, contradicting part 1). Therefore $f(x) > f(y)$ and so f is strictly decreasing. □

As a corollary of Lemma 6.3.1, we have

Lemma 6.3.2. *Let f be a hyperbolic function. Then f is a orientation-reversing homeomorphism of $\mathbb{R}^+$.*

We now consider strongly hyperbolic functions. Since strongly functions are strictly convex and differentiable, by Lemma B.3.4, they are continuously differentiable (are of class C^1).

In the three following lemmas, we derive some results on the limits of strongly hyperbolic functions

Lemma 6.3.3. *Let f be a strongly hyperbolic function. Then*

$$\lim_{x \to \infty} \frac{f'(x)}{f'(x+b)} = 1$$

for all $b > 0$.

Proof. Let $h(x) = \ln(|f'(x)|)$. For $b > 0$, we define $h_b : (0, \infty) \to \mathbb{R}$ by

$$h_b(x) = h(x) - h(x+b).$$

Since f is strongly hyperbolic, h is strictly convex and strictly decreasing. By Lemma B.2.2, h_b is strictly decreasing. Also, h_b is bounded below by 0. Therefore $\lim_{x \to \infty} h_b(x)$ exists and nonnegative. Since

$$h(x) - h(x+b) = \ln\left(\left|\frac{f'(x)}{f'(x+b)}\right|\right),$$

it follows that $\lim_{x\to\infty} \dfrac{f'(x)}{f'(x+b)}$ exists. By L'Hospital's Rule, we then have

$$\lim_{x\to\infty} \frac{f'(x)}{f'(x+b)} = \lim_{x\to\infty} \frac{f(x)}{f(x+b)} = 1.$$

The result now follows. □

Lemma 6.3.4. *Let f be a strongly hyperbolic function. Let $s, t \neq 0$. Then the following statements are true.*

(i)

$$\lim_{x\to+\infty} \frac{f'(x)}{f(x)} = 0.$$

(ii)

$$\lim_{x\to+\infty} \frac{f(x+s)-f(x)}{f'(x)} = s.$$

(iii)

$$\lim_{x\to+\infty} \frac{f(x+s)-f(x)}{f(x+t)-f(x)} = \frac{s}{t}.$$

Proof. (i) Let $b > 0$. For $x > b$, by the MVT, there exists $r \in (x-b, x)$ such that

$$f'(r) = \frac{f(x)-f(x-b)}{b}.$$

Since f is strictly convex, f' is strictly increasing, so that $f'(r) < f'(x)$. Dividing both sides by $f(x)$, we get

$$\frac{f(x)-f(x-b)}{bf(x)} < \frac{f'(x)}{f(x)} < 0.$$

Since f is strongly hyperbolic,

$$\lim_{x\to+\infty} \frac{f(x)-f(x-b)}{f(x)} = 1 - \lim_{x\to+\infty} \frac{f(x-b)}{f(x)} = 1-1 = 0.$$

The claim now follows from the Squeeze Theorem.

(ii) We assume $s > 0$. By changing variables, the case $s < 0$ can be proved similarly.

For $x > 0$, by the MVT, there exists $r \in (x, x+s)$ such that

$$f'(r) = \frac{f(x+s)-f(x)}{s}.$$

Since f' is strictly increasing and negative, we have

$$f'(x) < f'(r) < f'(x+s) < 0.$$

Hence

$$sf'(x) < f(x+s) - f(x) < sf'(x+s) < 0,$$

and therefore

$$\frac{sf'(x+s)}{f'(x)} < \frac{f(x+s)-f(x)}{f'(x)} < s.$$

The claim now follows from the Squeeze Theorem and Lemma 6.3.3.

(iii) We rewrite

$$\frac{f(x+s)-f(x)}{f(x+t)-f(x)} = \frac{f(x+s)-f(x)}{f'(x)} \frac{f'(x)}{f(x+t)-f(x)}.$$

The claim now follows from (ii). □

Lemma 6.3.5. *Let f be a strongly hyperbolic function. Then*

$$\liminf_{x \to 0+} \frac{f'(x)}{f(x)} = -\infty.$$

Proof. Let $g(x) \coloneqq \ln f(x)$. Then $g'(x) = \dfrac{f'(x)}{f(x)}$, and

$$\lim_{x \to 0+} g(x) = +\infty.$$

The claim now follows from Lemma B.1.1. □

6.3.2 More on strongly hyperbolic functions, part I

In this subsection, we study the roots of the function $\check{f} : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined as

$$\check{f}(x) = af(x+b) + c - f(x),$$

where f is a strongly hyperbolic function, and $a, b, c \in \mathbb{R}, a > 0$.

We first consider the derivative $\check{f}'(x)$ in Lemma 6.3.6. We then examine three cases of $\check{f}$ depending on the parameters a, b, c in Lemmas 6.3.7, 6.3.8, and 6.3.9.

Lemma 6.3.6. *Let f be a strongly hyperbolic function. Let $a > 0, b, c \in \mathbb{R}$. Let $\check{f}$: $(\max\{-b, 0\}, \infty) \to \mathbb{R}$ be defined by*

$$\check{f}(x) = af(x+b) + c - f(x).$$

Then the derivative

$$\check{f}'(x) = af'(x+b) - f'(x)$$

has at most one root, and if $\check{f}'$ has a root x_0, then $\check{f}'$ changes sign at x_0.

Proof. Since $\ln|f'(x)|$ is strictly convex and strictly decreasing, by Lemma B.2.2, the function $h : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined by

$$h(x) = \ln|f'(x + b)| + \ln a - \ln|f'(x)|$$

is strictly monotonic. Hence, h has at most one root, and if h has a root x_0, then h changes sign at x_0.

It is easy to see that $\check{f}'$ has a root x_0 if and only if h has x_0 as a root, and $\check{f}'(x) \lessgtr 0$ if and only if $h(x) \gtrless 0$. This completes the proof. □

When $b = c = 0$ and $a \neq 1$, the situation is trivial.

Lemma 6.3.7. *Let f be a strongly hyperbolic function. Let $1 \neq a > 0$. Then the function $\check{f} : (0, +\infty) \to \mathbb{R}$ defined by*

$$\check{f}(x) = af(x) - f(x)$$

has no roots.

The following lemma considers the special case when exactly one of b, c is zero. If $a = 1$, then it is easy to see that $\check{f}$ has no roots. We then further assume $a \neq 1$.

Lemma 6.3.8. *Let f be a strongly hyperbolic function. Let $1 \neq a > 0$, and either $b \neq 0, c = 0$ or $b = 0, c \neq 0$. Then the function $\check{f} : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined by*

$$\check{f}(x) = af(x + b) + c - f(x)$$

has at most 1 root. Furthermore, exactly one of the following statements is true.

(i) $\check{f}$ has exactly one root x_0 at which it changes sign. The derivative $\check{f}'$ is non-zero at x_0, and either

$$a > 1, b > 0, c = 0 \text{ or } a > 1, b = 0, c < 0 \text{ or } a < 1, b < 0, c = 0 \text{ or } a < 1, b = 0, c > 0.$$

(ii) $\check{f}$ has no root, and either

$$a < 1, b > 0, c = 0 \text{ or } a < 1, b = 0, c < 0 \text{ or } a > 1, b < 0, c = 0 \text{ or } a > 1, b = 0, c > 0.$$

Proof. We prove this lemma by checking the cases of a, b, c exhaustively. There are 4 cases depending on the sign of b, c. We only prove the cases $b > 0, c = 0$ and $b = 0, c > 0$. The cases $b < 0, c = 0$ and $b = 0, c < 0$ are similar.

Case 1: $b > 0, c = 0$. We have $\check{f}(x) = af(x+b) - f(x)$. If $a < 1$, then

$$\check{f}(x) < f(x+b) - f(x) < 0,$$

and so $\check{f}$ has no roots.

We now claim that if $a > 1$, then $\check{f}$ has exactly one root x_0 at which it changes sign. Let $g : (0, +\infty) \to \mathbb{R}$ be defined by

$$g(x) = \ln f(x+b) + \ln a - \ln f(x).$$

It is easy to see that $\check{f}$ has a root if and only if g has a root, and the sign of $\check{f}(x)$ is the same as the sign of $g(x)$. We have that g is continuous, $\lim_{x\to 0} g(x) = -\infty$, and $\lim_{x\to+\infty} g(x) = \ln a > 0$. Hence g, and thus $\check{f}$, has at least one root.

By Lemma 6.3.6, $\check{f}$ cannot have more than two roots. Suppose for a contradiction that $\check{f}$ has exactly two roots $x_0 < x_1$. By Lemma 6.3.6 and Rolle's Theorem, $\check{f}'$ is non-zero at these roots. By Lemma B.3.4, $\check{f}'$ is continuous and so $\check{f}$ is locally monotone at the roots. Hence $\check{f}$ must change sign at the two roots. It follows that g has two roots at which it changes sign.

Since $\lim_{x\to 0} g(x) = -\infty$, $g(x) < 0$ for $x \in (0, x_0)$. Since g changes sign at x_0 and has no roots between x_0 and x_1, we have $g(x) > 0$ for $x \in (x_0, x_1)$. Since g changes sign at x_1 and has no roots larger than x_1, we have $g(x) < 0$ for $x > x_1$. This contradicts $\lim_{a\to+\infty} g(x) = \ln a > 0$.

Therefore $\check{f}$ has exactly one root x_0. Then g also has exactly one root at x_0. By the IVT, g changes sign at x_0. Then $\check{f}$ also changes sign at x_0. This proves the claim.

Case 2: $b = 0, c > 0$. Then

$$\begin{aligned}\check{f}(x) &= af(x) + c - f(x)\\ &= (a-1)f(x) + c.\end{aligned}$$

If $a > 1$, then $\check{f} > 0$ and has no roots. If $a < 1$, then from Definition 6.2.2, $\check{f}$ has exactly one root at which it changes sign. □

We now deal with the case when both $b, c \neq 0$.

Lemma 6.3.9. *Let f be a strongly hyperbolic function. Let $a > 0, b, c \in \mathbb{R}\backslash\{0\}$. Then the function $\check{f} : (\max\{-b, 0\}, +\infty) \to \mathbb{R}$ defined by*

$$\check{f}(x) = af(x+b) + c - f(x)$$

has at most 2 roots. Furthermore, exactly one of the following statements is true.

(i) $\breve{f}$ *has exactly 2 roots* x_0 *and* x_1 *at which it changes sign. The derivative* $\breve{f}'$ *is non-zero at* x_0 *and* x_1*, and either*

$$a < 1, b < 0, c > 0 \text{ or } a > 1, b > 0, c < 0.$$

(ii) $\breve{f}$ *has exactly one root* x_0 *at which it does not change sign. The derivative* $\breve{f}'$ *also has a root at* x_0*, and either*

$$a < 1, b < 0, c > 0 \text{ or } a > 1, b > 0, c < 0.$$

(iii) $\breve{f}$ *has exactly one root* x_0 *at which it changes sign. The derivative* $\breve{f}'$ *is non-zero at* x_0*, and* $bc > 0$*.*

(iv) $\breve{f}$ *has no roots, and* $bc < 0$*.*

Proof. 1) From Lemma 6.3.6 and Rolle's Theorem, $\breve{f}$ cannot have more than two roots. Also, if $\breve{f}$ has two roots, then $\breve{f}'$ is non-zero at these roots. By Lemma B.3.4, f, and thus $\breve{f}$, is continuously differentiable. Hence $\breve{f}$ must change sign at the two roots.

Assume $\breve{f}$ has exactly one root x_0. If $\breve{f}'(x_0) \neq 0$, then by an argument similar to the above, $\breve{f}$ changes sign at x_0. In the case $\breve{f}(x_0) = \breve{f}'(x_0) = 0$, by Lemma 6.3.6, the derivative $\breve{f}'$ changes sign at x_0. This implies $\breve{f}$ has a local extremum at x_0 and therefore does not change sign.

If none of the above occurs, then it must be the case that $\breve{f}$ has no roots. Therefore, excluding the conditions on the parameters a, b, c, exactly one of the 4 statements is true.

2) We claim that if $bc > 0$, then $\breve{f}$ has exactly one root at which $\breve{f}$ changes sign. We assume that $b > 0, c > 0$; the case $b < 0, c < 0$ is similar. Then $\lim_{x\to 0} \breve{f}(x) = -\infty$, and $\lim_{x\to +\infty} \breve{f}(x) = c$. By the IVT, $\breve{f}$ has at least one root.

From part 1), $\breve{f}$ cannot have more than 2 roots. Suppose for a contradiction that $\breve{f}$ has exactly 2 roots $x_0 < x_1$. Since $\lim_{x\to 0} \breve{f}(x) = -\infty$, $\breve{f}(x) < 0$ for $x \in (0, x_0)$. Since $\breve{f}$ changes sign at x_0 and has no roots between x_0 and x_1, we have $\breve{f}(x) > 0$ for $x \in (x_0, x_1)$. Since $\breve{f}$ changes sign at x_1 and has no roots larger than x_1, we have $\breve{f}(x) < 0$ for $x > x_1$. This contradicts $\lim_{a\to +\infty} \breve{f}(x) = c > 0$. This proves the claim.

3) Assume $\breve{f}$ has exactly 2 roots x_0 and x_1 at which it changes sign. From part 2) and Lemma 6.3.8, it must be the case that $bc < 0$. Assume $b > 0, c < 0$. We rewrite

$$\breve{f}(x) = a\left(f(x+b) - f(x)\right) + c + (a-1)f(x).$$

By Lemma B.2.2, $f(x+b) - f(x)$ is strictly increasing. If $0 < a \leqslant 1$, then $\breve{f}$ is strictly increasing and thus has at most one root, which contradicts our assumption. Therefore $a > 1$.

Similarly, when $b < 0, c > 0$, we have $a < 1$.

4) Assume $\check{f}$ has exactly one root at which it changes sign. We can say there exists x^* such that $\check{f}(x^*) > 0$. Suppose for a contradiction that $b > 0, c < 0$. Then $\lim_{x\to 0} \check{f}(x) = -\infty$, and $\lim_{x\to+\infty} \check{f}(x) = c < 0$. By the IVT, there exist two roots $x_0 \in (0, x*)$ and $x_1 \in (x^*, \infty)$, which contradicts our assumption.

In a similar way, it cannot be the case $b < 0, c > 0$. Hence $bc > 0$.

5) When $\check{f}$ has exactly one root x_0 at which it does not change sign, the conditions on a, b, c follow in a similar manner as in part 3). When $\check{f}$ has no roots, then by part 2), $bc < 0$. This completes the proof. □

6.3.3 More on strongly hyperbolic functions, part II

We now generalise the results from Subsection 6.3.3 for our needs in Section 6.4, where we study the number and nature of intersections between two graphs $C_1 := \overline{f_{a_1,b_1,c_1}}$ and $C_2 := \overline{f_{a_2,b_2,c_2}}$ (for notations cf. Definition 6.2.3). As a preparation, in this subsection we consider the interaction between the convex branches of C_1 and C_2. The interaction between the concave branches of C_1 and C_2 is dependent on that of convex branches. We address this observation by investigating two functions $\check{f}$ and $\hat{f}$ simultaneously, where $\check{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R}$ is defined by

$$\check{f}(x) = a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2,$$

and $\hat{f} : (-\infty, \min\{-b_1, -b_2\}) \to \mathbb{R}$ is defined by

$$\hat{f}(x) = -m(a_1)f_2(-x - b_1) + c_1 + m(a_2)f_2(-x - b_2) - c_2.$$

We have two cases depending on b_i, c_i. These cases correspond to Lemmas 6.3.10 and 6.3.11.

Lemma 6.3.10. *Let f_1, f_2 be strongly hyperbolic functions. Let $a_1, a_2 > 0$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$. Assume $b_1 \neq b_2, c_1 = c_2$ or $b_1 = b_2, c_1 \neq c_2$. Let $\check{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R}$ be defined by*

$$\check{f}(x) = a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2.$$

Let $\hat{f} : (-\infty, \min\{-b_1, -b_2\}) \to \mathbb{R}$ be defined by

$$\hat{f}(x) = -m(a_1)f_2(-x - b_1) + c_1 + m(a_2)f_2(-x - b_2) - c_2.$$

(i) If $a_1 = a_2$, then both $\check{f}$ and $\hat{f}$ have no roots.

(ii) If $a_1 \neq a_2$, then either $\check{f}$ or $\hat{f}$, but not both, has exactly 1 root x_0 at which it changes sign. The corresponding derivative is non-zero at x_0.

Proof. We can assume $a_2 = 1, b_2 = c_2 = 0$.

(i) If $a_1 = 1$, then

$$\check{f}(x) = f_1(x + b_1) + c_1 - f_1(x),$$
$$\hat{f}(x) = -m(1)\left(f_2(-x - b_1) - f_2(-x)\right) + c_1.$$

The case $b_1 = 0, c_1 \neq 0$ is obvious. If $b_1 \neq 0, c_1 = 0$, then $\check{f}(x), \hat{f}(x) \neq 0$ for all x in the domain and thus have no roots.

(ii) Assume $a_1 \neq 1$. For $\hat{f}$, we rewrite

$$\begin{aligned}\frac{1}{m(1)}\hat{f}(x) &= \frac{m(a_1)}{m(1)} f_2(-x - b_1) + \frac{-c_1}{m(1)} - f_2(-x)\\ &= a^* f_2(x^* + b^*) + c^* - f_2(x^*)\end{aligned}$$

where $a^* = \dfrac{m(a_1)}{m(1)}, b^* = -b_1, c^* = \dfrac{-c_1}{m(1)}, x^* = -x$. On the other hand, by Lemma 6.3.8, we have 2 cases on $\check{f}$.

Case 1: $\check{f}$ has one root at which it changes sign, and

$$a_1 > 1, b_1 > 0, c_1 = 0 \text{ or } a_1 > 1, b_1 = 0, c_1 < 0 \text{ or}$$

$$a_1 < 1, b_1 < 0, c_1 = 0 \text{ or } a_1 < 1,_1 b = 0, c_1 > 0.$$

This implies

$$a^* > 1, b^* < 0, c^* = 0 \text{ or } a^* > 1, b^* = 0, c^* > 0 \text{ or}$$
$$a^* < 1, b^* > 0, c^* = 0 \text{ or } a^* < 1,^* b = 0, c^* < 0.$$

Comparing the cases in Lemma 6.3.8, we conclude that $\dfrac{1}{m(1)}\hat{f}(x)$, and consequently $\hat{f}(x)$, has no roots.

Case 2: $\check{f}$ has no roots. We show $\hat{f}$ has one root at which it changes sign. This can be proved similar to the previous case. ☐

Lemma 6.3.11. *Let f_1, f_2 be strongly hyperbolic functions. Let $a_1, a_2 > 0$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$. Assume $b_1 \neq b_2, c_1 \neq c_2$. Let $\check{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R}$ defined by*

$$\check{f}(x) = a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2.$$

Let $\hat{f} : (-\infty, \min\{-b_1, -b_2\}) \to \mathbb{R}$ be defined by

$$\hat{f}(x) = -m(a_1) f_2(-x - b_1) + c_1 + m(a_2) f_2(-x - b_2) - c_2.$$

Assume $\check{f}$ has at least 1 root. Then exactly one of the following is true.

(i) $\check{f}$ *has exactly 2 roots* x_0 *and* x_1 *at which it changes sign,* $\hat{f}$ *has no roots. The derivative* $\check{f}'$ *is non-zero at* x_0 *and* x_1*.*

(ii) $\check{f}$ *has exactly 1 root* x_0 *at which it does not change sign,* $\hat{f}$ *has no roots. The derivative* $\check{f}'$ *also has a root at* x_0*.*

(iii) $\check{f}$ *has exactly 1 root* x_0 *at which it changes sign,* $\hat{f}$ *also has 1 root* x_1 *at which it changes sign. The derivatives* $\check{f}'(x_0)$ *and* $\hat{f}'(x_1)$ *are non-zeros.*

Proof. The proof is based on Lemma 6.3.9 and can be carried out in a similar fashion as in the proof of Lemma 6.3.10. □

6.4 Verification of axioms

In this section, we prove Theorem 6.1.1, that is, $\mathcal{C}^-_{MS}(m, f_1, f_2)$ satisfies the Axiom of Joining and the Axiom of Touching (for the negative half of a toroidal circle plane).

We will verify the Axiom of Joining in two parts: we show the existence and the uniqueness of the circle going through three given non-parallel points p_1, p_2, p_3. Since we only consider the negative half, we will assume the given points are in *negative position* (cf. Definition 6.4.1). The verification of Axiom of Touching will also have two parts: the existence and the uniqueness of the circle C going through a given point q and touches a given circle D at a given point $p \in D$ non-parallel to q. For clarity, we prove each of these four parts in the following four subsections.

6.4.1 Existence of Joining

In this subsection, we prove Theorem 6.4.9, which states that, given a triple of points p_1, p_2, p_3 in negative position, there exists an element $C \in \mathcal{C}^-_{MS}(m, f_1, f_2)$ containing the triple. We define a triple of points in negative position as follows.

Definition 6.4.1. Three points are in *negative position* if they can be joined by an element of the negative half of the classical flat Minkowski plane $\mathcal{M}_C$.

For the clarity of the proof of Theorem 6.4.9, we categorise points in negative position into five types as follows.

Definition 6.4.2 (Negative position types)**.** Let p_1, p_2, p_3 be three points in negative position. We say they are in negative position type

1 if $p_1 = (\infty, \infty), p_2, p_3 \in \mathbb{R}^2$,

2 if $p_1 = (x_1, y_1), p_2 = (\infty, y_2), p_3 = (x_3, \infty),\ x_i, y_i \in \mathbb{R}$,

3 if $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, \infty),\ x_i, y_i \in \mathbb{R}$,

4 if $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (\infty, y_3),\ x_i, y_i \in \mathbb{R}$,

5 if $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3), x_i, y_i \in \mathbb{R}$.

It is easy to verify

Lemma 6.4.3. *Up to permutations, if three points are in negative position, then they are in exactly one of the five negative position types.*

We now show that for three points in each of these types, there is an element in $\mathcal{C}^-_{MS}(m, f_1, f_2)$ containing them. This is done in the following five lemmas. Since the points are in negative position, there are conditions on their coordinates. These conditions are listed as cases in the proof of each lemma (except Lemma 6.4.4).

Lemma 6.4.4. *Let p_1, p_2, p_3 be three points in negative position type 1. Then there exist $s_0 < 0, t_0 \in \mathbb{R}$ such that $\overline{l_{s_0,t_0}}$ contains p_1, p_2, p_3.*

Proof. From Definition 6.2.3, $\mathcal{C}^-_{MS}(m, f_1, f_2)$ contains non-vertical and non-horizontal lines with negative slope. The choice of $\overline{l_{s_0,t_0}}$ is the unique line containing p_2, p_3. □

Lemma 6.4.5. *Let p_1, p_2, p_3 be three points in negative position type 2. Then there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0,b_0,c_0}}$ contains p_1, p_2, p_3.*

Proof. Without loss of generality, we can assume $y_2 = x_3 = 0$ so that $p_2 = (\infty, 0), p_3 = (0, \infty)$. If $\overline{f_{a_0,b_0,c_0}}$ contains $(\infty, 0), (0, \infty)$, then it must be the case that $b_0 = c_0 = 0$. Since the points p_1, p_2, p_3 are in negative position, there are two cases depending on x_1, y_1.

Case 1: $x_1, y_1 > 0$. The equation

$$y_1 = af_1(x_1)$$

has a solution $a_0 = \dfrac{y_1}{f_1(x_1)}$, and we have $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1.

Case 2: $x_1, y_1 < 0$. Since m is an orientation-preserving homeomorphism, the equation

$$y_1 = -m(a)f_2(-x_1)$$

has a solution $a_0 = m\left(-\dfrac{y_1}{f_2(-x_1)}\right)$. Then $\overline{f_{a_0,b_0,c_0}}$ contains p_1 on its concave branch. □

Lemma 6.4.6. *Let p_1, p_2, p_3 be three points in negative position type 3. Then there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0,b_0,c_0}}$ contains p_1, p_2, p_3.*

Proof. We can assume $x_3 = 0, y_2 = 0, x_1 > x_2$. Since $p_3 = (0, \infty)$, if there exists $\overline{f_{a_0,b_0,c_0}}$ that contains p_1, p_2, p_3, then $b_0 = 0$. There are three cases depending on the positions of p_1 and p_2.

Case 1: $0 < x_2 < x_1, y_1 < 0$. We claim that there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1 and p_2. This is true if and only if the system

$$\begin{cases} y_1 = af_1(x_1) + c \\ 0 = af_1(x_2) + c \end{cases} \tag{6.1}$$

has a solution a_0, c_0. We show this is the case. The second equation gives $c = -af_1(x_2)$. Substituting this into the first equation, we obtain

$$y_1 = af_1(x_1) - af_1(x_2).$$

Let

$$a_0 = \frac{y_1}{f_1(x_1) - f_1(x_2)} > 0.$$

The choice of c_0 follows. This proves (6.1) has a solution and thus the claim.

Case 2: $x_2 < 0 < x_1, 0 < y_1$. We claim that there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1 and whose concave branch contains p_2, that is, the system

$$\begin{cases} y_1 = af_1(x_1) + c \\ 0 = -m(a)f_2(-x_2) + c \end{cases}$$

has a solution. It is sufficient to show the function $g : \mathbb{R}^+ \to \mathbb{R}$, defined by

$$g(a) = m(a)f_2(-x_2) + af_1(x_1) - y_1,$$

has a root. This is from the IVT and the following observations: g is continuous, $\lim_{a\to 0} g(a) = -y_1 < 0$, and $\lim_{a\to+\infty} g(a) = +\infty$.

Case 3: $x_2 < x_1 < 0, y_1 < 0$. One can show there exists $\overline{f_{a_0,b_0,c_0}}$ whose concave branch contains p_1 and p_2. This is similar to Case 1. □

Lemma 6.4.7. *Let p_1, p_2, p_3 be three points in negative position type 4. Then there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0,b_0,c_0}}$ contains p_1, p_2, p_3.*

Proof. We assume $y_3 = 0, x_2 = 0, y_1 > y_2$. Let $c_0 = 0$. There are three cases.

Case 1: $0 < y_2 < y_1, x_1 < 0$. We show there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1 and p_2, by showing the system

$$\begin{cases} y_1 = af_1(x_1 + b) \\ y_2 = af_1(b) \end{cases} \tag{6.2}$$

has a solution a_0, b_0. Eliminating the variable a, we get

$$\frac{y_1}{y_2} = \frac{f_1(x_1 + b)}{f_1(b)}. \tag{6.3}$$

We consider the function $g : (-x_1, \infty) \to \mathbb{R}$ defined by $g(b) = \dfrac{f_1(x_1 + b)}{f_1(b)}$. We have $g(b)$ continuous, $\lim_{b\to\infty} g(b) = 1$ by Definition 6.2.2, and $\lim_{b\to -x_1} g(b) = \infty$. By the IVT, there exists b_0 such that $g(b_0) = \dfrac{y_1}{y_2} > 1$, that is, b_0 is a root of (6.3). This shows (6.2) also has a solution.

Case 2: $y_2 < 0 < y_1, 0 < x_1$. We show there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1 and whose concave branch contains p_2, by showing the system

$$\begin{cases} y_1 = af_1(x_1 + b) \\ y_2 = -m(a)f_2(-b) \end{cases} \tag{6.4}$$

has a solution a_0, b_0. Substituting $a = \dfrac{y_1}{f_1(x_1 + b)}$ from the first equation into the second equation, we get

$$y_2 = -m\left(\frac{y_1}{f_1(x_1 + b)}\right) f_2(-b). \tag{6.5}$$

We consider the function $g : (-x_1, 0) \to \mathbb{R}$ defined by

$$g(b) = -m\left(\frac{y_1}{f_1(x_1 + b)}\right) f_2(-b).$$

We have $g(b)$ continuous, $\lim_{b\to -x_1+} g(b) = 0$, and $\lim_{b\to 0-} g(b) = -\infty$. Since $y_2 < 0$, by the IVT, (6.5) has a root b_0. This proves (6.4) has a solution.

Case 3: $y_2 < y_1 < 0$, $x_1 < 0$. One can show there exists $\overline{f_{a_0,b_0,c_0}}$ whose concave branch contains p_1 and p_2. This is similar to Case 1. □

Lemma 6.4.8. *Let p_1, p_2, p_3 be three points in negative position type 5. Then either there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0,b_0,c_0}}$ contains p_1, p_2, p_3; or, there exist $s_0 < 0, t_0 \in \mathbb{R}$ such that $\overline{l_{s_0,t_0}}$ contains p_1, p_2, p_3.*

Proof. We can assume $x_3 = y_3 = 0$ and $0 < x_1 < x_2$. For non-triviality, we also assume the three points are not collinear. There are four cases depending on the values of y_1, y_2.

Case 1: $y_2 < y_1 < 0$ and $y_1 < \frac{x_1}{x_2} y_2 < 0$. We claim that there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1, p_2, p_3. This is true if and only if the system

$$\begin{cases} y_1 = af_1(x_1 + b) + c \\ y_2 = af_1(x_2 + b) + c \\ 0 = af_1(b) + c \end{cases}$$

has a solution. The third equation gives $c = -af_1(b)$. Substitute this into the first two equations and rearrange, we get

$$a = \frac{y_1}{f_1(x_1 + b) - f_1(b)} = \frac{y_2}{f_1(x_2 + b) - f_1(b)}. \tag{6.6}$$

We consider the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(b) = \frac{f_1(x_2 + b) - f_1(b)}{f_1(x_1 + b) - f_1(b)}.$$

It is easy to check that g is continuous and $\lim_{b\to 0+} g(b) = 1$. Also, $\lim_{b\to\infty} g(b) = \frac{x_2}{x_1}$ by Lemma 6.3.4. The conditions of x_i, y_i imply that $1 < \frac{y_2}{y_1} < \frac{x_2}{x_1}$. By the IVT, there exists b_0 such that $g(b_0) = \frac{y_2}{y_1}$. This shows the second equality in (6.6) has a solution. From here the choice of $a > 0$ is evident. This proves the claim.

Case 2: $0 < y_2 < y_1$. We claim that there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1, p_2 and whose concave branch contains p_3, by showing the system

$$\begin{cases} y_1 = af_1(x_1 + b) + c \\ y_2 = af_1(x_2 + b) + c \\ 0 = -m(a)f_2(-b) + c \end{cases} \tag{6.7}$$

has a solution a_0, b_0, c_0. The third equation gives $c = m(a)f_2(-b)$. Taking the difference of the first two equations and rearrange, we get

$$a = \frac{y_1 - y_2}{f_1(x_1 + b) - f_1(x_2 + b)}.$$

We define $h : (-x_1, 0) \to \mathbb{R}$ by

$$h(b) = \frac{y_1 - y_2}{f_1(x_1 + b) - f_1(x_2 + b)}.$$

Substitute $a = h(b)$ into first equation of (6.7) gives

$$h(b)f_1(x_1 + b) + m(h(b))f_2(-b) - y_1 = 0.$$

Let $g : (-x_1, 0) \to \mathbb{R}$ be defined by

$$g(b) = h(b)f_1(x_1 + b) + m(h(b))f_2(-b) - y_1.$$

We show that $g(b) = 0$ has a root b_0. To do this we need to find the limits of $h(b)$ and $g(b)$ at the endpoints of the domain. We have

$$\lim_{b \to -x_1+} h(b) = 0, \lim_{b \to 0-} h(b) = \frac{y_2 - y_1}{f_1(x_1) - f_1(x_2)} > 0,$$

and

$$\lim_{b \to -x_1+} g(b) = -y_2 < 0, \lim_{b \to 0-} g(b) = +\infty.$$

Also, $g(b)$ is continuous, so by the IVT, $g(b)$ has at least a root $b_0 \in (-x_1, 0)$. This shows the system (6.7) has a solution and proves the claim.

Case 3: $y_1 < 0, y_2 > 0$. We show that there exists $\overline{f_{a_0,b_0,c_0}}$ whose concave branch contains p_1, p_3 and whose convex branch contains p_2. This is similar to Case 2.

Case 4: $y_2 < y_1 < 0$ and $\frac{x_1}{x_2} y_2 < y_1 < 0$. We show that there exists $\overline{f_{a_0,b_0,c_0}}$ whose concave branch contains p_1, p_2, p_3. This is similar to Case 1. □

From Lemmas 6.4.3, 6.4.4, 6.4.5, 6.4.6, 6.4.7, and 6.4.8, we have

Theorem 6.4.9 (Axiom of Joining, existence)**.** *Let $p_1, p_2, p_3 \in \mathbb{S}_1 \times \mathbb{S}_1$ be three points in negative position. Then there is at least one element in $\mathcal{C}^-_{MS}(m, f_1, f_2)$ that contains p_1, p_2, p_3.*

6.4.2 Uniqueness of Joining

We now show that two elements $C, D \in \mathcal{C}^-_{MS}(m, f_1, f_2)$ can have at most two intersections. The non-trivial case is when $C = \overline{f_{a_1,b_1,c_1}}$ and $D = \overline{f_{a_2,b_2,c_2}}$. This is treated in Lemmas 6.4.10 and 6.4.11.

Lemma 6.4.10. *Let $C = \overline{f_{a_1,b_1,c_1}}$ and $D = \overline{f_{a_2,b_2,c_2}}$. Assume the convex branch of C intersects the convex branch of D at two points. Then C and D have no other intersections.*

Proof. Assume $p = (x_p, y_p), q = (x_q, y_q)$ are on the convex branches of both C and D. Comparing the cases in Lemmas 6.3.10 and 6.3.11, we have $b_1 \neq b_2$ and $c_1 \neq c_2$, which show that C and D have no intersections at infinity. Also, the equation

$$a_1 f_1(x + b_1) + c_1 = a_2 f_1(x + b_2) + c_2$$

has two roots x_p, x_q. By Lemma 6.3.11, the equation

$$-m(a_1)f_2(-x-b_1)+c_1 = -m(a_2)f_2(-x-b_2)+c_2$$

has no roots. Hence, the concave branch of C does not intersect the concave branch of D.

It is easy to see that the convex branch of C cannot intersect the concave branch of D and vice versa. This completes the proof. □

Lemma 6.4.11. *Let $a_1, a_2 > 0, b_1, b_2, c_1, c_2 \in \mathbb{R}, (a_1, b_1, c_1) \neq (a_2, b_2, c_2)$. Then $C = \overline{f_{a_1,b_1,c_1}}$ and $D = \overline{f_{a_2,b_2,c_2}}$ have at most two points of intersection.*

Proof. We assume C and D have two intersections p, q and show that they can have no other intersections. There are three cases depending on the coordinates of p.

Case 1: $p = (b, \infty), b \in \mathbb{R}$. Then $b_1 = b_2 = -b$. If $q = (x_q, y_q) \in \mathbb{R}^2$, then the proof follows from Lemma 6.3.10. Otherwise, $q = (\infty, c)$ so that $c_1 = c_2 = c$. In this case, $a_1 \neq a_2$ and the proof follows from Lemma 6.3.7.

Case 2: $p = (\infty, c), c \in \mathbb{R}$. This case can be treated similarly to the previous case.

Case 3: $p = (x_p, y_p) \in \mathbb{R}^2$. By symmetry, we may also assume $q = (x_q, y_q) \in \mathbb{R}^2$. It is sufficient to show that if p, q are on convex branches of C and D, then C and D have no other intersections. This is proved in Lemma 6.4.10. □

It is easy to verify that if $C = \overline{f_{a_0,b_0,c_0}}$ and $D = \overline{l_{s_0,t_0}}$, then C and D have at most two points of intersection; and if $C = \overline{l_{s_0,t_0}}$ and $D = \overline{l_{s_1,t_1}}$, then C and D have at most one intersection. We then have the following.

Theorem 6.4.12 (Axiom of Joining, uniqueness)**.** *Two elements $C, D \in \mathcal{C}^-_{MS}(m, f_1, f_2)$ have at most two intersections.*

6.4.3 Existence of Touching

In this subsection, we show that, given an element $C \in \mathcal{C}^-_{MS}(m, f_1, f_2)$ and two points $p \in C, q \notin C, q \nparallel p$, there exists an element $D \in \mathcal{C}^-_{MS}(m, f_1, f_2)$ that contains p, q and touches C at p.

We recall two distinct elements C, D of $\mathcal{C}^-_{MS}(m, f_1, f_2)$ touch (combinatorially) at p if $C \cap D = \{p\}$. The main obstruction of the proof is the situation when C has the form $\overline{f_{a_0,b_0,c_0}}$. There are three cases depending on the position of the point p. These will be covered in Lemmas 6.4.14, 6.4.15, and 6.4.16. As a preparation, we have the following.

Lemma 6.4.13. *Let* $a_1, a_2 > 0, b_1, b_2, c_1, c_2 \in \mathbb{R}, (a_1, b_1, c_1) \neq (a_2, b_2, c_2)$*. If one of the following conditions holds, then* $D_1 = \overline{f_{a_1,b_1,c_1}}$ *and* $D_2 = \overline{f_{a_2,b_2,c_2}}$ *touch at a point* p*.*

(i) $p = (b, \infty)$*,* $a_1 = a_2$*,* $b_1 = b_2 = -b$ *and* $c_1 \neq c_2$*.*

(ii) $p = (\infty, c)$*,* $a_1 = a_2$*,* $b_1 \neq b_2$*, and* $c_1 = c_2 = c$*.*

(iii) $p = (x_p, y_p) \in \mathbb{R}^2$*,* $f_{a_1,b_1,c_1}(x_p) = f_{a_2,b_2,c_2}(x_p)$ *and* $f'_{a_1,b_1,c_1}(x_p) = f'_{a_2,b_2,c_2}(x_p)$*.*

Proof. 1) Assume the first condition holds. It is easy to check that $p \in D_1 \cap D_2$. By Lemma 6.3.10, D_1 and D_2 have no intersections on their convex and concave branches. They cannot have an intersection of the form (∞, c) either, because $c_1 \neq c_2$. Therefore p is the only common point of D_1 and D_2.

A similar conclusion can be made from the second condition.

2) Assume the third condition holds. Since $f_{a_1,b_1,c_1}(x_p) = f_{a_2,b_2,c_2}(x_p)$, the point p is an intersection of D_1 and D_2. If p is on the convex branch of D_1 and the concave branch of D_2, then it is easy to check $f'_{a_1,b_1,c_1}(x_p) = f'_{a_2,b_2,c_2}(x_p)$. So we further assume p is on the convex branches of D_1, D_2. Then the function $\breve{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R}$ defined by

$$\breve{f}(x) = a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2$$

has at least one root x_p such that $\breve{f}'(x_p) = 0$.

If $b_1 = b_2, c_1 \neq c_2$ or $b_1 \neq b_2, c_1 = c_2$, then from Lemma 6.3.10, the derivative $\breve{f}'(x_p)$ is nonzero, which contradicts the previous observation. Hence $b_1 \neq b_2, c_1 \neq c_2$. By Lemma 6.3.11, x_p is the only root of $\breve{f}$. This implies D_1 and D_2 have no other intersections on their convex branches. Also from Lemma 6.3.11, the equation

$$-m(a_1) f_2(-x - b_1) + c_1 = -m(a_2) f_2(-x - b_2) + c_2$$

has no roots, and so D_1 and D_2 have no intersections on their concave branches. From the conditions $b_1 \neq b_2, c_1 \neq c_2$ we conclude D_1 and D_2 have no intersections at infinity, so that p is the only common point D_1 and D_2. This completes the proof. □

Lemma 6.4.14. *Let* $p = (b_0, \infty)$ *and* $C = \overline{f_{a_0,b_0,c_0}}$*. Let* q *be a point such that* $q \notin C, q \nparallel p$*. Then there exist* $a_1 > 0, b_1, c_1 \in \mathbb{R}$ *such that* $D = \overline{f_{a_1,b_1,c_1}}$ *contains* p, q *and touches* C *at* p*.*

Proof. Let $a_1 = a_0, b_1 = b_0$. Depending on the position of q, we let c_1 be described as follows. In each case we verify $c_1 \neq c_0$.

Case 1: $q = (\infty, y_q)$. Since $q \notin C$, $y_q \neq c_0$. We let $c_1 = y_q$, so that $c_1 \neq c_0$.

Case 2: $q = (x_q, y_q), x_q > -b_1$. Let $c_1 = y_q - a_1 f_1(x_q + b_1)$. Since $q \notin C$,

$$y_q \neq a_0 f_1(x_q + b_0) + c_0.$$

In particular, $c_1 \neq c_0$.

Case 3: $q = (x_q, y_q), x_q < -b_1$. Let $c_1 = y_q + m(a_1) f_1(-x_q - b_1)$. Since $q \notin C$,

$$y_q \neq -m(a_0) f_1(-x_q - b_0) + c_0,$$

so that $c_1 \neq c_0$.

Let $D = \overline{f_{a_1,b_1,c_1}}$. Then D contains p, q, and by Lemma 6.4.13, touches C at p. □

Lemma 6.4.15. *Let $p = (\infty, c_0)$ and $C = \overline{f_{a_0,b_0,c_0}}$. Let q be a point such that $q \notin C, q \nparallel p$. Then there exist $a_1 > 0, b_1, c_1 \in \mathbb{R}$ such that $D = \overline{f_{a_1,b_1,c_1}}$ contains p, q and touches C at p.*

Proof. Let $a_1 = a_0, c_1 = c_0$. Depending on the position of q, we choose b_1 as follows.

Case 1: $q = (x_q, \infty)$. Since $q \notin C$, $x_q \neq b_0$. We let $b_1 = x_q$, so that $b_1 \neq b_0$.

Case 2: $q = (x_q, y_q), y_q > c_1$. Since f_1 is surjective on $\mathbb{R}^+$, there exists $b_1 \in (-x_q, \infty)$ satisfying

$$f_1(x_q + b_1) = \frac{y_q - c_1}{a_1} > 0,$$

so that $a_1 f_1(x_q + b_1) + c_1 = y_q$. Since $q \notin C$, we get $y_q \neq a_0(x_q + b_0) + c_0$. Rearranging gives us

$$f_1(x_q + b_0) \neq \frac{y_q - c_0}{a_0},$$

which implies $f_1(x_q + b_1) \neq f_1(x_q + b_0)$. As f_1 is injective on $\mathbb{R}^+$, it follows that $b_1 \neq b_0$.

Case 3: $q = (x_q, y_q), y_q < c_1$. In this case we let $b_1 \in (-\infty, -x_q)$ such that

$$f_1(-x_q - b_1) = \frac{y_q - c_1}{-m(a_1)} > 0.$$

With a similar reasoning to Case 2, we have $b_1 \neq b_0$.

Let $D = \overline{f_{a_1,b_1,c_1}}$. Then D contains p, q, and by Lemma 6.4.13, touches C at p. □

Lemma 6.4.16. *Let $p = (x_p, y_p) \in \mathbb{R}^2$ and $s < 0$. Let q be a point such that $q \nparallel p$. Then exactly one of the following is true.*

(i) There exists $t \in \mathbb{R}$ such that $D = \overline{l_{s,t}}$ contains p and q.

(ii) There exist $a_1 > 0, b_1, c_1 \in \mathbb{R}$ such that $D = \overline{f_{a_1,b_1,c_1}}$ contains p, q and satisfies

$$f'_{a_1,b_1,c_1}(x_p) = s.$$

Proof. We can assume $x_p = y_p = 0$. There are 8 cases depending on q.

Case 1: $q = (\infty, \infty)$. Graphs of the form $\overline{f_{a,b,c}}$ cannot contain q. Let $D = \overline{l_{s,0}}$. Then D contains p and q.

Case 2: $q = (x_q, \infty)$, $x_q \neq 0$. Let $b_1 = -x_q$. If $x_q < 0$, we consider the system

$$\begin{cases} 0 = af_1(b_1) + c \\ s = af_1'(b_1) \end{cases}$$

in variables a, c. The solution is $a_1 = \dfrac{s}{f_1'(b_1)}, c_1 = -a_1 f_1(b_1)$.

If $x_q > 0$, we consider the system

$$\begin{cases} 0 = -m(a)f_2(-b_1) + c \\ s = m(a)f_2'(-b_1) \end{cases}$$

in variables a, c. The solution is $a_1 = m^{-1}\left(\dfrac{s}{f_2'(b_1)}\right), c_1 = -m(a_1)f_1(b_1)$.

Case 3: $q = (\infty, y_q), y_q \neq 0$. Let $c_1 = y_q$. If $y_q < 0$, we consider the system

$$\begin{cases} 0 = af_1(b) + c_1 \\ s = af_1'(b) \end{cases} \tag{6.8}$$

in variables a, b. From the first equation we get $a = \dfrac{-c_1}{f_1(b)}$. Substitute this into the second equation we get

$$\frac{-s}{c_1} = \frac{f_1'(b)}{f_1(b)}. \tag{6.9}$$

Let $g : (0, +\infty) \to \mathbb{R}$ be defined by

$$g(b) = \frac{f_1'(b)}{f_1(b)}.$$

Since f_1 is continuously differentiable, g is continuous. By Lemma 6.3.4, $\lim_{b\to+\infty} g(b) = 0$. By Lemma 6.3.5, g is unbounded below. By the IVT, (6.9) and thus (6.8) has a solution. Then $D = \overline{f_{a_1,b_1,c_1}}$ contains p, q with p on its convex branch, and $f_{a_1,b_1,c_1}'(x_p) = s$.

If $y_q > 0$, we consider the system

$$\begin{cases} 0 = -m(a)f_2(-b) + c_1 \\ s = m(a)f_1'(-b) \end{cases}$$

in variables a, b. Similar to the above, this system also has a solution.

Case 4: $q = (x_q, y_q) \in \mathbb{R}^2$, where $y_q = sx_q$. Graphs of the form $\overline{f_{a,b,c}}$ cannot contain p, q and satisfies $f'_{a,b,c}(x_p) = s$. Let $D = \overline{l_{s,0}}$. Then D contains p and q.

Case 5: $q = (x_q, y_q) \in \mathbb{R}^2$, where $0 < x_q, sx_q < y_q < 0$, or $x_q < 0, 0 < sx_q < y_q$. We claim that there exists $\overline{f_{a_1,b_1,c_1}}$ containing p, q on its convex branch and $f'_{a_1,b_1,c_1}(x_p) = s$, that is, the system

$$\begin{cases} 0 = af_1(b) + c \\ y_q = af_1(x_q + b) + c \\ s = af'_1(b) \end{cases}$$

has a solution a_1, b_1, c_1. From the first equation, we get $c = -af_1(b)$. Substituting this into the remaining equations and rearranging, we have

$$a = \frac{y_q}{f_1(x_q + b) - f_1(b)} = \frac{s}{f'_1(b)}. \tag{6.10}$$

Let $g : (\max\{-x_q, 0\}, \infty) \to \mathbb{R}$ be defined by

$$g(b) = \frac{f_1(x_q + b) - f_1(b)}{f'_1(b)}.$$

We consider the case $0 < x_q, sx_q < y_q < 0$. It is easy to check that $g(b)$ is continuous and $\lim_{b\to 0+} g(b) = 0$. By Lemma 6.3.4, $\lim_{b\to\infty} g(b) = x_q$. By the IVT, there exists b_1 such that $g(b_1) = \dfrac{y_q}{s} \in (0, x_q)$. This shows (6.10) has a solution.

The case $x_q < 0, 0 < sx_q < y_q$ runs similarly to the above.

Case 6: $q = (x_q, y_q) \in \mathbb{R}^2$, where $x_q < 0, y_q < 0$. We claim that there exist a_1, b_1, c_1 such that $\overline{f_{a_1,b_1,c_1}}$ contains p on its convex branch, q on its concave branch, and $f'_{a_1,b_1,c_1}(x_p) = s$, that is, the system

$$\begin{cases} 0 = af_1(b) + c \\ y_q = -m(a)f_2(-x_q - b) + c \\ s = af'_1(b) \end{cases}$$

has a solution a_1, b_1, c_1. The third equation gives $a = \dfrac{s}{f'_1(b)}$. Substitute this into first equation and rearrange we get $c = -af_1(b) = -\dfrac{sf_1(b)}{f'_1(b)}$. Eliminating a, c in the second equation gives us

$$y_q = -m\left(\frac{s}{f'_1(b)}\right) f_2(-x_q - b) - s\frac{f_1(b)}{f'_1(b)}. \tag{6.11}$$

Let $g : (0, -x_q) \to \mathbb{R}$ be defined by

$$g(b) = -m\left(\frac{s}{f'_1(b)}\right) f_2(-x_q - b) - s\frac{f_1(b)}{f'_1(b)}.$$

Then $g(b)$ is continuous, $\liminf_{b \to 0+} g(b) = 0$, and $\lim_{b \to -x_q -} g(b) = -\infty$. Here we used

$$\liminf_{b \to 0^+} \frac{f(b)}{f'(b)} = 0$$

from Lemma 6.3.5. By the IVT, there exists $b_1 \in (0, -x_q)$ such that $g(b_1) = y_q$, that is, the equation (6.11) has a solution. This proves the claim.

Case 7: $q = (x_q, y_q) \in \mathbb{R}^2$, where $0 < x_q, y_q < sx_q < 0$, or $x_q < 0, 0 < y_q < sx_q$. We show there exists $\overline{f_{a_1,b_1,c_1}}$ containing p, q on its on concave branch. This is similar to Case 5.

Case 8: $q = (x_q, y_q) \in \mathbb{R}^2$, where $x_q > 0, y_q > 0$. We show there exists $\overline{f_{a_1,b_1,c_1}}$ containing p on its concave branch and q on its convex branch. This is similar to Case 6. □

If $C = \overline{l_{s_0,t_0}}$, then it is easy to construct an element D containing a point $p \in C$ and a point $q \notin C$ parallel to p. From Lemmas 6.4.13, 6.4.14, 6.4.15, and 6.4.16, we have

Theorem 6.4.17 (Axiom of Touching, existence)**.** *Given $C \in \mathcal{C}_{MS}^-(m, f_1, f_2), p \in C, q \notin C, q \nparallel p$, there exists $D \in \mathcal{C}_{MS}^-(m, f_1, f_2)$ that contains p, q and touches C at p.*

6.4.4 Uniqueness of Touching

We now show that, given an element $C \in \mathcal{C}_{MS}^-(m, f_1, f_2)$ and two points $p \in C, q \notin C, q \nparallel p$, there exists at most element $D \in \mathcal{C}_{MS}^-(m, f_1, f_2)$ that contains p, q and touches C at p. The intuitive strategy is to suppose that there are two circles D_1, D_2 satisfying the conditions, then show that $D_1 = D_2$. We first consider the case when $D_1 = \overline{f_{a_1,b_1,c_1}}$ and $D_2 = \overline{f_{a_2,b_2,c_2}}$ in the following lemma. As a side note, this is the converse of Lemma 6.4.13.

Lemma 6.4.18. *Let $D_1 = \overline{f_{a_1,b_1,c_1}}$ and $D_2 = \overline{f_{a_2,b_2,c_2}}$ touch at a point p.*

(i) If $p = (b, \infty)$, then $a_1 = a_2$, $b_1 = b_2 = b$, and $c_1 \neq c_2$.

(ii) If $p = (\infty, c)$, then $a_1 = a_2$, $b_1 \neq b_2$, and $c_1 = c_2 = c$.

(iii) If $p = (x_p, y_p) \in \mathbb{R}^2$, then $f'_{a_1,b_1,c_1}(x_p) = f'_{a_2,b_2,c_2}(x_p)$.

Proof. 1) If p is a point at infinity, then it is easy to see that $b_1 = b_2, c_1 \neq c_2$ or $b_1 \neq b_2, c_1 = c_2$. Suppose for a contradiction that $a_1 \neq a_2$. Since D_1 and D_2 touch at p, their convex branches have no intersections. This implies the equation

$$a_1 f_1(x + b_1) + c_1 = a_2 f_1(x + b_2) + c_2$$

has no roots. By Lemma 6.3.10, the equation

$$-m(a_1) f_2(-x - b_1) + c_1 = -m(a_2) f_2(-x - b_2) + c_2$$

has one root at which it changes sign. This implies D_1 and D_2 have one intersection on their concave branches, which is a contradiction. Therefore $a_1 = a_2$.

2) We now turn to the case $p = (x_p, y_p) \in \mathbb{R}^2$. Because p is the only intersection of D_1 and D_2, $b_1 \neq b_2, c_1 \neq c_2$.

It is easy to see that if p is on the convex branch of D_1 and the concave branch of D_2, then $f'_{a_1,b_1,c_1}(x_p) = f'_{a_2,b_2,c_2}(x_p)$.

Assume p is on the convex branches of D_1 and D_2. Let $\breve{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R}$ defined by

$$\breve{f}(x) = a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2.$$

By Lemma 6.3.11, it must be the case that $\breve{f}$ has one root x_0 at which it does not change sign. By Lemma 6.3.9, the derivative $\breve{f}'$ also has a root at x_0. This implies $f'_{a_1,b_1,c_1}(x_p) = f'_{a_2,b_2,c_2}(x_p)$.

The case p is on the concave branches of D_1 and D_2 is similar to the above. □

We now prove the main theorem of this subsection.

Theorem 6.4.19 (Axiom of Touching, uniqueness). *Let $C \in \mathcal{C}^-_{MS}(m, f_1, f_2)$, $p \in C$, $q \notin C$, $q \nparallel p$. Then there is at most one element $D \in \mathcal{C}^-_{MS}(m, f_1, f_2)$ that contains p, q and touches C at p.*

Proof. Assume there are $D_1, D_2 \in \mathcal{C}^-_{MS}(m, f_1, f_2)$ that contain p, q and touch C at p. We show that $D_1 = D_2$. There are four cases depending on the coordinates of p.

Case 1: $p = (\infty, \infty)$. Then C, D_1, D_2 are Euclidean lines. D_1 and D_2 are parallel to C and go through q so they must be the same line.

Case 2: $p = (b, \infty)$. Since C, D_1, D_2 contains p, they cannot be Euclidean lines. Let $C = \overline{f_{a_0,b_0,c_0}}$, $D_1 = \overline{f_{a_1,b_1,c_1}}$ and $D_2 = \overline{f_{a_2,b_2,c_2}}$. It follows that $b_1 = b_2 = b_0$. Since the two pairs C, D_1 and C, D_2 touch at p, by Lemma 6.4.18, $a_1 = a_2 = a_0$.

Suppose for a contradiction that $c_1 \neq c_2$. Then q must be either on the convex branches or the concave branches of both D_1, D_2. But this is impossible by the condition $a_1 = a_2$ and Lemma 6.3.10. Therefore $c_1 = c_2$ and $(a_1, b_1, c_1) = (a_2, b_2, c_2)$, so that $D_1 = D_2$.

Case 3: $p = (\infty, c)$. This can be treated similarly to Case 2.

Case 4: $p = (x_p, y_p) \in \mathbb{R}^2$. We only consider the non-trivial case when $C = \overline{f_{a_0,b_0,c_0}}$, $D_1 = \overline{f_{a_1,b_1,c_1}}$ and $D_2 = \overline{f_{a_2,b_2,c_2}}$.

Because C and D_1 touch at a finite point $p \in \mathbb{R}^2$, by Lemma 6.4.18, $f'_{a_0,b_0,c_0}(x_p) = f'_{a_1,b_1,c_1}(x_p)$. Similarly, we have $f'_{a_0,b_0,c_0}(x_p) = f'_{a_2,b_2,c_2}(x_p)$. It follows that $f'_{a_1,b_1,c_1}(x_p) = f'_{a_2,b_2,c_2}(x_p)$.

Suppose for a contradiction that $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$. By Lemma 6.3.10, it cannot be the case $b_1 \neq b_2, c_1 = c_2$ or $b_1 = b_2, c_1 \neq c_2$. If $b_1 \neq b_2, c_1 \neq c_2$, then we obtain a contradiction from Lemma 6.3.11. Hence $b_1 = b_2, c_1 = c_2$. It follows that $a_1 \neq a_2$. But this contradicts Lemma 6.3.7, because C and D have an intersection p. Therefore $(a_1, b_1, c_1) = (a_2, b_2, c_2)$ and so $D_1 = D_2$. This completes the proof. □

6.5 Isomorphism classes and automorphisms of strongly hyperbolic planes

As stated at the beginning of the chapter, determining isomorphism classes and automorphisms of modified strongly hyperbolic planes currently remains as future work. In this section, we only consider strongly hyperbolic planes.

We recall a strongly hyperbolic plane $\mathcal{M}_{SH}(f_i)$ is *normalised* if $f_1(1) = f_3(1) = 1$. Every strongly hyperbolic plane can be described in a normalised form. Throughout this section, we will assume $\mathcal{M}_{SH}(f_i)$ is normalised.

6.5.1 Isomorphisms

Let $\mathrm{Aut}(\mathcal{M}_{SH}(f_i))$ be the full automorphism group of $\mathcal{M}_{SH}(f_i)$. Let Σ^f be the connected component of $\mathrm{Aut}(\mathcal{M}_{SH}(f_i))$. From Definition 6.2.8, the group Φ_∞ is contained in Σ^f.

Let ϕ be an isomorphism between $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$. We have two cases depending on whether ϕ maps (∞, ∞) to (∞, ∞) or not. In the following lemma, we consider the second case.

Lemma 6.5.1. *If $\phi(\infty, \infty) = (x_0, y_0)$, $x_0, y_0 \in \mathbb{R} \cup \{\infty\}, (x_0, y_0) \neq (\infty, \infty)$, then both $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$ are isomorphic to the classical Minkowski plane.*

Proof. We have $\phi \Sigma^f \phi^{-1} = \Sigma^g$.

If $\dim \Sigma^f = \dim \Sigma^g = 3$, then by Lemma A.2.1, $\Sigma^f = \Sigma^g = \Phi_\infty$, and so $\phi \Phi_\infty \phi^{-1} = \Phi_\infty$. This is a contradiction, because the left hand side fixes (x_0, y_0), but the right hand side can only fix (∞, ∞).

If $\dim \Sigma^f = \dim \Sigma^g = 4$, then by Theorem 2.3.5, $\mathcal{M}_{SH}(f_i)$ is isomorphic to either a non-classical swapping half plane $\mathcal{M}(f, id)$ or a non-classical generalized Hartmann plane $\mathcal{M}(r_1, s_1; r_2, s_2)$. The former case cannot occur because the plane $\mathcal{M}(f, id)$ does not admit Φ_∞ as a group of automorphisms. In the latter case, we have

$$\Sigma^f = \Phi = \{(x, y) \mapsto (rx + a, sy + b) \mid r, s > 0, a, b \in \mathbb{R}\},$$

which fixes only the point (∞, ∞). A contradiction is obtained with an argument similar to the previous case.

Hence $\dim \Sigma^f = \dim \Sigma^g = 6$, and the planes are classical. $\square$

We now consider the case ϕ maps (∞, ∞) to (∞, ∞). Because translations in $\mathbb{R}^2$ are automorphisms of both planes $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$, we can also assume ϕ maps $(0, 0)$ to $(0, 0)$. Since ϕ induces an isomorphism between the two Desarguesian derived planes, we can represent ϕ by matrices in $\mathrm{GL}(2, \mathbb{R})$. From the fact that ϕ maps parallel classes to parallel classes, ϕ is of the form $\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$ or $\begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}$, for $r, s \in \mathbb{R}\backslash\{0\}$.

For $r', s' \in \mathbb{R}\backslash\{0\}$, we rewrite

$$\begin{bmatrix} r' & 0 \\ 0 & s' \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \cdot A,$$

and

$$\begin{bmatrix} 0 & r' \\ s' & 0 \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A,$$

where $r = |r'|, s = |s'|$, and

$$A \in \mathfrak{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Let the matrices in $\mathfrak{A}$ be A_1, A_2, A_3, A_4, respectively.

We note that ϕ is a composition of maps of the form A_i, $\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$, or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. To describe ϕ, it is sufficient to describe each of these maps. This is done in Lemmas 6.5.2, 6.5.3, 6.5.4.

Lemma 6.5.2 (The map $\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$)**.** *For $r, s > 0$, if*

$$\phi : \mathcal{P} \to \mathcal{P} : (x, y) \mapsto (rx, sy)$$

is an isomorphism between $\mathcal{M}_{SH}(f_i)$ *and* $\mathcal{M}_{SH}(g_i)$*, then*

$$(I) = \begin{cases} g_1(r) f_1\left(\frac{1}{r}\right) = 1,\ g_2(r) f_2\left(\frac{1}{r}\right) = 1 \\ g_1(x) = \dfrac{f_1(x/r)}{f_1(1/r)} \\ g_2(x) = \dfrac{f_2(x/r)}{f_1(1/r)} \\ g_3(x) = \dfrac{f_3(x/r)}{f_2(1/r)} \\ g_4(x) = \dfrac{f_4(x/r)}{f_2(1/r)} \end{cases}$$

holds. Conversely, if there exists $r > 0$ *such that* (I) *holds, then for every* $s > 0$*,*

$$\phi : \mathcal{P} \to \mathcal{P} : (x, y) \mapsto (rx, sy)$$

is an isomorphism between $\mathcal{M}_{SH}(f_i)$ *and* $\mathcal{M}_{SH}(g_i)$*.*

Proof. We prove the equations for $g_i, i = 1, 2$. The cases $i = 3, 4$ are similar. We have that ϕ maps the convex branch

$$\{(x, f_1(x)) \mid x > 0\}$$

onto

$$\{(x, sf_1(x/x)) \mid x > 0\}.$$

But because $r, s > 0$ (so that points in the first quadrant are mapped onto the first quadrant) and ϕ maps circles to circles, the convex branch is mapped onto

$$\{(x, ag_1(x)) \mid x > 0\},$$

for some $a > 0$. This implies

$$sf_1(x/r) = ag_1(x)$$

for all $x > 0$. For $x = 1, x = r$, the normalised condition gives

$$f_1\left(\frac{1}{r}\right) = \frac{1}{g_1(r)} = \frac{a}{s},$$

so that $g_1(r) f_1\left(\frac{1}{r}\right) = 1$ and $g_1(x) = \dfrac{f_1(x/r)}{f_1(1/r)}$.

Similarly, for the concave branch, we get

$$sf_2\left(\frac{x}{r}\right) = ag_2(x),$$

for $x > 0$. This gives $g_2(x) = \dfrac{f_2(x/r)}{f_1(1/r)}$.

The converse direction is easily verified. $\square$

Lemma 6.5.3 (The map A_i). *For $A \in \mathfrak{A}$, if*

$$\phi : \mathcal{P} \to \mathcal{P} : (x,y) \mapsto (x,y) \cdot A$$

is an isomorphism between $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$, then one of the following occurs.

(i) If $A = A_1$, then $(A1) = \begin{cases} g_1 = f_1 \\ g_2 = f_2 \\ g_3 = f_3 \\ g_4 = f_4 \end{cases}$ *holds.*

(ii) If $A = A_2$, then $(A2) = \begin{cases} g_1 = \dfrac{1}{f_4(1)} f_4 \\ g_2 = \dfrac{1}{f_4(1)} f_3 \\ g_3 = \dfrac{1}{f_2(1)} f_2 \\ g_4 = \dfrac{1}{f_2(1)} f_1 \end{cases}$ *holds.*

(iii) If $A = A_3$, then $(A3) = \begin{cases} g_1 = f_3 \\ g_2 = f_4 \\ g_3 = f_1 \\ g_4 = f_2 \end{cases}$ *holds.*

(iv) If $A = A_4$, then $(A4) = \begin{cases} g_1 = \dfrac{1}{f_2(1)} f_2 \\ g_2 = \dfrac{1}{f_2(1)} f_1 \\ g_3 = \dfrac{1}{f_4(1)} f_4 \\ g_4 = \dfrac{1}{f_4(1)} f_3 \end{cases}$ *holds.*

Proof. Direct calculations. □

Lemma 6.5.4 (The map $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$). *If $\phi : (x,y) \mapsto (y,x)$ is an isomorphism between $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$, then*

$$(II) = \begin{cases} g_1 = f_1^{-1} \\ g_2 = f_2^{-1} \\ g_3 = \dfrac{1}{f_4^{-1}(1)} f_4^{-1} \\ g_4 = \dfrac{1}{f_4^{-1}(1)} f_3^{-1} \end{cases}$$

holds.

Proof. Direct calculations. □

6.5.2 Group dimension classification

In this section, we classify strongly hyperbolic planes with respect to group dimension. Let $\mathcal{M}_C$ be the classical Minkowski plane (cf. Example 2.1.3). We denote

$$\Sigma^C := \langle PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R}), \{(x,y) \mapsto (y,x)\}\rangle$$

the full automorphism group of $\mathcal{M}_C$ (cf. [PS01] p.221).

Theorem 6.5.5. *A normalised plane $\mathcal{M}_{SH}(f_i)$ is isomorphic to the classical Minkowski plane $\mathcal{M}_C$ if and only if $f_i(x) = 1/x$, for $i = 1..4$.*

Proof. The "if" direction is straightforward. We prove the converse direction.

For $i = 1..4$, let $g_i(x) = 1/x$. Then $\mathcal{M}_{SH}(g_i)$ is isomorphic to $\mathcal{M}_C$ and $\Sigma^g \cong \Sigma^C$. Assume $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_C$ are isomorphic. Then $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$ are isomorphic. Let ϕ be an isomorphism between $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$.

Because $\Sigma^g \cong \Sigma^C$ is transitive on the torus, for $(x_0, y_0) \in \mathbb{S}_1 \times \mathbb{S}_1$, there exists an automorphism $\psi \in \Sigma^C$ such that $\psi(x_0, y_0) = (\infty, \infty)$. If $\phi((\infty, \infty)) = (x_0, y_0) \neq (\infty, \infty)$, then $\psi\phi$ is an isomorphism from $\mathcal{M}_{SH}(f_i)$ to $\mathcal{M}_{SH}(g_i)$ that maps (∞, ∞) to (∞, ∞).

From Lemmas 6.5.2, 6.5.3, 6.5.4, we then have $f_i(x) = 1/x$. □

We recall

$$\mathfrak{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Let $\mathfrak{A}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathfrak{A}$. Based on Theorem 6.5.5 and results from Subsection 6.5.1, we have the following.

Theorem 6.5.6. *Up to isomorphisms in $\mathfrak{A}$ and $\mathfrak{A}'$, two strongly hyperbolic planes $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$ are isomorphic if and only if there exists $r > 0$ such that (I) holds. (cf. Lemma 6.5.2).*

Proof. Let ϕ be an isomorphism between $\mathcal{M}_{SH}(f_i)$ and $\mathcal{M}_{SH}(g_i)$. If the planes are classical, then the proof follows from Theorem 6.5.5.

If the planes are nonclassical, from Lemma 6.5.1, ϕ maps (∞,∞) to (∞,∞). Up to isomorphisms in $\mathfrak{A}$ and $\mathfrak{A}'$, ϕ has the form $\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$, for some $r, s > 0$. The proof follows from Lemma 6.5.2. □

From Theorem 6.5.6, we are able to determine when a strongly hyperbolic plane $\mathcal{M}_{SH}(f_i)$ is isomorphic to a generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$ (cf. Example 2.1.8).

Theorem 6.5.7. *A nonclassical normalised plane $\mathcal{M}_{SH}(f_i)$ is isomorphic to a generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$ if and only if $f_1(x) = x^{-r_1}, f_2(x) = x^{-r_2}, f_3(x) = s_1^{-1}x^{-r_1}, f_4(x) = s_2^{-1}x^{-r_2}$, where $r_1, s_1, r_2, s_2 > 0$, $(r_1, s_1, r_2, s_2) \neq (1, 1, 1, 1)$.*

We now prove the main theorem of this section.

Theorem 6.5.8 (Group dimension classification)**.** *A normalised strongly hyperbolic plane $\mathcal{M}_{SH}(f_i)$ has group dimension*

6 if and only if $f_i(x) = x^{-1}$;

4 if and only if $f_1(x) = x^{-r_1}, f_2(x) = x^{-r_2}, f_3(x) = s_1^{-1}x^{-r_1}, f_4(x) = s_2^{-1}x^{-r_2}$, where $r_1, s_1, r_2, s_2 \in \mathbb{R}^+$, and $(r_1, s_1, r_2, s_2) \neq (1, 1, 1, 1)$;

3 in all other cases.

Proof. A strongly hyperbolic plane $\mathcal{M}_{SH}(f_i)$ has group dimension at least 3 because it admits the group Φ_∞ as a group of automorphisms.

The plane $\mathcal{M}_{SH}(f_i)$ has group dimension 6 if and only if it is classical. The form of f_i follows from Theorem 6.5.5. By Theorem 5.1.3, if $\mathcal{M}_{SH}(f_i)$ has group dimension 5 then it is isomorphic to the clasical plane. By Theorem 5.1.4, $\mathcal{M}_{SH}(f_i)$ has group dimension 4 if and only if it is isomorphic to either a non-classical swapping half plane $\mathcal{M}(f, id)$ or a non-classical generalized Hartmann plane $\mathcal{M}(r_1, s_1; r_2, s_2)$. The former case cannot occur, however, because the plane $\mathcal{M}(f, id)$ does not admit Φ_∞ as a group of automorphisms. In the latter case, the form of f_i follows from Theorem 6.5.7. □

6.5.3 The Klein-Kroll types

In this section we determine the Klein-Kroll types of strongly hyperbolic planes.

Theorem 6.5.9. *A strongly hyperbolic plane $\mathcal{M}_{SH}(f_i)$ has Klein-Kroll type*

VII.F.23 *if it is isomorphic to the classical flat Minkowski plane;*

III.C.19 *if it is isomorphic to a Hartmann plane* $\mathcal{M}_{GH}(r, 1; r, 1), r \in \mathbb{R}^+, r \neq 1$*;*

III.C.1 *in all other cases.*

Proof. Let Σ be the full automorphism group of $\mathcal{M}_{SH}(f_i)$. Let $p = (\infty, \infty)$. The group Σ has a subgroup

$$H = \{\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + a, y + b) \mid a, b \in \mathbb{R}\}$$

of Euclidean translations, which is transitive on $\mathcal{P}\backslash\{[p]_+ \cup [p]_-\}$. Hence a plane $\mathcal{M}_{SH}(f_i)$ has Klein-Kroll type at least III.C.

By Theorem 2.4.1, the only possible types are III.C.1, III.C.18, III.C.19, or VII.F.23. By Theorem 2.4.2, if $\mathcal{M}_{SH}(f_i)$ is of type III.C.18, then it is isomorphic to a proper Artzy-Groh plane $\mathcal{M}_{AG}(f, g)$ (cf. Example 2.1.7), where f and g are normalised and odd except when $f = g$ is inversely semi-multiplicative. In this case, $\Sigma = \Phi_1$, and in particular, $\dim \Sigma = 3$. On the other hand, Σ contains the group Φ_0. By Lemma A.2.1, $\Sigma = \Phi_0$, a contradiction. The result now follows from Theorem 2.4.2. □

6.6 Examples

In this section, we collect some examples of strongly hyperbolic functions. Those of the form $f(x) = ax^{-k}$, where $a, k > 0$, are strongly hyperbolic functions. These functions are used to construct generalised Hartmann planes. In the following we consider other examples.

Example 6.6.1. The function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f(x) = \frac{1}{x + \arctan(x)},$$

is strongly hyperbolic. We verify the conditions in Definition 6.2.2.

(i) $\lim_{x\to 0+} f(x) = 1/(0^+ + 0) = \infty$, $\lim_{x\to\infty} f(x) = 1/(\infty + \pi/2) = 0$.

(ii) The second derivative is

$$f''(x) = \frac{2(x^4 + 5x^2 + x \cdot \arctan(x) + 4)}{(x^2 + 1)^2(x + \arctan(x))^3} > 0.$$

Hence f is strictly convex.

(iii) For $b > 0$,

$$\frac{x + \pi/2}{x + b - \pi/2} \geqslant \frac{x + \arctan(x)}{x + b + \arctan(x + b)} \geqslant \frac{x - \pi/2}{x + b + \pi/2}.$$

Squeeze theorem gives

$$\lim_{x\to\infty} \frac{f(x+b)}{f(x)} = \lim_{x\to\infty} \frac{x + \arctan(x)}{x + b + \arctan(x + b)} = 1.$$

(iv) The derivative of $f(x)$ is

$$f'(x) = -\frac{\frac{1}{x^2+1} + 1}{(x + \arctan(x))^2},$$

which is continuous on $\mathbb{R}^+$.

(v) We have

$$\begin{aligned} \ln|f'(x)| &= \ln\left(\frac{1}{x^2+1} + 1\right) - 2\ln(x + \arctan(x)) \\ &= \ln(x^2+2) - \ln(x^2+1) - 2\ln(x + \arctan(x)). \end{aligned}$$

The second derivative of $\ln|f'(x)|$ is

$$\begin{aligned} \ln|f'(x)|'' &= -\frac{2(x^2-2)}{(x^2+2)^2} + \frac{2(x^2-1)}{(x^2+1)^2} + \frac{2(x^4 + 6x^2 + 2x\cdot\arctan(x) + 4)}{(x^2+1)^2(x + \arctan(x))^2} \\ &= \frac{2A}{(x^2+2)^2(x^2+1)^2(x+\arctan(x))^2}, \end{aligned}$$

where

$$\begin{aligned} A = {} & x^8 + 13x^6 + 8x^5\arctan(x) + 35x^4 + 3x^4\arctan(x)^2 + \\ & 14x^3\arctan(x) + 38x^2 + 3x^2\arctan(x)^2 + 4x\arctan(x) - 2\arctan(x)^2 + 16. \end{aligned}$$

All the terms of A are positive except the term $-2\arctan(x)^2$. But

$$-2\arctan(x)^2 + 16 > 0,$$

so $A > 0$. Therefore $\ln|f'(x)|'' > 0$ and so $\ln|f'(x)|$ is strictly convex.

Example 6.6.2. If f is a strongly hyperbolic function, then f^{-1} is not necessarily a strongly hyperbolic function. Consider $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f(x) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right).$$

The inverse is given by

$$f^{-1}(x) = \frac{1}{\sinh(x)}.$$

As mentioned in [Har81], for $b \neq 0$,

$$\lim_{x\to\infty} \frac{\sinh(x)}{\sinh(x+b)} = e^{-b} \neq 1,$$

and hence f^{-1} is not strongly hyperbolic. We now show that f is strongly hyperbolic.

(i) $\lim_{x\to 0+} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = 0$.

(ii) For each $b \in \mathbb{R}$, we have

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x+b)}{f(x)} &= \lim_{x\to\infty} \frac{f'(x+b)}{f'(x)} \\ &= \lim_{x\to\infty} \frac{x\sqrt{x^2+1}}{(x+b)\sqrt{(x+b)^2+1}} = 1. \end{aligned}$$

(iii) f is strictly convex. We have

$$f''(x) = \frac{2x^2+1}{x^2(x^2+1)^{3/2}} > 0.$$

(iv) The derivative

$$f'(x) = -\frac{1}{x\sqrt{x^2+1}}$$

is continuous on $\mathbb{R}^+$.

(v) $\ln|f'(x)|$ is strictly convex. The second derivative of $\ln|f'(x)|$ is

$$\frac{2x^4+x^2+1}{(x^3+x)^2},$$

which is positive.

Remark 6.6.3. The following two functions are not strongly hyperbolic functions:

$$f(x) = \frac{1}{x-1+\sin(x)/x},$$

$$g(x) = \frac{1}{x^3+\sin(x)}.$$

Direct calculations show that $\ln|f'(x)|$ and $\ln|g'(x)|$ are not strictly convex.

Chapter 7

On toroidal circle planes admitting groups of automorphisms fixing exactly one point

For ease of communication, in this chapter we refer to a toroidal circle plane admitting a group G of automorphisms as 'a plane with group G'.

The construction in Chapter 6 yields a family of flat Minkowski planes with group Φ_∞, namely strongly hyperbolic planes. In this chapter, we prove the converse: if a toroidal circle plane admits Φ_∞ as a group of automorphisms, then it is isomorphic to a strongly hyperbolic plane. Based on this we will also characterise planes with group Φ_0. This is the content of Section 7.2.

In Section 7.3, we characterise planes with group Φ_1. We show that these planes are precisely the Artzy-Groh planes in Example 2.1.7. We note that this section is independent from Section 7.2.

In Section 7.1, we describe the circle set of planes with 3-dimensional groups fixing exactly one point. The main result in this section, Theorem 7.1.7, provides a common framework which we use to obtain results in Sections 7.2 and 7.3.

Section 7.4 is devoted to the investigation of Polster planes. Based on the results from Chapter 5, Sections 7.2 and 7.3, we prove in Theorem 7.4.2 that the automorphism group of a Polster plane is of dimension 2.

7.1 The standard representation of planes with 3-dimensional groups fixing exactly one point

The goal of this section is to describe the circle set of planes with 3-dimensional groups fixing exactly one point. By Theorem 5.3.3, these planes admit the group Φ_d, for some $d \in \mathbb{R} \cup \{\infty\}$, as a group of automorphisms. In Subsection 7.1.1, we first show that a plane with such group must have a Desarguesian derived plane. We then describe the circle set in Subsection 7.1.2.

7.1.1 The existence of a Desarguesian derived plane

Let $\mathbb{T}$ be a toroidal circle plane with 3-dimensional group fixing exactly one point p. Let $\mathbb{T}_p$ be the derived $\mathbb{R}^2$-plane at p.

Lemma 7.1.1. *If two lines of $\mathbb{T}_p$ have no intersection, then they are derived from either two circles or two parallel classes of the same type.*

Proof. This follows from the definition of a derived plane. □

Lemma 7.1.2. *$\mathbb{T}_p$ is a skew parabola plane.*

Proof. The group Σ induces a 3-dimensional point-transitive group of automorphisms $\overline{\Sigma}$ on the derived plane $\mathbb{T}_p$. By Theorem 2.3.7, $\mathbb{T}_p$ is isomorphic to one of the following planes.

(i) the real hyperbolic plane $\mathrm{H}(\mathbb{R})$. But this case cannot occur because the automorphism group of $\mathrm{H}(\mathbb{R})$ is $\mathrm{PSL}(2, \mathbb{R})$ (cf. [Sal67b] Theorem 5.3).

(ii) hyperbolic arc planes. We show that these planes cannot be a derived plane of a toroidal circle plane.

The line set of a hyperbolic arc plane of type 1 consists of the horizontals, the verticals, and translations of the curve $C = \{(x, x^s) | x > 0\}$ for some $s \leqslant -1$. Since C intersects $y = 1$ but not $y = -1$, it follows from Lemma 7.1.1 that C and all the horizontals must be derived from circles. With a similar argument, the translations of C and the verticals are also derived from circles. This implies there are no lines derived from parallel classes, contradicting the definition of a derived plane.

For the hyperbolic arc planes of type 2, we see that all the lines of $\mathbb{T}_p$ are also derived from circles of $\mathbb{T}$. This also leads to a contradiction as above.

(iii) exponential arc planes. By applying Lemma 7.1.1 as in the case of hyperbolic arc planes, we conclude that $\mathbb{T}_p$ cannot be a exponential arc plane either.

(iv) skew parabola plane. Since we have ruled out other options, this is the only possible case.

The claim now follows. □

We now prove the main result of this subsection.

Theorem 7.1.3. *If a toroidal circle plane $\mathbb{T}$ admits a 3-dimensional connected group of automorphisms Σ fixing exactly one point p, then the derived plane $\mathbb{T}_p$ is Desarguesian.*

Proof. The group Σ induces a 3-dimensional group of automorphisms $\overline{\Sigma}$ on $\mathbb{T}_p$. From [Sal67b] Theorem 5.16 and 5.17 (or [Sal+95] Theorem 36.2 in the context of projective planes), if $\mathbb{T}_p$ is a proper skew parabola plane, then $\overline{\Sigma}$ fixes exactly one equivalence class of parallel lines. But since Σ fixes the sets $\mathcal{G}^{\pm}$, $\overline{\Sigma}$ has to fix at least two equivalence classes of parallel lines of $\mathbb{T}_p$ derived from the parallel classes of $\mathbb{T}$. Hence $\mathbb{T}_p$ cannot be a proper skew parabola plane. This completes the proof. □

7.1.2 The circle set in standard representation

In this subsection, we prove Theorem 7.1.7, which states that, if a toroidal circle plane $\mathbb{T}$ in standard representation admits Φ_d as a group of automorphisms, then $\mathcal{C}^-(f_1, f_2, \Phi_d)$ is the negative half of $\mathbb{T}$, for two hyperbolic functions f_1, f_2. We define the notation $\mathcal{C}^-(f_1, f_2, \Phi_d)$ as follows.

Definition 7.1.4 (Notation $\mathcal{C}^-(f_1, f_2, \Phi_d)$)**.** Let $d \in \mathbb{R} \cup \{\infty\}$. Let $f_1, f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be two continuous functions. Define $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ by

$$f(x) = \begin{cases} f_1(x) & \text{for } x > 0, \\ -f_2(-x) & \text{for } x < 0. \end{cases}$$

Let $C_f := \{(x, f(x)) \mid x \in \mathbb{R}\backslash\{0\}\} \cup \{(0, \infty), (\infty, 0)\}$. Let F be the set of images of C_f under Φ_d. Let $\overline{l_{s,t}} := \{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\}$, and $L := \{\overline{l_{s,t}} \mid s, t \in \mathbb{R}, s < 0\}$.

Define $\mathcal{C}^-(f_1, f_2, \Phi_d) := F \cup L$.

By Theorem 7.1.3, we have

Lemma 7.1.5. *If a toroidal circle plane $\mathbb{T}$ in standard representation admits Φ_d as a group of automorphisms, then there exist two functions $f_1, f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mathcal{C}^-(f_1, f_2, \Phi_d)$ is the negative half of $\mathbb{T}$.*

To prepare for the next lemma, we recall the definition of a hyperbolic function from Definition 6.2.1. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *hyperbolic* if it satisfies the following conditions.

(i) $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = 0$.

(ii) f is strictly convex.

Lemma 7.1.6. *If $\mathcal{C}^-(f_1, f_2, \Phi_d)$ is the negative half of a toroidal circle plane $\mathbb{T}$, then f_1, f_2 are hyperbolic functions.*

Proof. We show that f_1 is a hyperbolic function, the case f_2 is similar. Since circles are homeomorphisms of $\mathbb{S}^1$, $f_1(x)$ must be a continuous bijection from $\mathbb{R}^+$ onto $\mathbb{R}^+$. Also, because $\mathcal{C}^-(f_1, f_2, \Phi_d)$ is a negative half, $f_1(x)$ is strictly decreasing. Hence $\lim_{x\to 0^+} f_1(x) = \infty$ and $\lim_{x\to\infty} f_1(x) = 0$.

Suppose f_1 is not strictly convex. By Lemma B.2.3, there exists $x_1 > 0$ such that the function $g : \mathbb{R}^+\backslash\{x_1\} \to \mathbb{R}$ defined by

$$g(x) = \frac{f_1(x) - f_1(x_1)}{x - x_1},$$

is not increasing. Also, $g(x)$ is continuous, negative, with $\lim_{x\to\infty} g(x) = 0$, and thus $g(x)$ cannot be decreasing either. Hence g is not monotone, and therefore not injective; there exist $x_2, x_3 \in \mathbb{R}^+\backslash\{x_1\}$ such that $g(x_2) = g(x_3)$, that is,

$$\frac{f_1(x_1) - f_1(x_2)}{x_1 - x_2} = \frac{f_1(x_1) - f_1(x_3)}{x_1 - x_3}.$$

But then the straight line

$$y = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \cdot x + f(x_1) - \frac{f(x_1) - f(x_2)}{x_1 - x_2} \cdot x_1$$

has negative slope and intersects $y = f_1(x)$ at three points $(x_1, f_1(x_1))$, $(x_2, f_1(x_2))$ and $(x_3, f_1(x_3))$, contradicting the Axiom of Joining. □

From Lemmas 7.1.5 and 7.1.6, we have

Theorem 7.1.7. *If a toroidal circle plane $\mathbb{T}$ in standard representation admits Φ_d as a group of automorphisms, then there exist two hyperbolic functions f_1, f_2 such that $\mathcal{C}^-(f_1, f_2, \Phi_d)$ is the negative half of $\mathbb{T}$.*

7.2 Characterisation of strongly hyperbolic planes

The main results in this section are the following Theorems 7.2.1 and 7.2.2.

Theorem 7.2.1. *A toroidal circle plane $\mathbb{T}$ in standard representation is isomorphic to a strongly hyperbolic plane if and only if $\mathbb{T}$ admits Φ_∞ as a group of automorphisms.*

Let φ be the homeomorphism of $\mathbb{S}^1$ defined by

$$\varphi : (x, y) \mapsto (y, x).$$

If $\mathcal{C}$ is the circle set of a toroidal circle plane $\mathbb{T}$, then it is easy to verify that $\varphi(\mathcal{C})$, defined in the canonical way, is also the circle set of a toroidal circle plane. Let $\varphi(\mathbb{T})$ be the toroidal circle plane whose circle set is $\varphi(\mathcal{C})$. As a consequence of Theorem 7.2.1, we have

Theorem 7.2.2. *Let $\mathbb{T}$ be a toroidal circle plane. Then $\varphi(\mathbb{T})$ is isomorphic to a strongly hyperbolic plane if and only if $\mathbb{T}$ admits Φ_0 as a group of automorphisms.*

Our goal in this section is to verify Theorem 7.2.1. Subsection 7.2.1 contains preliminary results for the proof, which is presented in Subsection 7.2.2.

7.2.1 Preliminaries

In the next Subsection 7.2.2, we will prove, by means of contradiction, that if $\mathcal{C}^-(f_1, f_2, \Phi_\infty)$ is the negative half of $\mathbb{T}$, then f_1 and f_2 are strongly hyperbolic functions. By Theorem 7.1.7 in the previous section, we know that f_1 and f_2 are hyperbolic functions. In this subsection, we collect some technical results to verify that they satisfy the additional conditions of a strongly hyperbolic function.

In Lemmas 7.2.3 and 7.2.4, we assume the case f_i is a non-differentiable hyperbolic function. We then consider the case f_i is differentiable but $\ln|f_i'|$ is not strictly convex in Lemma 7.2.5. In both cases, we essentially show that the Axiom of Joining is violated.

Lemma 7.2.3. *Let f be a hyperbolic function. If f is not differentiable, then there exist $u, v \in \mathbb{R}$ such that the function $\widetilde{f} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by*

$$\widetilde{f}(x) = \ln|f_-'(x + u)| + v - \ln|f_-'(x)|$$

changes sign at least two times.

Proof. 1) Let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h(x) = \ln(|f'_-(x)|)$. By applying Lemma B.3.1 for f, we have that h is strictly decreasing. Also by Lemma B.3.2, h is left-continuous.

Since f is strictly convex, by Lemma B.3.3, any discontinuity of f'_- is a jump discontinuity. From the assumption that f is not differentiable, there exists x_0 at which f'_- has a jump discontinuity. Then h also has a jump discontinuity at x_0.

2) Let $\varepsilon = h(x_0^-) - h(x_0^+) > 0$. From the definition $h(x_0^-) = \lim_{a \to x_0^-} h(a)$, there exists $\delta_a > 0$ such that if $a \in (x_0 - \delta_a, x_0)$, $h(a) - h(x_0^-) < \varepsilon$. Since h is left-continuous, $h(a^-) - h(x_0^-) < \varepsilon$.

Similarly, there exists $\delta_b > 0$ such that if $b \in (x_0, x_0 + \delta_b)$, $h(x_0^+) - h(b) < \varepsilon$. Since h is strictly decreasing, $h(x_0^+) - h(b^+) < \varepsilon$.

Also because h is strictly decreasing, we have $h(x_0^+) > h(b^-)$ and $h(a^+) > h(x_0^-)$.

3) Let $\delta = \min\{\delta_a, \delta_b\} > 0$. Choose $a \in (x_0 - \delta, x_0)$ and let $b = 2x_0 - a \in (x_0, x_0 + \delta)$. For $u = a - x_0 < 0$, $v = -\varepsilon < 0$, we define $\widetilde{f} : (-u, \infty) \to \mathbb{R}$ as

$$\begin{aligned}\widetilde{f}(x) &= h(x+u) + v - h(x)\\ &= \ln|f'_-(x+u)| + v - \ln|f'_-(x)|.\end{aligned}$$

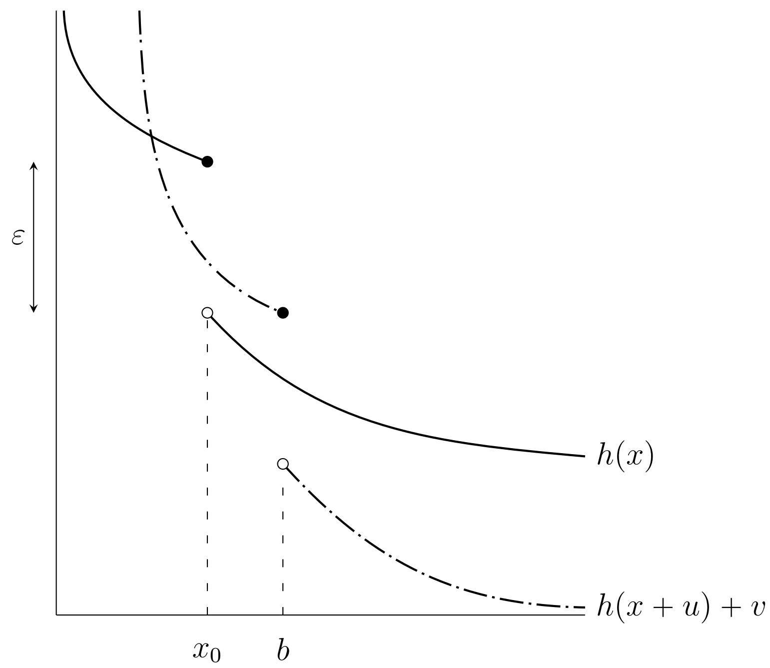

Figure 7.1

From part 2), we have

$$\begin{aligned}
\widetilde{f}(x_0^-) &= h(x_0^- + u^-) + v - h(x_0^-) = h(a^-) - \varepsilon - h(x_0^-) < 0, \\
\widetilde{f}(x_0^+) &= h(x_0^+ + u^+) + v - h(x_0^+) = h(a^+) - \varepsilon - h(x_0^+) = h(a^+) - h(x_0^-) > 0, \\
\widetilde{f}(b^-) &= h(b^- + u^-) + v - h(b^-) = h(x_0^-) - \varepsilon - h(b^-) = h(x_0^+) - h(b^-) > 0, \\
\widetilde{f}(b^+) &= h(b^+ + u^+) + v - h(b^+) = h(x_0^+) - \varepsilon - h(b^+) < 0.
\end{aligned}$$

Hence $\widetilde{f}$ changes sign at x_0 and b. For an illustration, compare Figure 7.1. The proof is complete here. □

Lemma 7.2.4. *Let f be a hyperbolic function. If there exist $u, v \in \mathbb{R}$ such that the function $\widetilde{f} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by*

$$\widetilde{f}(x) = \ln|f'_-(x+u)| + v - \ln|f'_-(x)|$$

changes sign at least 2 times, then there exist $a > 0, b, c \in \mathbb{R}$, $(a, b, c) \neq (1, 0, 0)$, such that the function $\breve{f} : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined by

$$\breve{f}(x) = af(x+b) + c - f(x)$$

has at least 3 roots.

Proof. 1) Assume there exist $u, v \in \mathbb{R}$ such that the function $\widetilde{f} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by

$$\widetilde{f}(x) = \ln|f'_-(x+u)| + v - \ln|f'_-(x)|$$

changes sign at x_0 and x_1. Let $a = e^v > 0$ so that $v = \ln a$. Also, let $b = u$. We consider the function $h : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined by

$$h(x) = af(x+b) - f(x).$$

Then h is continuous. The left derivative of h exists and is given by

$$h'_-(x) = af'_-(x+b) - f'_-(x).$$

It is easy to check that $\widetilde{f} \lesseqgtr 0$ if and only if $h'_-(x) \gtreqless 0$. This implies h'_- also changes sign at the same two points x_0 and x_1.

2) We now show that h has local extrema at x_0 and x_1. Since h'_- changes sign at x_0, there exists $r > 0$ such that, without loss of generality, $h'_-(x) < 0$ for all $x \in (x_0 - r, x_0)$, and $h'_-(x) > 0$ for all $x \in (x_0, x_0 + r)$. In this case, we show that h has a local minimum at x_0.

As $h'_-(x) < 0$ for all $x \in (x_0 - r, x_0)$, by Lemma B.1.3, h is decreasing on $(x_0 - r, x_0)$. Since h is continuous, $h(x) \geqslant \lim_{\xi \to x_0^-} h(\xi) = h(x_0)$ for all $x \in (x_0 - r, x_0)$. Similarly, h is increasing on $(x_0, x_0 + r)$ and $h(x) \geqslant h(x_0)$ for all $x \in (x_0, x_0 + r)$. Hence h has a local minimum at x_0.

A similar reasoning shows that h also has local extremum at x_1.

3) Assume $\max\{-b, 0\} = 0$ so that $\lim_{x\to\infty} h(x) = 0$ and $\lim_{x\to 0+} h(x) = -\infty$. Since h has at least 2 local extrema, it has a local minimum and a local maximum. Without loss of generality, we can assume h has a local minimum at x_0 and a local maximum at x_1 satisfying $h(x_0) < h(x_1)$. Let $c \in (h(x_0), h(x_1))$. Then by the IVT, $h(x) = c$ has at least 3 roots.

The case $\max\{-b, 0\} = -b$ is similar. The result now follows. □

Lemma 7.2.5. *Let f be a hyperbolic function. Assume f is differentiable. If the function $\ln|f'|$ is not strictly convex, then there exist $a > 0, b, c \in \mathbb{R}$, $(a, b, c) \neq (1, 0, 0)$, such that the function $\check{f} : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined by*

$$\check{f}(x) = af(x + b) + c - f(x)$$

has at least three roots.

Proof. Let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h(x) = \ln(|f'(x)|)$.

We note the following properties of h. By Lemma 6.3.1, f' is negative. By Lemma B.3.4, f' is continuous and strictly increasing. Hence h is continuous and strictly decreasing on $\mathbb{R}^+$. By Lemma B.1.1, $\liminf_{x\to 0^+} f'(x) = -\infty$. It follows that $\limsup_{x\to 0^+} h(x) = +\infty$.

We now have two cases depending on the convexity of h.

Case 1: If h is convex but not strictly convex, then by Lemma B.2.4, there exists an interval (u, v) on which h is affine. In particular, there exist $r, s \in \mathbb{R}$ such that for all $x \in (u, v)$,

$$h(x) = rx + s.$$

This implies

$$f'(x) = -e^{rx+s}.$$

Let $b \in (0, v - u)$ and $a = e^{-rb}$. Then for $x \in (u, v - b)$,

$$af'(x + b) = f'(x),$$

by direct calculations. Integrating both sides gives $af(x + b) + c = f(x)$, for some $c \in \mathbb{R}$. Define $\check{f}(x)$ as stated in the lemma. Then $\check{f}(x)$ has infinitely many roots.

Case 2: If h is not convex, then by Lemma B.2.2, there exists $u > 0$ such that the function $\hat{h} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$, defined by

$$\hat{h}(x) = h(x) - h(x+u),$$

is not decreasing. Also, since $\limsup_{x\to 0^+} h(x) = +\infty$, it follows that

$$\limsup_{x\to 0^+} \hat{h}(x) = +\infty,$$

and so $\hat{h}$ cannot be increasing either. Hence $\hat{h}$ is not monotone. Then there exists $a > 0$ such that the function $\hat{h} - \ln a$ has two roots at which it changes sign. Let $v = \ln a$, so that

$$\hat{h} - \ln a = \ln(|f'(x)|) - \ln(|f'(x+u)|) - v.$$

Since f is differentiable, $f' = f'_-$. It follows that the function $\widetilde{f} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by

$$\widetilde{f}(x) = \ln|f'_-(x+u)| + v - \ln|f'_-(x)|$$

changes sign at least two times. The claim now follows from Lemma 7.2.4. □

7.2.2 Proof of Theorem 7.2.1

If $\mathbb{T}$ admits Φ_∞ as a group of automorphisms, then by Theorem 7.1.7, $\mathcal{C}^-(f_1, f_2, \Phi_\infty)$ is the negative half of $\mathbb{T}$, for two hyperbolic functions f_1, f_2. We now prove that f_1 and f_2 are strongly hyperbolic functions in Lemmas 7.2.6 and 7.2.7, respectively.

Lemma 7.2.6. *If $\mathcal{C}^-(f_1, f_2, \Phi_\infty)$ is the negative half of $\mathbb{T}$, then f_1 is a strongly hyperbolic function.*

Proof. By Theorem 7.1.7, f_1 is a hyperbolic function. We now show that f_1 satisfies the remaining properties of a strongly hyperbolic function.

1) We show that for each $b \in \mathbb{R}$,

$$\lim_{x\to+\infty} \frac{f_1(x+b)}{f_1(x)} = 1.$$

We first prove the case $b > 0$. It is easy to check that a circle of $\mathcal{C}^-(f_1, f_2, \Phi_\infty)$ going through points $(0, 1)$ and $(\infty, 0)$ is the graph of

$$y = \frac{f(x+b)}{f_1(b)},$$

for some $b > 0$. From the Axiom of Joining, given three points $(0, 1)$ and $(\infty, 0)$ and (x, y) with $x > 0, 0 < y < 1$, there is a unique circle going through them, that is, the equation

$y = \dfrac{f(x+b)}{f_1(b)}$ has a unique solution $b > 0$. Since $x + b > 0$, $f(x+b) = f_1(x+b)$. So we define $h : \mathbb{R}^+ \to \mathbb{R}$ as

$$h(b) = \frac{f_1(x+b)}{f_1(b)}.$$

Then $h(b)$ is continuous, $h(b) \in (0,1)$ for all $b > 0$, and $\lim_{b\to 0^+} h(b) = 0$. To satisfy the Axiom of Joining, $h(b)$ must be strictly increasing and $\lim_{b\to\infty} h(b) = 1$. In particular,

$$\lim_{b\to\infty} \frac{f_1(x+b)}{f_1(b)} = 1.$$

Reverse the role of b and x gives the condition as stated.

The case $b = 0$ is trivial. When $b < 0$, we can rewrite

$$\frac{f_1(x+b)}{f_1(x)} = \left(\frac{f_1(x'-b)}{f_1(x')}\right)^{-1}$$

and apply the argument as in the case $b > 0$. The claim now follows.

2) By Lemmas 7.2.3 and 7.2.4, if f_1 is not differentiable, then the Axiom of Joining is violated. Also from Lemma 7.2.5, we see that $\ln|f_1'(x)|$ must be strictly convex. The result now follows. □

Lemma 7.2.7. *If $\mathcal{C}^-(f_1, f_2, \Phi_\infty)$ is the negative half of $\mathbb{T}$, then f_2 is a strongly hyperbolic function.*

Proof. Let $\varphi : \mathcal{P} \to \mathcal{P}$ be a homeomorphism defined by $\varphi : (x,y) \mapsto (-x,-y)$. Then φ induces an isomorphism between two toroidal circle planes $\mathbb{T}$ and $\varphi(\mathbb{T})$ in the canonical way. In particular, $\varphi(\mathcal{C}^-(f_1, f_2, \Phi_\infty)) = \mathcal{C}^-(f_2, f_1, \Phi_\infty)$ is the negative half of $\varphi(\mathbb{T})$. By applying Lemma 7.2.6 for $\varphi(\mathbb{T})$, f_2 is then strongly hyperbolic. □

Hence the set $\mathcal{C}^-(f_1, f_2, \Phi_\infty)$ coincides with the set $\mathcal{C}^-_{MS}(id, f_1, f_2)$ in Definition 6.2.3. In a similar way, we obtain a description for the positive half via strongly hyperbolic functions. Theorem 7.2.1 now follows.

7.3 Characterisation of Artzy-Groh planes

In this section, we extend [PS01] Theorem 4.4.13 by

Theorem 7.3.1. *A toroidal circle plane $\mathbb{T}$ in standard representation is isomorphic to an Artzy-Groh plane if and only if $\mathbb{T}$ admits Φ_1 as a group of automorphisms.*

The 'only if' direction follows from the construction of the Artzy-Groh plane (cf. Example 2.1.7). It remains to prove the 'if' direction.

Let $\mathbb{T}$ be a toroidal circle plane that admits Φ_1 as a group of automorphisms. By Lemma 7.1.7, $\mathcal{C}^-(f_1, f_2, \Phi_1)$ is the negative half of $\mathbb{T}$, for some hyperbolic functions f_1, f_2. We maintain the notations f and C_f from Definition 7.1.4 of $\mathcal{C}^-(f_1, f_2, \Phi_1)$. From the construction of $\mathcal{C}^-(f_1, f_2, \Phi_1)$, we have the following.

Lemma 7.3.2. *If a toroidal circle plane $\mathbb{T}$ admits Φ_1 as a group of automorphisms, then for an element $D \in \mathcal{C}^-(f_1, f_2, \Phi_1)$ not containing (∞, ∞), there exists a unique mapping $\phi \in \Phi_1$ such that $\phi(D) = C_f$.*

To show that $\mathcal{C}^-(f_1, f_2, \Phi_1)$ is the negative half of an Artzy-Groh plane, it is sufficient to show that f is differentiable. We start with the following definition of triangles and set of triangles $T(m_1, m_2, m_3)$.

Definition 7.3.3. A *triangle abc* is a triple $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)$ of non-collinear points in $\mathbb{R}^2$ satisfying $a_1 < b_1 < c_1$. For $m_1, m_2, m_3 < 0$ pairwise distinct, let $T(m_1, m_2, m_3)$ be the set of triangles abc such that

$$\begin{cases} \dfrac{a_2 - b_2}{a_1 - b_1} &= m_1 \\ \dfrac{b_2 - c_2}{b_1 - c_1} &= m_2 \\ \dfrac{a_2 - c_2}{a_1 - c_1} &= m_3. \end{cases}$$

Each point a, b, c is called a *vertex* of the triangle abc. Given a triangle abc, we denote by $\phi(abc)$ the triangle formed by the images of a, b, c under a mapping $\phi \in \Phi_1$. The existence of such a triangle comes from the fact that Φ_1 is connected and preserves the slope of lines. It is easy to check that Φ_1 is sharply transitive on $T(m_1, m_2, m_3)$. In particular, we have the following.

Lemma 7.3.4. *Let $m_1, m_2, m_3 < 0$ pairwise distinct. Let $abc \in T(m_1, m_2, m_3)$ and $\phi \in$*

Φ_1. Then $\phi(abc) \in T(m_1, m_2, m_3)$.

We now show that f is differentiable in Lemmas 7.3.5 and 7.3.6.

Lemma 7.3.5. *If f is not differentiable, then there exist $m_1, m_2, m_3 < 0$ such that $T(m_1, m_2, m_3)$ has no elements on C_f.*

Proof. For convenience, in this proof we denote C_f by C. We assume f_1 is not differentiable, the case f_2 is similar. Since f_1 is strictly convex and strictly decreasing, by Lemma B.3.3, there exists a point x_0 such that

$$f'_{1-}(x_0) < f'_{1+}(x_0) < 0.$$

Let $m_1, m_2, m_3 \in (f'_-(x_0), f'_+(x_0))$ pairwise distinct. Then $m_1, m_2, m_3 < 0$. Suppose there exists a triangle $abc \in T(m_1, m_2, m_3)$ that lies on C. We can assume its vertices are on the convex branch C' of C. Let $C_1 = \{(x, f_1(x)) \mid 0 < x \leqslant x_0\}$ and $C_2 = \{(x, f_1(x)) \mid x > x_0\}$. Then $C' = C_1 \cup C_2$. By the pigeonhole principle, either C_1 or C_2 contains at least two vertices of abc. This implies, for some $i \in \{1, 2, 3\}$, there exist, without loss of generality, $x_1 < x_2 \in (0, x_0]$ such that

$$\frac{f_1(x_2) - f_1(x_1)}{x_2 - x_1} = m_i.$$

By Lemma B.1.2, there exists $x_3 \in (x_1, x_2)$ such that $f'_{1-}(x_3) \leqslant m_i \leqslant f'_{1+}(x_3)$. By Lemma B.3.5, we get

$$f'_{1-}(x_3) \leqslant m_i \leqslant f'_{1+}(x_3) < f'_{1-}(x_0) \leqslant m_i \leqslant f'_{1+}(x_0),$$

which is a contradiction. The result now follows. □

Lemma 7.3.6. *If a toroidal circle plane $\mathbb{T}$ admits Φ_1 as a group of automorphisms, then f is differentiable.*

Proof. If f is not differentiable, then by Lemma 7.3.5, there exists $m_1, m_2, m_3 < 0$ such that $T(m_1, m_2, m_3)$ has no elements on C. Let $abc \in T(m_1, m_2, m_3)$. From the Axiom of Joining, there exists a circle $D \in \mathcal{C}^-(f_1, f_2, \Phi_1)$ containing abc. Since a, b, c are not collinear, D does not contain (∞, ∞). By Lemma 7.3.2, there exists $\phi \in \Phi_1$ such that $\phi(D) = C$, which implies $\phi(abc) \subset C$. But by Lemma 7.3.4, $\phi(abc) \in T(m_1, m_2, m_3)$, which contradicts our assumption. This completes the proof. □

Comparing with Example 2.1.7, we see that $\mathcal{C}^-(f_1, f_2, \Phi_1)$ is then the negative half of an Artzy-Groh plane. With the same work as above, we obtain a similar description of the positive half, and so $\mathbb{T}$ is an Artzy-Groh plane. This verifies Theorem 7.3.1.

7.4 On the automorphism group of a Polster plane

In this section, we determine the dimension of the automorphism group of a Polster plane (cf. Example 2.1.9) and describe its connected component.

We maintain the notation in Example 2.1.9. For convenience, we denote the restriction of $f_{a,0,0}$ to $\mathbb{R}^+$ by f_a, that is, for $a > 0$, let

$$f_a(x) = \begin{cases} af\left(\dfrac{x+1}{a}\right) & \text{for } x \geqslant x_a^*, \\ af\left(\dfrac{x}{a}\right) - 1 & \text{for } 0 < x \leqslant x_a^*, \end{cases}$$

where x_a^* satisfies the equation

$$af\left(\frac{x_a^*+1}{a}\right) = x_a^*.$$

Lemma 7.4.1. *Let $\mathbb{T}_P(f,g)$ be a Polster plane. For $d \in \mathbb{R} \cup \{\infty\}$, the group Φ_d is not a group of automorphims of $\mathbb{T}_P(f,g)$.*

Proof. A plane $\mathbb{T}_P(f,g)$ is not a flat Minkowksi plane since it does not satisfy the Axiom of Touching. By Theorems 7.2.1, 7.2.2, 7.3.1, for $d = 0, 1, \infty$, the group Φ_d is not a group of automorphims of $\mathbb{T}_P(f,g)$. Fix $d \in \mathbb{R}\backslash\{0,1\}$. Suppose for a contradiction that the group Φ_d is a group of automorphims of $\mathbb{T}_P(f,g)$.

For $1 \neq r > 0$, let $\phi : (x,y) \mapsto (rx, r^d y) \in \Phi_d$. Let $a_0 > 0$. Under ϕ, the set of points

$$\{(x, f_{a_0}(x)) \mid x \in \mathbb{R}^+\}$$

is mapped onto

$$\left\{\left(x, r^d f_{a_0}\left(\frac{x}{r}\right)\right) \mid x \in \mathbb{R}^+\right\}.$$

Since ϕ fixes $(0,\infty)$ and $(\infty,0)$, it fixes the set of circles going through these two points. This implies there exists $a_1 > 0$ such that, for all $x > 0$,

$$r^d f_{a_0}\left(\frac{x}{r}\right) = f_{a_1}(x).$$

It is necessary that $rx_{a_0}^* = x_{a_1}^*$ and $r^d f_{a_0}\left(\dfrac{rx_{a_0}^*}{r}\right) = f_{a_1}\left(x_{a_1}^*\right)$. In particular,

$$ra_0 f\left(\frac{x_{a_0}^*+1}{a_0}\right) = a_1 f\left(\frac{x_{a_1}^*+1}{a_1}\right), \tag{7.1}$$

and

$$r^d a_0 f\left(\frac{x_{a_0}^*+1}{a_0}\right) = a_1 f\left(\frac{x_{a_1}^*+1}{a_1}\right). \tag{7.2}$$

From (7.1) and (7.2), we get $r^d = r$, a contradiction. $\square$

We arrive at the main result of this section.

Theorem 7.4.2. *The full automorphism group* $\mathrm{Aut}(\mathbb{T}_P(f,g))$ *of a Polster plane* $\mathbb{T}_P(f,g)$ *has dimension 2. Its connected component is the translation group* $\mathbb{R}^2$.

Proof. For ease of notation, we denote $\mathrm{Aut}(\mathbb{T}_P(f,g))$ by Σ. Since Σ contains the group of translations $\mathbb{R}^2$, its dimension is at least 2. On the other hand, Σ cannot have dimension 4 and above, by Theorem 5.1.4. Suppose for a contradiction that $\dim \Sigma = 3$. From Theorem 5.3.3, Σ fixes exactly one point, and the group Φ_d is a subgroup of Σ, for some $d \in \mathbb{R} \cup \{\infty\}$. But this contradicts Lemma 7.4.1.

By Lemma A.2.1, the connected component of Σ is the translation group. The proof is complete. $\square$

Chapter 8

Conclusion

In this chapter we summarise our contributions to the literature. We conclude the thesis with a discussion of future work.

8.1 Contributions

There are a number of new lemmas, theorems, and corollaries in this thesis. Among them, the following are the main results.

In Chapter 3, we proved that toroidal circle planes are topological geometries (cf. Theorem 3.4.5). The topology H is the unique topology for the circle set such that the geometric operations are continuous; with it, the circle set is homeomorphic to $\mathrm{PGL}(2, \mathbb{R})$.

Based on the results from Chapter 3, in Chapter 4 we proved Theorem 4.0.1, which states that the automorphism group of a toroidal circle plane is a Lie group of dimension at most 6 with respect to the compact-open topology.

Theorem 4.0.1 is the basis for us to extend some results on flat Minkowski planes to toroidal circle planes. These are Theorems 5.1.1, 5.1.2, 5.1.3, and 5.1.4 in Chapter 5. Particularly, in Theorems 5.1.3 and 5.1.4, we determined all toroidal circle planes with group dimension at least 4.

In the remainder of Chapter 5, we considered toroidal circle planes with group dimension 3. We first proved that an almost simple group of automorphisms must be isomorphic to the group $\mathrm{PSL}(2, \mathbb{R})$ (cf. Theorem 5.2.6) and determined all possible actions of such a group (cf. Theorem 5.2.8). We then used these results to describe all possible 3-dimensional groups of automorphisms in Theorem 5.3.3.

In Chapter 6, we constructed a family of flat Minkowski planes called *modified strongly* *hyperbolic planes* (cf. Definition 6.2.7 and Theorem 6.0.1). A subfamily of these planes, called *strongly hyperbolic planes* (cf. Definition 6.2.8), has group dimension 3. We determined isomorphism classes and the Klein-Kroll types of strongly hyperbolic planes in Theorems 6.5.8 and 6.5.9, respectively.

In Chapter 7, we established a description of toroidal circle planes admitting Φ_d as a group of automorphisms (cf. Theorem 7.1.7). With this description, we then characterised two particular families of flat Minkowski planes with respect to their automorphism groups. These are strongly hyperbolic planes (constructed in Chapter 6) and Artzy-Groh planes (cf. Example 2.1.7), in Theorems 7.2.1 and 7.3.1, respectively. Based on Theorem 5.3.3 and these two characterisations, we were able to describe the automorphism group of Polster planes (cf. Example 2.1.9) in Theorem 7.4.2.

8.2 Future work

In Chapter 6, we only determined the isomorphism classes of strongly hyperbolic planes. It is hoped that a similar result is achieved for modified strongly hyperbolic planes.

Problem 8.2.1. *Determine isomorphism classes and possible Klein-Kroll types of modified strongly hyperbolic planes.*

The results in Chapter 5 lay the groundwork for a complete classification of toroidal circle planes with 3-dimensional groups of automorphisms. As described in Theorem 5.3.3, only certain groups are possible. One can start from there and construct additional planes admitting these groups.

Problem 8.2.2. *Determine all toroidal circle planes with 3-dimensional group of automorphisms.*

We have proved in Chapter 7 that Polster planes have 2-dimensional groups of automorphisms and are not in the classification above. Nevertheless, the construction has some interesting aspects. For a Polster plane, the convex branch of a circle is non-differentiable at one point. It is still an open problem if there is an upper bound for the number of non-differentiable points on convex branches. Also, the convex branch was modified by translations of $\mathbb{R}^2$. Perhaps modifying the convex branch with other transformations will give rise to new toroidal circle planes.

Problem 8.2.3. *Generalise the construction of convex branches of Polster planes.*

The classfication with respect to the Klein-Kroll types is specific to flat Minkowski planes. One can develop a similar classification for toroidal circle planes.

Problem 8.2.4. *Develop a classification of toroidal circle planes similar to that of flat Minkowski planes with respect to Klein-Kroll types.*

As mentioned in [Ste07], there is one particular Klein-Kroll type with no examples. This suggests a description of new flat Minkowski planes.

Problem 8.2.5. *Determine the existence (and construct examples) of flat Minkowski planes of Klein-Kroll type II.A.15.*

We end the thesis here with a reference to a further list of open problems in this area, [PS01] Section 4.9.

Appendix A

Some results from group theory

This appendix contains results on abstract groups, topological groups and transformation groups. References with related background are [Gal16] for abstract groups, [MZ55], [Pon66], [Bre72], [Str06] for topological groups and transformation groups. Other references are [Sal+95] Chapter 9, and [PS01] Appendix 2.

A.1 Abstract groups

Lemma A.1.1. *Let G be a group with two normal subgroups H and K. Assume $H \cap K = \{id\}$. Then G is isomorphic to a subgroup of $G/H \times G/K$. Furthermore, if $G = HK$, then $G \simeq H \times K$.*

Semi-direct products can be defined either internally or externally; both definitions are equivalent. The definition of external semi-direct products can be found in [Sal+07] 9.4. The following can be interpreted as the definition of internal semi-direct products or a property of external semi-direct products.

Lemma A.1.2. *Let G be a group with a normal subgroup H and a (not necessarily normal) subgroup K. If $H \cap K = \{id\}$ and $G = HK$, then $G \simeq H \rtimes K$.*

A.2 Topological groups

The definition of topological groups can be found in [Bre72] p.1, [Pon66] p.96, or [PS01] p.444. Some results on Lie groups are also included here, since they are topological groups.

In this thesis, we use the covering dimension 'dim'. Background on dimension can be found in [Sal+95] Section 92 and references therein.

Lemma A.2.1 (cf. [Sal+95] 93.12)**.** *If Δ is a closed subgroup of the locally compact, connected group Γ, and if* $\dim \Delta = \dim \Gamma < \infty$*, then* $\Delta = \Gamma$*.*

Lemma A.2.2 (cf. [EW06] p.28)**.** *Connected Lie groups (Lie algebras) of dimension* 1 *or* 2 *are solvable.*

Lemma A.2.3 (cf. [FH91] Exercise 7.11)**.** *Any discrete (0-dimensional) normal subgroup of a connected Lie group G is in the center $Z(G)$.*

'Simple groups' and 'almost simple groups' can have different meaning in different contexts. In this thesis, we use the term 'almost simple' as in the following definition.

Definition A.2.4 (Almost simple (simple, locally simple) Lie group)**.** An *almost simple* Lie group is a connected non-abelian Lie group which does not have nontrivial connected normal subgroups.

Equivalently, a connected non-abelian Lie group is almost simple if its Lie algebra is simple (cf. [Sal+95] 94.20). Any proper normal subgroup in an almost simple Lie group is discrete, and by Lemma A.2.3, is contained in its centre.

Almost simple Lie groups are classified. A classification up to dimension 52 can be found in [Sal+95] 94.33. For low dimension, we have the following, adapted from [PS01] Theorem A2.2.6.

Lemma A.2.5 (Almost simple Lie groups of dimension at most 6)**.** *Let G be an almost simple connected Lie group of dimension $n \leqslant 6$. Then $n \in \{3, 6\}$ and G is locally isomorphic to precisely one of the following simple groups.*

> *n=3:* $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{PSL}(2,\mathbb{R})$.
>
> *n=6:* $\mathrm{PSL}(2,\mathbb{C})$.

Groups locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ are finite covering groups $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ and the universal covering group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ of $\mathrm{PSL}(2,\mathbb{R})$. Groups locally isomorphic to $\mathrm{SO}(3,\mathbb{R})$ are described in the following.

Lemma A.2.6 (cf. [Zhe73] p.35 and p.37)**.** $\mathrm{SU}(2,\mathbb{C})$ *is the universal covering group* *and the double covering group of* $\mathrm{SO}(3,\mathbb{R})$*. In particular, if* G *is locally isomorphic to* $\mathrm{SO}(3,\mathbb{R})$*, then* G *is isomorphic to either* $\mathrm{SO}(3,\mathbb{R})$ *or* $\mathrm{SU}(2,\mathbb{C})$*.*

More references on the theory of covering spaces and covering groups are [Hat02] Section 1.3, [Bre72] Section 1.9, [Sal+95] 94.2, [PS01] p.449.

A.3 Transformation groups

The definition of transformation groups can be found in [MZ55] p.195, [Bre72] p.32, [Sal+95] Definition 96.1, or [PS01] p.452.

To prove Theorem 4.0.1 in Chapter 4, we applied a corollary of Szenthe's Theorem. A proof of this theorem was first presented by Szenthe [Sze74]. However, there was a gap in this proof. According to [HK15b], the gap was fixed independently and simultaneously by Antonyan and Dobrowolski [AD15], Hofmann and Kramer [HK15b], [HK15a], and A. A. George Michael [Geo12]. More comments on the history of this problem can be found in [HK15b], or [HM13] p. 608. A weaker version of this theorem can be found in [Sal+95] 96.14. The following is the original statement.

Lemma A.3.1 (Szenthe's Theorem)**.** *Let a* σ*-compact group* G *with compact* G/G^o *be an effective and transitive topological transformation group of a locally compact and locally contractible space* X*. Then* G *is a Lie group and* X *is homeomorphic to a coset space of* G*.*

What we used in Chapter 4 is the following corollary of Lemma A.3.1.

Lemma A.3.2 (cf. [PS01] Theorem A2.3.4)**.** *If* G *is a locally compact effective transformation group on a surface, then* G *is a Lie group.*

Results in Chapter 5 rely on arguments with the dimension of orbits and stabilisers, which are based on the following.

Lemma A.3.3 (cf. [Sal+95] 96.10, [PS01] Theorem A.2.3.6)**.** *If the Lie group* G *acts on a manifold* M*, then*

$$\dim G = \dim G_p + \dim G(p),$$

where G_p *and* $G(p)$ *are the stabiliser and orbit, respectively, of the point* $p \in M$*.*

By definition, groups of automorphisms of toroidal circle planes have induced actions on the sets $\mathcal{G}^{\pm}$, which are homeomorphic to $\mathbb{S}^1$. It is then important to know how these groups act on $\mathbb{S}^1$ and $\mathbb{R}$. The following is helpful for this purpose.

Lemma A.3.4 (cf. [Sal+95] 96.29)**.** *Consider a connected group Γ acting effectively on $\mathbb{R}$ or $\mathbb{S}^1$ (the only connected 1-manifolds).*

(a) If Γ has no fixed point, then Γ is transitive.

(b) Any non-trivial compact subgroup of Γ acts freely on $\mathbb{S}^1$; it cannot act on $\mathbb{R}$.

(c) If Γ is compact, then $\Gamma = id$, or $\Gamma \cong SO(2, \mathbb{R})$ and Γ is sharply transitive on $\mathbb{S}^1$.

As a consequence of Lemma A.3.4, we only have two possiblities for the action of the rotation group $\mathrm{SO}(2, \mathbb{R})$ on $\mathbb{S}^1$.

Lemma A.3.5. $\mathrm{SO}(2, \mathbb{R})$ *acts either trivially or transitively on* $\mathbb{S}^1$*.*

The following lemma describes all possible transitive and effective actions of transformation groups on 1-manifolds. We refer to this result as Brouwer's Theorem throughout the thesis. Although it is named after L. E. J. Brouwer, this theorem is different from the Brouwer's fixed-point theorem in topology. In some sources, a weaker version (for Lie groups) is stated, without a name, as a consequence of a result by Sophus Lie on Lie algebras, cf. [Nav11], [Ghy01] p.348 and [OV93] Theorem 2.1 p.218. According to [Sal+95] 96.30, this theorem is proved by Brouwer [Bro09], and a sketch of proof is provided there.

Lemma A.3.6 (Brouwer's Theorem)**.** *Let G be a locally compact, connected, effective and transitive transformation group on a connected 1-dimensional manifold M. Then G has dimension at most 3.*

(a) If $M \cong \mathbb{S}^1$, then G is isomorphic and acts equivalently to the rotation group $\mathrm{SO}(2, \mathbb{R})$ *or a finite covering group* $\mathrm{PSL}^{(k)}(2, \mathbb{R})$ *of the projective group* $\mathrm{PSL}(2, \mathbb{R})$*.*

(b) If $M \cong \mathbb{R}$, then G is isomorphic and acts equivalently to $\mathbb{R}$, the connected component L_2 *of the affine group of $\mathbb{R}$, or the simply connected covering group of* $\mathrm{PSL}(2, \mathbb{R})$*.*

Appendix B

Some results on real functions

In this appendix, we summarise some results on real functions that are used in Chapters 6 and 7. Section B.1 contains results on continuous functions. Section B.2 contains general properties of convex functions. Section B.3 deals with one-sided derivatives of convex functions.

Throughout this appendix, $I \subset \mathbb{R}$ will denote a nondegenerate interval.

B.1 Continuous functions

Lemma B.1.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function. If*

$$\lim_{x \to 0+} f(x) = +\infty,$$

then

$$\liminf_{x \to 0+} f'(x) = -\infty.$$

Proof. It is easy to check $\liminf_{x\to 0+} f'(x) \neq +\infty$. Suppose for a contradiction that $\liminf_{x\to 0+} f'(x)$ is finite. Then there exists $M \in \mathbb{R}$ such that $f'(x) > M$ for all $x \in (0, +\infty)$. Let $L < M$ and fix $x_0 > 0$. Since $f'(x_0) > L$ and

$$\lim_{x \to 0+} \frac{f(x) - f(x_0)}{x - x_0} = -\infty,$$

by the IVT, there exists $x_1 \in (0, x_0)$ such that

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = L.$$

By the MVT, there exists $x^* \in (x_1, x_0)$ such that $f'(x^*) = L < M$, a contradiction. $\square$

We recall the definitions of one-sided derivatives. The *left derivative* and the *right derivative* of a function f at a are defined by

$$f'_-(a) = \lim_{x \to a-} \frac{f(x) - f(a)}{x - a},$$

and

$$f'_+(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a},$$

if these limits exist. The analogue of Rolle's Theorem for one-sided derivatives can be found in [Art64] Theorem 1.2. The following is the analogue of the MVT for one-sided derivatives.

Lemma B.1.2 (cf. [Art64] Theorem 1.3)**.** *Let f be defined and continuous on $a \leqslant x \leqslant b$ and have one-sided derivatives in the interior. Then there exists a value ξ in the interior such that*

$$\frac{f(b) - f(a)}{b - a}$$

lies between $f'_-(\xi)$ and $f'_+(\xi)$.

Lemma B.1.3 (cf. [MV86] Theorem 1)**.** *Let f be a continuous function on $[a, b]$. If for each $x \in (a, b)$ one of the one-sided derivatives $f'_+(x)$ or $f'_-(x)$ exists, and is nonnegative (possibly $+\infty$), then f is increasing.*

B.2 Convex functions

Definitions of convex functions and midpoint convex functions can be found in [NP06].

Lemma B.2.1 ([NP06] Theorem 1.1.4)**.** *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is (strictly) convex if and only if f is (strictly) midpoint convex.*

Lemma B.2.2. *Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a continuous, strictly decreasing function. Then h is convex (resp. strictly convex) if and only if $h(x) - h(x + u)$ is decreasing (resp. strictly decreasing) for all $u > 0$.*

Proof. 1) Assume h is strictly convex. Fix $u > 0$. Suppose $0 < x < y$. We have $y, x + u \in (x, y + u)$. Then there exists $\alpha, \beta \in (0, 1)$ such that

$$x + u = \alpha \cdot x + (1 - \alpha) \cdot (y + u),$$

$$y = \beta \cdot x + (1 - \beta) \cdot (y + u).$$

Direct calculation shows that

$$\alpha = \frac{x - y}{x - y - u},$$

$$\beta = \frac{-u}{x-y-u},$$

so that $\alpha + \beta = 1$. From strict convexity,

$$h(x+u) < \alpha h(x) + (1-\alpha)h(y+u),$$

$$h(y) < \beta h(x) + (1-\beta)h(y+u).$$

Adding two inequalities together, we get

$$h(x+u) + h(y) < h(x) + h(y+u),$$

so that

$$h(y) - h(y+u) < h(x) - h(x+u).$$

Hence $h(x) - h(x+u)$ is strictly decreasing.

2) Assume for all $u > 0$, $h(x) - h(x+u)$ is strictly decreasing. Let $0 < x < y$.

Because $0 < x < (x+y)/2$, for $u := (x+y)/2 - x > 0$, we have

$$h(x) - h(x+u) > h\left(\frac{x+y}{2}\right) - h\left(\frac{x+y}{2} + u\right).$$

This gives

$$h(x) - h\left(\frac{x+y}{2}\right) > h\left(\frac{x+y}{2}\right) - h(y).$$

so that

$$h\left(\frac{x+y}{2}\right) < \frac{h(x)+h(y)}{2}.$$

Hence h is strictly midpoint convex. Since h is continuous, it is then strictly convex. □

Lemma B.2.3 (cf. [NP06] Theorem 1.3.1)**.** *Let $f : I \to \mathbb{R}$ be a function. Then f is convex (resp. strictly convex) if and only if for every $x_1 > 0$, the function $g : I \backslash \{x_1\} \to \mathbb{R}$ defined by*

$$g(x) = \frac{f(x) - f(x_1)}{x - x_1},$$

is nondecreasing (resp. increasing).

Lemma B.2.4 (cf. [Bul03] p.26 Corollary 3 (b), [LS15] p.160 Exercise 10.83)**.** *Let $f : I \to \mathbb{R}$ be a function. If f is convex but not strictly convex on the interval I, then I has a subinterval on which f is affine.*

B.3 Differentiability of convex functions

Lemma B.3.1 (cf. [RV73] Theorem B p.5)**.** *Let $f : I \to \mathbb{R}$ be a function. If f is (strictly) convex, then $f'_-(x)$ and $f'_+(x)$ exist and are (strictly) increasing on the interior of I.*

As a side note, a partial converse of Lemma B.3.1 can be found in [Art64] p.8 Problem E.

Lemma B.3.2 (cf. [BT89] Proposition A.38 (e))**.** *Let $f : I \to \mathbb{R}$ be a convex function. Then f'_- is left-continuous at every interior point of I.*

Lemma B.3.3. *Let $f : I \to \mathbb{R}$ be a convex function. If f is not differentiable then discontinuities of f'_- are jump discontinuities.*

Sketch of Proof. The left derivative f'_- must be increasing, and increasing functions can only have jump discontinuities. □

Lemma B.3.4 (cf. [Vak11] p.226 Exercise 6.13.4)**.** *Let $f : I \to \mathbb{R}$ be a convex and differentiable function. Then f' is increasing, and continuous on I.*

Lemma B.3.5 (cf. [RV73] p.5,6,7)**.** *Let $f : I \to \mathbb{R}$ be a (strictly) convex function. For $x, y \in I, x < y$, we have*

(i) $f'_-(x) \leqslant f'_+(x) \leqslant f'_-(y) \leqslant f'_+(y)$*. The second inequality is strict if f is strictly convex.*

(ii) $\lim_{y \to x+} f'_+(y) = f'_+(x)$ *and* $\lim_{y \to x-} f'_+(y) = f'_-(x)$.

(iii) $\lim_{y \to x+} f'_-(y) = f'_+(x)$ *and* $\lim_{y \to x-} f'_-(y) = f'_-(x)$.

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